曲线积分

A.第一型和第二型曲线积分

A.1 求积分$\int_L x^2\mathrm{d}s$,其中L为圆周$\begin{cases} x^2+y^2+z^2=a^2 \\ x+y+z=0 \end{cases}$.

Solution.

将$z=-x-y$代入球的方程有$x^2+xy+y^2=\dfrac{a^2}{2}$,即$\left(\dfrac{1}{2}x+y\right)^2+\left(\dfrac{\sqrt{3}}{2}x\right)^2=\dfrac{a^2}{2}$.做代换

$$\begin{cases} \frac{1}{2}x+y=\frac{\sqrt{2}}{2}a\cos\theta \\ \frac{\sqrt{3}}{2}x=\frac{\sqrt{2}}{2}a\sin\theta \end{cases},0\leqslant\theta\leqslant 2\pi$$

可得

$$\begin{cases} x=\frac{\sqrt{6}}{3}a\sin\theta \\ y=\frac{\sqrt{2}}{2}a\cos\theta-\frac{\sqrt{6}}{6}a\sin\theta \\ z=-\frac{\sqrt{2}}{2}a\cos\theta-\frac{\sqrt{6}}{6}a\sin\theta \end{cases}$$

于是

$$\mathrm{d}s=\sqrt{[x'(\theta)]^2+[y'(\theta)]^2+[z'(\theta)]^2}\mathrm{d}\theta=a\mathrm{d}\theta$$

于是

$$\int_L x^2\mathrm{d}s=\int_0^{2\pi}\frac{2}{3}a^3\sin^2\theta\mathrm{d}\theta=\frac{2}{3}a^3\pi$$

B.格林公式与第二型曲线积分与路径无关的条件 C.格林公式的推广与散度定理

C.1 设函数$u(x,y)$在有界闭区域D上有连续的二阶偏导数,D的边界L逐段光滑.试证明

$$\oint_{L^+}\frac{\partial u}{\partial \mathbf{n}}\mathrm{d}s=\iint_D\Delta u\mathrm{d}\sigma$$

其中$\dfrac{\partial u}{\partial \mathbf{n}}$表示$u(x,y)$沿$L$的外法线方向的方向导数,$\Delta$为Laplace算子.

Proof.

我们设L^+的单位切向量为$\mathbf{t}$,其方向余弦为$\cos\alpha,\cos\beta$.不难看出

$$\mathbf{n}\cdot\mathbf{t}=0,\mathbf{n}\times\mathbf{t}=\mathbf{k}$$

其中$\mathbf{k}$为z轴正方向的单位向量.设$\mathbf{n}=(a,b)$,则有

$$\mathbf{n}\times\mathbf{t}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a & b & 0\\ \cos\alpha & \cos\beta & 0\end{vmatrix}=(a\cos\beta-b\cos\alpha)\mathbf{k}$$

从而

$$\begin{cases}a\cos\alpha+b\cos\beta=0\\ a\cos\beta-b\cos\alpha=1\end{cases}$$

解得$\mathbf{n}=(\cos\beta,-\cos\alpha)$.由方向导数的定义可知

$$\begin{aligned}\oint_{L^+}\frac{\partial u}{\partial \mathbf{n}}\mathrm{d}s&=\oint_{L^+}\left(\frac{\partial u}{\partial x}\cos\beta-\frac{\partial u}{\partial y}\cos\alpha\right)\mathrm{d}s\\ &=\oint_{L^+}\frac{\partial u}{\partial x}\mathrm{d}y-\frac{\partial u}{\partial y}\mathrm{d}x\\ &=\iint_D\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)\right]\mathrm{d}x\mathrm{d}y\\ &=\iint_D\Delta u\mathrm{d}\sigma\end{aligned}$$

注:主要在于

C.2 设区域D的边界L为闭曲线L,某稳定流体(即任意一点的流速与时间无关,仅与该点的位置有关)在$\overline{D}=D+L$上的每一点(x,y)处的流速为

$$\mathbf{v}(x,y)=(P(x,y),Q(x,y))$$

其中$P(x,y),Q(x,y)$在$\overline{D}$上有连续的一阶偏导数.该流体通过闭曲线L的流量Φ定义为

$$\Phi=\oint_{L^+}\mathbf{v}\cdot\mathbf{n}\mathrm{d}s$$

其中$\mathbf{n}$为L的外法线方向的单位向量.试证明

$$\Phi=\iint_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{d}\sigma$$

Proof.

