北京大学数学科学学院2023-24高等数学A1期中考试

1.(20分)

求下列各极限.

(1) $\lim\limits_{n\to\infty}\dfrac{3^n}{n!}$

(2) $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{i}{(n+i)^3}$

(3) $\lim\limits_{x\to+\infty}\sin\left(\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\pi\right)$

(4) $\lim\limits_{n\to\infty}\left[\dfrac{1}{n^2}\sum\limits_{i=1}^{n}i\ln(n+i)-\dfrac{n+1}{2n}\ln n\right]$

Solution.

(1) 当$n\geqslant 6$时有

$$0<\frac{3^n}{n!}\leqslant\frac{3^5}{5!}\cdot\frac{3^{n-5}}{6^{n-5}}=\frac{6^5}{5!}\cdot\frac{1}{2^n}$$

而

$$\lim_{n\to\infty}\left(\frac{6^5}{5!}\cdot\frac{1}{2^n}\right)=\frac{6^5}{5!}\cdot\lim_{n\to\infty}\frac{1}{2^n}=0$$

夹逼可得

$$\lim_{n\to\infty}\frac{3^n}{n!}=0$$

(2) 由题意

$$0<\sum_{i=1}^{n}\frac{i}{(n+i)^3}<\sum_{i=1}^{n}\frac{i}{n^3}<\sum_{i=1}^{n}\frac{n}{n^3}=\frac{1}{n}$$

而

$$\lim_{n\to\infty}\frac{1}{n}=0$$

夹逼可得

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{(n+i)^3}=0$$

(3)

$$\begin{aligned}\lim_{x\to+\infty}\sin\left(\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\pi\right)&=\lim_{x\to+\infty}\sin\left(\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}\pi\right)\\&=\lim_{x\to+\infty}\sin\left(\frac{2}{\sqrt{1+\dfrac{1}{x}}+\sqrt{1-\dfrac{1}{x}}}\pi\right)\\&=\sin\pi\\&=0\end{aligned}$$

(4) 由题意有

$$\begin{aligned}\frac{1}{n^2}\sum_{i=1}^{n} i\ln(n+i) - \frac{n+1}{2n}\ln n &= \frac{1}{n^2}\sum_{i=1}^{n} i\ln(n+i) - \frac{1}{n^2}\sum_{i=1}^{n} i\ln n\\ &= \frac{1}{n^2}\sum_{i=1}^{n} i\ln\left(\frac{n+i}{n}\right)\\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}\ln\left(1+\frac{i}{n}\right)\end{aligned}$$

根据Riemann积分的定义有

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}\ln\left(1+\frac{i}{n}\right) &= \int_0^1 x\ln(1+x)\mathrm{d}x\\ &= \int_1^2 (x-1)\ln x\mathrm{d}x\\ &= \left(\frac{x^2}{2}\ln x - x\ln x - \frac{x^2}{4}+x\right)\Bigg|_1^2\\ &= \frac{1}{4}\end{aligned}$$

从而

$$\lim_{n\to\infty}\left[\frac{1}{n^2}\sum_{i=1}^{n} i\ln(n+i) - \frac{n+1}{2n}\ln n\right] = \frac{1}{4}$$

2.(20分)

计算下列各题并适当化简.

(1) 设$y = x\sqrt{1+x^2}+\ln\left(x+\sqrt{1+x^2}\right)$,求$\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(2) 设

$$y = \begin{cases} x^4\sin\dfrac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$$

求$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

(3) 设$y = \displaystyle\int_{\cot x}^{\tan x}\sqrt{1+t^2}\mathrm{d}t$,求$\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(4) 设$F(x) = f(x)-f''(x)+f^{(4)}(x)-\cdots+(-1)^n f^{(2n)}(x)$,其中$f(x)=x^n(1-x)^n$,
求$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(F'(x)\sin x - F(x)\cos x\right)$.

Solution.

