中值定理的应用

我们来看一些关于中值定理的例题.

Example 1(2021Fall PKU高等数学B期中考试).

注:这实际上就是Riemann引理.

证明:对于$[0,1]$上的任何连续函数$f(x)$, 都有$\lim\limits_{n\to\infty}\int_0^1 f(x)\sin(nx)\mathrm{d}x=0$.

注意:本题中没有假定$f(x)$的导函数$f'(x)$存在.

证明:由$f(x)$在$[0,1]$连续可得$f(x)$在$\left[\dfrac{j}{k},\dfrac{j+1}{k}\right]$上也连续,其中$k,j\in\mathbb{N}^*,0\leqslant j<m$.

记$f(x)$在$\left[\dfrac{j}{k},\dfrac{j+1}{k}\right]$的上下界分别为$M_j,m_j$.

由于$f(x)$在$[0,1]$连续,故$\displaystyle\int_0^1 f(x)\mathrm{d}x$存在.设$\displaystyle\int_0^1 f(x)\mathrm{d}x=A$,则Rieman和的极限有

$$\lim_{n\to\infty}\sum_{j=0}^{k-1}\frac{M_j}{k}=\lim_{n\to\infty}\sum_{j=0}^{k-1}\frac{m_j}{k}=A$$

从而

$$\lim_{n\to\infty}\sum_{j=0}^{k-1}\frac{M_j-m_j}{k}=A-A=0$$

即$\forall\varepsilon>0,\exists K>0,\text{s.t.}\forall k>K,\left|\displaystyle\sum_{j=0}^{k-1}\frac{M_j-m_j}{k}\right|<\dfrac{\varepsilon}{2}$.

由$f(x)$在$[0,1]$连续可得$\exists B>0,\text{s.t.}\,|f(x)|<B$.

现在,$\forall n\in\mathbb{N}^*$有

$$\begin{aligned}
\left|\int_0^1 f(x)\sin(nx)\mathrm{d}x\right| &= \left|\sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+1}{k}} f(x)\sin(nx)\mathrm{d}x\right|\\
&= \left|\sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+1}{k}}\left(f(x)-f\left(\frac{j}{k}\right)\right)\sin(nx)\mathrm{d}x+\int_{\frac{j}{k}}^{\frac{j+1}{k}} f\left(\frac{j}{k}\right)\sin(nx)\mathrm{d}x\right|\\
&\leqslant \sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+1}{k}}\left|f(x)-f\left(\frac{j}{k}\right)\right|\left|\sin(nx)\right|\mathrm{d}x+\sum_{j=0}^{k-1} f\left(\frac{j}{k}\right)\int_{\frac{j}{k}}^{\frac{j+1}{k}}\sin(nx)\mathrm{d}x\\
&\leqslant \sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+1}{k}}\left|f(x)-f\left(\frac{j}{k}\right)\right|\mathrm{d}x+\sum_{j=0}^{k-1} f\left(\frac{j}{k}\right)\cdot\frac{1}{n}\left(\cos\frac{j}{k}-\cos\frac{j+1}{k}\right)\\
&\leqslant \sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+1}{k}}\left|M_j-m_j\right|\mathrm{d}x+\frac{2Bk}{n}\\
&\leqslant \sum_{j=0}^{k-1}\frac{M_j-m_j}{k}+\frac{2Bk}{n}\\
&< \frac{\varepsilon}{2}+\frac{2Bk}{n}
\end{aligned}$$

从而$\forall \varepsilon > 0, \exists N = \max\left\{K, \dfrac{4Bk}{\varepsilon}\right\}$, s.t.$\forall n > N$,

$$\left|\lim_{n\to\infty}\int_0^1 f(x)\sin(nx)\mathrm{d}x\right| < \frac{\varepsilon}{2} + \frac{2Bk}{N} < \varepsilon$$

从而$\lim\limits_{n\to\infty}\displaystyle\int_0^1 f(x)\sin(nx)\mathrm{d}x = 0$,证毕.

还有一个与**Example 1**相似的命题.

Example 2.

设函数$f(x)$在$[0,\pi]$上连续.对于$n \in \mathbb{N}$,试证明

$$\lim_{n\to\infty}\int_0^\pi f(x)\left|\sin(nx)\right|\mathrm{d}x = \frac{2}{\pi}\int_0^\pi f(x)\mathrm{d}x$$

Proof.

