曲面积分

A.

A.1 给定三元函数$f(x,y,z)$,Laplace算子作用于f得到新的函数

$$\Delta f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

设Ω是三维有界闭区域,Ω的边界S是分段光滑的曲面.

设三元函数$u(x,y,z)$和$v(x,y,z)$在$\Omega\cup S$上有连续的二阶偏导数.

设$\mathbf{n}$表示S的单位外法向量.

(1) 试证明

$$\iint_S \frac{\partial u}{\partial \mathbf{n}}\mathrm{d}S=\iiint_\Omega \Delta u\mathrm{d}V$$

(2) 试证明

$$\iiint_\Omega (u\Delta v-v\Delta u)\,\mathrm{d}V=\iint_S\left(u\frac{\partial v}{\partial \mathbf{n}}-v\frac{\partial u}{\partial \mathbf{n}}\right)\mathrm{d}S$$

(3) 如果函数$f(x,y,z)$在Ω上满足$\Delta f=0$,则称为**调和函数**.

设u是调和函数,给定点$P_0\left(x_0,y_0,z_0\right)\in\Omega$,令$\mathbf{r}$是以$P_0$为起点,$S$上的点$P(x,y,z)$为终点的向量,试证明

$$u\left(x_0,y_0,z_0\right)=\frac{1}{4\pi}\iint_S\left(u\frac{\cos\left(\mathbf{r},\mathbf{n}\right)}{|\mathbf{r}|^2}+\frac{1}{|\mathbf{r}|}\frac{\partial u}{\partial \mathbf{n}}\right)$$

(4) 设u是Ω上的调和函数,令Ω是以$P_0\left(x_0,y_0,z_0\right)$为球心,以$R$为半径的球,试证明

$$u\left(x_0,y_0,z_0\right)=\frac{1}{4\pi R^2}\iint_S u\mathrm{d}S$$

即球心的函数值等于球面函数值的平均值.

Proof.

(1) 我们有

$$\begin{aligned}
\iint_S \frac{\partial u}{\partial \mathbf{n}}\mathrm{d}S&=\iint_S\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)\cdot\mathbf{n}\mathrm{d}S\\
&=\oiint_{S+}\frac{\partial u}{\partial x}\mathrm{d}y\mathrm{d}z+\frac{\partial u}{\partial x}\mathrm{d}z\mathrm{d}x+\frac{\partial u}{\partial x}\mathrm{d}x\mathrm{d}y\\
&=\iiint_\Omega\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right]\mathrm{d}V\\
&=\iiint_\Omega \Delta u\mathrm{d}V
\end{aligned}$$

(2) 我们有

$$\begin{aligned}\iint_S v\frac{\partial u}{\partial \mathbf{n}}\mathrm{d}S &= \iint_S \left(v\frac{\partial u}{\partial x}, v\frac{\partial u}{\partial y}, v\frac{\partial u}{\partial z}\right)\cdot \mathbf{n}\mathrm{d}S \\ &= \iiint_\Omega \left(v_x u_x + v u_{xx} + v_y u_y + v u_{yy} + v_z u_z + v u_{zz}\right)\mathrm{d}V \\ &= \iiint_\Omega \left(v_x u_x + v_y u_y + v_z u_z + v\Delta u\right)\mathrm{d}V\end{aligned}$$

同理

$$\iint_S u\frac{\partial v}{\partial \mathbf{n}}\mathrm{d}S = \iiint_\Omega \left(u_x v_x + u_y v_y + u_z v_z + u\Delta v\right)\mathrm{d}V$$

两式相减即可得

$$\iiint_\Omega (u\Delta v - v\Delta u)\,\mathrm{d}V = \iint_S \left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\right)\mathrm{d}S$$

(3) 令$v(x,y,z) = \dfrac{1}{|\mathbf{r}|} = \dfrac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$,则有

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{2(x-x_0)}{2|\mathbf{r}|^3}\right) = -\frac{|\mathbf{r}|^3 - 3|\mathbf{r}|^2\cdot\dfrac{(x-x_0)}{|\mathbf{r}|}\cdot(x-x_0)}{|\mathbf{r}|^6} = \frac{3(x-x_0)^2 - |\mathbf{r}|^2}{|\mathbf{r}|^5}$$

同理

$$\frac{\partial^2 v}{\partial y^2} = \frac{3(y-y_0)^2 - |\mathbf{r}|^2}{|\mathbf{r}|^5} \qquad \frac{\partial^2 v}{\partial z^2} = \frac{3(z-z_0)^2 - |\mathbf{r}|^2}{|\mathbf{r}|^5}$$

