Linear Algebra Done Right 6C

1. 设$v_1,\cdots,v_m\in V$.试证明

$$\{v_1,\cdots,v_m\}^{\perp}=(\operatorname{span}(v_1,\cdots,v_m))^{\perp}$$

Proof.

由于$\{v_1,\cdots,v_m\}\subseteq\operatorname{span}(v_1,\cdots,v_m)$,于是根据正交补的性质有$(\operatorname{span}(v_1,\cdots,v_m))^{\perp}\subseteq\{v_1,\cdots,v_m\}^{\perp}$.

对于任意$u\in\{v_1,\cdots,v_m\}^{\perp}$,都满足$\langle u,v_k\rangle=0$对任意$k\in\{1,\cdots,m\}$成立.

于是对于任意$v:=a_1v_1+\cdots+a_mv_m\in\operatorname{span}(v_1,\cdots,v_m)$有

$$\langle u,v\rangle=\left\langle u,\sum_{k=1}^{m}a_kv_k\right\rangle=\sum_{k=1}^{m}a_k\langle u,v_k\rangle=0$$

从而$u\in(\operatorname{span}(v_1,\cdots,v_m))^{\perp}$,因而$\{v_1,\cdots,v_m\}\subseteq(\operatorname{span}(v_1,\cdots,v_m))^{\perp}$.

综上可知$\{v_1,\cdots,v_m\}^{\perp}=(\operatorname{span}(v_1,\cdots,v_m))^{\perp}$.

2. 设U是V的子空间,且有一组基$u_1,\cdots,u_m$.向量组$u_1,\cdots,u_m,v_1,\cdots,v_n$是$V$的一组基.对上述$V$的基运用Gram-Schmidt过程得到$V$的规范正交基$e_1,\cdots,e_m,f_1,\cdots,f_n$.试证明:$e_1,\cdots,e_m$是$U$的规范正交基,$f_1,\cdots,f_n$是$U^{\perp}$的规范正交基.

Proof.

对$u_1,\cdots,u_m$应用Gram-Schmidt过程得到的$e_1,\cdots,e_m$自然是U的规范正交基.

对于任意$k\in\{1,\cdots,n\}$和任意$j\in\{1,\cdots,m\}$,都有$\langle f_k,e_j\rangle=0$,于是$f_k\in U^{\perp}$.

又因为$U\oplus U^{\perp}=V$,于是$\dim U^{\perp}=\dim V-\dim U=n$,因而$f_1,\cdots,f_n$是$U^{\perp}$的规范正交基.

3. 设U是$\mathbb{R}^4$的子空间,其定义为