北京大学数学科学学院2021-22高等数学B2期中考试

1.(10分) 计算二重积分

$$\iint_D \ln\left(1+x^2+y^2\right)\mathrm{d}x\mathrm{d}y \qquad D: x^2+y^2 \leqslant 1, x \geqslant 0, y \geqslant 0$$

Solution.

做极坐标变换$x = r\cos\theta, y = r\sin\theta$.于是变换后的积分区域$D': 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \dfrac{\pi}{2}$.于是

$$\begin{aligned}
\iint_D \ln\left(1+x^2+y^2\right)\mathrm{d}x\mathrm{d}y &= \iint_{D'} \ln\left(1+r^2\right) r\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^1 \ln\left(1+r^2\right) r\mathrm{d}r \\
&\xlongequal{t=r^2} \frac{1}{2}\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_1^2 \ln t\mathrm{d}t \\
&= \frac{\pi}{4}(\ln 2 - 1)
\end{aligned}$$

2.(10分) 计算三重积分

$$\iiint_\Omega \left(y^2+z^2\right)\mathrm{d}V \qquad \Omega: 0 \leqslant z \leqslant x^2+y^2 \leqslant 1$$

Solution.

做柱坐标变换$x = r\cos\theta, y = r\sin\theta, z = z$,于是变换后的积分区域

$$\Omega': 0 \leqslant z \leqslant r^2 \leqslant 1, 0 \leqslant \theta \leqslant 2\pi$$

于是

$$\begin{aligned}
\iiint_\Omega \left(y^2+z^2\right)\mathrm{d}V &= \iiint_{\Omega'} \left(r^2\sin^2\theta + z^2\right) r\mathrm{d}r\mathrm{d}z\mathrm{d}\theta \\
&= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r \int_0^{r^2} \left(r^2\sin^2\theta + z^2\right) r\mathrm{d}z \\
&= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 r\left(r^4\sin^2\theta + \frac{r^6}{3}\right)\mathrm{d}r \\
&= \int_0^{2\pi} \left(\frac{1}{6}\sin^2\theta + \frac{1}{24}\right)\mathrm{d}\theta \\
&= \frac{\pi}{6} + \frac{1}{24}\cdot 2\pi \\
&= \frac{\pi}{4}
\end{aligned}$$

3.(10分) 设曲线C为椭圆$\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$沿逆时针方向.计算曲线积分

$$\oint_C \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2}$$

Solution.

考虑椭圆$D:\dfrac{x^2}{16}+\dfrac{y^2}{9}\leqslant 1$和圆$D_\varepsilon: x^2+y^2\leqslant \varepsilon^2(\varepsilon>0)$.当$\varepsilon<3$时,$D_\varepsilon\subseteq D$.设$C_\varepsilon$为$D\backslash D_\varepsilon$的正向边界.
在$D\backslash D_\varepsilon$上应用Green公式,则有

$$\oint_{C+C_\varepsilon} \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2} = \iint_{D\backslash D_\varepsilon} \frac{(x^2+y^2-2x^2)+(x^2+y^2-2y^2)}{\left(x^2+y^2\right)^2}\mathrm{d}\sigma = 0$$

于是

$$\begin{aligned}\oint_C \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2} &= -\oint_{C_\varepsilon} \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2} \\ &\xlongequal{x=\varepsilon\cos t,y=\varepsilon\sin t} \int_0^{2\pi} \frac{\varepsilon^2\cos^2 t+\varepsilon^2\sin^2 t}{\varepsilon^2}\mathrm{d}t \\ &= \int_0^{2\pi}\mathrm{d}t \\ &= 2\pi\end{aligned}$$

4.(10分) 计算曲面积分

$$\iint_S \left(x^2y^2+y^2z^2+z^2x^2\right)\mathrm{d}S$$

其中S为圆锥面$z=\sqrt{x^2+y^2}$被柱面$x^2+y^2=1$所截下部分.

Solution.

