Contents

1 Integration

1.1 The Darboux Integral

Definition 1.1.1: Partitions

Let $[a, b]$ be an interval. An ordered collection of points $P = \{x_0, \ldots, x_n\}$ is called a partition of $[a, b]$ if each $x_i \in [a, b]$ and $x_0 = a, x_n = b$. Given an index $i \in \{1, \ldots, n\}$ we say that $[x_{i-1}, x_i]$ is the i^{th} subinterval of P and define $\Delta x_i = x_i - x_{i-1}$ to be its length. We define the norm of a partition $P = \{x_0, \ldots, x_n\}$, denoted $\|P\|$, to be the length of its longest subinterval, namely

$$\|P\| = \max_{1 \leq i \leq n} \Delta x_i$$

We denote the collection of partitions of $[a, b]$ as $\mathcal{P}_{[a,b]}$. Given $P, Q \in \mathcal{P}_{[a,b]}$, we say Q is a refinement of P, or Q refines P, if $P \subseteq Q$. Finally, we say that $P \cup Q$ is the common refinement of P and Q.

Definition 1.1.2: Upper and Lower Darboux Sums

Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $P = \{x_i\}_{i=1}^n \in \mathcal{P}_{[a,b]}$, we define the upper and lower Darboux sums of f on P, denoted $U(f, P)$ and $L(f, P)$ respectively, by

$$U(f, P) = \sum_{i=1}^{n} \left[\sup_{x \in [x_{i-1}, x_i]} f(x) \right] \Delta x_i \quad \text{and} \quad L(f, P) = \sum_{i=1}^{n} \left[\inf_{x \in [x_{i-1}, x_i]} f(x) \right] \Delta x_i$$

Additionally, we define the upper and lower Darboux integrals of f by

$$U(f) = \inf_{P \in \mathcal{P}_{[a,b]}} U(f, P) \quad \text{and} \quad L(f) = \sup_{P \in \mathcal{P}_{[a,b]}} L(f, P)$$

Lemma 1.1.3: Properties of Partitions

Let P and Q be partitions of an interval $[a, b]$. The following hold

1. $L(f, P) \leq U(f, P)$.
2. If P is a refinement of Q, namely $P \subseteq Q$, then
$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$
3. $L(f, P) \leq U(f, Q)$.

Proof. Let's write $P = \{x_0, \ldots, x_n\}$ where $x_0 = a$ and $x_n = b$.

1. This follows immediately as $\inf_{x \in [x_{i-1}, x_i]} f(x) \leq \sup_{x \in [x_{i-1}, x_i]} f(x)$ for any subinterval.

2. It suffices to prove this in the case where Q refines P by a single point, namely $Q = P \cup \{c\}$ where $c \notin P$. As $c \notin P$, there is an index $j \in \{1, \ldots, n\}$ for which $c \in [x_{j-1}, x_j]$. Note that as

$[x_{j-1}, c], [c, x_j] \subseteq [x_{j-1}, x_j]$, we have by properties of supremums and infimums that

$$\begin{aligned}\sup_{x\in[x_{j-1},x_j]} f(x)\Delta x_j &= \sup_{x\in[x_{j-1},x_j]} f(x)(x_j - c) + \sup_{x\in[x_{j-1},x_j]} f(x)(c - x_{j-1}) \\ &\geq \sup_{x\in[x_{j-1},c]} f(x)(x_j - c) + \sup_{x\in[c,x_j]} f(x)(c - x_{j-1})\end{aligned}$$

$$\begin{aligned}\inf_{x\in[x_{j-1},x_j]} f(x)\Delta x_j &= \inf_{x\in[x_{j-1},x_j]} f(x)(x_j - c) + \inf_{x\in[x_{j-1},x_j]} f(x)(c - x_{j-1}) \\ &\leq \inf_{x\in[x_{j-1},c]} f(x)(x_j - c) + \inf_{x\in[c,x_j]} f(x)(c - x_{j-1})\end{aligned}$$

The result follows as P and Q are identical on all other subintervals $[x_{i-1}, x_i]$ for $i \neq j$.

3. By considering the common refinement $P \cup Q$ of P and Q, we can apply 2 and the result follows. □

Definition 1.1.4: Darboux Integrability

A bounded function $f : [a, b] \to \mathbb{R}$ is called Darboux integrable if $L(f) = U(f)$, and the Darboux integral of f is defined to be their common value.

