北京大学数学科学学院2021-22高等数学B1期末考试

1.(10分)

证明:对于任意$x \in \mathbb{R}$,存在$\theta \in (0,1)$使得

$$\arctan x = \frac{x}{1+\theta^2 x^2}$$

成立.

Proof.

当$x=0$时,任取$\theta > 0$即可使等式成立.

当$x \neq 0$时,不妨设$x > 0$.令$f(u) = \arctan u$,于是$f(u)$在$\mathbb{R}$上连续且可导.

根据Lagrange中值定理,存在$\xi \in (0,x)$使得$f'(\xi) = \dfrac{f(x)-f(0)}{x-0}$.

即$\dfrac{1}{1+\xi^2} = \dfrac{\arctan x}{x}$.令$\theta = \dfrac{\xi}{x}$,就有$\arctan x = \dfrac{x}{1+(\theta x)^2}$.

当$x<0$时亦同理.于是命题得证.

2.(20分)

求出下面函数的极限.

(1) (10分) $\displaystyle\lim_{x\to 0}\frac{\tan^4 x}{\sqrt{1-\dfrac{x\sin x}{2}}-\sqrt{\cos x}}$.

(2) (10分) 设$n \in \mathbb{N}^*$.对于实序列$\{a_k\}_{k=1}^n$,求

$$\lim_{x\to 0}\left(\frac{\sum_{k=1}^n a_k^x}{n}\right)^{\frac{1}{x}}$$

Solution.

(1)

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan^4 x}{\sqrt{1-\dfrac{x\sin x}{2}}-\sqrt{\cos x}} &= \lim_{x\to 0}\frac{\tan^4 x}{1-\dfrac{x\sin x}{2}-\cos x}\cdot\left(\sqrt{1-\frac{x\sin x}{2}}+\sqrt{\cos x}\right)\\
&= \lim_{x\to 0}\frac{(x+o(x))^4}{1-\dfrac{x\left(x-\dfrac{x^3}{3}+o(x^3)\right)}{2}-\left(1-\dfrac{x^2}{2}+\dfrac{x^4}{24}+o(x^4)\right)}\\
&= \lim_{x\to 0}\frac{x^4+o(x^4)}{\dfrac{x^4}{8}+o(x^4)} = 8
\end{aligned}$$

(2) 我们有

$$\lim_{x\to 0}\frac{\ln\left(\sum\limits_{k=1}^{n}a_k^x\right)-\ln n}{x}=\lim_{x\to 0}\frac{\sum\limits_{k=1}^{n}a_k^x\ln a_k}{\sum\limits_{k=1}^{n}a_k^x}=\frac{\sum\limits_{k=1}^{n}\ln a_k}{n}$$

于是

$$\lim_{x\to 0}\left(\frac{\sum_{k=1}^{n}a_k^x}{n}\right)^{\frac{1}{x}}=\lim_{x\to 0}\exp\left(\frac{\ln\left(\sum_{k=1}^{n}a_k^x\right)-\ln n}{x}\right)=\mathrm{e}^{\frac{\sum_{k=1}^{n}\ln a_k}{n}}=\sqrt[n]{a_1\cdots a_k}$$

3.(15分)

设函数

$$f(x)=\frac{1-2x+5x^2}{(1-2x)(1+x^2)}$$

在$x=0$处的$2n+1$阶泰勒公式.

Solution.

我们有

$$f(x)=-\frac{2x}{1+x^2}+\frac{1}{1-2x}$$

令$g(x)=\dfrac{1}{1+x}$,对$g(x)$泰勒展开有

$$g(x)=1-x+x^2-x^3+\cdots=\sum_{i=0}^{n}(-x)^i$$

于是

$$\begin{aligned}f(x)&=-2xg(x^2)+g(-2x)\\&=-2x\sum_{i=0}^{n}(-x^2)^i+\sum_{i=0}^{2n+1}(2x)^i+o(x^{2n+1})\\&=1+\sum_{i=1}^{n}\left((2x)^{2i}+(2^{2i+1}-2(-1)^i)x^{2i+1}\right)+o(x^{2n+1})\end{aligned}$$

4.(10分)

定义三元函数$f:\mathbb{R}^3\to\mathbb{R}$为

$$f(x,y,z)=\begin{cases}\dfrac{xyz}{x^2+y^2+z^2},(x,y,z)\neq(0,0,0)\\0,(x,y,z)=(0,0,0)\end{cases}$$

回答下列问题.

