# 北京大学数学科学学院2022-23高等数学B1期中考试

**1.(20分)**

**(1)** **(6分)** 求序列极限

$$\lim_{n\to\infty}\sqrt[n]{2+\cos n}$$

**(2)** **(7分)** 求序列极限

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sin\left(\frac{i}{n}-\frac{1}{2n^i}\right)$$

**(3)** **(7分)** 求函数极限

$$\lim_{x\to 0}\left(1+\tan^2 x\right)^{\frac{1}{\sin^2 x}}$$

**Solution.**

**(1)** **Solution.**

由$-1\leqslant\cos n\leqslant 1$有

$$\sqrt[n]{1}\leqslant\sqrt[n]{2+\cos n}\leqslant\sqrt[n]{3}$$

而

$$\lim_{n\to\infty}\sqrt[n]{1}=\lim_{n\to\infty}\sqrt[n]{3}=1$$

夹逼可得

$$\lim_{n\to\infty}\sqrt[n]{2+\cos n}=1$$

**(2)** **Solution.**

注意到

$$\frac{i-1}{n}<\frac{i}{n}-\frac{1}{2n^i}<\frac{i}{n}$$

依Riemann积分的定义有

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sin\left(\frac{i}{n}-\frac{1}{2n^i}\right)=\int_0^1\sin x\mathrm{d}x=1-\cos 1$$

**(3)** **Solution.**

$$\begin{aligned}\lim_{x\to 0}\left(1+\tan^2 x\right)^{\frac{1}{\sin^2 x}}&=\lim_{x\to 0}\left(1+\tan^2 x\right)^{1+\frac{1}{\tan^2 x}}\\&=\lim_{x\to 0}\left(1+\tan^2 x\right)^{\frac{1}{\tan^2 x}}\cdot\lim_{x\to 0}\left(1+\tan^2 x\right)\\&=\mathrm{e}\cdot 1\\&=\mathrm{e}\end{aligned}$$

**2.(20分)**

**(1)** **(6分)** 设$x > 0$,求出函数

$$f(x) = x^{\sqrt{x}}$$

的导函数$f'(x)$.

**(2)** **(7分)** 设$x < 1$,求出函数

$$g(x) = \int_0^{\sin x} \frac{\mathrm{d}t}{\sqrt{1-t^3}}$$

的导函数$g'(x)$.

**(3)** **(7分)** 设$x \neq \pm 1$,求出函数

$$h(x) = \frac{1}{x^2-1}$$

的四阶导函数$h^{(4)}(x)$.

**Solution.**

**(1)** **Solution.**

置$y = \ln(f(x)) = \sqrt{x}\ln x$,则

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}f(x)}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}\mathrm{e}^y}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \mathrm{e}^y \left( \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} \right) \\ &= \frac{x^{\sqrt{x}}(\ln x + 2)}{2\sqrt{x}} \end{aligned}$$

**(2)** **Solution.**

置$y = \sin x$,则有

$$\begin{aligned} g'(x) &= \frac{\mathrm{d}g(x)}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \frac{1}{\sqrt{1-y^3}} \cdot \cos x \\ &= \frac{\cos x}{\sqrt{1-\sin^3 x}} \end{aligned}$$

**(3)** **Solution.**

由

$$h(x) = \frac{1}{x^2-1} = \frac{1}{2}\left( \frac{1}{x-1} - \frac{1}{x+1} \right)$$

有

$$h^{(4)}(x) = \frac{1}{2}\left[ \left( \frac{1}{x-1} \right)^{(4)} - \left( \frac{1}{x+1} \right)^{(4)} \right]$$

$$
\begin{aligned}
&= \frac{1}{2}\left[4!\,(x-1)^{-5} - 4!\,(x+1)^{-5}\right] \\
&= \frac{12}{(x-1)^5} - \frac{12}{(x+1)^5}
\end{aligned}
$$