根据**C.1**可知$\mathbf{n}=(\cos\beta,-\cos\alpha)$.于是

$$
\begin{aligned}
\Phi &= \oint_{L^+} \mathbf{v}\cdot\mathbf{n}\mathrm{d}s \\
&= \oint_{L^+} (P\cos\beta - Q\cos\alpha)\mathrm{d}s \\
&= \oint_{L^+} (P\mathrm{d}y - Q\mathrm{d}x) \\
&= \iint_D \left(\frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y}\right)\right)\mathrm{d}\sigma \\
&= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mathrm{d}\sigma
\end{aligned}
$$

于是命题得证.上述命题的另一形式为

$$
\oint_{L^+} (P,Q)\cdot\mathbf{n}\mathrm{d}s = \oint_{L^+} -Q\mathrm{d}x + P\mathrm{d}y = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mathrm{d}\sigma
$$

其中$\dfrac{\partial P}{\partial x}+\dfrac{\partial Q}{\partial y}$称为向量场$\mathbf{v}$的**散度**,因此格林公式在物理上也被称为**散度定理**.
散度定理指出,稳定流体通过某一闭曲线的流量,等于其散度在该闭曲线所包的区域上的二重积分之值.

C.3 设函数$u(x,y)$和$v(x,y)$在有界闭区域D上有连续的二阶偏导数,D的边界L逐段光滑.

(1) 试证明

$$
\iint_D v\Delta u\mathrm{d}\sigma = \oint_{L^+} v\frac{\partial u}{\partial \mathbf{n}}\mathrm{d}s - \iint_D \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}\sigma
$$

其中$\mathbf{n}$为L的外法线方向的单位向量.

(2) 试证明

$$
\iint_D (u\Delta v - v\Delta u)\,\mathrm{d}\sigma = \oint_{L^+} \left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\right)\mathrm{d}s
$$

Proof.

(1) 我们有

$$
\frac{\partial u}{\partial \mathbf{n}} = \left(\frac{\partial u}{\partial x}\cos(\mathbf{n},y), -\frac{\partial u}{\partial y}\cos(\mathbf{n},x)\right)
$$

于是

$$
\begin{aligned}
\oint_{L^+} v\frac{\partial u}{\partial \mathbf{n}}\mathrm{d}s &= \oint_{L^+} v\frac{\partial u}{\partial x}\mathrm{d}y - v\frac{\partial u}{\partial y}\mathrm{d}x \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)\right]\mathrm{d}\sigma \\
&= \iint_D \left[v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right)\right]\mathrm{d}\sigma
\end{aligned}
$$

移项即可得到欲证等式.

(2) 与**(1)**同理有

$$\iint_D u\Delta v \mathrm{d}\sigma = \oint_{L^+} u\frac{\partial v}{\partial \mathbf{n}}\mathrm{d}s - \iint_D \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}\sigma$$

相减即有

$$\iint_D (u\Delta v - v\Delta u)\,\mathrm{d}\sigma = \oint_{L^+}\left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\right)\mathrm{d}s$$

C.4 设D是有界平面区域,其边界L分段光滑,定点$P_0(x_0,y_0)\notin L$.设L上一点$P(x,y)$,向量$\mathbf{n}_P$为P处L的外侧法向量.定义向量$\mathbf{r}_P=\overrightarrow{P_0P}$,定义函数$f(x,y)$为

$$f(P)=\frac{\cos(\mathbf{r}_P,\mathbf{n}_P)}{|\mathbf{r}_P|}$$

计算曲线积分$\displaystyle\oint_L f(x,y)\mathrm{d}s$.

