北京大学数学科学学院2022-23高等数学A2期中考试

1.(32分) 指出下列各积分的积分类型,并计算其积分值,其中

$$D_1 = [0,1] \subset \mathbb{R} \qquad D_2 = [0,1]\times[0,1] \subset \mathbb{R}^2 \qquad D_3 = [0,1]\times[0,1]\times[0,1] \subset \mathbb{R}^3$$

记$\partial\Omega$表示Ω的边界.记$S_1 = \partial D_2, S_2 = \partial D_3$,$S_1^+$为逆时针方向的$S_1$,$S_2^+$为外法线方向的$S_2$.

(1) $\displaystyle\int_{D_1} x\mathrm{d}x$

(2) $\displaystyle\oint_{S_1} xy\mathrm{d}s$

(3) $\displaystyle\oiint_{S_2} xyz\mathrm{d}S$

(4) $\displaystyle\iint_{D_2} xy\mathrm{d}x\mathrm{d}y$

(5) $\displaystyle\oint_{S_1^+} 2xy\mathrm{d}x + \left(x^2+y^2\right)\mathrm{d}y$

(6) $\displaystyle\iiint_{D_3} x^6y^{16}z^{16}\mathrm{d}x\mathrm{d}y\mathrm{d}z$

(7) $\displaystyle\oiint_{S_2^+} \left(\frac{x}{2}+z^3\sin y^2\right)\mathrm{d}y\mathrm{d}z + \left(\frac{y}{3}+\mathrm{e}^{x\cos z}\right)\mathrm{d}z\mathrm{d}x + \left(\frac{z}{6}+\arctan(xy)\right)\mathrm{d}x\mathrm{d}y$

(8) $\displaystyle\oint_{\Gamma^+} x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z$,其中$\Gamma^+$是由$(0,0,0)$出发,依次经过点$(1,0,0),(1,1,0),(1,1,1),(0,1,1),(0,1,0)$后回到$(0,0,0)$的直线段构成.

Solution.

(1) 定积分.我们有

$$\int_{D_1} x\mathrm{d}x = \left(\frac{1}{2}x^2\right)\Big|_0^1 = \frac{1}{2}$$

(2) 第一型曲线积分.我们只需考虑线段$L_1: x=1, 0\leqslant y\leqslant 1$和$L_2: y=1, 0\leqslant x\leqslant 1$即可.此时有

$$\oint_{S_1} xy\mathrm{d}s = \int_{L_1} xy\mathrm{d}s + \int_{L_2} xy\mathrm{d}s = \int_0^1 y\mathrm{d}y + \int_0^1 x\mathrm{d}x = 1$$

(3) 第一型曲面积分.我们只需考虑面$D_1: x=1, 0\leqslant y,z\leqslant 1$,$D_2: y=1, 0\leqslant x,z\leqslant 1$和$D_3: z=1, 0\leqslant x,y\leqslant 1$即可.于是

$$\oiint_{S_2} xyz\mathrm{d}S = \iint_{D_1} yz\mathrm{d}S + \iint_{D_2} xz\mathrm{d}S + \iint_{D_3} xy\mathrm{d}S = 3\iint_{[0,1]\times[0,1]} xy\mathrm{d}\sigma = \frac{3}{4}$$

(4) 二重积分.我们有

$$\iint_{D_2} xy\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x\int_0^1 xy\mathrm{d}y = \frac{1}{2}\int_0^1 x\mathrm{d}x = \frac{1}{4}$$

(5) 第二型曲线积分.在D_2上运用Green公式有

$$\oint_{S_1^+} 2xy\mathrm{d}x + \left(x^2+y^2\right)\mathrm{d}y = \iint_{D_2} (2x-2x)\,\mathrm{d}\sigma = 0$$

(6) 三重积分.我们有

$$\iiint_{D_3} x^6y^{16}z^{16}\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 x^6y^{16}z^{16}\mathrm{d}z = \frac{1}{7}\cdot\frac{1}{17}\cdot\frac{1}{17} = \frac{1}{2023}$$

(7) 第二型曲面积分.在D_3上运用Gauss公式有

$$\begin{aligned}
&\oiint_{S_2^+} \left(\frac{x}{2}+z^3\sin y^2\right)\mathrm{d}y\mathrm{d}z + \left(\frac{y}{3}+\mathrm{e}^{x\cos z}\right)\mathrm{d}z\mathrm{d}x + \left(\frac{z}{6}+\arctan(xy)\right)\mathrm{d}x\mathrm{d}y \\
=& \iiint_{D_3} \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)\mathrm{d}V \\
=& \iiint_{D_3} \mathrm{d}V = 1
\end{aligned}$$

(8) 第二型曲线积分.考虑两个正方形$\Omega_1: z=0, 0\leqslant x,y\leqslant 1$和$\Omega_2: y=1, 0\leqslant x,z\leqslant 1$.则$\Omega=\Omega_1\cup\Omega_2$的边界就是$\Gamma^+$.运用Stokes公式有

$$\oint_{\Gamma^+} x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z = \iint_{\Omega^+} 0\mathrm{d}y\mathrm{d}z + 0\mathrm{d}z\mathrm{d}x + 0\mathrm{d}x\mathrm{d}y = 0$$

2.(12分) 求二重积分

$$I = \iint_D \left|y-x^2\right|\mathrm{d}x\mathrm{d}y$$

其中$D=\{(x,y)\in\mathbb{R}^2: -1\leqslant x\leqslant 1, 0\leqslant y\leqslant 1\}$.