(1) **(5分)** 求函数$f(x,y,z)$在$(0,0,0)$处的三个偏导数.

(2) **(5分)** $f(x,y,z)$在$(0,0,0)$处是否可微?试证明之.

Solution.

(1) 我们有

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0,0)} = \lim_{x\to 0}\frac{f(x,0,0)-f(0,0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

同理$\left.\frac{\partial f}{\partial y}\right|_{(0,0,0)} = \left.\frac{\partial f}{\partial z}\right|_{(0,0,0)} = 0.$

(2) 记$g(x,y,z) = \dfrac{\partial f}{\partial z}$.

当$(x,y,z)\neq(0,0,0)$时,$g(x,y,z) = \dfrac{xy(x^2+y^2+z^2)-2z(xyz)}{(x^2+y^2+z^2)^2} = \dfrac{xy(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}$.

当$(x,y,z)=(0,0,0)$时,$g(x,y,z)=0$.

令$x=y=kz$,于是$\lim\limits_{(x,y,z)\to(0,0,0)} g(x,y,z) = \dfrac{k^2(2k^2-1)}{(2k^2+1)^2}$.

于是$\lim\limits_{(x,y,z)\to(0,0,0)g(x,y,z)}$不存在,因而$f(x)$在$(0,0,0)$处的偏导数不连续,因而$f$不可微.

5.(15分)

设$f,g:\mathbb{R}\to\mathbb{R}$都有连续的二阶导数.对于任意$x,y\in\mathbb{R}$, 定义$h(x,y) = xg\left(\dfrac{y}{x}\right)+f\left(\dfrac{y}{x}\right)$, 试计算$x^2h_{xx}(x,y)+2xyh_{yx}(x,y)+y^2h_{yy}(x,y)$.

Solution.

我们有

$$h_x(x,y) = g\left(\frac{y}{x}\right) - \frac{y}{x}g'\left(\frac{y}{x}\right) - \frac{y}{x^2}f'\left(\frac{y}{x}\right)$$

$$h_y(x,y) = g'\left(\frac{y}{x}\right) + \frac{1}{x}f'\left(\frac{y}{x}\right)$$

$$h_{xx}(x,y) = -\frac{y}{x^2}g'\left(\frac{y}{x}\right) + \frac{y}{x^2}g'\left(\frac{y}{x}\right) + \frac{y^2}{x^3}g''\left(\frac{y}{x}\right) + \frac{2y}{x^3}f'\left(\frac{y}{x}\right) + \frac{y^2}{x^4}f''\left(\frac{y}{x}\right)$$

$$h_{yy}(x,y) = \frac{1}{x}g''\left(\frac{y}{x}\right) + \frac{1}{x^2}f''\left(\frac{y}{x}\right)$$

$$h_{yx}(x,y) = -\frac{y}{x^2}g''\left(\frac{y}{x}\right) - \frac{1}{x^2}f'\left(\frac{y}{x}\right) - \frac{y}{x^3}f''\left(\frac{y}{x}\right)$$

于是

$$\begin{aligned}&x^2h_{xx}(x,y)+2xyh_{yx}(x,y)+y^2h_{yy}(x,y)\\&=\frac{y^2}{x}g''\left(\frac{y}{x}\right)+\frac{2y}{x}f'\left(\frac{y}{x}\right)+\frac{y^2}{x^2}f''\left(\frac{y}{x}\right)+\frac{y^2}{x}g''\left(\frac{y}{x}\right)+\frac{y^2}{x^2}f''\left(\frac{y}{x}\right)-\frac{2y^2}{x}g''\left(\frac{y}{x}\right)-\frac{2y}{x}f'\left(\frac{y}{x}\right)-\frac{2y^2}{x^2}f''\left(\frac{y}{x}\right)\\&=0\end{aligned}$$

6.(20分)

设函数$F:\mathbb{R}^3\to\mathbb{R}$为

$$F(x,y,z)=x^3+(y^2-1)z^3-xyz$$

回答下列问题.

(1) **(5分)** 证明:存在$\mathbb{R}^2$上$(1,1)$的邻域D使得D上由$F(x,y,z)\equiv 0$确定唯一隐函数$z=f(x,y)$且$f(1,1)=1$.