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1+x^2}+\dfrac{x^2}{\sqrt{1+x^2}}+\dfrac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\dfrac{x}{\sqrt{1+x^2}}\right) = 2\sqrt{1+x^2}$

(2) 当$x \neq 0$时有

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 \sin\frac{1}{x} + x^4 \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) = 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 \sin\frac{1}{x} - 4x\cos\frac{1}{x} - 2x\cos\frac{1}{x} + \sin\frac{1}{x} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 \sin\frac{1}{x} - 6x\cos\frac{1}{x} + \sin\frac{1}{x}$$

而

$$\lim_{x\to 0}\frac{4x^3 \sin\frac{1}{x} - x^2\cos\frac{1}{x} - 0}{x-0} = \lim_{x\to 0}\left(4x^2\sin\frac{1}{x} - x\cos\frac{1}{x}\right) = 0$$

从而

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \begin{cases} 12x^2\sin\frac{1}{x} - 6x\cos\frac{1}{x} + \sin\frac{1}{x},\ x \neq 0 \\ 0,\ x = 0 \end{cases}$$

(3) 由题意

$$y = \int_{\cot x}^{\tan x} \sqrt{1+t^2}\mathrm{d}t = \int_0^{\tan x} \sqrt{1+t^2}\mathrm{d}t - \int_0^{\cot x} \sqrt{1+t^2}\mathrm{d}t$$

从而

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{1+\tan^2 x}}{\cos^2 x} + \frac{\sqrt{1+\cot^2 x}}{\sin^2 x} = \frac{1}{|\cos^3 x|} + \frac{1}{|\sin^3 x|}$$

(4) 由题意

$$F''(x) = f''(x) - f^{(4)}(x) + \cdots + (-1)^{2n+2} f^{(2n+2)}(x)$$

从而

$$F(x) + F''(x) = f(x) + (-1)^{2n+2} f^{(2n+2)}(x)$$

而

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F'(x)\sin x - F(x)\cos x\right) = \left(F(x) + F''(x)\right)\sin x$$

而$f(x) = x^n(1-x)^n$为$2n$次多项式,从而$f^{(2n+2)}(x) = 0$.

故

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F'(x)\sin x - F(x)\cos x\right) = x^n(1-x)^n \sin x$$

3.(15分)

计算下列不定积分.

(1) $\displaystyle\int \sqrt{1+x^2}\mathrm{d}x$

(2) $\displaystyle\int \frac{\arctan \mathrm{e}^x}{\mathrm{e}^x + \mathrm{e}^{-x}}\mathrm{d}x$

(3) 设$y = y(x)$是由方程$y^2(x-y) = x^2$确定的隐函数,求$\displaystyle\int \frac{\mathrm{d}x}{y^2}$.

Solution

(1) 置$I=\int\sqrt{1+x^2}\mathrm{d}x$,则

$$\begin{aligned}
I&=\int\sqrt{1+x^2}\mathrm{d}x\\
&=x\sqrt{1+x^2}-\int x\mathrm{d}\left(\sqrt{1+x^2}\right)\\
&=x\sqrt{1+x^2}-\int\frac{x^2+1-1}{\sqrt{1+x^2}}\mathrm{d}x\\
&=x\sqrt{1+x^2}+\int\frac{\mathrm{d}x}{\sqrt{1+x^2}}-I
\end{aligned}$$

从而

$$\int\sqrt{1+x^2}\mathrm{d}x=\frac{1}{2}\left(x\sqrt{1+x^2}+\int\frac{\mathrm{d}x}{\sqrt{1+x^2}}\right)=\frac{x}{2}\sqrt{x^2+1}+\frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)+C$$

