北京大学数学科学学院2021-22高等数学B1期中模拟

1.(10分)

多选题,错选或少选均不得分,无需写出解答过程.

(1) **(5分)** 选出下列选项中总是正确的式子.

A. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\mathrm{d}x < \frac{\pi}{2}$

B. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\mathrm{d}x > 1$

C. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\mathrm{d}x > \frac{1}{2} + \frac{\pi}{4}$

D. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\mathrm{d}x > \frac{1}{2}\int_0^{\pi} \frac{\sin x}{x}\mathrm{d}x$

(2) **(5分)** 设$f(x)$是定义在$[1,+\infty)$上的非负单调递减的连续函数.定义$s_n = \displaystyle\sum_{k=1}^{n} f(k)$,选出下列选项中总是正确的式子.

A. $s_n \leqslant \displaystyle\int_1^{n} f(x)\mathrm{d}x$

B. $s_n \leqslant f(1) + \displaystyle\int_1^{n} f(x)\mathrm{d}x$

C. $s_n \geqslant \displaystyle\int_1^{n+1} f(x)\mathrm{d}x$

D. $s_n \geqslant f(1) + \displaystyle\int_1^{n+1} f(x)\mathrm{d}x$

(1) **Solution.**

注意到$\dfrac{\sin x}{x} < 1$,于是$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\mathrm{d}x < \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2}$,于是**A**项正确.

注意到$\dfrac{\sin x}{x} > \dfrac{2}{\pi}$,于是$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\mathrm{d}x > \int_0^{\frac{\pi}{2}} \frac{2\mathrm{d}x}{\pi} = 1$,于是**B**项正确.

割线放缩可得$\dfrac{\sin x}{x} > \dfrac{1-\frac{\pi}{2}}{\frac{\pi}{2}}x + 1$,于是$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\mathrm{d}x > \frac{\frac{\pi}{2}\left(1+\frac{\pi}{2}\right)}{2}$,于是**C**项正确.

注意到$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\mathrm{d}x - \frac{1}{2}\int_0^{\pi} \frac{\sin x}{x}\mathrm{d}x = \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\mathrm{d}x - \int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{x}\mathrm{d}x\right) > 0$,于是**D**项正确.

(2) **Solution.**

注意到

$$\forall k \in \mathbb{N}^*, \forall x \in (k, k+1), f(k) \geqslant f(x) \geqslant f(k+1)$$

于是

$$\forall k \in \mathbb{N}^*, f(k) \geqslant \int_k^{k+1} f(x)\mathrm{d}x \geqslant f(k+1)$$

于是

$$s_n \geqslant \sum_{i=1}^{n} \int_i^{i+1} f(x)\mathrm{d}x = \int_1^{n+1} f(x)\mathrm{d}x$$

且

$$s_n - f(1) \leqslant \sum_{i=1}^{n-1} \int_i^{i+1} = \int_1^{n} f(x)\mathrm{d}x$$

于是选择**BC**项.

2.(18分)

计算下列极限.

(1) **(6分)** 计算序列极限

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}i^{2021}}{n^{2022}}$$

(2) **(6分)** 计算函数极限

$$\lim_{x\to+\infty}\left(\sin\frac{1}{x^{2022}}+\cos\frac{1}{x^{1011}}\right)^{x^{2022}}$$

(3) **(6分)** 计算函数极限

$$\lim_{x\to0}\frac{\sin 3x-3\sin x}{x^3}$$

(1) **Solution.**

注意到

$$\frac{\sum_{i=1}^{n}i^{2021}}{n^{2022}}=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{2021}$$

于是根据Riemann积分的定义有

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}i^{2021}}{n^{2022}}=\int_0^1 x^{2021}\mathrm{d}x=\left.\frac{x^{2022}}{2022}\right|_0^1=\frac{1}{2022}$$

(2) **Solution.**

作变量代换$u=x^{1011}$.于是

$$\begin{aligned}\lim_{x\to+\infty}\left(\sin\frac{1}{x^{2022}}+\cos\frac{1}{x^{1011}}\right)^{x^{2022}}&=\lim_{u\to+\infty}\left(\sin\frac{1}{u^2}+\cos\frac{1}{u}\right)^{u^2}\\&=\lim_{u\to+\infty}\left(1-2\sin^2\frac{1}{2u}+\sin\frac{1}{u^2}\right)^{\frac{1}{2\sin^2\frac{1}{2u}-\sin\frac{1}{u^2}}\cdot u^2\left(2\sin^2\frac{1}{2u}+\sin\frac{1}{u^2}\right)}\end{aligned}$$

