Linear Algebra Done Right 3F

1. 证明:线性泛函不是满射就是零映射.

Proof.

设$\phi \in \mathcal{L}(V, \mathbb{F})$.
若存在$v \in V$使得$\phi(v) \neq 0$,那么对于任意$f \in \mathbb{F}$都有存在$\lambda := \dfrac{f}{\phi(v)} \in \mathbb{F}$使得$\phi(\lambda v) = f$,于是$\phi$是满射.
若对于任意$v \in V$都有$\phi(v) = 0$,那么显然ϕ是零映射.

2. 给出$\mathbb{R}^{[0,1]}$上的三个不同的线性泛函.

Solution.

$\phi_1 : \forall f \in \mathbb{R}^{[0,1]}, \phi_1(f) = 0.$

$\phi_2 : \forall f \in \mathbb{R}^{[0,1]}, \phi_2(f) = f(0).$

$\phi_3 : \forall f \in \mathbb{R}^{[0,1]}, \phi_3(f) = f(1).$

3. 设V是有限维的,且$v \in V (v \neq \mathbf{0})$.证明:存在$\phi \in V'$使得$\phi(v) = 1$.

Proof.

将v扩展为V的一组基$v_1, \cdots, v_m$,其中$v_1 = v$.这基的对偶基$\phi_1, \cdots, \phi_m$就满足$\phi_1(v) = 1$.

4. 设V是有限维的,且U是V的子空间,$U \neq V$.证明:存在$\phi \in V'$使得$\phi(u) = 0$对任意$u \in U$成立且$\phi \neq \mathbf{0}$.

Proof.

设U的一组基$u_1, \cdots, u_m$,将其扩展为V的一组基$u_1, \cdots, u_m, v_1, \cdots, v_n$.由于$U \neq V$,于是$n \geqslant 1$.
选取这组基的对偶基$\phi_1, \cdots, \phi_{m+n}$.令$\phi = \phi_{m+1}$,于是对于任意$u \in U$都有

$$\phi(u) = \phi(a_1 u_1 + \cdots + a_m u_m) = 0$$

这ϕ即符合题设.

5. 设$T \in \mathcal{L}(V, W), w_1, \cdots, w_m$是range T的基.于是对于任意$v \in V$,都存在唯一的$\phi_1(v), \cdots, \phi_m(v)$使得

$$Tv = \phi_1(v) w_1 + \cdots + \phi_m(v) w_m$$

从而定义了从V到$\mathbb{F}$的函数$\phi_1, \cdots, \phi_m$.证明函数$\phi_1, \cdots, \phi_m$中的每个都是V上的线性泛函.

Proof.

我们只需证明$\phi_1,\cdots,\phi_m$是V到$\mathbb{F}$的线性映射.

首先,$T(\mathbf{0})=\mathbf{0}$,又$w_1,\cdots,w_m$线性无关,于是$\phi_k(\mathbf{0})=0$对于任意$1\leqslant k\leqslant m$均成立.

考虑$Tv_k=w_k,1\leqslant k\leqslant m$,各$v_k$自然是线性无关的.根据题设,$\phi_k(v_k)=1$,其余$\phi_j(v_k)=0$.

对于任意$u,v\in V$有$T(u+v)=Tu+Tv$.两边展开有

$$\sum_{k=1}^{m}\phi_k(u+v)w_k=\sum_{k=1}^{m}\phi_k(u)w_k+\sum_{k=1}^{m}\phi_k(v)w_k=\sum_{k=1}^{m}\left(\phi_k(u)+\phi_k(v)\right)w_k$$

由于$w_1,\cdots,w_m$线性无关,因此表出$T(u+v)$的方式是唯一的.对比系数可得$\phi_k(u+v)=\phi_k(u)+\phi_k(v)$,于是各$\phi_k$满足可加性.

对于任意$v\in V$和任意$\lambda\in\mathbb{F}$都有$T(\lambda v)=\lambda Tv$.两边展开有

$$\sum_{k=1}^{m}\phi_k(\lambda v)w_k=\lambda\sum_{k=1}^{m}\phi_k(v)w_k$$

同理,对比系数可知$\phi_k(\lambda v)=\lambda\phi_k(v)$,于是各$\phi_k$满足齐次性.

综上可知$\phi_1,\cdots,\phi_m$是V上的线性泛函,命题得证.

6. 设$\phi,\beta\in V'$.证明:$\operatorname{null}\phi\subseteq\operatorname{null}\beta$当且仅当存在$c\in\mathbb{F}$使得$\beta=c\phi$.

