北京大学数学科学学院2024-25高等数学B1期中模拟

(1) Solution.

$$\begin{aligned}\lim_{n\to\infty}\left|\cos\left(\pi\sqrt{n^2+1}\right)\right| &= \lim_{n\to\infty}\left|\cos\left(\pi\sqrt{n^2+1}-n\pi\right)\right| \\ &= \lim_{n\to\infty}\left|\cos\frac{\pi}{\sqrt{n^2+1}+n}\right| \\ &= \cos 0 \\ &= 1\end{aligned}$$

(2) Solution.

$$\begin{aligned}\lim_{x\to+\infty}\left(\frac{x^2-1}{x^2+1}\right)^{x^2} &= \lim_{x\to\infty}\left(1-\frac{2}{x^2+1}\right)^{\frac{x^2+1}{2}\cdot\frac{2x^2}{x^2+1}} \\ &= \mathrm{e}^{-2}\end{aligned}$$

(3) Solution.

当$x\neq 0$时有

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 2x\sin\frac{1}{x}+x^2\left(-\frac{1}{x^2}\cos\frac{1}{x}\right) = 2x\sin\frac{1}{x}-\cos\frac{1}{x}$$

又

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{x} = \lim_{x\to 0}x\sin\frac{1}{x} = 0$$

于是

$$f'(x) = \begin{cases} 2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}, x\neq 0 \\ 0, x=0\end{cases}$$

Solution.

我们有

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \mathrm{e}^t\left(\sin 2t+2\cos 2t\right) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \mathrm{e}^t\left(\cos t-\sin t\right)\end{aligned}$$

于是

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^t\left(\cos t-\sin t\right)}{\mathrm{e}^t\left(\sin 2t+2\cos 2t\right)} = \frac{\cos t-\sin t}{\sin 2t+2\cos 2t}$$

于是

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{-(\sin t+\cos t)(\sin 2t+2\cos 2t)-(\cos t-\sin t)(2\cos 2t-4\sin 2t)}{\left(\sin 2t+2\cos 2t\right)^2}\cdot\frac{1}{\mathrm{e}^t\left(\sin 2t+2\cos 2t\right)} \\ &= \frac{\cos t\left(-5-4\cos 2t+3\sin 2t\right)}{\mathrm{e}^t\left(\sin 2t+2\cos 2t\right)^3}\end{aligned}$$

Solution.

置$u=x-1$,于是

$$\int \frac{x^3+1}{x(x-1)^3}\mathrm{d}x = \int \frac{u^3+3u^2+3u+2}{u^3(u+1)}\mathrm{d}u$$

设

$$\frac{u^3+3u^2+3u+2}{u^3(u+1)} = \frac{A}{u+1} + \frac{Bu^2+Cu+D}{u^3}$$

于是

$$\begin{cases} A+B=1 \\ B+C=3 \\ C+D=3 \\ D=2 \end{cases}$$

解得$A=-1, B=2, C=1, D=2$. 于是

$$\begin{aligned} \int \frac{x^3+1}{x(x-1)^3}\mathrm{d}x &= \int \left(\frac{2}{u^3} + \frac{1}{u^2} + \frac{2}{u} - \frac{1}{u+1}\right)\mathrm{d}u \\ &= -\frac{1}{(x-1)^2} - \frac{1}{x-1} + 2\ln|x-1| - \ln|x| + C \end{aligned}$$

Solution.

(1) Solution.

由题意可得

$$f(t) = \int_0^{x(t)} y\mathrm{d}x = \int_0^{\arcsin t} \sec x\mathrm{d}x$$

于是

$$f'(t) = \sec(\arcsin t)\cdot(\arcsin t)' = \frac{1}{\sqrt{1-t^2}\sqrt{1-t^2}} = \frac{1}{1-t^2}$$

(2) Solution.

由**(1)**的结果可得

$$\int f'(t)\mathrm{d}t = \int \frac{\mathrm{d}t}{1-t^2} = \frac{1}{2}\int\left(\frac{1}{t+1} - \frac{1}{t-1}\right)\mathrm{d}t = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + C$$

又

$$f(0) = \int_0^{\arcsin 0} \sec x\mathrm{d}x = 0$$

于是

$$f(t) = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right|$$

Proof.

若$f(x)$在$[0,1]$上没有零点,那么不妨设$\forall x\in[0,1], f(x)>0$,于是

$$\int_0^1 f(x)\mathrm{d}x>0$$

这与题设不符($f(x)<0$时亦同理),于是$f(x)$在$[0,1]$上至少有一个零点,设$f(x_1)=0$.

若$f(x)$在$[0,1]$上仅有一个零点x_1,那么不妨设$x\in[0,x_1)$时$f(x)<0$,$x\in(x_1,1]$时$f(x)>0$.

