二重积分练习

1. 求积分$I=\iint_D\left(x^2+2y\right)\mathrm{d}x\mathrm{d}y$,其中$D$为曲线$y=x^2,y=\sqrt{x}$围成的区域.

Solution.

我们有

$$\begin{aligned}\iint_D\left(x^2+2y\right)\mathrm{d}x\mathrm{d}y&=\int_0^1\left[\int_{x^2}^{\sqrt{x}}\left(x^2+2y\right)\mathrm{d}y\right]\mathrm{d}x\\&=\int_0^1\left(x^2\left(\sqrt{x}-x^2\right)+x-x^4\right)\mathrm{d}x\\&=\int_0^1\left(x+x^{\frac{5}{2}}-2x^4\right)\\&=\left(\frac{1}{2}x^2+\frac{2}{7}x^{\frac{7}{2}}-\frac{2}{5}x^5\right)\Big|_0^1\\&=\frac{27}{70}\end{aligned}$$

2. 求积分$I=\iint_D\sin y^3\mathrm{d}x\mathrm{d}y$,其中$D$是曲线$y=\sqrt{x}$,直线$y=2$和$x=0$围成的区域.

Solution.

我们有

$$\begin{aligned}\iint_D\sin y^3\mathrm{d}x\mathrm{d}y&=\int_0^2\left[\int_0^{y^2}\sin y^3\mathrm{d}x\right]\mathrm{d}y\\&=\int_0^2 y^2\sin y^3\mathrm{d}y\\&=\frac{1}{3}\int_0^8\sin y^3\mathrm{d}y^3\\&=\frac{1-\cos 8}{3}\end{aligned}$$

3. 求积分$I=\iint_D\left(4-x^2-y^2\right)^{-\frac{1}{2}}\mathrm{d}x\mathrm{d}y$,其中$D$是单位圆$x^2+y^2\leqslant 1$在第一象限的部分.

Solution.

做代换$x=r\cos\theta, y=r\sin\theta$,于是$D'=\left\{(r,\theta):0\leqslant r\leqslant 1, 0\leqslant\theta\leqslant\dfrac{\pi}{2}\right\}$.于是

$$\begin{aligned}\iint_D\left(4-x^2-y^2\right)^{-\frac{1}{2}} &= \iint_{D'}\left(4-r^2\right)^{-\frac{1}{2}} r\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^1\left[\int_0^{\frac{\pi}{2}}\left(4-r^2\right)^{-\frac{1}{2}} r\mathrm{d}\theta\right]\mathrm{d}r \\ &= \frac{\pi}{2}\int_0^1\frac{r\mathrm{d}r}{\sqrt{4-r^2}} \\ &= \frac{\pi}{4}\int_0^1\frac{\mathrm{d}r^2}{\sqrt{4-r^2}} \\ &= \frac{\pi}{4}\left(-2\sqrt{4-r^2}\right)\Big|_0^1 \\ &= \frac{2-\sqrt{3}}{2}\pi\end{aligned}$$

4. 求积分$I=\iint_D(x+y)\mathrm{d}x\mathrm{d}y$,其中$D$是由$y^2=2x, x+y=4, x+y=12$围成的区域.

Solution.

Method I.

注意到积分区域D可以恰好可以分为两部分

$$D_1=\{(x,y)|2\leqslant x\leqslant 8, 4-x\leqslant y\leqslant\sqrt{2x}\}$$

$$D_2=\{(x,y)|8\leqslant x\leqslant 18, -\sqrt{2x}\leqslant y\leqslant 12-x\}$$

于是

$$\begin{aligned}\iint_{D_1}(x+y)\mathrm{d}x\mathrm{d}y &= \int_2^8\mathrm{d}x\int_{4-x}^{\sqrt{2x}}(x+y)\mathrm{d}y \\ &= \int_2^8\left(\frac{1}{2}x^2+\sqrt{2}x^{\frac{3}{2}}+x-8\right)\mathrm{d}x \\ &= \frac{826}{5} \\ \iint_{D_2}(x+y)\mathrm{d}x\mathrm{d}y &= \int_8^{18}\mathrm{d}x\int_{-\sqrt{2x}}^{12-x}(x+y)\mathrm{d}y \\ &= \int_8^{18}\left(-\frac{1}{2}x^2+\sqrt{2}x^{\frac{3}{2}}-x+72\right) \\ &= \frac{5678}{15}\end{aligned}$$

于是

$$\iint_D(x+y)\mathrm{d}x\mathrm{d}y=\frac{826}{5}+\frac{5678}{15}=\frac{8156}{15}$$

Method II.

