Lecture 3 Triple integral(三重积分)

L.3.1 求$I=\iiint_{\Omega}\left(y^2+z^2\right)\mathrm{d}V$,其中$\Omega=\{(x,y,z)|0\leqslant z\leqslant x^2+y^2\leqslant 1\}$.

Solution.

做柱坐标变换,则变换后的积分区域$\Omega'=\{(r,\theta,z)|0\leqslant r\leqslant 1,0\leqslant\theta\leqslant 2\pi,0\leqslant z\leqslant r^2\}$.我们有

$$\begin{aligned}
I&=\iiint_{\Omega}\left(y^2+z^2\right)\mathrm{d}V\\
&=\iiint_{\Omega'}\left(r^2\sin^2\theta+z^2\right)r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z\\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\mathrm{d}r\int_0^{r^2}\left(r^3\sin^2\theta+rz^2\right)\mathrm{d}z\\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\left(r^5\sin^2\theta+\frac{1}{3}r^7\right)\mathrm{d}r\\
&=\int_0^{2\pi}\left(\frac{1}{6}\sin^2\theta+\frac{1}{24}\right)\mathrm{d}\theta\\
&=\frac{\pi}{4}
\end{aligned}$$

L.3.2 求$I=\iiint_{\Omega}z(x^2+y^2+z^2)\mathrm{d}V$,其中$\Omega$为球体$x^2+y^2+z^2\leqslant 2z$.

Solution.

注意到$\Omega=\{(x,y,z)|x^2+y^2+(z-1)^2\leqslant 1\}$.

做球坐标变换,可得变换后的积分区域为$\Omega'=\{(\rho,\theta,\varphi)|0\leqslant r\leqslant 2\cos\varphi,0\leqslant\theta\leqslant 2\pi,0\leqslant\varphi\leqslant\frac{\pi}{2}\}$.于是

$$\begin{aligned}
I&=\iiint_{\Omega}z(x^2+y^2+z^2)\mathrm{d}V\\
&=\iiint_{\Omega'}\rho\cos\varphi\cdot\rho^2\cdot\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi\\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{2\cos\varphi}\rho^5\sin\varphi\cos\varphi\mathrm{d}\rho\\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\frac{32\cos^7\varphi\sin\varphi}{3}\mathrm{d}\varphi\\
&\xlongequal{t=\cos\varphi}2\pi\int_0^1\frac{32t^7\mathrm{d}t}{3}\\
&=\frac{8\pi}{3}
\end{aligned}$$