非常好数分 爱来自上交

1.

数列$\{x_n\}$有$\lim\limits_{n\to\infty} x_n = A$. 正项数列$\{y_n\}$有$\lim\limits_{n\to\infty} \dfrac{y_n}{\sum_{i=1}^n y_i} = 0$.

试证明: $\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^n y_i x_{n+1-i}}{\sum_{i=1}^n y_i} = A$.

Proof.

记序列$\{a_n\}$满足$a_n = x_n - A$.则$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} x_n - A = 0$.

则$\forall \varepsilon_a > 0, \exists N_a \in \mathbb{N}^*, \text{s.t.} \forall n \geqslant N_a, |a_n| \leqslant \varepsilon_a$

且$\exists M_a \in \mathbb{R}, \text{s.t.} \forall n \in \mathbb{N}^*, |a_n| < M_a$.

又$\lim\limits_{n\to\infty} \dfrac{y_n}{\sum_{i=1}^n y_i} = 0$,则$\forall \varepsilon_y > 0, \exists N_y \in \mathbb{N}^*, \text{s.t.} \forall n \geqslant N_y, \left|\dfrac{y_n}{\sum_{i=1}^n y_i}\right| < \varepsilon_y$

则$n > N_a$且$n + 2 - N_a > N_y$时有

$$\begin{aligned}
\left|\frac{\sum_{i=1}^n y_i x_{n+1-i}}{\sum_{i=1}^n y_i} - A\right| &= \left|\frac{\sum_{i=1}^n y_i a_{n+1-i}}{\sum_{i=1}^n y_i}\right| \\
&= \left|\frac{\sum_{i=1}^{n+1-N_a} y_i a_{n+1-i}}{\sum_{i=1}^n y_i} + \frac{\sum_{i=n+2-N_a}^n y_i a_{n+1-i}}{\sum_{i=1}^n y_i}\right| \\
&\leqslant \left|\frac{\sum_{i=1}^{n+1-N_a} y_i}{\sum_{i=1}^n y_i}\varepsilon_a + \frac{\sum_{i=n+2-N_a}^n y_i}{\sum_{i=1}^n y_i} M_a\right| \\
&\leqslant \varepsilon_a + M_a\varepsilon_b
\end{aligned}$$

从而对于任意$\varepsilon > 0$,取$0 < \varepsilon_a < \varepsilon, 0 < \varepsilon_b < \dfrac{\varepsilon - \varepsilon_a}{M_a}$和对应的$N_a, N_y$

令$N = N_a + N_y + 2$,则$\forall n \geqslant N$有

$$\begin{aligned}
\left|\frac{\sum_{i=1}^n y_i x_{n+1-i}}{\sum_{i=1}^n y_i} - A\right| &\leqslant \varepsilon_a + M_a\varepsilon_b \\
&< \varepsilon_a + M_a \cdot \frac{\varepsilon - \varepsilon_a}{M_a} \\
&= \varepsilon
\end{aligned}$$

从而$\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^n y_i x_{n+1-i}}{\sum_{i=1}^n y_i} = A$,证毕.

2.

数列$\{x_n\}$满足对于$\{x_n\}$的任意子列$\{x_{n_k}\}$均有$\lim\limits_{k\to\infty} \dfrac{\sum_{i=1}^k x_{n_i}}{k} = 1$.

试证明: $\lim\limits_{n\to\infty} x_n = 1$.

Proof.
采取反证法.假定$\{x_n\}$不收敛或不收敛于1,则有

$$\exists\varepsilon > 0, \forall N \in \mathbb{N}^*, s.t. \exists n \geqslant N, |x_n - 1| > \varepsilon$$

也即$\{x_n\}$中有无穷多项x_i满足$|x_n - 1| > \varepsilon$.
我们将这些项分为$\{x_n\}$的两个子序列$\left\{x_{n_+}\right\}, \left\{x_{n_-}\right\}$, 满足

$$\forall x_i \in \left\{x_{n_+}\right\}, x_i > 1 + \varepsilon$$
$$\forall x_j \in \left\{x_{n_-}\right\}, x_j < 1 - \varepsilon$$

则$\left\{x_{n_+}\right\}, \left\{x_{n_-}\right\}$中至少有一个为无穷序列.
当$\left\{x_{n_+}\right\}$或$\left\{x_{n_-}\right\}$为无穷序列时有

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k} x_{n_{+,i}}}{k} \geqslant \lim_{k\to\infty} \frac{\sum_{i=1}^{k} 1+\varepsilon}{k} = 1 + \varepsilon > 1$$
$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k} x_{n_{-,i}}}{k} \leqslant \lim_{k\to\infty} \frac{\sum_{i=1}^{k} 1-\varepsilon}{k} = 1 - \varepsilon < 1$$

而根据题意,若$\left\{x_{n_+}\right\}$或$\left\{x_{n_-}\right\}$为无穷序列,则有

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k} x_{n_{+,i}}}{k} = \lim_{k\to\infty} \frac{\sum_{i=1}^{k} x_{n_{-,i}}}{k} = 1$$

矛盾.故$\lim\limits_{n\to\infty} x_n = 1$,证毕.

3.
正项数列$\{x_n\}$满足$\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^{n} x_i}{n} = a, a \in \mathbb{R}$.
试证明: $\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^{n} x_i^2}{n^2} = 0$.

Proof.
我们有

$$\begin{aligned}\lim_{n\to\infty} \frac{x_n}{n} &= \lim_{n\to\infty} \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{n-1}{n} \cdot \frac{\sum_{i=1}^{n-1} x_i}{n-1} \right) \\ &= a - 1 \cdot a = 0\end{aligned}$$

根据收敛序列的有界性,$\exists M \in \mathbb{R}$, s.t.$\dfrac{\sum_{i=1}^{n} x_i}{n} < M$
则

$$0 < \frac{\sum_{i=1}^{n} x_i^2}{n^2} = M \cdot \frac{\sum_{i=1}^{n} \frac{x_i^2}{n}}{Mn} < M \cdot \frac{\sum_{i=1}^{n} \frac{x_i^2}{i}}{\sum_{i=1}^{n} x_i}$$

依Stolz定理

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}\frac{x_i^2}{i}}{\sum_{i=1}^{n}x_i}=\lim_{n\to\infty}\frac{\frac{x_n^2}{n}}{x_n}=\lim_{n\to\infty}\frac{x_n}{n}=0$$

依夹逼准则$\lim\limits_{n\to\infty}\frac{\sum_{i=1}^{n}x_i^2}{n^2}=0$,证毕.