北京大学数学科学学院2024-25高等数学A1期中考试

1.(16分)

解答下列各题.

(1) **(8分)** 若

$$\lim_{x\to 1}\frac{x^2+ax+b}{\sin\left(x^2-1\right)}=\frac{4}{9}$$

求参数a,b的值.

(2) **(8分)** 设函数$f(x)$在开区间(c,d)上连续.试证明: 对于任意$x_1,x_2,\cdots,x_n\in(c,d)$,存在$\xi\in(c,d)$使得$f(\xi)=\dfrac{1}{n}\sum\limits_{i=1}^{n}f(x_i)$.

(1) **Solution.**

注意到

$$\lim_{x\to 1}\frac{x^2-1}{\sin\left(x^2-1\right)}=\lim_{y\to 0}\frac{y}{\sin y}=1$$

$$\lim_{x\to 1}\frac{x-1}{\sin\left(x^2-1\right)}=\lim_{x\to 1}\frac{x^2-1}{\sin\left(x^2-1\right)}\cdot\frac{1}{x+1}=1\cdot\frac{1}{2}=\frac{1}{2}$$

于是

$$\lim_{x\to 1}=\frac{x^2+ax+b}{\sin\left(x^2-1\right)}=\lim_{x\to 1}\frac{x^2-1+a\left(x-1\right)+\left(b-a-1\right)}{\sin\left(x^2-1\right)}$$

于是

$$\begin{cases} b-a-1=0 \\ 1+\dfrac{1}{2}a=\dfrac{4}{9}\end{cases}$$

解得$a=-\dfrac{10}{9},b=\dfrac{1}{9}$.

(2) **Proof.**

根据连续函数的有界性,不妨设$M_f=\max\limits_{1\leqslant i\leqslant n}f(x_i)$在$i=a$处取到, $m_f=\min\limits_{1\leqslant i\leqslant n}f(x_i)$在$i=b$处取到,于是

$$\frac{1}{n}\sum_{i=1}^{n}m_f\leqslant\frac{1}{n}\sum_{i=1}^{n}f(x_i)\leqslant\frac{1}{n}\sum_{i=1}^{n}M_f$$

即

$$f(x_b)=m_f\leqslant\frac{1}{n}\sum_{i=1}^{n}f(x_i)\leqslant M_f=f(x_a)$$

根据连续函数的介值定理,$\exists\xi$满足$x_a\gtreqless\xi\gtreqless x_b$, s.t.$f(\xi)=\dfrac{1}{n}\sum\limits_{i=1}^{n}f(x_i)$,证毕.

2.(16分)

解答下列各题.

(1) **(8分)** 设函数

$$f(x)=\sqrt{x^2+1}\arctan x-\ln\left(x+\sqrt{x^2+1}\right)$$

求$\mathrm{d}f(x)$.

(2) **(8分)** 求函数

$$y=\frac{1}{4\sqrt{2}}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}-\frac{1}{2\sqrt{2}}\arctan\frac{\sqrt{2}x}{x^2-1}$$

的一阶导数y'.

(1) **Solution.**

我们有

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}=\frac{x\arctan x}{\sqrt{x^2+1}}+\frac{1}{\sqrt{x^2+1}}-\frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)=\frac{x\arctan x}{\sqrt{x^2+1}}$$

于是

$$\mathrm{d}f(x)=\frac{x\arctan x}{\sqrt{x^2+1}}\mathrm{d}x$$

(2) **Solution.**

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{1}{4\sqrt{2}}\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\cdot\frac{2\sqrt{2}\left(1-x^2\right)}{\left(x^2-2\sqrt{2}x+1\right)^2}-\frac{2}{\dfrac{x^2}{\left(x^2-1\right)^2}+1}\cdot\frac{-\sqrt{2}\left(x^2+1\right)}{\left(x^2-1\right)^2}\right)\\&=\frac{1}{4\sqrt{2}}\left(\frac{2\sqrt{2}\left(1-x^2\right)}{x^4+1}+\frac{2\sqrt{2}\left(1+x^2\right)}{x^4+1}\right)\\&=\frac{1}{x^4+1}\end{aligned}$$

3.(18分)

设函数

$$f(x)=\left(x^2-3x+2\right)^{100}\cos\frac{\pi x^2}{4}$$

(1) **(5分)** 设函数$u(x),v(x)$任意阶可导.对于正整数n,写出函数$y=u(x)v(x)$的n阶导数的Leibniz公式.

