非常好数分 爱来自上交

1. 数列$\{x_n\}$有$\lim\limits_{n\to\infty} x_n = A$. 正项数列$\{y_n\}$有$\lim\limits_{n\to\infty} \dfrac{y_n}{\sum_{i=1}^{n} y_i} = 0$.

试证明: $\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^{n} y_i x_{n+1-i}}{\sum_{i=1}^{n} y_i} = A$.

证明:记序列$\{a_n\}$满足$a_n = x_n - A$.则$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} x_n - A = 0$.

记序列$\{b_n\}$满足$b_n = \dfrac{y_n}{\sum_{i=1}^{n} y_n}$.

则$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$

即

$$\forall \varepsilon_a > 0, \exists N_a \in \mathbb{N}^*\ s.t. \forall n \geqslant N_a, |a_n| \leqslant \varepsilon_a$$

$$\forall \varepsilon_b > 0, \exists N_b \in \mathbb{N}^*\ s.t. \forall n \geqslant N_b, |b_n| \leqslant \varepsilon_b$$

根据收敛序列的有界性有$\exists M_a \in \mathbb{R}\ s.t. \forall n \in \mathbb{N}^*, |a_n| \leqslant M_a$.同理$\exists M_b \in \mathbb{R}\ s.t. \forall n \in \mathbb{N}^*, |b_n| \leqslant M_b$.

现在,对于任意$\varepsilon > 0$,取$0 < \varepsilon_a < \dfrac{\varepsilon}{M_b}, 0 < \varepsilon_b < \dfrac{\varepsilon - \varepsilon_a M_b}{M_a}$和对应的$N_a, N_b$

令$N = N_a + N_b + 2$,则$\forall n \geqslant N$有

$$\begin{aligned}
\left|\frac{\sum_{i=1}^{n} y_i x_{n+1-i}}{\sum_{i=1}^{n} y_i} - A\right| &= \left|\frac{\sum_{i=1}^{n} y_i (a_{n+1-i} + A)}{\sum_{i=1}^{n} y_i} - A\right| \\
&= \left|\frac{\sum_{i=1}^{n} y_i a_{n+1-i}}{\sum_{i=1}^{n} y_i}\right| \leqslant \frac{\sum_{i=1}^{n} y_i \left|a_{n+1-i}\right|}{\sum_{i=1}^{n} y_i} \\
&\leqslant \sum_{i=1}^{n} \frac{y_i \left|a_{n+1-i}\right|}{\sum_{j=1}^{i} y_j} = \sum_{i=1}^{n} b_i \left|a_{n+1-i}\right| \\
&= \sum_{i=1}^{n-N_a+1} b_i \left|a_{n+1-i}\right| + \sum_{i=n-N_a+2}^{n} b_i \left|a_{n+1-i}\right| \\
&\leqslant \varepsilon_a M_b + \varepsilon_b M_a \\
&\leqslant \varepsilon_a M_b + M_a \cdot \frac{\varepsilon - \varepsilon_a M_b}{M_a} \\
&= \varepsilon
\end{aligned}$$

从而$\left|\dfrac{\sum_{i=1}^{n} y_i x_{n+1-i}}{\sum_{i=1}^{n} y_i} - A\right| \leqslant \varepsilon$

从而$\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^{n} y_i x_{n+1-i}}{\sum_{i=1}^{n} y_i} = A$,证毕.

2. 数列$\{x_n\}$满足对于$\{x_n\}$的任意子列$\{x_{n_k}\}$均有$\lim\limits_{k\to\infty}\dfrac{\sum_{i=1}^{k}x_{n_i}}{k}=1$.

试证明: $\lim\limits_{n\to\infty}x_n=1$.

证明:采取反证法.假定$\{x_n\}$不收敛或不收敛于1,则有

$$\exists\varepsilon>0,\forall N\in\mathbb{N}^*,s.t.\exists n\geqslant N,|x_n-1|>\varepsilon$$

也即$\{x_n\}$中有无穷多项x_i满足$|x_n-1|>\varepsilon$.

我们将这些项分为$\{x_n\}$的两个子序列$\left\{x_{n_+}\right\},\left\{x_{n_-}\right\}$, 满足

$$\forall x_i\in\left\{x_{n_+}\right\},x_i>1+\varepsilon$$
$$\forall x_j\in\left\{x_{n_-}\right\},x_j<1-\varepsilon$$

则$\left\{x_{n_+}\right\},\left\{x_{n_-}\right\}$中至少有一个为无穷序列.

当$\left\{x_{n_+}\right\}$或$\left\{x_{n_-}\right\}$为无穷序列时有

$$\lim_{k\to\infty}\frac{\sum_{i=1}^{k}x_{n_{+,i}}}{k}\geqslant\lim_{k\to\infty}\frac{\sum_{i=1}^{k}1+\varepsilon}{k}=1+\varepsilon>1$$
$$\lim_{k\to\infty}\frac{\sum_{i=1}^{k}x_{n_{-,i}}}{k}\leqslant\lim_{k\to\infty}\frac{\sum_{i=1}^{k}1-\varepsilon}{k}=1-\varepsilon<1$$

而根据题意,若$\left\{x_{n_+}\right\}$或$\left\{x_{n_-}\right\}$为无穷序列,则有

$$\lim_{k\to\infty}\frac{\sum_{i=1}^{k}x_{n_{+,i}}}{k}=\lim_{k\to\infty}\frac{\sum_{i=1}^{k}x_{n_{-,i}}}{k}=1$$

矛盾.故$\lim\limits_{n\to\infty}x_n=1$,证毕.

3. 正项数列$\{x_n\}$满足$\lim\limits_{n\to\infty}\dfrac{\sum_{i=1}^{n}x_i}{n}=a, a\in\mathbb{R}$.

试证明: $\lim\limits_{n\to\infty}\dfrac{\sum_{i=1}^{n}x_i^2}{n^2}=0$.

证明:我们有

$$\begin{aligned}\lim_{n\to\infty}\frac{x_n}{n}&=\lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n}x_i}{n}-\frac{n-1}{n}\cdot\frac{\sum_{i=1}^{n-1}x_i}{n-1}\right)\\&=a-1\cdot a=0\end{aligned}$$

根据收敛序列的有界性,$\exists M\in\mathbb{R}\ s.t.\dfrac{\sum_{i=1}^{n}x_i}{n}<M$

则

$$0<\frac{\sum_{i=1}^{n}x_i^2}{n^2}=M\cdot\frac{\sum_{i=1}^{n}\frac{x_i^2}{n}}{Mn}<M\cdot\frac{\sum_{i=1}^{n}\frac{x_i^2}{i}}{\sum_{i=1}^{n}x_i}$$

依Stolz定理

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}\frac{x_i^2}{i}}{\sum_{i=1}^{n}x_i}=\lim_{n\to\infty}\frac{\frac{x_n^2}{n}}{x_n}=\lim_{n\to\infty}\frac{x_n}{n}=0$$

依夹逼准则$\lim\limits_{n\to\infty}\dfrac{\sum_{i=1}^{n}x_i^2}{n^2}=0$,证毕.