反三角函数及其在不定积分中的运用

前言:在高中时,我们已经熟知正弦,余弦,正切三种三角函数.在高等数学中, 我们将会学习更多三角函数以及它们的反函数,并以此为工具解决一些问题.
首先,我们扩展一下所学的三角函的类型.

六种三角函数

$$\begin{aligned} &\text{正弦函数} \quad \sin x && \text{正割函数} \quad \sec x = \frac{1}{\cos x} \\ &\text{余弦函数} \quad \cos x && \text{余割函数} \quad \csc x = \frac{1}{\sin x} \\ &\text{正切函数} \quad \tan x && \text{余切函数} \quad \cot x = \frac{1}{\tan x} \end{aligned}$$

根据反函数的求导法则,我们可以写出反三角函数的导数.

反三角函数的导数

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \qquad (\arctan x)' = \frac{1}{1+x^2}$$

现在,我们来进行一系列不定积分的推导.

Example 1.
求不定积分

$$\int \tan x \mathrm{d}x$$

Solution.

$$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = -\int \frac{\mathrm{d}(\cos x)}{\cos x} = -\ln|\cos x| + C$$

Example 2.

求不定积分

$$\int \frac{\mathrm{d}x}{a^2 - x^2}$$

Solution.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{a^2 - x^2} &= \frac{1}{2a} \int \left(\frac{1}{a+x} + \frac{1}{a-x} \right) \mathrm{d}x \\
&= \frac{1}{2a} \left(\int \frac{1}{a+x} \mathrm{d}(a+x) - \int \frac{1}{a-x} \mathrm{d}(a-x) \right) \\
&= \frac{1}{2a} \left(\ln|a+x| - \ln|a-x| \right) + C \\
&= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C
\end{aligned}$$

Example 3.

求不定积分

$$\int \frac{\mathrm{d}x}{a^2 + x^2}$$

Solution.

$$\int \frac{\mathrm{d}x}{a^2 + x^2} = \frac{1}{a^2} \int \frac{\mathrm{d}x}{1 + \left(\frac{x}{a}\right)^2} = \frac{1}{a} \int \frac{\mathrm{d}\frac{x}{a}}{1 + \left(\frac{x}{a}\right)^2} = \frac{1}{a} \arctan \frac{x}{a} + C$$

Example 4.

求不定积分

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}}$$

Solution.

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \int \frac{\mathrm{d}\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \arcsin\frac{x}{a} + C$$

Example 5.

求不定积分

$$\int \frac{\mathrm{d}x}{\sin x}$$

Solution.

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{\sin x \mathrm{d}x}{\sin^2 x} = -\int \frac{\mathrm{d}(\cos x)}{1-\cos^2 x} = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$$

Example 6.

求不定积分

$$\int \frac{\mathrm{d}x}{\cos x}$$

Solution.

$$\int \frac{\mathrm{d}x}{\cos x} = \int \frac{\cos x \mathrm{d}x}{\cos^2 x} = \int \frac{\mathrm{d}(\sin x)}{1-\sin^2 x} = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C = \ln|\sec x + \tan x| + C$$

Example 7.

求不定积分

$$\int \sqrt{a^2-x^2}\mathrm{d}x$$

Solution(Method I).

采取换元法.令$x = a\sin t$,则$\mathrm{d}x = a\cos t\mathrm{d}t$.

$$\begin{aligned}\int\sqrt{a^2-x^2}\mathrm{d}x &= \int a\cos t\cdot a\cos t\mathrm{d}t = a^2\int\cos^2 t\mathrm{d}t\\ &= \frac{a^2}{2}\int(1+\cos 2t)\mathrm{d}t\\ &= \frac{a^2}{2}\left(t+\frac{1}{2}\sin 2t\right)+C\\ &= \frac{a^2}{2}\left(\arcsin\frac{x}{a}+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right)+C\\ &= \frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C\end{aligned}$$

Solution(Method II).

