Taylor展开

Example 1.

求函数

$$f(x)=\sqrt{1-2x+x^3}-\sqrt{1-3x+x^2}$$

在$x=0$处的三阶泰勒展开式.

Solution(Method I).

函数$g(x)=\sqrt{1+x}$在$x=0$处的三阶泰勒展开式为

$$f(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+o\left(x^3\right)$$

于是$g(x)$的三阶泰勒展开式为

$$\begin{aligned}
g(x)=&\left(1+\frac{x^3-2x}{2}-\frac{\left(x^3-2x\right)^2}{8}+\frac{\left(x^3-2x\right)^3}{16}+o\left(\left(x^3-2x\right)^3\right)\right)\\
&-\left(1+\frac{x^2-3x}{2}-\frac{\left(x^2-3x\right)^2}{8}+\frac{\left(x^2-3x\right)^3}{16}+o\left(\left(x^2-3x\right)^3\right)\right)\\
=&\left(1+\frac{x^3-2x}{2}-\frac{x^2}{2}+\frac{x^3}{2}+o(x^3)\right)-\left(1+\frac{x^2-3x}{2}-\frac{-6x^3+9x^2}{8}-\frac{27x^3}{16}+o(x^3)\right)\\
=&\frac{1}{2}x+\frac{1}{8}x^2+\frac{15}{16}x^3+o(x^3)
\end{aligned}$$

Solution(Method II).

您当然可以求导,这里就不再赘述了.

Example 2(2019Winter PKU高等数学B期末考试).

设$f(x)$在$\mathbb{R}$上有三阶导数,且存在$M_0,M_3>0$使得$\forall x\in\mathbb{R},|f(x)|\leqslant M_0,\left|f^{(3)}(x)\right|\leqslant M_3$.

试证明存在$M_1,M_2>0$使得$\forall x\in\mathbb{R},|f'(x)|\leqslant M_1,|f''(x)|\leqslant M_2$.

进一步的,试证明$M_1\leqslant 4M_0^{\frac{2}{3}}M_3^{\frac{1}{3}},M_2\leqslant 4M_0^{\frac{1}{3}}M_3^{\frac{2}{3}}$.

Proof.

将$f(x)$在$x=x_0$处泰勒展开.

$$f(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2f''(x_0)}{2!}+\frac{(x-x_0)^3f^{(3)}(\xi)}{3!},x_0\leqslant\xi\leqslant x$$

对于任意$x_0\in\mathbb{R}$,取$x=x_0+h,x_0-h$有

$$\begin{cases} f(x_0+h)-f(x_0)=hf'(x_0)+\dfrac{h^2f''(x_0)}{2}+\dfrac{h^3f^{(3)}(\xi_+)}{6} \\ f(x_0-h)-f(x_0)=-hf'(x_0)+\dfrac{h^2f''(x_0)}{2}-\dfrac{h^3f^{(3)}(\xi_-)}{6} \end{cases}$$

两式相加和相减可知

$$\begin{cases} f(x_0+h)+f(x_0-h)-2f(x_0)=h^2f''(x_0)+\dfrac{h^3}{6}\left(f^{(3)}(\xi_+)-f^{(3)}(\xi_-)\right) \\ f(x_0+h)-f(x_0-h)=2hf'(x_0)+\dfrac{h^3}{6}\left(f^{(3)}(\xi_+)+f^{(3)}(\xi_-)\right) \end{cases}$$

进而

$$|f''(x_0)|\leqslant\left(\frac{1}{h^2}\left(4M_0+\frac{h^3}{3}M_3\right)\right)_{\min}=3\sqrt[3]{\frac{4M_0}{h^2}\cdot\frac{M_3h}{6}\cdot\frac{M_3h}{6}}=\sqrt[3]{3}M_0^{\frac{1}{3}}M_3^{\frac{2}{3}}$$

同理

$$|f'(x_0)|\leqslant\sqrt[3]{\frac{9}{8}}M_0^{\frac{2}{3}}M_3^{\frac{1}{3}}$$

于是命题得证.

Example 3.

详见Hardy-Littlewood引理.