

北京大学数学科学学院2023-24高等数学B1期中考试

1.(10分)

求序列极限

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{ne}\right)^n$$

Solution.

置 $t = ne$, 则

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{ne}\right)^n = \lim_{t \rightarrow \infty} \left(1 + \frac{1}{t}\right)^{\frac{t}{e}} = \left(\lim_{t \rightarrow \infty} \left(1 + \frac{1}{t}\right)^t\right)^{\frac{1}{e}} = e^{\frac{1}{e}}$$

2.(10分)

设 $[x]$ 为不超过 x 的最大整数, 求函数极限

$$\lim_{x \rightarrow +\infty} x \sin \frac{1}{[x]}$$

Solution.

由 $[x] \leq x < [x] + 1$ 有

$$[x] \sin \frac{1}{[x]} \leq x \sin \frac{1}{[x]} < ([x] + 1) \sin \frac{1}{[x]}$$

置 $y = \frac{1}{[x]}$, 则有

$$\lim_{x \rightarrow +\infty} x \sin \frac{1}{[x]} = \lim_{y \rightarrow 0^+} \frac{\sin y}{y} = 1$$

从而

$$\lim_{x \rightarrow +\infty} [x] \sin \frac{1}{[x]} = 1$$

$$\lim_{x \rightarrow +\infty} ([x] + 1) \sin \frac{1}{[x]} = 1 + \lim_{x \rightarrow +\infty} \sin \frac{1}{[x]} = 1$$

由夹逼准则可知

$$\lim_{x \rightarrow +\infty} x \sin \frac{1}{[x]} = 1$$

3.(10分)

设 $x > 0$, 求函数

$$f(x) = \int_0^{\ln x} \sqrt{1 + e^t} dt$$

的导函数.

Solution.

置 $y = \ln x$, 则

$$\frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx} = \frac{d \int_0^y \sqrt{1+e^t} dt}{dy} \cdot \frac{1}{x} = \frac{\sqrt{1+e^y}}{x} = \frac{\sqrt{1+x}}{x}$$

4.(10分)

求不定积分

$$\int \frac{4x^2 + 4x - 11}{(2x-1)(2x+3)(2x-5)} dx$$

Solution.

设

$$\begin{aligned} \frac{4x^2 + 4x - 11}{(2x-1)(2x+3)(2x-5)} &= \frac{A}{2x-1} + \frac{B}{2x+3} + \frac{C}{2x-5} \\ &= \frac{A(4x^2 - 4x - 15) + B(4x^2 - 12x + 5) + C(4x^2 + 4x - 3)}{(2x-1)(2x+3)(2x-5)} \\ &= \frac{4(A+B+C)x^2 + 4(C-A-3B)x + (5B-15A-3C)}{(2x-1)(2x+3)(2x-5)} \end{aligned}$$

从而

$$\begin{cases} A+B+C=1 \\ C-A-3B=1 \\ 5B-15A-3C=-11 \end{cases}$$

解得 $A = \frac{1}{2}, B = -\frac{1}{4}, C = \frac{3}{4}$.

从而

$$\begin{aligned} \int \frac{4x^2 + 4x - 11}{(2x-1)(2x+3)(2x-5)} dx &= \int \left(\frac{1}{2} \cdot \frac{1}{2x-1} - \frac{1}{4} \cdot \frac{1}{2x+3} + \frac{3}{4} \cdot \frac{1}{2x-5} \right) dx \\ &= -\frac{1}{2} \int \frac{dx}{2x-1} + \frac{1}{4} \int \frac{dx}{2x+3} + \frac{3}{4} \int \frac{dx}{2x-5} \\ &= -\frac{1}{4} \ln |2x-1| + \frac{1}{8} \ln |2x+3| + \frac{3}{8} \ln |2x-5| + C \end{aligned}$$

5.(10分)

求欧氏平面直角坐标系中曲线

$$y = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln(x + \sqrt{x^2-1})$$

在 $x=1$ 到 $x=2$ 的弧长.