(2) 置$u=\mathrm{e}^x$,则$\dfrac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{e}^x=u$.于是

$$\begin{aligned}
\int\frac{\arctan\mathrm{e}^x}{\mathrm{e}^x+\mathrm{e}^{-x}}\mathrm{d}x&=\int\frac{\arctan u}{u\left(u+u^{-1}\right)}\mathrm{d}u\\
&=\int\frac{\arctan u}{u^2+1}\mathrm{d}u\\
&=\int\arctan u\mathrm{d}\left(\arctan u\right)\\
&=\frac{1}{2}\left(\arctan\mathrm{e}^x\right)^2+C
\end{aligned}$$

(3) 置$t=\dfrac{x}{y}$,于是$x=\dfrac{t^3}{t-1},y=\dfrac{t^2}{t-1},\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{3t^2(t-1)-t^3}{(t-1)^2}=\dfrac{2t^3-3t^2}{(t-1)^2}$ 则

$$\begin{aligned}
\int\frac{\mathrm{d}x}{y^2}&=\int\frac{(t-1)^2}{t^4}\cdot\frac{2t^3-3t^2}{(t-1)^2}\mathrm{d}t\\
&=\int\left(\frac{2}{t}-\frac{3}{t^2}\right)\mathrm{d}t\\
&=2ln\left|t\right|+\frac{3}{t}+C\\
&=2ln\left|x\right|-2ln\left|y\right|+\frac{3x}{y}+C
\end{aligned}$$

4.(10分)

试确定实数a,b使得函数

$$f(x)=\lim_{n\to\infty}\frac{x^{2n-1}+ax^2+bx}{x^{2n}+1}$$

成为$\mathbb{R}$上的连续函数.

Solution.

当$|x|>1$时,有

$$f(x)=\lim_{n\to\infty}\frac{x^{2n-1}+ax^2+bx}{x^{2n}+1}=\lim_{n\to\infty}\left(\frac{1}{x+\frac{1}{x^{2n-1}}}+\frac{a}{x^{2n-2}+\frac{1}{x^2}}+\frac{b}{x^{2n-1}+\frac{1}{x}}\right)=\frac{1}{x}$$

当$|x|<1$时,有

$$f(x)=\lim_{n\to\infty}\frac{x^{2n-1}+ax^2+bx}{x^{2n}+1}=ax^2+bx$$

故$f(x)$在$x=1$处的左右极限分别为

$$\lim_{x\to 1^+}=1,\ \lim_{x\to 1^-}=a+b$$

在$x=-1$处的左右极限分别为

$$\lim_{x\to -1^-}=-1,\ \lim_{x\to -1^+}=a-b$$

由于$f(x)$在$\mathbb{R}$上连续,则有

$$\begin{cases} a+b=1 \\ a-b=-1 \end{cases}$$

从而$a=0,b=1$.此时$f(1)=1,f(-1)=-1$,成立.

5.(15分)

计算下列定积分.

(1) $\displaystyle\int_0^1\frac{\sqrt{x}}{1+\sqrt{x}}\mathrm{d}x$

(2) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2 x}{1+\mathrm{e}^x}\mathrm{d}x$

(3) $\displaystyle\int_0^{\pi}\left(\int_0^x\frac{\sin t}{\pi-t}\mathrm{d}t\right)\mathrm{d}x$

Solution.

(1) 置$t=\sqrt{x}$,则$\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{1}{2\sqrt{x}}=\dfrac{1}{2t}$.于是

$$\int_0^1\frac{\sqrt{x}}{1+\sqrt{x}}\mathrm{d}x=2\int_0^1\frac{t^2\mathrm{d}t}{1+t}=2\left(\int_0^1(t-1)\mathrm{d}t+\int_0^1\frac{\mathrm{d}t}{1+t}\right)=2\ln 2-1$$