采取分部积分法.

$$\begin{aligned}\int\sqrt{a^2-x^2}\mathrm{d}x &= x\sqrt{a^2-x^2}-\int x\mathrm{d}\left(\sqrt{a^2-x^2}\right)\\ &= x\sqrt{a^2-x^2}+\int\frac{x^2\mathrm{d}x}{\sqrt{a^2-x^2}}\\ &= x\sqrt{a^2-x^2}+\int\frac{(x^2-a^2)+a^2}{\sqrt{a^2-x^2}}\mathrm{d}x\\ &= x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\mathrm{d}x+a^2\int\frac{\mathrm{d}x}{\sqrt{a^2-x^2}}\end{aligned}$$

可知

$$\begin{aligned}\int\sqrt{a^2-x^2}\mathrm{d}x &= \frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\int\frac{\mathrm{d}x}{\sqrt{a^2-x^2}}\right)\\ &= \frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C\end{aligned}$$

Example 8.

求不定积分

$$\int\frac{\mathrm{d}x}{\sqrt{a^2+x^2}}$$

Solution.

采取换元法.令$x=a\tan t$,则$\mathrm{d}x=\dfrac{a\mathrm{d}t}{\cos^2 t}$.

$$\begin{aligned}\int\frac{\mathrm{d}x}{\sqrt{a^2+x^2}}&=\int\frac{\dfrac{a\mathrm{d}t}{\cos^2 t}}{a\cdot\dfrac{1}{\cos t}}=\int\frac{\mathrm{d}t}{\cos t}\\&=\frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right|+C\\&=\ln\left|\frac{1}{\cos t}+\tan t\right|+C\\&=\ln\left|\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right|+C\end{aligned}$$

亦可以写作$\ln\left|x+\sqrt{a^2+x^2}\right|+C$.

Example 9.

求不定积分

$$\int\sqrt{a^2+x^2}\mathrm{d}x$$

Solution

采取换元法和分部积分法结合的方法.置$x=a\tan t$,则$\mathrm{d}x=\dfrac{a\mathrm{d}t}{\cos^2 t}$,从而

$$\int\sqrt{a^2+x^2}\mathrm{d}x=\int\frac{a}{\cos t}\cdot\frac{a\mathrm{d}t}{\cos^2 t}=\int\frac{a^2\mathrm{d}t}{\cos^3 t}$$

我们记$I=\displaystyle\int\frac{\mathrm{d}t}{\cos^3 t}$,则

$$\begin{aligned}I&=\int\frac{\mathrm{d}t}{\cos^3 t}\\&=\int\frac{\mathrm{d}\left(\tan t\right)}{\cos t}\\&=\frac{\tan t}{\cos t}-\int\tan t\mathrm{d}\left(\frac{1}{\cos t}\right)\\&=\frac{\tan t}{\cos t}-\int\frac{\sin t}{\cos t}\cdot\frac{\sin t}{\cos^2 t}\mathrm{d}t\\&=\frac{\tan t}{\cos t}-\int\frac{1-\cos^2 t}{\cos^3 t}\mathrm{d}t\\&=\frac{\tan t}{\cos t}+\int\frac{\mathrm{d}t}{\cos t}-I\end{aligned}$$

则有

$$\begin{aligned}\int\sqrt{a^2+x^2}\mathrm{d}x=a^2I&=\frac{a^2}{2}\left(\frac{\tan t}{\cos t}+\int\frac{\mathrm{d}t}{\cos t}\right)\\&=\frac{a^2}{2}\left(\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}}+\ln\left|\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right|\right)+C\\&=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln\left|\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right|+C\end{aligned}$$

亦可以写作$\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|+C$.

Example 10.

求不定积分

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$$

Solution.

