北京大学数学科学学院2022-23高等数学B2期中模拟考试

1.(10分) 设D是由直线$x=1,y=x,y=2x$所围成的有界闭区域,求二重积分

$$\iint_D (\sqrt{x}+y)\mathrm{d}x\mathrm{d}y$$

Solution.

我们有

$$\begin{aligned}\iint_D (\sqrt{x}+y)\mathrm{d}x\mathrm{d}y &= \int_0^1 \mathrm{d}x \int_x^{2x} \left(\sqrt{x}+y\right)\mathrm{d}y \\ &= \int_0^1 \left(x\sqrt{x}+\frac{3}{2}x^2\right)\mathrm{d}x \\ &= \left(\frac{2}{5}x^{\frac{5}{2}}+\frac{1}{2}x^3\right)\Big|_0^1 \\ &= \frac{9}{10}\end{aligned}$$

2.(10分) 求三重积分

$$\iiint_\Omega \frac{1}{\sqrt{x^2+y^2+z^2}}\mathrm{d}V$$

其中Ω是球$x^2+y^2+z^2\leqslant 2x$.

Solution.

积分区域Ω即$\{(x,y,z):0\leqslant x\leqslant 2, 0\leqslant y^2+z^2\leqslant 2x-x^2\}$.

做球坐标变换$x=\rho\cos\varphi, y=\rho\cos\theta\sin\varphi, z=\rho\sin\theta\sin\varphi$,则积分区域变换为

$$\Omega'=\left\{(\rho,\theta,\varphi):0\leqslant\theta\leqslant 2\pi, 0\leqslant\varphi\leqslant\frac{\pi}{2}, 0\leqslant\rho\leqslant 2\cos\varphi\right\}$$

于是

$$\begin{aligned}\iiint_\Omega \frac{1}{\sqrt{x^2+y^2+z^2}}\mathrm{d}V &= \iiint_{\Omega'} \frac{\rho^2\sin\varphi}{\rho}\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta \\ &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{2\cos\varphi}\rho\sin\varphi\mathrm{d}\rho \\ &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}} 2\cos^2\varphi\sin\varphi\mathrm{d}\varphi \\ &= 2\pi\cdot\frac{2}{3} \\ &= \frac{4\pi}{3}\end{aligned}$$

3.(10分) 求曲线积分

$$\int_L \frac{\mathrm{d}y-\mathrm{d}x}{x-y+1}$$

其中L是圆周$x^2+y^2-2x=0$在$y\leqslant 0$的部分,沿逆时针方向.

Solution.

考虑二元函数

$$P(x,y)=-\frac{1}{x-y+1}\quad Q(x,y)=\frac{1}{x-y+1}$$

将半圆周按方向补成半圆周S,在半圆$D:\{(x,y):(x-1)^2+y^2\leqslant 1,y\leqslant 0\}$上根据Green公式有

$$\begin{aligned}\oint_S P\mathrm{d}x+Q\mathrm{d}y&=\iint_D\left(\frac{\partial Q}{\partial y}-\frac{\partial P}{\partial x}\right)\mathrm{d}\sigma\\&=\iint_D\frac{1-1}{(x-y+1)^2}\mathrm{d}\sigma\\&=0\end{aligned}$$

于是可将路径变换为半圆的直径,方向向右.于是

$$\oint_L P\mathrm{d}x+Q\mathrm{d}y=\int_0^2\frac{\mathrm{d}x}{x+1}=\ln 3$$

4.(15分) 求曲面积分

$$\iint_S y(x-z)\mathrm{d}y\mathrm{d}z+x^2\mathrm{d}z\mathrm{d}x+\left(y^2+xz\right)\mathrm{d}x\mathrm{d}y$$

其中S是球面$x^2+y^2+z^2=5$在$z\geqslant 1$的部分,取外侧.

Solution.

该球面的单位外法向量为$\mathbf{n}=\dfrac{1}{\sqrt{5}}(x,y,z)$.于是

$$\begin{aligned}&\iint_S y(x-z)\mathrm{d}y\mathrm{d}z+x^2\mathrm{d}z\mathrm{d}x+\left(y^2+xz\right)\mathrm{d}x\mathrm{d}y\\=&\frac{1}{\sqrt{5}}\iint_S\left(x^2y-xyz+x^2y+zy^2+xz^2\right)\mathrm{d}S\\=&\frac{1}{\sqrt{5}}\iint_S\left(2x^2y+y^2z+z^2x-xyz\right)\mathrm{d}S\end{aligned}$$

考虑到积分区域关于Oxz平面和Oyz平面对称,于是只需考虑y^2z一项.