This definition should hopefully seem natural, in general, we have that $U(f, P)$ is an over approximation of the integral and $L(f, P)$ and under approximation. We would want the integral to be the value that our over and under approximations "converge" to. Note that this same thought motivates why statement 2 in Lemma 1.1.3 is true, when we add more points to our partition, our over and under approximations become more accurate.

Lemma 1.1.5: Integrability Criterion

A function $f : [a, b] \to \mathbb{R}$ is Darboux integrable if and only if for every $\varepsilon > 0$, there is a partition $P \in \mathcal{P}_{[a,b]}$ such that $U(f, P) - L(f, P) < \varepsilon$. Moreover, the Darboux integral I is the unique value such that $L(f, P) \leq I \leq U(f, P)$ for all $P \in \mathcal{P}_{[a,b]}$.

This should also feel relatively natural. Given a partition P, the quantity $U(f, P) - L(f, P)$ represents the "error" of our approximations using the partition P. Thus, if we can always choose a partition to make our error arbitrarily small, it follows that our over and under approximations converge.

1.2 Properties of The Darboux Integral

Lemma 1.2.1

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then

$$\sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x) \geq \sup_{x\in[a,b]} |f(x)| - \inf_{x\in[a,b]} |f(x)|$$

Proof. Recall that we have proven for a bounded function f, we can write the oscillation as

$$\sup_{x\in[a,b]} |f(x) - f(y)| = \sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x)$$

Thus, given $x, y \in [a, b]$, we have $||f(x)| - |f(y)|| \leq |f(x) - f(y)|$ by the reverse triangle inequality, and so it follows that

$$\begin{aligned}\sup_{x\in[a,b]} |f(x)| - \inf_{x\in[a,b]} |f(x)| = \sup_{x\in[a,b]} ||f(x)| - |f(y)|| &\leq \sup_{x\in[a,b]} |f(x) - f(y)| \\ &= \sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x)\end{aligned}$$

□

Theorem 1.2.2: Linearity

Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions, then for any $\alpha \in \mathbb{R}$, the function $\alpha f + g : [a, b] \to \mathbb{R}$ is integrable and moreover

$$\int_a^b (\alpha f + g) = \alpha \int_a^b f + \int_a^b g$$

Proof. This is one of the situations were it is more convenient to use the Riemman integral rather than the Darboux integral. As they are equivalent, we are free to use whichever we like. In this case, the result follows almost identically to the proofs of the linearity limit laws when we use the Riemann integral. □

Theorem 1.2.3: Additivity of Domain

If $f : [a, b] \to \mathbb{R}$ is integrable on $[a, c]$ and on $[c, b]$ for some $c \in (a, b)$, then f is integrable and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof. We first show that f is integrable. Let $\varepsilon > 0$, as $f_1 := f|_{[a,c]}$ and $f_2 := f_{[c,b]}$ are integrable, there are partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ such that

$$U(f_1, P_1) - L(f_1, P_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(f_2, P_2) - L(f_2, P_2) < \frac{\varepsilon}{2}$$

Define $P = P_1 \cup P_2$ which is a partition of $[a, b]$ by construction. It then follows that

$$\begin{aligned} U(f, P) - L(f, P) &= (U(f_1, P_1) + U(f_2, P_2)) - (L(f_1, P_1) + L(f_2, P_2)) \\ &= (U(f_1, P_1) - L(f_1, P_1)) + (U(f_2, P_2) - L(f_2, P_2)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, f is integrable. Now let's prove the integral equality. We aim to use uniqueness as stated in Lemma 1.1.5. Let P be a partition of $[a, b]$, and set $\widetilde{P} = P \cup \{c\}$ so that we may write $\widetilde{P} = P_1 \cup P_2$ where P_1 and P_2 are partitions of $[a, c]$ and $[c, b]$ respectively. By Lemma 1.1.5 we know that

$$L(f_1, P_1) \leq \int_a^c f \leq U(f_1, P_1) \quad \text{and} \quad L(f_2, P_2) \leq \int_c^b f \leq U(f_2, P_2)$$

By adding these two inequalities together, and applying Lemma 1.1.3, we have

$$\begin{aligned} L(f_1, P_1) + L(f_2, P_2) \leq \int_a^c f + \int_c^b f \leq U(f_1, P_1) + U(f_2, P_2) \\ L(f, \widetilde{P}) \leq \int_a^c f + \int_c^b f \leq U(f, \widetilde{P}) \end{aligned}$$