Solution.

$$\begin{aligned}y' &= \frac{1}{2} \left(\sqrt{x^2 - 1} + \frac{x^2}{\sqrt{x^2 - 1}} - \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(1 + \frac{x}{\sqrt{x^2 - 1}} \right) \right) \\&= \frac{1}{2} \left(\sqrt{x^2 - 1} + \frac{x^2}{\sqrt{x^2 - 1}} - \frac{1}{\sqrt{x^2 - 1}} \right) \\&= \sqrt{x^2 - 1}\end{aligned}$$

故

$$s = \int_1^2 \sqrt{1 + y'^2} dx = \int_1^2 x dx = \frac{x^2}{2} \Big|_1^2 = \frac{3}{2}$$

6.(10分)

设欧氏空间中 V 是曲线弧 $y = \frac{\ln x}{\sqrt{2\pi}} (1 \leq x \leq 2)$ 与直线 $x = 1, x = 2$ 围成的曲边三角形绕 x 轴旋转一周形成的旋转体,求 V 的体积.

Solution.

$$\begin{aligned}V &= \pi \int_1^2 y^2 dx = \pi \int_1^2 \frac{(\ln x)^2}{2\pi} dx = \frac{1}{2} \int_1^2 (\ln x)^2 dx \\&= \frac{1}{2} \left(x (\ln x)^2 \Big|_1^2 + \int_1^2 x d(\ln x)^2 \right) \\&= \frac{1}{2} \left(2 (\ln 2)^2 + \int_1^2 2 \ln x dx \right) \\&= (\ln 2)^2 + (x \ln x - x) \Big|_1^2 \\&= (\ln 2)^2 + 2 \ln 2 - 1\end{aligned}$$

7.(10分)

无穷序列 $\{a_n\}, \{b_n\}$ 满足 $0 < b_1 < a_1$,且有以下递推关系

$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \sqrt{a_n b_n}$$

试证明 $\lim_{n \rightarrow \infty} a_n$ 存在.

Proof.

据均值不等式有 $a_{n+1} = \frac{a_n + b_n}{2} \geq \sqrt{a_n b_n} = b_{n+1}$,当且仅当 $a_n = b_n$ 时取等.

由 $a_1 > b_1 > 0$ 有 $\forall n \in \mathbb{N}^*, a_n > b_n > 0$.

从而

$$a_{n+1} - a_n = \frac{a_n + b_n}{2} - a_n = \frac{b_n - a_n}{2} < 0$$

从而 $\{a_n\}$ 递减且有界, 故 $\lim_{n \rightarrow \infty} a_n$ 存在.

8.(20分)

本题中每个小问都要求给出证明和计算过程.

(1) (2分) 试证明: 当 $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ 时有

$$-1 < \frac{4 \sin x}{3 + \sin^2 x} < 1$$

(2) (8分) 当 $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ 时, 求函数

$$f(x) = \arcsin \left(\frac{4 \sin x}{3 + \sin^2 x} \right)$$

的导函数.

(3) (10分) 试证明

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{4 \cos^2 x + \sin^2 x}} = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\frac{9}{4} \cos^2 x + 2 \sin^2 x}}$$

Solution.

(1) **Proof.**

记 $\phi(x) = \frac{4 \sin x}{3 + \sin^2 x}$, 则 $\phi(-x) = \phi(x)$. 当 $x = 0$ 时原式显然成立.

当 $x \in \left(0, \frac{\pi}{2}\right)$ 时 $\sin x \in (0, 1)$, 则 $\phi(x) = \frac{4}{\sin x + \frac{3}{\sin x}} < \frac{4}{4} = 1$.

同理当 $x \in \left(-\frac{\pi}{2}, 0\right)$ 时 $\phi(x) > -1$. 综上可知原命题成立.

(2) **Solution.**

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1 - \left(\frac{4 \sin x}{3 + \sin^2 x}\right)^2}} \cdot \frac{4 \cos x (3 + \sin^2 x) - 4 \sin x (2 \sin x \cos x)}{(3 + \sin^2 x)^2} \\ &= \frac{3 + \sin^2 x}{\sqrt{\sin^4 x - 10 \sin^2 x + 9}} \cdot \frac{4 \cos x (3 - \sin^2 x)}{(3 + \sin^2 x)^2} \\ &= \frac{4 \cos x (3 - \sin^2 x)}{\sqrt{1 - \sin^2 x} \cdot \sqrt{9 - \sin^2 x} \cdot (3 + \sin^2 x)} \\ &= \frac{4 (3 - \sin^2 x)}{(3 + \sin^2 x) \sqrt{9 - \sin^2 x}} \end{aligned}$$

