北京大学数学科学学院2022-23高等数学B2期中考试

1.(10分) 求常微分方程

$$\left(xy-x^3y^3\right)\mathrm{d}x+\left(1+x^2\right)\mathrm{d}y=0$$

的满足$y(0)=1$的解.

Solution.

对原方程变形可得

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^3y^3-xy}{1+x^2}$$

令$u=\dfrac{1}{y^2}$,则有

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{x^3y^3-xy}{1+x^2}\cdot\left(-\frac{2}{y^3}\right)=\frac{2ux-2x^3}{1+x^2}$$

即

$$\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{2ux}{1+x^2}=-\frac{2x^3}{1+x^2}$$

对应齐次方程为

$$\frac{\mathrm{d}u}{u}=\frac{2x\mathrm{d}x}{1+x^2}$$

通解为

$$u=C(1+x^2)$$

现在将$u=C(x)\,(1+x^2)$代入原方程可得

$$C'(x)\left(1+x^2\right)=-\frac{2x^3}{1+x^2}$$

于是

$$C(x)=-\frac{1}{1+x^2}-\ln(1+x^2)+C$$

于是

$$u(x)=-1-\left(1+x^2\right)\ln(1+x^2)+C\left(1+x^2\right)$$

$y(0)=1$要求$u(0)=1$,于是$C=2$,即原方程的解为

$$y=\frac{1}{\sqrt{1+2x^2+(1+x^2)\ln\left(1+x^2\right)}}$$

2.(10分) 求常微分方程

$$x^2y''+3xy'+4y=0(x>0)$$

的满足$y(1)=y'(1)=1$的解.

Solution.

做代换$x=\mathrm{e}^t$,于是

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=\mathrm{e}^{-t}\frac{\mathrm{d}y}{\mathrm{d}t}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}t}{\mathrm{d}x}=\mathrm{e}^{-2t}\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2}-\frac{\mathrm{d}y}{\mathrm{d}t}\right)$$

于是

$$y_t''-y_t'+3y_t'+4y=0$$

这是一个齐次方程,对应的特征根为$\lambda_{1,2}=-1\pm\sqrt{3}\mathrm{i}$,其通解为

$$y=\mathrm{e}^{-t}\left(C_1\cos\sqrt{3}t+C_2\sin\sqrt{3}t\right)$$

于是原方程的通解为

$$y=\frac{C_1\cos\sqrt{3}\ln x+C_2\sin\sqrt{3}\ln x}{x}$$

要求$y(1)=1$则有$C_1=1$.而

$$y_x'=\mathrm{e}^{-t}y_t'=\mathrm{e}^{-2t}\left(\left(\sqrt{3}C_2-C_1\right)\cos\sqrt{3}t-\left(\sqrt{3}C_1+C_2\right)\sin\sqrt{3}t\right)$$

要求$y'(1)=1$就有$\sqrt{3}C_2-C_1=1$.于是$C_1=1,C_2=\dfrac{2\sqrt{3}}{3}$,于是原方程的解为

$$y=\frac{3\cos\sqrt{3}\ln x+2\sqrt{3}\sin\sqrt{3}\ln x}{3x}$$

3.(10分) 求常微分方程

$$y''+y'-2y=x+\mathrm{e}^x+\sin x$$

的满足$y(0)=-\dfrac{7}{20},y'(0)=\dfrac{38}{15}$的解.

Solution.

原方程对应的齐次方程的特征根为$\lambda_1=1,\lambda_2=-2$,于是原方程的通解为

$$y=C_1\mathrm{e}^x+C_2\mathrm{e}^{-2x}$$

考虑方程$y''+y'-2y=x$的特解,设$y=Ax^2+Bx+C$,则

$$2A+(2Ax+B)-2\left(Ax^2+Bx+C\right)=x$$

于是$A=0,B=-\frac{1}{2},C=-\frac{1}{4}$.
考虑方程$y''+y'-2y=\mathrm{e}^x$的特解,设$y=Ax\mathrm{e}^x$,则有

$$A\mathrm{e}^x\left(x+2+x+1-2x\right)=\mathrm{e}^x$$

于是$A=\frac{1}{3}$.
考虑方程$y''+y'-2y=\sin x$的特解,设$y=A\cos x+B\sin x$,则有

$$(-B-A-2B)\sin x+(-A+B-2A)\cos x=\sin x$$

于是$A=-\frac{1}{10},B=-\frac{3}{10}$.
于是原方程的通解为

$$y=C_1\mathrm{e}^x+C_2\mathrm{e}^{-2x}-\frac{1}{2}x-\frac{1}{4}+\frac{1}{3}x\mathrm{e}^x-\frac{1}{10}\cos x-\frac{3}{10}\sin x$$

