北京大学数学科学学院2023-24高等数学B1期中考试

1.(20分)

(1) (6分) 求序列极限

$$\lim_{n\to\infty} \sqrt[n]{2 + \cos n}$$

(2) (7分) 求序列极限

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sin\left(\frac{i}{n} - \frac{1}{2n^i}\right)$$

(3) (7分) 求函数极限

$$\lim_{x\to 0} \left(1 + \tan^2 x\right)^{\frac{1}{\sin^2 x}}$$

Solution.

(1) Solution.

由 $-1 \leqslant \cos n \leqslant 1$ 有

$$\sqrt[n]{1} \leqslant \sqrt[n]{2 + \cos n} \leqslant \sqrt[n]{3}$$

而

$$\lim_{n\to\infty} \sqrt[n]{1} = \lim_{n\to\infty} \sqrt[n]{3} = 1$$

夹逼可得

$$\lim_{n\to\infty} \sqrt[n]{2 + \cos n} = 1$$

(2) Solution.

注意到

$$\frac{i-1}{n} < \frac{i}{n} - \frac{1}{2n^i} < \frac{i}{n}$$

依Riemann积分的定义有

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sin\left(\frac{i}{n} - \frac{1}{2n^i}\right) = \int_0^1 \sin x \, \mathrm{d}x = 1 - \cos 1$$

(3) Solution.

$$
\begin{aligned}
\lim_{x\to 0} \left(1 + \tan^2 x\right)^{\frac{1}{\sin^2 x}} &= \lim_{x\to 0} \left(1 + \tan^2 x\right)^{1 + \frac{1}{\tan^2 x}} \\
&= \lim_{x\to 0} \left(1 + \tan^2 x\right)^{\frac{1}{\tan^2 x}} \cdot \lim_{x\to 0} \left(1 + \tan^2 x\right) \\
&= \mathrm{e} \cdot 1 \\
&= \mathrm{e}
\end{aligned}
$$

2.(20分)

(1) (6分) 设$x > 0$,求出函数

$$f(x) = x^{\sqrt{x}}$$

的导函数$f'(x)$.

(2) (7分) 设$x < 1$,求出函数

$$g(x) = \int_0^{\sin x} \frac{\mathrm{d}t}{\sqrt{1-t^3}}$$

的导函数$g'(x)$.

(3) (7分) 设$x \neq \pm 1$,求出函数

$$h(x) = \frac{1}{x^2 - 1}$$

的四阶导函数$h^{(4)}(x)$.

Solution.

(1) Solution.

置$y = \ln(f(x)) = \sqrt{x}\ln x$,则

$$f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}f(x)}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}\mathrm{e}^y}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$
$$= \mathrm{e}^y \left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} \right)$$
$$= \frac{x^{\sqrt{x}}(\ln x + 2)}{2\sqrt{x}}$$

(2) Solution.

置$y = \sin x$,则有

$$g'(x) = \frac{\mathrm{d}g(x)}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$
$$= \frac{1}{\sqrt{1-y^3}} \cdot \cos x$$
$$= \frac{\cos x}{\sqrt{1-\sin^3 x}}$$

(3) Solution.

由

$$h(x) = \frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1} \right)$$

有

$$h^{(4)}(x) = \frac{1}{2}\left[\left(\frac{1}{x-1} \right)^{(4)} - \left(\frac{1}{x+1} \right)^{(4)} \right]$$

$$= \frac{1}{2} \left[4! \left(x - 1 \right)^{-5} - 4! \left(x + 1 \right)^{-5} \right]$$
$$= \frac{12}{(x-1)^5} - \frac{12}{(x+1)^5}$$

3.(15分)

求不定积分

$$\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)(x-1)^5}}$$

Solution.

