Cauchy命题与Stolz定理

关于数列$\{a_n\}$前n项的均值与$\{a_n\}$的关系,我们有如下命题:

若数列$\{a_n\}$满足$\lim_{n\to\infty} a_n = A$,则$\lim_{n\to\infty} \dfrac{\sum_{i=1}^n a_i}{n} = A$.

这就是Cauchy命题.下面我们给出这个命题的证明.

证明:由$\lim_{n\to\infty} a_n = A$有

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, s.t. \forall n \geqslant N, |a_n - A| \leqslant \varepsilon.$$

记满足上述条件的ε, N为ε_a, N_a.记$n < N_a$时$|a_n - A|$取得的最大值为M_a.则有

$$\sum_{i=1}^n |a_i - A| \leqslant M_a(N_a - 1) + \varepsilon_a(n + 1 - N_a) = (M_a - \varepsilon_a)(N_a - 1) + n\varepsilon_a$$

现在我们证明$\lim_{n\to\infty} \dfrac{\sum_{i=1}^n a_i}{n} = A$.

$\forall \varepsilon > 0$,取$0 < \varepsilon_a < \varepsilon$和对应的$N_a, M_a$.

要使$\left|\dfrac{\sum_{i=1}^n a_i}{n} - A\right| \leqslant \varepsilon$,根据绝对值三角不等式有

$$\left|\frac{\sum_{i=1}^n a_i}{n} - A\right| = \frac{1}{n}\left|\sum_{i=1}^n (a_i - A)\right| \leqslant \frac{1}{n}\sum_{i=1}^n |a_i - A|$$

又$n \geqslant N_a$时

$$\frac{1}{n}\sum_{i=1}^n |a_i - A| \leqslant \frac{(M_a - \varepsilon_a)(N_a - 1)}{n} + \varepsilon_a$$

令

$$\frac{(M_a - \varepsilon_a)(N_a - 1)}{n} + \varepsilon_a \leqslant \varepsilon$$

解得

$$n \geqslant \frac{(M_a - \varepsilon_a)(N_a - 1)}{\varepsilon - \varepsilon_a}$$

则$\exists N = \max\left\{\left[\dfrac{(M_a - \varepsilon_a)(N_a - 1)}{\varepsilon - \varepsilon_a}\right] + 1, N_a\right\}$,$s.t.$

$$\forall n \geqslant N, \left|\frac{\sum_{i=1}^n a_i}{n} - A\right| \leqslant \frac{1}{n}\sum_{i=1}^n |a_i - A| < \frac{(M_a - \varepsilon_a)(N_a - 1) + N\varepsilon_a}{N} \leqslant \varepsilon$$

从而原命题得证.

例1(24.10.09 SJTU数分小测):

数列$\{x_n\}$满足对于$\{x_n\}$的任意子列$\{x_{n_k}\}$均有$\lim_{k\to\infty}\dfrac{\sum_{i=1}^{k}x_{n_i}}{k}=1$.**试证明:**$\lim_{n\to\infty}x_n=1$.

证明:采取反证法.假定$\{x_n\}$不收敛或不收敛于1,则有

$$\exists\varepsilon>0,\forall N\in\mathbb{N}^*,s.t.\exists n\geqslant N,|x_n-1|>\varepsilon$$

也即$\{x_n\}$中有无穷多项x_i满足$|x_n-1|>\varepsilon$.

我们将这些项分为$\{x_n\}$的两个子序列$\left\{x_{n_+}\right\},\left\{x_{n_-}\right\}$, 满足

$$\forall x_i\in\left\{x_{n_+}\right\},x_i>1+\varepsilon$$
$$\forall x_j\in\left\{x_{n_-}\right\},x_j<1-\varepsilon$$

则$\left\{x_{n_+}\right\},\left\{x_{n_-}\right\}$中至少有一个为无穷序列.

当$\left\{x_{n_+}\right\}$或$\left\{x_{n_-}\right\}$为无穷序列时有

$$\lim_{k\to\infty}\frac{\sum_{i=1}^{k}x_{n_{+,i}}}{k}\geqslant\lim_{k\to\infty}\frac{\sum_{i=1}^{k}1+\varepsilon}{k}=1+\varepsilon>1$$
$$\lim_{k\to\infty}\frac{\sum_{i=1}^{k}x_{n_{-,i}}}{k}\leqslant\lim_{k\to\infty}\frac{\sum_{i=1}^{k}1-\varepsilon}{k}=1-\varepsilon<1$$

而根据题意,若$\left\{x_{n_+}\right\}$或$\left\{x_{n_-}\right\}$为无穷序列,则有

$$\lim_{k\to\infty}\frac{\sum_{i=1}^{k}x_{n_{+,i}}}{k}=\lim_{k\to\infty}\frac{\sum_{i=1}^{k}x_{n_{-,i}}}{k}=1$$

矛盾.故$\lim_{n\to\infty}x_n=1$.

例2(24.10.09 SJTU数分小测):

正项数列$\{x_n\}$满足$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}x_i}{n}=a,a\in\mathbb{R}$.**试证明:**$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}x_i^2}{n^2}=0$.

证明(解法一):我们有

$$\begin{aligned}\lim_{n\to\infty}\frac{x_n}{n}&=\lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n}x_i}{n}-\frac{n-1}{n}\cdot\frac{\sum_{i=1}^{n-1}x_i}{n-1}\right)\\&=a-1\cdot a=0\end{aligned}$$

根据收敛序列的有界性,$\exists M\in\mathbb{R}\ s.t.\frac{\sum_{i=1}^{n}x_i}{n}<M$

则

$$0<\frac{\sum_{i=1}^{n}x_i^2}{n^2}=M\cdot\frac{\sum_{i=1}^{n}\frac{x_i^2}{n}}{Mn}<M\cdot\frac{\sum_{i=1}^{n}\frac{x_i^2}{i}}{\sum_{i=1}^{n}x_i}$$

依Stolz定理

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}\frac{x_i^2}{i}}{\sum_{i=1}^{n}x_i}=\lim_{n\to\infty}\frac{\frac{x_n^2}{n}}{x_n}=\lim_{n\to\infty}\frac{x_n}{n}=0$$

依夹逼准则$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}x_i^2}{n^2}=0$,证毕.