分$x>a$和$x<-a$两种情况考虑. 当$x>a$时,设$x=\frac{a}{\cos t}$,其中$t\in\left(0,\frac{\pi}{2}\right)$.则有

$$\begin{aligned}\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}&=\int\frac{1}{\sqrt{a^2\tan^2 t}}\cdot\frac{a\sin t}{\cos^2 t}\mathrm{d}t\\&=\int\frac{\mathrm{d}t}{\cos t}\\&=\ln\left|\frac{1}{\cos t}+\tan t\right|+C\end{aligned}$$

此时我们有$\tan t=\sqrt{\frac{1}{\cos^2 t}-1}=\frac{1}{a}\sqrt{x^2-a^2}$.
当$x<-a$时,令$x=-\frac{a}{\cos t}$可得到相同的结果.
综上有$\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+C$.

Example 11.

求不定积分

$$\int\sqrt{x^2-a^2}\mathrm{d}x$$

Solution(Method I).

采取分部积分法.

$$\begin{aligned}\int\sqrt{x^2-a^2}\mathrm{d}x&=x\sqrt{x^2-a^2}-\int x\mathrm{d}\left(\sqrt{x^2-a^2}\right)\\&=x\sqrt{x^2-a^2}-\int\frac{x^2\mathrm{d}x}{\sqrt{x^2-a^2}}\\&=x\sqrt{x^2-a^2}-\int\frac{\left(x^2-a^2+a^2\right)\mathrm{d}x}{\sqrt{x^2-a^2}}\\&=x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}\mathrm{d}x-a^2\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\end{aligned}$$

则有

$$\begin{aligned}\int\sqrt{x^2-a^2}\mathrm{d}x&=\frac{1}{2}\left(x\sqrt{x^2-a^2}-a^2\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\right)\\&=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C\end{aligned}$$

Solution(Method II).

采取换元法.当$x>a$时,置$x=\dfrac{a}{\cos t},t\in\left(0,\dfrac{\pi}{2}\right)$.则

$$\begin{aligned}\int\sqrt{x^2-a^2}\mathrm{d}x&=\int\sqrt{a^2\tan t}\cdot\frac{a\sin t}{\cos^2 t}\mathrm{d}t\\&=a^2\int\frac{\sin^2 t}{\cos^3 t}\mathrm{d}t\\&=a^2\left(\int\frac{\mathrm{d}t}{\cos^3 t}-\int\frac{\mathrm{d}t}{\cos t}\right)\end{aligned}$$

由**Example 9.**可知$\displaystyle\int\frac{\mathrm{d}t}{\cos^3 t}=\frac{1}{2}\left(\frac{\tan t}{\cos t}+\int\frac{\mathrm{d}t}{\cos t}\right)$ 从而

$$\begin{aligned}\int\sqrt{x^2-a^2}\mathrm{d}x&=\frac{a^2}{2}\left(\frac{\tan t}{\cos t}-\int\frac{\mathrm{d}t}{\cos t}\right)\\&=\frac{a^2}{2}\left(\frac{\sqrt{1-\dfrac{a^2}{x^2}}}{\dfrac{a^2}{x^2}}+\ln\left|\frac{x}{a}+\sqrt{1-\frac{a^2}{x^2}}\right|\right)+C_0\\&=\frac{x^2}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C\end{aligned}$$

当$x<-a$时亦可以得到相同的结果.于是

$$\int\sqrt{x^2-a^2}\mathrm{d}x=\frac{x^2}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C$$

Integral Table

1. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

2. $\displaystyle\int \sqrt{x^2 \pm a^2}\mathrm{d}x = \frac{x^2}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

3. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C$

4. $\displaystyle\int \sqrt{a^2 - x^2} = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$

5. $\displaystyle\int \frac{\mathrm{d}x}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

6. $\displaystyle\int \frac{\mathrm{d}x}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$

7. $\displaystyle\int \tan x\mathrm{d}x = -\ln|\cos x| + C$

8. $\displaystyle\int \frac{\mathrm{d}x}{\sin x} = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$

9. $\displaystyle\int \frac{\mathrm{d}x}{\cos x} = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C = \ln|\sec x + \tan x| + C$