

相乘序列的极限

设序列 $\{a_n\}, \{b_n\}$ 满足 $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$,
则有

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n a_i b_{n+1-i}}{n} = 0$$

Analysis.

我们把和式拆成两部分,有

$$\frac{\sum_{i=1}^n a_i b_{n+1-i}}{n} = \frac{\sum_{i=1}^j a_i b_{n+1-i}}{n} + \frac{\sum_{i=j+1}^n a_i b_{n+1-i}}{n}$$

注意到两部分中分别可以令 a_i 和 b_i 很小,且 $\{a_n\}, \{b_n\}$ 有界,放缩即可证明.

Proof.

由题意

$$\forall \varepsilon_a > 0, \exists N_a \in \mathbb{N}^*, \text{s.t.} \forall n \geq N_a, |a_n| \leq \varepsilon_a$$

$$\forall \varepsilon_b > 0, \exists N_b \in \mathbb{N}^*, \text{s.t.} \forall n \geq N_b, |b_n| \leq \varepsilon_b$$

且由收敛序列有界可知

$$\exists M_a \in \mathbb{R}, \text{s.t.} \forall n \in \mathbb{N}^*, |a_n| \leq M_a$$

$$\exists M_b \in \mathbb{R}, \text{s.t.} \forall n \in \mathbb{N}^*, |b_n| \leq M_b$$

对于给定的 $\varepsilon_a, \varepsilon_b$ 和对应的 $N_a, N_b, \forall n > N_a + N_b$ 有

$$\begin{aligned} \left| \frac{\sum_{i=1}^n a_i b_{n+1-i}}{n} \right| &= \left| \frac{\sum_{i=1}^{N_a-1} a_i b_{n+1-i}}{n} + \frac{\sum_{i=N_a}^n a_i b_{n+1-i}}{n} \right| \\ &\leq \left| M_a \cdot \frac{\sum_{i=n+2-N_a}^n b_i}{n} + M_b \cdot \frac{\sum_{i=N_a}^n a_i}{n} \right| \\ &\leq M_a \cdot \frac{N_a-1}{n} \cdot \varepsilon_b + M_b \cdot \frac{n-N_a+1}{n} \cdot \varepsilon_a \\ &\leq M_a \varepsilon_b + M_b \varepsilon_a \end{aligned}$$

现在,对于任意 $\varepsilon > 0$,取 $0 < \varepsilon_a < \varepsilon, 0 < \varepsilon_b < \frac{\varepsilon - M_b \varepsilon_a}{M_a}$ 和对应的 N_a, N_b

令 $N = N_a + N_b + 1$,则 $\forall n > N$ 有

$$\left| \frac{\sum_{i=1}^n a_i b_{n+1-i}}{n} \right| \leq M_a \varepsilon_b + M_b \varepsilon_a < \varepsilon$$

从而 $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n a_i b_{n+1-i}}{n} = 0$, 证毕.

Enhanced Proposition.

设序列 $\{a_n\}, \{b_n\}$ 满足 $\lim_{n \rightarrow \infty} a_n = A, \lim_{n \rightarrow \infty} b_n = B, A, B \in \mathbb{R}$,
则有

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n a_i b_{n+1-i}}{n} = AB$$

Proof.

我们记序列 $\{x_n\}$ 满足 $x_n = a_n - A$, 序列 $\{y_n\}$ 满足 $y_n = b_n - B$.

则 $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 0$.

据Cauchy命题有 $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n x_i}{n} = \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n y_i}{n} = 0$.

则

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n a_i b_{n+1-i}}{n} &= \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n (x_i + A)(y_{n+1-i} + B)}{n} \\ &= AB + A \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n y_i}{n} + B \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n x_i}{n} + \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n x_i y_{n+1-i}}{n} \\ &= AB \end{aligned}$$

从而原命题得证.

例1(24.10.09 SJTU数分小测):

数列 $\{x_n\}$ 有 $\lim_{n \rightarrow \infty} x_n = A$. 正项数列 $\{y_n\}$ 有 $\lim_{n \rightarrow \infty} \frac{y_n}{\sum_{i=1}^n y_i} = 0$.

试证明: $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n y_i x_{n+1-i}}{\sum_{i=1}^n y_i} = A$.

Proof.

记序列 $\{a_n\}$ 满足 $a_n = x_n - A$. 则 $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} x_n - A = 0$.

则 $\forall \varepsilon_a > 0, \exists N_a \in \mathbb{N}^*, \text{s.t. } \forall n \geq N_a, |a_n| \leq \varepsilon_a$

且 $\exists M_a \in \mathbb{R}, \text{s.t. } \forall n \in \mathbb{N}^*, |a_n| < M_a$.

