Linear Algebra Done Right 7A

1. 设$n \in \mathbb{N}^*$,定义$T \in \mathcal{L}(\mathbb{F}^n)$为

$$T(z_1, \cdots, z_n) = (0, z_1, \cdots, z_{n-1})$$

求$T^*(z_1, \cdots, z_n)$的表达式.

Solution.

令$v = (a_1, \cdots, a_n), w = (b_1, \cdots, b_n)$,于是

$$\begin{aligned}
\langle Tv, w\rangle &= \langle (0, a_1, \cdots, a_{n-1}), (b_1, \cdots, b_n)\rangle \\
&= \sum_{k=1}^{n-1} a_k b_{k+1} \\
&= \langle (a_1, \cdots, a_n), (b_2, \cdots, b_n, 0)\rangle \\
&= \langle v, T^* w\rangle
\end{aligned}$$

于是

$$T^*(z_1, \cdots, z_n) = (z_2, \cdots, z_n, 0)$$

2. 设$T \in \mathcal{L}(V, W)$.试证明

$$T = \mathbf{0} \Leftrightarrow T^* = \mathbf{0} \Leftrightarrow TT^* = \mathbf{0} \Leftrightarrow T^*T = \mathbf{0}$$

Proof.

我们有

$$\begin{aligned}
T = \mathbf{0} &\Leftrightarrow \operatorname{null} T = V, \operatorname{range} T = \{\mathbf{0}\} \\
&\Leftrightarrow (\operatorname{range} T^*)^{\perp} = V, (\operatorname{null} T^*)^{\perp} = \{\mathbf{0}\} \\
&\Leftrightarrow \operatorname{range} T^* = \{\mathbf{0}\}, \operatorname{null} T = V \\
&\Leftrightarrow T^* = \mathbf{0}
\end{aligned}$$

从$T = \mathbf{0}$出发推出$TT^* = T^*T = \mathbf{0}$是容易的.现在假设$T^*T = \mathbf{0}$,于是对任意$v \in V$有

$$TT^* = \mathbf{0} \Rightarrow TT^*v = \mathbf{0} \Rightarrow \langle T^*Tv, v\rangle = 0 \Rightarrow \langle Tv, Tv\rangle = 0 \Rightarrow Tv = \mathbf{0}$$

这表明$T = \mathbf{0}$.

现在假设$TT^* = \mathbf{0}$,同理可推出$T^* = \mathbf{0}$,从而$T = \mathbf{0}$.

于是上述四个命题等价.

3. 设$T \in \mathcal{L}(V)$和$\lambda \in \mathbb{F}$.试证明:λ是T的特征值,当且仅当$\overline{\lambda}$是T^*的特征值.

Proof.

我们有

$$\begin{aligned} \lambda\text{是}T\text{的特征值} &\Leftrightarrow T - \lambda I\text{不是满射} \\ &\Leftrightarrow \text{range } (T - \lambda I) \neq V \\ &\Leftrightarrow \left(\text{null } (T - \lambda I)^*\right)^\perp \neq V \\ &\Leftrightarrow \left(\text{null } T^* - \overline{\lambda} I\right)^\perp \neq V \\ &\Leftrightarrow \text{null } \left(T^* - \overline{\lambda} I\right) \neq \{\mathbf{0}\} \\ &\Leftrightarrow T^* - \overline{\lambda} I\text{不是单射} \\ &\Leftrightarrow \overline{\lambda}\text{是}T^*\text{的特征值} \end{aligned}$$

4. 设$T \in \mathcal{L}(V)$且U是T的子空间.试证明U在T下不变,当且仅当$U^\perp$在T^*下不变.

Proof.

首先假设U在T下不变.对于任意$u \in U$和$w \in U^\perp$,我们有

$$\langle Tu, w \rangle = \langle u, T^* w \rangle$$

由于U在T下不变,于是$Tu \in U$.因此$\langle Tu, w \rangle = 0$,于是$\langle u, T^* w \rangle = 0$.

