北京大学数学科学学院2023-24高等数学B1期末考试

1.(10分)

求极限

$$\lim_{x \to 0} \frac{\sin\left(x - \int_0^x \sqrt{1+t^2}\mathrm{d}t\right)}{x^3}$$

Solution.

我们有

$$\lim_{x \to 0} \frac{\sin\left(x - \int_0^x \sqrt{1+t^2}\mathrm{d}t\right)}{x^3} = \lim_{x \to 0} \frac{\cos\left(x - \int_0^x \sqrt{1+t^2}\mathrm{d}t\right)\left(1 - \sqrt{1+x^2}\right)}{3x^2}$$

$$= 1 \cdot \lim_{x \to 0} \frac{-x^2}{3x^2\left(1 + \sqrt{1-x^2}\right)}$$

$$= 1 \cdot \left(-\frac{1}{3}\right) \cdot \frac{1}{2}$$

$$= -\frac{1}{6}$$

2.(10分)

设函数$f : [0,7] \to \mathbb{R}$为

$$f(x) = x^3 - 6x^2 + 9x - 1$$

称区间$[a,b]$是f的单调区间,当$0 \leqslant a < b \leqslant 7$且限制在$[a,b]$上的$f$严格单调.求$f$的长度最大的单调区间.

Solution.

对$f(x)$求导有

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$$

于是当$x \in [0,1)$或$x \in (3,7]$时$f(x) > 0.x \in (1,3)$时$f(x) < 0.x = 1$或3时$f(x) = 0$.

因而$f(x)$在$[0,1]$单调递增,在$[1,3]$单调递减,在$[3,7]$单调递增.

于是f的长度最大的单调区间为$[3,7]$,其长度为4.

3.(10分)

设欧氏空间$\mathbb{R}^3$中的平面$T : 2x - y + 3z = 6$.设T与x,y,z三轴的交点分别为A,B,C.以原点$O(0,0,0)$为球心,与T相切的球面记作S.

(1) (5分) 求△ABC的面积.

(2) (5分) 求球面S与T相切的点的坐标.

Solution.

(1) 分别令x, y, z三者中的两者为0,可解得$A(3, 0, 0), B(0, -6, 0), C(0, 0, 2)$.于是

$$S_{\triangle ABC} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\begin{vmatrix} i & j & k \\ -3 & -6 & 0 \\ -3 & 0 & 2 \end{vmatrix} = \frac{1}{2}\left|(12, 6, -18)\right| = 3\sqrt{14}$$

(2) 设切点为$P(x, y, z)$.设$\vec{u}$为T的法向量,根据T的一般式可知$\vec{u} = (2, -1, 3)$.

由于S于T相切,因此$\overrightarrow{OP} \perp T$,于是$\overrightarrow{OP} // \vec{u}$.于是可以列出方程组

$$\begin{cases} 2x - y + 3z = 6 \\ \dfrac{x}{2} = -\dfrac{y}{1} = \dfrac{z}{3} \end{cases}$$

解得$x = \dfrac{6}{7}, y = -\dfrac{3}{7}, z = \dfrac{9}{7}$.于是$P\left(\dfrac{6}{7}, -\dfrac{3}{7}, \dfrac{9}{7}\right)$.

4.(10分)

设二元函数$z = f(x, y)$是由方程$F(x, y, z) = z^3 + ze^x + y = 0$确定的隐函数.求$z = f(x, y)$在$(0, 2)$处函数值下降最快的方向上的单位向量.

Solution.

首先求F的偏导,有

$$\frac{\partial F}{\partial x} = ze^x \qquad \frac{\partial F}{\partial y} = 1 \qquad \frac{\partial F}{\partial z} = 3z^2 + e^x$$

当$(x, y) = (0, 2)$时$F(x, y, z) = 0$有$z^3 + z + 2 = 0$,即$(z + 1)(z^2 - z + 2) = 0$.故这方程仅有$z = -1$一实根.

当$(x, y, z) = (0, 2, -1)$时$F_z(0, 2, -1) = 4 \neq 0$.于是根据隐函数存在定理,$F(x, y, z) = 0$在$(0, 2)$处附近确定唯一的隐函数$z = f(x, y)$,并且有

$$\frac{\partial f}{\partial x} = -\frac{F_x(0, 2, -1)}{F_z(0, 2, -1)} = \frac{1}{4} \qquad \frac{\partial f}{\partial y} = -\frac{F_y(0, 2, -1)}{F_z(0, 2, -1)} = -\frac{1}{4}$$

于是f在$(0, 2)$处的梯度向量为$\left(\dfrac{1}{4}, -\dfrac{1}{4}\right)$.

