反三角函数及其在不定积分中的运用

前言:在高中时,我们已经熟知正弦,余弦,正切三种三角函数.在高等数学中, 我们将会学习更多三角函数以及它们的反函数,并以此为工具解决一些问题.

首先,我们引入以下三种新的三角函数.

$$\text{正割函数}\sec x = \frac{1}{\cos x},\text{余割函数}\csc x = \frac{1}{\sin x},\text{余切函数}\cot x = \frac{1}{\tan x}.$$

现在,我们首先来求反三角函数的导数.根据反函数的求导法则有$f'(x) = \dfrac{1}{g'(f(x))}$,其中$g(x)$是$f(x)$的反函数.

我们以$y = \arcsin x$为例. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\cos y} = \dfrac{1}{\cos(\arcsin(x))} = \dfrac{1}{\sqrt{1-x^2}}$. 同理可得$\dfrac{\mathrm{d}\arccos x}{\mathrm{d}x} = -\dfrac{1}{\sqrt{1-x^2}}$.
而$\dfrac{\mathrm{d}\arctan x}{\mathrm{d}x} = \dfrac{1}{\cos^2(\arctan x)} = \dfrac{1}{1+x^2}$. 同理可得$\dfrac{\mathrm{d}\ \mathrm{arccot}\ x}{\mathrm{d}x} = -\dfrac{1}{1+x^2}$.
然后,我们就可以开始求不定积分的漫漫征程了.

例1:求不定积分$\int \tan x \mathrm{d}x$.

解:

$$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = -\int \frac{\mathrm{d}(\cos x)}{\cos x} = -\ln|\cos x| + C$$

例2:求不定积分$\int \dfrac{\mathrm{d}x}{a^2-x^2}$.

解:

$$\begin{aligned}\int \frac{\mathrm{d}x}{a^2-x^2} &= \frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)\mathrm{d}x \\ &= \frac{1}{2a}\left(\int\frac{1}{a+x}\mathrm{d}(a+x) - \int\frac{1}{a-x}\mathrm{d}(a-x)\right) \\ &= \frac{1}{2a}\left(\ln|a+x| - \ln|a-x|\right) + C \\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C\end{aligned}$$

例3:求不定积分$\int \dfrac{\mathrm{d}x}{a^2+x^2}$.

解:

$$\begin{aligned}\int \frac{\mathrm{d}x}{a^2+x^2} &= \frac{1}{a^2}\int\frac{\mathrm{d}x}{1+\left(\frac{x}{a}\right)^2} \\ &= \frac{1}{a}\int\frac{\mathrm{d}\frac{x}{a}}{1+\left(\frac{x}{a}\right)^2} \\ &= \frac{1}{a}\arctan\frac{x}{a} + C\end{aligned}$$

例4:求不定积分$\int \dfrac{\mathrm{d}x}{\sqrt{a^2-x^2}}$.

解:

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} &= \int\frac{\mathrm{d}\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \\ &= \arcsin\frac{x}{a}\end{aligned}$$

例5:求不定积分$\int \frac{\mathrm{d}x}{\sin x}$.

解:

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sin x} &= \int \frac{\sin x \mathrm{d}x}{\sin^2 x} \\ &= -\int \frac{\mathrm{d}(\cos x)}{1-\cos^2 x} \\ &= \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C\end{aligned}$$

例6:求不定积分$\int \frac{\mathrm{d}x}{\cos x}$.

解:

$$\begin{aligned}\int \frac{\mathrm{d}x}{\cos x} &= \int \frac{\cos x \mathrm{d}x}{\cos^2 x} \\ &= \int \frac{\mathrm{d}(\sin x)}{1-\sin^2 x} \\ &= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

例7:求不定积分$\int \sqrt{a^2-x^2}\mathrm{d}x$.

解法一:采取换元法.令$x=a\sin t$,则$\mathrm{d}x = a\cos t\ \mathrm{d}t$.

$$\begin{aligned}\int \sqrt{a^2-x^2}\mathrm{d}x &= \int a\cos t \cdot a\cos t\mathrm{d}t = a^2\int \cos^2 t\mathrm{d}t \\ &= \frac{a^2}{2}\int (1+\cos 2t)\mathrm{d}t \\ &= \frac{a^2}{2}\left(t+\frac{1}{2}\sin 2t\right) + C \\ &= \frac{a^2}{2}\left(\arcsin\frac{x}{a} + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right) + C \\ &= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C\end{aligned}$$

解法二:采取分部积分法.

$$\begin{aligned}\int \sqrt{a^2-x^2}\mathrm{d}x &= x\sqrt{a^2-x^2} - \int x\mathrm{d}\left(\sqrt{a^2-x^2}\right) \\ &= x\sqrt{a^2-x^2} + \int \frac{x^2\mathrm{d}x}{\sqrt{a^2-x^2}} \\ &= x\sqrt{a^2-x^2} + \int \frac{(x^2-a^2)+a^2}{\sqrt{a^2-x^2}}\mathrm{d}x \\ &= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\mathrm{d}x + a^2\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}\end{aligned}$$

可知

$$\begin{aligned}\int \sqrt{a^2-x^2}\mathrm{d}x &= \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}\right) \\ &= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C\end{aligned}$$

例8:求不定积分$\int \frac{\mathrm{d}x}{\sqrt{a^2+x^2}}$.

