Lecture 3 Triple integral(三重积分)

L.3.1 求$I=\iiint_{\Omega}\left(y^2+z^2\right)\mathrm{d}V$,其中$\Omega=\{(x,y,z)|0\leqslant z\leqslant x^2+y^2\leqslant 1\}$.

Solution.

做柱坐标变换,则变换后的积分区域$\Omega'=\{(r,\theta,z)|0\leqslant r\leqslant 1,0\leqslant\theta\leqslant 2\pi,0\leqslant z\leqslant r^2\}$.我们有

$$\begin{aligned}
I&=\iiint_{\Omega}\left(y^2+z^2\right)\mathrm{d}V\\
&=\iiint_{\Omega'}\left(r^2\sin^2\theta+z^2\right)r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z\\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\mathrm{d}r\int_0^{r^2}\left(r^3\sin^2\theta+rz^2\right)\mathrm{d}z\\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\left(r^5\sin^2\theta+\frac{1}{3}r^7\right)\mathrm{d}r\\
&=\int_0^{2\pi}\left(\frac{1}{6}\sin^2\theta+\frac{1}{24}\right)\mathrm{d}\theta\\
&=\frac{\pi}{4}
\end{aligned}$$

L.3.2 求$I=\iiint_{\Omega}z(x^2+y^2+z^2)\mathrm{d}V$,其中$\Omega$为球体$x^2+y^2+z^2\leqslant 2z$.

Solution.

注意到$\Omega=\{(x,y,z)|x^2+y^2+(z-1)^2\leqslant 1\}$.

做球坐标变换,可得变换后的积分区域为$\Omega'=\{(\rho,\theta,\varphi)|0\leqslant r\leqslant 2\cos\varphi,0\leqslant\theta\leqslant 2\pi,0\leqslant\varphi\leqslant\frac{\pi}{2}\}$.于是

$$\begin{aligned}
I&=\iiint_{\Omega}z(x^2+y^2+z^2)\mathrm{d}V\\
&=\iiint_{\Omega'}\rho\cos\varphi\cdot\rho^2\cdot\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi\\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{2\cos\varphi}\rho^5\sin\varphi\cos\varphi\mathrm{d}\rho\\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\frac{32\cos^7\varphi\sin\varphi}{3}\mathrm{d}\varphi\\
&\xlongequal{t=\cos\varphi}2\pi\int_0^1\frac{32t^7\mathrm{d}t}{3}\\
&=\frac{8\pi}{3}
\end{aligned}$$

L.3.3 设$n\in\mathbb{N}^*$,记n维空间单位球$\sum\limits_{k=1}^{n}x_k^2\leqslant 1$的体积为$\alpha(n)$.计算$\alpha(4)$,并写出序列$\alpha(n)$的递推表达式.

Solution.

当$n\geqslant 2$时,设各维度的变量为$x_1,\cdots,x_n$,积分区域为$\Omega_n=\left\{(x_1,\cdots,x_n):\sum\limits_{k=1}^{n}x_k^2\leqslant 1\right\}$.我们有

$$\alpha(n)=\underbrace{\int\cdots\int_{\Omega_n}}_{n\text{重积分}}\mathrm{d}V_n=\int_{-1}^{1}\mathrm{d}x_n\underbrace{\int\cdots\int_{D_{x_n}}}_{(n-1)\text{重积分}}\mathrm{d}V_{n-1}$$

其中$D_{x_n}=\left\{(x_1,\cdots,x_{n-1}):\sum\limits_{k=1}^{n-1}x_k^2\leqslant 1-x_n^2\right\}$,这是$n-1$维空间上半径为$\sqrt{1-x_n^2}$的球.
做变换$u_k=\sqrt{1-x_n^2}$其中$k=1,\cdots,n-1$,变换后的积分区域即为Ω_{n-1}.
这一变换的Jacobi行列式$|J|=\left(\sqrt{1-x_n^2}\right)^n$,于是

$$\underbrace{\int\cdots\int_{D_{x_n}}}_{(n-1)\text{重积分}}\mathrm{d}V_{n-1}=\underbrace{\int\cdots\int_{\Omega_{n-1}}}_{(n-1)\text{重积分}}|J|\mathrm{d}V_{n-1}=\alpha(n-1)\left(1-x_n^2\right)^{\frac{n-1}{2}}$$

于是

$$\begin{aligned}\alpha(n)&=\alpha(n-1)\int_{-1}^{1}\left(1-x_n^2\right)^{\frac{n-1}{2}}\mathrm{d}x_n\\&\xlongequal{t=\arcsin x_n}\alpha(n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^n t\mathrm{d}t\end{aligned}$$

最后的积分我们已经在上学期推导过,故此给出结论

$$\alpha(n)=\begin{cases}\dfrac{2(n-1)!!}{n!!}\alpha(n-1),n\text{为奇数}\\\dfrac{\pi(n-1)!!}{n!!}\alpha(n-1),n\text{为偶数}\end{cases}$$

据此亦可推出$\alpha(4)=\dfrac{3\pi}{8}\alpha(3)=\dfrac{\pi^2}{2}$.