(2) **(10分)** 对于正整数n满足$1\leqslant n\leqslant 100$,求$f^{(n)}(1)$.

(3) **(3分)** 求$f^{(101)}(2)$.

(1) Solution.

$$(u(x)v(x))^{(n)} = \sum_{i=0}^{n} C_n^i u^{(i)}(x)v^{(n-i)}(x)$$

(2) Solution.

设$u(x) = \left(x^2 - 3x + 2\right)^{100}, v(x) = \cos \frac{\pi}{4}x^2$,于是$f(x) = u(x)v(x)$.

设$\alpha(x) = (x-1)^{100}, \beta(x) = (x-2)^{100}$,于是$u(x) = \alpha(x)\beta(x)$.

注意到$n < 100$时,$\alpha^{(n)}(1) = \frac{100!}{(100-n)!}(1-1)^{100-n} = 0$,于是 $n < 100$时

$$u^{(n)}(1) = \sum_{i=1}^{n} C_n^i \alpha^{(i)}(1)\beta^{(n-i)}(1) = 0$$

而$n = 100$时

$$u^{(100)}(1) = \alpha^{(100)}(1)\beta(1) = 100! \cdot (1-2)^{100} = 100!$$

于是

$$f^{(n)}(1) = \begin{cases} 0, 1 \leqslant n < 100 \\ \frac{\sqrt{2}}{2} \cdot 100!, n = 100 \end{cases}$$

(3) Solution.

注意到当且仅当$n = 100$时$\beta^{(100)}(2) = 100!$,否则$\beta^{(n)}(2) = 0$.于是

$$u^{(101)}(2) = 101\alpha^{(1)}(2)\beta^{(100)}(2) = 101 \cdot 100(1-2)^{99} \cdot 100! = -100 \cdot 101!$$

又$v^{(1)}(x) = -\frac{\pi x}{2} \sin \frac{\pi}{4}x^2$,于是$v^{(1)}(2) = 0$.

综上可知

$$\begin{aligned} f^{(101)}(2) &= u^{(101)}(2)v(2) + 101u^{(100)}(2)v^{(1)}(2) \\ &= u^{(101)}(2)v(2) \\ &= 100 \cdot 101! \end{aligned}$$

4.(16分)

计算下列积分.

(1) (5分) $A = \int_0^{2\pi} |\sin x - \cos x| \, \mathrm{d}x$.

(2) (3分) $B = \int_0^{2\pi} \sqrt{1 + \sin 2x} \mathrm{d}x$.

(3) (5分) $I = \int \sqrt{\mathrm{e}^x - 1} \mathrm{d}x$.

(4) (3分) $J = \int \frac{x\mathrm{e}^x}{\sqrt{\mathrm{e}^x - 1}} \mathrm{d}x$.

(1) Solution.

$$\begin{aligned}
A &= \int_0^{2\pi} \left|\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)\right| \mathrm{d}x \\
&= \sqrt{2}\int_{-\frac{\pi}{4}}^{\frac{7\pi}{4}} |\sin u|\,\mathrm{d}u \\
&= \sqrt{2}\int_0^{2\pi} |\sin u|\,\mathrm{d}u \\
&= 2\sqrt{2}\int_0^{\pi} \sin u\mathrm{d}u \\
&= 2\sqrt{2}\left(-\cos u\right)\Big|_0^{\pi} \\
&= 4\sqrt{2}
\end{aligned}$$

(2) Solution.

$$\begin{aligned}
B &= \int_0^{2\pi} \sqrt{\sin^2 x+\cos^2 x+2\sin x\cos x}\mathrm{d}x \\
&= \int_0^{2\pi} |\sin x+\cos x|\,\mathrm{d}x \\
&= \sqrt{2}\int_0^{2\pi} \left|\sin\left(x+\frac{\pi}{4}\right)\right| \mathrm{d}x \\
&= 4\sqrt{2}
\end{aligned}$$

(3) Solution.