我们有

$$\int_0^\pi f(x)\left|\sin(nx)\right|\mathrm{d}x = \sum_{k=1}^{n}\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} f(x)\left|\sin(nx)\right|\mathrm{d}x$$

根据积分第一中值定理,$\exists \xi_k \in \left[\dfrac{k\pi}{n}, \dfrac{(k+1)\pi}{n}\right]$, s.t. $\displaystyle\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} f(x)\left|\sin(nx)\right|\mathrm{d}x = f(\xi_k)\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}}\left|\sin(nx)\right|\mathrm{d}x$

令$u = nx$,则

$$\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}}\left|\sin(nx)\right|\mathrm{d}x = \frac{1}{n}\int_{k\pi}^{(k+1)\pi}\left|\sin u\right|\mathrm{d}u = \frac{2}{n}$$

于是

$$\begin{aligned}\lim_{n\to\infty}\int_0^\pi f(x)\left|\sin(nx)\right|\mathrm{d}x &= \lim_{n\to\infty}\sum_{k=1}^{n}\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} f(x)\left|\sin(nx)\right|\mathrm{d}x \\ &= \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n} f(\xi_k) \\ &= \frac{2}{\pi}\lim_{n\to\infty}\frac{\pi}{n}\sum_{k=1}^{n} f(\xi_k) \\ &= \frac{2}{\pi}\int_0^\pi f(x)\mathrm{d}x\end{aligned}$$

Example 3.

设函数$f(x)$在$[0,1]$连续,在$(0,1)$可导. 试证明:对于任意$A, B > 0$和$n \in \mathbb{N}^*$,在$[0,1]$上存在严格递增的序

列$\theta_0, \cdots, \theta_n$使得

$$(A+B)^n = \sum_{k=0}^{n} \frac{1}{f'(\theta_k)} C_n^k A^k B^{n-k}$$

Proof.

题设式子两端同时除以$(A+B)^n$有

$$1 = \sum_{k=0}^{n} \frac{1}{f'(\theta_k)} C_n^k \left(\frac{A}{A+B}\right)^k \left(\frac{B}{A+B}\right)^{n-k}$$

令$\dfrac{A}{A+B} = a, \dfrac{B}{A+B} = b$,于是$0 < a, b < 1$且$a + b = 1$.不妨设$0 < a < b < 1$. 只需证

$$\sum_{k=0}^{n} \frac{1}{f'(\theta_k)} C_n^k a^k b^{n-k} = 1$$

即可.

由Lagrange中值定理,对于任意$\alpha, \beta \in [0,1]$且$\alpha < \beta$,总存在$\xi \in (\alpha, \beta)$使得$f'(\xi) = \dfrac{f(\beta) - f(\alpha)}{\beta - \alpha}$.

对上式稍作变形即可得$\beta - \alpha = \dfrac{f(\beta) - f(\alpha)}{f'(\xi)}$.

根据二项式定理,我们有

$$1 = (a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$$

取$y_k = \sum\limits_{i=0}^{k} C_n^i a^i b^{n-i} \in (0,1]$,于是$y_k - y_{k-1} = C_n^k a^k b^{n-k} > 0$,于是$\{y_k\}_{i=0}^{n}$单调递增.

由于$f'(x) > 0$,于是$f(x)$单调递增,进而取$x_k = f^{-1}(y_k)$也保证其单调递增.

于是对任意$0 \leqslant k \leqslant n$,在区间$[x_{k-1}, x_k]$(不妨令$x_{-1} = y_{-1} = 0$)应用Lagrange中值定理可知

$$\exists \theta_k \in (x_{k-1}, x_k), \text{s.t.} x_k - x_{k-1} = \frac{y_k - y_{k-1}}{f'(\theta_k)} = \frac{1}{f'(\theta_k)} C_n^k a^k b^{n-k}$$

对上述等式求和有

$$\sum_{k=0}^{n} (x_k - x_{k-1}) = \sum_{k=0}^{n} \frac{1}{f'(\theta_k)} C_n^k a^k b^{n-k}$$

我们注意到$x_{-1} = 0, x_n = f^{-1}(y_n) = f^{-1}(1) = 1$,于是

$$\sum_{k=0}^{n} \frac{1}{f'(\theta_k)} C_n^k a^k b^{n-k} = 1$$

命题得证