做代换$\begin{cases} u=x+y \\ v=y \end{cases}$,则$|J|=1$.原积分区域为$y^2\leqslant 2x, 4\leqslant x+y\leqslant 12$.

代入u,v可得$v^2+2v-2u\leqslant 0, 4\leqslant u\leqslant 12$.

于是积分区域为$D'=\{(u,v)|4\leqslant u\leqslant 12, -\sqrt{2u+1}-1\leqslant v\leqslant\sqrt{2u+1}-1\}$.

于是我们有

$$\begin{aligned}
I&=\iint_D(x+y)\mathrm{d}x\mathrm{d}y=\iint_{D'}u\mathrm{d}u\mathrm{d}v\\
&=\int_4^{12}\mathrm{d}u\int_{-\sqrt{2u+1}-1}^{\sqrt{2u+1}-1}u\mathrm{d}v=\int_4^{12}2u\sqrt{2u+1}\mathrm{d}u\\
&\xlongequal{t=\sqrt{2u+1}}\int_3^5(t^2-1)t\cdot t\mathrm{d}t=\left(\frac{1}{5}t^5-\frac{1}{3}t^3\right)\Big|_3^5\\
&=\frac{8156}{15}
\end{aligned}$$

5. 二维正态分布函数

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{\left(x-\mu_1\right)^2}{\sigma_1^2}+\frac{\left(y-\mu_2\right)^2}{\sigma_1^2}-2\rho\frac{\left(x-\mu_1\right)\left(y-\mu_2\right)}{\sigma_1\sigma_2}\right)\right]$$

试证明下列重积分.

(1) 试证明

$$\iint_{\mathbb{R}^2}xf(x,y)\mathrm{d}x\mathrm{d}y=\mu_1$$

(2) 试证明

$$\iint_{\mathbb{R}^2}x^2f(x,y)\mathrm{d}x\mathrm{d}y=\mu_1^2+\sigma_1^2$$

(3) 试证明

$$\iint_{\mathbb{R}^2}\left(x+y\right)^2f(x,y)\mathrm{d}x\mathrm{d}y=\left(\mu_1+\mu_2\right)^2+\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2$$

Proof.

(1) 做代换$u=\dfrac{x-\mu_1}{\sigma_1}, v=\dfrac{y-\mu_2}{\sigma_2}$,则有

$$\iint_{\mathbb{R}^2}xf(x,y)\mathrm{d}x\mathrm{d}y=\iint_{\mathbb{R}^2}\left(\sigma_1u+\mu_1\right)\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{u^2+v^2-2\rho uv}{2\left(1-\rho^2\right)}\right)\mathrm{d}u\mathrm{d}v$$

令

$$A(u,v)=u\exp\left(-\frac{u^2+v^2-2\rho uv}{2\left(1-\rho^2\right)}\right)$$

注意到

$$A(-u,-v)=-u\exp\left(-\frac{u^2+v^2-2\rho uv}{2\left(1-\rho^2\right)}\right)=-A(u,v)$$

于是$A(u,v)$关于原点$(0,0)$中心对称,于是

$$\iint_{\mathbb{R}^2}A(u,v)\mathrm{d}u\mathrm{d}v=0$$

再做一次代换$s=u+v,t=u-v$,于是

$$\begin{aligned}\iint_{\mathbb{R}^2}\exp\left(-\frac{u^2+v^2-2\rho uv}{2\left(1-\rho^2\right)}\right)&=\frac{1}{2}\iint_{\mathbb{R}^2}\exp\left(-\frac{\dfrac{s^2+t^2}{2}-\dfrac{\rho\left(s^2-t^2\right)}{2}}{2\left(1-\rho^2\right)}\right)\mathrm{d}s\mathrm{d}t\\&=\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}s\int_{-\infty}^{+\infty}\exp\left(-\frac{s^2}{4(1+\rho)}\right)\exp\left(-\frac{t^2}{4(1-\rho)}\right)\mathrm{d}t\\&=\frac{1}{2}\cdot 2\sqrt{\pi(1-\rho)}\int_{-\infty}^{+\infty}\exp\left(-\frac{s^2}{4(1+\rho)}\right)\mathrm{d}s\\&=2\pi\sqrt{1-\rho^2}\end{aligned}$$

