曲线积分练习

A.第一型和第二型曲线积分

A.1 求积分 $\displaystyle\int_L x^2 \mathrm{d}s$,其中 L 为圆周 $\begin{cases} x^2 + y^2 + z^2 = a^2 \\ x + y + z = 0 \end{cases}$.

Solution.

将 $z = -x - y$ 代入球的方程有 $x^2 + xy + y^2 = \dfrac{a^2}{2}$,即 $\left(\dfrac{1}{2}x + y\right)^2 + \left(\dfrac{\sqrt{3}}{2}x\right)^2 = \dfrac{a^2}{2}$.做代换

$$\begin{cases} \frac{1}{2}x + y = \frac{\sqrt{2}}{2}a\cos\theta \\ \frac{\sqrt{3}}{2}x = \frac{\sqrt{2}}{2}a\sin\theta \end{cases}, 0 \leqslant \theta \leqslant 2\pi$$

可得

$$\begin{cases} x = \frac{\sqrt{6}}{3}a\sin\theta \\ y = \frac{\sqrt{2}}{2}a\cos\theta - \frac{\sqrt{6}}{6}a\sin\theta \\ z = -\frac{\sqrt{2}}{2}a\cos\theta - \frac{\sqrt{6}}{6}a\sin\theta \end{cases}$$

于是

$$\mathrm{d}s = \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2 + [z'(\theta)]^2}\mathrm{d}\theta = a\mathrm{d}\theta$$

于是

$$\int_L x^2 \mathrm{d}s = \int_0^{2\pi} \frac{2}{3}a^3 \sin^2\theta \mathrm{d}\theta = \frac{2}{3}a^3\pi$$

B.格林公式与第二型曲线积分与路径无关的条件 C.格林公式的推广与散度定理

C.1 设函数 $u(x,y)$ 在有界闭区域 D 上有连续的二阶偏导数,D 的边界 L 逐段光滑.试证明

$$\oint_{L^+} \frac{\partial u}{\partial \mathbf{n}}\mathrm{d}s = \iint_D \Delta u \mathrm{d}\sigma$$

其中 $\dfrac{\partial u}{\partial \mathbf{n}}$ 表示 $u(x,y)$ 沿 L 的外法线方向的方向导数,Δ 为 Laplace 算子.

Proof.

我们设 L^+ 的单位切向量为 $\mathbf{t}$,其方向余弦为 $\cos\alpha, \cos\beta$.不难看出

$$\mathbf{n} \cdot \mathbf{t} = 0, \mathbf{n} \times \mathbf{t} = \mathbf{k}$$

其中**k**为z轴正方向的单位向量.设$\mathbf{n} = (a, b)$,则有

$$\mathbf{n} \times \mathbf{t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & 0 \\ \cos\alpha & \cos\beta & 0 \end{vmatrix} = (a\cos\beta - b\cos\alpha)\mathbf{k}$$

从而

$$\begin{cases} a\cos\alpha + b\cos\beta = 0 \\ a\cos\beta - b\cos\alpha = 1 \end{cases}$$

解得$\mathbf{n} = (\cos\beta, -\cos\alpha)$.由方向导数的定义可知

$$\begin{aligned} \oint_{L^+} \frac{\partial u}{\partial \mathbf{n}}\mathrm{d}s &= \oint_{L^+} \left(\frac{\partial u}{\partial x}\cos\beta - \frac{\partial u}{\partial y}\cos\alpha \right)\mathrm{d}s \\ &= \oint_{L^+} \frac{\partial u}{\partial x}\mathrm{d}y - \frac{\partial u}{\partial y}\mathrm{d}x \\ &= \iint_D \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \right]\mathrm{d}x\mathrm{d}y \\ &= \iint_D \Delta u\mathrm{d}\sigma \end{aligned}$$

注:主要在于

C.2 设区域D的边界L为闭曲线L,某稳定流体(即任意一点的流速与时间无关,仅与该点的位置有关)在$\overline{D} = D + L$上的每一点(x, y)处的流速为

$$\mathbf{v}(x, y) = (P(x, y), Q(x, y))$$

其中$P(x, y), Q(x, y)$在$\overline{D}$上有连续的一阶偏导数.该流体通过闭曲线L的流量Φ定义为

$$\Phi = \oint_{L^+} \mathbf{v} \cdot \mathbf{n}\mathrm{d}s$$

其中$\mathbf{n}$为L的外法线方向的单位向量.试证明

$$\Phi = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)\mathrm{d}\sigma$$

Proof.

