Lecture 9 Sequence series(数项级数)

L.9.1 设p是正实数,判断正项级数

$$\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)^{p}\ln\frac{n+1}{n-1}$$

的敛散性.

Proof.

考虑到

$$\left(\sqrt{n+1}-\sqrt{n}\right)^{p}\ln\frac{n+1}{n-1}=\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)^{p}\ln\left(1+\frac{2}{n-1}\right)\sim\frac{2^{1-p}}{n^{1+\frac{p}{2}}}$$

又因为$\displaystyle\sum_{n=1}^{\infty}\frac{2^{1-p}}{n^{1+\frac{p}{2}}}$收敛,于是原正项级数收敛.

L.9.2 判断正项级数

$$\sum_{n=1}^{\infty}\left(n^{\frac{1}{n}}-\sin\frac{1}{n}\right)^{n^2}$$

的敛散性.

提示:使用Cauchy判别法,得到的极限借助Taylor公式计算.

Proof.

置$u_n=\left(n^{\frac{1}{n}}-\sin\frac{1}{n}\right)^{n^2}$,则

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{u_n}&=\lim_{n\to\infty}\left(n^{\frac{1}{n}}-\sin\frac{1}{n}\right)^{n}\\
&=\lim_{n\to\infty}\exp\left[n\ln\left(n^{\frac{1}{n}}-\sin\frac{1}{n}\right)\right]\\
&=\lim_{n\to\infty}\exp\left[n\left(n^{\frac{1}{n}}-\sin\frac{1}{n}-1\right)\right]\\
&=\lim_{n\to\infty}\exp\left[n\left(n^{\frac{1}{n}}-1\right)\right]\cdot\exp\left(-n\sin\frac{1}{n}\right)\\
&=\lim_{n\to\infty}\exp\left(\ln n+o(\ln n)-1\right)\\
&=+\infty
\end{aligned}$$

其中

$$\ln\left(n^{\frac{1}{n}}-\sin\frac{1}{n}\right)\sim n^{\frac{1}{n}}-\sin\frac{1}{n}-1\quad(n\to\infty)$$

$$n^{\frac{1}{n}}=\mathrm{e}^{\frac{\ln n}{n}}=1+\frac{\ln n}{n}+o\left(\frac{\ln n}{n}\right)$$

从而$\sum\limits_{n=1}^{\infty} u_n$发散.

L.9.3 判断一般项级数
$$\sum_{n=1}^{\infty}\left(\frac{\sin n}{n}\sum_{k=1}^{n}\frac{1}{k}\right)$$
的敛散性.

Proof.
首先有$\sum\limits_{n=1}^{\infty}\left|\dfrac{\sin n}{n}\right|$发散,又$\sum\limits_{n=1}^{\infty} > 1$,因此这级数不绝对收敛.
下面证明这级数条件收敛.置$a_n = \dfrac{1}{n}\sum\limits_{k=1}^{n}\dfrac{1}{k}, b_n = \sin n$.则
$$a_n - a_{n+1} = \left(\frac{1}{n} - \frac{1}{n+1}\right)\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{(n+1)^2} \geqslant \frac{1}{n^2+n} - \frac{1}{(n+1)^2} > 0$$
于是a_n单调递减,又$0 < a_n \leqslant \dfrac{1}{n}\sum\limits_{k=1}^{n}\dfrac{1}{n} = \dfrac{1}{n}$,根据夹逼准则有$\lim\limits_{n\to\infty} a_n = 0$.
考虑$\{b_n\}$的部分和$B_n = \sum\limits_{k=1}^{n} b_k$,则有
$$\begin{aligned}
B_n &= \sum_{k=1}^{n}\sin n \\
&= \frac{1}{\sin\frac{1}{2}}\left(\sin 1\sin\frac{1}{2} + \cdots + \sin n\sin\frac{1}{2}\right) \\
&= \frac{1}{2\sin\frac{1}{2}}\left(\cos\frac{1}{2} - \cos\frac{3}{2} + \cdots + \cos\frac{n-1}{2} - \cos\frac{n+1}{2}\right) \\
&= \frac{1}{2\sin\frac{1}{2}}\left(\cos\frac{1}{2} - \cos\frac{n+1}{2}\right)
\end{aligned}$$
于是
$$|B_n| = \left|\frac{1}{2\sin\frac{1}{2}}\left(\cos\frac{1}{2} - \cos\frac{n+1}{2}\right)\right| \leqslant \left|\frac{1}{\sin\frac{1}{2}}\right|$$
于是该部分和序列有界.
根据Dirichlet判别法,$\sum\limits_{n=1}^{\infty}\left(\dfrac{\sin n}{n}\sum\limits_{k=1}^{n}\dfrac{1}{k}\right) = \sum\limits_{n=1}^{\infty} a_n b_n$条件收敛.

