相乘序列的极限

设序列$\{a_n\},\{b_n\}$满足$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$,
则有

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i b_{n+1-i}}{n} = 0$$

Analysis.
我们把和式拆成两部分,有

$$\frac{\sum_{i=1}^{n} a_i b_{n+1-i}}{n} = \frac{\sum_{i=1}^{j} a_i b_{n+1-i}}{n} + \frac{\sum_{i=j+1}^{n} a_i b_{n+1-i}}{n}$$

注意到两部分中分别可以令a_i和b_i很小,且$\{a_n\},\{b_n\}$有界,放缩即可证明.

Proof.
由题意

$$\forall \varepsilon_a > 0, \exists N_a \in \mathbb{N}^*, \text{s.t.} \forall n \geqslant N_a, |a_n| \leqslant \varepsilon_a$$

$$\forall \varepsilon_b > 0, \exists N_b \in \mathbb{N}^*, \text{s.t.} \forall n \geqslant N_b, |b_n| \leqslant \varepsilon_b$$

且由收敛序列有界可知

$$\exists M_a \in \mathbb{R}, \text{s.t.} \forall n \in \mathbb{N}^*, |a_n| \leqslant M_a$$

$$\exists M_b \in \mathbb{R}, \text{s.t.} \forall n \in \mathbb{N}^*, |a_n| \leqslant M_b$$

对于给定的$\varepsilon_a,\varepsilon_b$和对应的$N_a,N_b$,$\forall n > N_a + N_b$有

$$\begin{aligned}
\left|\frac{\sum_{i=1}^{n} a_i b_{n+1-i}}{n}\right| &= \left|\frac{\sum_{i=1}^{N_a-1} a_i b_{n+1-i}}{n} + \frac{\sum_{i=N_a}^{n} a_i b_{n+1-i}}{n}\right| \\
&\leqslant \left|M_a \cdot \frac{\sum_{i=n+2-N_a}^{n} b_i}{n} + M_b \cdot \frac{\sum_{i=N_a}^{n} a_i}{n}\right| \\
&\leqslant M_a \cdot \frac{N_a - 1}{n} \cdot \varepsilon_b + M_b \cdot \frac{n - N_a + 1}{n} \cdot \varepsilon_a \\
&\leqslant M_a \varepsilon_b + M_b \varepsilon_a
\end{aligned}$$

现在,对于任意$\varepsilon > 0$,取$0 < \varepsilon_a < \varepsilon, 0 < \varepsilon_b < \dfrac{\varepsilon - M_b \varepsilon_a}{M_a}$和对应的$N_a,N_b$
令$N = N_a + N_b + 1$,则$\forall n > N$有

$$\left|\frac{\sum_{i=1}^{n} a_i b_{n+1-i}}{n}\right| \leqslant M_a \varepsilon_b + M_b \varepsilon_a < \varepsilon$$

从而$\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^{n} a_i b_{n+1-i}}{n} = 0$,证毕.

Enhanced Proposition.

设序列$\{a_n\}$, $\{b_n\}$满足$\lim\limits_{n\to\infty} a_n = A$, $\lim\limits_{n\to\infty} b_n = B, A, B \in \mathbb{R}$,

则有

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i b_{n+1-i}}{n} = AB$$

Proof.

我们记序列$\{x_n\}$满足$x_n = a_n - A$,序列$\{y_n\}$满足$y_n = b_n - B$.

则$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_n = 0$.

据Cauchy命题有$\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^{n} x_i}{n} = \lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^{n} y_i}{n} = 0$.

则

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i b_{n+1-i}}{n} &= \lim_{n\to\infty} \frac{\sum_{i=1}^{n} (x_i + A)(y_{n+1-i} + B)}{n} \\
&= AB + A \lim_{n\to\infty} \frac{\sum_{i=1}^{n} y_i}{n} + B \lim_{n\to\infty} \frac{\sum_{i=1}^{n} x_i}{n} + \lim_{n\to\infty} \frac{\sum_{i=1}^{n} x_i y_{n+1-i}}{n} \\
&= AB
\end{aligned}$$

从而原命题得证.

例1(24.10.09 SJTU数分小测):

数列$\{x_n\}$有$\lim\limits_{n\to\infty} x_n = A$. 正项数列$\{y_n\}$有$\lim\limits_{n\to\infty} \dfrac{y_n}{\sum_{i=1}^{n} y_i} = 0$.

试证明: $\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^{n} y_i x_{n+1-i}}{\sum_{i=1}^{n} y_i} = A$.

Proof.

记序列$\{a_n\}$满足$a_n = x_n - A$.则$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} x_n - A = 0$.

则$\forall \varepsilon_a > 0, \exists N_a \in \mathbb{N}^*, \text{s.t.} \forall n \geqslant N_a, |a_n| \leqslant \varepsilon_a$

且$\exists M_a \in \mathbb{R}, \text{s.t.} \forall n \in \mathbb{N}^*, |a_n| < M_a$.

