北京大学数学科学学院2023-24高等数学B1期中考试

1.(10分)

求序列极限

$$\lim_{n\to\infty}\left(1+\frac{1}{n\mathrm{e}}\right)^n$$

Solution.

置$t=n\mathrm{e}$,则

$$\lim_{n\to\infty}\left(1+\frac{1}{n\mathrm{e}}\right)^n=\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{\frac{t}{\mathrm{e}}}=\left(\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t\right)^{\frac{1}{\mathrm{e}}}=\mathrm{e}^{\frac{1}{\mathrm{e}}}$$

2.(10分)

设$[x]$为不超过x的最大整数,求函数极限

$$\lim_{x\to+\infty}x\sin\frac{1}{[x]}$$

Solution.

由$[x]\leqslant x<[x]+1$有

$$[x]\sin\frac{1}{[x]}\leqslant x\sin\frac{1}{[x]}<([x]+1)\sin\frac{1}{[x]}$$

置$y=\dfrac{1}{x}$,则有

$$\lim_{x\to+\infty}x\sin\frac{1}{x}=\lim_{y\to0^+}\frac{\sin y}{y}=1$$

从而

$$\lim_{x\to+\infty}[x]\sin\frac{1}{[x]}=1$$

$$\lim_{x\to+\infty}([x]+1)\sin\frac{1}{[x]}=1+\lim_{x\to+\infty}\sin\frac{1}{[x]}=1$$

由夹逼准则可知

$$\lim_{x\to+\infty}x\sin\frac{1}{[x]}=1$$

3.(10分)

设$x>0$,求函数

$$f(x)=\int_0^{\ln x}\sqrt{1+\mathrm{e}^t}\mathrm{d}t$$

的导函数.

Solution.

置$y=\ln x$,则

$$\frac{\mathrm{d}f}{\mathrm{d}x}=\frac{\mathrm{d}f}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}\int_0^y\sqrt{1+\mathrm{e}^t}\mathrm{d}t}{\mathrm{d}y}\cdot\frac{1}{x}=\frac{\sqrt{1+\mathrm{e}^y}}{x}=\frac{\sqrt{1+x}}{x}$$

4.(10分)

求不定积分

$$\int\frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)}\mathrm{d}x$$

Solution.

设

$$\begin{aligned}\frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)}&=\frac{A}{2x-1}+\frac{B}{2x+3}+\frac{C}{2x-5}\\&=\frac{A(4x^2-4x-15)+B(4x^2-12x+5)+C(4x^2+4x-3)}{(2x-1)(2x+3)(2x-5)}\\&=\frac{4(A+B+C)x^2+4(C-A-3B)x+(5B-15A-3C)}{(2x-1)(2x+3)(2x-5)}\end{aligned}$$

从而

$$\begin{cases}A+B+C=1\\C-A-3B=1\\5B-15A-3C=-11\end{cases}$$

解得$A=\dfrac{1}{2},B=-\dfrac{1}{4},C=\dfrac{3}{4}$.

从而

$$\begin{aligned}\int\frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)}\mathrm{d}x&=\int\left(\frac{1}{2}\cdot\frac{1}{2x-1}-\frac{1}{4}\cdot\frac{1}{2x+3}+\frac{3}{4}\cdot\frac{1}{2x-5}\right)\mathrm{d}x\\&=-\frac{1}{2}\int\frac{\mathrm{d}x}{2x-1}+\frac{1}{4}\int\frac{\mathrm{d}x}{2x+3}+\frac{3}{4}\int\frac{\mathrm{d}x}{2x-5}\\&=-\frac{1}{4}\ln|2x-1|+\frac{1}{8}\ln|2x+3|+\frac{3}{8}\ln|2x-5|+C\end{aligned}$$

5.(10分)

求欧氏平面直角坐标系中曲线

$$y=\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)$$

在$x=1$到$x=2$的弧长.