Proof.

$\Rightarrow$:令$c=0$即可.此时$\beta=\mathbf{0}$,$\operatorname{null}\beta=V$,必然有$\operatorname{null}\phi\subseteq\operatorname{null}\beta$.

$\Leftarrow$:对于任意$v\in\operatorname{null}\phi$,都有$\beta(v)=c\phi(v)=c\cdot 0=0$,于是$v\in\operatorname{null}\beta$,因此$\operatorname{null}\phi\subseteq\operatorname{null}\beta$.

7. 设$V_1,\cdots,V_m$是向量空间.证明$(V_1\times\cdots\times V_m)'$与$V_1'\times\cdots\times V_m'$同构.

Proof.

在**3E.3.**中令$W=\mathbb{F}$即得证.

8. 设$v_1,\cdots,v_n$是V的基,$\phi_1,\cdots,\phi_n$是V'的对偶基.定义$\Gamma:V\to\mathbb{F}^n$和$\Lambda:\mathbb{F}^n\to V$为

$$\Gamma(v)=(\phi_1(v),\cdots,\phi_n(v))\ \text{和}\ \Lambda(a_1,\cdots,a_n)=a_1v_1+\cdots+a_nv_n$$

试证明Γ和Λ互为彼此的逆.

Proof.

对于任意$v \in V$有

$$(\Lambda \circ \Gamma)(v) = \Lambda(\Gamma(v)) = \phi_1(v)v_1 + \cdots + \phi_n(v)v_n = v$$

于是$\Lambda\Gamma = I$.对于任意$(c_1, \cdots, c_n) \in \mathbb{F}^n$有

$$(\Gamma \circ \Lambda)(c_1, \cdots, c_n) = \Gamma(\Lambda(c_1, \cdots, c_n)) = \Gamma(c_1v_1 + \cdots + c_nv_n) = (c_1, \cdots, c_n)$$

于是$\Gamma\Lambda = I$.于是两者互为对方的逆.

9. 设$m \in \mathbb{N}^*$,证明$\mathcal{P}_m(\mathbb{R})$的基$1, x, \cdots, x^m$的对偶基是$\phi_0, \cdots, \phi_m$.
其中对任意$0 \leqslant k \leqslant m$和任意$p \in \mathcal{P}_m(\mathbb{R})$有$\phi_k(p) = \dfrac{p^{(k)}(0)}{k!}$.

Proof.

考虑$p \in \mathcal{P}_m(\mathbb{R})$在$x = 0$处的$m$阶泰勒多项式和其Lagrange余项.

$$p(x) = \sum_{k=0}^{m} \frac{p^{(k)}(0)x^k}{k!} + \frac{p^{(m+1)}(\xi)x^{m+1}}{(m+1)!}$$

考虑到p的次数最高为m,于是必然有$p^{(k+1)}(\xi) = 0$.于是上式可以写为

$$p(x) = \sum_{k=0}^{m} \frac{p^{(k)}(0)x^k}{k!}$$

令$\phi_k(p) = \dfrac{p^{(k)}(0)}{k!}$,即有

$$p(x) = \phi_0(p)p_0 + \cdots + \phi_m(p)p_m$$

于是命题得证.

10. 设$m \in \mathbb{N}^*$.

(1) 证明$1, x-5, \cdots, (x-5)^m$是$\mathcal{P}_m(\mathbb{R})$的一组基.

(2) 写出上面这组基的对偶基.

Solution.

(1) 记$p_k = (x-5)^k, 0 \leqslant k \leqslant m$.设一组标量$c_0, \cdots, c_m$使得

$$\mathbf{0} = c_0p_0 + \cdots + c_mp_m$$

当且仅当$c_0 = \cdots = c_m = 0$时上式成立.否则,根据代数基本定理,至多存在m个根使得右边为0,这并不是零映射.

于是$p_0, \cdots, p_m$是长度为$m+1$的线性无关组.

又$\mathcal{P}_m(\mathbb{R})$的标准基$1, x, \cdots, x^m$长度为$m+1$.于是$p_0, \cdots, p_m$是$\mathcal{P}_m(\mathbb{R})$的一组基.

(2) 这基的对偶基为$\phi_0, \cdots, \phi_m$,满足$\phi_k(p) = \dfrac{p^{(k)}(5)}{k!}$.

证明方法与**9.**类似.

11. 设$v_1, \cdots, v_n$是V的基,$\phi_1, \cdots, \phi_m$是V'的相应的对偶基.设$\psi \in V'$,证明

$$\psi = \psi(v_1)\phi_1 + \cdots + \psi(v_n)\phi_n$$

Proof.