Finally, as $L(f,P) \leq L(f,\widetilde{P}) \leq U(f,\widetilde{P}) \leq U(f,P)$, the result follows by Lemma 1.1.5. □

Theorem 1.2.4: Monotonicity

If $f, g : [a,b] \to \mathbb{R}$ are integrable functions such that $f(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f \leq \int_a^b g$$

Proof. Define $h := g - f$, by construction, $h(x) \geq 0$ for all $x \in [a,b]$ and by linearity h is integrable. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$. On each subinterval $[x_{i-1}, x_i] \subseteq [a,b]$, we have

$$\inf_{x \in [x_{i-1}, x_i]} f(x) \geq \inf_{x \in [a,b]} f(x) \geq 0 \quad \implies \quad 0 \leq \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x) = L(f,P)$$

Thus, as $L(f,P) \geq 0$ for all partitions P, it follows by using linearity once again that

$$0 \leq \sup_P L(f,P) = \int_a^b g = \int_a^b g - \int_a^b f \quad \implies \quad \int_a^b f \leq \int_a^b g$$ □

Theorem 1.2.5: Subnormality

If $f : [a,b] \to \mathbb{R}$ is integrable, then $|f| : [a,b] \to \mathbb{R}$ is integrable and

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Proof. To prove that $|f|$ is integrable, we can make use of Lemma 1.2.1. Let $\varepsilon > 0$, by the integrability of f there is a partition P of $[a,b]$ for which

$$U(f,P) - L(f,P) < \varepsilon$$

We claim that the same partition works for $|P$. Indeed, write $P = \{x_0, \dots, x_n\}$, by Lemma 1.2.1

$$\begin{aligned} U(|f|,P) - L(|f| < P) &= \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} |f(x)| - \inf_{x \in [x_{i-1}, x_i]} |f(x)| \right) \Delta x_i \\ &\leq \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right) \Delta x_i \\ &= U(f,P) - L(f,P) \\ &< \varepsilon \end{aligned}$$

Thus, $|f|$. To prove the integral equality, note that $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a,b]$. By using monotonicity and linearity, it follows that

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f| \quad \implies \quad \left| \int_a^b f \right| \leq \int_a^b |f|$$ □

1.3 Sufficient Conditions for Integrability

In this section we will present several sufficient conditions for the Darboux integrability of a bounded function. It's worth noting that there does exist a necessary and sufficient condition for integrability, though we do not have the tools to state or prove it.

Proposition 1.3.1: Monotonicity

If $f : [a, b] \to \mathbb{R}$ is monotone, then f is integrable.

Proof. We can start by making a useful observation and first assume f is monotonically increasing. Given a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$, on each subinterval $[x_{i-1}, x_i]$, as f is increasing, we have

$$\sup_{x\in[x_{i-1},x_i]} f(x) = f(x_i) \quad \text{and} \quad \inf_{x\in[x_{i-1},x_i]} f(x) = f(x_{i-1})$$

If we were to sum over the differnces between the supremum and infimum on each subinterval, all but $f(b)$ and $-f(a)$ would cancel out. Thet trick from here is to choose a partition whose length is about some quantity involving ε and $f(b) - f(a)$. Indeed, given $\varepsilon > 0$, pick a partition P of $[a, b]$ for which

$$\|P\| < \frac{\varepsilon}{f(b) - f(a)}$$

Note that if $f(b) = f(a)$, we have that f is constant and hence integrable, so we may assume that $f(b) > f(a)$. Write $P = \{x_0, \dots, x_n\}$, then from our observation above, it follows that

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} \left(\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \right) \Delta x_i \\
&\leq \|P\| \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}) \\
&= \|P\|(f(b) - f(a)) \\
&< (f(b) - f(a)) \frac{\varepsilon}{f(b) - f(a)} \\
&= \varepsilon
\end{aligned}$$

Finally, if f is monotonically decreasing, we can apply the above to $g := -f$ and use linearity. $\square$

Theorem 1.3.2: Continuity

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable.