于是

$$\begin{cases} y(0)=C_1+C_2-\frac{1}{4}-\frac{1}{10}=-\frac{7}{20} \\ y'(0)=C_1-2C_2-\frac{1}{2}+\frac{1}{3}-\frac{3}{10}=\frac{38}{15} \end{cases}$$

解得$C_1=1,C_2=-1$,于是原方程的解为

$$y=\mathrm{e}^x-\mathrm{e}^{-2x}-\frac{1}{2}x-\frac{1}{4}+\frac{1}{3}x\mathrm{e}^x-\frac{1}{10}\cos x-\frac{3}{10}\sin x$$

4.(10分) 设关于R的函数

$$I(R)=\oint_{x^2+y^2=R^2}\frac{x\mathrm{d}y-y\mathrm{d}x}{\left(x^2+xy+y^2\right)^2}$$

试证明

$$\lim_{R\to\infty}I(R)=0$$

其中积分方向为圆周的逆时针方向.

Proof.

考虑二元函数

$$P(x,y)=\frac{-y}{\left(x^2+xy+y^2\right)^2}\qquad Q(x,y)=\frac{x}{\left(x^2+xy+y^2\right)^2}$$

积分曲线$S_R: x^2+y^2=R^2$的单位切向量为$\mathbf{n}=\frac{1}{R}(-y,x)$,于是

$$\begin{aligned}
I(R) &= \oint_{S_R^+} (P,Q)\cdot \mathbf{n}\mathrm{d}s \\
&= \oint_S \frac{x^2+y^2}{R\left(x^2+xy+y^2\right)^2}\mathrm{d}s \\
&= \int_0^{2\pi} \frac{R}{\left(R^2\left(1+\cos t\sin t\right)\right)^2}\mathrm{d}t \\
&\leqslant \int_0^{2\pi} \frac{4}{R^3}\mathrm{d}t \\
&= \frac{8\pi}{R^3}
\end{aligned}$$

于是

$$0\leqslant I(R)\leqslant \frac{8\pi}{R^3}$$

由夹逼准证可知

$$\lim_{R\to+\infty} I(R)=0$$

5.(10分) 设L为空间曲线

$$\begin{cases} x^2+y^2=1 \\ x+z=1 \end{cases}$$

其方向为自z轴正方向向负方向看的逆时针方向.计算曲线积分

$$\int_L \left(y-z+\sin^2 x\right)\mathrm{d}x+(z-x+\sin^2 y)\mathrm{d}y+\left(x-y+\sin^2 z\right)\mathrm{d}z$$

Solution.

令$P(x,y,z)=y-z+\sin^2 x, Q(x,y,z)=z-x+\sin^2 y, R(x,y,z)=x-y+\sin^2 z$.于是$P,Q,R$在$\mathbb{R}^3$上可微.$L$围成的空间曲面$S: z=1-x, x^2+y^2\leqslant 1$,其单位外法向量为$\mathbf{n}=\dfrac{(1,0,1)}{\sqrt{2}}$.

于是据Stokes公式有

$$\begin{aligned}
\int_L P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z &= \iint_{S^+} \begin{vmatrix} \mathrm{d}x\mathrm{d}y & \mathrm{d}y\mathrm{d}z & \mathrm{d}z\mathrm{d}x \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= -2\iint_{S^+} \mathrm{d}x\mathrm{d}y+\mathrm{d}y\mathrm{d}z+\mathrm{d}z\mathrm{d}x \\
&= -4\iint_{x^2+y^2\leqslant 1} \mathrm{d}\sigma \\
&= -4\pi
\end{aligned}$$

6.(10分) 设D是单位圆$x^2+y^2\leqslant 1$,求积分

$$\iint_D (x+y+xy)^2\,\mathrm{d}\sigma$$

Solution.

