北京大学数学科学学院2023-24高等数学B2期中考试

1.(10分) 设$L=\{(x,y)\in\mathbb{R}^2: x^2+y^2=1, y\geqslant 0\}$,求曲线积分

$$\int_L (3+x)\mathrm{d}s$$

Solution.

做代换$x=\cos t, y=\sin t$,其中$0\leqslant t\leqslant \pi$.于是

$$\int_L (3+x)\mathrm{d}s=\int_0^{\pi}(3+\cos t)\mathrm{d}t=3\pi$$

2.(10分) 设E是曲线$\left\{(x,y)\in\mathbb{R}^2: x^2+\dfrac{y^2}{4}=1\right\}$沿逆时针方向.求第二型曲线积分

$$\oint_E \frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}$$

Solution.

令

$$P(x,y)=\frac{-y}{x^2+y^2}\qquad Q(x,y)=\frac{x}{x^2+y^2}$$

设E所围的区域为D,令$D_\varepsilon=\{(x,y)\in\mathbb{R}^2: x^2+y^2\leqslant\varepsilon^2\}$,其中$\varepsilon>0$,记$D_\varepsilon$的正向边界为$S_\varepsilon^+$.在区域$D\backslash D_\varepsilon$上运用Green公式有

$$\begin{aligned}&\oint_E \frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}+\oint_{S_\varepsilon^-}\frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}\\=&\iint_{D\backslash D_\varepsilon}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y\\=&\iint_{D\backslash D_\varepsilon}\frac{y^2-x^2+(x^2-y^2)}{(x^2+y^2)^2}\mathrm{d}x\mathrm{d}y\\=&0\end{aligned}$$

又S_ε^+的单位切向量$\mathbf{n}=\dfrac{1}{\varepsilon}(-y,x)$,于是

$$\oint_E \frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}=\oint_{S_\varepsilon^+}\frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}=\oint_{S_\varepsilon}\frac{1}{\varepsilon}\cdot\frac{x^2+y^2}{x^2+y^2}\mathrm{d}s=\oint_{S_\varepsilon}\frac{\mathrm{d}s}{\varepsilon}=\frac{2\pi\varepsilon}{\varepsilon}=2\pi$$

3.(10分) 设D是由直线$y=0,y=2,y=x,y=x+2$围成的有界闭区域,求二重积分

$$\iint_D \left(\frac{1}{2}x-y\right)\mathrm{d}x\mathrm{d}y$$

Solution.

做代换$u=y-x,v=y$,于是积分区域变为$D'=\{(u,v)\in\mathbb{R}^2:0\leqslant u,v\leqslant 2\}$.

变换的Jacobi行列式

$$|D|=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}=\begin{vmatrix}-1 & 1\\ 0 & 1\end{vmatrix}=1$$

于是

$$\begin{aligned}\iint_D \left(\frac{1}{2}x-y\right)\mathrm{d}x\mathrm{d}y &= \iint_{D'} -\frac{1}{2}(v+u)\mathrm{d}v\mathrm{d}u\\ &= -\frac{1}{2}\int_0^2\mathrm{d}u\int_0^2(u+v)\mathrm{d}v\\ &= -\frac{1}{2}\int_0^2(2u+2)\mathrm{d}u\\ &= -4\end{aligned}$$

4.(10分) 设曲面$M=\{(x,y,z)\in\mathbb{R}^3:x^2+z^2=1,x^2+y^2\leqslant 1,x\geqslant 0,y\geqslant 0,z\geqslant 0\}$,求曲面积分

$$\iint_M x\mathrm{d}S$$

Solution.

