北京大学数学科学学院2024-25高等数学B1期中模拟

1.(15分)

解答下列问题.

(1) **(5分)** 求序列极限

$$\lim_{n\to\infty}\left|\cos\left(\sqrt{n^2+1}\pi\right)\right|$$

(2) **(5分)** 求函数极限

$$\lim_{x\to+\infty}\left(\frac{x^2-1}{x^2+1}\right)^{x^2}$$

(3) **(5分)** 求函数

$$f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, x\neq 0 \\ 0, x=0 \end{cases}$$

的导函数$f'(x)$.

(1) **Solution.**

$$\begin{aligned}\lim_{n\to\infty}\left|\cos\left(\pi\sqrt{n^2+1}\right)\right| &= \lim_{n\to\infty}\left|\cos\left(\pi\sqrt{n^2+1}-n\pi\right)\right| \\ &= \lim_{n\to\infty}\left|\cos\frac{\pi}{\sqrt{n^2+1}+n}\right| \\ &= \cos 0 \\ &= 1\end{aligned}$$

(2) **Solution.**

$$\begin{aligned}\lim_{x\to+\infty}\left(\frac{x^2-1}{x^2+1}\right)^{x^2} &= \lim_{x\to\infty}\left(1-\frac{2}{x^2+1}\right)^{\frac{x^2+1}{2}\cdot\frac{2x^2}{x^2+1}} \\ &= \mathrm{e}^{-2}\end{aligned}$$

(3) **Solution.**

当$x\neq 0$时有

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}=2x\sin\frac{1}{x}+x^2\left(-\frac{1}{x^2}\cos\frac{1}{x}\right)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$$

又

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{x^2\sin\dfrac{1}{x}}{x}=\lim_{x\to 0}x\sin\frac{1}{x}=0$$

于是

$$f'(x)=\begin{cases} 2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}, x\neq 0 \\ 0, x=0 \end{cases}$$

2.(10分)

设参数方程

$$\begin{cases} x = \mathrm{e}^t \sin 2t \\ y = \mathrm{e}^t \cos t, x = 0 \end{cases}$$

确定函数$y = f(x)$.求$\dfrac{\mathrm{d}y}{\mathrm{d}x}, \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

Solution.

我们有

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^t (\sin 2t + 2\cos 2t)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{e}^t (\cos t - \sin t)$$

于是

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^t (\cos t - \sin t)}{\mathrm{e}^t (\sin 2t + 2\cos 2t)} = \frac{\cos t - \sin t}{\sin 2t + 2\cos 2t}$$

于是

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{-(\sin t + \cos t)(\sin 2t + 2\cos 2t) - (\cos t - \sin t)(2\cos 2t - 4\sin 2t)}{(\sin 2t + 2\cos 2t)^2 \cdot \mathrm{e}^t (\sin 2t + 2\cos 2t)} \\ &= \frac{\cos t (-5 - 4\cos 2t + 3\sin 2t)}{\mathrm{e}^t (\sin 2t + 2\cos 2t)^3} \end{aligned}$$

3.(10分)

求不定积分

$$\int \frac{x^3 + 1}{x(x-1)^3} \mathrm{d}x$$

Solution.

置$u = x - 1$,于是

$$\int \frac{x^3 + 1}{x(x-1)^3} \mathrm{d}x = \int \frac{u^3 + 3u^2 + 3u + 2}{u^3(u+1)} \mathrm{d}u$$

设

$$\frac{u^3 + 3u^2 + 3u + 2}{u^3(u+1)} = \frac{A}{u+1} + \frac{B}{u} + \frac{C}{u^2} + \frac{D}{u^3}$$

于是

$$
\begin{cases}
A+B=1 \\
B+C=3 \\
C+D=3 \\
D=2
\end{cases}
$$

解得$A=-1, B=2, C=1, D=2$. 于是

$$
\begin{aligned}
\int \frac{x^3+1}{x(x-1)^3}\mathrm{d}x &= \int\left(\frac{2}{u^3}+\frac{1}{u^2}+\frac{2}{u}-\frac{1}{u+1}\right)\mathrm{d}u \\
&= -\frac{1}{(x-1)^2}-\frac{1}{x-1}+2\ln|x-1|-\ln|x|+C
\end{aligned}
$$

4.(10分)

求心形线

$$
r=a(1+\cos\theta), a>0, 0\leqslant\theta\leqslant 2\pi
$$

的弧长.

Solution.