对于任意$v := b_1 v_1 + \cdots + b_n v_n \in V$有$\phi_k(v) = b_k$.于是

$$\begin{aligned}
\psi(v) &= \psi(b_1 v_1 + \cdots + b_n v_n) \\
&= b_1 \psi(v_1) + \cdots + b_n \psi(v_n) \\
&= \psi(v_1)\phi_1(v) + \cdots + \psi(v_n)\phi_n(v) \\
&= (\psi(v_1)\phi_1 + \cdots + \psi(v_n)\phi_n)(v)
\end{aligned}$$

于是$\psi = \psi(v_1)\phi_1 + \cdots + \psi(v_n)\phi_n$,命题得证.

12. 设$S, T \in \mathcal{L}(V, W)$.

(1) 证明$(S+T)' = S' + T'$.

(2) 证明$(\lambda T)' = \lambda T'$对任意$\lambda \in \mathbb{F}$都成立.

Proof.

(1) 对任意$\phi \in W'$,有

$$(S+T)'(\phi) = \phi \circ (S+T) = \phi \circ S + \phi \circ T = S'(\phi) + T'(\phi) = (S' + T')(\phi)$$

于是$(S+T)' = S' + T'$.

(2) 对任意$\phi \in W'$和任意$\lambda \in \mathbb{F}$有

$$(\lambda T)'(\phi) = \phi \circ (\lambda T) = \lambda(\phi \circ T) = \lambda T'(\phi)$$

于是$(\lambda T)' = \lambda T'$.

13. 证明V上的恒等算子的对偶映射是V'上的恒等算子.

Proof.

设$I \in \mathcal{L}(V)$是V上的恒等映射.

对于任意$\phi \in V'$和任意$v \in V$,都有

$$(I'(\phi))(v) = (\phi \circ I)(v) = \phi(I(v)) = \phi(v)$$

这表明$I'(\phi) = \phi$,于是I'是V'上的恒等映射.命题得证.

14. 定义$T : \mathbb{R}^3 \to \mathbb{R}^2$为$T(x, y, z) = (4z + 5y + 6z, 7x + 8y + 9z)$.

设ϕ_1, ϕ_2为$\mathbb{R}^2$的标准基的对偶基,ψ_1, ψ_2, ψ_3为$\mathbb{R}^3$的标准基的对偶基.

(1) 描述$T'(\phi_1), T'(\phi_2)$这两个线性泛函.

(2) 将$T'(\phi_1), T'(\phi_2)$分别写成ψ_1, ψ_2, ψ_3的线性组合.

Solution.

(1) $T'(\phi_1) : (x, y, z) \mapsto 4x + 5y + 6z$,$T'(\phi_2) : (x, y, z) \mapsto 7x + 8y + 9z$.

(2) $T'(\phi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3, T'(\phi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$.

15. 定义$T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$为$(Tp)(x) = x^2 p(x) + p''(x)$对任意$x \in \mathbb{R}$成立.

(1) 设$\phi \in \mathcal{P}(\mathbb{R})'$定义为$\phi(p) = p'(4)$.描述$\mathcal{P}(\mathbb{R})$上的线性泛函$T'(\phi)$.

(2) 设$\phi \in \mathcal{P}(\mathbb{R})'$定义为$\phi(p) = \displaystyle\int_0^1 p\mathrm{d}x$.计算$(T'(\phi))(x^3)$.

Solution.

(1) 对任意$p(x) \in \mathcal{P}(\mathbb{R})$有

$$(T'(\phi))(p(x)) = (\phi \circ T)(p(x)) = \phi(x^2 p(x) + p''(x)) = (x^2 p + p'')'(4) = 8p(4) + 16p'(4) + p'''(4)$$

于是$T'(\phi)$就将任意的p映射到$8p(4) + 16p'(4) + p'''(4)$.

(2) 我们有

$$(T'(\phi))(x^3)=\phi(T(x^3))=\int_0^1(x^5+6x)\mathrm{d}x=\frac{19}{6}$$

16. 设W是有限维的,$T\in\mathcal{L}(V,W)$.证明$T=\mathbf{0}$当且仅当$T'=\mathbf{0}$.

Proof.

$\Rightarrow$:对任意$\phi\in W'$和任意$v\in V$,有$(T'(\phi))(v)=(\phi\circ T)(v)=\phi(\mathbf{0})=\mathbf{0}$,于是$T'=\mathbf{0}$.