根据**C.1**可知$\mathbf{n} = (\cos\beta, -\cos\alpha)$.于是

$$\begin{aligned}
\Phi &= \oint_{L^+} \mathbf{v}\cdot\mathbf{n}\mathrm{d}s \\
&= \oint_{L^+} (P\cos\beta - Q\cos\alpha)\mathrm{d}s \\
&= \oint_{L^+} (P\mathrm{d}y - Q\mathrm{d}x) \\
&= \iint_D \left(\frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y}\right)\right)\mathrm{d}\sigma \\
&= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mathrm{d}\sigma
\end{aligned}$$

于是命题得证.上述命题的另一形式为

$$\oint_{L^+}(P,Q)\cdot\mathbf{n}\mathrm{d}s = \oint_{L^+} -Q\mathrm{d}x + P\mathrm{d}y = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mathrm{d}\sigma$$

其中$\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$称为向量场$\mathbf{v}$的**散度**,因此格林公式在物理上也被称为**散度定理**.
散度定理指出,稳定流体通过某一闭曲线的流量,等于其散度在该闭曲线所包的区域上的二重积分之值.

C.3 设函数$u(x,y)$和$v(x,y)$在有界闭区域D上有连续的二阶偏导数,D的边界L逐段光滑.

(1) 试证明

$$\iint_D v\Delta u\mathrm{d}\sigma = \oint_{L^+} v\frac{\partial u}{\partial\mathbf{n}}\mathrm{d}s - \iint_D \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}\sigma$$

其中$\mathbf{n}$为L的外法线方向的单位向量.

(2) 试证明

$$\iint_D (u\Delta v - v\Delta u)\mathrm{d}\sigma = \oint_{L^+} \left(u\frac{\partial v}{\partial\mathbf{n}} - v\frac{\partial u}{\partial\mathbf{n}}\right)\mathrm{d}s$$

Proof.

(1) 我们有

$$\frac{\partial u}{\partial\mathbf{n}} = \left(\frac{\partial u}{\partial x}\cos(\mathbf{n},y), -\frac{\partial u}{\partial y}\cos(\mathbf{n},x)\right)$$

于是

$$\begin{aligned}
\oint_{L^+} v\frac{\partial u}{\partial\mathbf{n}}\mathrm{d}s &= \oint_{L^+} v\frac{\partial u}{\partial x}\mathrm{d}y - v\frac{\partial u}{\partial y}\mathrm{d}x \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)\right]\mathrm{d}\sigma \\
&= \iint_D \left[v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right)\right]\mathrm{d}\sigma
\end{aligned}$$

移项即可得到欲证等式.

(2) 与**(1)**同理有

$$\iint_D u\Delta v\mathrm{d}\sigma = \oint_{L^+} u\frac{\partial v}{\partial \mathbf{n}}\mathrm{d}s - \iint_D \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}\sigma$$

相减即有

$$\iint_D (u\Delta v - v\Delta u)\,\mathrm{d}\sigma = \oint_{L^+} \left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\right)\mathrm{d}s$$

C.4 设D是有界平面区域,其边界L分段光滑,定点$P_0(x_0,y_0) \notin L$.设L上一点$P(x,y)$,向量$\mathbf{n}_P$为P处L的外侧法向量.定义向量$\mathbf{r}_P = \overrightarrow{P_0P}$,定义函数$f(x,y)$为

$$f(P) = \frac{\cos(\mathbf{r}_P, \mathbf{n}_P)}{|\mathbf{r}_P|}$$

计算曲线积分$\oint_L f(x,y)\mathrm{d}s$.

