Linear Algebra Done Right 3D

1. 设$T \in \mathcal{L}(V,W)$可逆,证明T^{-1}可逆,且有$(T^{-1})^{-1} = T$.

Proof.

假设$v_1, v_2 \in V$且$Tv_1 = w_1, Tv_2 = w_2$.于是$T^{-1}w_1 = v_1, T^{-1}w_2 = v_2$.

从而$T^{-1}w_1 = T^{-1}w_2$必有$v_1 = v_2$,于是必有$w_1 = Tv_1 = Tv_2 = w_2$,即T^{-1}是单射.

对于任意$v \in V$都有$T^{-1}(Tv) = v$,于是range $T^{-1} = V$,进而T^{-1}是满射.

综上,T^{-1}可逆.下面证明$(T^{-1})^{-1} = T$.注意到$T^{-1}T = I, TT^{-1} = I$.

于是根据定义可知T是T^{-1}的逆,进而$T = (T^{-1})^{-1}$.

命题得证.

2. 设$T \in \mathcal{L}(U,V)$和$S \in \mathcal{L}(V,W)$可逆,试证明:$ST \in \mathcal{L}(U,W)$可逆,且$(ST)^{-1} = T^{-1}S^{-1}$.

Proof.

我们有

$$(T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = T^{-1}T = I$$

又有

$$(ST)(T^{-1}S^{-1}) = STT^{-1}S^{-1} = SS^{-1} = I$$

于是ST是可逆的且$(ST)^{-1} = T^{-1}S^{-1}$.命题得证.

3. 设V是有限维的,$T \in \mathcal{L}(V)$.证明下列命题是等价的.

(1) T是可逆的.

(2) 对于V的任意一组基$v_1, \cdots, v_n$,$Tv_1, \cdots, Tv_n$是V的基.

(3) 存在V的一组基$v_1, \cdots, v_n$使得$Tv_1, \cdots, Tv_n$是V的基.

Proof.

(1)⇒(2):由于T可逆,于是T既是单射又是满射.令$v := a_1v_1 + \cdots + a_nv_n \in V$,于是有

$$Tv = a_1Tv_1 + \cdots + a_nTv_n$$

由T是单射可知null $T = \mathbf{0}$.$Tv = \mathbf{0}$当且仅当$v = \mathbf{0}$,即$a_1 = \cdots = a_n = 0$.于是$Tv_1, \cdots, Tv_n$线性无关.又其长度为$\dim V$,于是$Tv_1, \cdots, Tv_n$是V的基.

(2)$\Rightarrow$**(3)**:这是显然成立的.

(3)$\Rightarrow$**(1)**:我们只需证明T是单射,即null $T=\mathbf{0}$.令$v:=a_1v_1+\cdots+a_nv_n\in V$,于是

$$Tv=a_1Tv_1+\cdots+a_nTv_n$$

由于$Tv_1,\cdots,Tv_n$是V的一组基,于是$Tv=\mathbf{0}$当且仅当$a_1=\cdots=a_n=0$,即$v=\mathbf{0}$.

于是null $T=\mathbf{0}$,于是T是单射,进而T可逆.

如此,这三个命题便等价.

4. 设V是有限维的且$\dim V>1$.试证明:从V到自身的不可逆线性映射构成的集合不是$\mathcal{L}(V)$的子空间.

Proof.

假定V的一组基为$v_1,\cdots,v_n$.构造线性映射$T_1,T_2\in\mathcal{L}(V)$如下

$$\begin{cases}T_1v_1=\mathbf{0}\\T_1v_k=v_k,2\leqslant k\leqslant n\end{cases}\text{与}\begin{cases}T_2v_n=\mathbf{0}\\T_2v_k=v_k,1\leqslant k<n\end{cases}$$

于是T_1,T_2均是不可逆的.然而,对于任意$v:=a_1v_1+\cdots+a_nv_n$有

$$\begin{aligned}(T_1+T_2)v&=(T_1+T_2)(a_1v_1+\cdots+a_nv_n)\\&=a_1v_1+2a_2v_2+\cdots+2a_{n-1}v_{n-1}+a_nv_n\end{aligned}$$

可知$(T_1+T_2)v=\mathbf{0}$当且仅当$a_1=\cdots=a_n=0$,即$v=\mathbf{0}$.