于是

$$\begin{aligned}\iint_{\mathbb{R}^2}xf(x,y)\mathrm{d}x\mathrm{d}y&=\iint_{\mathbb{R}^2}\frac{\sigma_1A(u,v)+\mu_1\exp\left(-\dfrac{u^2+v^2-2\rho uv}{2\left(1-\rho^2\right)}\right)\mathrm{d}u\mathrm{d}v}{2\pi\sqrt{1-\rho^2}}\\&=\frac{\sigma_1\cdot 0+\mu_1\cdot 2\pi\sqrt{1-\rho^2}}{2\pi\sqrt{1-\rho^2}}\\&=\mu_1\end{aligned}$$

(2) 仍做**(1)**中的代换,则有

$$\iint_{\mathbb{R}^2}x^2f(x,y)\mathrm{d}x\mathrm{d}y=\iint_{\mathbb{R}^2}\left(\sigma_1^2u^2+2\sigma_1\mu_1u+\mu_1^2\right)\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{u^2+v^2-2\rho uv}{2\left(1-\rho^2\right)}\right)\mathrm{d}u\mathrm{d}v$$

第一个括号中的常数项和一次项我们已经计算过对应的积分,现在做代换$s=u+v,t=u-v$,考虑积分

$$\begin{aligned}\iint_{\mathbb{R}^2}u^2\exp\left(-\frac{u^2+v^2-2\rho uv}{2\left(1-\rho^2\right)}\right)&=\frac{1}{2}\iint_{\mathbb{R}^2}\frac{(s+t)^2}{4}\exp\left(-\frac{s^2}{4(1+\rho)}\right)\exp\left(-\frac{t^2}{4(1-\rho)}\right)\mathrm{d}s\mathrm{d}t\\&=\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}s\int_{-\infty}^{+\infty}\frac{(s+t)^2}{4}\exp\left(-\frac{s^2}{4(1+\rho)}\right)\exp\left(-\frac{t^2}{4(1-\rho)}\right)\mathrm{d}t\\&=\frac{1}{2}\int_{-\infty}^{+\infty}\left(\frac{s^2}{4}\cdot 2\sqrt{\pi(1-\rho)}+\sqrt{\pi}\,(1-\rho)^{\frac{3}{2}}\right)\exp\left(-\frac{s^2}{4(1+\rho)}\right)\mathrm{d}s\\&=\frac{1}{2}\left[2\sqrt{\pi(1-\rho)}\cdot\sqrt{\pi}\,(1+\rho)^{\frac{3}{2}}+\sqrt{\pi}\,(1-\rho)^{\frac{3}{2}}\cdot 2\sqrt{\pi(1+\rho)}\right]\\&=\pi\sqrt{1-\rho^2}\,(1-\rho+1+\rho)\\&=2\pi\sqrt{1-\rho^2}\end{aligned}$$

类似地可得

$$\iint_{\mathbb{R}^2}x^2f(x,y)\mathrm{d}x\mathrm{d}y=\mu_1^2+\sigma_1^2$$

(3) 仍做**(1)**中的代换,有

$$\iint_{\mathbb{R}^2}(x+y)^2 f(x,y)\mathrm{d}x\mathrm{d}y=\iint_{\mathbb{R}^2}(\sigma_1 u+\sigma_2 v+\mu_1+\mu_2)^2\frac{\exp\left(-\frac{u^2+v^2-2\rho uv}{2\left(1-\rho^2\right)}\right)}{2\pi\sqrt{1-\rho^2}}\mathrm{d}u\mathrm{d}v$$

注意到

$$\iint_{\mathbb{R}^2}uv\exp\left(-\frac{u^2+v^2-2\rho uv}{2\left(1-\rho^2\right)}\right)=4\pi\rho\sqrt{1-\rho^2}$$

于是

$$\iint_{\mathbb{R}^2}(x+y)^2 f(x,y)\mathrm{d}x\mathrm{d}y=\sigma_1^2+\sigma_2^2+2\rho\sigma_1+\sigma_2+(\mu_1+\mu_2)^2$$