**3.(15分)**

求不定积分

$$
\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)(x-1)^5}}
$$

**Solution.**

置$t = \sqrt[3]{\dfrac{x+1}{x-1}} = \left(1 + \dfrac{2}{x-1}\right)^{\frac{1}{3}}$,则

$$
\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}x} &= \frac{1}{3\left(1+\dfrac{2}{x-1}\right)^{\frac{2}{3}}} \cdot \left(-\frac{2}{(x-1)^2}\right) \\
&= -\frac{2}{3}(x+1)^{-\frac{2}{3}}(x-1)^{-\frac{4}{3}}
\end{aligned}
$$

从而

$$
t\mathrm{d}t = -\frac{2}{3}(x+1)^{-\frac{1}{3}}(x+1)^{-\frac{5}{3}}\mathrm{d}x
$$

从而

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)(x-1)^5}} &= \int -\frac{3}{2}t\mathrm{d}t \\
&= -\frac{3}{4}t^2 + C \\
&= -\frac{3}{4}\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}} + C, C\text{为积分常数}
\end{aligned}
$$

**4.(15分)**

设$K$是曲线弧$y = \mathrm{e}^x \ (0 \leqslant x \leqslant 1)$与直线$x = 0, x = 1, y = 0$围成的曲边梯形绕$x$轴旋转一周形成的旋转体,求$K$的侧面积.

**Solution.**

由题意可得$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x=y$,则

$$\begin{aligned}S&=2\pi\int_0^1 y\sqrt{1+y'^2}\mathrm{d}x\\&=2\pi\int_1^{\mathrm{e}}\sqrt{1+y^2}\mathrm{d}y\end{aligned}$$

而

$$\begin{aligned}\int\sqrt{1+y^2}\mathrm{d}y&=y\sqrt{1+y^2}-\int y\mathrm{d}\sqrt{1+y^2}\\&=y\sqrt{1+y^2}-\int\frac{y^2\mathrm{d}y}{\sqrt{1+y^2}}\\&=y\sqrt{1+y^2}-\int\sqrt{y^2+1}\mathrm{d}y+\int\frac{\mathrm{d}y}{\sqrt{y^2+1}}\end{aligned}$$

从而

$$\begin{aligned}\int\sqrt{1+y^2}\mathrm{d}y&=\frac{1}{2}\left(y\sqrt{y^2+1}+\int\frac{\mathrm{d}y}{\sqrt{y^2+1}}\right)\\&=\frac{1}{2}\left(y\sqrt{y^2+1}+\ln\left|y+\sqrt{1+y^2}\right|\right)+C\end{aligned}$$

从而

$$\begin{aligned}S&=\pi\left(y\sqrt{y^2+1}+\ln\left|y+\sqrt{y^2+1}\right|\right)\Big|_1^{\mathrm{e}}\\&=\pi\left(\mathrm{e}\sqrt{1+\mathrm{e}^2}+\ln\left(\mathrm{e}+\sqrt{1+\mathrm{e}^2}\right)-\sqrt{2}-\ln 2\right)\end{aligned}$$

**5.(10分)**

设$a,b,c\in\mathbb{R}$且$a,b,c>0$,$f:\mathbb{R}\to\mathbb{R}$在$\mathbb{R}$上连续,且

$$f(0)=-a,\lim_{x\to-\infty}f(x)=b,\lim_{x\to+\infty}f(x)=c$$

求证$f(x)=0$在$\mathbb{R}$上至少有两个不相等的实根$r_1,r_2$.

**Proof.**

由$\lim\limits_{x\to-\infty}f(x)=b$可知

$$\forall\varepsilon>0,\exists A\in\mathbb{R},\text{s.t.}\forall x<A,|f(x)-b|<\varepsilon$$

取$\varepsilon\in(0,b)$和对应的$A$,则$0<b-\varepsilon<f(x)<b+\varepsilon$.

从而$\exists x<0, \text{s.t.} f(x)>0>f(0)=-a$.

根据连续函数的介值定理,必然$\exists \xi_1\in(-\infty,0), \text{s.t.} f(\xi_1)=0$.