$\Leftarrow$:对任意$\phi\in W'$和任意$v\in V$,有$\phi(T(v))=(T'(\phi))(v)=\mathbf{0}(v)=\mathbf{0}$,于是$Tv=\mathbf{0}$,即$T=\mathbf{0}$.

17. 设V和W是有限维的,$T\in\mathcal{L}(V,W)$.证明T可逆当且仅当T'可逆.

Proof.

由于T可逆,于是$\dim V=\dim W=\dim V'=\dim W'$.从而

$$T\text{可逆}\Leftrightarrow T\text{是单射}\Leftrightarrow T'\text{是满射}\Leftrightarrow T'\text{可逆}$$

18. 设V和W是有限维的,证明:将$T\in\mathcal{L}(V,W)$映射到$T'\in\mathcal{L}(W',V')$的映射是$\mathcal{L}(V,W)$到$\mathcal{L}(W',V')$的同构.

Proof.

设$\Phi:T\mapsto T'$.据**12.**可知Φ是线性的,据**16.**可知$\operatorname{null}\Phi=\mathbf{0}$,故$\Phi$是单射.

又$\dim\mathcal{L}(V,W)=(\dim V)(\dim W)=(\dim W')(\dim V')=\dim\mathcal{L}(W',V')$,于是$\Phi$是这两个空间的同构映射.

19. 设$U\subseteq V$,解释为何$U^0=\{\phi\in V':U\subseteq\operatorname{null}\phi\}$.

Proof.

我们知道U^0的定义是$\phi(u)=\mathbf{0}$对所有$u\in U$成立.这只要使$U\subseteq\operatorname{null}\phi$即可.

20. 设V是有限维的,且U是V的子空间.证明:$U=\{v\in V:\phi(v)=0\text{对任意}\phi\in U^0\text{都成立}\}$.

Proof.

设$W=\{v\in V:\phi(v)=0$对任意$\phi\in U^0$都成立$\}$.

我们知道对任意$\phi\in U^0$和任意$u\in U$有$\phi(u)=0$,于是$U\subseteq W$.

设$u_1,\cdots,u_m$是U的一组基,将其扩展为V的一组基$u_1,\cdots,u_m,v_1,\cdots,v_n$.令$\phi\in V'$满足

$$\phi(u_1)=\cdots=\phi(u_m)=0,\phi(v_1)=\cdots=\phi(v_n)=1$$

于是对于任意$u\in U$有$\phi(u)=0$,即$\phi\in U^0$.

而对于任意$v:a_1u_1+\cdots+a_mu_m+b_1v_1+\cdots+b_nv_n\in V$且$v\notin U$有

$$\phi(v)=\sum_{i=1}^n b_n\neq 0$$

即$v\notin W$,从而$W\subseteq U$.综上可知$U=W$.

21. 设V是有限维的,且U和W是V的子空间.

(1) 证明:$W^0\subseteq U^0$当且仅当$U\subseteq W$.

(2) 证明:$W^0=U^0$当且仅当$U=W$.

Proof.

设$A_U=\{v\in V:\phi(v)=\mathbf{0}$对任意$\phi\in U^0$都成立$\}$.

(1) $\Rightarrow$:$W^0\subseteq U^0$即任意$\phi\in W^0$有$\phi\in U^0$,于是任意$v\in A_U$有$\phi(v)=\mathbf{0}$,即$\phi\in A_W$.

这表明$A_U\subseteq A_W$,结合**20.**即$U\subseteq W$.

$\Leftarrow$:对于任意$\phi\in W^0$和任意$v\in U\subseteq W$有$\phi(v)=\mathbf{0}$,即$\phi\in U^0$.于是$W^0\subseteq U^0$.

(2) 我们有

$$W^0=U^0\Leftrightarrow W^0\subseteq U^0,U^0\subseteq W^0\Leftrightarrow U\subseteq W,W\subseteq U\Leftrightarrow U=W$$

22. 设V是有限维的,且U和W是V的子空间.

(1) 证明:$(U+W)^0=U^0\cap W^0$.

(2) 证明:$(U\cap W)^0=U^0+W^0$.

Proof.

(1) 假定$\phi\in(U+W)^0$,于是对于任意$u\in U,w\in W$有

$$\phi(u+w)=\phi(u)+\phi(w)=0$$

于是对任意$u \in U, w \in W$有$\phi(u) = \phi(w) = 0$,即$\phi \in U^0$且$\phi \in W^0$,即$(U+W)^0 \in U^0 \cap W^0$.

