# Lecture 13 Integrals with parameters(含参变量积分)

**L.13.1** 计算含参变量$\alpha$的积分

$$\int_0^1 \frac{\ln(1+\alpha x)}{1+x^2}\mathrm{d}x$$

其中参数$\alpha \geqslant 0$.由此计算积分

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x$$

**Solution.**

首先注意到对任意$x \in [0,1]$和$\alpha \geqslant 0$都有函数

$$f(x,\alpha) = \frac{\ln(1+\alpha x)}{1+x^2}$$

二元连续.令$I(\alpha) = \int_0^1 f(x,\alpha)\mathrm{d}x$,就有

$$\begin{aligned}
I'(\alpha) &= \int_0^1 \frac{\partial f}{\partial \alpha}(x,\alpha)\mathrm{d}x = \int_0^1 \frac{x}{(1+x^2)(1+\alpha x)}\mathrm{d}x \\
&= \frac{1}{1+\alpha^2}\int_0^1 \left(\frac{x+\alpha}{1+x^2} - \frac{\alpha}{1+\alpha x}\right)\mathrm{d}x \\
&= \frac{1}{1+\alpha^2}\left(\frac{\ln 2}{2} + \frac{\pi\alpha}{4} - \ln(1+\alpha)\right) \\
&= \frac{2\ln 2 - 4\ln(1+\alpha) + \pi\alpha}{4(1+\alpha^2)}
\end{aligned}$$

于是

$$\begin{aligned}
I(\alpha) = I(0) + \int_0^\alpha I'(t)\mathrm{d}t &= \int_0^\alpha \frac{2\ln 2 - 4\ln(1+t) + \pi t}{4(1+t^2)}\mathrm{d}t \\
&= \frac{\ln 2}{2}\arctan\alpha + \frac{\pi}{8}\ln\left(1+\alpha^2\right) - \int_0^\alpha \frac{\ln(1+t)}{1+t^2}\mathrm{d}t
\end{aligned}$$

特别地,当$\alpha = 1$时有

$$I(1) = \frac{\pi\ln 2}{8} + \frac{\pi\ln 2}{8} + I(1)$$

于是

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x = I(1) = \frac{\pi\ln 2}{8}$$

**L.13.2** 计算无穷积分

$$\int_0^{+\infty} \frac{\sin^3 x}{x}\mathrm{d}x$$

**Solution.**

由于

$$\sin 3x = 3\sin x - 4\sin^3 x$$

于是

$$\int_0^{+\infty} \frac{\sin^3 x}{x}\mathrm{d}x = \int_0^{+\infty} \frac{3\sin x - \sin 3x}{4x}\mathrm{d}x = \frac{3}{4}\int_0^{+\infty} \frac{\sin x}{x}\mathrm{d}x - \frac{1}{4}\int_0^{+\infty} \frac{\sin 3x}{3x}\mathrm{d}(3x) = \frac{\pi}{4}$$

**L.13.3** 设参数$A > 0$,考虑含参变量$y$的无穷积分

$$I(y) = \int_0^{+\infty} \frac{\sin(xy)}{x}\mathrm{d}x$$

试证明:

**(1)** $I(y)$在$y \in [A, +\infty)$一致收敛.

**(2)** $I(y)$在$y \in [0, +\infty)$不一致收敛.

**Proof.**

**(1)** 设$f(x,y) = \sin(xy), g(x,y) = \dfrac{1}{x}$.注意到$g(x,y)$对$x$单调递减且趋于0,又对任意$0 < b < c$有

$$\left|\int_b^c f(x,y)\mathrm{d}x\right| = \frac{1}{y}\left|\cos(cy) - \cos(by)\right| \leqslant \frac{2}{A}$$

即$\displaystyle\int_b^c f(x,y)\mathrm{d}x$对$y \in [A, +\infty)$一致有界.根据Dirichlet判别法可知$I(y)$在$y \in [A, +\infty)$一致收敛.

**(2)** 采取反证法.假定$I(y)$在$y \in [0, +\infty)$一致收敛,由Cauchy收敛准则可知对任意$\varepsilon > 0$和$y \in [0, +\infty)$,总存在$N \in \mathbb{R}$使得对任意$A, A' > N$都有

$$\left|\int_A^{A'} \frac{\sin(xy)}{x}\mathrm{d}x\right| < \varepsilon$$

令$u = xy$,代换后就有

$$\left|\int_A^{A'} \frac{\sin(xy)}{x}\mathrm{d}x\right| = \left|\int_{Ay}^{A'y} \frac{\sin(u)}{u}\mathrm{d}u\right|$$

令$y = \dfrac{1}{A}, A' = 2A$,则有

$$\left|\int_A^{A'} \frac{\sin(xy)}{x}\mathrm{d}x\right| = \left|\int_1^2 \frac{\sin u}{u}\mathrm{d}u\right|$$

右式是一个确定的正数,不可能比任意的$\varepsilon$都小.因此$I(y)$在$y \in [0, +\infty)$不一致收敛.

**L.13.4** 试证明含参变量$y$的无穷积分

$$I(y) = \int_0^{+\infty} \frac{x}{2 + x^y}\mathrm{d}x$$

在$y \in (2,+\infty)$连续.

**Proof.**

只需证明对任意$y_0 \in (2,0)$,$I(y)$在$y \in [y_0,+\infty)$一致收敛即可.我们有

$$\int_0^1 \frac{x}{2+x^y}\mathrm{d}x \leqslant \int_0^1 \frac{x}{2}\mathrm{d}x = \frac{1}{4}$$

收敛.又有

$$\int_1^{+\infty} \frac{x}{2+x^y}\mathrm{d}x < \int_1^{+\infty} \frac{x}{2+x^2}\mathrm{d}x < \int_1^{+\infty} \frac{1}{x^{y_0-1}}\mathrm{d}x$$

收敛,于是对任意$y_0 \in (2,+\infty)$都有$I(y)$在$[y_0,+\infty]$连续,即$I(y)$在$(2,+\infty)$连续.