于是T_1+T_2是单射,进而它可逆,因而这个集合对加法不封闭,不是$\mathcal{L}(V)$的子空间.

命题得证.

5. 设V是有限维的,U是V的子空间,$S\in\mathcal{L}(U,V)$.试证明:存在可逆的$T\in\mathcal{L}(V)$使得$\forall u\in U,Tu=Su$,当且仅当S是单射.

Proof.

$\Rightarrow$:对于$u\in U\subseteq V$,$Su=\mathbf{0}$当且仅当$Tu=\mathbf{0}$.

由于T是单射,于是null $T=\mathbf{0}$,进而null $S=\mathbf{0}$,于是S也是单射.

$\Leftarrow$:设$u_1,\cdots,u_m$是U的一组基,将其扩展为V的一组基$u_1,\cdots,u_m,v_1,\cdots,v_n$.

因为S是单射,于是$Su_1,\cdots,Su_m$线性无关.将其扩展为V的一组基$Su_1,\cdots,Su_m,w_1,\cdots,w_n$.

令T满足$\begin{cases}Tu_k=Su_k,1\leqslant k\leqslant m\\Tv_j=w_j,1\leqslant j\leqslant n\end{cases}$.于是$T$是满射,进而$T$可逆.

命题得证.

6. 设W是有限维的,$S,T\in\mathcal{L}(V,W)$.试证明:$\text{null }S=\text{null }T$,当且仅当存在可逆的$E\in\mathcal{L}(W)$使得$S=ET$.

Proof.

$\Rightarrow$:因为W是有限维的,不妨设$\text{range }T$的一组基为$w_1,\cdots,w_m$.

于是存在线性无关的$v_1,\cdots,v_m$使得$\forall 1\leqslant k\leqslant m,Tv_m=w_m$.

现在证明$V=\text{null }T\oplus\text{span}(v_1,\cdots,v_m)$.对于任意$v\in V$,都存在$a_1,\cdots,a_m\in\mathbb{F}$使得

$$Tv=a_1w_1+\cdots+a_mw_m$$

于是$T(v-a_1v_1-\cdots-a_mv_m)=\mathbf{0}$,即$(v-a_1v_1-\cdots-a_mv_m)\in\text{null }T$.

于是$v=(v-a_1v_1-\cdots-a_mv_m)+(a_1v_1+\cdots+a_mv_m)$.又因为$v$的任意性,因而$V=\text{null }T+\text{span}(v_1,\cdots,v_m)$.

现在我们证明直和的条件成立,即$\text{null }T\cap\text{span}(v_1,\cdots,v_m)=\{\mathbf{0}\}$.

假定$v:=a_1v_1+\cdots+a_mv_m\in\text{null }T$,于是

$$Tv=a_1Tv_1+\cdots+a_mTv_m=a_1w_1+\cdots+a_mw_m=\mathbf{0}$$

由于$w_1,\cdots,w_m$是W的基,于是上式成立当且仅当$a_1=\cdots=a_m=0$,即$v=\mathbf{0}$,即$\text{null }T\cap\text{span}(v_1,\cdots,v_m)=\{\mathbf{0}\}$.从而$V=\text{null }T\oplus\text{span}(v_1,\cdots,v_m)$.

由于$v_1,\cdots,v_m$线性无关,于是$Sv_1,\cdots,Sv_m$也是线性无关的.

