Linear Algebra Done Right 8A

1. 设$T \in \mathcal{L}(V)$.试证明:若$\dim \operatorname{null} T^4 = 8, \dim \operatorname{null} T^6 = 9$,那么对任意整数$m \geqslant 5$都有$\dim \operatorname{null} T^m = 9$.

Proof.

根据**8.1**有$\operatorname{null} T^4 \subseteq \operatorname{null} T^5 \subseteq \operatorname{null} T^6$,于是$8 \leqslant \dim \operatorname{null} T^5 \leqslant 9$.

如果$\dim \operatorname{null} T^5 = 8$,则有$\operatorname{null} T^4 = \operatorname{null} T^5$,根据**8.2**应有$\operatorname{null} T^6 = \operatorname{null} T^4$,矛盾.

于是$\dim \operatorname{null} T^5 = 9$.

2. 设$T \in \mathcal{L}(V)$,$m \in \mathbb{N}^*$,$v \in V$,$T^{m-1}v \neq \mathbf{0}$且$T^m v = \mathbf{0}$.试证明:$v, Tv, \cdots, T^{m-1}v$线性无关.

Proof.

考虑$a_0, \cdots, a_{m-1} \in \mathbb{F}$使得

$$a_0 v + a_1 Tv + \cdots + a_{m-1} T^{m-1} v = \mathbf{0}$$

在等式两端作用T^{m-1}可得$a_0 T^{m-1} v = \mathbf{0}$,由于$T^{m-1}v \neq \mathbf{0}$,则$a_0 = 0$.

于是我们有

$$a_1 Tv + \cdots + a_{m-1} T^{m-1} v = \mathbf{0}$$

在等式两端作用T^{m-2},同理可得$a_1 = 0$.

依次递推可知$a_0 = \cdots = a_{m-1} = 0$,于是$v, Tv, \cdots, T^{m-1}v$线性无关.

3. 设$T \in \mathcal{L}(V)$.试证明$V = \operatorname{null} T \oplus \operatorname{range} T$当且仅当$\operatorname{null} T = \operatorname{null} T^2$.

Proof.

$\Rightarrow$:显然$\operatorname{null} T \subseteq \operatorname{null} T^2$.又有

$$T^2 v = \mathbf{0} \Rightarrow T(Tv) = \mathbf{0} \Rightarrow$$