二重积分练习

1. 求积分$I=\iint_D\left(x^2+2y\right)\mathrm{d}x\mathrm{d}y$,其中$D$为曲线$y=x^2,y=\sqrt{x}$围成的区域.

Solution.

我们有

$$\begin{aligned}\iint_D\left(x^2+2y\right)\mathrm{d}x\mathrm{d}y&=\int_0^1\left[\int_{x^2}^{\sqrt{x}}\left(x^2+2y\right)\mathrm{d}y\right]\mathrm{d}x\\&=\int_0^1\left(x^2\left(\sqrt{x}-x^2\right)+x-x^4\right)\mathrm{d}x\\&=\int_0^1\left(x+x^{\frac{5}{2}}-2x^4\right)\\&=\left(\frac{1}{2}x^2+\frac{2}{7}x^{\frac{7}{2}}-\frac{2}{5}x^5\right)\Big|_0^1\\&=\frac{27}{70}\end{aligned}$$

2. 求积分$I=\iint_D\sin y^3\mathrm{d}x\mathrm{d}y$,其中$D$是曲线$y=\sqrt{x}$,直线$y=2$和$x=0$围成的区域.

Solution.

我们有

$$\begin{aligned}\iint_D\sin y^3\mathrm{d}x\mathrm{d}y&=\int_0^2\left[\int_0^{y^2}\sin y^3\mathrm{d}x\right]\mathrm{d}y\\&=\int_0^2 y^2\sin y^3\mathrm{d}y\\&=\frac{1}{3}\int_0^8\sin y^3\mathrm{d}y^3\\&=\frac{1-\cos 8}{3}\end{aligned}$$

3. 求积分$I=\iint_D\left(4-x^2-y^2\right)^{-\frac{1}{2}}\mathrm{d}x\mathrm{d}y$,其中$D$是单位圆$x^2+y^2\leqslant 1$在第一象限的部分.

Solution.

做代换$x=r\cos\theta,y=r\sin\theta$,于是$D'=\left\{(r,\theta):0\leqslant r\leqslant 1,0\leqslant\theta\leqslant\dfrac{\pi}{2}\right\}$.于是

$$\begin{aligned}\iint_D\left(4-x^2-y^2\right)^{-\frac{1}{2}}&=\iint_{D'}\left(4-r^2\right)^{-\frac{1}{2}}r\mathrm{d}r\mathrm{d}\theta\\&=\int_0^1\left[\int_0^{\frac{\pi}{2}}\left(4-r^2\right)^{-\frac{1}{2}}r\mathrm{d}\theta\right]\mathrm{d}r\\&=\frac{\pi}{2}\int_0^1\frac{r\mathrm{d}r}{\sqrt{4-r^2}}\\&=\frac{\pi}{4}\int_0^1\frac{\mathrm{d}r^2}{\sqrt{4-r^2}}\\&=\frac{\pi}{4}\left(-2\sqrt{4-r^2}\right)\Big|_0^1\\&=\frac{2-\sqrt{3}}{2}\pi\end{aligned}$$

4. 求积分$I=\iint_D(x+y)\mathrm{d}x\mathrm{d}y$,其中$D$是由$y^2=2x,x+y=4,x+y=12$围成的区域.

Solution.

Method I.

注意到积分区域D可以恰好可以分为两部分

$$D_1=\{(x,y)|2\leqslant x\leqslant 8,4-x\leqslant y\leqslant\sqrt{2x}\}$$

$$D_2=\{(x,y)|8\leqslant x\leqslant 18,-\sqrt{2x}\leqslant y\leqslant 12-x\}$$

于是

$$\begin{aligned}\iint_{D_1}(x+y)\mathrm{d}x\mathrm{d}y&=\int_2^8\mathrm{d}x\int_{4-x}^{\sqrt{2x}}(x+y)\mathrm{d}y\\&=\int_2^8\left(\frac{1}{2}x^2+\sqrt{2}x^{\frac{3}{2}}+x-8\right)\mathrm{d}x\\&=\frac{826}{5}\\\iint_{D_2}(x+y)\mathrm{d}x\mathrm{d}y&=\int_8^{18}\mathrm{d}x\int_{-\sqrt{2x}}^{12-x}(x+y)\mathrm{d}y\\&=\int_8^{18}\left(-\frac{1}{2}x^2+\sqrt{2}x^{\frac{3}{2}}-x+72\right)\\&=\frac{5678}{15}\end{aligned}$$

于是

$$\iint_D(x+y)\mathrm{d}x\mathrm{d}y=\frac{826}{5}+\frac{5678}{15}=\frac{8156}{15}$$

Method II.

做代换$\begin{cases} u = x+y \\ v = y \end{cases}$,则$|J| = 1$.原积分区域为$y^2 \leqslant 2x, 4 \leqslant x+y \leqslant 12$.

代入u, v可得$v^2 + 2v - 2u \leqslant 0, 4 \leqslant u \leqslant 12$.

于是积分区域为$D' = \{(u,v) | 4 \leqslant u \leqslant 12, -\sqrt{2u+1} - 1 \leqslant v \leqslant \sqrt{2u+1} - 1\}$.

于是我们有

$$\begin{aligned} I &= \iint_D (x+y)\mathrm{d}x\mathrm{d}y = \iint_{D'} u\mathrm{d}u\mathrm{d}v \\ &= \int_4^{12} \mathrm{d}u \int_{-\sqrt{2u+1}-1}^{\sqrt{2u+1}-1} u\mathrm{d}v = \int_4^{12} 2u\sqrt{2u+1}\mathrm{d}u \\ &\xlongequal{t=\sqrt{2u+1}} \int_3^5 (t^2-1)t \cdot t\mathrm{d}t = \left(\frac{1}{5}t^5 - \frac{1}{3}t^3\right)\Big|_3^5 \\ &= \frac{8156}{15} \end{aligned}$$