Solution.

$$\begin{aligned}y'&=\frac{1}{2}\left(\sqrt{x^2-1}+\frac{x^2}{\sqrt{x^2-1}}-\frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{x}{\sqrt{x^2-1}}\right)\right)\\&=\frac{1}{2}\left(\sqrt{x^2-1}+\frac{x^2}{\sqrt{x^2-1}}-\frac{1}{\sqrt{x^2-1}}\right)\\&=\sqrt{x^2-1}\end{aligned}$$

故

$$s=\int_1^2\sqrt{1+y'^2}\mathrm{d}x=\int_1^2 x\mathrm{d}x=\left.\frac{x^2}{2}\right|_1^2=\frac{3}{2}$$

6.(10分)

设欧氏空间中V是曲线弧$y=\dfrac{\ln x}{\sqrt{2\pi}}(1\leqslant x\leqslant 2)$与直线$x=1,x=2$围成的曲边三角形绕$x$轴旋转一周形成的旋转体,求$V$的体积.

Solution.

$$\begin{aligned}V&=\pi\int_1^2 y^2\mathrm{d}x=\pi\int_1^2\frac{(\ln x)^2\,\mathrm{d}x}{2\pi}=\frac{1}{2}\int_1^2(\ln x)^2\,\mathrm{d}x\\&=\frac{1}{2}\left(x\left(\ln x\right)^2\Big|_1^2+\int_1^2 x\mathrm{d}\left(\ln x\right)^2\right)\\&=\frac{1}{2}\left(2\left(\ln 2\right)^2+\int_1^2 2\ln x\mathrm{d}x\right)\\&=\left(\ln 2\right)^2+\left(x\ln x-x\right)|_1^2\\&=\left(\ln 2\right)^2+2\ln 2-1\end{aligned}$$

7.(10分)

无穷序列$\{a_n\}$,$\{a_n\}$满足$0 < b_1 < a_1$,且有以下递推关系

$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \sqrt{a_n b_n}$$

试证明$\lim\limits_{n\to\infty} a_n$存在.

Proof.

据均值不等式有$a_{n+1} = \dfrac{a_n + b_n}{2} \geqslant \sqrt{a_n b_n} = b_{n+1}$,当且仅当$a_n = b_n$时取等.

由$a_1 > b_1 > 0$有$\forall n \in \mathbb{N}^*, a_n > b_n > 0$.

从而

$$a_{n+1} - a_n = \frac{a_n + b_n}{2} - a_n = \frac{b_n - a_n}{2} < 0$$

从而$\{a_n\}$递减且有界,故$\lim\limits_{n\to\infty} a_n$存在.

8.(20分)

本题中每个小问都要求给出证明和计算过程.

(1) **(2分)** 试证明:当$x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$时有

$$-1 < \frac{4\sin x}{3 + \sin^2 x} < 1$$

(2) **(8分)** 当$x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$时,求函数

$$f(x) = \arcsin\left(\frac{4\sin x}{3 + \sin^2 x}\right)$$

的导函数.

(3) **(10分)** 试证明

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{4\cos^2 x + \sin^2 x}} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{\frac{9}{4}\cos^2 x + 2\sin^2 x}}$$

Solution.

(1) **Proof.**

记$\phi(x) = \dfrac{4\sin x}{3 + \sin^2 x}$,则$\phi(-x) = \phi(x)$.当$x = 0$时原式显然成立.

当$x \in \left(0, \dfrac{\pi}{2}\right)$时$\sin x \in (0, 1)$,则$\phi(x) = \dfrac{4}{\sin x + \dfrac{3}{\sin x}} < \dfrac{4}{4} = 1$.

同理当$x \in \left(-\dfrac{\pi}{2}, 0\right)$时$\phi(x) > -1$.综上可知原命题成立.

(2) Solution.

$$
\begin{aligned}
f'(x) &= \frac{1}{\sqrt{1-\left(\frac{4\sin x}{3+\sin^2 x}\right)^2}} \cdot \frac{4\cos x\left(3+\sin^2 x\right)-4\sin x\left(2\sin x\cos x\right)}{\left(3+\sin^2 x\right)^2} \\
&= \frac{3+\sin^2 x}{\sqrt{\sin^4 x-10\sin^2 x+9}} \cdot \frac{4\cos x\left(3-\sin^2 x\right)}{\left(3+\sin^2 x\right)^2} \\
&= \frac{4\cos x\left(3-\sin^2 x\right)}{\sqrt{1-\sin^2 x}\cdot\sqrt{9-\sin^2 x}\cdot\left(3+\sin^2 x\right)} \\
&= \frac{4\left(3-\sin^2 x\right)}{\left(3+\sin^2 x\right)\sqrt{9-\sin^2 x}}
\end{aligned}
$$