曲面方程为$z=\sqrt{1-x^2}$,投影区域为$D=\{(x,y)\in\mathbb{R}^2:0\leqslant x,y,x^2+y^2\leqslant 1\}$.于是

$$\begin{aligned}\iint_M x\mathrm{d}S &= \iint_D x\sqrt{1+z_x^2}\mathrm{d}\sigma\\ &= \iint_D \frac{x}{\sqrt{1-x^2}}\mathrm{d}x\mathrm{d}y\\ &= \int_0^1\mathrm{d}x\int_0^{\sqrt{1-x^2}}\frac{x}{\sqrt{1-x^2}}\mathrm{d}y\\ &= \int_0^1 x\mathrm{d}x\\ &= \frac{1}{2}\end{aligned}$$

5.(10分) 求一阶常微分方程初值问题

$$y' = x + y^2, y(0) = 0$$

的皮卡序列的前两项y_1, y_2.

Solution.

所求的初值问题与积分方程

$$y = \int_0^x \left(x + y^2\right) \mathrm{d}x$$

等价.将$y = y_0(x) \equiv 0$代入上式右端,得到

$$y_1(x) = \int_0^x x\mathrm{d}x = \frac{1}{2}x^2$$

再将$y = y_1(x) = \dfrac{1}{2}x^2$代入上式右端,得到

$$y_2(x) = \int_0^x \left(x + \frac{1}{4}x^4\right) \mathrm{d}x = \frac{1}{2}x^2 + \frac{1}{20}x^5$$

6.(10分) 求二阶常微分方程

$$y'' - 2y' + y = \mathrm{e}^x$$

的通解.

Solution.

对应的齐次方程$y'' - 2y' + y = 0$的特征根为$\lambda_1 = \lambda_2 = 1$,于是方程的通解为

$$y = C_1\mathrm{e}^x + C_2x\mathrm{e}^x$$

设方程的特解为$y = Ax^2\mathrm{e}^x$,代入原方程有

$$A\mathrm{e}^x \left(x^2 + 4x + 2 - 2x^2 - 4x + x^2\right) = \mathrm{e}^x$$

于是$A = \dfrac{1}{2}$,因此原方程的通解为

$$y = C_1\mathrm{e}^x + C_2x\mathrm{e}^x + \frac{1}{2}x^2\mathrm{e}^x$$

7.(10分) 设有界闭区域$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 \leqslant z \leqslant 3 - 2x^2 - y^2\}$,$S^-$是$V$的边界内侧,求曲面积分

$$\iint_{S^-} \left(x^2 + y\sin z\right) \mathrm{d}y\mathrm{d}z - (2y + z\cos x)\,\mathrm{d}z\mathrm{d}x + (-2xz + x\sin y)\,\mathrm{d}x\mathrm{d}y$$

Solution.

令

$$P(x,y,z)=x^2+y\sin z \qquad Q(x,y,z)=-2y-z\cos x \qquad R(x,y,z)=-2xz+x\sin y$$

在V上运用Gauss公式有

$$\begin{aligned}&\iint_{S^+} Q\mathrm{d}y\mathrm{d}z+Q\mathrm{d}z\mathrm{d}x+R\mathrm{d}x\mathrm{d}y\\=&\iiint_V\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}V\\=&\iiint_V(2x-2-2x)\,\mathrm{d}V\\=&-2\iiint_V\mathrm{d}V\end{aligned}$$

考虑到V在Oxy平面上的投影为$D=\{(x,y)\in\mathbb{R}^2:x^2+y^2\leqslant 1\}$,于是

$$\begin{aligned}\iiint_V\mathrm{d}V&=\iint_D\mathrm{d}\sigma\int_{x^2+2y^2}^{3-2x^2-y^2}\mathrm{d}z\\&=\iint_D 3\left(1-x^2-y^2\right)\mathrm{d}\sigma\\&=3\int_0^{2\pi}\mathrm{d}\theta\int_0^1\left(1-r^2\right)r\mathrm{d}r\\&=\frac{3\pi}{2}\end{aligned}$$

于是

$$\iint_{S^-}Q\mathrm{d}y\mathrm{d}z+Q\mathrm{d}z\mathrm{d}x+R\mathrm{d}x\mathrm{d}y=2\iiint_V\mathrm{d}V=3\pi$$