于是

$$\Delta v = \frac{3(x-x_0)^2 + 3(y-y_0)^2 + 3(z-z_0)^2 - 3|\mathbf{r}|^2}{|\mathbf{r}|^5} = 0$$

现在处理等式右端,我们有

$$\frac{\partial v}{\partial \mathbf{n}} = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\right)\cdot\mathbf{n} = -\frac{1}{|\mathbf{r}|^3}(x-x_0, y-y_0, z-z_0)\cdot\mathbf{n} = -\frac{\mathbf{r}\cdot\mathbf{n}}{|\mathbf{r}|^3} = -\frac{\cos(\mathbf{r},\mathbf{n})}{|\mathbf{r}|^2}$$

注意到v在(x_0,y_0,z_0)处没有定义.因此取半径为ε,球心为P_0的球Ω',记其外表面为S',那么$\Omega\backslash\Omega'$的表面的单位外法向量$\mathbf{n}$指向Ω'的内侧,即$\mathbf{n} = -(x-x_0, y-y_0, z-z_0)$.

命此为S'的定向,于是有

$$\begin{aligned}\iint_{S'}\left(v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}}\right)\mathrm{d}S &= \iint_{S'}\left(u\frac{\cos(\mathbf{r},\mathbf{n})}{|\mathbf{r}|^2} + \frac{1}{|\mathbf{r}|}\frac{\partial u}{\partial \mathbf{n}}\right)\mathrm{d}S \\ &= \iint_{S'}\left(-\frac{u}{\varepsilon^2} + \frac{1}{\varepsilon}\frac{\partial u}{\partial \mathbf{n}}\right)\mathrm{d}S \\ &= -\frac{1}{\varepsilon^2}\iint_{S'} u\mathrm{d}S\end{aligned}$$

其中在Ω'上应用**(1)**的结论可得

$$-\iint_{S'}\frac{\partial u}{\partial \mathbf{n}} = \iiint_{\Omega'}\Delta u\mathrm{d}V = 0$$

又有

$$\left|\iint_{S'} u\mathrm{d}S - 4\pi\varepsilon^2 u(x_0,y_0,z_0)\right| = \left|\iint_{S'}(u - u(x_0,y_0,z_0))\,\mathrm{d}S\right| \leqslant \iint_{S'}|u - u(x_0,y_0,z_0)|\,\mathrm{d}S$$

而

$$\lim_{\varepsilon\to 0} u(x,y,z)=u\left(x_0,y_0,z_0\right)$$

于是夹逼可得

$$u\left(x_0,y_0,z_0\right)=\frac{1}{4\pi\varepsilon^2}\iint_{S'} u\mathrm{d}S \quad (\varepsilon\to 0)$$

现在,在$\Omega\backslash\Omega'$上运用**(2)**的结论可得

$$0=\iint_{S+S'}\left(u\frac{\cos\left(\mathbf{r},\mathbf{n}\right)}{|\mathbf{r}|^2}+\frac{1}{|\mathbf{r}|}\frac{\partial u}{\partial\mathbf{n}}\right)=\iint_{S}\left(u\frac{\cos\left(\mathbf{r},\mathbf{n}\right)}{|\mathbf{r}|^2}+\frac{1}{|\mathbf{r}|}\frac{\partial u}{\partial\mathbf{n}}\right)+\iint_{S'}\left(u\frac{\cos\left(\mathbf{r},\mathbf{n}\right)}{|\mathbf{r}|^2}+\frac{1}{|\mathbf{r}|}\frac{\partial u}{\partial\mathbf{n}}\right)$$

命$\varepsilon\to 0$,于是可得

$$\begin{aligned}\iint_{S}\left(u\frac{\cos\left(\mathbf{r},\mathbf{n}\right)}{|\mathbf{r}|^2}+\frac{1}{|\mathbf{r}|}\frac{\partial u}{\partial\mathbf{n}}\right)&=-\iint_{S'}\left(u\frac{\cos\left(\mathbf{r},\mathbf{n}\right)}{|\mathbf{r}|^2}+\frac{1}{|\mathbf{r}|}\frac{\partial u}{\partial\mathbf{n}}\right)\\&=\frac{1}{\varepsilon^2}\iint_{S'}u\mathrm{d}S\\&=4\pi u\left(x_0,y_0,z_0\right)\end{aligned}$$

于是

$$u\left(x_0,y_0,z_0\right)=\frac{1}{4\pi}\iint_{S}\left(u\frac{\cos\left(\mathbf{r},\mathbf{n}\right)}{|\mathbf{r}|^2}+\frac{1}{|\mathbf{r}|}\frac{\partial u}{\partial\mathbf{n}}\right)$$

命题得证.

(4) 在**(3)**中令Ω为题中所述的球即可.