现在我们扩展$w_1,\cdots,w_m,e_1,\cdots,e_n$为$W$的一组基,扩展$Sv_1,\cdots,Sv_m$为$Sv_1,\cdots,Sv_m,f_1,\cdots,f_n$为$W$的另一组基.定义$E\in\mathcal{L}(W)$满足

$$\begin{cases}Ew_k=Sv_k,1\leqslant k\leqslant m\\ Ee_j=f_j,1\leqslant j\leqslant n\end{cases}$$

我们已经证明了$V=\text{null }T\oplus\text{span}(v_1,\cdots,v_m)$,又$\text{null }T=\text{null }S$,于是对于任意$v\in V$都可将其表示为$v=v_{null}+a_1v_1+\cdots+a_nv_n$.于是

$$\begin{aligned}(ET)v&=E(Tv)\\&=E(Tv_{null}+a_1Tv_1+\cdots+a_mTv_m)\\&=E(a_1w_1+\cdots+a_mw_m)\\&=\mathbf{0}+a_1Sv_1+\cdots+a_mSv_m\\&=S(v_{null}+a_1v_1+\cdots+a_mv_m)\\&=Sv\end{aligned}$$

又E是满射,于是E可逆.

$\Leftarrow$:假定$\text{null }S\neq\text{null }T$.

若$\exists v\in V,\text{s.t.}Sv\neq\mathbf{0},Tv=\mathbf{0}$,则有$(ET)v=E(Tv)=E(\mathbf{0})\neq\mathbf{0}$,这与$E$是线性映射矛盾.

若$\exists v\in V,\text{s.t.}Sv=\mathbf{0},Tv\neq\mathbf{0}$,则有$(ET)v=E(Tv)=\mathbf{0}$,这与$E$可逆矛盾(此时$E$不是单射).

于是$\text{null }S=\text{null }T$.

7. 设V是有限维的,$S, T \in \mathcal{L}(V, W)$.证明:range $S =$ range T当且仅当存在可逆的$E \in \mathcal{L}(V)$使得$S = TE$.

Proof.

$\Rightarrow$:设range T(即range S)的一组基为$w_1, \cdots, w_m$.

于是根据线性映射引理,存在一组线性无关的$v_1, \cdots, v_m \in V$使得$\forall 1 \leqslant k \leqslant m, Tv_k = w_k$.

也存在一组线性无关的$u_1, \cdots, u_m \in V$使得$\forall 1 \leqslant k \leqslant m, Su_k = w_k$.

将$v_1, \cdots, v_m$扩展为V的一组基$v_1, \cdots, v_m, e_1, \cdots, e_n$.

将$u_1, \cdots, u_m$扩展为V的另一组基$u_1, \cdots, u_m, f_1, \cdots, f_n$.

令$E \in \mathcal{L}(V)$满足

$$\begin{cases} Eu_k = v_k, 1 \leqslant k \leqslant m \\ Ef_j = e_j, 1 \leqslant j \leqslant n \end{cases}$$

于是对于任意$v := a_1u_1 + \cdots + a_mu_m + b_1f_1 + \cdots + b_nf_n \in V$有

$$\begin{aligned} (TE)v &= T(Ev) \\ &= T(a_1Eu_1 + \cdots + a_mEu_m + b_1Ef_1 + \cdots + b_nEf_n) \\ &= T(a_1v_1 + \cdots + a_nv_n + b_1e_1 + \cdots + b_ne_n) \\ &= a_1w_1 + \cdots + a_nw_n \\ &= S(a_1u_1 + \cdots + a_mu_m + b_1f_1 + \cdots + b_nf_n) \\ &= Sv \end{aligned}$$

于是$S = TE$.又E是满射,于是E可逆.

$\Leftarrow$:对于任意$v \in V$都有$Ev \in V$,又$Sv = T(Ev)$,这表明range $S \subseteq$ range T.

又对于任意$Ev \in V$都有对应的$v \in V$,又$Sv = T(Ev)$,这表明range $T \subseteq$ range S.