置$t=\sqrt{\mathrm{e}^x-1}$,则$x=\ln\left(t^2+1\right)$, $\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{2t}{t^2+1}$.于是

$$\begin{aligned}
I &= \int \sqrt{\mathrm{e}^x-1}\mathrm{d}x = \int t\cdot\frac{2t\mathrm{d}t}{t^2+1} \\
&= 2\int \frac{t^2+1-1}{t^2+1}\mathrm{d}t \\
&= 2\left(\int \mathrm{d}t-\int\frac{\mathrm{d}t}{t^2+1}\right) \\
&= 2\left(t-\arctan t\right)+C \\
&= 2\sqrt{\mathrm{e}^x-1}-2\arctan\sqrt{\mathrm{e}^x-1}+C
\end{aligned}$$

(4) Solution.

置$t=\sqrt{\mathrm{e}^x-1}$,则$x=\ln\left(t^2+1\right)$, $\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{2t}{t^2+1}$.于是

$$\begin{aligned}
J &= \int \frac{x\mathrm{e}^x}{\sqrt{\mathrm{e}^x+1}}\mathrm{d}x = \int \frac{\ln\left(t^2+1\right)\cdot\left(t^2+1\right)}{t}\cdot\frac{2t\mathrm{d}t}{t^2+1} \\
&= 2\int \ln\left(t^2+1\right)\mathrm{d}t \\
&= 2\int x\mathrm{d}t = 2\left(xt-\int t\mathrm{d}x\right) \\
&= 2x\sqrt{\mathrm{e}^x-1}-4\sqrt{\mathrm{e}^x-1}+4\arctan\sqrt{\mathrm{e}^x-1}+C
\end{aligned}$$

5.(14分)

解答下列各题.

(1) **(5分)** 设函数$f(x)$在$x=a$可导,试证明

$$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{h}=2f'(a)$$

(2) **(2分)** 举例说明:即使$f(x)$在$x=a$处连续且$\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a-h)}{h}$存在,也不能保证$f'(a)$存在.

(3) **(5分)** 设函数$f(x)$在$x=a$可导,对于常数$k\neq 0,1$,试证明

$$\lim_{h\to 0}\frac{f(a+kh)-f(a-h)}{h}=(k-1)f'(a)$$

(4) **(2分)** 举例说明:对于常数$k\neq 0,1$,即使$\lim\limits_{h\to 0}\dfrac{f(a+kh)-f(a-h)}{h}$存在,也不能保证$f'(a)$存在.

(1) **Solution.**

由$f(x)$在$x=a$处可导,可知

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$$

于是

$$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{h}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}=2f'(a)$$

(2) **Solution.**

取$f(x)=|x|$,易知

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^-}f(x)=f(0)=0$$

即$f(x)$在$x=0$处连续.取$a=0$,于是

$$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{h}=\lim_{h\to 0}\frac{|h|-|-h|}{h}=0$$

然而

$$\lim_{h\to 0+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0+}\frac{h}{h}=1$$

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1$$

于是$f(x)$在$x=0$处的左右导数不相等,因此$f'(0)$不存在.

(3) **Solution.**

由$f(x)$在$x=a$处可导,可知

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$$

于是

$$\begin{aligned}
\lim_{h\to0}\frac{f(a+kh)-f(a+h)}{h} &= \lim_{h\to0}\frac{f(a+kh)-f(a)}{h}-\lim_{h\to0}\frac{f(a+h)-f(a)}{h}\\
&= k\lim_{h\to0}\frac{f(a+kh)-f(a)}{kh}-\lim_{h\to0}\frac{f(a+h)-f(a)}{h}\\
&= kf'(a)-f'(a)\\
&= (k-1)f'(a)
\end{aligned}$$

(4) Solution.

