二重积分练习

1. 求积分$I=\iint_D\left(x^2+2y\right)\mathrm{d}x\mathrm{d}y$,其中$D$为曲线$y=x^2, y=\sqrt{x}$围成的区域.

Solution.

我们有

$$\begin{aligned}
\iint_D\left(x^2+2y\right)\mathrm{d}x\mathrm{d}y &= \int_0^1\left[\int_{x^2}^{\sqrt{x}}\left(x^2+2y\right)\mathrm{d}y\right]\mathrm{d}x \\
&= \int_0^1\left(x^2\left(\sqrt{x}-x^2\right)+x-x^4\right)\mathrm{d}x \\
&= \int_0^1\left(x+x^{\frac{5}{2}}-2x^4\right) \\
&= \left(\frac{1}{2}x^2+\frac{2}{7}x^{\frac{7}{2}}-\frac{2}{5}x^5\right)\Big|_0^1 \\
&= \frac{27}{70}
\end{aligned}$$

2. 求积分$I=\iint_D\sin y^3\mathrm{d}x\mathrm{d}y$,其中$D$是曲线$y=\sqrt{x}$,直线$y=2$和$x=0$围成的区域.

Solution.

我们有

$$\begin{aligned}
\iint_D\sin y^3\mathrm{d}x\mathrm{d}y &= \int_0^2\left[\int_0^{y^2}\sin y^3\mathrm{d}x\right]\mathrm{d}y \\
&= \int_0^2 y^2\sin y^3\mathrm{d}y \\
&= \frac{1}{3}\int_0^8\sin y^3\mathrm{d}y^3 \\
&= \frac{1-\cos 8}{3}
\end{aligned}$$

3. 求积分$I=\iint_D\left(4-x^2-y^2\right)^{-\frac{1}{2}}\mathrm{d}x\mathrm{d}y$,其中$D$是单位圆$x^2+y^2\leqslant 1$在第一象限的部分.

Solution.

做代换$x=r\cos\theta,y=r\sin\theta$,于是$D'=\left\{(r,\theta):0\leqslant r\leqslant 1,0\leqslant\theta\leqslant\dfrac{\pi}{2}\right\}$.于是

$$\begin{aligned}
\iint_D\left(4-x^2-y^2\right)^{-\frac{1}{2}} &= \iint_D'\left(4-r^2\right)^{-\frac{1}{2}}r\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^1\left[\int_0^{\frac{\pi}{2}}\left(4-r^2\right)^{-\frac{1}{2}}r\mathrm{d}\theta\right]\mathrm{d}r \\
&= \frac{\pi}{2}\int_0^1\frac{r\mathrm{d}r}{\sqrt{4-r^2}} \\
&= \frac{\pi}{4}\int_0^1\frac{\mathrm{d}r^2}{\sqrt{4-r^2}} \\
&= \frac{\pi}{4}\left(-2\sqrt{4-r^2}\right)\Big|_0^1 \\
&= \frac{2-\sqrt{3}}{2}\pi
\end{aligned}$$