于是range $T =$ range S.

8. 设V和W都是有限维的,且$S, T \in \mathcal{L}(V, W)$.证明:存在可逆的$E_1 \in \mathcal{L}(V)$和$E_2 \in \mathcal{L}(W)$使得$S = E_2TE_1$,当且仅当$\dim \operatorname{null} S = \dim \operatorname{null} T$.

Proof.

$\Rightarrow$:因为$S = E_2TE_1$,又E_2可逆,于是$(E_2)^{-1}S = TE_1$.

根据**6.**可得null $S =$ null TE_1.又$\dim \operatorname{range} TE_1 = \dim \operatorname{range} T$,于是

$$\dim \operatorname{null} S = \dim \operatorname{null} TE_1 = \dim V - \dim \operatorname{range} TE_1 = \dim V - \dim \operatorname{range} T = \dim \operatorname{null} T$$

这就证明了$\dim \operatorname{null} S = \dim \operatorname{null} T$.

$\Leftarrow$:设null S的一组基为$u_1, \cdots, u_m$,null T的一组基为$v_1, \cdots, v_m$.

将它们分别扩展为V的基$u_1,\cdots,u_m$,$e_1,\cdots,e_n$和$v_1,\cdots,v_m$,$f_1,\cdots,f_n$.

而$Se_1,\cdots,Se_n$在W中线性无关(若否,则存在$e\in\operatorname{span}(e_1,\cdots,e_n)\neq\mathbf{0}$使得$Se=\mathbf{0}$,于是$e\in\operatorname{null}S$,这与各$u$与各$e$线性无关相悖).于是将$Se_1,\cdots,Se_n$扩展为$W$的一组基$Se_1,\cdots,Se_n,x_1,\cdots,x_p$.

同理将$Tf_1,\cdots,Tf_n$扩展为W的一组基$Tf_1,\cdots,Tf_n,y_1,\cdots,y_p$.

构造E_1,E_2满足

$$\begin{cases}E_1u_k=v_k,1\leqslant k\leqslant m\\E_1e_j=f_j,1\leqslant j\leqslant n\end{cases}\text{与}\begin{cases}E_2(Tf_j)=Se_j,1\leqslant j\leqslant n\\E_1x_i=y_i,1\leqslant i\leqslant p\end{cases}$$

于是对于任意$v:=a_1e_1+\cdots+a_ne_n+b_1u_1+\cdots+b_mu_m\in V$有

$$\begin{aligned}(E_2TE_1)v&=E_2T(E_1v)\\&=E_2T(a_1f_1+\cdots+a_nf_n+b_1v_1+\cdots+b_nv_n)\\&=E_2(a_1Tf_1+\cdots+a_nTf_n)\\&=a_1Se_1+\cdots+a_nSe_n+\mathbf{0}\\&=S(a_1e_1+\cdots+a_ne_n+b_1u_1+\cdots+b_mu_m)\\&=Sv\end{aligned}$$

从而$S=E_2TE_1$.又因为E_1,E_2都是基到基的映射,于是它们都可逆.

9. 设V是有限维的,且$T\in\mathcal{L}(V,W)$是满射.试证明:存在V的子空间U使得$T|_U$是由U映成W的同构.

Proof.

取W的一组基$w_1,\cdots,w_n$.由于T是满射,于是存在$v_1,\cdots,v_n\in V$使得$\forall 1\leqslant k\leqslant n,Tv_k=w_k$.

下面证明$v_1,\cdots,v_n$线性无关.设$v:=a_1v_1+\cdots+a_nv_n\in V$,于是

$$Tv=a_1Tv_1+\cdots+a_nTv_n=a_1w_1+\cdots+a_nw_n$$

由于$w_1,\cdots,w_n$是W的基,于是$Tv=\mathbf{0}$(即$v=\mathbf{0}$)当且仅当$a_1=\cdots=a_n=0$,于是$v_1,\cdots,v_n$线性无关.

