Lecture 6 Surface integral(曲面积分)

L.6.1 求曲面积分$\iint_S x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y$,其中$S$是抛物面$z = x^2 + y^2$被平面$z = 4$截出的有限部分,积分方向为外侧.

Solution.

令$T = \{(x, y, z)|x^2 + y^2 \leqslant 4, z = 4\}$,$D = \{(x, y)|x^2 + y^2 \leqslant 4\}$.

令Ω为S和T围成的区域,不难发现Ω的外侧边界即为$S^+ \cup T^+$.于是根据高斯公式有

$$\iint_{S^+} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y + \iint_{T^+} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y = 3\iiint_\Omega \mathrm{d}V$$

而

$$\iiint_\Omega \mathrm{d}V = \int_0^{2\pi} \mathrm{d}\theta \int_0^2 \mathrm{d}r \int_{r^2}^4 r\mathrm{d}z = 8\pi$$

$$\iint_{T^+} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y = 4\iint_D \mathrm{d}\sigma = 16\pi$$

于是

$$\iint_{S^+} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y = 4\iint_D \mathrm{d}\sigma = 3\cdot 8\pi - 16\pi = 8\pi$$

L.6.2 求曲面积分$\iint_S x^3\mathrm{d}y\mathrm{d}z + y^3\mathrm{d}z\mathrm{d}x + z^3\mathrm{d}x\mathrm{d}y$,其中$S$是单位球在第一卦限的部分,积分方向为外侧.

Solution.

Method I.

由于$\mathrm{d}x\mathrm{d}y$表示有向面积微元,当$z < 0$时$\mathrm{d}x\mathrm{d}y$也取负值,故$z^3\mathrm{d}x\mathrm{d}y$一项的积分关于$Oxy$平面对称.

同理可知其余两项也关于各自对应的平面的对称性,于是可将积分区域补全为单位球Ω的外侧边界.于是

$$\begin{aligned}\iint_S x^3\mathrm{d}y\mathrm{d}z + y^3\mathrm{d}z\mathrm{d}x + z^3\mathrm{d}x\mathrm{d}y &= \frac{1}{8}\iiint_\Omega 3\left(x^2 + y^2 + z^2\right)\mathrm{d}V \\ &= \frac{3}{8}\int_0^{2\pi} \mathrm{d}\theta \int_0^\pi \mathrm{d}\varphi \int_0^1 \rho^4 \sin\varphi \mathrm{d}\rho \\ &= \frac{3\pi}{10}\end{aligned}$$

Method II.

注意到单位球在(x,y,z)处向外的法向量为$\mathbf{n}=(x,y,z)$.令S在Oxy平面的投影为D,于是

$$\begin{aligned}
\iint_S x^3\mathrm{d}y\mathrm{d}z+y^3\mathrm{d}z\mathrm{d}x+z^3\mathrm{d}x\mathrm{d}y &= \iint_S \left(x^4+y^4+z^4\right)\mathrm{d}S \\
&= \iint_D \frac{x^4+y^4+\left(1-x^2-y^2\right)^2}{\sqrt{1-x^2-y^2}}\mathrm{d}\sigma \\
&= \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 \frac{r^5\left(1+\sin^4\theta+\cos^4\theta\right)-2r^3+r}{\sqrt{1-r^2}}\mathrm{d}r \\
&\xlongequal{r=\sin u} \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{\frac{\pi}{2}} \sin^5 u\left(1+\sin^4\theta+\cos^4\theta\right)-2\sin^3 u+\sin u\mathrm{d}u \\
&= \int_0^{\frac{\pi}{2}}\left[\frac{8}{15}\left(1+\sin^4\theta+\cos^4\theta\right)-\frac{4}{3}+1\right]\mathrm{d}\theta \\
&= \frac{1}{5}\cdot\frac{\pi}{2}+\frac{8}{15}\cdot 2\cdot\frac{3\pi}{16} \\
&= \frac{3\pi}{10}
\end{aligned}$$

L.6.3 求曲面积分$\iint_S x^2\mathrm{d}y\mathrm{d}z+y^2\mathrm{d}z\mathrm{d}x+xy\mathrm{d}x\mathrm{d}y$,其中$S$是空间区域$\Omega:\frac{(x-1)^2}{4}+\frac{(y-1)^2}{9}\leqslant z\leqslant 1$的外侧边界.

Solution.

