极限练习

Problem 1.

求序列极限

$$\lim_{n\to\infty}\prod_{i=1}^{n}\left(1+\frac{i}{n^2}\right)$$

Solution.

熟知当$x\in(1,+\infty)$时有

$$1-\frac{1}{x}<\ln x<x-1$$

我们对原式两端取对数后可得

$$\sum_{i=1}^{n}\left(1-\frac{1}{1+\frac{i}{n^2}}\right)<\sum_{i=1}^{n}\ln\left(1+\frac{i}{n^2}\right)<\sum_{i=1}^{n}\frac{i}{n^2}$$

即

$$\sum_{i=1}^{n}\frac{i}{n^2+i}<\sum_{i=1}^{n}\ln\left(1+\frac{i}{n^2}\right)<\frac{n+1}{2n}$$

又$i\in[1,n]$时

$$\frac{i}{n^2+n}<\frac{i}{n^2+i}<\frac{i}{n^2}$$

于是

$$\frac{1}{2}=\sum_{i=1}^{n}\frac{i}{n^2+n}<\sum_{i=1}^{n}\frac{i}{n^2+i}<\sum_{i=1}^{n}\frac{i}{n^2}=\frac{1}{2}+\frac{1}{2n}$$

夹逼可得

$$\lim_{n\to\infty}\sum_{i=1}^{n}\ln\left(1+\frac{i}{n^2}\right)=\frac{1}{2}$$

于是

$$\lim_{n\to\infty}\prod_{i=1}^{n}\left(1+\frac{i}{n^2}\right)=\sqrt{e}$$

Problem 2.

设$S_n=1^1+2^2+3^3+\cdots+n^n$,试证明:当$n\geqslant 2$时有

$$n^n\left(1+\frac{1}{4(n-1)}\right)\leqslant S_n<n^n\left(1+\frac{2}{\mathrm{e}(n-1)}\right)$$

Proof.

记$A_n=\dfrac{S_n}{n^n}$,则

$$\begin{aligned}A_n&=\frac{\sum_{i=1}^{n}i^i}{n^n}=1+\frac{\sum_{i=1}^{n-1}i^i}{n^n}\\&=1+\frac{(n-1)^{n-1}}{n^n}\cdot\frac{\sum_{i=1}^{n-1}i^i}{(n-1)^{n-1}}\\&=1+\frac{(n-1)^{n-1}}{n^n}A_{n-1}\\&=1+\frac{(n-1)^{n-1}}{n^n}\left(1+\frac{(n-2)^{n-2}}{(n-1)^{n-1}}A_{n-2}\right)\end{aligned}$$

一方面,我们有

$$\begin{aligned}A&=1+\frac{(n-1)^{n-1}}{n^n}\left(1+\frac{(n-2)^{n-2}}{(n-1)^{n-1}}A_{n-2}\right)\\&\leqslant 1+\frac{(n-1)^{n-1}}{n^n}\left(1+\frac{(n-2)^{n-2}}{(n-1)^{n-1}}\cdot\frac{(n-2)(n-2)^{n-2}}{(n-2)^{n-2}}\right)\\&=1+\left(1-\frac{1}{n}\right)^n\cdot\frac{1}{n-1}\left(1+\frac{(n-2)^{n-1}}{(n-1)^{n-1}}\right)\\&<1+\frac{1}{\mathrm{e}(n-1)}(1+1)\\&=1+\frac{2}{\mathrm{e}(n-1)}\end{aligned}$$

另一方面,我们有

$$\begin{aligned}A&=1+\frac{(n-1)^{n-1}}{n^n}\left(1+\frac{(n-2)^{n-2}}{(n-1)^{n-1}}A_{n-2}\right)\\&\geqslant 1+\frac{(n-1)^{n-1}}{n^n}\\&=1+\left(1-\frac{1}{n}\right)^n\cdot\frac{1}{n-1}\\&\geqslant 1+\left(1-\frac{1}{2}\right)^2\cdot\frac{1}{n-1}\\&=1+\frac{1}{4(n-1)}\end{aligned}$$

于是

$$1+\frac{1}{4(n-1)}<\frac{S_n}{n^n}<1+\frac{2}{\mathrm{e}(n-1)}$$

于是原命题得证.