Solution.

设P处沿L正方向的单位切向量为$(\cos\alpha, \cos\beta)$,那么$\mathbf{n}_P = (\cos\beta, -\cos\alpha)$.于是我们有

$$f(x,y) = \frac{\cos(\mathbf{r}_P, \mathbf{n}_P)}{\mathbf{r}_P} = \frac{\mathbf{r}_P \cdot \mathbf{n}_P}{|\mathbf{r}_P|^2 |\mathbf{n}_P|}$$
$$= \frac{(x-x_0)\cos\beta - (y-y_0)\cos\alpha}{(x-x_0)^2 + (y-y_0)^2}$$

于是

$$\oint_L f(x,y)\mathrm{d}s = \oint_L \frac{(x-x_0)\cos\beta - (y-y_0)\cos\alpha}{(x-x_0)^2 + (y-y_0)^2}\mathrm{d}s$$
$$= \oint_{L^+} \frac{(x-x_0)\mathrm{d}y - (y-y_0)\mathrm{d}x}{(x-x_0)^2 + (y-y_0)^2}$$

令$A(x,y) = \dfrac{-(y-y_0)}{(x-x_0)^2 + (y-y_0)^2}$, $B(x,y) = \dfrac{x-x_0}{(x-x_0)^2 + (y-y_0)^2}$.于是我们有

$$\frac{\partial A}{\partial y} = -\frac{1}{(x-x_0)^2 + (y-y_0)^2} + \frac{2(y-y_0)^2}{[(x-x_0)^2 + (y-y_0)^2]^2} = \frac{(y-y_0)^2 - (x-x_0)^2}{[(x-x_0)^2 + (y-y_0)^2]^2}$$

同理可得

$$\frac{\partial B}{\partial x} = \frac{-(x-x_0)^2 + (y-y_0)^2}{[(x-x_0)^2 + (y-y_0)^2]^2}$$

若$P_0 \notin D$,那么A, B在D上有连续的一阶偏导数,从而根据格林公式有

$$\oint_L f(x,y)\mathrm{d}s = \oint_{L^+} A\mathrm{d}x + B\mathrm{d}y = \iint_D \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)\mathrm{d}\sigma = 0$$

若$P_0 \in D$,那么考虑P_0的邻域$E = \{(x,y)|(x-x_0)^2 + (y-y_0)^2 \leqslant \varepsilon^2\}$,其中$\varepsilon > 0$.令$\varepsilon$充分小至$E \subset D$.令$E$的边界为$L_E$,从而$A, B$在$D\backslash E$上有连续的一阶偏导数.我们有

$$0 = \iint_{D\backslash E} \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)\mathrm{d}\sigma = \oint_{L^+} (A\mathrm{d}x + B\mathrm{d}y) + \oint_{L_E^-} (A\mathrm{d}x + B\mathrm{d}y)$$

做代换$x = \varepsilon\cos\theta + x_0, y = \varepsilon\sin\theta + y_0$,环路$L_E{}^+$即$\theta$从0变化至$2\pi$的路径.于是

$$\oint_{L^+} A\mathrm{d}x + B\mathrm{d}y = \oint_{L_E{}^+} A\mathrm{d}x + B\mathrm{d}y$$
$$= \int_0^{2\pi} \left[\frac{-\varepsilon\sin\theta}{\varepsilon^2} \cdot (-\varepsilon\sin\theta) + \frac{\varepsilon\cos\theta}{\varepsilon^2} \cdot \varepsilon\cos\theta \right] \mathrm{d}\theta$$
$$= \int_0^{2\pi} \mathrm{d}\theta = 2\pi$$

于是所求积分为

$$\oint_L f(x,y)\mathrm{d}s = \begin{cases} 0, P_0 \notin D \\ 2\pi, P \in D \end{cases}$$