令$U=\operatorname{span}(v_1,\cdots,v_n)$.下面证明$T|_U:U\to W$是同构.

我们已经知道$\operatorname{null}T|_U=\mathbf{0}$,即$T|_U$是单射.又$\dim U=\dim W$,于是$T|_U$可逆,进而它是$U$到$W$的同构.

如此,命题得证.

10. 设V和W是有限维的,且U是V的子空间.令$\mathcal{E}=\{T\in\mathcal{L}(V,W):U\subseteq\operatorname{null}T\}$.

(1) 试证明$\mathcal{E}$是$\mathcal{L}(V,W)$的子空间.

(2) 求$\dim\mathcal{E}$关于$\dim U,\dim V,\dim W$的表达式.

(1) Proof.

对于任意$T_1, T_2 \in \mathcal{E}$和任意$u \in U$都有

$$(T_1 + T_2)u = T_1 u + T_2 u = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

这表明$T_1 + T_2 \in \mathcal{E}$,于是$\mathcal{E}$对加法封闭.

对于任意$T = \in \mathcal{E}$,$\lambda \in \mathbb{F}$和任意$u \in U$都有

$$(\lambda T)u = \lambda(Tu) = \lambda \mathbf{0} = \mathbf{0}$$

这表明$\lambda T \in \mathcal{E}$,于是$\mathcal{E}$对标量乘法封闭.

于是$\mathcal{E}$是$\mathcal{L}(V, W)$的子空间,命题得证.

(2) Proof.

设U的一组基$u_1, \cdots, u_m$,将其扩展为V的一组基$u_1, \cdots, u_m, v_1, \cdots, v_n$.

不妨记$\dim W = p$,取W的一组基$w_1, \cdots, w_p$.考虑$p \times (m+n)$矩阵$A = \mathcal{M}(T)$.

于是$T \in \mathcal{E}$表明各$Tu_k = \mathbf{0}$,即A的前m列均为0,而后n列的元素不定.

注意到$\mathcal{M}$是$\mathcal{E}$到$\mathcal{M}(T)$的同构.如此,我们有

$$\dim \mathcal{E} = \dim \mathbb{F}^{p,n} = pn = (\dim V - \dim U) \cdot \dim W$$

11. 设V是有限维的,$S, T \in \mathcal{L}(V)$.试证明:ST可逆,当且仅当S和T都可逆.

Proof.

$\Rightarrow$:不妨记$(ST)^{-1} = R$.对于任意$v \in V$都有$v = Iv = R(ST)v = RS(Tv)$.

令$Tv = \mathbf{0}$.则$v = RS(\mathbf{0}) = \mathbf{0}$,这表明null $T = \{\mathbf{0}\}$,于是T是单射,即T是可逆的.

对于任意$v \in V$,又有$v = Iv = (STR)v = S(TRv)$.

这表明$\forall v \in V, \exists u := TRv \in V$, s.t.$Su = v$,即range $S = V$,于是S是满射,即S是可逆的.

$\Leftarrow$:在**2.**中令各向量空间均为V即可得证.

12. 设V是有限维的,且$S, T, U \in \mathcal{L}(V)$,且$STU = I$.试证明:$T$可逆,且$T^{-1} = US$.

Proof.

对于任意$v \in V$都有$STUv = ST(Uv) = v$.

令$Uv = \mathbf{0}$,于是$v = ST(\mathbf{0}) = \mathbf{0}$,于是null $U = \{\mathbf{0}\}$,进而U可逆(U是单射).

又$STUv = S(TUv) = v$,于是range $S = V$,即S也可逆(S是满射).

由$STU = I$且S, U均可逆可知$TU = S^{-1}, ST = U^{-1}$.于是

$$T(US) = (TU)S = S^{-1}S = I, (US)T = U(ST) = UU_{-1} = I$$

于是T可逆,并且$T^{-1} = US$.
命题得证.