由题意

$$\int_0^1 f(x)\mathrm{d}x=\int_0^{x_1} f(x)\mathrm{d}x+\int_{x_1}^1 f(x)\mathrm{d}x$$

于是

$$\begin{aligned}\int_0^1 xf(x)\mathrm{d}x&=\int_0^{x_1} xf(x)\mathrm{d}x+\int_{x_1}^1 xf(x)\mathrm{d}x\\&>x_1\int_0^{x_1} f(x)\mathrm{d}x+x_1\int_{x_1}^1 f(x)\mathrm{d}x\\&=x_1\left(\int_0^{x_1} f(x)\mathrm{d}x+\int_{x_1}^1 f(x)\mathrm{d}x\right)\\&=x_1\int_0^1 f(x)\mathrm{d}x=0\end{aligned}$$

这与题设不符,于是$f(x)$在$[0,1]$上至少有两个零点.

(1) **Proof.**

置$f(x)=\dfrac{1}{x}$,于是

$$\forall k\in\mathbb{N}^*,\forall x\in(k,k+1),\frac{1}{k}>f(x)>\frac{1}{k+1}$$

对上式积分有

$$\frac{1}{k+1}<\int_k^{k+1} f(x)\mathrm{d}x<\frac{1}{k}$$

对上式求和有

$$\sum_{k=1}^{n-1}\frac{1}{k+1}<\int_1^n f(x)\mathrm{d}x<\sum_{k=1}^{n-1}\frac{1}{k}$$

于是

$$x_n+\ln n-1<\left.(\ln x)\right|_1^n<x_n+\ln n-\frac{1}{n}$$

所以

$$\frac{1}{n}<x_n<1$$

从而$\{x_n\}$有界.我们又有

$$x_{n+1}-x_n=\frac{1}{n+1}+\ln\frac{n}{n+1}=\ln\frac{n}{n+1}+1-\frac{n}{n+1}<0$$

于是$\{x_n\}$递减,从而$\lim\limits_{n\to\infty}x_n<0$存在.

(2) Proof.

不难发现

$$I_1 = \int_0^{\frac{\pi}{2}} \sin t \mathrm{d}t = 1$$

而

$$\begin{aligned}
I_n - I_{n-1} &= \int_0^{\frac{\pi}{2}} \frac{\sin^2 nt - \sin^2(n-1)t}{\sin t}\mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin(n+(n-1))t \cdot \sin(n-(n-1))t}{\sin t}\mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \sin(2n-1)t\mathrm{d}t \\
&= \frac{1}{2n-1}\int_0^{n\pi-\frac{\pi}{2}} \sin u\mathrm{d}u \\
&= \frac{1}{2n-1}
\end{aligned}$$

于是

$$I_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$$

设$t_n = x_n + \ln n = \sum\limits_{i=1}^{n} \frac{1}{i}$,于是$I_n = t_{2n} - \frac{1}{2}t_n$. 于是

$$\begin{aligned}
\frac{I_n}{\ln n} &= \frac{x_{2n} + \ln 2n}{\ln n} - \frac{x_n + \ln n}{2\ln n} \\
&= \frac{x_{2n}}{\ln 2n} - \frac{x_n}{2\ln n} + \frac{\ln 2}{\ln n} + \frac{1}{2}
\end{aligned}$$

注意到$\forall n \in \mathbb{N}^*, x_n < 1$.于是

$$\lim_{n\to\infty} \frac{I_n}{\ln n} = \frac{1}{2}$$

Proof.

我们有

$$\int_0^{\pi} f(x)\left|\sin(nx)\right|\mathrm{d}x = \sum_{k=1}^{n} \int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} f(x)\left|\sin(nx)\right|\mathrm{d}x$$

根据积分中值定理,$\exists \xi_k \in \left[\frac{k\pi}{n}, \frac{(k+1)\pi}{n}\right]$, s.t. $\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} f(x)\left|\sin(nx)\right|\mathrm{d}x = f(\xi_k)\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} \left|\sin(nx)\right|\mathrm{d}x$

令$u = nx$,则

$$\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} \left|\sin(nx)\right|\mathrm{d}x = \frac{1}{n}\int_{k\pi}^{(k+1)\pi} \left|\sin u\right|\mathrm{d}u = \frac{2}{n}$$

于是

$$
\begin{aligned}
\lim_{n\to\infty}\int_0^\pi f(x)\left|\sin(nx)\right|\mathrm{d}x &= \lim_{n\to\infty}\sum_{k=1}^{n}\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} f(x)\left|\sin(nx)\right|\mathrm{d}x \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n} f(\xi_k) \\
&= \frac{2}{\pi}\lim_{n\to\infty}\frac{\pi}{n}\sum_{k=1}^{n} f(\xi_k) \\
&= \frac{2}{\pi}\int_0^\pi f(x)\mathrm{d}x
\end{aligned}
$$