同理亦可知$\exists \xi_2\in(0,+\infty), \text{s.t.} f(\xi_2)=0$.

从而原命题得证.

**6.(20分)**

设

$$A(r)=\int_0^{2\pi}\ln(1-2r\cos x+r^2)\mathrm{d}x$$

**(1) (12分)** 试证明$\forall r\in(-1,1), A(r^2)=2A(r)$.

**(2) (4分)** 试证明$A(r)$在$\left(-\frac{1}{2},\frac{1}{2}\right)$上有界.

**(3) (4分)** 试计算$r\in(-1,1)$时$A(r)$的值.

**Solution**

**(1) Proof.**

注意到

$$1-2r\cos x+r^2=1+2r\cos(2\pi-x)+r^2$$

从而

$$\begin{aligned}\int_0^{2\pi}\ln(1-2r\cos x+r^2)\mathrm{d}x&=\int_{2\pi}^{0}\ln(1+2r\cos x+r^2)\mathrm{d}(2\pi-x)\\&=\int_0^{2\pi}\ln(1+2r\cos x+r^2)\mathrm{d}x\end{aligned}$$

从而

$$\begin{aligned}2A(r)&=\int_0^{2\pi}\left(\ln(1-2r\cos x+r^2)+\ln(1+2r\cos x+r^2)\right)\mathrm{d}x\\&=\int_0^{2\pi}\ln(r^4+2r^2+1-4r^2\cos^2x)\mathrm{d}x\\&=\int_0^{2\pi}\ln(1-2r^2\cos 2x+r^4)\mathrm{d}x\end{aligned}$$

置$u=2x$,注意到

$$1-2r^2\cos u+r^4=1-2r^2\cos(4\pi-u)+r^4$$

从而

$$\begin{aligned}\int_0^{2\pi}\ln(1-2r^2\cos 2x+r^4)\mathrm{d}x&=\frac{1}{2}\int_0^{4\pi}\ln(1-2r^2\cos u+r^4)\mathrm{d}u\\&=\frac{1}{2}\left(\int_0^{2\pi}\ln(1-2r^2\cos u+r^4)\mathrm{d}u+\int_{2\pi}^{4\pi}\ln(1-2r^2\cos u+r^4)\mathrm{d}u\right)\\&=\int_0^{2\pi}\ln(1-2r^2\cos u+r^4)\mathrm{d}u\end{aligned}$$

从而原命题得证.

**(2) Proof.**

不难发现

$$1-2r\cos x+r^2=(1-r\cos x)^2+r^2(1-\cos^2 x)$$

当$r\in\left(-\frac{1}{2},\frac{1}{2}\right)$, $x\in(0,2\pi)$时有

$$\frac{1}{2}<1-r\cos x<\frac{3}{2}, 0<r^2<\frac{1}{4}, 0\leqslant\cos^2 x\leqslant 1$$

从而

$$\frac{1}{4}<1-2r\cos x+r^2<\frac{5}{2}$$

即

$$\int_0^{2\pi}\ln\frac{1}{4}\mathrm{d}x<\int_0^{2\pi}\ln(1-2r\cos x+r^2)\mathrm{d}x<\int_0^{2\pi}\ln\frac{5}{2}\mathrm{d}x$$

即

$$-4\pi\ln 2<A(r)<2\pi(\ln 5-\ln 2)$$

从而$A(r)$有界,原命题得证.

**(3) Solution.**

对于任意$r\in(-1,1)$,重复应用**(1)**的结论有

$$A(r)=\frac{A\left(r^2\right)}{2}=\cdots=\frac{A\left(r^{2n}\right)}{2^n}$$

当$n$充分大时$r^{2n}\in\left(-\frac{1}{2},\frac{1}{2}\right)$,从而$A\left(r^{2n}\right)$有界.

对上式取极限有

$$A(r)=\lim_{n\to\infty}\frac{A\left(r^{2n}\right)}{2^n}=0$$

从而$\forall r\in(-1,1), A(r)=0$.