13. 若**12.**中V不一定是有限维的,试证明其结论不一定成立.

Proof.
不妨令

$$S(x_1, x_2, x_3, \cdots) = (x_1, x_2, x_3, \cdots)$$
$$T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, \cdots)$$
$$U(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots)$$

此时仍有$STU = I$,然而T不是满射,因而也就不是可逆的.

14. 证明或给出一反例:如果V是有限维向量空间且$R, S, T \in \mathcal{L}(V)$使得$RST$是满射,那么$S$是单射.

Proof.
由于$RST \in \mathcal{L}(V)$,又其为满射,于是RST可逆.据**12.**,S也是可逆的,于是S为单射.

15. 设$T \in \mathcal{L}(V)$,且$v_1, \cdots, v_m \in V$使得$Tv_1, \cdots, Tv_m$张成V.试证明$v_1, \cdots, v_m$张成V.

Proof.
由$V = \operatorname{span}(Tv_1, \cdots, Tv_m)$可知range $T = V$,于是T是满射,进而T可逆.
对于任意$v \in V$,都有一组标量$a_1, \cdots, a_m$满足$Tv = a_1(Tv_1) + \cdots + a_m(Tv_m)$.
两边取逆即有$v = a_1v_1 + \cdots + a_mv_m$,即$V = \operatorname{span}(v_1, \cdots, v_m)$.
命题得证.

16. 试证明:从$\mathbb{F}^{n,1}$到$\mathbb{F}^{m,1}$的每个线性映射都能由一个矩阵乘法给出.换言之,证明:如果$T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$,那么存在$m \times n$矩阵$A$使得$Tx = Ax$对任意$x \in \mathbb{F}^{n,1}$成立.

Proof.

17. 设V是有限维的,且$S \in \mathcal{L}(V)$.定义$\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$如下

$$\mathcal{A}(T) = ST$$

对任意$T \in \mathcal{L}(V)$成立.

(1) 证明$\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.

(2) 证明$\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

Proof.

(1) 设null S的一组基为$u_1, \cdots, u_m$,将其扩展为V的一组基$v_1, \cdots, v_n$(其中前m项即为$u_1, \cdots, u_m$).
对于任意$T \in \operatorname{null} \mathcal{A}$,只需range $T \subseteq \operatorname{null} S$,即null $\mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$
于是$\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.命题得证.

(2) 我们知道$\dim \mathcal{L}(V) = (\dim V)^2$.
又$\dim \operatorname{range} S + \dim \operatorname{null} S = \dim V$,$\dim \operatorname{range} \mathcal{A} + \dim \operatorname{null} \mathcal{A} = \dim(\mathcal{L}(V)) = (\dim V)^2$.
于是$\dim \operatorname{range} \mathcal{A} = (\dim V)^2 - \dim \operatorname{null} \mathcal{A} = (\dim V)(\dim V - \dim \operatorname{null} S) = (\dim V)(\dim \operatorname{range} S)$.
命题得证.

18. 证明V和$\mathcal{L}(\mathbb{F}, V)$是同构的.

Proof.
我们有

$$\dim(\mathcal{L}(\mathbb{F}, V)) = (\dim(\mathbb{F}))(\dim V) = \dim V$$

于是V和$\mathcal{L}(\mathbb{F}, V)$同构.

19. 设V是有限维的,且$T \in \mathcal{L}(V)$.证明:T关于V的任意基的矩阵相同,当且仅当T是恒等算子的标量倍.

Proof.
$\Rightarrow$:设$v_1, \cdots, v_n$是V的一组基.
设$A = \mathcal{M}(T, (v_1, v_2, \cdots, v_n)), B = \mathcal{M}(T, (v_2, v_1, \cdots, v_n))$.
设$Tv_1 = A_{1,1}v_1 + \cdots + A_{n,1}v_n$,$Tv_2 = A_{1,2}v_1 + \cdots + A_{n,2}v_n$.