又

$$\lim_{u\to+\infty}u^2\cdot2\sin^2\frac{1}{2u}=\frac{1}{2}\left(\lim_{u\to+\infty}2u\sin\frac{1}{2u}\right)^2=\frac{1}{2}$$

$$\lim_{u\to+\infty}u^2\sin\frac{1}{u^2}=1$$

于是

$$\lim_{x\to+\infty}\left(\sin\frac{1}{x^{2022}}+\cos\frac{1}{x^{1011}}\right)^{x^{2022}}=\mathrm{e}^{-\left(\frac{1}{2}+1\right)}=\mathrm{e}^{-\frac{3}{2}}$$

(3) **Solution.**

根据三倍角公式有

$$\sin 3x=3\sin x-4\sin^3 x$$

于是

$$\lim_{x\to 0}\frac{\sin 3x-3\sin x}{x^3}=-4\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3=-4$$

3.(12分)

计算下列积分.

(1) **(6分)** 计算定积分

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos x\arctan\left(\mathrm{e}^x\right)}{1+\sin^2 x}\mathrm{d}x$$

(2) **(6分)** 计算不定积分

$$\int\frac{2x^2+2x+13}{(x-2)\left(1+x^2\right)^2}\mathrm{d}x$$

(1) **Solution.**

置$f(x)=\dfrac{\cos x\arctan\left(\mathrm{e}^x\right)}{1+\sin^2 x}\mathrm{d}x$,则

$$f(-x)=\frac{\cos(-x)\arctan\left(\dfrac{1}{\mathrm{e}^x}\right)}{1+(\sin x)^2}=\frac{\cos x\left(\dfrac{\pi}{2}-\arctan\mathrm{e}^x\right)}{1+\sin^2 x}$$

从而$f(x)+f(-x)=\dfrac{\pi}{2}\cdot\dfrac{\cos x}{1+\sin^2 x}$.于是

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f(x)\mathrm{d}x=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\cos x}{1+\sin^2 x}\mathrm{d}x=\frac{\pi}{2}\int_0^1\frac{\mathrm{d}u}{u^2+1}=\frac{\pi^2}{8}$$

(2) **Solution.**

设

$$\frac{2x^2+2x+13}{(x-2)\left(1+x^2\right)^2}=\frac{A}{x-2}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$$

两端同乘$x-2$后令$x=2$,于是$A=\dfrac{8+4+13}{5^2}=1$. 则

$$2x^2+2x+13=x^4+2x^2+1+\left(x^2+1\right)\left(Bx^2+Cx-2Bx-2C\right)+(Dx+E)(x-2)$$

解得$B=-1,C=-2,D=-3,E=-4$.

而

$$\int\frac{x+2}{x^2+1}\mathrm{d}x=\frac{1}{2}\int\frac{\mathrm{d}x^2}{x^2+1}+2\int\frac{\mathrm{d}x}{x^2+1}$$

$$\int\frac{3x+4}{\left(x^2+1\right)^2}\mathrm{d}x=\frac{3}{2}\int\frac{\mathrm{d}x^2}{\left(x^2+1\right)^2}+2\int\frac{\left(1+x^2\right)+\left(1-x^2\right)}{\left(x^2+1\right)^2}\mathrm{d}x$$

于是

$$\int \frac{2x^2+2x+13}{(x-2)\left(1+x^2\right)^2}\mathrm{d}x = \int \frac{\mathrm{d}x}{x-2} - 4\int \frac{\mathrm{d}x}{x^2+1} - \frac{1}{2}\int \frac{\mathrm{d}x^2}{x^2+1} - \frac{3}{2}\int \frac{\mathrm{d}x^2}{\left(x^2+1\right)^2} - 2\int \frac{1-x^2}{\left(1+x^2\right)^2}\mathrm{d}x$$

$$= \ln|x-2| - 4\arctan x - \frac{1}{2}\ln(x^2+1) + \frac{3}{2\left(x^2+1\right)} - \frac{2x}{x^2+1} + C$$

$$= -\frac{1}{2}\ln\left(x^2+1\right) + \ln|x-2| - 4\arctan x + \frac{3-4x}{2\left(x^2+1\right)} + C$$

4.(8分)

设函数$f(x)$满足

$$f(x)=\begin{cases} axe^x+bx^x, x>1 \\ |x|, x\leqslant 1 \end{cases}$$

求所有可能的参数a,b使得$f(x)$在$x=1$处可导.

