不定积分练习

Problem 1.

求不定积分

$$\int \frac{\sqrt{x^2-a^2}}{x}\mathrm{d}x$$

Solution(Method I).

采取换元法.置$\dfrac{a}{x}=\sin t$,则

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-a\cdot\frac{1}{\sin^2 x}\cdot\cos x=-\frac{a\cos x}{\sin^2 x}$$

从而

$$\begin{aligned}\int \frac{\sqrt{x^2-a^2}}{x}\mathrm{d}x&=\int\frac{-a\cos^2 t\mathrm{d}t}{\sin^2 t}\\&=-a\int\frac{1-\sin^2 t}{\sin^2 t}\mathrm{d}t\\&=-a\left(\int\frac{\mathrm{d}t}{\sin^2 t}-\int\mathrm{d}t\right)\\&=-a\left(-\cot t-t\right)+C\end{aligned}$$

当$x>a$时有$t\in\left(0,\dfrac{\pi}{2}\right)$,于是

$$\cot t=\sqrt{\frac{\cos^2 t}{\sin^2 t}}=\sqrt{\frac{1-\left(\frac{a}{x}\right)^2}{\left(\frac{a}{x}\right)^2}}=\frac{\sqrt{x^2-a^2}}{a}$$

当$x<-a$时亦可知$\cot t=-\dfrac{\sqrt{x^2-a^2}}{a}$. 于是

$$\int\frac{\sqrt{x^2-a^2}}{x}\mathrm{d}x=\begin{cases}\sqrt{x^2-a^2}+a\arcsin\dfrac{a}{x}+C,x>a\\-\sqrt{x^2-a^2}+a\arcsin\dfrac{a}{x}+C,x<-a\end{cases}$$

Solution(Method II).

采取换元法.置$\dfrac{a}{x}=\cos t$,则$\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{a\sin t}{\cos^2 t}$. 从而

$$\begin{aligned}\int\frac{\sqrt{x^2-a^2}}{x}\mathrm{d}x&=\int\frac{a\sin^2t\mathrm{d}t}{\cos^2t}\\&=a\int\frac{1-\cos^2t}{\cos^2t}\mathrm{d}t\\&=a\left(\int\frac{\mathrm{d}t}{\cos^2t}-\int\mathrm{d}t\right)\\&=a\left(\tan t-t\right)+C\end{aligned}$$

于是可以得到与**Method I**相似的结果.

$$\int\frac{\sqrt{x^2-a^2}}{x}\mathrm{d}x=\begin{cases}\sqrt{x^2-a^2}-a\arccos\dfrac{a}{x}+C,x>a\\-\sqrt{x^2-a^2}-a\arccos\dfrac{a}{x}+C,x<-a\end{cases}$$

Solution(Method III).

直接采取分部积分法.

$$\begin{aligned}\int\frac{\sqrt{x^2-a^2}}{x}\mathrm{d}x&=\sqrt{x^2-a^2}-\int x\mathrm{d}\left(\frac{\sqrt{x^2-a^2}}{x}\right)\\&=\sqrt{x^2-a^2}-\int x\cdot\frac{x\cdot\dfrac{x}{\sqrt{x^2-a^2}}-\sqrt{x^2-a^2}}{x^2}\mathrm{d}x\\&=\sqrt{x^2-a^2}-\int\frac{a^2x\mathrm{d}x}{x^2\sqrt{x^2-a^2}}\\&=\sqrt{x^2-a^2}-\frac{a^2}{2}\int\frac{\mathrm{d}x^2}{x^2\sqrt{x^2-a^2}}\end{aligned}$$

置$u=\sqrt{x^2-a^2}$,则

$$\begin{aligned}\int\frac{\mathrm{d}x^2}{x^2\sqrt{x^2-a^2}}&=\int\frac{\mathrm{d}u^2}{(u^2+a^2)u}\\&=2\int\frac{\mathrm{d}u}{u^2+a^2}\\&=\frac{2}{a}\arctan\frac{u}{a}+C\end{aligned}$$

从而

$$\int\frac{\sqrt{x^2-a^2}}{x}\mathrm{d}x=\sqrt{x^2-a^2}-a\arctan\frac{\sqrt{x^2-a^2}}{a}+C$$

Problem 2.

求不定积分

$$\int\frac{\mathrm{d}x}{x^6\sqrt{1+x^2}}$$

Solution.

置$x=\tan t, t\in\left(0,\dfrac{\pi}{2}\right)$,从而$\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{1}{\cos^2 t}$.

从而

$$\begin{aligned}\int\frac{\mathrm{d}x}{x^6\sqrt{1+x^2}}&=\int\frac{\mathrm{d}t}{\tan^6 t\cdot\cos^2 t\cdot\dfrac{1}{\cos t}}\\&=\int\frac{\cos^5 t}{\sin^6 t}\mathrm{d}t\end{aligned}$$

置$u=\sin t$,则$u^2=\dfrac{\sin^2 t}{\sin^2 t+\cos^2 t}=\dfrac{\tan^2 t}{\tan^2 t+1}=\dfrac{x^2}{x^2+1}$.