根据极坐标中图形的弧长公式有

$$
\begin{aligned}
r &= \int_0^{2\pi}\sqrt{r^2+r'^2}\mathrm{d}\theta \\
&= a\int_0^{2\pi}\sqrt{\cos^2\theta+2\cos\theta+1+\sin^2\theta}\mathrm{d}\theta \\
&= \sqrt{2}a\int_0^{2\pi}\sqrt{1+\cos\theta}\mathrm{d}\theta \\
&= 2a\int_0^{2\pi}\left|\cos\frac{\theta}{2}\right|\mathrm{d}\theta \\
&= 8a\int_0^{\frac{\pi}{2}}\cos\frac{\theta}{2}\mathrm{d}\left(\frac{\theta}{2}\right) \\
&= 8a
\end{aligned}
$$

5.(10分)

设$t\in(0,1)$,$f(t)$表示曲线$y=\sec t$与直线$x=0, y=0$和$x=\arcsin t$围成的封闭图形的面积.

(1) **(5分)** 求$f(t)$的导数$f'(t)$.

(2) **(5分)** 求$f(t)$.

(1) Solution.

由题意可得

$$f(t)=\int_0^{x(t)} y\mathrm{d}x=\int_0^{\arcsin t}\sec x\mathrm{d}x$$

于是

$$f'(t)=\sec(\arcsin t)\cdot(\arcsin t)'=\frac{1}{\sqrt{1-t^2}\sqrt{1-t^2}}=\frac{1}{1-t^2}$$

(2) Solution.

由**(1)**的结果可得

$$\int f'(t)\mathrm{d}t=\int\frac{\mathrm{d}t}{1-t^2}=\frac{1}{2}\int\left(\frac{1}{t+1}-\frac{1}{t-1}\right)\mathrm{d}t=\frac{1}{2}\ln\left|\frac{t+1}{t-1}\right|+C$$

又

$$f(0)=\int_0^{\arcsin 0}\sec x\mathrm{d}x=0$$

于是

$$f(t)=\frac{1}{2}\ln\left|\frac{t+1}{t-1}\right|$$

6.(10分)

设函数$f(x)$在$[0,1]$上连续,且有

$$\int_0^1 f(x)\mathrm{d}x=\int_0^1 xf(x)\mathrm{d}x=0$$

试证明:$f(x)$在$[0,1]$上至少有两个零点.

Proof.

若$f(x)$在$[0,1]$上没有零点,那么不妨设$\forall x\in[0,1]$, $f(x)>0$,于是

$$\int_0^1 f(x)\mathrm{d}x>0$$

这与题设不符($f(x)<0$时亦同理),于是$f(x)$在$[0,1]$上至少有一个零点,设$f(x_1)=0$.

若$f(x)$在$[0,1]$上仅有一个零点x_1,那么不妨设$x\in[0,x_1)$时$f(x)<0$,$x\in(x_1,1]$时$f(x)>0$.

由题意

$$\int_0^1 f(x)\mathrm{d}x=\int_0^{x_1} f(x)\mathrm{d}x+\int_{x_1}^1 f(x)\mathrm{d}x$$

于是

$$\int_0^{x_1} f(x)\mathrm{d}x=-\int_{x_1}^1 f(x)\mathrm{d}x<0$$

又$x<x_1$时$0>xf(x)>x_1f(x)$,$x>x_1$时$0<x_1f(x)<xf(x)$.于是

$$\begin{aligned}\int_0^1 xf(x)\mathrm{d}x &= \int_0^{x_1} xf(x)\mathrm{d}x + \int_{x_1}^1 xf(x)\mathrm{d}x \\ &> x_1\int_0^{x_1} f(x)\mathrm{d}x + x_1\int_{x_1}^1 f(x)\mathrm{d}x \\ &= x_1\left(\int_0^{x_1} f(x)\mathrm{d}x + \int_{x_1}^1 f(x)\mathrm{d}x\right) \\ &= x_1\int_0^1 f(x)\mathrm{d}x = 0\end{aligned}$$

这与题设不符,于是$f(x)$在$[0,1]$上至少有两个零点.

7.(20分)

解答下列问题.

(1) **(8分)** 设序列$\{x_n\}$满足$x_n=\sum\limits_{i=1}^n \frac{1}{i}-\ln n$,试证明$\lim\limits_{n\to\infty} x_n$存在且有限.

(2) **(12分)** 对于$n\in\mathbb{N}$,定义

$$I_n=\int_0^{\frac{\pi}{2}} \frac{\sin^2 nt}{\sin t}\mathrm{d}t$$

求序列极限$\lim\limits_{n\to\infty}\frac{I_n}{\ln n}$.

(1) **Proof.**

置$f(x)=\frac{1}{x}$,于是

$$\forall k\in\mathbb{N}^*,\forall x\in(k,k+1),\frac{1}{k}>f(x)>\frac{1}{k+1}$$

对上式积分有

$$\frac{1}{k+1}<\int_k^{k+1} f(x)\mathrm{d}x<\frac{1}{k}$$

对上式求和有

$$\sum_{k=1}^{n-1}\frac{1}{k+1}<\int_1^n f(x)\mathrm{d}x<\sum_{k=1}^{n-1}\frac{1}{k}$$

于是

$$x_n+\ln n-1<\left.(\ln x)\right|_1^n<x_n+\ln n-\frac{1}{n}$$

所以

$$\frac{1}{n}<x_n<1$$

从而$\{x_n\}$有界.我们又有

$$x_{n+1}-x_n=\frac{1}{n+1}+\ln\frac{n}{n+1}=\ln\frac{n}{n+1}+1-\frac{n}{n+1}<0$$

于是$\{x_n\}$递减,从而$\lim\limits_{n\to\infty} x_n$存在.