Solution.

首先需要$f(x)$在$x=1$处连续,于是$\lim\limits_{x\to1^+}=\lim\limits_{x\to1^-}=f(1)=1$,即$ae+b=1$. 又要求$f(x)$在$x=1$处的左右导数相同.易知

$$\lim_{\Delta x\to0^-}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to0^-}\frac{x+\Delta x-x}{\Delta x}=1$$

又$x>1$时

$$f'(x)=ae^x(x+1)+bx^x(\ln x+1)$$

于是$f'_+(1)=2ae+b=1$.

综上可以解得$a=0,b=1$.

5.(12分)

设函数$f(x)$是定义在$\mathbb{R}$上的以1为周期的连续函数,试证明:$\exists c\in\mathbb{R}$, s.t. $f(c)=f(c+\pi)$.

Proof.

置$F(x)=f(x+\pi)-f(x)$.根据连续函数的有界性,可知$\exists c_1,c_2\in[0,1)$, s.t. $f(c_1)\leqslant f(x)\leqslant f(c_2)$,于是

$$F(c_1)=f(c_1+\pi)-f(c_1)\geqslant 0$$

$$F(c_2)=f(c_2+\pi)-f(c_2)\leqslant 0$$

根据连续函数的介值定理,$\exists c$介于c_1,c_2之间,满足$F(c)=0$,即$f(c)=f(c+\pi)$.

6.(8分)

计算曲线$y=\int_0^x\sqrt{\sin x}\mathrm{d}x$在$x\in[0,\pi]$部分的弧长.

Solution

根据弧长公式有

$$
\begin{aligned}
s&=\int_0^\pi\sqrt{1+y'^2}\mathrm{d}x\\
&=\int_0^\pi\sqrt{1+\sin x}\mathrm{d}x\\
&=\int_0^\pi\sqrt{\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2}}\mathrm{d}x\\
&=\int_0^\pi\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|\mathrm{d}x\\
&=\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)\Big|_0^\pi\\
&=4
\end{aligned}
$$

7.(12分)

考虑方程$x=\tan x$的正实根.

(1) **(4分)** 试证明:$x=\tan x$有无穷多个正实根.

(2) **(8分)** 将$x=\tan x$的正实根从小到大排列成序列$\{x_n\}$,试证明$\lim\limits_{n\to\infty}(x_{n+1}-x_n)=\pi$.

(1) **Proof.**

设$f(x)=\tan x-x$,对于任意$k\in\mathbb{N}^*$,在区间$\left[k\pi,k\pi+\frac{\pi}{2}\right)$上总有

$$f(k\pi)=0-k\pi<0$$

又$\lim\limits_{x\to\left(k\pi+\frac{\pi}{2}\right)^-}f(x)=\lim\limits_{x\to\left(k\pi+\frac{\pi}{2}\right)^-}\tan x-k\pi-\frac{\pi}{2}=+\infty.$

即$\exists x\in\left[k\pi,k\pi+\frac{\pi}{2}\right)$, s.t. $f(x)>0$.

于是根据连续函数的介值定理可知$\exists x_0\in\left[k\pi,k\pi+\frac{\pi}{2}\right)$, s.t. $f(x_0)=0$.

又知这样的k有无穷多个,于是$x=\tan x$有无穷多个正实根.

(2) **Proof.**

由**(1)**的证明已知$n\pi<x_n<n\pi+\frac{\pi}{2}$.

设$t_n=x_n-n\pi\in\left(0,\frac{\pi}{2}\right)$,于是$\tan t_n=\tan x_n=x_n>n\pi$.

对于任意$\varepsilon > 0$,取$N = \left[\dfrac{\tan\left(\dfrac{\pi}{2} - \varepsilon\right)}{\pi}\right] + 1$,于是$\forall n > N$有

$$\left|t_n - \frac{\pi}{2}\right| = \frac{\pi}{2} - t_n < \frac{\pi}{2} - \arctan \pi N < \frac{\pi}{2} - \left(\frac{\pi}{2} - \varepsilon\right) < \varepsilon$$

从而$\lim\limits_{n\to\infty} t_n = \dfrac{\pi}{2}$.

于是$\lim\limits_{n\to\infty} (x_{n+1} - x_n) = \pi + \lim\limits_{n\to\infty} (t_{n+1} - t_n) = \pi$.

8.(12分)

设$n \in \mathbb{N}^*$,定义序列$\{x_n\}$满足$x_n = \sqrt[n]{n}$.