记S在Oxy平面的投影为$D: x^2+y^2 \leqslant 4$,则有

$$\begin{aligned}\frac{1}{\sqrt{5}}\iint_S y^2 z\mathrm{d}S &= \frac{1}{\sqrt{5}}\iint_D y^2\sqrt{5-x^2-y^2}\sqrt{1+\frac{x^2+y^2}{5-x^2-y^2}}\mathrm{d}x\mathrm{d}y \\ &= \iint_D y^2\mathrm{d}x\mathrm{d}y \\ &= \int_0^{2\pi}\mathrm{d}\theta\int_0^2 r^3\sin^2\theta\mathrm{d}r \\ &= 2\pi\end{aligned}$$

于是

$$\iint_S y(x-z)\mathrm{d}y\mathrm{d}z + x^2\mathrm{d}z\mathrm{d}x + \left(y^2+xz\right)\mathrm{d}x\mathrm{d}y = 2\pi$$

5.(10分) 求常微分方程

$$xy' + 2y = \sin x$$

满足$y(\pi) = \dfrac{1}{\pi}$的特解.

Solution.

原方程两端同乘x,则有

$$x^2y' + 2xy = x\sin x$$

令$u = x^2y$,则有

$$u' = x\sin x$$

两端积分可得

$$u = -x\cos x + \sin x + C$$

当$x=0$时,$y=0$,不是满足题设条件的特解.当$x \neq 0$时,原方程的通解为

$$y = -\frac{x\cos x - \sin x}{x^2} + C$$

令$y(\pi) = \dfrac{1}{\pi}$,可得$C=0$,于是

$$y = \frac{\sin x - x\cos x}{x^2}$$

6.(10分) 求常微分方程

$$y'' + 2y' = 3 + 4\sin 2x$$

的通解.

Solution.

该方程对应的齐次方程的特征根为$\lambda_1 = 0, \lambda_2 = -2$.于是通解的形式为

$$y = C_1 + C_2 \mathrm{e}^{-2x}$$

考虑方程$y'' + 2y' = 3$的特解,设$y = Ax^2 + Bx$,则有

$$A = 0 \quad B = \frac{3}{2}$$

考虑方程$y'' + 2y' = 4\sin 2x$的特解,设$y = A\sin 2x + B\cos 2x$,则有

$$\begin{cases} -4A - 4B = 4 \\ -4B + 4A = 0 \end{cases}$$

从而解得

$$A = -\frac{1}{2} \quad B = -\frac{1}{2}$$

于是方程的通解为

$$y = C_1 + C_2 \mathrm{e}^{-2x} + \frac{3}{2}x - \frac{1}{2}\sin 2x - \frac{1}{2}\cos 2x$$

7.(10分) 设参数$a > 0$,$\mathbb{R}^3$中的圆柱$x^2 + z^2 = a^2$和$y^2 + z^2 = a^2$相交的区域为Ω,求Ω的体积.

Solution.

由于对称性,考虑第一卦限Ω'即可.对于特定的z,Ω'在第一卦限占的区域为

$$D_z = \left\{(x, y) : 0 \leqslant x, y \leqslant \sqrt{a^2 - z^2}\right\}$$

于是

$$\begin{aligned} V_{\Omega'} &= \iiint_{\Omega'} \mathrm{d}V \\ &= \int_0^a \mathrm{d}z \iint_{D_z} \mathrm{d}\sigma \\ &= \int_0^a \left(a^2 - z^2\right) \mathrm{d}z \\ &= \frac{2a^3}{3} \end{aligned}$$

于是

$$V = 8V_{\Omega'} = \frac{16}{3}a^3$$

8.(10分) 设$u(x,y)$是闭矩形$D:[a,b]\times[c,d]$上的连续函数,在D上存在连续的二阶偏导数,并且$u(x,y)=0$对D的边界上任意一点都成立,试证明

$$\iint_D |u(x,y)|^2\,\mathrm{d}\sigma \leqslant \left(\iint_D \left|\frac{\partial u}{\partial x}\right|\mathrm{d}\sigma\right)\left(\iint_D \left|\frac{\partial u}{\partial y}\right|\mathrm{d}\sigma\right)$$

Solution.

