

古早计算大赛题目

试证明:球坐标系下Laplace算子为

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

Proof.

我们知道笛卡尔坐标系下的Laplace算子为

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

以及笛卡尔坐标系向球坐标系的变换

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases}$$

考虑函数 $f(x, y, z) : \mathbb{R}^3 \rightarrow \mathbb{R}$, 则有

$$\begin{aligned} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) &= \frac{1}{r^2} \left(2r \frac{\partial f}{\partial r} + r^2 \frac{\partial^2 f}{\partial r^2} \right) = \frac{2}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} \\ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) &= \frac{1}{r^2 \sin \theta} \left(\cos \theta \frac{\partial f}{\partial \theta} + \sin \theta \frac{\partial^2 f}{\partial \theta^2} \right) = \frac{1}{r^2 \tan \theta} \frac{\partial f}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} \end{aligned}$$

而

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial r} \right) = \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r} \right)$$

我们知道

$$\frac{\partial}{\partial r} \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} \right) = \frac{\partial^2 x}{\partial r^2} \cdot \frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2} \cdot \left(\frac{\partial x}{\partial r} \right)^2$$

而

$$\begin{aligned} \frac{\partial x}{\partial r} &= \sin \theta \cos \phi, \quad \frac{\partial^2 x}{\partial r^2} = 0 \\ \frac{\partial y}{\partial r} &= \sin \theta \sin \phi, \quad \frac{\partial^2 y}{\partial r^2} = 0 \\ \frac{\partial z}{\partial r} &= \cos \theta, \quad \frac{\partial^2 z}{\partial r^2} = 0 \end{aligned}$$

代入可得

$$\begin{aligned} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) &= \sin^2 \theta \cos^2 \phi \frac{\partial^2 f}{\partial x^2} + \sin^2 \theta \sin^2 \phi \frac{\partial^2 f}{\partial y^2} + \sin^2 \theta \frac{\partial^2 f}{\partial z^2} \\ &\quad + \frac{2}{r} \left(\sin \theta \cos \phi \frac{\partial f}{\partial x} + \sin \theta \sin \phi \frac{\partial f}{\partial y} + \cos \theta \frac{\partial f}{\partial z} \right) \end{aligned}$$

又

$$\frac{\partial x}{\partial \theta} = r \cos \theta \cos \phi, \quad \frac{\partial^2 x}{\partial \theta^2} = -r \sin \theta \cos \phi$$

$$\frac{\partial y}{\partial \theta} = r \cos \theta \sin \phi, \frac{\partial^2 y}{\partial \theta^2} = -r \sin \theta \sin \phi$$

$$\frac{\partial z}{\partial \theta} = -r \sin \theta, \frac{\partial^2 z}{\partial \theta^2} = -r \cos \theta$$

于是

$$\begin{aligned} \frac{1}{r^2 \tan \theta} \frac{\partial f}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} &= \cos^2 \theta \cos^2 \phi \frac{\partial^2 f}{\partial x^2} + \cos^2 \theta \sin^2 \phi \frac{\partial^2 f}{\partial y^2} + \cos^2 \theta \frac{\partial^2 f}{\partial z^2} \\ &+ \frac{1}{r} \left(\frac{\cos 2\theta}{\sin \theta} \cos \phi \frac{\partial f}{\partial x} + \frac{\cos 2\theta}{\sin \theta} \sin \phi \frac{\partial f}{\partial y} - 2 \cos \theta \frac{\partial f}{\partial z} \right) \end{aligned}$$

又

$$\frac{\partial x}{\partial \phi} = -r \sin \theta \sin \phi, \frac{\partial^2 x}{\partial \phi^2} = -r \sin \theta \cos \phi$$

$$\frac{\partial y}{\partial \phi} = r \sin \theta \cos \phi, \frac{\partial^2 y}{\partial \phi^2} = -r \sin \theta \sin \phi$$

$$\frac{\partial z}{\partial \phi} = \frac{\partial^2 z}{\partial \phi^2} = 0$$

于是

$$\frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} = \sin^2 \phi \frac{\partial^2 f}{\partial x^2} + \cos^2 \phi \frac{\partial^2 f}{\partial y^2} - \frac{1}{r} \left(\frac{\cos \phi}{\sin \theta} \frac{\partial f}{\partial x} + \frac{\sin \phi}{\sin \theta} \frac{\partial f}{\partial y} \right)$$

将三项相加,考虑如下几项偏导数的系数

$$\frac{\partial f}{\partial x} : \frac{1}{r} \left(2 \sin \theta \cos \phi + \frac{\cos 2\theta}{\sin \theta} \cos \phi - \frac{\cos \phi}{\sin \theta} \right) = \frac{\cos \phi}{r \sin \theta} (2 \sin^2 \theta - 1 + \cos 2\theta) = 0$$

$$\frac{\partial f}{\partial y} : \frac{1}{r} \left(2 \sin \theta \sin \phi + \frac{\cos 2\theta}{\sin \theta} \sin \phi - \frac{\sin \phi}{\sin \theta} \right) = \frac{\sin \phi}{r \sin \theta} (2 \sin^2 \theta - 1 + \cos 2\theta) = 0$$

$$\frac{\partial f}{\partial z} : \frac{2}{r} \cos \theta - \frac{1}{r} \cdot 2 \cos \theta = 0$$

$$\frac{\partial^2 f}{\partial x^2} : \sin^2 \theta \cos^2 \phi + \cos^2 \theta \cos^2 \phi + \sin^2 \phi = 1$$

$$\frac{\partial^2 f}{\partial y^2} : \sin^2 \theta \sin^2 \phi + \cos^2 \theta \sin^2 \phi + \cos^2 \phi = 1$$

$$\frac{\partial^2 f}{\partial y^2} : \sin^2 \theta + \cos^2 \theta = 1$$

于是

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

从而原命题得证.