北京大学数学科学学院2023-24高等数学B1期末考试

1.(10分)

求极限

$$\lim_{x\to 0}\frac{\sin\left(x-\int_0^x\sqrt{1+t^2}\mathrm{d}t\right)}{x^3}$$

Solution.

我们有

$$\begin{aligned}\lim_{x\to 0}\frac{\sin\left(x-\int_0^x\sqrt{1+t^2}\mathrm{d}t\right)}{x^3}&=\lim_{x\to 0}\frac{\cos\left(x-\int_0^x\sqrt{1+t^2}\mathrm{d}t\right)\left(1-\sqrt{1+x^2}\right)}{3x^2}\\&=1\cdot\lim_{x\to 0}\frac{-x^2}{3x^2\left(1+\sqrt{1-x^2}\right)}\\&=1\cdot\left(-\frac{1}{3}\right)\cdot\frac{1}{2}\\&=-\frac{1}{6}\end{aligned}$$

2.(10分)

设函数$f:[0,7]\to\mathbb{R}$为

$$f(x)=x^3-6x^2+9x-1$$

称区间$[a,b]$是f的单调区间,当$0\leqslant a<b\leqslant 7$且限制在$[a,b]$上的$f$严格单调.求$f$的长度最大的单调区间.

Solution.

对$f(x)$求导有

$$f'(x)=3x^2-12x+9=3(x-1)(x-3)$$

于是当$x\in[0,1)$或$x\in(3,7]$时$f(x)>0$.$x\in(1,3)$时$f(x)<0$.$x=1$或3时$f(x)=0$.

因而$f(x)$在$[0,1]$单调递增,在$[1,3]$单调递减,在$[3,7]$单调递增.

于是f的长度最大的单调区间为$[3,7]$,其长度为4.

3.(10分)

设欧氏空间$\mathbb{R}^3$中的平面$T:2x-y+3z=6$.设T与x,y,z三轴的交点分别为A,B,C.以原点$O(0,0,0)$为球心,与T相切的球面记作S.

(1) **(5分)** 求$\triangle ABC$的面积.

(2) **(5分)** 求球面S与T相切的点的坐标.

Solution.

(1) 分别令x,y,z三者中的两者为0,可解得$A(3,0,0),B(0,-6,0),C(0,0,2)$.于是

$$S_{\triangle ABC}=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\begin{vmatrix} i & j & k \\ -3 & -6 & 0 \\ -3 & 0 & 2 \end{vmatrix}=\frac{1}{2}\left|(12,6,-18)\right|=3\sqrt{14}$$

(2) 设切点为$P(x,y,z)$.设$\vec{u}$为T的法向量,根据T的一般式可知$\vec{u}=(2,-1,3)$.
由于S于T相切,因此$\overrightarrow{OP}\perp T$,于是$\overrightarrow{OP}//\vec{u}$.于是可以列出方程组

$$\begin{cases} 2x-y+3z=6 \\ \dfrac{x}{2}=-\dfrac{y}{1}=\dfrac{z}{3} \end{cases}$$

解得$x=\dfrac{6}{7},y=-\dfrac{3}{7},z=\dfrac{9}{7}$.于是$P\left(\dfrac{6}{7},-\dfrac{3}{7},\dfrac{9}{7}\right)$.

4.(10分)

设二元函数$z=f(x,y)$是由方程$F(x,y,z)=z^3+ze^x+y=0$确定的隐函数.求$z=f(x,y)$在$(0,2)$处函数值下降最快的方向上的单位向量.

Solution.

首先求F的偏导,有

$$\frac{\partial F}{\partial x}=z\mathrm{e}^x\qquad\frac{\partial F}{\partial y}=1\qquad\frac{\partial F}{\partial z}=3z^2+\mathrm{e}^x$$

当$(x,y)=(0,2)$时$F(x,y,z)=0$有$z^3+z+2=0$,即$(z+1)(z^2-z+2)=0$.故这方程仅有$z=-1$一实根.
当$(x,y,z)=(0,2,-1)$时$F_z(0,2,-1)=4\neq 0$.于是根据隐函数存在定理,$F(x,y,z)=0$在$(0,2)$处附近确定唯一的隐函数$z=f(x,y)$,并且有

$$\frac{\partial f}{\partial x}=-\frac{F_x(0,2,-1)}{F_z(0,2,-1)}=\frac{1}{4}\qquad\frac{\partial f}{\partial y}=-\frac{F_y(0,2,-1)}{F_z(0,2,-1)}=-\frac{1}{4}$$

于是f在$(0,2)$处的梯度向量为$\left(\dfrac{1}{4},-\dfrac{1}{4}\right)$.
取负梯度后单位化有$\left(-\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)$,此即所求向量.

5.(10分)

求函数$f(x,y)=x^y$在$(1,1)$处的二阶泰勒多项式.

Solution.