(2) Proof.

不难发现

$$I_1 = \int_0^{\frac{\pi}{2}} \sin t \mathrm{d}t = 1$$

而

$$\begin{aligned}
I_n - I_{n-1} &= \int_0^{\frac{\pi}{2}} \frac{\sin^2 nt - \sin^2 (n-1)t}{\sin t} \mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin(n+(n-1))t \cdot \sin(n-(n-1))t}{\sin t} \mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \sin(2n-1)t \mathrm{d}t \\
&= \frac{1}{2n-1} \int_0^{n\pi - \frac{\pi}{2}} \sin u \mathrm{d}u \\
&= \frac{1}{2n-1}
\end{aligned}$$

于是

$$I_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$$

设$t_n = x_n + \ln n = \sum\limits_{i=1}^{n} \frac{1}{i}$,于是$I_n = t_{2n} - \frac{1}{2} t_n$. 于是

$$\begin{aligned}
\frac{I_n}{\ln n} &= \frac{x_{2n} + \ln 2n}{\ln n} - \frac{x_n + \ln n}{2 \ln n} \\
&= \frac{x_{2n}}{\ln 2n} - \frac{x_n}{2 \ln n} + \frac{\ln 2}{\ln n} + \frac{1}{2}
\end{aligned}$$

注意到$\forall n \in \mathbb{N}^*, x_n < 1$.于是

$$\lim_{n\to\infty} \frac{I_n}{\ln n} = \frac{1}{2}$$

8.(15分)

设函数$f(x)$在$[0,\pi]$上连续.对于$n \in \mathbb{N}$,试证明

$$\lim_{n\to\infty} \int_0^{\pi} f(x) \left|\sin(nx)\right| \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} f(x) \mathrm{d}x$$

Proof.

首先证明

$$\forall [a,b] \subset [0,\pi], \exists \xi \in [a,b], \text{s.t.} \int_a^b f(x) \left|\sin nx\right| \mathrm{d}x = f(\xi) \int_a^b \left|\sin nx\right| \mathrm{d}x$$

由于$f(x)$在$[a,b]$上连续,于是其有界.不妨设$m=\min\limits_{a\leqslant x\leqslant b}f(x),M=\max\limits_{a\leqslant x\leqslant b}f(x)$,于是

$$m\left|\sin nx\right|\leqslant f(x)\left|\sin nx\right|\leqslant M\left|\sin nx\right|$$

对上式作定积分有

$$m\int_a^b\left|\sin nx\right|\leqslant\int_a^b f(x)\left|\sin nx\right|\leqslant M\int_a^b\left|\sin nx\right|$$

即

$$m\leqslant\frac{\int_a^b f(x)\left|\sin nx\right|}{\int_a^b\left|\sin nx\right|}\leqslant M$$

于是根据介值定理,存在$\xi\in[a,b]$使得$\displaystyle\int_a^b f(x)\left|\sin nx\right|\mathrm{d}x=f(\xi)\int_a^b\left|\sin nx\right|\mathrm{d}x$.

我们有

$$\int_0^\pi f(x)\left|\sin(nx)\right|\mathrm{d}x=\sum_{k=1}^n\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}}f(x)\left|\sin(nx)\right|\mathrm{d}x$$

根据上述引理,$\exists\xi_k\in\left[\dfrac{k\pi}{n},\dfrac{(k+1)\pi}{n}\right]$, s.t. $\displaystyle\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}}f(x)\left|\sin(nx)\right|\mathrm{d}x=f(\xi_k)\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}}\left|\sin(nx)\right|\mathrm{d}x$

令$u=nx$,则

$$\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}}\left|\sin(nx)\right|\mathrm{d}x=\frac{1}{n}\int_{k\pi}^{(k+1)\pi}\left|\sin u\right|\mathrm{d}u=\frac{2}{n}$$

于是

$$\begin{aligned}\lim_{n\to\infty}\int_0^\pi f(x)\left|\sin(nx)\right|\mathrm{d}x&=\lim_{n\to\infty}\sum_{k=1}^n\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}}f(x)\left|\sin(nx)\right|\mathrm{d}x\\&=\lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^n f(\xi_k)\\&=\frac{2}{\pi}\lim_{n\to\infty}\frac{\pi}{n}\sum_{k=1}^n f(\xi_k)\\&=\frac{2}{\pi}\int_0^\pi f(x)\mathrm{d}x\end{aligned}$$