考虑Newton-Lebniz公式,对任意$(x,y)\in D$有

$$u(x,y)=\int_a^x \frac{\partial u}{\partial x}(t,y)\mathrm{d}t=\int_c^y \frac{\partial u}{\partial y}(x,s)\mathrm{d}s$$

于是

$$|u(x,y)|\leqslant \int_a^x \left|\frac{\partial u}{\partial x}(t,y)\right|\mathrm{d}t \leqslant \int_a^b \left|\frac{\partial u}{\partial x}(t,y)\right|\mathrm{d}t$$

同理

$$|u(x,y)|\leqslant \int_c^y \left|\frac{\partial u}{\partial y}(x,s)\right|\mathrm{d}s \leqslant \int_c^d \left|\frac{\partial u}{\partial y}(x,s)\right|\mathrm{d}s$$

于是

$$|u(x,y)|^2\leqslant \left(\int_a^b \left|\frac{\partial u}{\partial x}(t,y)\right|\mathrm{d}t\right)\left(\int_c^d \left|\frac{\partial u}{\partial y}(x,s)\right|\mathrm{d}s\right)$$

对$(x,y)\in D$做二重积分有

$$\begin{aligned}\iint_D |u(x,y)|^2\,\mathrm{d}\sigma &\leqslant \left(\int_c^d \mathrm{d}y\int_a^b \left|\frac{\partial u}{\partial x}(t,y)\right|\mathrm{d}t\right)\left(\int_a^b \mathrm{d}x\int_c^d \left|\frac{\partial u}{\partial y}(x,s)\right|\mathrm{d}s\right)\\ &\leqslant \left(\iint_D \left|\frac{\partial u}{\partial x}\right|\mathrm{d}\sigma\right)\left(\iint_D \left|\frac{\partial u}{\partial y}\right|\mathrm{d}\sigma\right)\end{aligned}$$

9.(15分) 在如下的常微分方程

$$\begin{cases} u''(x)+\lambda u(x)=0\\ -a_1u'(0)+a_2u(0)=b_1u'(L)+b_2u(L)=0\end{cases}$$

中,我们想求解定义在$[0,L]$上的函数$u(x)$,其中参数a_1,a_2,b_1,b_2,L都是给定的函数.

对于大多数参数λ,上述常微分方程并没有解.如果对于某个λ,上述方程存在不恒等于0的解,我们就成λ为一个**本征值**,对应的解$u_\lambda(x)$称为**本征函数**.据此,回答下列问题.

(1) 当$a_1=-1,b_1=1,a_2=b_2=0$时,试证明0是本征值.

(2) 对于$a_1=-1,b_1=1,a_2=b_2=0$以及$a_1=b_1=0,a_2=b_2=1$的情形,分别求解所有本征值和每个本征值对应的所有本征函数.

(3) 如果$a_1, a_2, b_1, b_2 > 0$,试不通过求解方程证明所有本征值都是正数.

(4) 如果$a_1, a_2, b_1, b_2 > 0$,设λ和μ是两个不同的本征值,对应本征函数分别为$u_\lambda(x)$和$u_\mu(x)$.试证明

$$\int_0^L u_\lambda(x) u_\mu(x) \mathrm{d}x = 0$$

Proof.

(1) 由题意有

$$u'(0) = u'(L) = 0$$

不妨令$u(x) = C(C \neq 0)$,于是$u''(x) = 0$且有$u'(0) = u'(L) = 0$.

于是0是方程的本征值.

(2) 对于$-a_1 = b_1 = 1, a_2 = b_2 = 0$的情形有$u'(0) = u'(L) = 0$.

对于$a_1 = b_1 = 0, a_2 = b_2 = 1$的情形有$u(0) = u(L) = 0$.

若$\lambda \leqslant 0$,则特征根为重根$\sqrt{-\lambda}$,通解为$u(x) = \mathrm{e}^{\sqrt{-\lambda}x}\left(C_1 x + C_2\right)$,其中$C_1, C_2$不全为0.

此时又有$u'(x) = \mathrm{e}^{\sqrt{-\lambda}x}\left(\sqrt{-\lambda}C_1 x + \sqrt{-\lambda}C_2 + C_1\right)$.

要求$u'(0) = u'(L) = 0$,则$\sqrt{-\lambda}C_2 + C_1 = \mathrm{e}^{\sqrt{-\lambda}L}\left(\sqrt{-\lambda}C_1 L + \sqrt{-\lambda}C_2 + C_1\right)$,这只有$\lambda = 0$时才成立.

要求$u(0) = u(L) = 0$,则$C_2 = \mathrm{e}^{\sqrt{-\lambda}L}\left(C_1 L + C_2\right)$,这也只有$\lambda = 0$时才成立.

