极限练习

Problem 1.

求序列极限

$$\lim_{n\to\infty}\prod_{i=1}^{n}\left(1+\frac{i}{n^2}\right)$$

Solution.

熟知当$x\in(1,+\infty)$时有

$$1-\frac{1}{x}<\ln x<x-1$$

我们对原式两端取对数后可得

$$\sum_{i=1}^{n}\left(1-\frac{1}{1+\frac{i}{n^2}}\right)<\sum_{i=1}^{n}\ln\left(1+\frac{i}{n^2}\right)<\sum_{i=1}^{n}\frac{i}{n^2}$$

即

$$\sum_{i=1}^{n}\frac{i}{n^2+i}<\sum_{i=1}^{n}\ln\left(1+\frac{i}{n^2}\right)<\frac{n+1}{2n}$$

又$i\in[1,n]$时

$$\frac{i}{n^2+n}<\frac{i}{n^2+i}<\frac{i}{n^2}$$

于是

$$\frac{1}{2}=\sum_{i=1}^{n}\frac{i}{n^2+n}<\sum_{i=1}^{n}\frac{i}{n^2+i}<\sum_{i=1}^{n}\frac{i}{n^2}=\frac{1}{2}+\frac{1}{2n}$$

夹逼可得

$$\lim_{n\to\infty}\sum_{i=1}^{n}\ln\left(1+\frac{i}{n^2}\right)=\frac{1}{2}$$

于是

$$\lim_{n\to\infty}\prod_{i=1}^{n}\left(1+\frac{i}{n^2}\right)=\sqrt{e}$$

Problem 2.

设$S_n=1^1+2^2+3^3+\cdots+n^n$,试证明:当$n\geqslant 2$时有

$$n^n\left(1+\frac{1}{4(n-1)}\right)\leqslant S_n<n^n\left(1+\frac{2}{\mathrm{e}(n-1)}\right)$$

Proof.

记$A_n=\dfrac{S_n}{n^n}$,则

$$\begin{aligned}A_n&=\frac{\sum_{i=1}^{n}i^i}{n^n}=1+\frac{\sum_{i=1}^{n-1}i^i}{n^n}\\&=1+\frac{(n-1)^{n-1}}{n^n}\cdot\frac{\sum_{i=1}^{n-1}i^i}{(n-1)^{n-1}}\\&=1+\frac{(n-1)^{n-1}}{n^n}A_{n-1}\\&=1+\frac{(n-1)^{n-1}}{n^n}\left(1+\frac{(n-2)^{n-2}}{(n-1)^{n-1}}A_{n-2}\right)\end{aligned}$$

一方面,我们有

$$\begin{aligned}A&=1+\frac{(n-1)^{n-1}}{n^n}\left(1+\frac{(n-2)^{n-2}}{(n-1)^{n-1}}A_{n-2}\right)\\&\leqslant 1+\frac{(n-1)^{n-1}}{n^n}\left(1+\frac{(n-2)^{n-2}}{(n-1)^{n-1}}\cdot\frac{(n-2)(n-2)^{n-2}}{(n-2)^{n-2}}\right)\\&=1+\left(1-\frac{1}{n}\right)^n\cdot\frac{1}{n-1}\left(1+\frac{(n-2)^{n-1}}{(n-1)^{n-1}}\right)\\&<1+\frac{1}{\mathrm{e}(n-1)}(1+1)\\&=1+\frac{2}{\mathrm{e}(n-1)}\end{aligned}$$

另一方面,我们有

$$\begin{aligned}A&=1+\frac{(n-1)^{n-1}}{n^n}\left(1+\frac{(n-2)^{n-2}}{(n-1)^{n-1}}A_{n-2}\right)\\&\geqslant 1+\frac{(n-1)^{n-1}}{n^n}\\&=1+\left(1-\frac{1}{n}\right)^n\cdot\frac{1}{n-1}\\&\geqslant 1+\left(1-\frac{1}{2}\right)^2\cdot\frac{1}{n-1}\\&=1+\frac{1}{4(n-1)}\end{aligned}$$

于是

$$1+\frac{1}{4(n-1)}<\frac{S_n}{n^n}<1+\frac{2}{\mathrm{e}(n-1)}$$

于是原命题得证.