取负梯度后单位化有$\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$,此即所求向量.

5.(10分)

求函数$f(x,y) = x^y$在$(1,1)$处的二阶泰勒多项式.

Solution.

我们有

$$\frac{\partial f}{\partial x} = yx^{y-1} \qquad \frac{\partial f}{\partial y} = x^y \ln x$$

$$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2} \qquad \frac{\partial^2 f}{\partial x \partial y} = x^{y-1}\left(1 + y\ln x\right) \qquad \frac{\partial^2 f}{\partial y^2} = x^y\left(\ln x\right)^2$$

代入$x = 1, y = 1$可知

$$f(x,y) = 1 + (x-1) + (x-1)(y-1)$$

6.(10分)

设D是由直线$x+y = 2\pi$,x轴和y轴围成的有界闭区域.求D上的二元函数$f(x,y) = \sin x + \sin y - \sin(x+y)$达到最大值的$D$中所有点.

Solution.

我们有

$$\frac{\partial f}{\partial x} = \cos x - \cos(x+y) \qquad \frac{\partial f}{\partial y} = \cos y - \cos(x+y)$$

令$\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$,有

$$\cos x = \cos y = \cos(x+y)$$

若$x = y$,则有$\cos x = \cos 2x = 2\cos^2 x - 1$,解得$\cos x = -\dfrac{1}{2}$或$1$.

由于$x, y \geqslant 0$且$x + y \leqslant 2\pi$,于是$x = y = \dfrac{2\pi}{3}$或$x = y = 0$.此时有

$$f\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{2} \qquad f(0,0) = 0$$

若$x + y = 2\pi$,又要求$\cos x = \cos(x+y)$,这是无解的.

于是D中所有使$f(x,y)$取得最大值的点为$\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3}\right)$.

7.(10分)

回答下列问题.

(1) (2分) 举例说明:当z是(x,y)的函数,也是(t,u)的函数时,$x \equiv t \not\Rightarrow \dfrac{\partial z}{\partial x} \equiv \dfrac{\partial z}{\partial t}$.

(2) (8分) 给定方程

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = z^2$$

作变量代换

$$x = t, y = \frac{t}{1+tu}, z = \frac{t}{1+tW}$$

试证明:

$$\frac{\partial W}{\partial t} = 0$$

Solution.

(1) 令$z = x + y = tu + t$,于是$\frac{\partial z}{\partial x} = 1, \frac{\partial z}{\partial t} = 1 + u.$显然当$x = t$时不一定有$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial t}.$

(2)

8.(15分)

证明下列恒等式.

(1) (3分) 对于任意$x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$,有

$$2\int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \int_0^{2x\sqrt{1-x^2}} \frac{\mathrm{d}t}{\sqrt{1-t^2}}$$

(2) (12分) 对于任意$x \in \left(-\frac{1}{\sqrt[4]{6}}, \frac{1}{\sqrt[4]{6}}\right)$,有

$$2\int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^4}} = \int_0^{\frac{2x\sqrt{1-x^4}}{1+x^4}} \frac{\mathrm{d}t}{\sqrt{1-t^4}}$$

Proof.

(1) 令$F(x) = 2\int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^2}}, G(x) = \int_0^{2x\sqrt{1-x^2}} \frac{\mathrm{d}t}{\sqrt{1-t^2}}.$于是$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{2}{\sqrt{1-x^2}}.$

$$\frac{\mathrm{d}G}{\mathrm{d}x} = \frac{2}{\sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}} \cdot \left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right)$$

$$= \frac{2}{\sqrt{4x^4 - 4x^2 + 1}} \cdot \frac{1-2x^2}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

即$F'(x) = G'(x).$又$F(0) = G(0) = 0.$令$H(x) = F(x) - G(x).$

根据Lagrange中值定理,对任意$x \in \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$且$x \neq 0$,存在$\xi \in (x, 0)$或$(0, x)$使得

$$H'(\xi) = \frac{H(x) - H(0)}{x} = 0$$

即$H(x) = H(0) = 0$,于是$F(x) = G(x)$对任意$x \in \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$成立,从而命题得证.