对于任意$\psi \in V'$满足$\psi \in U^0$且$\psi \in W^0$.对于任意$v \in U+W$,可写作$v = u + w$,其中$u \in U, w \in W$.于是

$$\psi(v) = \psi(u+w) = \psi(u) + \psi(w) = 0 + 0 = 0$$

即$\psi \in (U+W)^0$,从而$U^0 + W^0 \subseteq (U+W)^0$.

综上可知$(U+W)^0 = U^0 \cap W^0$.

(2) 假定$\phi \in (U \cap W)^0$.设$\alpha \in U^0, \beta \in W^0$满足

23. 设V是有限维的,$\phi_1, \cdots, \phi_m \in V'$.证明下面三个集合彼此相等.

(1) $\operatorname{span}(\phi_1, \cdots, \phi_m)$.

(2) $\left((\operatorname{null} \phi_1) \cap \cdots \cap (\operatorname{null} \phi_m)\right)^0$.

(3) $\{\phi \in V' : (\operatorname{null} \phi_1) \cap \cdots \cap (\operatorname{null} \phi_m) \subseteq \operatorname{null} \phi\}$

Proof.

据**19.**可知**(2)**与**(3)**相等.

根据**22.**可知

$$\begin{aligned}\left((\operatorname{null} \phi_1) \cap \cdots \cap (\operatorname{null} \phi_m)\right)^0 &= (\operatorname{null} \phi_1)^0 + \left((\operatorname{null} \phi_2) \cap \cdots \cap (\operatorname{null} \phi_m)\right)^0 \\ &= \operatorname{span}(\phi_1) + \left((\operatorname{null} \phi_2) \cap \cdots \cap (\operatorname{null} \phi_m)\right)^0\end{aligned}$$

反复递推可知

$$\left((\operatorname{null} \phi_1) \cap \cdots \cap (\operatorname{null} \phi_m)\right)^0 = \operatorname{span}(\phi_1) + \cdots + \operatorname{span}(\phi_m) = \operatorname{span}(\phi_1, \cdots, \phi_m)$$

即**(1)**与**(2)**相等.于是题中的三个集合相等.

24. 设V是有限维的,且$v_1, \cdots, v_m \in V$.定义$\Gamma \in \mathcal{L}(V', \mathbb{F}^m)$为$\Gamma(\phi) = (\phi(v_1), \cdots, \phi(v_m))$.

(1) 证明$v_1, \cdots, v_m$张成V当且仅当Γ是单射.

(2) 证明$v_1, \cdots, v_m$线性无关当且仅当Γ是满射.

Proof.

令$e_1, \cdots, e_m$是$\mathbb{F}^m$的标准基,令$\psi_1, \cdots, \psi_m$是这标准基的对偶基.

令$\Psi \in \mathcal{L}(\mathbb{F}^m, (\mathbb{F}^m)')$为$\Psi(e_k) = \psi_k$.令$T \in \mathcal{L}(\mathbb{F}^m, V)$为$Te_k = v_k$.

对于任意$\phi \in V'$和任意$1 \leqslant k \leqslant m$有

$$\begin{aligned}(T'(\phi))(e_k) &= \phi(Te_k) = \phi(v_k) \\ &= \sum_{j=1}^{m} \phi(v_j)\psi_j(e_k) \\ &= \left[\Psi\left(\phi(v_1), \cdots, \phi(v_m)\right)\right](e_k) \\ &= \left[\Psi(\Gamma(\phi))\right](e_k)\end{aligned}$$

于是$\Psi \circ \Gamma = T'$.由于Ψ是可逆的,于是T'和Γ的单射和满射性互相等价.

(1) 我们有

$$V = \operatorname{span}(v_1, \cdots, v_m) \Leftrightarrow T\text{是满射} \Leftrightarrow T'\text{是单射} \Leftrightarrow \Gamma\text{是单射}$$

(2) 我们有

$$v_1, \cdots, v_m\text{线性无关} \Leftrightarrow T\text{是单射} \Leftrightarrow T'\text{是满射} \Leftrightarrow \Gamma\text{是满射}$$

25. 设V是有限维的,且$\phi_1, \cdots, \phi_m \in V'$.定义$\Gamma \in \mathcal{L}(V, \mathbb{F}^m)$为$\Gamma(v) = (\phi_1(v), \cdots, \phi_m(v))$.

(1) 证明$\phi_1, \cdots, \phi_m$张成V'当且仅当Γ是单射.

(2) 证明$\phi_1, \cdots, \phi_m$线性无关当且仅当Γ是满射.

Proof.