Problem 3.

对于实数$p>1$,序列$\{a_n\}$满足

$$a_n=\sum_{i=1}^{n}\frac{1}{i^p}$$

试证明$\{a_n\}$收敛.

Proof.

我们仿照证明调和级数发散的方法放缩之,则$n \geqslant 2$时有

$$\begin{aligned}\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p} &< \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{(2^n-1)^p} \\ &< \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{2^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\cdots+\frac{2^{n-1}}{2^{(n-1)p}} \\ &= \sum_{i=1}^{n}\frac{2^{i-1}}{2^{p(i-1)}}=\sum_{i=1}^{n}\left(2^{1-p}\right)^{i-1} \\ &= \frac{1-2^{(1-p)n}}{1-2^{1-p}}<\frac{1}{1-2^{1-p}}\end{aligned}$$

又$\{a_n\}$单调递增.于是$\{a_n\}$存在极限.

Problem 4.

设序列$\{a_n\}$满足

$$a_n=\sum_{i=1}^{n}\frac{1}{\sqrt{i}}-2\sqrt{n}$$

试证明$\{a_n\}$收敛.

Analysis.

根据朴素的认识,我们有

$$\sum_{i=1}^{n}\frac{1}{\sqrt{i}} \simeq \int_1^n \frac{1}{\sqrt{x}}\mathrm{d}x=2\sqrt{n}-2$$

我们要做的就是通过放缩证明求和与积分的差距越来越小.

Proof.

构造函数$f(x)=\dfrac{1}{\sqrt{x}}$,则$\forall k \in \mathbb{N}^*$有

$$f(k+1)<f(x)<f(k)$$

于是

$$\int_k^{k+1} f(x)\mathrm{d}x<\int_k^{k+1} f(k)\mathrm{d}x=\frac{1}{\sqrt{k}}$$

$$\int_k^{k+1} f(x)\mathrm{d}x>\int_k^{k+1} f(k+1)\mathrm{d}x=\frac{1}{\sqrt{k+1}}$$

于是

$$\sum_{i=1}^{n}\frac{1}{\sqrt{i}}>\sum_{i=1}^{n}\int_i^{i+1} f(x)\mathrm{d}x=\int_1^{n+1} f(x)\mathrm{d}x=2\sqrt{n+1}-2$$

$$\sum_{i=1}^{n}\frac{1}{\sqrt{i}}<\sum_{i=1}^{n}\int_{i-1}^{i} f(x)\mathrm{d}x=\int_0^{n} f(x)\mathrm{d}x=2\sqrt{n}$$

于是

$$2\sqrt{n+1}-2\sqrt{n}-2<a_n<0$$

即

$$-2<a_n<0$$

又

$$a_{n+1}-a_n=\frac{1}{\sqrt{n+1}}-2\sqrt{n+1}+2\sqrt{n}=\frac{1}{\sqrt{n+1}}-\frac{2}{\sqrt{n+1}+\sqrt{n}}<0$$

即$\{a_n\}$单调递减.于是原命题得证.

Problem 5.

设序列$\{a_n\}$满足

$$a_n=\underbrace{\sqrt{1+\sqrt{1+\cdots+\sqrt{1}}}}_{n\text{重根号}}$$

试证明$\lim\limits_{n\to\infty}a_n$存在,并求其值.

Proof.

观察递推式,我们可以得到

$$a_{n+1}=\sqrt{1+a_n}$$

下面证明$\forall n\in\mathbb{N}^*,1\leqslant a_n<\dfrac{\sqrt{5}+1}{2}$.

首先,$n=1$时$a_1=1$,成立.