(2) 令$F(x) = 2\displaystyle\int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^4}}, G(x) = \displaystyle\int_0^{\frac{2x\sqrt{1-x^4}}{1+x^4}} \frac{\mathrm{d}t}{\sqrt{1-t^4}}$.于是$\dfrac{\mathrm{d}F}{\mathrm{d}x} = \dfrac{2}{\sqrt{1-x^4}}$.

$$\frac{\mathrm{d}G}{\mathrm{d}x} = \frac{2}{\sqrt{1 - \left(\frac{2x\sqrt{1-x^4}}{1+x^4}\right)^4}} \cdot \left(\frac{\left(\sqrt{1-x^4} - \frac{2x^4}{\sqrt{1-x^4}}\right)(1+x^4) - 4x^4\sqrt{1-x^4}}{(1+x^4)^2}\right)$$

$$= \frac{2}{\sqrt{\frac{x^{16} - 12x^{12} + 38x^8 - 12x^4 + 1}{(1+x^4)^4}}} \cdot \frac{x^8 - 6x^4 + 1}{\sqrt{1-x^4}\,(1+x^4)^2}$$

$$= \frac{2}{\sqrt{\left(\frac{x^8 - 6x^4 + 1}{(1+x^4)^2}\right)^2}} \cdot \frac{x^8 - 6x^4 + 1}{\sqrt{1-x^4}\,(1+x^4)^2}$$

$$= \frac{2}{\sqrt{1-x^4}}$$

于是$F'(x) = G'(x)$,又$F(0) = G(0) = 0$.与**(1)**同理可知$F(x) = G(x)$,命题得证.

9.(15分)

设函数$P(x)$在$[0,1]$连续,有$P(0) = 0, P(1) = 1.P(x)$在$(0,1)$可导,且对任意$x \in (0,1)$有$P'(x) > 0$.任意取定$A, B \in \mathbb{R}, n \in \mathbb{N}^*$.试证明:在$(0,1)$上存在$\theta_0, \cdots, \theta_n \in \mathbb{R}$使得

$$(A+B)^n = \sum_{k=0}^{n} \frac{1}{P'(\theta_k)} \frac{n!}{k!(n-k)!} A^{n-k} B^k$$

并且

$$0 < \theta_0 < \cdots < \theta_n < 1$$

Proof.

设$a = \dfrac{A}{A+B}, b = \dfrac{B}{A+B}$.于是要证的等式即

$$1 = \sum_{k=0}^{n} \frac{1}{P'(\theta_k)} \frac{n!}{k!(n-k)!} a^{n-k} b^k$$

由于对于任意$x \in (0,1), P'(x) > 0$,因而$P(x)$严格单调递增,因而$P(x)$是单射.

据Lagrange中值定理,对于任意$0 \leqslant \alpha < \beta \leqslant 1$,存在$\xi \in (\alpha, \beta)$使得

$$P'(\xi) = \frac{P(\beta) - P(\alpha)}{\beta - \alpha}$$

即

$$\frac{1}{P'(\xi)} = \frac{\beta - \alpha}{P(\beta) - P(\alpha)}$$

现在,考虑序列$\{\phi_i\}_{i=0}^{n+1}$为

$$\phi_0 = 0, \phi_i = \sum_{k=0}^{i-1} \frac{n! a^{n-k} b^k}{k!(n-k!)}$$

于是$\{\phi_i\}$严格递增.记序列$\{\psi_i\}_{i=0}^{n+1}$使得$P(\psi_i) = \phi_i$.由于$P(x)$严格单调递增,于是ψ_i也是严格单调递增序列.
又因为$P(0) = 0 = \phi_0, P(1) = 1 = \phi_{n+1}$,于是$\psi_0 = 0, \psi_{n+1} = 1$.
现在,对于任意$k \in [0, n]$,根据Lagrange中值定理都存在$\xi_k \in (\psi_k, \psi_{k+1})$使得

$$P'(\xi_k) = \frac{\psi_{k+1} - \psi_k}{P(\psi_{k+1}) - P(\psi_k)} = \frac{\psi_{k+1} - \psi_k}{\phi_{k+1} - \phi_k}$$

于是我们有

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{P'(\theta_k)} \frac{n!}{k!(n-k)!} a^{n-k} b^k &= \sum_{k=0}^{n} \frac{\psi_{k+1} - \psi_k}{\phi_{k+1} - \phi_k} \frac{n!}{k!(n-k)!} a^{n-k} b^k \\
&= \sum_{k=0}^{n} (\psi_{k+1} - \psi_k) \\
&= \psi_{n+1} - \psi_0 \\
&= 1
\end{aligned}$$

于是命题得证.