由于x与u同号,则$u=\dfrac{x}{\sqrt{1+x^2}}$.

于是

$$\begin{aligned}\int\frac{\cos^5 t}{\sin^6 t}\mathrm{d}t&=\int\frac{\cos^4 t\left(\cos t\mathrm{d}t\right)}{\sin^6 t}\\&=\int\frac{\left(1-u^2\right)^2\mathrm{d}u}{u^6}\\&=\int\left(\frac{1}{u^6}-\frac{2}{u^4}+\frac{1}{u^2}\right)\mathrm{d}u\\&=-\frac{1}{5u^5}+\frac{2}{3u^3}-\frac{1}{u}+C\\&=-\frac{\left(x^2+1\right)^{\frac{5}{2}}}{5x^5}+\frac{2\left(x^2+1\right)^{\frac{3}{2}}}{3x^3}-\frac{\left(x^2+1\right)^{\frac{1}{2}}}{x}+C\end{aligned}$$

Problem 3.

求不定积分

$$\int\frac{\mathrm{d}x}{\sqrt[3]{(x+1)(x-1)^5}}$$

Solution.

置$t=\sqrt[3]{\dfrac{x+1}{x-1}}$,则

$$\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{3}\left(\frac{x+1}{x-1}\right)^{-\frac{2}{3}}\frac{-2}{(x-1)^2}=-\frac{2}{3}\frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}}$$

从而

$$\begin{aligned}\int\frac{\mathrm{d}x}{\sqrt[3]{(x+1)(x-1)^5}}&=-\frac{3}{2}\int t\mathrm{d}t\\&=-\frac{3}{4}t^2+C\\&=-\frac{3}{4}\left(\frac{x+1}{x-1}\right)^{\frac{2}{3}}+C\end{aligned}$$

Problem 4.

求不定积分

$$\int \frac{x\arccos x}{\left(1-x^2\right)^2}\mathrm{d}x$$

Solution(Method I).

置$t=\arccos x$,则$x=\cos t$,从而$\dfrac{\mathrm{d}x}{\mathrm{d}t}=-\sin t$.置$u=\sin t$.

从而

$$\begin{aligned}
\int \frac{x\arccos x}{\left(1-x^2\right)^2}\mathrm{d}x &= -\int \frac{t\cos t\sin t\mathrm{d}t}{\sin^4 t} \\
&= -\int \frac{t\mathrm{d}\left(\sin t\right)}{\sin^3 t} = -\int \frac{t\mathrm{d}u}{u^3} \\
&= \frac{1}{2}\int t\mathrm{d}\left(\frac{1}{u^2}\right) = \frac{1}{2}\left(\frac{t}{u^2}-\int \frac{\mathrm{d}t}{u^2}\right) \\
&= \frac{1}{2}\left(\frac{\arccos x}{1-x^2}-\int \frac{\mathrm{d}t}{\sin^2 t}\right) \\
&= \frac{1}{2}\left(\frac{\arccos x}{1-x^2}+\cot t\right)+C \\
&= \frac{1}{2}\left(\frac{\arccos x}{1-x^2}+\frac{x}{\sqrt{1-x^2}}\right)+C
\end{aligned}$$

Solution(Method II).

直接分部积分有

$$\begin{aligned}
\int \frac{x\arccos x}{\left(1-x^2\right)^2}\mathrm{d}x &= \frac{1}{2}\int \arccos x\mathrm{d}\left(\frac{1}{1-x^2}\right) \\
&= \frac{1}{2}\left(\frac{\arccos x}{1-x^2}-\int \frac{\mathrm{d}\arccos x}{1-x^2}\right) \\
&= \frac{1}{2}\left(\frac{\arccos x}{1-x^2}+\int \frac{\mathrm{d}x}{\left(1-x^2\right)^{\frac{3}{2}}}\right)
\end{aligned}$$

置$x=\sin t$,则有

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right)^{\frac{3}{2}}} = \int \frac{\mathrm{d}\sin t}{\cos^3 t} = \int \frac{\mathrm{d}t}{\cos^2 t} = \tan t + C = \frac{x}{\sqrt{1-x^2}}+C$$

于是

$$\int \frac{x\arccos x}{\left(1-x^2\right)^2}\mathrm{d}x = \frac{1}{2}\left(\frac{\arccos x}{1-x^2}+\frac{x}{\sqrt{1-x^2}}\right)+C$$

Problem 5.