又 $\lim_{n \rightarrow \infty} \frac{y_n}{\sum_{i=1}^n y_i} = 0$, 则 $\forall \varepsilon_y > 0, \exists N_y \in \mathbb{N}^*, \text{s.t. } \forall n \geq N_y, \left| \frac{y_n}{\sum_{i=1}^n y_i} \right| < \varepsilon_y$

则 $n > N_a$ 且 $n + 2 - N_a > N_y$ 时有

$$\begin{aligned} \left| \frac{\sum_{i=1}^n y_i x_{n+1-i}}{\sum_{i=1}^n y_i} - A \right| &= \left| \frac{\sum_{i=1}^n y_i a_{n+1-i}}{\sum_{i=1}^n y_i} \right| \\ &= \left| \frac{\sum_{i=1}^{n+1-N_a} y_i a_{n+1-i}}{\sum_{i=1}^n y_i} + \frac{\sum_{i=n+2-N_a}^n y_i a_{n+1-i}}{\sum_{i=1}^n y_i} \right| \\ &\leq \left| \frac{\sum_{i=1}^{n+1-N_a} y_i}{\sum_{i=1}^n y_i} \varepsilon_a + \frac{\sum_{i=n+2-N_a}^n y_i}{\sum_{i=1}^n y_i} M_a \right| \end{aligned}$$

$$\leq \varepsilon_a + M_a \varepsilon_b$$

从而对于任意 $\varepsilon > 0$, 取 $0 < \varepsilon_a < \varepsilon$, $0 < \varepsilon_b < \frac{\varepsilon - \varepsilon_a}{M_a}$ 和对应的 N_a, N_y

令 $N = N_a + N_y + 2$, 则 $\forall n \geq N$ 有

$$\begin{aligned} \left| \frac{\sum_{i=1}^n y_i x_{n+1-i}}{\sum_{i=1}^n y_i} - A \right| &\leq \varepsilon_a + M_a \varepsilon_b \\ &< \varepsilon_a + M_a \cdot \frac{\varepsilon - \varepsilon_a}{M_a} \\ &= \varepsilon \end{aligned}$$

从而 $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n y_i x_{n+1-i}}{\sum_{i=1}^n y_i} = A$, 证毕.

例2(2023Fall PKU高等数学B期中考试):

序列 $\{a_n\}$ 有 $\lim_{n \rightarrow \infty} a_n = A, 0 < q < 1$.

试证明: $\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i q^{n-i} = \frac{A}{1-q}$.

证明: 记序列 $\{x_n\}$ 满足 $x_n = a_n - A$.

则

$$\begin{aligned} \sum_{i=1}^n a_i q^{n-i} &= \sum_{i=1}^n (x_i + A) q^{n-i} \\ &= \sum_{i=1}^n x_i q^{n-i} + A \sum_{i=1}^n q^{n-i} \\ &= \sum_{i=1}^n x_i q^{n-i} + A \cdot \frac{1 - q^n}{1 - q} \end{aligned}$$

下面证明 $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i q^{n-i} = 0$.

由于 $\lim_{n \rightarrow \infty} x_n = 0$, 则 $\forall \varepsilon_x > 0, \exists N_x \in \mathbb{N}^*, \text{s.t. } \forall n \geq N_x, |x_n| < \varepsilon_x$ 且 $\exists M_x \in \mathbb{R}, \text{s.t. } \forall n \in \mathbb{N}^*, |x_n| < M_x$.
则 $\forall n > N_x$ 有

$$\begin{aligned} \left| \sum_{i=1}^n x_i q^{n-i} \right| &= \left| \sum_{i=1}^{N_x-1} x_i q^{n-i} + \sum_{i=N_x}^n x_i q^{n-i} \right| \\ &= M_x \sum_{i=1}^{N_x-1} q^{n-i} + \varepsilon_x \sum_{i=N_x}^n q^{n-i} \\ &= M_x \cdot \frac{q^{n-N_x+1} - q^n}{1 - q} + \varepsilon_x \cdot \frac{1 - q^{n-N_x+1}}{1 - q} \\ &< M_x q^n \cdot \frac{1 - q^{N_x-1}}{q^{N_x-1}(1 - q)} + \frac{\varepsilon_x}{1 - q} \end{aligned}$$

$$\begin{aligned}
&< \frac{M_x q^n}{q^{N_x}(1-q)} + \frac{\varepsilon_x}{1-q} \\
&= \frac{1}{1-q} (M_x q^{n-N_x} + \varepsilon_x)
\end{aligned}$$

对于任意 $\varepsilon > 0$,取 $0 < \varepsilon_x < (1-q)\varepsilon$ 和对应的 N_x .

令 $N = \left\lceil \log_q \frac{(1-q)\varepsilon - \varepsilon_x}{M_x} \right\rceil + N_x$,则 $\forall n > N$ 有

$$\begin{aligned}
\left| \sum_{i=1}^n x_i q^{n-i} \right| &< \frac{1}{1-q} (M_x q^{n-N_x} + \varepsilon_x) \\
&< \frac{1}{1-q} (M_x q^{N-N_x} + \varepsilon_x) \\
&\leq \frac{1}{1-q} ((1-q)\varepsilon - \varepsilon_x + \varepsilon_x) \\
&= \varepsilon
\end{aligned}$$

从而原命题得证.