取$f(x)=\begin{cases} x,|x|>0\\ 1,x=0\end{cases}$

取$a=0$,则有

$$\lim_{h\to0}\frac{f(a+kh)-f(a+h)}{h}=\lim_{h\to0}\frac{kh-h}{h}=k-1$$

然而$\lim\limits_{x\to0}f(x)=0\neq f(0)$,于是$f(x)$在$x=0$处不连续,$f'(0)$不存在.

6.(20分)

证明下列各题.

(1) (10分) 设序列$\{x_n\}_{n=1}^{\infty}$有极限$\lim\limits_{n\to\infty}x_n=a$.试用序列极限的定义证明

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}x_i=a$$

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(2) (6分) 设序列$\{x_n\}_{n=1}^{\infty}$有极限$\lim\limits_{n\to\infty}x_n=a$.试证明

$$\lim_{n\to\infty}\sqrt[n]{x_1x_2\cdots x_n}=a$$

(3) (4分) 设序列$\{x_n\}_{n=1}^{\infty}$满足$\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{i=1}^{n}x_i=a$,又有$\lim\limits_{n\to\infty}n\left(x_n-x_{n-1}\right)=0$.试证明$\lim\limits_{n\to\infty}x_n=a$.

(1) Proof.

由$\lim\limits_{n\to\infty}x_n=a$可知

$$\forall\varepsilon_x>0,\exists N_x\in\mathbb{N}^*,\text{s.t.}\forall n>N_x,|x_n-a|<\varepsilon_x$$

现在,对于任意$\varepsilon>0$,取$\varepsilon_x=\dfrac{\varepsilon}{2}$和对应的$N_x$,并令$M_x=\max\limits_{1\leqslant i\leqslant N_x}|x_i-a|$.

于是取$N=\max\left\{\left[\dfrac{2N_xM_x}{\varepsilon}\right]+1,N_x\right\}$,对于任意$n>N$有

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n}x_i-a\right|&=\left|\frac{1}{n}\sum_{i=1}^{n}(x_i-a)\right|\leqslant\frac{1}{n}\sum_{i=1}^{n}|x_i-a|\\&=\frac{1}{n}\left(\sum_{i=1}^{N_x}|x_i-a|+\sum_{i=N_x+1}^{n}|x_i-a|\right)\\&<\frac{1}{n}\left(N_xM_x+(n-N_x)\,\varepsilon_x\right)\\&<\frac{N_xM_x}{n}+\varepsilon_x\\&<\varepsilon\end{aligned}$$

于是$\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{i=1}^{n}x_i=a$,原命题得证.

(2) Proof.

取$t_n=\ln x_n$,于是$\lim\limits_{n\to\infty}t_i=\ln a$.

根据**(1)**的结论有$\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{i=1}^{n}t_i=\lim\limits_{n\to\infty}t_i=\ln a$.

于是

$$a=\exp\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}t_i\right)=\lim_{n\to\infty}\sqrt[n]{\prod_{i=1}^{n}\mathrm{e}^{\ln x_i}}=\lim_{n\to\infty}\sqrt[n]{x_1x_2\cdots x_n}$$

原命题得证.

(3) Proof.

置$S_n=\dfrac{1}{n}\sum\limits_{i=1}^{n}x_i$,于是$\lim\limits_{n\to\infty}\dfrac{S_n}{n}=a$.

置$t_i=i(x_{i+1}-x_i)$,则$\lim\limits_{n\to\infty}t_n=\lim\limits_{n\to\infty}\dfrac{n}{n+1}\cdot(n+1)\,(x_{n+1}-x_n)=1\cdot0=0$.

根据**(1)**的结论有$\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{i=1}^{n}t_i=\lim\limits_{n\to\infty}t_i=0$.

于是

$$\begin{aligned}\lim_{n\to\infty}\left(x_n-\frac{S_n}{n}\right)&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(x_n-x_i)\\&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}i\,(x_{i+1}-x_i)\\&=\lim_{n\to\infty}\frac{n-1}{n}\cdot\frac{1}{n-1}\sum_{i=1}^{n-1}t_i\\&=1\cdot0\\&=0\end{aligned}$$

从而$\lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}\left(x_n-\dfrac{S_n}{n}+\dfrac{S_n}{n}\right)=0+a=a$,原命题得证.