由于$A=B$,于是$Tv_2=A_{1,1}v_2+A_{2,1}v_1+\cdots+A_{n,1}v_n$,$Tv_1=A_{1,2}v_2+A_{2,2}v_1+\cdots+A_{n,2}v_n$.
比较参数可得

$$\begin{cases} A_{1,1}=A_{2,2} \\ A_{2,1}=A_{1,2} \end{cases}$$

同样地令$C=\mathcal{M}(T,(v_1,2v_2,\cdots,v_n))$,则有$Tv_1=A_{1,1}v_1+2A_{2,1}v_2+\cdots+A_{n,1}v_n$.比较参数可得

$$A_{2,1}=2A_{2,1}$$

于是$A_{2,1}=A_{1,2}=0$.将这样的操作应用于任意的$v_j,v_k(1\leqslant j,k\leqslant n)$可知

$$\begin{cases} A_{k,k}=A_{1,1},1\leqslant k\leqslant n \\ A_{j,k}=A_{j,k}=0,1\leqslant j<k\leqslant n \end{cases}$$

这表明$\mathcal{T}$除对角线的元素相同且不一定为0外,其余元素均为0.这自然是恒等矩阵I的标量倍.
进而我们知道T是恒等算子的标量倍.
$\Leftarrow$:对于任意V的一组基$v_1,\cdots,v_n$,记$T=\lambda I$,有

$$Tv_k=\lambda v_k$$

这表明$\mathcal{T}$的第k列除了第k行的元素为λ以外其余元素均为0.自然我们知道$\mathcal{M}(T)$是I的标量倍.

20. 设$q\in\mathcal{P}(\mathbb{R})$.试证明存在$p\in\mathcal{P}(\mathbb{R})$使得

$$\forall x\in\mathbb{R},q(x)=(x^2+x)p''(x)+2xp'(x)+p(3)$$

Proof.
设q的次数为m.定义线性映射$T:p(x)\mapsto(x^2+x)p''(x)+2xp'(x)+p(3)$.
假定$r\in\mathbb{R}^m$使得$Tr=\mathbf{0}$.那么一定有$r''(x)=0,r'(x)=0,r(3)=0$.这表明当且仅当$r=\mathbf{0}$时$Tr=\mathbf{0}$.
于是$T\in\mathcal{L}(\mathbb{R}^m)$是单射,进而它可逆并且是满射.于是对于任意$q\in\mathbb{R}^m$,都存在$p\in\mathbb{R}^m$使得$Tp=q$,于是命题得证.

21. 设$n\in\mathbb{N}^*$,且$A_{j,k}\in\mathbb{F}(j,k=1,\cdots,n)$.证明下面两个命题等价.

(1) 平凡解$x_1=\cdots=x_n=0$是下面的齐次方程组的唯一解.

$$\begin{cases} \sum\limits_{k=1}^{n}A_{1,k}x_k=0 \\ \vdots \\ \sum\limits_{k=1}^{n}A_{n,k}x_k=0 \end{cases}$$

(2) 对于任意$c_1, \cdots, c_n \in \mathbb{F}$,下列方程组都有解.

$$\begin{cases} \sum_{k=1}^{n} A_{1,k} x_k = c_1 \\ \qquad \vdots \\ \sum_{k=1}^{n} A_{n,k} x_k = c_n \end{cases}$$

Proof.
设$\mathbb{F}^n$的标准基$v_1, \cdots, v_n$.记$n \times n$矩阵A的第j行第k列元素由题设的$A_{j,k}$确定,代表了线性映射$T \in \mathcal{L}(\mathbb{R}^n)$.
假设向量$u := x_1 v_1 + \cdots + x_n v_n \in \mathbb{R}^n$.于是

$$\begin{aligned} Tu &= x_1 T v_1 + \cdots + x_n T v_n \\ &= x_1 \sum_{j=1}^{n} A_{j,1} v_j + \cdots + x_n \sum_{j=1}^{n} A_{j,n} \\ &= \left(\sum_{k=1}^{n} A_{1,k} x_1 \right) v_1 + \cdots + \left(\sum_{k=1}^{n} A_{n,k} x_n \right) v_n \end{aligned}$$

