

## 不定积分练习

### Problem 1.

求不定积分

$$\int \frac{\sqrt{x^2 - a^2}}{x} dx$$

### Solution(Method I).

采取换元法. 置  $\frac{a}{x} = \sin t$ , 则

$$\frac{dx}{dt} = -a \cdot \frac{1}{\sin^2 t} \cdot \cos t = -\frac{a \cos t}{\sin^2 t}$$

从而

$$\begin{aligned} \int \frac{\sqrt{x^2 - a^2}}{x} dx &= \int \frac{-a \cos^2 t dt}{\sin^2 t} \\ &= -a \int \frac{1 - \sin^2 t}{\sin^2 t} dt \\ &= -a \left( \int \frac{dt}{\sin^2 t} - \int dt \right) \\ &= -a (-\cot t - t) + C \end{aligned}$$

当  $x > a$  时有  $t \in (0, \frac{\pi}{2})$ , 于是

$$\cot t = \sqrt{\frac{\cos^2 t}{\sin^2 t}} = \sqrt{\frac{1 - \left(\frac{a}{x}\right)^2}{\left(\frac{a}{x}\right)^2}} = \frac{\sqrt{x^2 - a^2}}{a}$$

当  $x < -a$  时亦可知  $\cot t = -\frac{\sqrt{x^2 - a^2}}{a}$ . 于是

$$\int \frac{\sqrt{x^2 - a^2}}{x} dx = \begin{cases} \sqrt{x^2 - a^2} + a \arcsin \frac{a}{x} + C, & x > a \\ -\sqrt{x^2 - a^2} + a \arcsin \frac{a}{x} + C, & x < -a \end{cases}$$

### Solution(Method II).

采取换元法. 置  $\frac{a}{x} = \cos t$ , 则  $\frac{dx}{dt} = \frac{a \sin t}{\cos^2 t}$ . 从而

$$\begin{aligned}
\int \frac{\sqrt{x^2 - a^2}}{x} dx &= \int \frac{a \sin^2 t}{\cos^2 t} dt \\
&= a \int \frac{1 - \cos^2 t}{\cos^2 t} dt \\
&= a \left( \int \frac{dt}{\cos^2 t} - \int dt \right) \\
&= a (\tan t - t) + C
\end{aligned}$$

于是可以得到与**Method I**相似的结果.

$$\int \frac{\sqrt{x^2 - a^2}}{x} dx = \begin{cases} \sqrt{x^2 - a^2} - a \arccos \frac{a}{x} + C, x > a \\ -\sqrt{x^2 - a^2} - a \arccos \frac{a}{x} + C, x < -a \end{cases}$$

### Solution(Method III).

直接采取分部积分法.

$$\begin{aligned}
\int \frac{\sqrt{x^2 - a^2}}{x} dx &= \sqrt{x^2 - a^2} - \int x d \left( \frac{\sqrt{x^2 - a^2}}{x} \right) \\
&= \sqrt{x^2 - a^2} - \int x \cdot \frac{x \cdot \frac{x}{\sqrt{x^2 - a^2}} - \sqrt{x^2 - a^2}}{x^2} dx \\
&= \sqrt{x^2 - a^2} - \int \frac{a^2 x dx}{x^2 \sqrt{x^2 - a^2}} \\
&= \sqrt{x^2 - a^2} - \frac{a^2}{2} \int \frac{dx^2}{x^2 \sqrt{x^2 - a^2}}
\end{aligned}$$

置  $u = \sqrt{x^2 - a^2}$ , 则

$$\begin{aligned}
\int \frac{dx^2}{x^2 \sqrt{x^2 - a^2}} &= \int \frac{du^2}{(u^2 + a^2)u} \\
&= 2 \int \frac{du}{u^2 + a^2} \\
&= \frac{2}{a} \arctan \frac{u}{a} + C
\end{aligned}$$

从而

$$\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arctan \frac{\sqrt{x^2 - a^2}}{a} + C$$

### Problem 2.

求不定积分

$$\int \frac{dx}{x^6 \sqrt{1 + x^2}}$$

**Solution.**

置  $x = \tan t, t \in (0, \frac{\pi}{2})$ , 从而  $\frac{dx}{dt} = \frac{1}{\cos^2 t}$ .

从而

$$\begin{aligned}\int \frac{dx}{x^6 \sqrt{1+x^2}} &= \int \frac{dt}{\tan^6 t \cdot \cos^2 t \cdot \frac{1}{\cos t}} \\ &= \int \frac{\cos^5 t}{\sin^6 t} dt\end{aligned}$$

置  $u = \sin t$ , 则  $u^2 = \frac{\sin^2 t}{\sin^2 t + \cos^2 t} = \frac{\tan^2 t}{\tan^2 t + 1} = \frac{x^2}{x^2 + 1}$ .