(2) **(5分)** 求出在$(1,1)$处函数$z=f(x,y)$减少最快的方向上的单位向量$\vec{v}$.

(3) **(10分)** 设$\mathbb{R}^3$中平面$x+2y-2z=1$的z分量为正的法向量记为$\vec{u}$.向量$(\vec{v},0)\in\mathbb{R}^3$.求$\vec{u}$与$(\vec{v},0)$的夹角余弦.

Solution.

(1) 首先求一阶偏导数.

$$\frac{\partial F}{\partial x}=3x^2-yz\qquad \frac{\partial F}{\partial y}=2z^3y-xz\qquad \frac{\partial F}{\partial z}=3(y^2-1)z^2-xy$$

于是F在$(1,1,1)$附近有连续的偏导数且$\left.\dfrac{\partial F}{\partial z}\right|_{(1,1,1)}=-1\neq 0$.

根据隐函数存在定理,存在$(1,1,1)$的邻域使得$F(x,y,z)\equiv 0$确定唯一隐函数$z=f(x,y)$,且$f(1,1)=1$.

(2) 我们有

$$F_x(1,1,1)=2\qquad F_y(1,1,1)=1$$

于是

$$\frac{\partial f}{\partial x}=-\frac{F_x(1,1,1)}{F_z(1,1,1)}=2\qquad \frac{\partial f}{\partial y}=-\frac{F_y(1,1,1)}{F_z(1,1,1)}=1$$

于是$\vec{v}=-\,\mathbf{grad}f|_{(1,1)}=-\dfrac{(2,1)}{|(2,1)|}=\left(-\dfrac{2\sqrt{5}}{5},-\dfrac{\sqrt{5}}{5}\right)$.

(3) 法向量$\vec{u}=(-1,-2,2)$.于是

$$\cos\langle\vec{u},(\vec{v},0)\rangle=\frac{\vec{u}\cdot(\vec{v},0)}{|\vec{u}||(\vec{v},0)|}=\frac{4}{3\cdot\sqrt{5}}=-\frac{4\sqrt{5}}{15}$$

7.(10分)

给定正整数$n\geqslant 3$,求单位圆的内接n边形面积的最大值.

Proof.

我们将这n边形记作$A_1A_2\cdots A_n$,单位圆的圆心记作O.令$\theta_k=\angle A_kOA_{k+1}(1\leqslant k<n),\theta_n=\angle A_1OA_n$.

考虑将多边形分成n个三角形.于是

$$S_{\triangle A_kOA_k+1}=\frac{1}{2}r^2\sin\theta_k=\frac{1}{2}\sin\theta_k$$

其中,当$\theta>\pi$时这三角形的面积为负.由于这在图形中表示需要减去这部分面积,于是这是不矛盾的.

令$S(\theta_1,\cdots,\theta_n)=\dfrac{1}{2}\displaystyle\sum_{k=1}^{n}\sin\theta_k$.我们要求$\displaystyle\sum_{k=1}^{n}\theta_k=2\pi$的约束条件下$S$的最大值.

令$\phi(\theta_1,\cdots,\theta_n)=\displaystyle\sum_{k=1}^{n}\theta_k-2\pi$.

构造辅助函数$F(\theta_1,\cdots,\theta_n,\phi)=S(\theta_1,\cdots,\theta_n)-\lambda\phi(\theta_1,\cdots,\theta_n)$.求$F$对$\theta_1,\cdots,\theta_n,\lambda$的偏导数可得

$$F_{\theta_k}(\theta_1,\cdots,\theta_n,\lambda)=\frac{1}{2}\cos\theta_k-\lambda,\forall 1\leqslant k\leqslant n$$

$$F_{\phi}(\theta_1,\cdots,\theta_n,\lambda)=\phi(\theta_1,\cdots,\theta_n)=0$$

于是当各$F_{\theta_k}=0$时F取到最大值.

此时$\cos\theta_k=2\lambda$.为使得各$\cos\theta_k$相等,又$n\geqslant 3$,于是$\theta_1=\cdots=\theta_n=\dfrac{2\pi}{n}$.

于是面积的最大值为$S=\dfrac{n}{2}\sin\dfrac{2\pi}{n}$.