解:采取换元法.令$x=a\tan t$,则$\mathrm{d}x=\frac{a\mathrm{d}t}{\cos^2 t}$.

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{a^2+x^2}} &= \int \frac{\frac{a\mathrm{d}t}{\cos^2 t}}{a\cdot\frac{1}{\cos t}} = \int \frac{\mathrm{d}t}{\cos t}\\ &= \frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right|+C\\ &= \ln\left|\frac{1}{\cos t}+\tan t\right|+C\\ &= \ln\left|\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right|+C\end{aligned}$$

亦可以写作$\ln\left|x+\sqrt{a^2+x^2}\right|+C$.

例9:求不定积分$\int \sqrt{a^2+x^2}\mathrm{d}x$.

解:采取换元法和分部积分法结合的方法.$x=a\tan t$,则$\mathrm{d}x=\frac{a\mathrm{d}t}{\cos^2 t}$.

$$\int \sqrt{a^2+x^2}\mathrm{d}x = \int \frac{a}{\cos t}\cdot\frac{a\mathrm{d}t}{\cos^2 t} = \int \frac{a^2\mathrm{d}t}{\cos^3 t}$$

我们记$I=\int \frac{\mathrm{d}t}{\cos^3 t}$,则

$$\begin{aligned}I = \int \frac{\mathrm{d}t}{\cos^3 t} &= \int \frac{\mathrm{d}\left(\tan t\right)}{\cos t}\\ &= \frac{\tan t}{\cos t}-\int \tan t\mathrm{d}\left(\frac{1}{\cos t}\right)\\ &= \frac{\tan t}{\cos t}-\int \frac{\sin t}{\cos t}\cdot\frac{\sin t}{\cos^2 t}\mathrm{d}t\\ &= \frac{\tan t}{\cos t}-\int \frac{1-\cos^2 t}{\cos^3 t}\mathrm{d}t\\ &= \frac{\tan t}{\cos t}+\int \frac{\mathrm{d}t}{\cos t}-I\end{aligned}$$

则有

$$\begin{aligned}\int \sqrt{a^2+x^2}\mathrm{d}x = a^2 I &= \frac{a^2}{2}\left(\frac{\tan t}{\cos t}+\int \frac{\mathrm{d}t}{\cos t}\right)\\ &= \frac{a^2}{2}\left(\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}}+\ln\left|\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right|\right)+C\\ &= \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln\left|\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right|+C\end{aligned}$$

亦可以写作$\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|+C$.

例10:求不定积分$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$.

解:分$x>a$和$x<-a$两种情况考虑.当$x>a$时,设$x=\dfrac{a}{\cos t}$,其中$t\in\left(0,\dfrac{\pi}{2}\right)$.则有

$$\begin{aligned}\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}&=\int\frac{1}{\sqrt{a^2\tan^2 t}}\cdot\frac{a\sin t}{\cos^2 t}\mathrm{d}t\\&=\int\frac{\mathrm{d}t}{\cos t}\\&=\ln\left|\frac{1}{\cos t}+\tan t\right|+C\end{aligned}$$

此时我们有$\tan t=\sqrt{\dfrac{1}{\cos^2 t}-1}=\dfrac{1}{a}\sqrt{x^2-a^2}$.

当$x<-a$时,令$x=-\dfrac{a}{\cos t}$可得到相同的结果.

综上有$\int\dfrac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+C$.

例11:求不定积分$\int\sqrt{x^2-a^2}\mathrm{d}x$.

解:我们直接采取分部积分法.

$$\begin{aligned}\int\sqrt{x^2-a^2}\mathrm{d}x&=x\sqrt{x^2-a^2}-\int x\mathrm{d}\left(\sqrt{x^2-a^2}\right)\\&=x\sqrt{x^2-a^2}-\int\frac{x^2\mathrm{d}x}{\sqrt{x^2-a^2}}\\&=x\sqrt{x^2-a^2}-\int\frac{\left(x^2-a^2+a^2\right)\mathrm{d}x}{\sqrt{x^2-a^2}}\\&=x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}\mathrm{d}x-a^2\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\end{aligned}$$

则有

$$\begin{aligned}\int\sqrt{x^2-a^2}\mathrm{d}x&=\frac{1}{2}\left(x\sqrt{x^2-a^2}-a^2\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\right)\\&=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C\end{aligned}$$

总结:整理上述积分可以得出

$$\begin{aligned}\int\tan x\mathrm{d}x&=-\ln|\cos x|+C\\\int\frac{\mathrm{d}x}{\sin x}&=\frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right|+C\\\int\frac{\mathrm{d}x}{\cos x}&=\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C\\&=\ln|\sec x+\tan x|+C\\\int\frac{\mathrm{d}x}{a^2-x^2}&=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C\\\int\frac{\mathrm{d}x}{a^2+x^2}&=\frac{1}{a}\arctan\frac{x}{a}+C\\\int\frac{\mathrm{d}x}{\sqrt{x^2\pm a^2}}&=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C\\\int\frac{\mathrm{d}x}{\sqrt{a^2-x^2}}&=\arcsin\frac{x}{a}+C\\\int\sqrt{x^2\pm a^2}\mathrm{d}x&=\frac{x}{2}\sqrt{x^2\pm a^2}\pm\frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|+C\\\int\sqrt{a^2-x^2}\mathrm{d}x&=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C\end{aligned}$$