由于  $x$  与  $u$  同号, 则  $u = \frac{x}{\sqrt{1+x^2}}$ .

于是

$$\begin{aligned}\int \frac{\cos^5 t}{\sin^6 t} dt &= \int \frac{\cos^4 t (\cos t dt)}{\sin^6 t} \\ &= \int \frac{(1-u^2)^2 du}{u^6} \\ &= \int \left( \frac{1}{u^6} - \frac{2}{u^4} + \frac{1}{u^2} \right) du \\ &= -\frac{1}{5u^5} + \frac{2}{3u^3} - \frac{1}{u} + C \\ &= -\frac{(x^2+1)^{\frac{5}{2}}}{5x^5} + \frac{2(x^2+1)^{\frac{3}{2}}}{3x^3} - \frac{(x^2+1)^{\frac{1}{2}}}{x} + C\end{aligned}$$

**Problem 3.**

求不定积分

$$\int \frac{dx}{\sqrt[3]{(x+1)(x-1)^5}}$$

**Solution.**

置  $t = \sqrt[3]{\frac{x+1}{x-1}}$ , 则

$$\frac{dt}{dx} = \frac{1}{3} \left( \frac{x+1}{x-1} \right)^{-\frac{2}{3}} \frac{-2}{(x-1)^2} = -\frac{2}{3} \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}}$$

从而

$$\begin{aligned}\int \frac{dx}{\sqrt[3]{(x+1)(x-1)^5}} &= -\frac{3}{2} \int t dt \\ &= -\frac{3}{4} t^2 + C \\ &= -\frac{3}{4} \left( \frac{x+1}{x-1} \right)^{\frac{2}{3}} + C\end{aligned}$$

**Problem 4.**

求不定积分

$$\int \frac{x \arccos x}{(1-x^2)^2} dx$$

**Solution(Method I).**

置  $t = \arccos x$ , 则  $x = \cos t$ , 从而  $\frac{dx}{dt} = -\sin t$ . 置  $u = \sin t$ .

从而

$$\begin{aligned} \int \frac{x \arccos x}{(1-x^2)^2} dx &= - \int \frac{t \cos t \sin t dt}{\sin^4 t} \\ &= - \int \frac{t d(\sin t)}{\sin^3 t} = - \int \frac{t du}{u^3} \\ &= \frac{1}{2} \int t d\left(\frac{1}{u^2}\right) = \frac{1}{2} \left( \frac{t}{u^2} - \int \frac{dt}{u^2} \right) \\ &= \frac{1}{2} \left( \frac{\arccos x}{1-x^2} - \int \frac{dt}{\sin^2 t} \right) \\ &= \frac{1}{2} \left( \frac{\arccos x}{1-x^2} + \cot t \right) + C \\ &= \frac{1}{2} \left( \frac{\arccos x}{1-x^2} + \frac{x}{\sqrt{1-x^2}} \right) + C \end{aligned}$$

**Solution(Method II).**

直接分部积分有

$$\begin{aligned} \int \frac{x \arccos x}{(1-x^2)^2} dx &= \frac{1}{2} \int \arccos x d\left(\frac{1}{1-x^2}\right) \\ &= \frac{1}{2} \left( \frac{\arccos x}{1-x^2} - \int \frac{d \arccos x}{1-x^2} \right) \\ &= \frac{1}{2} \left( \frac{\arccos x}{1-x^2} + \int \frac{dx}{(1-x^2)^{\frac{3}{2}}} \right) \end{aligned}$$

置  $x = \sin t$ , 则有

$$\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} = \int \frac{d \sin t}{\cos^3 t} = \int \frac{dt}{\cos^2 t} = \tan t + C = \frac{x}{\sqrt{1-x^2}} + C$$

于是

$$\int \frac{x \arccos x}{(1-x^2)^2} dx = \frac{1}{2} \left( \frac{\arccos x}{1-x^2} + \frac{x}{\sqrt{1-x^2}} \right) + C$$

**Problem 5.**

求不定积分

$$\int x \ln(x + \sqrt{1+x^2}) dx$$

**Solution.**

$$\begin{aligned} \int x \ln(x + \sqrt{1+x^2}) dx &= x^2 \ln(x + \sqrt{1+x^2}) - \int x d(x \ln(x + \sqrt{1+x^2})) \\ &= x^2 \ln(x + \sqrt{1+x^2}) - \int x \left( \ln(x + \sqrt{1+x^2}) + \frac{x \left(1 + \frac{x}{\sqrt{1+x^2}}\right)}{x + \sqrt{1+x^2}} \right) dx \\ &= x^2 \ln(x + \sqrt{1+x^2}) - \int x \ln(x + \sqrt{1+x^2}) dx - \int \frac{x^2 dx}{\sqrt{1+x^2}} \end{aligned}$$