Proof. Let $\varepsilon > 0$, recall that a continuous function on a compact set is necessarily uniformly continuous, and so there is $\delta > 0$ such that for all $x, y \in [a, b]$

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a}$$

Let P be a partition of $[a, b]$ such that $\|P\| < \delta$ and write $P = \{x_0, \dots, x_n]$. As f is continuous and each $[x_{i-1}, x_i]$ is compact, by the Extreme Value Theorem, there is $t_i, s_i \in [x_{i-1}, x_i]$ such that

$$\sup_{x\in[x_{i-1},x_i]} f(x) = f(t_i) \quad \text{and} \quad \inf_{x\in[x_{i-1},x_i]} f(x) = f(s_i)$$

Moreover, as $s_i, t_i \in [x_{i-1}, x_i]$, we have $|s_i - t_i| \leq \Delta x_i < \delta$ and thus

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} (f(t_i) - f(s_i))\Delta x_i < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i \\ &= \frac{\varepsilon}{b-a}(b-a) \\ &= \varepsilon \end{aligned}$$

□

Proposition 1.3.3: Extension

If $f, g : [a,b] \to \mathbb{R}$ are bounded functions such that f is integrable on $[t,b]$ for all $t \in (a,b)$ and g is integrable on $[a,s]$ for all $s \in (a,b)$, then f and g are integrable and moreover

$$\int_a^b f = \lim_{t \to a^+} \int_t^b f \quad \text{and} \quad \int_a^b g = \lim_{s \to b^-} \int_a^s g$$

Proof. Let $\varepsilon > 0$, let $\omega = \sup_{x,y \in [a,b]} |f(x) - f(y)|$ denote the oscillation of f. Note that if $\omega = 0$, then f is a constant function and the result follows immediately. Assuming $\omega \neq 0$, define

$$\delta = \min\left\{\frac{\varepsilon}{2\omega}, \frac{b-a}{2}\right\} > 0$$

Set $x_1 = a + \delta$. Note that $x_1 \in (a,b)$ by construction, and so by assumption, f is integrable on $[x_1, b]$. It follows that there is a partition Q of $[x_1, b]$ such that

$$U(f,Q) - L(f,Q) < \frac{\varepsilon}{2}$$

Refine Q by $\{a\}$ so that $P := Q \cup \{a\}$ is a partition of $[a,b]$. By construction, it follows that

$$\begin{aligned} U(f,P) - L(f,P) &= \left(\sup_{x \in [a,x_1]} f(x) - \inf_{x \in [a,x_1]} f(x)\right)(x_1 - a) + U(f,Q) - L(f,Q) \\ &< \left(\sup_{x \in [a,b]} f(x) - \inf_{x \in [a,b]} f(x)\right)\delta + \frac{\varepsilon}{2} \\ &\leq \omega\left(\frac{\varepsilon}{2\omega}\right) + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

We now show that the integral of f can be obtained by taking limits. Let $\varepsilon > 0$, as f is bounded, there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Define $\delta = \frac{\varepsilon}{M} > 0$, given $t \in (a, a+\delta)$, by additivity of domain, subnormality and monotonicity, it follows that

$$\left|\int_a^b f - \int_t^b f\right| = \left|\int_a^t f\right| \leq \int_a^t |f| \leq M(t-a) < M\left(\frac{\varepsilon}{M}\right) = \varepsilon$$

Here we have taken for granted the straight forward result that $\int_a^b 1 = b - a$ for any $a < b \in \mathbb{R}$. An almost identical proof will establish the result for g. □

Corollary 1.3.4: Discontinuous Functions

If $f : [a,b] \to \mathbb{R}$ has finitely many points of discontinuity, then f is integrable.

Proof. We can combine the results of the previous two theorems. Enumerate the discontinuities of f as $\{d_1, \ldots, d_n\}$ and write $a = d_0, b = d_{n+1}$. Then we can partition $[a, b]$ into subintervals of the form $[d_{i-1}, d_i]$. For such a subinterval, let $m_i = \frac{1}{2}(d_i + d_{i-1})$ denote the midpoint. For any $t \in (d_{i-1}, m_i)$, we have that f is continuous on $[t, m_i]$ and hence integrable, thus buy Proposition 1.3.3 it follows that f is integrable on $[d_{i-1}, m_i]$. Similarly, for any $s \in (m_i, d_i)$, we have that f is continuous on $[m_i, s]$ and hence integrable, and thus f is integrable on $[m_i, d_i]$ again by Proposition 1.3.3. Finally, as f is integrable on $[d_{i-1}, m_i]$ and $[m_i, d_i]$, it follows by additivity of domain that f is integrable on $[d_{i-1}, d_i]$ for each i, and thus by additivity of domain once again it follows that f is integrable on $[a, b]$. $\square$