我们有

$$\frac{\partial f}{\partial x}=yx^{y-1}\qquad\frac{\partial f}{\partial y}=x^y\ln x$$

$$\frac{\partial^2 f}{\partial x^2}=y(y-1)x^{y-2}\qquad\frac{\partial^2 f}{\partial x\partial y}=x^{y-1}\left(1+y\ln x\right)\qquad\frac{\partial^2 f}{\partial y^2}=x^y\left(\ln x\right)^2$$

代入$x=1,y=1$可知

$$f(x,y)=1+(x-1)+(x-1)(y-1)$$

6.(10分)

设D是由直线$x+y=2\pi$,x轴和y轴围成的有界闭区域.求D上的二元函数$f(x,y)=\sin x+\sin y-\sin(x+y)$达到最大值的$D$中所有点.

Solution.

我们有

$$\frac{\partial f}{\partial x}=\cos x-\cos(x+y)\qquad\frac{\partial f}{\partial y}=\cos y-\cos(x+y)$$

令$\dfrac{\partial f}{\partial x}=\dfrac{\partial f}{\partial y}=0$,有

$$\cos x=\cos y=\cos(x+y)$$

若$x=y$,则有$\cos x=\cos 2x=2\cos^2 x-1$,解得$\cos x=-\dfrac{1}{2}$或1.

由于$x,y\geqslant 0$且$x+y\leqslant 2\pi$,于是$x=y=\dfrac{2\pi}{3}$或$x=y=0$.此时有

$$f\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)=\frac{3\sqrt{3}}{2}\qquad f(0,0)=0$$

若$x+y=2\pi$,又要求$\cos x=\cos(x+y)$,这是无解的.

于是D中所有使$f(x,y)$取得最大值的点为$\left(\dfrac{2\pi}{3},\dfrac{2\pi}{3}\right)$.

7.(10分)

回答下列问题.

(1) **(2分)** 举例说明:当z是(x,y)的函数,也是(t,u)的函数时,$x\equiv t\nRightarrow\dfrac{\partial z}{\partial x}\equiv\dfrac{\partial z}{\partial t}$.

(2) **(8分)** 给定方程

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=z^2$$

作变量代换

$$x=t,y=\frac{t}{1+tu},z=\frac{t}{1+tW}$$

试证明:

$$\frac{\partial W}{\partial t}=0$$

Solution.

(1) 令$z=x+y=tu+t$,于是$\frac{\partial z}{\partial x}=1,\frac{\partial z}{\partial t}=1+u$.显然当$x=t$时不一定有$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial t}$.

(2) 由$z=\frac{t}{1+tW}$有$W(t,u)=\frac{1}{z(x(t,u),y(t,u))}-\frac{1}{t}$.于是

$$\begin{aligned}\frac{\partial W}{\partial t}&=-\frac{1}{z^2}\cdot\frac{\partial z}{\partial t}+\frac{1}{t^2}\\&=-\frac{1}{z^2}\left(\frac{\partial z}{\partial x}+\frac{1}{(1+tu)^2}\frac{\partial z}{\partial y}\right)+\frac{1}{t^2}\\&=-\frac{1}{z^2}\left(\frac{\partial z}{\partial x}+\left(\frac{y}{x}\right)^2\frac{\partial z}{\partial y}\right)+\frac{1}{x^2}\\&=\frac{1}{x^2}-\frac{1}{x^2}\\&=0\end{aligned}$$

于是命题得证.

8.(15分)

证明下列恒等式.

(1) **(3分)** 对于任意$x\in\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$,有

$$2\int_0^x\frac{\mathrm{d}t}{\sqrt{1-t^2}}=\int_0^{2x\sqrt{1-x^2}}\frac{\mathrm{d}t}{\sqrt{1-t^2}}$$

(2) **(12分)** 对于任意$x\in\left(-\frac{1}{\sqrt[4]{6}},\frac{1}{\sqrt[4]{6}}\right)$,有

$$2\int_0^x\frac{\mathrm{d}t}{\sqrt{1-t^4}}=\int_0^{\frac{2x\sqrt{1-x^4}}{1+x^4}}\frac{\mathrm{d}t}{\sqrt{1-t^4}}$$

Proof.

(1) 令$F(x)=2\int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^2}},G(x)=\int_0^{2x\sqrt{1-x^2}}\frac{\mathrm{d}t}{\sqrt{1-t^2}}$.于是$\frac{\mathrm{d}F}{\mathrm{d}x}=\frac{2}{\sqrt{1-x^2}}$.

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}x}&=\frac{2}{\sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}}\cdot\left(\sqrt{1-x^2}-\frac{x^2}{\sqrt{1-x^2}}\right)\\&=\frac{2}{\sqrt{4x^4-4x^2+1}}\cdot\frac{1-2x^2}{\sqrt{1-x^2}}=\frac{2}{\sqrt{1-x^2}}\end{aligned}$$

即$F'(x)=G'(x)$.又$F(0)=G(0)=0$.令$H(x)=F(x)-G(x)$.