又$\lim\limits_{n\to\infty} \dfrac{y_n}{\sum_{i=1}^{n} y_i} = 0$,则$\forall \varepsilon_y > 0, \exists N_y \in \mathbb{N}^*, \text{s.t.} \forall n \geqslant N_y, \left| \dfrac{y_n}{\sum_{i=1}^{n} y_i} \right| < \varepsilon_y$

则$n > N_a$且$n + 2 - N_a > N_y$时有

$$\begin{aligned}
\left| \frac{\sum_{i=1}^{n} y_i x_{n+1-i}}{\sum_{i=1}^{n} y_i} - A \right| &= \left| \frac{\sum_{i=1}^{n} y_i a_{n+1-i}}{\sum_{i=1}^{n} y_i} \right| \\
&= \left| \frac{\sum_{i=1}^{n+1-N_a} y_i a_{n+1-i}}{\sum_{i=1}^{n} y_i} + \frac{\sum_{i=n+2-N_a}^{n} y_i a_{n+1-i}}{\sum_{i=1}^{n} y_i} \right| \\
&\leqslant \left| \frac{\sum_{i=1}^{n+1-N_a} y_i}{\sum_{i=1}^{n} y_i} \varepsilon_a + \frac{\sum_{i=n+2-N_a}^{n} y_i}{\sum_{i=1}^{n} y_i} M_a \right|
\end{aligned}$$

$$\leqslant \varepsilon_a + M_a\varepsilon_b$$

从而对于任意$\varepsilon > 0$,取$0 < \varepsilon_a < \varepsilon, 0 < \varepsilon_b < \dfrac{\varepsilon - \varepsilon_a}{M_a}$和对应的$N_a, N_y$
令$N = N_a + N_y + 2$,则$\forall n \geqslant N$有

$$\begin{aligned}\left|\frac{\sum_{i=1}^{n} y_i x_{n+1-i}}{\sum_{i=1}^{n} y_i} - A\right| &\leqslant \varepsilon_a + M_a\varepsilon_b \\ &< \varepsilon_a + M_a \cdot \frac{\varepsilon - \varepsilon_a}{M_a} \\ &= \varepsilon\end{aligned}$$

从而$\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^{n} y_i x_{n+1-i}}{\sum_{i=1}^{n} y_i} = A$,证毕.

例2(2023Fall PKU高等数学B期中考试):
序列$\{a_n\}$有$\lim\limits_{n\to\infty} a_n = A$,$0 < q < 1$.
试证明: $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} a_i q^{n-i} = \dfrac{A}{1-q}$.

Proof.
记序列$\{x_n\}$满足$x_n = a_n - A$.
则

$$\begin{aligned}\sum_{i=1}^{n} a_i q^{n-i} &= \sum_{i=1}^{n} (x_i + A) q^{n-i} \\ &= \sum_{i=1}^{n} x_i q^{n-i} + A\sum_{i=1}^{n} y_n \\ &= \sum_{i=1}^{n} x_i q^{n-i} + A \cdot \frac{1-q^n}{1-q}\end{aligned}$$

下面证明$\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} x_i q^{n-i} = 0$.
由于$\lim\limits_{n\to\infty} x_n = 0$,则$\forall \varepsilon_x > 0, \exists N_x \in \mathbb{N}^*, \text{s.t.} \forall n \geqslant N_x, |x_n| < \varepsilon$ 且$\exists M_x \in \mathbb{R}, \text{s.t.} \forall n \in \mathbb{N}^*, |x_n| < M_x$.
则$\forall n >$有

$$\begin{aligned}\left|\sum_{i=1}^{n} x_i q^{n-i}\right| &= \left|\sum_{i=1}^{N_x-1} x_i q^{n-i} + \sum_{i=N_x}^{n} x_i q^{n-i}\right| \\ &= M_x \sum_{i=1}^{N_x-1} q^{n-i} + \varepsilon_x \sum_{i=N_x}^{n} q^{n-i} \\ &= M_x \cdot \frac{q^{n-N_x+1} - q^n}{1-q} + \varepsilon_x \cdot \frac{1-q^{n-N_x+1}}{1-q} \\ &< M_x q^n \cdot \frac{1-q^{N_x-1}}{q^{N_x-1}(1-q)} + \frac{\varepsilon_x}{1-q}\end{aligned}$$

$$
\begin{aligned}
&< \frac{M_x q^n}{q^{N_x}(1-q)} + \frac{\varepsilon_x}{1-q} \\
&= \frac{1}{1-q}\left(M_x q^{n-N_x} + \varepsilon_x\right)
\end{aligned}
$$

对于任意$\varepsilon > 0$,取$0 < \varepsilon_x < (1-q)\varepsilon$和对应的$N_x$.

令$N = \left[\log_q \dfrac{(1-q)\varepsilon - \varepsilon_x}{M_x}\right] + N_x$,则$\forall n > N$有

$$
\begin{aligned}
\left|\sum_{i=1}^{n} x_i q^{n-i}\right| &< \frac{1}{1-q}\left(M_x q^{n-N_x} + \varepsilon_x\right) \\
&< \frac{1}{1-q}\left(M_x q^{N-N_x} + \varepsilon_x\right) \\
&\leqslant \frac{1}{1-q}((1-q)\varepsilon - \varepsilon_x + \varepsilon_x) \\
&= \varepsilon
\end{aligned}
$$

从而原命题得证.