Lecture 5 Curvilinear integral(曲线积分)

L.5.1 设L为椭圆$\Gamma:\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$的逆时针方向,求曲线积分$I=\displaystyle\oint_L\frac{x\mathrm{d}y-y\mathrm{d}x}{x^2+y^2}$.

Solution.

Method I.

做代换$x=4\cos\theta,y=3\sin\theta$,则$L$对应$\theta$从0变化至$2\pi$.

我们有$\mathrm{d}x=-4\sin\theta\mathrm{d}\theta,\mathrm{d}y=3\cos\theta\mathrm{d}\theta$,于是

$$\begin{aligned}
I=\oint_L\frac{x\mathrm{d}y-y\mathrm{d}x}{x^2+y^2}&=\int_0^{2\pi}\frac{12\cos^2\theta+12\sin^2\theta}{16\cos^2\theta+9\sin^2\theta}\mathrm{d}\theta\\
&\xlongequal{t=\tan\theta}4\int_0^{+\infty}\frac{12\mathrm{d}t}{16+9t^2}\\
&=4\left(\arctan\left(\frac{3}{4}t\right)\right)\Bigg|_0^{+\infty}\\
&=2\pi
\end{aligned}$$

Method II.

令$P(x,y)=\dfrac{-y}{x^2+y^2},Q(x,y)=\dfrac{x}{x^2+y^2}$.于是

$$\frac{\partial Q}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}\qquad\frac{\partial P}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

考虑区域$D=\left\{(x,y):\dfrac{x^2}{16}+\dfrac{y^2}{9}\leqslant 1\right\}$和区域$E=\{(x,y):x^2+y^2\leqslant\varepsilon^2\}$,其中$0<\varepsilon<3$.
显然$E\subset D$,从而在$D\backslash E$上P,Q有连续的一阶偏导数.设E的边界为M,则根据格林公式有

$$\oint_L P\mathrm{d}x+Q\mathrm{d}y+\oint_{M^-}P\mathrm{d}x+Q\mathrm{d}y=\iint_{D\backslash E}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}\sigma=0$$

对E的边界M做代换$x=\varepsilon\cos t,y=\varepsilon\sin t$,于是

$$\begin{aligned}
I=\oint_L P\mathrm{d}x+Q\mathrm{d}y=\oint_{M^+}&P\mathrm{d}x+Q\mathrm{d}y\\
&=\int_0^{2\pi}\frac{\cos^2t+\sin^2t}{\cos^2t+\sin^2t}\mathrm{d}t\\
&=2\pi
\end{aligned}$$

L.5.2 求曲线积分$I=\displaystyle\int_L\frac{(x+y)\mathrm{d}x+(x-y)\mathrm{d}y}{3(x^2+y^2)}$,其中$L$是曲线$x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$在$x$轴上方部分的逆时针方向.

Solution.

做代换$x=\cos^3 t, y=\sin^3 t$,则积分曲线L对应t从0变化至π.于是

$$\begin{aligned}
I&=\int_L \frac{(x+y)\mathrm{d}x+(x-y)\mathrm{d}y}{3(x^2+y^2)}\\
&=\int_0^\pi \frac{-3\left(\cos^3 t+\sin^3 t\right)\cos^2 t\sin t+3\left(\cos^3 t-\sin^3 t\right)\sin^2 t\cos t}{3\left(\cos^6 t+\sin^6 t\right)}\mathrm{d}t\\
&=\int_0^\pi \frac{\sin^2 t\cos^4 t-\sin^4 t\cos^2 t-\sin^5 t\cos t-\cos^5 t\sin t}{\cos^6 t+\sin^6 t}\mathrm{d}t
\end{aligned}$$