L.3.4 设$\Omega=\{(x,y,z)|0\leqslant x+y-z\leqslant 1,0\leqslant y+z-x\leqslant 1,0\leqslant x+z-y\leqslant 1\}$是六个平面围成的区域.求重积分$\iiint_\Omega(x+y-z)(y+z-x)(x+z-y)\mathrm{d}V$.

Solution.

做变换$u=x+y-z,v=y+z-x,w=x+z-y$,则变换后的积分区域为$\Omega'=\{(u,v,w)|0\leqslant u,v,w\leqslant 1\}$.

该变换的逆变换为$x=\dfrac{u+w}{2},y=\dfrac{u+v}{2},z=\dfrac{v+w}{2}$,其Jacobi行列式为

$$|J|=\begin{vmatrix}\frac{1}{2}&0&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&0\\0&\frac{1}{2}&\frac{1}{2}\end{vmatrix}=\frac{1}{4}$$

于是

$$\iiint_\Omega(x+y-z)(y+z-x)(x+z-y)\mathrm{d}V=\iiint_{\Omega'}|J|uvw\mathrm{d}V=\frac{1}{4}\cdot\left(\frac{1}{2}\right)^3=\frac{1}{32}$$

L.3.5 设参数$a,b,c>0$,求曲面$\left(\dfrac{x}{a}+\dfrac{y}{b}\right)^2+\left(\dfrac{z}{c}\right)^2=1$围成的空间图形的体积.

Solution.

这是一道错题,因为题目中的曲面不封闭.

L.3.6 求$I=\displaystyle\iiint_\Omega(x+y+z)^2\mathrm{d}V$,其中$\Omega$为$x^2+y^2\leqslant 2z$和$x^2+y^2+z^2\leqslant 3$围成的区域.

Solution.

我们有

$$I=\iiint_\Omega(x+y+z)^2\mathrm{d}V=\iiint_\Omega\left(x^2+y^2+z^2+2xy+2yz+2xz\right)\mathrm{d}V$$

注意到积分区域关于x轴和y轴对称,于是xz,yz,xy三项的积分值为0.

做柱坐标变换,变换后的积分区域为$\Omega'=\left\{(r,\theta,z)|0\leqslant r\leqslant\sqrt{2},0\leqslant\theta\leqslant 2\pi,\dfrac{r^2}{2}\leqslant z\leqslant\sqrt{3-r^2}\right\}$.于是

$$\begin{aligned}I&=\iiint_\Omega(x^2+y^2+z^2)\mathrm{d}V\\&=\iiint_{\Omega'}r(r^2+z^2)\mathrm{d}r\mathrm{d}\theta\mathrm{d}z\\&=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{2}}\mathrm{d}r\int_{\frac{r^2}{2}}^{\sqrt{3-r^2}}r^3+rz^2\mathrm{d}z\\&=2\pi\int_0^{\sqrt{2}}\left[\left(\frac{2}{3}r^2+1\right)r\sqrt{3-r^2}-\frac{r^5}{2}-\frac{r^7}{24}\right]\mathrm{d}r\end{aligned}$$

我们有

$$\int_0^{\sqrt{2}}\left(\frac{r^5}{2}+\frac{r^7}{24}\right)\mathrm{d}r=\left.\left(\frac{r^6}{12}+\frac{r^8}{192}\right)\right|_0^{\sqrt{2}}=\frac{3}{4}$$

$$\int_0^{\sqrt{2}}r\sqrt{3-r^2}\mathrm{d}r\xlongequal{u=r^2}\frac{1}{2}\int_0^2\sqrt{3-u}\mathrm{d}u=\sqrt{3}-\frac{1}{3}$$

$$\int_0^{\sqrt{2}} \frac{2}{3} r^3 \sqrt{3-r^2} \mathrm{d}r \xlongequal{t=\sqrt{3-x^2}} \frac{2}{3} \int_1^{\sqrt{3}} t^2 \left(3-t^2\right) \mathrm{d}t = \frac{12\sqrt{3}-8}{15}$$

于是

$$I = 2\pi \left(\frac{12\sqrt{3}-8}{15} + \sqrt{3} - \frac{1}{3} - \frac{3}{4} \right) = 2\pi \left(\frac{9\sqrt{3}}{5} - \frac{97}{60} \right) = \frac{\left(108\sqrt{3}-97\right)\pi}{30}$$