求不定积分

$$\int x\ln\left(x+\sqrt{1+x^2}\right)\mathrm{d}x$$

Solution.

$$\begin{aligned}
\int x\ln\left(x+\sqrt{1+x^2}\right)\mathrm{d}x &= x^2\ln\left(x+\sqrt{1+x^2}\right)-\int x\mathrm{d}\left(x\ln\left(x+\sqrt{1+x^2}\right)\right)\\
&= x^2\ln\left(x+\sqrt{1+x^2}\right)-\int x\left(\ln\left(x+\sqrt{1+x^2}\right)+\frac{x\left(1+\frac{x}{\sqrt{1+x^2}}\right)}{x+\sqrt{1+x^2}}\right)\mathrm{d}x\\
&= x^2\ln\left(x+\sqrt{1+x^2}\right)-\int x\ln\left(x+\sqrt{1+x^2}\right)\mathrm{d}x-\int\frac{x^2\mathrm{d}x}{\sqrt{1+x^2}}
\end{aligned}$$

而

$$\int\frac{x^2\mathrm{d}x}{\sqrt{1+x^2}}=\int x\mathrm{d}\left(\sqrt{1+x^2}\right)=x\sqrt{1+x^2}-\int\sqrt{1+x^2}\mathrm{d}x$$

又

$$\int\frac{x^2\mathrm{d}x}{\sqrt{1+x^2}}=\int\frac{\left(x^2+1-1\right)\mathrm{d}x}{\sqrt{1+x^2}}=\int\sqrt{1+x^2}\mathrm{d}x-\int\frac{\mathrm{d}x}{\sqrt{1+x^2}}$$

两式相加有

$$2\int\frac{x^2\mathrm{d}x}{\sqrt{1+x^2}}=x\sqrt{1+x^2}-\int\frac{\mathrm{d}x}{\sqrt{1+x^2}}$$

从而

$$\begin{aligned}
\int x\ln\left(x+\sqrt{1+x^2}\right)\mathrm{d}x &= \frac{1}{2}\left(x^2\ln\left(x+\sqrt{1+x^2}\right)-\int\frac{x^2\mathrm{d}x}{\sqrt{1+x^2}}\right)\\
&= \frac{1}{2}x^2\ln\left(x+\sqrt{1+x^2}\right)-\frac{1}{4}x\sqrt{1+x^2}+\frac{1}{4}\int\frac{\mathrm{d}x}{\sqrt{1+x^2}}\\
&= \frac{1}{2}x^2\ln\left(x+\sqrt{1+x^2}\right)-\frac{1}{4}x\sqrt{1+x^2}+\frac{1}{4}\ln\left(x+\sqrt{x^2+1}\right)+C
\end{aligned}$$

Problem 6.

求不定积分

$$\int\frac{\mathrm{e}^{\arctan x}\mathrm{d}x}{\left(1+x^2\right)^{\frac{3}{2}}}$$

Solution(Method I).

注意到$\dfrac{\mathrm{d}\left(\mathrm{e}^{\arctan x}\right)}{\mathrm{d}x}=\dfrac{\mathrm{e}^{\arctan x}}{1+x^2}$,从而

$$\int\frac{\mathrm{e}^{\arctan x}\mathrm{d}x}{\left(1+x^2\right)^{\frac{3}{2}}}=\int\frac{\mathrm{d}\left(\mathrm{e}^{\arctan x}\right)}{\sqrt{1+x^2}}=\frac{\mathrm{e}^{\arctan x}}{\sqrt{1+x^2}}-\int\frac{x\mathrm{e}^{\arctan x}\mathrm{d}x}{\left(1+x^2\right)^{\frac{3}{2}}}$$

而

$$\int\frac{x\mathrm{e}^{\arctan x}\mathrm{d}x}{\left(1+x^2\right)^{\frac{3}{2}}}=\int\frac{x\mathrm{d}\left(\mathrm{e}^{\arctan x}\right)}{\sqrt{1+x^2}}=\frac{x\mathrm{e}^{\arctan x}}{\sqrt{1+x^2}}-\int\frac{\mathrm{e}^{\arctan x}\mathrm{d}x}{\left(1+x^2\right)^{\frac{3}{2}}}$$

两式相加可得

$$\int\frac{\mathrm{e}^{\arctan x}\mathrm{d}x}{\left(1+x^2\right)^{\frac{3}{2}}}=\frac{\mathrm{e}^{\arctan x}(1+x)}{2\sqrt{1+x^2}}+C$$

Solution(Method II).

置$x=\tan t$,则

$$\begin{aligned}\int\frac{\mathrm{e}^{\arctan x}\mathrm{d}x}{\left(1+x^2\right)^{\frac{3}{2}}}&=\int\cos^3 t\mathrm{e}^t\mathrm{d}(\tan t)\\&=\int\mathrm{e}^t\cos t\mathrm{d}t\\&=\int\cos t\mathrm{d}\left(\mathrm{e}^t\right)\\&=\mathrm{e}^t\cos t+\int\mathrm{e}^t\sin t\mathrm{d}t\\&=\mathrm{e}^t\cos t+\mathrm{e}^t\sin t-\int\mathrm{e}^t\cos t\mathrm{d}t\end{aligned}$$

从而

$$\begin{aligned}\int\frac{\mathrm{e}^{\arctan x}\mathrm{d}x}{\left(1+x^2\right)^{\frac{3}{2}}}&=\int\mathrm{e}^t\cos t\mathrm{d}t\\&=\frac{1}{2}\left(\mathrm{e}^t\cos t+\mathrm{e}^t\sin t\right)\\&=\frac{\mathrm{e}^{\arctan x}(1+x)}{2\sqrt{1+x^2}}+C\end{aligned}$$

Problem 7.

求不定积分

$$\int\frac{x\ln x}{\left(1+x^2\right)^2}\mathrm{d}x$$

Solution.