令$e_1, \cdots, e_m$是$\mathbb{F}^m$的标准基,令$\psi_1, \cdots, \psi_m$是这标准基的对偶基.令$\Psi \in \mathcal{L}(\mathbb{F}^m, (\mathbb{F}^m)')$为$\Psi(e_k) = \psi_k$.
对于任意$(x_1, \cdots, x_m) \in \mathbb{F}^m$和任意$v \in V$有

$$\begin{aligned}\left[\Gamma'(\Psi(x_1, \cdots, x_m))\right](v) &= \Psi(x_1, \cdots, x_m) \circ \Gamma(v) \\ &= (x_1\psi_1 + \cdots + x_m\psi_m)(\phi_1(v), \cdots, \phi_m(v)) \\ &= x_1\phi_1(v) + \cdots + x_m\phi_m(v) \\ &= (x_1\phi_1 + \cdots + x_m\phi_m)(v)\end{aligned}$$

这表明$\Gamma' \circ \Psi : \mathbb{F}^m \to V'$由上式给出.由于$\Psi$是可逆的,于是$\Gamma' \circ \Psi$和$\Gamma'$的单射和满射性互相等价.

(1) 我们有

$$V' = \operatorname{span}(\phi_1, \cdots, \phi_m) \Leftrightarrow \Gamma' \circ \Psi\text{是满射} \Leftrightarrow \Gamma'\text{是单射} \Leftrightarrow \Gamma\text{是单射}$$

(2) 我们有

$$\phi_1, \cdots, \phi_m\text{线性无关} \Leftrightarrow \Gamma' \circ \Psi\text{是单射} \Leftrightarrow \Gamma'\text{是满射} \Leftrightarrow \Gamma\text{是满射}$$

26. 设V是有限维的,且Ω是V'的子空间.证明:$\Omega = \{v \in V : \phi(v) = 0$对任意$\phi \in \Omega$成立$\}^0$.

Proof.

设$U = \{v \in V : \phi(v) = 0$对任意$\phi \in \Omega$成立$\}$.设$\phi_1, \cdots, \phi_m$为$\Omega$的一组基.

根据U的定义可知$U \subseteq ((\text{null } \phi_1) \cap \cdots \cap (\text{null } \phi_m))$.

现在,对于任意$v \in ((\text{null } \phi_1) \cap \cdots \cap (\text{null } \phi_m))$和给定的$\phi \in \Omega$有

$$\phi(v) = (a_1\phi_1 + \cdots + a_m\phi_m)(v) = a_1\phi_1(v) + \cdots + a_m\phi_m(v) = 0$$

这表明$v \in U$.于是$((\text{null } \phi_1) \cap \cdots \cap (\text{null } \phi_m)) \subseteq U$.根据**23.**可知

$$U^0 = ((\text{null } \phi_1) \cap \cdots \cap (\text{null } \phi_m))^0 = \text{span}(\phi_1, \cdots, \phi_m) = \Omega$$

命题得证.

27. 设$T \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}))$且$\text{null } T' = \text{span}(\phi)$,其中$\phi \in (\mathcal{P}(\mathbb{R}))'$,定义为$\phi(p) = p(8)$.试证明

$$\text{range } T = \{p \in \mathcal{P}_5(\mathbb{R}) : p(8) = 0\}$$

Proof.

据**20.**有$\text{range } T = \{p \in \mathcal{P}_5(\mathbb{R}) : \psi(p) = 0$对任意$\psi \in (\text{range } T)^0$成立$\}$.

又$(\text{range } T)^0 = \text{null } T' = \text{span}(\phi)$,于是

$$\begin{aligned}\psi(p) = 0\text{对任意}\psi \in (\text{range } T)^0\text{成立} &\Leftrightarrow \psi(p) = 0\text{对任意}\psi \in \text{span}(\phi)\text{成立}\\ &\Leftrightarrow \lambda\phi(p) = 0\text{对任意}\lambda \in \mathbb{F}\text{成立}\\ &\Leftrightarrow \phi(p) = 0 \Leftrightarrow p(8) = 0\end{aligned}$$

于是$\text{range } T = \{p \in \mathcal{P}_5(\mathbb{R}) : p(8) = 0\}$.命题得证.

28. 设V是有限维的,且$\phi_1, \cdots, \phi_m \in V'$线性无关.证明$\dim((\text{null } \phi_1) \cap \cdots \cap (\text{null } \phi_m)) = (\dim V) - m$.

Proof.

令$U = (\text{null } \phi_1) \cap \cdots \cap (\text{null } \phi_m)$.