Solution.

设P处沿L正方向的单位切向量为$(\cos\alpha,\cos\beta)$,那么$\mathbf{n}_P=(\cos\beta,-\cos\alpha)$.于是我们有

$$\begin{aligned} f(x,y) &= \frac{\cos(\mathbf{r}_P,\mathbf{n}_P)}{\mathbf{r}_P} = \frac{\mathbf{r}_P\cdot\mathbf{n}_P}{|\mathbf{r}_P|^2|\mathbf{n}_P|} \\ &= \frac{(x-x_0)\cos\beta-(y-y_0)\cos\alpha}{(x-x_0)^2+(y-y_0)^2} \end{aligned}$$

于是

$$\begin{aligned} \oint_L f(x,y)\mathrm{d}s &= \oint_L \frac{(x-x_0)\cos\beta-(y-y_0)\cos\alpha}{(x-x_0)^2+(y-y_0)^2}\mathrm{d}s \\ &= \oint_{L^+}\frac{(x-x_0)\mathrm{d}y-(y-y_0)\mathrm{d}x}{(x-x_0)^2+(y-y_0)^2} \end{aligned}$$

令$A(x,y)=\dfrac{-(y-y_0)}{(x-x_0)^2+(y-y_0)^2}$, $B(x,y)=\dfrac{x-x_0}{(x-x_0)^2+(y-y_0)^2}$.于是我们有

$$\frac{\partial A}{\partial y} = -\frac{1}{(x-x_0)^2+(y-y_0)^2}+\frac{2(y-y_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}=\frac{(y-y_0)^2-(x-x_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}$$

同理可得

$$\frac{\partial B}{\partial x}=\frac{-(x-x_0)^2+(y-y_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}$$

若$P_0\notin D$,那么A,B在D上有连续的一阶偏导数,从而根据格林公式有

$$\oint_L f(x,y)\mathrm{d}s=\oint_{L^+}A\mathrm{d}x+B\mathrm{d}y=\iint_D\left(\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}\right)\mathrm{d}\sigma=0$$

若$P_0\in D$,那么考虑P_0的邻域$E=\{(x,y)|(x-x_0)^2+(y-y_0)^2\leqslant\varepsilon^2\}$,其中$\varepsilon>0$.令$\varepsilon$充分小至$E\subset D$.

令E的边界为L_E,从而A,B在$D\backslash E$上有连续的一阶偏导数.我们有

$$0=\iint_{D\backslash E}\left(\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}\right)\mathrm{d}\sigma=\oint_{L^+}(A\mathrm{d}x+B\mathrm{d}y)+\oint_{L_E{}^-}(A\mathrm{d}x+B\mathrm{d}y)$$

做代换$x=\varepsilon\cos\theta+x_0,y=\varepsilon\sin\theta+y_0$,环路${L_E}^+$即$\theta$从0变化至$2\pi$的路径.于是

$$\begin{aligned}\oint_{L^+}A\mathrm{d}x+B\mathrm{d}y&=\oint_{{L_E}^+}A\mathrm{d}x+B\mathrm{d}y\\&=\int_0^{2\pi}\left[\frac{-\varepsilon\sin\theta}{\varepsilon^2}\cdot(-\varepsilon\sin\theta)+\frac{\varepsilon\cos\theta}{\varepsilon^2}\cdot\varepsilon\cos\theta\right]\mathrm{d}\theta\\&=\int_0^{2\pi}\mathrm{d}\theta=2\pi\end{aligned}$$

于是所求积分为

$$\oint_L f(x,y)\mathrm{d}s=\begin{cases}0,P_0\notin D\\2\pi,P\in D\end{cases}$$