L.9.4 设$\{a_n\}$是单调递增的有界序列,试证明:级数
$$\sum_{n=1}^{\infty}\left(1 - \frac{a_n}{a_{n+1}}\right)$$

收敛.

注:$\{a_n\}$似应为单调递增的**正项**有界序列.

Proof.

不妨设$\lim\limits_{n\to\infty} a_n = A$.

Method I.

我们有

$$\sum_{n=1}^{\infty}\left(1-\frac{a_n}{a_{n+1}}\right)=\sum_{n=1}^{\infty}\frac{1}{a_{n+1}}\cdot(a_{n+1}-a_n)\leqslant\int_{a_1}^{A}\frac{1}{x}\mathrm{d}x=\ln A-\ln a_1$$

于是原正项级数收敛.

Method II.

令$u_n=\dfrac{1}{a_{n+1}}, v_n=a_{n+1}-a_n$,则$\{u_n\}$单调递减且有下界$\dfrac{1}{A}$

又$\sum\limits_{n=1}^{\infty} v_n=\lim\limits_{n\to\infty} a_{n+1}-a_1=A-a_1$收敛.

于是根据Abel判别法可知原正项级数收敛.

L.9.5 设级数$\sum\limits_{n=1}^{\infty} a_n$满足$a_n>0$且$\lim\limits_{n\to\infty} n\ln\dfrac{a_n}{a_{n+1}}=1$.试举例说明级数$\sum\limits_{n=1}^{\infty} a_n$可能收敛也可能发散.

Proof.

令$a_n=\dfrac{1}{n(\ln n)^p}$,则有

$$\begin{aligned}\lim_{n\to\infty} n\ln\frac{a_n}{a_{n+1}}&=\lim_{n\to\infty} n\ln\left(\frac{(n+1)(\ln(n+1))^p}{n(\ln n)^p}\right)\\&=\lim_{n\to\infty}\left[n\ln\left(1+\frac{1}{n}\right)+np\ln\left(\frac{\ln(n+1)}{\ln n}\right)\right]\\&=\lim_{n\to\infty}\left[\ln\left(1+\frac{1}{n}\right)^n\right]+\left[np\cdot\frac{\ln\left(1+\dfrac{1}{n}\right)}{\ln n}\right]\\&=\lim_{n\to\infty}\ln\mathrm{e}+\frac{p\ln\mathrm{e}}{\ln n}\\&=1\end{aligned}$$

其中

$$\ln\left(\frac{\ln(n+1)}{\ln n}\right)\sim\frac{\ln(n+1)}{\ln n}-1=\frac{\ln\left(1+\dfrac{1}{n}\right)}{\ln n}$$

而$0<p\leqslant 1$时这级数发散,$p>1$时这级数收敛,符合题目条件.

L.9.6 我们知道调和级数$\sum_{n=1}^{\infty}\frac{1}{n}$是发散的.现在我们去除调和级数中所有分母包含数字9的项,如$\frac{1}{9},\frac{1}{19}$等,得到一个新的级数$\sum_{n=1}^{\infty}\frac{1}{p_n}$.试证明这个新级数收敛.

Proof.

将所有自然数按位数分类.所有k位的数字中不出现9的一共有$8\cdot 9^{k-1}$个,这些数中最小的是10^{k-1}.于是

$$\sum_{n=1}^{\infty}\frac{1}{p_n}\leqslant\sum_{k=1}^{\infty}\frac{1}{10^{k-1}}\cdot\left(8\cdot 9^{k-1}\right)=8\cdot\sum_{k=1}^{\infty}\left(\frac{9}{10}\right)^{k-1}=80$$

根据比较判别法可知$\sum_{n=1}^{\infty}\frac{1}{p_n}$收敛.