8.(15分) 设$r>0$,$f:(-r,r)\to\mathbb{R}$连续,$f(0)=0$,且f在$x=0$处可导.对于$t>0$,定义

$$V(t)=\left\{(x,y,z)\in\mathbb{R}^3:x^2+16y^2+\frac{z^2}{25}\leqslant t^2\right\}$$

试证明

$$\lim_{t\to 0}\frac{1}{t^5}\iiint_{V(t)}f\left(x^2+16y^2+\frac{z^2}{25}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\pi f'(0)$$

Solution.

做代换

$$x=\rho\cos\theta\sin\varphi \qquad y=\frac{1}{4}\rho\sin\theta\sin\varphi \qquad z=5\rho\cos\varphi$$

于是区域$V(t)$变换为

$$V'(t)=\{(\rho,\theta,\varphi):0\leqslant\rho\leqslant t,0\leqslant\theta\leqslant 2\pi,0\leqslant\varphi\leqslant\pi\}$$

变换的Jacobi行列式

$$|J|=\frac{5}{4}\rho^2\sin\varphi$$

于是

$$\begin{aligned}&\iiint_{V(t)} f\left(x^2+16y^2+\frac{z^2}{25}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\=&\iiint_{V'(t)}\frac{5}{4}f\left(\rho^2\right)\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta\\=&\frac{5}{4}\int_0^t\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}f\left(\rho^2\right)\rho^2\sin\varphi\mathrm{d}\varphi\\=&5\pi\int_0^t\rho^2 f\left(\rho^2\right)\mathrm{d}\rho\end{aligned}$$

由于f在$x=0$处可导,于是

$$\lim_{x\to 0}\frac{f\left(x^2\right)-f(0)}{x^2}=\lim_{x\to 0}\frac{f\left(x^2\right)}{x^2}=f'(0)$$

于是

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{t^5}\iiint_{V(t)} f\left(x^2+16y^2+\frac{z^2}{25}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\=&\lim_{t\to 0}\frac{5\pi}{t^5}\int_0^t\rho^2 f\left(\rho^2\right)\mathrm{d}\rho\\=&\lim_{t\to 0}\frac{\pi t^2 f\left(t^2\right)}{t^4}\\=&\pi f'(0)\end{aligned}$$

9.(15分) 求出所有可导的$f:\mathbb{R}\to\mathbb{R}$使得

$$f'(x)=xf(x)+x\int_0^1 tf(t)\mathrm{d}t$$

Solution.
令$I=\displaystyle\int_0^1 tf(t)\mathrm{d}t$,再令$u=f(x)+I$,原方程即

$$u'=ux$$

移项积分可得方程的解为$u=C\mathrm{e}^{\frac{1}{2}x^2}$.这样即有

$$f(x)=C\mathrm{e}^{\frac{1}{2}x^2}-\int_0^1 tf(t)\mathrm{d}t$$

于是

$$
\begin{aligned}
I=\int_0^1 xf(x)\mathrm{d}x &= \int_0^1\left(Cx\mathrm{e}^{\frac{1}{2}x^2}-Ix\right)\mathrm{d}x\\
&=\left(\frac{1}{2}C\mathrm{e}^{\frac{1}{2}x^2}-\frac{1}{2}Ix^2\right)\bigg|_0^1\\
&=\frac{1}{2}C\left(\sqrt{\mathrm{e}}-1\right)-\frac{1}{4}I
\end{aligned}
$$

于是

$$
I=\frac{2}{3}C\left(\sqrt{\mathrm{e}}-1\right)
$$

代入$u=f(x)+I$即有

$$
f(x)=C\mathrm{e}^{\frac{1}{2}x^2}-\frac{2}{3}C\left(\sqrt{\mathrm{e}}-1\right),C\in\mathbb{R}
$$

即为原方程的解.