There were a few common mistakes with these questions in particular, so I've drafted some more detailed sample solutions. Please let me know if you have any questions.

Question 2. Set $\alpha = \sqrt[3]{1+\sqrt{3}}$. Determine $[\mathbb{Q}(\alpha);\mathbb{Q}(\sqrt{3})]$ and a basis for the extension.

Proof. Write $x = \alpha$ so that $x^3 = 1+\sqrt{3}$, and hence $m_1(x) = x^3 - 1 - \sqrt{3} \in \mathbb{Q}(\sqrt{3})[x]$ is a monic polynomial containing α as a root. Note that by the monotonicity of x^3, we have that α is the only root of $m_1(x)$ over $\mathbb{R}$ and hence over $\mathbb{Q}(\sqrt{3})$. It will then suffice to show $\alpha \notin \mathbb{Q}(\sqrt{3})$ to show $m_1(x)$ is irreducible, and hence the minimal polynomial. Suppose towards a contradiction $\alpha \in \mathbb{Q}(\sqrt{3})$. Then $\mathbb{Q}(\sqrt{3})$ is a field containing $\mathbb{Q}$ and α, so by minimality we have $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{3})$, namely $\mathbb{Q}(\alpha) \leq \mathbb{Q}(\sqrt{3})$ as $\mathbb{Q}$-vector spaces. However, we claim that $[\mathbb{Q}(\alpha);\mathbb{Q}] = 6$. Indeed, to see this, write $x = \alpha$ and hence $x^3 - 1 = \sqrt{3}$, so that $m_2(x) = x^6 - 2x^3 - 2 \in \mathbb{Q}[x]$ is a monic polynomial containing α as a root, and moreover $m_2(x)$ is irreducible over $\mathbb{Q}$ by Eisenstien with $p = 2$. Thus, $m_2(x)$ is the minimal polynomial for α over $\mathbb{Q}$, and hence

$$[\mathbb{Q}(\alpha);\mathbb{Q}] = \deg(m_2(x)) = 6$$

which is a contradiction as $[\mathbb{Q}(\sqrt{3});\mathbb{Q}] = 2$. Thus, $\alpha \notin \mathbb{Q}(\sqrt{3})$ and hence $m_1(x)$ is the minimal polynomial for α over $\mathbb{Q}(\sqrt{3})$. Finally, we have that $[\mathbb{Q}(\alpha);\mathbb{Q}(\sqrt{3})] = \deg(m_2(x)) = 3$, and a basis is given by $\beta = \{1, \alpha, \alpha^2\}$. $\square$

Question 3. If $a, b \in \mathbb{R}$ for which $z = a + bi$ is algebraic over $\mathbb{Q}$, then a and b are algebraic over $\mathbb{Q}$.

Proof. As z is algebraic over $\mathbb{Q}$, there is a polynomial $q(x) \in \mathbb{Q}[x]$ for which $q(z) = 0$. Recall that we have $q(\overline{z}) = 0$ by Assignment 3, and note that $p(x) = x^2 + 1 \in \mathbb{Q}[x]$ satisfies $p(i) = 0$. Thus each of $z, \overline{z}, i$ are algebraic over $\mathbb{Q}$, and hence over any extension of $\mathbb{Q}$. Finally, $E = \mathbb{Q}(z, \overline{z}, i)$ is a finite, and hene algebraic, extension of $\mathbb{Q}$ by the degree Lemma, as

$$[\mathbb{Q}(z, \overline{z}, i) : \mathbb{Q}] \leq \deg(p(x)) \deg(q(x))^2 < \infty$$

Thus, a and b are algebraic over $\mathbb{Q}$ as $a = \frac{z+\overline{z}}{2} \in E$ and $b = \frac{x-\overline{z}}{2i} \in E$. $\square$