若$\lambda > 0$,则特征根为共轭复根$\pm\sqrt{\lambda}i$,通解为$u(x) = C_1 \sin\sqrt{\lambda}x + C_2 \cos\sqrt{\lambda}x$,其中$C_1, C_2$不全为0.

要求$u'(0) = u'(L) = 0$,则$C_1 = C_1 \cos\sqrt{\lambda}L - C_2 \sin\sqrt{\lambda}L = 0$.于是$L = \dfrac{k\pi}{\sqrt{\lambda}}$.

要求$u(0) = u(L) = 0$,则$C_2 = C_1 \sin\sqrt{\lambda}L + C_2 \cos\sqrt{\lambda}L$,于是$L = \dfrac{k\pi}{\sqrt{\lambda}}$.

于是对于$-a_1 = b_1 = 1, a_2 = b_2 = 0$的情形,本征值$\lambda \geqslant 0$,对应的本征函数为

$$u(x) = \begin{cases} C(C \neq 0), \lambda = 0 \\ C\cos\dfrac{k\pi x}{L}(C \neq 0, k \in \mathbb{Z}), \lambda > 0 \end{cases}$$

对于$a_1 = b_1 = 0, a_2 = b_2 = 1$的情形,本征值$\lambda \geqslant 0$,对应的本征函数为

$$u(x) = \begin{cases} C(C \neq 0), \lambda = 0 \\ C\sin\dfrac{k\pi x}{L}(C \neq 0, k \in \mathbb{Z}), \lambda > 0 \end{cases}$$

(3) 在$u''(x) + \lambda u(x) = 0$两端同乘$u(x)$后在$[0, L]$上定积分有

$$\begin{aligned} 0 &= \int_0^L u''(x)u(x)\mathrm{d}x + \lambda\int_0^L (u(x))^2\,\mathrm{d}x \\ &= u(x)u'(x)|_0^L - \int_0^L (u'(x))^2\,\mathrm{d}x + \lambda\int_0^L (u(x))^2\,\mathrm{d}x \\ &= -\frac{b_2}{b_1}(u(L))^2 - \frac{a_2}{a_1}(u(0))^2 - \int_0^L (u'(x))^2\,\mathrm{d}x + \lambda\int_0^L (u(x))^2\,\mathrm{d}x \end{aligned}$$

前三项都是非正的,这就要求$\lambda \geqslant 0$.

又$\lambda = 0$时要求

$$(u(L))^2 = (u(0))^2 = 0 \qquad (u'(x))^2 \equiv 0$$

于是$u(x) \equiv 0$,这与λ是本征值相悖.

于是必有$\lambda > 0$.

(4) 我们有

$$\begin{cases} u''_\lambda(x) + \lambda u_\lambda(x) = 0 \\ u''_\mu(x) + \mu u_\mu(x) = 0 \end{cases}$$

分别同乘$u_\mu(x)$和$u_\lambda(x)$在$[0, L]$上定积分有

$$\begin{cases} \int_0^L u''_\lambda(x) u_\mu(x) \mathrm{d}x + \lambda \int_0^L u_\lambda(x) u_\mu(x) \mathrm{d}x = 0 \\ \int_0^L u''_\mu(x) u_\lambda(x) \mathrm{d}x + \mu \int_0^L u_\mu(x) u_\lambda(x) \mathrm{d}x = 0 \end{cases}$$

而

$$\begin{aligned} & \int_0^L u''_\lambda(x) u_\mu(x) \mathrm{d}x - \int_0^L u''_\mu(x) u_\lambda(x) \mathrm{d}x \\ = & u_\mu(x) u'_\lambda(x) \Big|_0^L - \int_0^L u'_\lambda(x) u'_\mu(x) \mathrm{d}x - \left(u_\lambda(x) u'_\mu(x) \Big|_0^L - \int_0^L u'_\mu(x) u'_\lambda(x) \mathrm{d}x \right) \\ = & u_\mu(x) u'_\lambda(x) - u'_\mu(x) u_\lambda(x) \Big|_0^L \\ = & 0 \end{aligned}$$

于是

$$(\lambda - \mu) \int_0^L u_\lambda(x) u_\mu(x) \mathrm{d}x = 0$$

如果后面的定积分不为0,这就要求$\lambda = \mu$,与题设不符.于是

$$\int_0^L u_\lambda(x) u_\mu(x) \mathrm{d}x = 0$$