做极坐标变换$x=r\cos\theta, y=r\sin\theta$,于是积分区域变为$D':0\leqslant r\leqslant 1, 0\leqslant\theta\leqslant 2\pi$.于是

$$\begin{aligned}\iint_D (x+y+xy)^2\,\mathrm{d}\sigma &= \iint_D \left(x^2+y^2+x^2y^2+2xy+2x^2y+2xy^2\right)\mathrm{d}\sigma\\ &= \iint_D \left(x^2+y^2+x^2y^2\right)\mathrm{d}\sigma\\ &= \int_0^{2\pi}\mathrm{d}\theta\int_0^1 \left(r^2+r^4\sin^2\theta\cos^2\theta\right) r\mathrm{d}r\\ &= \int_0^{2\pi}\left(\frac{1}{4}+\frac{1}{6}\sin^2\theta\cos^2\theta\right)\mathrm{d}\theta\\ &\xlongequal{t=2\theta} \frac{\pi}{2}+\frac{1}{48}\int_0^{4\pi}\sin^2 t\mathrm{d}t\\ &= \frac{13}{24}\pi\end{aligned}$$

7.(10分) 设平面闭区域D由直线$y=x$和曲线$y=x^3$围成,求积分

$$\iint_D \left(\frac{3x^2\sin y}{y}+2\mathrm{e}^{x^2}\right)\mathrm{d}\sigma$$

Solution.

首先考虑第一项,有

$$\begin{aligned}\iint_D \frac{3x^2\sin y}{y}\mathrm{d}x\mathrm{d}y &= \int_{-1}^0 \mathrm{d}y\int_{\sqrt[3]{y}}^{y}\frac{3x^2\sin y}{y}\mathrm{d}x+\int_0^1\mathrm{d}y\int_y^{\sqrt[3]{y}}\frac{3x^2\sin y}{y}\mathrm{d}x\\ &\xlongequal{t=x^3} \int_{-1}^0\mathrm{d}y\int_y^{y^3}\frac{\sin y}{y}\mathrm{d}t+\int_0^1\mathrm{d}y\int_{y^3}^{y}\frac{\sin y}{y}\mathrm{d}t\\ &= \int_{-1}^0 \sin y\left(y^2-1\right)\mathrm{d}y+\int_0^1\sin y\left(1-y^2\right)\mathrm{d}y\\ &= 2\int_0^1 \sin y\left(1-y^2\right)\mathrm{d}y\\ &= 2\left(y^2\cos y-2y\sin y-3\cos y\right)\Big|_0^1\\ &= 6-4\cos 1-4\sin 1\end{aligned}$$

再考虑第二项,有

$$
\begin{aligned}
\iint_{D} 2\mathrm{e}^{x^{2}} \mathrm{d}\sigma &= \int_{-1}^{0} \mathrm{d}x \int_{x}^{x^{3}} 2\mathrm{e}^{x^{2}} \mathrm{d}y + \int_{0}^{1} \mathrm{d}x \int_{x^{3}}^{x} 2\mathrm{e}^{x^{2}} \mathrm{d}y \\
&= 4 \int_{0}^{1} \left(x - x^{3}\right) \mathrm{e}^{x^{2}} \mathrm{d}x \\
&\xlongequal{t=x^{2}} 2 \int_{0}^{1} (1-t) \mathrm{e}^{t} \mathrm{d}t \\
&= 2 \left(\mathrm{e}^{t}(2-t)\right)\Big|_{0}^{1} \\
&= 2\mathrm{e} - 4
\end{aligned}
$$

于是

$$
\iint_{D} \left(\frac{3x^{2}\sin y}{y} + 2\mathrm{e}^{x^{2}}\right) \mathrm{d}\sigma = 2\mathrm{e} + 2 - 4\cos 1 - 4\sin 1
$$

8.(10分) 设空间闭区域Ω由曲面$z=\sqrt{1+x^{2}+y^{2}}, z=\sqrt{3\left(1+x^{2}+y^{2}\right)}$和$x^{2}+y^{2}=1$围成,求积分

$$
\iiint_{\Omega} \frac{(x+y+z)^{2}\sqrt{1+x^{2}+y^{2}}}{\left(x^{2}+y^{2}+z^{2}\right)\left(1+x^{2}+y^{2}+z^{2}\right)} \mathrm{d}V
$$

Solution.

