MAT401: Polynomial Equations and Fields — Midterm Practice

Topics: Rings, Ideals, Quotients, Fields, Homomorphisms.

Question 1. Denote by $C(\mathbb{R})$ the ring of continuous functions $f : \mathbb{R} \to \mathbb{R}$ under pointwise addition and multiplication. For points $a_1, \dots, a_n \in \mathbb{R}$, define

$$Z(a_1, \dots, a_n) = \{f \in C(\mathbb{R}) : f(a_i) = 0 \ \forall i = 1, \dots, n\}$$

Prove that $Z(a_1, \dots, a_n) \trianglelefteq C(\mathbb{R})$. Is $Z(a_1, \dots, a_n)$ necessarily a prime ideal for any choice of points?

Question 2. Let R be an integral domain. Prove that $R[x]$, the ring of polynomials in the variable x with coefficients over R, is also an integral domain.

Question 3. If R is a ring and K is a field for which there is a ring isomorphism $f : K \to R$, prove that R is a field, and hence f is a field isomorphism.

Question 4. Let R be a ring, with I and J ideals of R.

(a) Prove the ideal test: A nonempty subset I of R is an ideal of R if and only $x - y \in I$ and $ax \in I$ for all $x, y \in I$ and $a \in R$. (This essentially combines the subring test with the definition of an ideal)

(b) Show that $I + J := \{x + y : x \in I, y \in J\}$ is an ideal of R.

(c) Show that $IJ = \{\sum_{i=1}^{n} x_i y_i : n \in \mathbb{N}, x_i \in I, y_i \in J\}$ is an ideal of R.

Question 5. Fix a prime $p \in \mathbb{N}$, and recall that $\sqrt{p} \notin \mathbb{Q}$. Define the ideal $I_p = \langle x^2 - p \rangle$.

(a) Prove that $\mathbb{Q}[x]/I_p$ is a field.

(b) Prove that $\mathbb{Q}[x]/I_p \cong \mathbb{Q}(\sqrt{p}) = \{a + b\sqrt{p} : a, b \in \mathbb{Q}\}$. *Hint: Associate* $[x]$ *and* $\sqrt{p}$*.*

(c) **Bonus:** Prove that $\mathbb{Q}[x]/I_p \cong \mathbb{Q}[x]/I_q$ if and only if $p = q$. (This is diving into material we'll see later on in the course, so don't worry about it too much now)

Question 6. Let R, S be commutative rings with unity, and $f : R \to S$ a surjective homomorphism.

(a) Prove that for any ideals $I \trianglelefteq R$ and $J \trianglelefteq S$, we have $f(I) \trianglelefteq S$ and $f^{-1}(J) \trianglelefteq R$.

(b) Suppose now that $f : R \to S$ is an isomorhpism, and consider the corresponding induced maps $F : R/I \to S/f(I)$ and $G : R/f^{-1}(J) \to S/J$ by

$$F(r + I) = f(r) + f(I) \quad \text{and} \quad G(r + f^{-1}(J)) = f(r) + J$$

Prove that F and G are isomorphisms.

(c) **Bonus:** Suppose $I \trianglelefteq R, J \trianglelefteq S$ are prime. Are either of $f(I) \trianglelefteq S, f^{-1}(J) \trianglelefteq R$ prime? What if we replace prime with maximal? (Don't worry too much about this one, it is much easier to prove with an additional result you don't know yet)

Question 7. Let R and S be rings with unity. Prove that $R \oplus S$ cannot be a field.

Question 8. An ideal I in a ring R with unity is called *reducible* if there are ideals $J_1, J_2 \trianglelefteq R$ for which $I = J_1 \cap J_2$ and $I \subsetneq J_1, J_2$, and is called irreducible otherwise. Prove that an ideal I of R is prime if and only if it is irreducible. (This furthers illustrates the idea that prime ideals behave like prime numbers)

Question 9. Let $\mathbb{R}[x, y]$ be the ring of polynomials in the variables x and y over $\mathbb{R}$. Prove that $\mathbb{R}[x, y]/\langle y \rangle \cong \mathbb{R}[x]$. *Hint: Use the first isomorphism theorem.*

Question 10. Let R be a ring, and $I, J \trianglelefteq R$ for which $I \subseteq J$. Prove that $I \trianglelefteq J$ and use the map $f : R/I \to R/J$ defined by $f(x + I) = x + J$ to show that

$$J/I \trianglelefteq R/I \quad \text{and} \quad R/J \cong (R/I)/(J/I)$$

You'll have to argue that the map is a well-defined homomorphism first, then look at the kernel and make use of a theorem.