曲面方程为$z=g(x,y)=\sqrt{x^2+y^2}, 0\leqslant x^2+y^2\leqslant 1$.于是

$$\sqrt{1+g_x^2+g_y^2} = \sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2} = \sqrt{2}$$

于是

$$\begin{aligned}\iint_S \left(x^2y^2+y^2z^2+z^2x^2\right)\mathrm{d}S &= \iint_S \left(x^2+y^2+\left(x^2+y^2\right)^2\right)\sqrt{2}\mathrm{d}x\mathrm{d}y \\ &= \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1\left(r^2+r^4\right)r\mathrm{d}r \\ &= \frac{5\sqrt{2}\pi}{6}\end{aligned}$$

5.(15分) 计算曲面积分

$$\oiint_S x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y$$

其中S为抛物面$z = x^2 + y^2$被平面$z = 4$所截部分的外侧.

Solution.

考虑空间区域$\Omega : 0 \leqslant x^2 + y^2 \leqslant z \leqslant 4$,则$\Omega$的外侧为$S$和圆$D : 0 \leqslant x^2 + y^2 \leqslant 4, z = 4$.

在Ω上应用Gauss公式,则有

$$\begin{aligned}\oiint_{S+D} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y &= \iiint_\Omega 3\mathrm{d}V \\ &= 3\int_0^{2\pi}\mathrm{d}\theta\int_0^2\mathrm{d}r\int_{r^2}^4 r\mathrm{d}z \\ &= 3\int_0^{2\pi}\mathrm{d}\theta\int_0^2\left(4r - r^3\right)\mathrm{d}r \\ &= 24\pi\end{aligned}$$

而D的单位外法向量为$(0,0,1)$,于是

$$\begin{aligned}\oiint_D x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y &= \oiint_D (x,y,z)\cdot(0,0,1)\,\mathrm{d}S \\ &= \iint_D z\mathrm{d}S \\ &= 16\pi\end{aligned}$$

于是

$$\oiint_S x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y = 24\pi - 16\pi = 8\pi$$

6.(10分) 求常微分方程

$$y' = xy + 3x + 2y + 6$$

的所有解.

Solution.

做换元$v = x + 2, u = y + 3$,于是$\dfrac{\mathrm{d}u}{\mathrm{d}v} = \dfrac{\mathrm{d}y}{\mathrm{d}x} = y'$.于是原微分方程即为

$$\frac{\mathrm{d}u}{\mathrm{d}v} = uv$$

如果$u\equiv 0$,那么显然这是该微分方程的解.

如果$u\neq 0$,则有

$$\frac{\mathrm{d}u}{u}=v\mathrm{d}v$$

两边积分可得

$$\ln|u|=\frac{1}{2}v^2+C$$

即

$$u=C\mathrm{e}^{\frac{1}{2}v^2}\quad (C\neq 0)$$

合并两种情况并回代可得原方程的解为

$$y=C\mathrm{e}^{\frac{(x+2)^2}{2}}-3\quad (C\in\mathbb{R})$$

7.(15分) 求常微分方程

$$y''-4y'+3y-4\mathrm{e}^x=0$$

的通解.

Solution.

该常微分方程对应的齐次方程为

$$\lambda^2-4\lambda+3=0$$

于是特征根为

$$\lambda_1=1\quad \lambda_2=3$$

于是设

$$y=C_1(x)\mathrm{e}^x+C_2(x)\mathrm{e}^{3x}$$

代入原方程则有

$$\begin{cases} C_1'(x)\mathrm{e}^x+C_2'(x)\mathrm{e}^{3x}=0\\ C_1'(x)\mathrm{e}^x+3C_2'(x)\mathrm{e}^{3x}=4\mathrm{e}^x\end{cases}$$

解得

$$C_1'(x)=-2\quad C_2'(x)=2\mathrm{e}^{-2x}$$

于是

$$C_1(x)=-2x+C_1\quad C_2(x)=-\mathrm{e}^{-2x}+C_2$$

于是原微分方程的通解为

$$y=C_1\mathrm{e}^x+C_2\mathrm{e}^{3x}-\mathrm{e}^x(1+2x)$$

8.(10分) 设平面有界闭区域为

$$D=\left\{(x,y)\in\mathbb{R}^2:\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\right\}\qquad a,b>0$$

设曲线L为D的边界,函数$P(x,y),Q(x,y)$在D上有连续的一阶偏导数.记$\mathbf{F}=(P,Q)$,$\mathbf{n}$为曲线L的单位外法向量.试证明

$$\oint_{L^+}\mathbf{F}\cdot\mathbf{n}\mathrm{d}s=\iint_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

Proof.