Solution.

考虑到被积函数$|y-x^2|$和积分区域D都关于y轴对称,于是仅需考虑$D': [0,1]\times[0,1]$即可.我们有

$$\begin{aligned}
I' &= \iint_{D'} \left|y-x^2\right|\mathrm{d}x\mathrm{d}y \\
&= \int_0^1 \mathrm{d}x \left(\int_0^{x^2} \left(x^2-y\right)\mathrm{d}y + \int_{x^2}^1 \left(y-x^2\right)\mathrm{d}y\right) \\
&= \int_0^1 \left(x^4-x^2+\frac{1}{2}\right)\mathrm{d}x \\
&= \frac{11}{30}
\end{aligned}$$

于是

$$I = 2I' = \frac{11}{15}$$

3.(12分) 计算由封闭曲面

$$S=\left\{(x,y,z)\in\mathbb{R}^3:\left(x^2+\frac{y^2}{4}+\frac{z^2}{5}\right)^2\leqslant x\right\}$$

围成区域的体积.

Solution.

做变换

$$x=\rho\cos\varphi\qquad y=2\rho\sin\varphi\cos\theta\qquad z=\sqrt{5}\rho\sin\varphi\sin\theta$$

这变换的Jacobi矩阵

$$|J|=2\sqrt{5}\rho^2\sin\varphi$$

曲面S围成的区域变换后即为

$$\Omega=\left\{(\rho,\theta,\varphi):0\leqslant\varphi\leqslant\frac{\pi}{2},0\leqslant\theta\leqslant 2\pi,0\leqslant\rho^3\leqslant\cos\varphi\right\}$$

于是

$$\begin{aligned}V&=\iiint_\Omega|J|\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi\\&=2\sqrt{5}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{\sqrt[3]{\cos\varphi}}\rho^2\sin\varphi\mathrm{d}\rho\\&=\frac{2\sqrt{5}}{3}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi\mathrm{d}\varphi\\&\xlongequal{u=\sin\varphi}\frac{2\sqrt{5}}{3}\int_0^{2\pi}\mathrm{d}\theta\int_0^1u\mathrm{d}u\\&=\frac{2\sqrt{5}\pi}{3}\end{aligned}$$

4.(12分) 设S^+是单位球面$x^2+y^2+z^2=1$的外侧,试求曲面积分

$$I=\oiint_{S^+}\frac{x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y}{(x^2+4y^2+9z^2)^{\frac{3}{2}}}$$

Solution.

考虑球体$\Omega:x^2+y^2+z^2\leqslant 1$和小椭球$\Omega':x^2+4y^2+9z^2\leqslant\varepsilon^2$,$\varepsilon<1$,并记$S_\varepsilon^+$为$\Omega_\varepsilon$的表面的外侧.

在$\Omega\backslash\Omega'$上运用Gauss公式有

$$\begin{aligned}
&\oiint_{S^+\cup S_\varepsilon^-}\frac{x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y}{(x^2+4y^2+9z^2)^{\frac{3}{2}}}\\
=&\iiint_{\Omega\backslash\Omega'}\frac{3\left(x^2+4y^2+9z^2\right)^{\frac{3}{2}}-\left(2x^2+8y^2+18z^2\right)\cdot\dfrac{3}{2}\sqrt{x^2+4y^2+9z^2}}{\left(x^2+4y^2+9z^2\right)^3}\mathrm{d}V\\
=&0
\end{aligned}$$

于是

$$\begin{aligned}
\oiint_{S^+}\frac{x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y}{(x^2+4y^2+9z^2)^{\frac{3}{2}}}&=\oiint_{S_\varepsilon^+}\frac{x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y}{(x^2+4y^2+9z^2)^{\frac{3}{2}}}\\
&=\frac{1}{\varepsilon^3}\oiint_{S_\varepsilon^+}x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y\\
&=\frac{1}{\varepsilon^3}\iiint_{\Omega_\varepsilon}3\mathrm{d}V=\frac{3}{\varepsilon^3}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{4\pi\varepsilon^3}{3}\\
&=\frac{2\pi}{3}
\end{aligned}$$

5.(12分) 设$f(t)$是$[0,1]$上的可积函数,满足

$$\int_0^1 f(t)\mathrm{d}t=1\qquad\int_0^1 tf(t)\mathrm{d}t=2\qquad\int_0^1 t^2f(t)\mathrm{d}t=3$$

试求累次积分

$$I=\int_0^1\mathrm{d}x\int_0^x\mathrm{d}y\int_0^y f(z)\mathrm{d}z$$

Solution.

