北京大学数学科学学院2023-24高等数学A1期末考试

1.(20分)

求下列函数的极限.

(1) **(10分)** 求极限$\lim_{x\to 0}\frac{1}{x}\left((1+x)^{\frac{1}{x}}-\mathrm{e}\right)$.

(2) **(10分)** 设函数$f(x)$在$x=0$处$n+1$阶可导,且满足

$$f(0)=f'(0)=\cdots=f^{(n-1)}=0 \qquad f^{(n)}(0)=a$$

求极限$\lim_{x\to 0}\frac{f\left(\mathrm{e}^{x}-1\right)-f(x)}{x^{n+1}}$.

Solution.

(1) 令$u=(1+x)^{\frac{1}{x}}$,于是$\ln u=\frac{\ln(1+x)}{x}$,于是

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}\ln u}\cdot\frac{\mathrm{d}\ln u}{\mathrm{d}x}=(1+x)^{\frac{1}{x}}\frac{x-(x+1)\ln(x+1)}{x^2(x+1)}$$

因此

$$\begin{aligned}\lim_{x\to 0}\frac{1}{x}\left((1+x)^{\frac{1}{x}}-\mathrm{e}\right)&=\lim_{x\to 0}(1+x)^{\frac{1}{x}}\cdot\lim_{x\to 0}\frac{x-(x+1)\ln(x+1)}{x^2(x+1)}\\&=\mathrm{e}\cdot\lim_{x\to 0}\frac{\frac{1}{(x+1)^2}-\frac{1}{1+x}}{2x}\\&=\mathrm{e}\cdot\lim_{x\to 0}\frac{1-(1+x)}{2x(x+1)^2}\\&=-\frac{\mathrm{e}}{2}\end{aligned}$$

(2) 设$f^{(n+1)}(0)=b$.考虑$f(x)$在$x=0$处的泰勒展开

$$f(x)=f(0)+\sum_{k=1}^{n+1}\frac{f^{(k)}(0)x^k}{k!}+o(x^{n+1})=\frac{ax^n}{n!}+\frac{bx^{n+1}}{(n+1)!}+o(x^{n+1})$$

因为$\mathrm{e}^x-1\sim x$,于是

$$f(\mathrm{e}^x-1)-f(x)=\frac{a}{n!}\left((\mathrm{e}^x-1)^n-x^n\right)+\frac{b}{(n+1)!}\left((\mathrm{e}^x-1)^{n+1}-x^{n+1}\right)+o(x^{n+1})$$

于是

$$\begin{aligned}\lim_{x\to 0}\frac{f(\mathrm{e}^x-1)-f(x)}{x^{n+1}}&=\frac{a}{n!}\lim_{x\to 0}\frac{(\mathrm{e}^x-1)^n-x^n}{x^{n+1}}+\frac{b}{(n-1)!}\lim_{x\to 0}\left[\left(\frac{\mathrm{e}^x-1}{x}\right)^{n+1}-1+\frac{o(x^{n+1})}{x^{n+1}}\right]\\&=\frac{a}{n!}\lim_{x\to 0}\frac{\left(x+\frac{x^2}{2}+o(x^2)\right)^n-x^n}{x^{n+1}}+0\\&=\frac{a}{n!}\lim_{x\to 0}\frac{x^n+\frac{n}{2}x^{n+1}+o(x^{n+1})-x^n}{x^{n+1}}\\&=\frac{a}{2(n-1)!}\end{aligned}$$

2.(20分)

回答下列问题.

(1) **(10分)** 设函数$F(u,v)$有连续的二阶偏导数,$z=z(x,y)$是由方程$F(x-z,y-z)=0$确定的隐函数.计算并化简

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial^2 z}{\partial y^2}$$

(2) **(10分)** 给定方程组

$$\begin{cases} xy+yz^2+4=0 \\ x^2y+yz-z^2+5=0 \end{cases}$$

试讨论上述方程在$P_0\left(1,-2,1\right)$处能确定的隐函数,并计算其在P_0处的导数.

Solution.