(2) 置$t=-x$,于是

$$
\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\mathrm{e}^x}\mathrm{d}x &= \int_{-\frac{\pi}{2}}^{0} \frac{\sin^2 x}{1+\mathrm{e}^x}\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\mathrm{e}^x}\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{1+\mathrm{e}^{-t}}\mathrm{d}t + \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\mathrm{e}^x}\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{\sin^2 x}{1+\mathrm{e}^x} - \frac{\sin^2 x}{1+\mathrm{e}^{-x}}\right)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \sin^2 x\mathrm{d}x \\
&= \frac{\pi}{4}
\end{aligned}
$$

(3) 置$f(x)=\int_0^x \frac{\sin t}{\pi - t}\mathrm{d}t$,则

$$
\begin{aligned}
\int_0^{\pi}\left(\int_0^x \frac{\sin t}{\pi-t}\mathrm{d}t\right)\mathrm{d}x &= \int_0^{\pi} f(x)\mathrm{d}x \\
&= xf(x)|_0^{\pi} - \int_0^{\pi} xf'(x)\mathrm{d}x \\
&= \pi\int_0^{\pi}\frac{\sin t}{\pi-t}\mathrm{d}t - \int_0^{\pi}\frac{x\sin x}{\pi-x} \\
&= \int_0^{\pi}\frac{(\pi-x)\sin x}{\pi-x}\mathrm{d}x \\
&= \int_0^{\pi}\sin x\mathrm{d}x \\
&= 2
\end{aligned}
$$

6.(10分)

设$f(x)$在$[0,1]$上Riemann可积,求

$$
\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}f\left(\frac{i}{n}\right)
$$

Solution.

我们有

$$
\frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}f\left(\frac{i}{n}\right) = \frac{1}{n}\sum_{i=1}^{\left[\frac{n}{2}\right]} f\left(\frac{2i-1}{n}\right) - \frac{1}{n}\sum_{i=1}^{\left[\frac{n}{2}\right]} f\left(\frac{2i}{n}\right)
$$

根据Riemann积分的定义有

$$
\int_0^1 f(x)\mathrm{d}x = \lim_{n\to\infty}\frac{1}{m}\sum_{i=1}^{m} f\left(\frac{i}{m}\right) = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{\left[\frac{n}{2}\right]} f\left(\frac{2i}{n}\right)
$$

于是

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}f\left(\frac{i}{n}\right) &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{\left[\frac{n}{2}\right]}f\left(\frac{2i-1}{n}\right)-\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{\left[\frac{n}{2}\right]}f\left(\frac{2i}{n}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{\left[\frac{n}{2}\right]}f\left(\frac{2i-1}{n}\right)+\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{\left[\frac{n}{2}\right]}f\left(\frac{2i}{n}\right)-2\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{\left[\frac{n}{2}\right]}f\left(\frac{2i}{n}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{i}{n}\right)-\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{\left[\frac{n}{2}\right]}f\left(\frac{2i}{n}\right)\\
&= \int_0^1 f(x)\mathrm{d}x-\int_0^1 f(x)\mathrm{d}x\\
&= 0
\end{aligned}$$

7.(10分)

设$f(x)$在$[0,+\infty)$上连续,$f(0)=0$,且$\forall x>0, 0<f(x)<x$. 令

$$a_1=f(1), a_2=f(a_1),\cdots,a_n=f(a_{n-1}), n=2,3,\cdots$$

证明: $\lim\limits_{n\to\infty}a_n=0$.

Proof.

由$\forall x>0, 0<f(x)<x$有$\forall n\in\mathbb{N}^*, 0<a_{n+1}=f(a_n)<a_n$.

即$\{a_n\}$单调递减且有下界0.不妨设$\lim\limits_{n\to\infty}a_n=A$.

由$f(x)$在$[0,+\infty)$连续,对递推式求极限有

$$A=\lim_{n\to\infty}f\left(a_{n-1}\right)=\lim_{n\to\infty}f\left(a_n\right)=f(A)$$

由题意可知当且仅当$x=0$时$f(x)=x$.于是$A=0$,即$\lim\limits_{n\to\infty}a_n=0$,原命题得证.