注意到$\sin^2 t\cos^4 t, \sin^4 t\cos^2 t$关于直线$t=\dfrac{\pi}{2}$对称,而$\sin^5 t\cos t, \cos^5 t\sin t$关于点$\left(\dfrac{\pi}{2},0\right)$中心对称,于是

$$\begin{aligned}
I&=2\int_0^{\frac{\pi}{2}} \frac{\sin^2 t\cos^4 t-\sin^4 t\cos^2 t}{\cos^6 t+\sin^6 t}\mathrm{d}t\\
&=2\left(\int_0^{\frac{\pi}{2}}\frac{\sin^2 t\cos^4 t}{\sin^6 t+\cos^6 t}\mathrm{d}t-\int_{-\frac{\pi}{2}}^0\frac{\sin^2 t\cos^4 t}{\sin^6 t+\cos^6 t}\mathrm{d}t\right)\\
&=0
\end{aligned}$$

L.5.3 求曲线积分$I=\displaystyle\int_L \left(x^2y^3+x^3y^2\right)\mathrm{d}s$,其中$L$为单位圆.

提示:将第一型曲线积分化为第二型曲线积分,然后使用格林公式.

Solution.

Method I.

在单位圆$x^2+y^2=1$上有$x\mathrm{d}x+y\mathrm{d}y=0$且$\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}=\mathrm{d}s$,于是$\mathrm{d}x=-y\mathrm{d}s, \mathrm{d}y=x\mathrm{d}s$.

$$\begin{aligned}
I&=\int_L \left(x^2y^3+x^3y^2\right)\mathrm{d}s\\
&=\int_L -x^3y\mathrm{d}x+xy^3\mathrm{d}y\\
&=\iint_{x^2+y^2\leqslant 1}\left(x^3+y^3\right)\mathrm{d}\sigma\\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^4\left(\sin^3\theta+\cos^3\theta\right)\mathrm{d}r\\
&=\frac{1}{5}\int_0^{2\pi}\left(\sin^3\theta+\cos^3\theta\right)\mathrm{d}\theta\\
&=0
\end{aligned}$$

Method II. 直接做代换$x=\cos\theta, y=\sin\theta$,则$\mathrm{d}s=\sqrt{\sin^2\theta+\cos^2\theta}\mathrm{d}\theta=\mathrm{d}\theta$.于是

$$I=\int_0^{2\pi}\left(\sin^3\theta\cos^2\theta+\sin^2\theta\cos^3\theta\right)\mathrm{d}\theta$$

而$\sin^3\theta\cos^2\theta$关于$(\pi,0)$中心对称;$\sin^2\theta\cos^3\theta$关于$\theta=\pi$轴对称,关于$\left(\dfrac{\pi}{2},0\right)$和$\left(\dfrac{3\pi}{2},0\right)$中心对称.
于是$I=0$.

L.5.4 回答下列问题.

(1) 设曲线C的弧长为L,试证明

$$\left|\int_C P(x,y)\mathrm{d}x+Q(x,y)\mathrm{d}y\right|\leqslant ML$$

其中$M=\max\limits_{(x,y)\in C}\sqrt{[P(x,y)]^2+[Q(x,y)]^2}$.

(2) 试证明

$$\lim_{R\to+\infty}\oint_{x^2+y^2=R^2}\frac{y\mathrm{d}x-x\mathrm{d}y}{\left(x^2+xy+y^2\right)^2}=0$$

注:本问在讲义上为$R\to 0$,实际上有误,应更正为$R\to+\infty$.

Proof.

(1) 设C在点(x,y)处沿积分路径的单位切向量为$\mathbf{n}=(\cos\alpha,\cos\beta)$,则此处的弧微分满足

$$\mathrm{d}x=\cos\alpha\mathrm{d}s\qquad \mathrm{d}y=\cos\beta\mathrm{d}y$$

于是

$$\begin{aligned}\left|\int_C P(x,y)\mathrm{d}x+Q(x,y)\mathrm{d}y\right|&\leqslant\int_C|P(x,y)|\mathrm{d}x+|Q(x,y)|\mathrm{d}y\\&=\int_C\left(|P(x,y)|\cos\alpha+|Q(x,y)|\cos\beta\right)\mathrm{d}s\\&\leqslant\int_C\sqrt{[P(x,y)]^2+[Q(x,y)]^2}\sqrt{\cos^2\alpha+\cos^2\beta}\mathrm{d}s\\&=\int_C\sqrt{[P(x,y)]^2+[Q(x,y)]^2}\mathrm{d}s\\&\leqslant\int_C M\mathrm{d}s\\&=ML\end{aligned}$$

于是命题得证.