(3) **Proof.**

$$\frac{1}{\sqrt{4 \cos^2 x + \sin^2 x}} = \frac{1}{\sqrt{4 - 3 \sin^2 x}}$$
$$\frac{1}{\sqrt{\frac{9}{4} \cos^2 x + 2 \sin^2 x}} = \frac{1}{\sqrt{\frac{9}{4} - \frac{1}{4} \sin^2 x}} = \frac{2}{\sqrt{9 - \sin^2 x}}$$

从而

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\frac{9}{4} \cos^2 x + 2 \sin^2 x}} &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\frac{9}{4} - \frac{1}{4} \sin^2 x}} = \frac{2}{\sqrt{9 - \sin^2 x}} \\ &= \int_0^{\frac{\pi}{2}} \frac{f'(x) (3 + \sin^2 x)}{2 (3 - \sin^2 x)} dx \\ &= \int_{f(0)}^{f(\frac{\pi}{2})} \frac{(3 + \sin^2 x)}{2 (3 - \sin^2 x)} df(x) \end{aligned}$$

而

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{4 \cos^2 x + \sin^2 x}} &= \int_{f(0)}^{f(\frac{\pi}{2})} \frac{df(x)}{\sqrt{4 - 3 \sin^2 f(x)}} \\ &= \int_{f(0)}^{f(\frac{\pi}{2})} \frac{df(x)}{\sqrt{4 - 3 \sin^2 f(x)}} \\ &= \int_{f(0)}^{f(\frac{\pi}{2})} \frac{df(x)}{\sqrt{4 - 3 \left(\frac{4 \sin x}{3 + \sin^2 x} \right)^2}} \\ &= \int_{f(0)}^{f(\frac{\pi}{2})} \frac{(3 + \sin^2 x) df(x)}{2 \sqrt{\sin^4 x + 6 \sin^2 x + 9 - 12 \sin^2 x}} \\ &= \int_{f(0)}^{f(\frac{\pi}{2})} \frac{(3 + \sin^2 x)}{2 (3 - \sin^2 x)} df(x) \end{aligned}$$

从而原命题得证.

9.(10分)

设函数 $f : [0, 1] \rightarrow \mathbb{R}, g : [0, 1] \rightarrow \mathbb{R}$ 在 $[0, 1]$ 上连续, 满足 $f(0) = g(0), \sin(f(1)) = \sin(g(1)), \cos(f(1)) = \cos(g(1))$, 且

$$\forall x \in [0, 1], (\cos(f(x)) + \cos(g(x)))^2 + (\sin(f(x)) + \sin(g(x)))^2 \neq 0$$

证明: $f(1) = g(1)$.

Proof.

置 $h(x) = f(x) - g(x)$, 则 $h(x)$ 在 $[0, 1]$ 连续. 下面证明 $h(1) = 0$.

由题意

$$\sin h(1) = \sin(f(1) - g(1)) = \sin f(1) \cos g(1) - \sin g(1) \cos f(1) = 0$$

$$\cos h(1) = \cos(f(1) - g(1)) = \cos f(1) \cos g(1) + \sin f(1) \sin g(1) = 1$$

从而 $\exists n \in \mathbb{N}^*, \text{s.t. } h(1) = 2n\pi$.

下面采取反证法说明 $n = 0$.

若 $n > 0$, 则有

$$h(0) = 0 < \pi \leq 2n\pi = h(1)$$

据介值定理, $\exists a \in [0, 1], \text{s.t. } h(a) = \pi$, 从而

$$\sin g(a) = \sin f(a) \cos h(a) - \sin h(a) \cos f(a) = -\sin f(a)$$

$$\cos g(a) = \cos f(a) \cos h(a) + \sin f(a) \sin h(a) = -\cos f(a)$$

则

$$(\cos(f(x)) + \cos(g(x)))^2 + (\sin(f(x)) + \sin(g(x)))^2 = 0$$

与题设矛盾.

若 $n < 0$, 同理亦可推出矛盾.

从而 $n = 0$, 即 $f(1) = g(1)$, 原命题得证.