根据Lagrange中值定理,对任意$x\in\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$且$x\neq 0$,存在$\xi\in(x,0)$或$(0,x)$使得

$$H'(\xi)=\frac{H(x)-H(0)}{x}=0$$

即$H(x)=H(0)=0$,于是$F(x)=G(x)$对任意$x\in\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$成立,从而命题得证.

(2) 令$F(x)=2\int_0^x\frac{\mathrm{d}t}{\sqrt{1-t^4}},G(x)=\int_0^{\frac{2x\sqrt{1-x^4}}{1+x^4}}\frac{\mathrm{d}t}{\sqrt{1-t^4}}$.于是$\frac{\mathrm{d}F}{\mathrm{d}x}=\frac{2}{\sqrt{1-x^4}}$.

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}x}&=\frac{2}{\sqrt{1-\left(\frac{2x\sqrt{1-x^4}}{1+x^4}\right)^4}}\cdot\left(\frac{\left(\sqrt{1-x^4}-\frac{2x^4}{\sqrt{1-x^4}}\right)(1+x^4)-4x^4\sqrt{1-x^4}}{\left(1+x^4\right)^2}\right)\\&=\frac{2}{\sqrt{\frac{x^{16}-12x^{12}+38x^8-12x^4+1}{(1+x^4)^4}}}\cdot\frac{x^8-6x^4+1}{\sqrt{1-x^4}\left(1+x^4\right)^2}\\&=\frac{2}{\sqrt{\left(\frac{x^8-6x^4+1}{(1+x^4)^2}\right)^2}}\cdot\frac{x^8-6x^4+1}{\sqrt{1-x^4}\left(1+x^4\right)^2}\\&=\frac{2}{\sqrt{1-x^4}}\end{aligned}$$

于是$F'(x)=G'(x)$,又$F(0)=G(0)=0$.与**(1)**同理可知$F(x)=G(x)$,命题得证.

9.(15分)

设函数$P(x)$在$[0,1]$连续,有$P(0)=0,P(1)=1$.$P(x)$在$(0,1)$可导,且对任意$x\in(0,1)$有$P'(x)>0$.任意取定$A,B\in\mathbb{R},n\in\mathbb{N}^*$.试证明:在$(0,1)$上存在$\theta_0,\cdots,\theta_n\in\mathbb{R}$使得

$$(A+B)^n=\sum_{k=0}^{n}\frac{1}{P'(\theta_k)}\frac{n!}{k!(n-k)!}A^{n-k}B^k$$

并且

$$0<\theta_0<\cdots<\theta_n<1$$

Proof.

设$a=\dfrac{A}{A+B},b=\dfrac{B}{A+B}$.于是要证的等式即

$$1=\sum_{k=0}^{n}\frac{1}{P'(\theta_k)}\frac{n!}{k!(n-k)!}a^{n-k}b^k$$

由于对于任意$x\in(0,1),P'(x)>0$,因而$P(x)$严格单调递增,因而$P(x)$是单射.

据Lagrange中值定理,对于任意$0\leqslant\alpha<\beta\leqslant 1$,存在$\xi\in(\alpha,\beta)$使得

$$P'(\xi)=\frac{P(\beta)-P(\alpha)}{\beta-\alpha}$$

即

$$\frac{1}{P'(\xi)}=\frac{\beta-\alpha}{P(\beta)-P(\alpha)}$$

现在,考虑序列$\{\phi_i\}_{i=0}^{n+1}$为

$$\phi_0=0,\phi_i=\sum_{k=0}^{i-1}\frac{n!a^{n-k}b^k}{k!(n-k!)}$$

于是$\{\phi_i\}$严格递增.记序列$\{\psi_i\}_{i=0}^{n+1}$使得$P(\psi_i)=\phi_i$.由于$P(x)$严格单调递增,于是ψ_i也是严格单调递增序列.

又因为$P(0)=0=\phi_0,P(1)=1=\phi_{n+1}$,于是$\psi_0=0,\psi_{n+1}=1$.

现在,对于任意$k\in[0,n]$,根据Lagrange中值定理都存在$\xi_k\in(\psi_k,\psi_{k+1})$使得

$$P'(\xi_k)=\frac{\psi_{k+1}-\psi_k}{P(\psi_{k+1})-P(\psi_k)}=\frac{\psi_{k+1}-\psi_k}{\phi_{k+1}-\phi_k}$$

于是我们有

$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{P'(\theta_k)}\frac{n!}{k!(n-k)!}a^{n-k}b^k&=\sum_{k=0}^{n}\frac{\psi_{k+1}-\psi_k}{\phi_{k+1}-\phi_k}\frac{n!}{k!(n-k)!}a^{n-k}b^k\\&=\sum_{k=0}^{n}(\psi_{k+1}-\psi_k)\\&=\psi_{n+1}-\psi_0\\&=1\end{aligned}$$

于是命题得证.