根据高斯公式有

$$\iint_S x^2\mathrm{d}y\mathrm{d}z+y^2\mathrm{d}z\mathrm{d}x+xy\mathrm{d}x\mathrm{d}y=\iiint_\Omega(2x+2y)\mathrm{d}V$$

做代换$u=x-1,v=y-1$,则积分区域$\Omega'=\{(u,v,z)|\frac{u^2}{4}+\frac{v^2}{9}\leqslant z\leqslant 1\}$.于是

$$\iiint_\Omega(2x+2y)\mathrm{d}V=\iiint_{\Omega'}(2u+2v+4)\mathrm{d}V$$

由于积分区域关于Ouz平面和Ovz平面对称,于是

$$\iiint_{\Omega'}(2u+2v+4)\mathrm{d}V=4\iiint_{\Omega'}\mathrm{d}V\xlongequal{u=2r\cos\theta,v=3r\sin\theta}4\int_0^{2\pi}\mathrm{d}\theta\int_0^1\mathrm{d}r\int_{r^2}^1 6r\mathrm{d}z=12\pi$$

从而原曲面积分的值为12π.

L.6.4 求曲线积分$\oint_{\Gamma_h}(y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z$,其中$\Gamma_h$为平面$x+y+z=h$(参数$h\in(-1,1)$)截取单位球所得的曲线,取从$z$轴负方向向正方向看的逆时针方向.

Solution.

截取的曲线围成一个圆$S:\begin{cases} x+y+z=h \\ x^2+y^2+z^2=1 \end{cases}$ 根据斯托克斯公式,取S的向下的一侧S^-,则有

$$\oint_{\Gamma_h}(y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z=-2\iint_{S^-}(y+z)\mathrm{d}y\mathrm{d}z+(z+x)\mathrm{d}z\mathrm{d}x+(x+y)\mathrm{d}x\mathrm{d}y$$

由于平面$x+y+z=h$的单位法向量为$\dfrac{1}{\sqrt{3}}(1,1,1)$,于是

$$\iint_{S^+}(y+z)\mathrm{d}y\mathrm{d}z+(z+x)\mathrm{d}z\mathrm{d}x+(x+y)\mathrm{d}x\mathrm{d}y=\frac{2}{\sqrt{3}}\iint_S(x+y+z)\mathrm{d}S=\frac{2h}{\sqrt{3}}\left(1-\frac{1}{3}h^2\right)\pi$$

于是

$$\oint_{\Gamma_h}(y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z=\frac{4h}{\sqrt{3}}\left(1-\frac{1}{3}h^2\right)$$

L.6.5 设S是一光滑闭曲面,围成区域Ω.设函数$u(x,y,z)$和$v(x,y,z)$在$\Omega\cup S$上有连续的二阶偏导数.

(1) 设n_1为单位法向量$\mathbf{n}$在x轴方向上的分量,试证明

$$\iiint_\Omega v\frac{\partial u}{\partial x}\mathrm{d}V=-\iiint_\Omega u\frac{\partial v}{\partial x}\mathrm{d}V+\iint_S uvn_1\mathrm{d}S$$

(2) 设$\mathbf{n}$为单位外法向量,试证明

$$\iiint_\Omega \nabla u\nabla v\mathrm{d}V=-\iiint_\Omega u\Delta v\mathrm{d}V+\iint_S u\frac{\partial v}{\partial \mathbf{n}}\mathrm{d}S$$

Proof.

(1) 我们有$n_1\mathrm{d}S=\mathrm{d}y\mathrm{d}z$,于是

$$\iint_S uvn_1\mathrm{d}S=\iint_S uv\mathrm{d}y\mathrm{d}z=\iiint_\Omega\frac{\partial(uv)}{\partial x}\mathrm{d}V=\iiint_\Omega\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)\mathrm{d}V$$

移项即可得欲证等式.

(2) 我们有

$$\begin{aligned}\iint_S u\frac{\partial v}{\partial \mathbf{n}}\mathrm{d}S&=\iint_S u\frac{\partial v}{\partial x}\mathrm{d}y\mathrm{d}z+u\frac{\partial v}{\partial y}\mathrm{d}z\mathrm{d}x+u\frac{\partial v}{\partial z}\mathrm{d}x\mathrm{d}y\\&=\iiint_\Omega\left[\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(u\frac{\partial v}{\partial z}\right)\right]\mathrm{d}V\\&=\iiint_\Omega\left(\nabla u\nabla v+u\Delta v\right)\mathrm{d}V\end{aligned}$$

移项即可得欲证等式.