据**23.**可知$\text{span}(\phi_1,\cdots,\phi_m)=U^0$.又有$\dim V=\dim U+\dim U^0$.于是

$$\begin{aligned}\dim U&=\dim V-\dim U^0\\&=\dim V-\dim(\text{span}(\phi_1,\cdots,\phi_m))\\&=\dim V-m\end{aligned}$$

于是命题得证.

29. 设V和W是有限维的,且$T\in\mathcal{L}(V,W)$.

(1) 证明:如果$\phi\in W'$且$\text{null }T'=\text{span}(\phi)$,那么$\text{range }T=\text{null }\phi$.

(2) 证明:如果$\psi\in V'$且$\text{range }T'=\text{span}(\psi)$,那么$\text{null }T=\text{null }\psi$.

Proof.

(1) 据**20.**有$\text{range }T=\{w\in W:\psi(w)=0\text{对任意}\psi\in(\text{range }T)^0\text{成立}\}$.

又$(\text{range }T)^0=\text{null }T'=\text{span}(\phi)$,于是

$$\begin{aligned}\psi(w)=0\text{对任意}\psi\in(\text{range }T)^0\text{成立}&\Leftrightarrow\psi(w)=0\text{对任意}\psi\in\text{span}(\phi)\text{成立}\\&\Leftrightarrow\lambda\phi(w)=0\text{对任意}\lambda\in\mathbb{F}\text{成立}\\&\Leftrightarrow\phi(w)=0\Leftrightarrow w\in\text{null }\phi\end{aligned}$$

于是$\text{range }T=\text{null }\phi$,命题得证.

(2) 据**20.**有$\text{null }T=\{v\in V:\phi(v)=0\text{对任意}\phi\in(\text{null }T)^0\text{成立}\}$.

又$(\text{null }T)^0=\text{range }T'=\text{span}(\psi)$,于是

$$\begin{aligned}\psi(v)=0\text{对任意}\phi\in(\text{null }T)^0\text{成立}&\Leftrightarrow\phi(v)=0\text{对任意}\phi\in\text{span}(\psi)\text{成立}\\&\Leftrightarrow\lambda\psi(v)=0\text{对任意}\lambda\in\mathbb{F}\text{成立}\\&\Leftrightarrow\psi(v)=0\Leftrightarrow v\in\text{null }\psi\end{aligned}$$

于是$\text{null }T=\text{null }\psi$,命题得证.

30. 设V是有限维的,且$\phi_1,\cdots,\phi_m$是V'的一个基.试证明存在V的基使得其对偶基为$\phi_1,\cdots,\phi_m$.

Proof.

令$U_k=\displaystyle\bigcap_{j\in\{1,\cdots,m\}\setminus\{k\}}(\text{null }\phi_j)$.

据**28.**可知$\dim U_k = \dim V - (m-1) = 1$.又$\bigcap\limits_{j=1}^{m} (\text{null } \phi_j) = \{\mathbf{0}\}$.

于是一定存在$u_k \in U_k$使得$u_k \notin \text{null } \phi_k$,即$\phi_k(u_k) \neq 0$.定义$v_k = \dfrac{u_k}{\phi_k(u_k)}$,显然$v_k$可以作为$U_k$的基.

设$v = a_1 v_1 + \cdots + a_m v_m \in V$,于是$\phi_k(v) = a_k$.

这表明$v = \mathbf{0}$当且仅当$a_1 = \cdots = a_m = 0$,于是$v_1, \cdots, v_m$线性无关,又其长度为m,于是这向量组是V的基.

对于任意$1 \leqslant j \leqslant m$且$j \neq m$有

$$\begin{cases} \phi_k(v_k) = 1 \\ \phi_j(v_k) = 0, \forall j \neq k \end{cases}$$

于是$\phi_1, \cdots, \phi_m$是$v_1, \cdots, v_m$的对偶基.命题得证.

31. 设U是V的子空间,令$i : U \to V$为包含映射.

(1) 证明:null $i' = U^0$.

(2) 证明:如果V是有限维的,那么range $i' = U'$.

(3) 证明:如果V是有限维的,那么$\tilde{i'}$是V'/U^0映到U'的同构映射.

Proof.

(1) 我们有null $i' = (\text{range } i)^0 = U^0$.

(2) 对于任意$\phi \in U'$,都可以被扩充为V上的线性泛函ψ.
i'的定义表明$i'(\psi) = \psi \circ i = \phi$,由此$\phi \in \text{range } i'$,于是range $i' = U'$.