(1) 设$G(x,y,z)=F(x-z,y-z)$.于是有

$$G_x(x,y,z)=F_u(x-z,y-z)\qquad G_y(x,y,z)=F_v(x-z,y-z)$$

$$G_z(x-z,y-z)=-F_u(x-z,y-z)-F_v(x-z,y-z)$$

于是根据隐函数存在定理,$G(x,y,z)=F(x-z,y-z)=0$确定的隐函数$z=z(x,y)$满足

$$\frac{\partial z}{\partial x}=-\frac{G_x}{G_z}=\frac{F_u}{F_u+F_v}\qquad \frac{\partial z}{\partial y}=-\frac{G_y}{G_z}=\frac{F_v}{F_u+F_v}$$

其中F_u,F_v均指代$F_u(x-z,y-z),F_v(x-z,y-z)$.于是有

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{F_u+F_v}{F_u+F_v}=1$$

将上式对x求偏导有

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}=0$$

对y求偏导有

$$\frac{\partial^2 z}{\partial y^2}+\frac{\partial^2 z}{\partial y\partial x}=0$$

于是

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial^2 z}{\partial y^2}=0$$

(2) 设$F(x,y,z)=xy+yz^2+4,G(x,y,z)=x^2y+yz-z^2+5$.计算$F,G$在$P_0$处的各偏导有

$$F_x=-2\qquad F_y=2\qquad F_z=-4\qquad G_x=-4\qquad G_y=2\qquad G_z=-4$$

计算雅可比行列式有

3.(20分)

求函数$f(x,y)=(y-x^2)(y-x^3)$的极值.

Solution.

对$f(x,y)$求偏导有

$$\frac{\partial f}{\partial x}=5x^4-3yx^2-2yx \qquad \frac{\partial f}{\partial y}=2y-x^2-x^3$$

令$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$,有

$$\begin{cases} x(5x^3-3xy-2y)=0 \\ 2y-x^2-x^3=0 \end{cases}$$

将$y=\frac{x^2+x^3}{2}$代入第一个方程中有$x^3\left(-\frac{3}{2}x^2+\frac{5}{2}x-1\right)=0$.于是$x=0$或1或$\frac{2}{3}$.
而

$$f_{xx}(x,y)=20x^3-6xy-2y \qquad f_{xy}=-3x^2-2x \qquad f_{yy}=2$$

当$(x,y)=(0,0)$时$B^2=AC$,于是$(0,0)$目前无法判断是否是极值点.为此,令$y=0$,则$f(x,y)=x^5$.
于是在$(0,0)$的任意邻域D内总存在使$f(x,0)$与$f(-x,0)$异号的$x\neq 0$.于是$(0,0)$不是$f(x,y)$的极值点.
当$(x,y)=(1,1)$时$B^2<AC$,于是$(1,1)$不是极值点.
当$(x,y)=\left(\frac{2}{3},\frac{10}{27}\right)$时$B^2>AC$,且$A<0$,于是$\left(\frac{2}{3},\frac{10}{27}\right)$是$f(x,y)$的极大值点,此时$f(x,y)=-\frac{4}{729}$.
由于$f(x,y)$在$\mathbb{R}^2$上连续且可微,于是$f(x,y)$没有其它的极值.
于是$f(x,y)$在$\left(\frac{2}{3},\frac{10}{27}\right)$取到极小值$-\frac{4}{729}$,没有极大值.

4.(20分)

回答下列问题.

(1) **(10分)** 设函数$f(x,y)$在点$(0,0)$的某邻域内有定义且在$(0,0)$处连续.若极限$\lim\limits_{(x,y)\to(0,0)}\frac{f(x,y)}{x^2+y^2}$存在,试证明$f(x,y)$在$(0,0)$处可微.

(2) **(10分)** 欧氏空间$\mathbb{R}^3$中平面$T:x+y+z=1$截圆柱面$S:x^2+y^2=1$得一椭圆周R.求R上到原点最近和最远的点.

Solution.

(1) 由题意可知$\lim\limits_{(x,y)\to(0,0)}f(x,y)=0$.又$f(x,y)$在$(0,0)$处连续,于是$f(0,0)=0$.

不妨令$\lim\limits_{(x,y)\to(0,0)}\frac{f(x,y)}{x^2+y^2}=t$.

取$(x,y)\to(0,0)$的特殊路径$x\to 0,y=0$可知

$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}x\cdot\frac{f(x,0)}{x^2+0^2}=0$$

同理$\lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y}=0$.因而$f_x(0,0)=f_y(0,0)=0$.
于是

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-xf_x(0,0)-yf_y(0,0)-f(0,0)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{f(x,y)}{x^2+y^2}\cdot\sqrt{x^2+y^2}=0$$

因而$f(x,y)=f_(0,0)+xf_x(0,0)+yf_y(0,0)+o(\rho)$,其中$\rho=\sqrt{x^2+y^2}\to 0$.