做柱坐标变换$z=z, x=r\cos\theta, y=r\sin\theta$,于是积分区域变为

$$
\Omega^{\prime}: 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi, \sqrt{1+r^{2}} \leqslant z \leqslant \sqrt{3}\sqrt{1+r^{2}}
$$

于是

$$
\begin{aligned}
&\iiint_{\Omega} \frac{(x+y+z)^{2}\sqrt{1+x^{2}+y^{2}}}{\left(x^{2}+y^{2}+z^{2}\right)\left(1+x^{2}+y^{2}+z^{2}\right)} \mathrm{d}V \\
=&\iiint_{\Omega^{\prime}} \frac{\left(r^{2}+z^{2}+r^{2}\sin\theta\cos\theta+2rz(\sin\theta+\cos\theta)\right)\sqrt{1+r^{2}}}{\left(r^{2}+z^{2}\right)\left(1+r^{2}+z^{2}\right)} r\mathrm{d}V \\
=&\int_{0}^{2\pi} \mathrm{d}\theta \int_{0}^{1} \mathrm{d}r \int_{\sqrt{1+r^{2}}}^{\sqrt{3}\sqrt{1+r^{2}}} \frac{r\sqrt{1+r^{2}}}{\left(z^{2}+r^{2}+1\right)} \mathrm{d}z \\
=&\int_{0}^{2\pi} \mathrm{d}\theta \int_{0}^{1} \mathrm{d}r \left(r \arctan\frac{z}{\sqrt{1+r^{2}}}\right)\Big|_{\sqrt{1+r^{2}}}^{\sqrt{3}\sqrt{1+r^{2}}} \\
=&\int_{0}^{2\pi} \mathrm{d}\theta \int_{0}^{1} \frac{\pi}{12} r\mathrm{d}r \\
=&\frac{\pi^{2}}{12}
\end{aligned}
$$

9.(10分) 设Γ由闭曲线$x^2+y^2=9(y\geqslant 0)$和$\dfrac{x^2}{9}+\dfrac{y^2}{16}=1(y\leqslant 0)$组成,方向沿逆时针方向,求曲线积分

$$\oint_{\Gamma}\left(\frac{y^2+y+4x^2}{4x^2+y^2}+\sin x^2\right)\mathrm{d}x+\left(\frac{4x^2-x+y^2}{4x^2+y^2}+\sin y^2\right)\mathrm{d}y$$

Solution.

设Ω为Γ围成的闭区域.

首先令

$$P(x,y)=\frac{y}{4x^2+y^2}\qquad Q(x,y)=\frac{-x}{4x^2+y^2}$$

再令

$$A(x,y)=1+\sin x^2\qquad B(x,y)=1+\sin y^2$$

于是原曲线积分即为

$$\oint_{\Gamma}(A+P)\mathrm{d}x+(B+Q)\mathrm{d}y$$

首先有

$$\frac{\partial A}{\partial y}=\frac{\partial B}{\partial x}=0$$

在$\mathbb{R}^2$上成立,于是据Green公式有

$$\oint_{L}A\mathrm{d}x+B\mathrm{d}y=\iint_{\Omega}0\mathrm{d}\sigma=0$$

考虑小椭圆$\Omega_\varepsilon:4x^2+y^2\leqslant\varepsilon^2$和其边界$S_\varepsilon$,其中$0<\varepsilon<3$.在$\Omega\backslash\Omega_\varepsilon$上根据Green公式有

$$\begin{aligned}\oint_{\Gamma+S_\varepsilon^-}P\mathrm{d}x+Q\mathrm{d}y&=\iint_{\Omega\backslash\Omega_\varepsilon}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}\sigma\\&=\iint_{\Omega\backslash\Omega_\varepsilon}-\frac{(4x^2+y^2)-8x^2+(4x^2+y^2)-2y^2}{(4x^2+y^2)^2}\mathrm{d}\sigma\\&=0\end{aligned}$$