由于$\mathbf{n}=(n_x,n_y)$是L的单位外法向量,于是L^+的单位切向量为$(-n_y,n_x)$.于是

$$\oint_{L^+}\mathbf{F}\cdot\mathbf{n}\mathrm{d}s=\oint_{L^+}(P,Q)\cdot(n_x,n_y)\,\mathrm{d}s=\oint_{L^+}(-Q,P)\cdot(-n_y,n_x)\,\mathrm{d}s=\oint_{L^+}-Q\mathrm{d}x+P\mathrm{d}y$$

在D上运用Green公式可得

$$\oint_{L^+}-Q\mathrm{d}x+P\mathrm{d}y=\iint_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

于是

$$\oint_{L^+}\mathbf{F}\cdot\mathbf{n}\mathrm{d}s=\iint_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

命题得证.

9.(10分) 设$f(x)$为$\mathbb{R}$上的连续函数.试证明

$$\iint_S f(x+y+z)\mathrm{d}S=2\pi\int_{-1}^{1}f\left(\sqrt{3}\xi\right)\mathrm{d}\xi$$

其中S为单位球面$x^2+y^2+z^2=1$.

Proof.

对球面上的点(x,y,z)做正交变换$T:(x,y,z)\mapsto(u,v,w)$.设T关于$\mathbb{R}^3$的规范正交基的矩阵为A.

现在,令$u=\dfrac{x+y+z}{\sqrt{3}}$,则

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{3}}\\A_{2,1}&A_{2,2}&A_{2,3}\\A_{3,1}&A_{3,2}&A_{3,3}\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}$$

根据Schmidt正交化过程可以求解出矩阵A.

由于正交变换后依然保持$u^2+v^2+w^2=1$,我们对变换后的坐标进行换元可得

$$\begin{cases} u=\xi \\ v=\sqrt{1-\xi^2}\cos\theta \\ w=\sqrt{1-\xi^2}\sin\theta \end{cases}$$

于是

$$\begin{pmatrix} x_\theta \\ y_\theta \\ z_\theta \end{pmatrix} = A^{-1}\begin{pmatrix} u_\theta \\ v_\theta \\ w_\theta \end{pmatrix} = A^{-1}\begin{pmatrix} 0 \\ -\sqrt{1-\xi^2}\sin\theta \\ \sqrt{1-\xi^2}\cos\theta \end{pmatrix}$$

以及

$$\begin{pmatrix} x_\xi \\ y_\xi \\ z_\xi \end{pmatrix} = A^{-1}\begin{pmatrix} u_\xi \\ v_\xi \\ w_\xi \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ -\dfrac{\xi}{\sqrt{1-\xi^2}}\cos\theta \\ -\dfrac{\xi}{\sqrt{1-\xi^2}}\sin\theta \end{pmatrix}$$

于是

$$E=\begin{pmatrix} x_\theta & y_\theta & z_\theta \end{pmatrix}\begin{pmatrix} x_\theta \\ y_\theta \\ z_\theta \end{pmatrix} = 1-\xi^2$$

$$F=\begin{pmatrix} x_\theta & y_\theta & z_\theta \end{pmatrix}\begin{pmatrix} x_\xi \\ y_\xi \\ z_\xi \end{pmatrix} = 0$$

$$G=\begin{pmatrix} x_\xi & y_\xi & z_\xi \end{pmatrix}\begin{pmatrix} x_\xi \\ y_\xi \\ z_\xi \end{pmatrix} = \frac{1}{1-\xi^2}$$

变换后的积分区域为$D:-1\leqslant\xi\leqslant 1, 0\leqslant\theta\leqslant 2\pi$.于是

$$\begin{aligned}\iint_S f(x+y+z)\mathrm{d}S &= \iint_D f\left(\sqrt{3}\xi\right)\sqrt{EG-F^2}\mathrm{d}\sigma \\ &= \int_0^{2\pi}\mathrm{d}\theta\int_{-1}^{1} f\left(\sqrt{3}\xi\right)\mathrm{d}\xi \\ &= 2\pi\int_{-1}^{1} f\left(\sqrt{3}\xi\right)\mathrm{d}\xi\end{aligned}$$