于是**(1)**中的方程等价于$Tu = \mathbf{0}$.由**(1)**的题设可知$Tu = \mathbf{0}$当且仅当$x_1 = \cdots = x_n = 0$,即$u = \mathbf{0}$.
于是null $T = \{\mathbf{0}\}$,即T是单射.
(2)中的方程等价于对于任意$v := c_1 v_1 + \cdots + c_n v_n \in V$,都存在$u \in V$使得$Tu = v$,即$T$是满射.
由于$T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$,于是$T$的单射性和满射性等价.从而**(1)**和**(2)**等价.

22. 设$T \in \mathcal{L}(V)$且$v_1, \cdots, v_n$是V的一组基.试证明:$\mathcal{M}(T, (v_1, \cdots, v_n))$可逆当且仅当$T$可逆.

Proof.
设$A = \mathcal{M}(T, (v_1, \cdots, v_n))$.
$\Rightarrow$:设$B = \mathcal{M}(S)$满足$AB = BA = I$. 自然有$\mathcal{M}(I, (v_1, \cdots, v_n)) = \mathcal{M}(T, (v_1, \cdots, v_n))\mathcal{M}(S, (v_1, \cdots, v_n))$.
即$TS = I$,同理$ST = I$.于是T可逆.
$\Leftarrow$:设$B = \mathcal{M}(T^{-1}, (v_1, \cdots, v_n))$.同理不难得出$AB = BA = I$,于是$\mathcal{M}(T, (v_1, \cdots, v_n))$可逆.

23. 设$u_1, \cdots, u_n$和$v_1, \cdots, v_n$是V的两组基.令$T \in \mathcal{L}(V)$使得$\forall 1 \leqslant k \leqslant n, Tv_k = u_k$.试证明

$$\mathcal{M}(T, (v_1, \cdots, v_n)) = \mathcal{M}(I, (u_1, \cdots, u_n), (v_1, \cdots, v_n))$$

Proof.

设$A = \mathcal{M}(T,(v_1,\cdots,v_n)), B = \mathcal{M}(T,(u_1,\cdots,u_n)), C = \mathcal{M}(I,(v_1,\cdots,v_n),(u_1,\cdots,u_n))$.

由于T是基到基的映射,于是T是可逆的.

由题意$\mathcal{M}(T,(v_1,\cdots,v_n),(u_1,\cdots,u_n)) = I$.

又$\mathcal{M}(T,(v_1,\cdots,v_n),(u_1,\cdots,u_n))\mathcal{M}(T^{-1},(u_1,\cdots,u_n),(v_1,\cdots,v_n)) = I$.

于是$\mathcal{M}(T^{-1},(u_1,\cdots,u_n),(v_1,\cdots,v_n)) = I$.于是

$$\begin{aligned}\mathcal{M}(T,(v_1,\cdots,v_n)) &= I\mathcal{M}(T,(v_1,\cdots,v_n))\\ &= \mathcal{M}(T^{-1},(u_1,\cdots,u_n),(v_1,\cdots,v_n))\mathcal{M}(T,(v_1,\cdots,v_n))\\ &= \mathcal{M}(I,(u_1,\cdots,u_n),(v_1,\cdots,v_n))\end{aligned}$$

命题得证.

24. 设A和B是相同大小的方阵且$AB = I$.试证明$BA = I$.

Proof.

我们有$(BA)B = B(AB) = BI = B$,又有$A(BA) = (AB)A = IA = I$.

于是$BA = I$.