(3) Proof.

$$
\frac{1}{\sqrt{4\cos^2 x+\sin^2 x}} = \frac{1}{\sqrt{4-3\sin^2 x}}
$$

$$
\frac{1}{\sqrt{\frac{9}{4}\cos^2 x+2\sin^2 x}} = \frac{1}{\sqrt{\frac{9}{4}-\frac{1}{4}\sin^2 x}} = \frac{2}{\sqrt{9-\sin^2 x}}
$$

从而

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{\frac{9}{4}\cos^2 x+2\sin^2 x}} &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\frac{9}{4}-\frac{1}{4}\sin^2 x}} = \frac{2}{\sqrt{9-\sin^2 x}} \\
&= \int_0^{\frac{\pi}{2}} \frac{f'(x)\left(3+\sin^2 x\right)}{2\left(3-\sin^2 x\right)}\mathrm{d}x \\
&= \int_{f(0)}^{f\left(\frac{\pi}{2}\right)} \frac{\left(3+\sin^2 x\right)}{2\left(3-\sin^2 x\right)}\mathrm{d}f(x)
\end{aligned}
$$

而

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{4\cos^2 x+\sin^2 x}} &= \int_{f(0)}^{f\left(\frac{\pi}{2}\right)} \frac{\mathrm{d}f(x)}{\sqrt{4-3\sin^2 f(x)}} \\
&= \int_{f(0)}^{f\left(\frac{\pi}{2}\right)} \frac{\mathrm{d}f(x)}{\sqrt{4-3\sin^2 f(x)}} \\
&= \int_{f(0)}^{f\left(\frac{\pi}{2}\right)} \frac{\mathrm{d}f(x)}{\sqrt{4-3\left(\frac{4\sin x}{3+\sin^2 x}\right)^2}} \\
&= \int_{f(0)}^{f\left(\frac{\pi}{2}\right)} \frac{\left(3+\sin^2 x\right)\mathrm{d}f(x)}{2\sqrt{\sin^4 x+6\sin^2 x+9-12\sin^2 x}} \\
&= \int_{f(0)}^{f\left(\frac{\pi}{2}\right)} \frac{\left(3+\sin^2 x\right)}{2\left(3-\sin^2 x\right)}\mathrm{d}f(x)
\end{aligned}
$$

从而原命题得证.

9.(10分)

设函数$f:[0,1]\to\mathbb{R},g:[0,1]\to\mathbb{R}$在$[0,1]$上连续,满足$f(0)=g(0),\sin(f(1))=\sin(g(1)),\cos(f(1))=\cos(g(1))$,且

$$\forall x\in[0,1],(\cos(f(x))+\cos(g(x)))^2+(\sin(f(x))+\sin(g(x)))^2\neq 0$$

证明:$f(1)=g(1)$.

Proof.

置$h(x)=f(x)-g(x)$,则$h(x)$在$[0,1]$连续.下面证明$h(1)=0$.

由题意

$$\sin h(1)=\sin\left(f(1)-g(1)\right)=\sin f(1)\cos g(1)-\sin g(1)\cos f(1)=0$$

$$\cos h(1)=\cos\left(f(1)-g(1)\right)=\cos f(1)\cos g(1)+\sin f(1)\sin g(1)=1$$

从而$\exists n\in\mathbb{N}^*$,s.t.$h(1)=2n\pi$.

下面采取反证法说明$n=0$.

若$n>0$,则有

$$h(0)=0<\pi\leqslant 2n\pi=h(1)$$

据介值定理,$\exists a\in[0,1]$,s.t.$h(a)=\pi$,从而

$$\sin g(a)=\sin f(a)\cos h(a)-\sin h(a)\cos f(a)=-\sin f(a)$$

$$\cos g(a)=\cos f(a)\cos h(a)+\sin f(a)\sin h(a)=-\cos f(a)$$

则

$$(\cos(f(x))+\cos(g(x)))^2+(\sin(f(x))+\sin(g(x)))^2=0$$

与题设矛盾.

若$n<0$,同理亦可推出矛盾.

从而$n=0$,即$f(1)=g(1)$,原命题得证.