(3) 我们有$\dim(V'/U^0) = \dim V' - (\dim V - \dim U) = \dim U = \dim U'$.只需证明$\tilde{i'}$是单射即可.
对任意$\phi + U^0 \in V'/U^0$有

$$\tilde{i'}\left(\phi + U^0\right) = i'(\phi) = \phi \circ i$$

由于$i \neq \mathbf{0}$,于是$\tilde{i'}\left(\phi + U^0\right) = \mathbf{0}_{U'}$当且仅当$\phi = \mathbf{0}_{V'}$,即$\phi + U^0 = \mathbf{0}_{V'/U^0}$.于是$\tilde{i'}$是单射.
综上可知$\tilde{i'}$是V'/U^0映到U'的同构映射.

32. V的**双重对偶空间**记为V'',定义为V'的对偶空间.定义$\Lambda : V \to V''$为

$$(\Lambda v)(\phi) = \phi(v)$$

对任意$v \in V$和$\phi \in V'$都成立.

(1) 证明:Λ是V到V''的线性映射.

(2) 证明:如果$T \in \mathcal{L}(V)$,那么$T'' \circ \Lambda = \Lambda \circ T$.

(3) 证明:如果V是有限维的,那么Λ是V到V''的同构映射.

Proof.

(1) 以下的$\phi \in V'$是任意选取的.

$(\Lambda(\mathbf{0})_V)(\phi) = \phi(\mathbf{0}_V) = 0$,于是$\Lambda(\mathbf{0}_V) = \mathbf{0}_{V''}$.

对于任意$u, v \in V$有

$$(\Lambda(u+v))(\phi) = \phi(u+v) = \phi(u) + \phi(v) = (\Lambda(u))(\phi) + (\Lambda(v))(\phi) = (\Lambda(u) + \Lambda(v))(\phi)$$

于是$\Lambda(u+v) = \Lambda(u) + \Lambda(v)$,这表明$\Lambda$满足可加性.

对于任意$v \in V$和$\lambda \in \mathbb{F}$有

$$(\Lambda(\lambda v))(\phi) = \phi(\lambda v) = \lambda\phi(v) = \lambda \cdot (\Lambda(v))(\phi) = (\lambda\Lambda(v))(\phi)$$

于是$\Lambda(\lambda v) = \lambda(\Lambda(v))$,这表明$\Lambda$满足齐次性.

于是Λ是V到V''的线性映射.

(2) 对于任意$v \in V$和任意$\phi \in V'$有

$$\begin{aligned}[(T'' \circ \Lambda)(v)](\phi) &= [T''(\Lambda(v))](\phi) = [(\Lambda(v)) \circ T'](\phi) = (\Lambda(v))(T'(\phi)) \\ &= (T'(\phi))(v) = (\phi \circ T)(v) = \phi(Tv) = [\Lambda(Tv)](\phi) \\ &= [(\Lambda \circ T)(v)](\phi)\end{aligned}$$

这表明$T'' \circ \Lambda = \Lambda \circ T$,命题得证.

(3) 首先有$\dim V = \dim V' = \dim V''$.

设$v_1, \cdots, v_m$是V的一组基,$\phi_1, \cdots, \phi_m$是其对偶基.令$v := a_1v_1 + \cdots + a_nv_n \in V$使得$\Lambda v = \mathbf{0}_{V''}$.

即对于任意$\phi \in V'$,$\phi(v) = 0$.而对于任意$1 \leqslant k \leqslant m$有

$$\phi_k(v) = \phi_k(a_1v_1 + \cdots + a_nv_n) = a_k$$

于是$a_1 = \cdots = a_m = 0$.这表明$\Lambda v = \mathbf{0}$当且仅当$v = \mathbf{0}$,即$\text{null } \Lambda = \{\mathbf{0}\}$.于是$\Lambda$是单射.

综上可知Λ是V到V''的同构.

33. 设U是V的子空间.令$\pi : V \to V/U$是商映射.

(1) 证明:π'是单射.

(2) 证明:$\text{range } \pi' = U^0$.

(3) 证明:π'是$(V/U)'$映到U^0的同构映射.

Proof.

(1) 根据商映射的定义,对于任意$v + U \in V/U$都存在$v \in V$使得$\pi(v) = v + U$.于是π是满射,进而π'是单射.

(2) 对于任意$u \in U$有$u + U = \mathbf{0}_V + U = \mathbf{0}_{V/U}$,即$\text{null } \pi = U$.于是$\text{range } \pi' = (\text{null } \pi)^0 = U^0$.

(3) 由**(1)**可知π'是单射,由**(2)**可知π'是满射.于是π'是$(V/U)'$映到U^0的同构映射.