考虑区域$D_1:0\leqslant z\leqslant y\leqslant x\leqslant 1$,于是

$$I_1=\int_0^1\mathrm{d}x\int_0^x\mathrm{d}y\int_0^y f(z)\mathrm{d}z=\iiint_{D_1}f(z)\mathrm{d}V$$

考虑区域$D_2:0\leqslant z\leqslant x\leqslant y\leqslant 1$,于是

$$I_2=\int_0^1\mathrm{d}x\int_x^1\mathrm{d}y\int_0^y f(z)\mathrm{d}z=\iiint_{D_2}f(z)\mathrm{d}V$$

注意到被积函数$f(z)$关于平面$x=y$对称,而D_1和D_2也关于平面$x=y$对称,于是$I_1=I_2$.考虑积分

$$\begin{aligned}
I_3=I_1+I_2=\iiint_{D_1\cup D_2}f(z)\mathrm{d}V&=\int_0^1\mathrm{d}z\iint_{[z,1]\times[z,1]}f(z)\mathrm{d}x\mathrm{d}y\\
&=\int_0^1(1-z)^2f(z)\mathrm{d}z=3-2\cdot 2+1=0
\end{aligned}$$

于是

$$I=\frac{1}{2}I_3=0$$

6.(10分) 设

$$F(t)=\iiint_{x^2+y^2+z^2\leqslant t^2} f(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

其中$f(s)$连续,在$s=0$处可导,并且满足$f(0)=0, f'(0)=10$.求极限

$$\lim_{t\to 0^+}\frac{F(t)}{t^5}$$

Solution.

做球坐标变换

$$x=\rho\sin\varphi\cos\theta \qquad y=\rho\sin\varphi\sin\theta \qquad z=\rho\cos\theta$$

则

$$\begin{aligned}
F(t)&=\iiint_{x^2+y^2+z^2\leqslant t^2} f(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&=\iiint_{0\leqslant\rho\leqslant t} f\left(\rho^2\right)\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi\\
&=\int_0^t\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\rho^2 f\left(\rho^2\right)\sin\varphi\mathrm{d}\varphi\\
&=4\pi\int_0^t\rho^2 f\left(\rho^2\right)\mathrm{d}\rho
\end{aligned}$$

于是

$$\begin{aligned}
\lim_{t\to 0^+}\frac{F(t)}{t^5}&=\lim_{t\to 0^+}\frac{4\pi\displaystyle\int_0^t\rho^2 f\left(\rho^2\right)\mathrm{d}\rho}{t^5}\\
&=\lim_{t\to 0^+}\frac{\pi t^2 f\left(t^2\right)}{t^4}\\
&=\lim_{t\to 0^+}\pi\cdot\frac{f\left(t^2\right)-f(0)}{t^2}\\
&=\pi f'(0)=10\pi
\end{aligned}$$

7.(10分) 设$f(x,y)$是$\mathbb{R}^2$上的非负连续函数.对于$r>0,\rho>0$,令

$$I_r=\iint_{x^2+y^2\leqslant r^2} f(x,y)\mathrm{d}x\mathrm{d}y \qquad J_\rho=\iint_{-\rho\leqslant x,y\leqslant\rho} f(x,y)\mathrm{d}x\mathrm{d}y$$

试证明:当极限$\lim\limits_{r\to+\infty} I_r$与极限$\lim\limits_{\rho\to+\infty} J_\rho$之一存在且有限时,另一个极限必然也存在且有限,并且两者相等.

Proof.

首先假定$\lim\limits_{r\to+\infty} I_r$存在有限,不妨记为$I$.我们有

$$\left\{(x,y): x^2+y^2 \leqslant r^2\right\} \subset \{(x,y): -r \leqslant x, y \leqslant r\} \subset \left\{(x,y): x^2+y^2 \leqslant 2r^2\right\}$$

又因为$f(x,y)$非负,于是

$$I_r \leqslant J_r \leqslant I_{\sqrt{2}r}$$

两边取极限,有$\lim\limits_{r\to+\infty} I_{\sqrt{2}r} = \lim\limits_{r\to+\infty} I_r = I$.由夹逼准则可知

$$I \leqslant \lim_{r\to+\infty} J_r \leqslant I$$

于是

$$\lim_{r\to+\infty} J_r = I$$

现在假定$\lim\limits_{r\to\infty} J_r$存在有限,不妨记为$J$.我们有

$$\{(x,y): -\rho \leqslant x, y \leqslant \rho\} \subset \left\{(x,y): x^2+y^2 \leqslant 2\rho^2\right\} \subset \left\{(x,y): -\sqrt{2}\rho \leqslant x, y \leqslant \sqrt{2}\rho\right\}$$

同理夹逼可得

$$J \leqslant \lim_{\rho\to+\infty} I_{\sqrt{2}\rho} \leqslant J$$

于是

$$\lim_{\rho\to+\infty} I_\rho = J$$

从而命题得证.