又有

$$\begin{aligned}\oint_{\Gamma}P\mathrm{d}x+Q\mathrm{d}y=\oint_{S_\varepsilon^+}P\mathrm{d}x+Q\mathrm{d}y\\&=\frac{1}{\varepsilon^2}\oint_{S}y\mathrm{d}x-x\mathrm{d}y\\&=-\frac{2}{\varepsilon^2}\iint_{\Omega_\varepsilon}\mathrm{d}\sigma\\&=-\pi\end{aligned}$$

于是

$$\oint_{\Gamma}\left(\frac{y^2+y+4x^2}{4x^2+y^2}+\sin x^2\right)\mathrm{d}x+\left(\frac{4x^2-x+y^2}{4x^2+y^2}+\sin y^2\right)\mathrm{d}y=-\pi$$

10.(10分) 设曲面S是柱体$\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leqslant 1, 0 \leqslant z \leqslant 1\}$的表面的外侧.

(1) 求曲面积分

$$\iint_S (y-z)|x|\mathrm{d}y\mathrm{d}z + (z-x)|y|\mathrm{d}z\mathrm{d}x + (x-y)z\mathrm{d}x\mathrm{d}y$$

(2) 求曲面积分

$$\iint_S (y-z)x^2\mathrm{d}y\mathrm{d}z + (z-x)y^2\mathrm{d}z\mathrm{d}x + (x-y)z^2\mathrm{d}x\mathrm{d}y$$

(3) 求曲面积分

$$\iint_S (y-z)x^3\mathrm{d}y\mathrm{d}z + (z-x)y^3\mathrm{d}z\mathrm{d}x + (x-y)z^3\mathrm{d}x\mathrm{d}y$$

Solution.

S可分为三个部分,即

$$D_1 = \left\{(x,y,z) \in \mathbb{R}^3 : z = 0, x^2+y^2 \leqslant 1\right\}$$

$$D_2 = \left\{(x,y,z) \in \mathbb{R}^3 : z = 1, x^2+y^2 \leqslant 1\right\}$$

$$D_3 = \left\{(x,y,z) \in \mathbb{R}^3 : 0 \leqslant z \leqslant 1, x^2+y^2 = 1\right\}$$

三个面的单位外法向量分别为$\mathbf{n}_1 = (0,0,-1), \mathbf{n}_2 = (0,0,1), \mathbf{n}_3 = (x,y,0)$.

各区域都关于Oxz和Oyz平面对称,也关于z轴对称.

(1) 在D_1面上有

$$I_1 = \iint_{D_1} -1(x-y)z\mathrm{d}S = 0$$

在D_2面上有

$$I_2 = \iint_{D_1} -1(x-y)z\mathrm{d}S = -\iint_{x^2+y^2\leqslant 1} (x-y)\mathrm{d}\sigma = 0$$

在D_3面上有

$$I_3 = \iint_{D_3} x|x|(y-z) + y|y|(z-x)\mathrm{d}S = 0$$

于是

$$\iint_S (y-z)|x|\mathrm{d}y\mathrm{d}z + (z-x)|y|\mathrm{d}z\mathrm{d}x + (x-y)z\mathrm{d}x\mathrm{d}y = 0$$

(2) 同理不难有$I_1 = I_2 = 0$,而

$$I_3 = \iint_{D_3} (y-z)x^3 + (z-x)y^3\mathrm{d}S = 0$$

于是

$$\iint_S (y-z)x^2\mathrm{d}y\mathrm{d}z + (z-x)y^2\mathrm{d}z\mathrm{d}x + (x-y)z^2\mathrm{d}x\mathrm{d}y = 0$$

(3) 仍然不难有$I_1 = I_2 = 0$,而

$$I_3 = \iint_{D_3} (y-z)x^4 + (z-x)y^4 \mathrm{d}S = \iint_{D_3} x^4 y - xy^4 + z\left(y^4 - x^4\right) \mathrm{d}S = 0$$

于是

$$\iint_S (y-z)x^3 \mathrm{d}y\mathrm{d}z + (z-x)y^3 \mathrm{d}z\mathrm{d}x + (x-y)z^3 \mathrm{d}x\mathrm{d}y = 0$$