而

$$\int \frac{x^2 dx}{\sqrt{1+x^2}} = \int x d(\sqrt{1+x^2}) = x\sqrt{1+x^2} - \int \sqrt{1+x^2} dx$$

又

$$\int \frac{x^2 dx}{\sqrt{1+x^2}} = \int \frac{(x^2 + 1 - 1) dx}{\sqrt{1+x^2}} = \int \sqrt{1+x^2} dx - \int \frac{dx}{\sqrt{1+x^2}}$$

两式相加有

$$2 \int \frac{x^2 dx}{\sqrt{1+x^2}} = x\sqrt{1+x^2} - \int \frac{dx}{\sqrt{1+x^2}}$$

从而

$$\begin{aligned} \int x \ln(x + \sqrt{1+x^2}) dx &= \frac{1}{2} \left( x^2 \ln(x + \sqrt{1+x^2}) - \int \frac{x^2 dx}{\sqrt{1+x^2}} \right) \\ &= \frac{1}{2} x^2 \ln(x + \sqrt{1+x^2}) - \frac{1}{4} x\sqrt{1+x^2} + \frac{1}{4} \int \frac{dx}{\sqrt{1+x^2}} \\ &= \frac{1}{2} x^2 \ln(x + \sqrt{1+x^2}) - \frac{1}{4} x\sqrt{1+x^2} + \frac{1}{4} \ln(x + \sqrt{x^2+1}) + C \end{aligned}$$

**Problem 6.**

求不定积分

$$\int \frac{e^{\arctan x} dx}{(1+x^2)^{\frac{3}{2}}}$$

**Solution(Method I).**

注意到  $\frac{d(e^{\arctan x})}{dx} = \frac{e^{\arctan x}}{1+x^2}$ , 从而

$$\int \frac{e^{\arctan x} dx}{(1+x^2)^{\frac{3}{2}}} = \int \frac{d(e^{\arctan x})}{\sqrt{1+x^2}} = \frac{e^{\arctan x}}{\sqrt{1+x^2}} - \int \frac{x e^{\arctan x} dx}{(1+x^2)^{\frac{3}{2}}}$$

而

$$\int \frac{x e^{\arctan x} dx}{(1+x^2)^{\frac{3}{2}}} = \int \frac{x d(e^{\arctan x})}{\sqrt{1+x^2}} = \frac{x e^{\arctan x}}{\sqrt{1+x^2}} - \int \frac{e^{\arctan x} dx}{(1+x^2)^{\frac{3}{2}}}$$

两式相加可得

$$\int \frac{e^{\arctan x} dx}{(1+x^2)^{\frac{3}{2}}} = \frac{e^{\arctan x}(1+x)}{2\sqrt{1+x^2}} + C$$

**Solution(Method II).**

置  $x = \tan t$ , 则

$$\begin{aligned} \int \frac{e^{\arctan x} dx}{(1+x^2)^{\frac{3}{2}}} &= \int \cos^3 t e^t d(\tan t) \\ &= \int e^t \cos t dt \\ &= \int \cos t d(e^t) \\ &= e^t \cos t + \int e^t \sin t dt \\ &= e^t \cos t + e^t \sin t - \int e^t \cos t dt \end{aligned}$$

从而

$$\begin{aligned} \int \frac{e^{\arctan x} dx}{(1+x^2)^{\frac{3}{2}}} &= \int e^t \cos t dt \\ &= \frac{1}{2} (e^t \cos t + e^t \sin t) \\ &= \frac{e^{\arctan x}(1+x)}{2\sqrt{1+x^2}} + C \end{aligned}$$

**Problem 7.**

求不定积分

$$\int \frac{x \ln x}{(1+x^2)^2} dx$$

**Solution.**

**Problem 8.**

求不定积分

$$\int \frac{dx}{x(x^5+1)}$$

**Solution.**

拆分原式可得

$$\begin{aligned}\int \frac{dx}{x(x^5+1)} &= \int \frac{(x^5+1)-x^5}{x(x^5+1)} dx \\ &= \int \frac{dx}{x} - \int \frac{x^4 dx}{x^5+1} \\ &= \int \frac{dx}{x} - \frac{1}{5} \int \frac{d(x^5)}{x^5+1} \\ &= \ln|x| - \frac{1}{5} \ln|x^5+1| + C\end{aligned}$$