MAT401: Polynomial Equations and Fields — Exam Practice

Topics: Rings, Fields, Galois Theory.

Ring Theory

Question 1. Let R be a commutative ring and define the *nilradical of R* by

$$\mathrm{Nil}(R) = \{r \in R : r^n = 0 \text{ for some } n \in \mathbb{N}\}$$

namely, the ideal of nilpotent elements in R. Prove that $\mathrm{Nil}(R/\mathrm{Nil}(R))$ is trivial.

Question 2. Let R be a finite, commutative ring with unity. Prove that every prime ideal is maximal.

Question 3. Let R be a ring with unity such that $|R| = 401^2$. Prove that R is commutative.

Question 4. Show by example that there is a commutative ring R and a maximal ideal $I \trianglelefteq R$ that is *not* prime. *Note:* Of course, R cannot contain unity.

Question 5. If R is a commutative ring for which every proper ideal is prime, must R be a field?

Field Theory

Question 1. Let F be a field for which $\mathrm{char}(F) = p$ for a prime p, and $q(x) \in F[x]$ an irreducible polynomial of degree n. Prove that $K := F[x]/\langle q(x)\rangle$ is a field of characteristic p with p^n elements. *Note:* This is a way that we can construct fields of order p^n for any prime p and $n \in \mathbb{N}$.

Question 2. Let p and q be distinct primes.

(a) Prove that $\mathbb{Q}(\sqrt{p}) \cong \mathbb{Q}(\sqrt{q})$ if and only if $p = q$.

(b) Prove that $\mathbb{Q}(\sqrt{p} + \sqrt{q}) = \mathbb{Q}(\sqrt{p}, \sqrt{q})$.

Question 3. Let F be a field, $q(x) \in F[x]$ a polynomial of degree n, and K a splitting field of $q(x)$ over F. Prove that $[K : F] \leq n!$

Question 4. Let F be a field, and $\alpha \in F$ algebraic over F for which $[F(\alpha) : F] = p$, for some prime p. Prove that for any $k \in \{1, \ldots, p-1\}$, we have $F(\alpha^k) = F(\alpha)$. *Hint:* Use the degree lemma.

Question 5. Let K be an extension of a field F for which $[K : F] = 401$. Prove that there is $\alpha \in K$ for which $K = F(\alpha)$.

Galois Theory

Question 1. Let K be an extension of a field F.

(a) If M is an intermediate extension, namely $F \subseteq M \subseteq K$, state and prove a relationship between $\mathrm{Gal}(K/M)$ and $\mathrm{Gal}(K/F)$.

(b) If $H_1 \leq H_2$ are subgroups of $\mathrm{Gal}(K/F)$, state and prove a relationship between the fixed fields of H_1 and H_2.

Question 2. Prove that $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}+\sqrt{5})/\mathbb{Q}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Question 3. Compute the Galois group of $f(x) = x^4 - 4x^2 + 2$ over $\mathbb{Q}$.

Question 4. Let $\alpha = i\frac{\sqrt{3}}{2} - \frac{1}{2}$ and $\beta = \sqrt[3]{2}$. Prove that $\mathrm{Gal}(\mathbb{Q}(\alpha, \beta)/\mathbb{Q}) \cong S_3$.

Question 5. Compute the Galois group of $f(x) = x^5 - 2$ over $\mathbb{Q}$.