

有理分式的待定系数技巧

我们经常做到一些奇奇怪怪的有理分式积分.

Problem 1(2023Fall PKU高等数学期中考试)

求不定积分

$$\int \frac{4x^2 + 4x - 11}{(2x - 1)(2x + 3)(2x - 5)} dx$$

通常,我们会采取下面的做法.

Solution(Method I).

设

$$\begin{aligned} \frac{4x^2 + 4x - 11}{(2x - 1)(2x + 3)(2x - 5)} &= \frac{A}{2x - 1} + \frac{B}{2x + 3} + \frac{C}{2x - 5} \\ &= \frac{A(4x^2 - 4x - 15) + B(4x^2 - 12x + 5) + C(4x^2 + 4x - 3)}{(2x - 1)(2x + 3)(2x - 5)} \\ &= \frac{4(A + B + C)x^2 + 4(C - A - 3B)x + (5B - 15A - 3C)}{(2x - 1)(2x + 3)(2x - 5)} \end{aligned}$$

从而

$$\begin{cases} A + B + C = 1 \\ C - A - 3B = 1 \\ 5B - 15A - 3C = -11 \end{cases}$$

解得 $A = \frac{1}{2}, B = -\frac{1}{4}, C = \frac{3}{4}$.

从而

$$\begin{aligned} \int \frac{4x^2 + 4x - 11}{(2x - 1)(2x + 3)(2x - 5)} dx &= \int \left(\frac{1}{2} \cdot \frac{1}{2x - 1} - \frac{1}{4} \cdot \frac{1}{2x + 3} + \frac{3}{4} \cdot \frac{1}{2x - 5} \right) dx \\ &= -\frac{1}{2} \int \frac{dx}{2x - 1} + \frac{1}{4} \int \frac{dx}{2x + 3} + \frac{3}{4} \int \frac{dx}{2x - 5} \\ &= -\frac{1}{4} \ln |2x - 1| + \frac{1}{8} \ln |2x + 3| + \frac{3}{8} \ln |2x - 5| + C \end{aligned}$$

然而,这样求系数的方法十分麻烦.我们可以采取一种更简便的方法.

Solution(Method II).

设

$$\frac{4x^2 + 4x - 11}{(2x - 1)(2x + 3)(2x - 5)} = \frac{A}{2x - 1} + \frac{B}{2x + 3} + \frac{C}{2x - 5}$$

两端同乘 $2x - 1$ 有

$$\frac{4x^2 + 4x - 11}{(2x + 3)(2x - 5)} = A + \frac{B(2x - 1)}{2x + 3} + \frac{C(2x - 1)}{2x - 5}$$

令 $2x - 1 = 0$, 即 $x = \frac{1}{2}$, 于是 $A = \frac{4 \cdot \left(\frac{1}{2}\right)^2 + 4 \cdot \frac{1}{2} - 11}{4 \cdot (-4)} = \frac{1}{2}$

用同样的方法可以得到 $B = -\frac{1}{4}$, $C = \frac{3}{4}$. 从而拆分后进行积分.