置$t = \sqrt[3]{\dfrac{x+1}{x-1}} = \left(1 + \dfrac{2}{x-1} \right)^{\frac{1}{3}}$,则

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{3 \left(1 + \dfrac{2}{x-1} \right)^{\frac{2}{3}}} \cdot \left(-\frac{2}{(x-1)^2} \right)$$
$$= -\frac{2}{3}(x+1)^{-\frac{2}{3}}(x-1)^{-\frac{4}{3}}$$

从而

$$t\mathrm{d}t = -\frac{2}{3}(x+1)^{-\frac{1}{3}}(x+1)^{-\frac{5}{3}}\mathrm{d}x$$

从而

$$\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)(x-1)^5}} = \int -\frac{3}{2}t\mathrm{d}t$$
$$= -\frac{3}{4}t^2 + C$$
$$= -\frac{3}{4} \left(\frac{x+1}{x-1} \right)^{\frac{2}{3}} + C, C\text{为积分常数}$$

4.(15分)

设K是曲线弧$y = \mathrm{e}^x \ (0 \leqslant x \leqslant 1)$与直线$x = 0, x = 1, y = 0$围成的曲边梯形绕$x$轴旋转一周形成的旋转体,求$K$的侧面积.

Solution.

由题意可得 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x = y$,则

$$S = 2\pi \int_0^1 y\sqrt{1 + y'^2}\mathrm{d}x$$
$$= 2\pi \int_1^{\mathrm{e}} \sqrt{1 + y^2}\mathrm{d}y$$

而

$$\int \sqrt{1 + y^2}\mathrm{d}y = y\sqrt{1 + y^2} - \int y\mathrm{d}\sqrt{1 + y^2}$$
$$= y\sqrt{1 + y^2} - \int \frac{y^2\mathrm{d}y}{\sqrt{1 + y^2}}$$
$$= y\sqrt{1 + y^2} - \int \sqrt{y^2 + 1}\mathrm{d}y + \int \frac{\mathrm{d}y}{\sqrt{y^2 + 1}}$$

从而

$$\int \sqrt{1 + y^2}\mathrm{d}y = \frac{1}{2}\left(y\sqrt{y^2 + 1} + \int \frac{\mathrm{d}y}{\sqrt{y^2 + 1}}\right)$$
$$= \frac{1}{2}\left(y\sqrt{y^2 + 1} + \ln\left|y + \sqrt{1 + y^2}\right|\right) + C$$

从而

$$S = \pi\left(y\sqrt{y^2 + 1} + \ln\left|y + \sqrt{y^2 + 1}\right|\right)\Big|_1^{\mathrm{e}}$$
$$= \pi\left(\mathrm{e}\sqrt{1 + \mathrm{e}^2} + \ln\left(\mathrm{e} + \sqrt{1 + \mathrm{e}^2}\right) - \sqrt{2} - \ln 2\right)$$

5.(10分)

设 $a, b, c \in \mathbb{R}$ 且 $a, b, c > 0$, $f : \mathbb{R} \to \mathbb{R}$ 在 $\mathbb{R}$ 上连续,且

$$f(0) = -a, \lim_{x \to -\infty} f(x) = b, \lim_{x \to +\infty} f(x) = c$$

求证 $f(x) = 0$ 在 $\mathbb{R}$ 上至少有两个不相等的实根 r_1, r_2.

Proof.

由 $\lim\limits_{x \to -\infty} f(x) = b$ 可知

$$\forall \varepsilon > 0, \exists A \in \mathbb{R}, \text{s.t.} \forall x < A, |f(x) - b| < \varepsilon$$

取 $\varepsilon \in (0, b)$ 和对应的 A,则 $0 < b - \varepsilon < f(x) < b + \varepsilon$.

6.(20分)

设

$$A(r) = \int_0^{2\pi} \ln(1 - 2r\cos x + r^2)\mathrm{d}x$$

(1) (12分) 试证明$\forall r \in (-1, 1), A(r^2) = 2A(r).$

(2) (4分) 试证明$A(r)$在$\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$上有界.

(3) (4分) 试计算$r \in (-1, 1)$时$A(r)$的值.

Solution

(1) Proof.