(2) 令$P(x,y)=\dfrac{y}{\left(x^2+xy+y^2\right)^2},Q(x,y)=-\dfrac{x}{\left(x^2+xy+y^2\right)^2}$.于是

$$[P(x,y)]^2+[Q(x,y)]^2=\frac{x^2+y^2}{\left(x^2+xy+y^2\right)^4}=\frac{R^2}{\left(R^2+xy\right)^4}\leqslant\frac{16}{R^6}$$

在曲线$C:x^2+y^2=R^2$上运用**(1)**的结论有

$$0\leqslant\left|\oint_C P\mathrm{d}x+Q\mathrm{d}y\right|\leqslant 2\pi R\cdot\frac{4}{R^3}=\frac{8\pi}{R^2}$$

运用夹逼准则可得

$$\lim_{R\to+\infty}\oint_{x^2+y^2=R^2}\frac{y\mathrm{d}x-x\mathrm{d}y}{\left(x^2+xy+y^2\right)^2}=0$$

L.5.5 设平面正方形区域$D=[0,\pi]\times[0,\pi]$,记L为D的正向边界.

(1) 试证明

$$\oint_L x\mathrm{e}^{\sin y}\mathrm{d}y-y\mathrm{e}^{-\sin x}\mathrm{d}x=\oint_L x\mathrm{e}^{-\sin y}\mathrm{d}y-y\mathrm{e}^{\sin x}\mathrm{d}x$$

(2) 试证明

$$\oint_L x\mathrm{e}^{\sin y}\mathrm{d}y-y\mathrm{e}^{-\sin x}\mathrm{d}x\geqslant 2\pi^2$$

提示:考虑重积分的对称性.

Proof.

(1) 根据格林公式和积分区域的对称性有

$$\begin{aligned}
\oint_L x\mathrm{e}^{\sin y}\mathrm{d}y-y\mathrm{e}^{-\sin x}\mathrm{d}x&=\iint_D\left(\mathrm{e}^{\sin y}+\mathrm{e}^{-\sin x}\right)\mathrm{d}\sigma\\
&=\int_0^\pi\mathrm{d}x\int_0^\pi\left(\mathrm{e}^{\sin y}+\mathrm{e}^{-\sin x}\right)\mathrm{d}y\\
&=\pi\int_0^\pi\left(\mathrm{e}^{\sin t}+\mathrm{e}^{-\sin t}\right)\mathrm{d}t\\
&=\int_0^\pi\mathrm{d}y\int_0^\pi\left(\mathrm{e}^{\sin x}+\mathrm{e}^{-\sin y}\right)\mathrm{d}x\\
&=\iint_D\left(\mathrm{e}^{\sin x}+\mathrm{e}^{-\sin y}\right)\mathrm{d}\sigma\\
&=\oint_L x\mathrm{e}^{-\sin y}\mathrm{d}y-y\mathrm{e}^{\sin x}\mathrm{d}x
\end{aligned}$$

于是命题得证.

(2) 我们有

$$\begin{aligned}
\oint_L x\mathrm{e}^{\sin y}\mathrm{d}y-y\mathrm{e}^{-\sin x}\mathrm{d}x&=\pi\int_0^\pi\left(\mathrm{e}^{\sin t}+\mathrm{e}^{-\sin t}\right)\mathrm{d}t\\
&\geqslant\pi\int_0^\pi 2\sqrt{\mathrm{e}^{\sin t}\cdot\mathrm{e}^{-\sin t}}\mathrm{d}t\\
&=\pi\int_0^\pi 2\mathrm{d}t\\
&=2\pi^2
\end{aligned}$$

于是命题得证.