假定$n=k$时上述命题成立,那么当$n=k+1$时有

$$\sqrt{1+1}\leqslant a_{k+1}<\sqrt{1+\frac{\sqrt{5}+1}{2}}$$

即

$$\sqrt{2}\leqslant a_{k+1}<\frac{\sqrt{5}}{2}$$

于是

$$\frac{a_{n+1}}{a_n}=\frac{\sqrt{1+a_n}}{a_n}=\sqrt{\frac{1}{a_n^2}+\frac{1}{a_n}}>1$$

由此可得$\{a_n\}$递增且有上界.设$\lim_{n\to\infty}a_n=A$,对递推式两边求极限有

$$A=\sqrt{1+A}$$

解得$A=\dfrac{\sqrt{5}+1}{2}$,于是$\lim\limits_{n\to\infty}a_n=\dfrac{\sqrt{5}+1}{2}$.

Problem 6.

设序列$\{a_n\}$满足

$$a_n=\underbrace{\sqrt{1+\sqrt{2+\cdots+\sqrt{n}}}}_{n\text{重根号}}$$

试证明$\lim\limits_{n\to\infty}a_n$存在.

Solution(Method I).

我们有

$$1<\sqrt{1+\sqrt{2+\cdots+\sqrt{n}}}<\sqrt{4+\sqrt{16+\cdots+\sqrt{2^{2^n}}}}=\frac{1}{2}\sqrt{1+\sqrt{1+\cdots+\sqrt{1}}}<\sqrt{5}+1$$

又$\{a_n\}$显然递增.于是原命题得证.

Solution(Method II).

注意到$\forall n>4,\sqrt{2n}\leqslant n-1$.

于是

$$\begin{aligned}a_n&=\sqrt{1+\sqrt{2+\cdots+\sqrt{n-1+\sqrt{n}}}}\\&<\sqrt{1+\sqrt{2+\cdots+\sqrt{n-1+\sqrt{2n}}}}\\&\leqslant\sqrt{1+\sqrt{2+\cdots+\sqrt{n-2+\sqrt{2(n-1)}}}}\\&<\cdots\\&<\sqrt{1+\sqrt{2+\sqrt{3+2\sqrt{4}}}}\end{aligned}$$

即$\{a_n\}$有上界.又$\{a_n\}$显然递增,于是原命题得证.

Problem 7.

设序列$\{a_n\}$满足

$$a_n=\underbrace{\sqrt[n]{n+\sqrt[n]{n+\cdots+\sqrt[n]{n}}}}_{n\text{重根号}}$$

试证明$\lim\limits_{n\to\infty}a_n$存在,并求其值.

Proof.

注意到$n>2$时$\sqrt[n]{n+2}<2$,于是

$$\begin{aligned}
a_n &= \sqrt[n]{n+\sqrt[n]{n+\cdots+\sqrt[n]{n+\sqrt[n]{n}}}} \\
&< \sqrt[n]{n+\sqrt[n]{n+\cdots+\sqrt[n]{n+2}}} \\
&< \cdots \\
&< \sqrt[n]{n+2}
\end{aligned}$$

即

$$\sqrt[n]{n} < a_n < \sqrt[n]{n+2} \quad \forall n > 2$$

对上述不等式两边求极限,夹逼可得$\lim\limits_{n\to\infty} a_n = 1$.

Problem 8.
设$x_n = \dfrac{1}{n^2}\sum\limits_{k=0}^{n} \ln\dbinom{n}{k}$,求$\lim\limits_{n\to\infty} x_n$.