(1) **(6分)** 用$\varepsilon - N$语言证明$\lim\limits_{n\to\infty} x_n = 1$.

(2) **(6分)** 求所有的正实数a满足$\lim\limits_{n\to\infty} n(x_n - 1)^a$收敛.

(1) **Proof.**

记$t_n = \sqrt[n]{n} - 1$,于是$n = (t+1)^n$,作部分二项展开有

$$n = 1 + nt_n + \frac{n^2 - n}{2} t_n^2 + \cdots > \frac{n^2 - n}{2} t_n^2$$

于是$t_n < \sqrt{\dfrac{2n}{n^2 - n}} = \dfrac{1}{\sqrt{n-2}}$.

对于任意$\varepsilon > 0$,取$N = \left[\dfrac{1}{\varepsilon^2}\right] + 3$,于是$\forall n > N$有

$$|a_n - 1| = |t_n| < \frac{1}{N-2} < \varepsilon$$

于是$\lim\limits_{n\to\infty} a_n = 1$,得证.

(2) **Proof.**

仍然记$t_n = \sqrt[n]{n} - 1$,于是$n = 1 + nt_n + \cdots + t_n^n < C_n^k t_n^k$.

于是$t_n < \left(\dfrac{n}{C_n^k}\right)^{\frac{1}{k}} = n^{-\frac{k-1}{k}} \cdot \left(\dfrac{n^k}{C_n^k}\right)^{\frac{1}{k}}$.

对于$\forall a > 1$,总有

$$0 < nt_n^a < n\left(n^{-\frac{k-1}{k}} \cdot \left(\frac{n^k}{C_n^k}\right)^{\frac{1}{k}}\right)^a = n^{\left(1 - \frac{a(k-1)}{k}\right)} \cdot \left(\frac{n^k}{C_n^k}\right)^{\frac{a}{k}}$$

任取k满足$1 - \dfrac{a(k-1)}{k} < 1$,对上式取极限可知$\lim\limits_{n\to\infty} nt_n^a = 0$.

对于$\forall a \leqslant 1$,任取$m \in \mathbb{N}^*$有$\lim\limits_{n\to\infty} n\left(\sqrt[n]{m} - 1\right) = \lim\limits_{n\to\infty} \dfrac{m^{\frac{1}{n}} - 1}{\frac{1}{n}} = \ln m$.

于是$\forall m \in \mathbb{N}^*$, $\lim\limits_{n\to\infty} nt_n^a \geqslant \lim\limits_{n\to\infty} nt_n > \ln m$.于是$\lim\limits_{n\to\infty} n\left(\sqrt[n]{n} - 1\right) = +\infty$.

综上,当且仅当$a > 1$时$\lim\limits_{n\to\infty} n\left(x_n - 1\right)^a$存在收敛.

9.(8分)

给定正整数a,定义$f_a(x)=\left(x+\sqrt{x^2+1}\right)^a$,求所有自然数$n$满足$f_a^{(n)}(0)=0$.

Solution.

注意到

$$f_a'(x)=a\left(x+\sqrt{x^2+1}\right)^{a-1}\cdot\left(1+\frac{x}{\sqrt{x^2+1}}\right)=\frac{a\left(x+\sqrt{x^2+1}\right)^a}{\sqrt{x^2+1}}=\frac{a}{\sqrt{x^2+1}}f_a(x)$$

设$y=f_a(x)$,于是

$$\sqrt{1+x^2}y'=ay$$

即

$$\left(1+x^2\right)y'^2=a^2y^2$$

对该式两端求导有

$$2xy'^2+2\left(1+x^2\right)y'y''=2a^2yy'$$

即

$$xy'+\left(1+x^2\right)y''=a^2y$$

等式两端求n阶导有

$$xy^{(n+1)}+ny^{(n)}+\left(1+x^2\right)y^{(n+2)}+2nxy^{(n+1)}+\left(n^2-n\right)y^{(n)}=a^2y^{(n)}$$

代入$x=0$有

$$y^{(n+2)}=\left(a^2-n^2\right)y^{(n)}$$

由$f_a(0)=1,f_a'(0)=a$可知

$$\begin{cases} f_a^{(2k+1)}(0)=\prod_{i=1}^{k}\left(a^2-(2i-1)^2\right)a \\ f_a^{(2k+2)}(0)=\prod_{i=0}^{k}\left(a^2-(2i)^2\right) \end{cases}$$

于是当$n\geqslant a+2$且n与a的奇偶性相同时$f_a^{(n)}(0)=0$.