古早计算大赛题目

试证明:球坐标系下Laplace算子为

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Solution.

我们知道笛卡尔坐标系下的Laplace算子为

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

以及笛卡尔坐标系向球坐标系的变换

$$\begin{cases}x=r\sin\theta\cos\phi\\ y=r\sin\theta\sin\phi\\ z=r\cos\theta\end{cases}$$

考虑函数$f(x,y,z):\mathbb{R}^3\to\mathbb{R}$,则有

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)=\frac{1}{r^2}\left(2r\frac{\partial f}{\partial r}+r^2\frac{\partial^2 f}{\partial r^2}\right)=\frac{2}{r}\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial r^2}$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)=\frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial f}{\partial\theta}+\sin\theta\frac{\partial^2 f}{\partial\theta^2}\right)=\frac{1}{r^2\tan\theta}\frac{\partial f}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}$$

而

$$\frac{\partial^2 f}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right)=\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}\right)$$

我们知道

$$\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}\right)=\frac{\partial^2 x}{\partial r^2}\cdot\frac{\partial f}{\partial x}+\frac{\partial^2 f}{\partial x^2}\cdot\left(\frac{\partial x}{\partial r}\right)^2$$

而

$$\frac{\partial x}{\partial r}=\sin\theta\cos\phi,\frac{\partial^2 x}{\partial r^2}=0$$
$$\frac{\partial y}{\partial r}=\sin\theta\sin\phi,\frac{\partial^2 y}{\partial r^2}=0$$
$$\frac{\partial z}{\partial r}=\cos\theta,\frac{\partial^2 z}{\partial r^2}=0$$

代入可得

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)&=\sin^2\theta\cos^2\phi\frac{\partial^2 f}{\partial x^2}+\sin^2\theta\sin^2\phi\frac{\partial^2 f}{\partial y^2}+\sin^2\theta\frac{\partial^2 f}{\partial z^2}\\&+\frac{2}{r}\left(\sin\theta\cos\phi\frac{\partial f}{\partial x}+\sin\theta\sin\phi\frac{\partial f}{\partial y}+\cos\theta\frac{\partial f}{\partial z}\right)\end{aligned}$$

又

$$\frac{\partial x}{\partial\theta}=r\cos\theta\cos\phi,\frac{\partial^2 x}{\partial\theta^2}=-r\sin\theta\cos\phi$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \frac{\partial^2 y}{\partial \theta^2} = -r\sin\theta\sin\phi$$

$$\frac{\partial z}{\partial \theta} = -r\sin\theta, \frac{\partial^2 z}{\partial \theta^2} = -r\cos\theta$$

于是

$$\begin{aligned}\frac{1}{r^2\tan\theta}\frac{\partial f}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} &= \cos^2\theta\cos^2\phi\frac{\partial^2 f}{\partial x^2} + \cos^2\theta\sin^2\phi\frac{\partial^2 f}{\partial y^2} + \cos^2\theta\frac{\partial^2 f}{\partial z^2} \\ &+ \frac{1}{r}\left(\frac{\cos 2\theta}{\sin\theta}\cos\phi\frac{\partial f}{\partial x} + \frac{\cos 2\theta}{\sin\theta}\sin\phi\frac{\partial f}{\partial y} - 2\cos\theta\frac{\partial f}{\partial z}\right)\end{aligned}$$

又

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi, \frac{\partial^2 x}{\partial \phi^2} = -r\sin\theta\cos\phi$$

$$\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi, \frac{\partial^2 y}{\partial \phi^2} = -r\sin\theta\sin\phi$$

$$\frac{\partial z}{\partial \phi} = \frac{\partial^2 z}{\partial \phi^2} = 0$$

于是

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} = \sin^2\phi\frac{\partial^2 f}{\partial x^2} + \cos^2\phi\frac{\partial^2 f}{\partial y^2} - \frac{1}{r}\left(\frac{\cos\phi}{\sin\theta}\frac{\partial f}{\partial x} + \frac{\sin\phi}{\sin\theta}\frac{\partial f}{\partial y}\right)$$

将三项相加,考虑如下几项偏导数的系数

$$\begin{aligned}\frac{\partial f}{\partial x} &: \frac{1}{r}\left(2\sin\theta\cos\phi + \frac{\cos 2\theta}{\sin\theta}\cos\phi - \frac{\cos\phi}{\sin\theta}\right) = \frac{\cos\phi}{r\sin\theta}\left(2\sin^2\theta - 1 + \cos 2\theta\right) = 0 \\ \frac{\partial f}{\partial y} &: \frac{1}{r}\left(2\sin\theta\sin\phi + \frac{\cos 2\theta}{\sin\theta}\sin\phi - \frac{\sin\phi}{\sin\theta}\right) = \frac{\sin\phi}{r\sin\theta}\left(2\sin^2\theta - 1 + \cos 2\theta\right) = 0 \\ \frac{\partial f}{\partial z} &: \frac{2}{r}\cos\theta - \frac{1}{r}\cdot 2\cos\theta = 0 \\ \frac{\partial^2 f}{\partial x^2} &: \sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\phi = 1 \\ \frac{\partial^2 f}{\partial y^2} &: \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi + \cos^2\phi = 1 \\ \frac{\partial^2 f}{\partial y^2} &: \sin^2\theta + \cos^2\theta = 1\end{aligned}$$

于是

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

从而原命题得证.

Solution.

考虑$f(x)$的n个根$x_1, \cdots, x_n$满足$x_1 \leqslant \cdots \leqslant x_n$.

若$x_k < x_{k+1}$,则有$f(x_k) = f(x_{k+1}) = 0$.据Rolle中值定理可知存在$\xi_k \in (x_k, x_{k+1})$使得$f'(\xi) = 0$.

若$x_{k-1} < x_k = x_{k+1} = \cdots = x_{k+j-1} < x_{k+j}$,即$x_k$是$f(x) = 0$的$j$重根,则可将$f(x)$写作$f(x) = (x - x_k)^j g(x)$.

于是$f'(x) = (jg(x) + (x - x_j)g'(x))(x - x_k)^{j-1}$,因而$f'(x_k) = 0$在$x = x_k$处至少有$j - 1$重根.

于是我们证明了$f'(x)$在$[a, b]$上至少有$n - 1$个根(包括重根).

如此递推可知$f^{(n-1)}(x)$在$[a, b]$上至少有1个根.