Solution.
置$a_n = \sum\limits_{k=0}^{n} \ln\dbinom{n}{k}$, $b_n = n^2$.
于是据Stolz定理有

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$

而

$$a_{n+1}-a_n = \ln\left(\frac{(n+1)!^{n+2}\prod_{k=0}^{n} k!^2}{n!^{n+1}\prod_{k=0}^{n+1} k!^2}\right) = \ln\frac{(n+1)!^n}{n!^{n+1}} = \ln\frac{(n+1)^n}{n!}$$

于是

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{1}{2n+1}\ln\frac{(n+1)^n}{n!}$$

置$c_n = \ln\dfrac{(n+1)^n}{n!}$, $d_n = 2n+1$.
于是据Stolz定理有

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} &= \lim_{n\to\infty} \frac{c_n}{d_n} = \lim_{n\to\infty} \frac{c_n-c_{n-1}}{d_n-d_{n-1}} \\
&= \frac{1}{2}\lim_{n\to\infty} \ln\left(\frac{(n+1)^n}{n!}\cdot\frac{(n-1)!}{n^{n-1}}\right) \\
&= \frac{1}{2}\lim_{n\to\infty} \ln\left[\left(\frac{n+1}{n}\right)^n\right] \\
&= \frac{1}{2}\ln \mathrm{e} = \frac{1}{2}
\end{aligned}$$

Problem 9.

序列$\{a_n\}$满足$\lim\limits_{n\to\infty} a_n\sum\limits_{i=1}^{n}a_i^2=1$,试证明: $\lim\limits_{n\to\infty}\sqrt[3]{3n}a_n=1$.

Proof.

置$S_n=\sum\limits_{i=1}^{n}a_i^2$,于是

$$\lim_{n\to\infty}\sqrt[3]{3n}a_n=\lim_{n\to\infty}\sqrt[3]{3na_n^3}=\lim_{n\to\infty}\sqrt[3]{\frac{3n}{S_n^3}\cdot(a_nS_n)^3}=\lim_{n\to\infty}\sqrt[3]{\frac{3n}{S_n^3}}$$

依Stolz定理有

$$\begin{aligned}\lim_{n\to\infty}\frac{S_n^3}{3n}&=\lim_{n\to\infty}\frac{S_n^3-S_{n-1}^3}{3n-3(n-1)}\\&=\lim_{n\to\infty}\frac{(S_n-S_{n-1})(S_n^2+S_nS_{n-1}+S_{n-1}^2)}{3}\\&=\lim_{n\to\infty}\frac{a_n^2(S_n^2+S_nS_{n-1}+S_{n-1}^2)}{3}\\&=\lim_{n\to\infty}(a_nS_n)^2\frac{1+\frac{S_{n-1}}{S_n}+\left(\frac{S_{n-1}}{S_n}\right)^2}{3}\end{aligned}$$

下面证明$\lim\limits_{n\to\infty}a_n=0$.

若否,则$\exists\varepsilon>0,\forall N\in\mathbb{N}^*,\exists n>N,\text{s.t.}\ |a_n|>\varepsilon$,也即$\{a_n\}$中有无穷多项$a_i$满足$|a_i|>\varepsilon$.

我们取这样的$\left[\dfrac{1}{\varepsilon^3}\right]+2$个$a_i$,记$\max\limits_{|a_i|>\varepsilon}\{i\}=N_0$,于是当$n>N_0$且$|a_n|>\varepsilon$时总有

$$a_n\sum_{i=1}^{n}a_i^2>\varepsilon\sum_{i=1}^{\left[\frac{1}{\varepsilon^3}\right]+2}\varepsilon^2>1+\varepsilon^3$$

于是$\exists\varepsilon'=\varepsilon^3>0,\forall N'\in\mathbb{N}^*,\exists n=\max\{N',N_0\}$ 且 $|a_n|>\varepsilon,\text{s.t.}\ |a_nS_n-1|>\varepsilon'$.

这与题设中$\lim\limits_{n\to\infty}a_n\sum\limits_{i=1}^{n}a_i^2=1$矛盾,于是$\lim\limits_{n\to\infty}a_n=0$. 从而

$$\lim_{n\to\infty}\frac{S_{n-1}}{S_n}=1-\lim_{n\to\infty}\frac{a_n^2}{S_n}=1-\lim_{n\to\infty}a_n^3=1$$

从而

$$\lim_{n\to\infty}\frac{S_n^3}{3n}=1$$

于是原命题得证.