L.5.6 设D是有界平面区域,其边界L分段光滑,定点$P_0(x_0,y_0)\notin L$.设L上一点$P(x,y)$,向量$\mathbf{n}_P$为P处L的外侧法向量.定义向量$\mathbf{r}_P=\overrightarrow{P_0P}$,定义函数$f(x,y)$为

$$f(P)=\frac{\cos(\mathbf{r}_P,\mathbf{n}_P)}{|\mathbf{r}_P|}$$

计算曲线积分$\displaystyle\oint_L f(x,y)\mathrm{d}s$.

Solution.

设P处沿L正方向的单位切向量为$(\cos\alpha,\cos\beta)$,那么$\mathbf{n}_P=(\cos\beta,-\cos\alpha)$.于是我们有

$$\begin{aligned}f(x,y)=\frac{\cos(\mathbf{r}_P,\mathbf{n}_P)}{\mathbf{r}_P}&=\frac{\mathbf{r}_P\cdot\mathbf{n}_P}{|\mathbf{r}_P|^2|\mathbf{n}_P|}\\&=\frac{(x-x_0)\cos\beta-(y-y_0)\cos\alpha}{(x-x_0)^2+(y-y_0)^2}\end{aligned}$$

于是

$$\begin{aligned}\oint_L f(x,y)\mathrm{d}s&=\oint_L\frac{(x-x_0)\cos\beta-(y-y_0)\cos\alpha}{(x-x_0)^2+(y-y_0)^2}\mathrm{d}s\\&=\oint_{L^+}\frac{(x-x_0)\mathrm{d}y-(y-y_0)\mathrm{d}x}{(x-x_0)^2+(y-y_0)^2}\end{aligned}$$

令$A(x,y)=\dfrac{-(y-y_0)}{(x-x_0)^2+(y-y_0)^2},B(x,y)=\dfrac{x-x_0}{(x-x_0)^2+(y-y_0)^2}$.于是我们有

$$\frac{\partial A}{\partial y}=-\frac{1}{(x-x_0)^2+(y-y_0)^2}+\frac{2(y-y_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}=\frac{(y-y_0)^2-(x-x_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}$$

同理可得

$$\frac{\partial B}{\partial x}=\frac{-(x-x_0)^2+(y-y_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}$$

若$P_0\notin D$,那么A,B在D上有连续的一阶偏导数,从而根据格林公式有

$$\oint_L f(x,y)\mathrm{d}s=\oint_{L^+}A\mathrm{d}x+B\mathrm{d}y=\iint_D\left(\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}\right)\mathrm{d}\sigma=0$$

若$P_0\in D$,那么考虑P_0的邻域$E=\{(x,y)|(x-x_0)^2+(y-y_0)^2\leqslant\varepsilon^2\}$,其中$\varepsilon>0$.令$\varepsilon$充分小至$E\subset D$.令$E$的边界为$L_E$,从而$A,B$在$D\backslash E$上有连续的一阶偏导数.我们有

$$0=\iint_{D\backslash E}\left(\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}\right)\mathrm{d}\sigma=\oint_{L^+}(A\mathrm{d}x+B\mathrm{d}y)+\oint_{L_E{}^-}(A\mathrm{d}x+B\mathrm{d}y)$$

做代换$x=\varepsilon\cos\theta+x_0,y=\varepsilon\sin\theta+y_0$,环路$L_E{}^+$即$\theta$从$0$变化至$2\pi$的路径.于是

$$\begin{aligned}\oint_{L^+}A\mathrm{d}x+B\mathrm{d}y&=\oint_{L_E{}^+}A\mathrm{d}x+B\mathrm{d}y\\&=\int_0^{2\pi}\left[\frac{-\varepsilon\sin\theta}{\varepsilon^2}\cdot(-\varepsilon\sin\theta)+\frac{\varepsilon\cos\theta}{\varepsilon^2}\cdot\varepsilon\cos\theta\right]\mathrm{d}\theta\\&=\int_0^{2\pi}\mathrm{d}\theta=2\pi\end{aligned}$$

于是所求积分为

$$\oint_L f(x,y)\mathrm{d}s=\begin{cases}0, P_0\notin D\\2\pi, P\in D\end{cases}$$