于是$f(x,y)$在$(0,0)$处可微.

(2) 假定$x^2+y^2=1$.设距离为$d(x,y,z)=\sqrt{x^2+y^2+z^2}=\sqrt{1+z^2}$.不妨令$x=\sin\theta,y=\cos\theta$,则有

$$z=1-x-y=1-\sin\theta-\cos\theta$$

令$z(\theta)=(1-\sin\theta-\cos\theta)^2$,于是$z'(\theta)=2(1-\sin\theta-\cos\theta)(\sin\theta-\cos\theta)$.令$z'(\theta)=0$可知$\theta=\dfrac{\pi}{4},\dfrac{5\pi}{4},0,\dfrac{\pi}{2}$.
于是我们有

$$z\left(\frac{\pi}{4}\right)=\left(\sqrt{2}-1\right)^2\qquad z\left(\frac{5\pi}{4}\right)=\left(\sqrt{2}+1\right)^2\qquad z\left(\frac{\pi}{2}\right)=0\qquad z(0)=0$$

距离原点最近的点为$(1,0,0)$和$(0,1,0)$,距离原点最远的点为$\left(-\dfrac{\sqrt{2}}{2},-\dfrac{\sqrt{2}}{2},1+\sqrt{2}\right)$.

5.(20分)

回答下列问题.

(1) **(10分)** 设$f(x)$是一个定义在$\mathbb{R}$上的周期为$T\neq 0$的无穷阶光滑函数.试证明:对于任意$k\in\mathbb{N}^*$,总存在$\xi\in\mathbb{R}$使得$f^{(k)}(\xi)=0$.

(2) **(10分)** 设函数$f(u,v)$有连续的偏导数$f_u(u,v)$和$f_v(u,v)$且满足$f(x,1-x)=1$.试证明:在单位圆周$S:u^2+v^2=1$上至少存在两个不同的点(u_1,v_1)和(u_2,v_2)使得$v_if_u(u_i,v_i)=u_if_v(u_i,v_i)$,其中$i=1,2$.

Proof.

(1) 首先,若$f(x)$是周期为T的周期函数,则$f'(x)$也是周期为T的周期函数.这可以由

$$f'(x+T)=\lim_{\Delta x\to 0}\frac{f(x+T+\Delta x)-f(x+T)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x)$$

得到.于是归纳可得对于任意$k\in\mathbb{N}^*$,$f^{(k)}(x)$都是周期为T的周期函数.

于是对于任意$k\in\mathbb{N}^*$,考虑某一$x\in\mathbb{R}$,则有$f^{(k-1)}(x)=f^{(k-1)}(x+T)$.

根据Rolle中值定理,存在$\xi \in (x, x+T)$使得$f^{(k)}(\xi) = 0$,命题得证.

(2) 考虑单位圆周上的点$T(\cos\theta, \sin\theta)$.令$g(\theta) = f(\cos\theta, \sin\theta)$.于是

$$g'(\theta) = -\sin\theta f_u(\cos\theta, \sin\theta) + \cos\theta f_v(\cos\theta, \sin\theta)$$

注意到$g(0) = f(1,0) = 1, g\left(\frac{\pi}{2}\right) = f(0,1) = 1, g(2\pi) = f(1,0) = 1$.
由于$f_u(u,v), f_v(u,v)$均是连续函数,于是$g(\theta)$也是连续函数.
在$\left[0, \frac{\pi}{2}\right]$应用Rolle中值定理可知存在$\phi_1 \in \left(0, \frac{\pi}{2}\right)$使得$g'(\phi_1) = 0$,即

$$\sin\phi_1 f_u(\cos\phi_1, \sin\phi_1) = \cos\phi_1 f_v(\cos\phi_1, \sin\phi_1)$$

同理在$\left[\frac{\pi}{2}, 2\pi\right]$应用Rolle中值定理可知存在$\phi_2 \in \left(\frac{\pi}{2}, 2\pi\right)$使得

$$\sin\phi_2 f_u(\cos\phi_2, \sin\phi_2) = \cos\phi_2 f_v(\cos\phi_2, \sin\phi_2)$$

于是单位圆周上至少存在两个点$(\cos\phi_1, \sin\phi_1)$和$(\cos\phi_2, \sin\phi_2)$满足题意.