由于上式对任意$u \in U$成立,于是$T^* w \in U^\perp$,即$U^\perp$在T^*下不变.

由于$(T^*)^* = T$, $\left(U^\perp\right)^\perp = U$,于是将上述条件中的$U$和$T$分别替换为$U^\perp$和$T^*$即可证得另一方向.

5. 设$T \in \mathcal{L}(V, W)$,$e_1, \cdots, e_n$为V的规范正交基,$f_1, \cdots, f_m$为W的规范正交基.试证明:

$$||Te_1||^2 + \cdots + ||Te_n||^2 = ||T^* f_1||^2 + \cdots + ||T^* f_m||^2$$

Proof.

由于$f_1, \cdots, f_m$是W的规范正交基,于是对于任意$k \in \{1, \cdots, n\}$有

$$||Te_k||^2 = |\langle Te_k, f_1 \rangle|^2 + \cdots + |\langle Te_k, f_m \rangle|^2$$

同理,对于任意$j \in \{1, \cdots, m\}$有

$$||T^* f_j||^2 = |\langle T^* f_j, e_1\rangle|^2 + \cdots + |\langle T^* f_j, e_n\rangle|^2$$

又因为$\langle Te_k, f_j\rangle = \langle e_k, T^* f_j\rangle$,于是$|\langle Te_k, f_j\rangle|^2 = |\langle T^* f_j, e_k\rangle|^2$.于是

$$\sum_{k=1}^{n} ||Te_k||^2 = \sum_{k=1}^{n}\sum_{j=1}^{m} |\langle Te_k, f_j\rangle|^2 = \sum_{j=1}^{m}\sum_{k=1}^{n} |\langle T^* f_j, e_k\rangle|^2 = \sum_{j=1}^{m} ||T^* f_j||^2$$

6. 设$T \in \mathcal{L}(V, W)$,试证明下列命题.

(1) T是单射,当且仅当T^*是满射.

(2) T是满射,当且仅当T^*是单射.

Proof.

(1) 由于$\text{null } T = (\text{range } T^*)^\perp$,我们有

$$T\text{是单射} \Leftrightarrow \text{null } T = \mathbf{0} \Leftrightarrow \text{range } T^* = V \Leftrightarrow T^*\text{是满射}$$

(2) 由于$(T^*)^* = T$,于是交换**(1)**中的T和T^*即可.

7. 设$T \in \mathcal{L}(V, W)$,试证明下列命题.

(1) $\dim \text{null } T^* = \dim \text{null } T + \dim W - \dim V$.

(2) $\dim \text{range } T^* = \dim \text{range } T$.

Proof.

(1) 我们有

$$\dim \text{null } T^* = \dim (\text{range } T)^\perp = \dim W - \dim \text{range } T = \dim W - \dim V + \dim \text{null } T$$

(2) 我们有

$$\dim \text{range } T^* = \dim (\text{null } T)^\perp = \dim V - \dim \text{null } T = \dim \text{range } T$$

8. 设A为$m \times n$矩阵,试证明其行秩等于列秩.

Proof.

记$\overline{v}$为列向量v的复共轭.考虑A的列向量的张成空间的基$v_1, \cdots, v_l$,我们有

$$a_1\overline{v_1} + \cdots + a_l\overline{v_l} = \mathbf{0} \Rightarrow \overline{a_1}v_1 + \cdots + \overline{a_l}v_l = \mathbf{0} \Leftrightarrow \overline{a_1} = \cdots = \overline{a_l} = 0 \Leftrightarrow a_1 = \cdots = a_l = 0$$

从而$\overline{v_1}, \cdots, \overline{v_l}$线性无关.于是$\overline{A}$的列秩不小于$A$的列秩.交换两者可知$\overline{A}$与$A$的列秩相等.

现在我们假设A是线性映射$T \in \mathcal{L}(\mathbb{F}^m, \mathbb{F}^n)$对应于标准基的矩阵.于是

$$A\text{的列秩} = \dim \operatorname{range} T = \dim \operatorname{range} T^* = A^*\text{的列秩} = A^{\mathrm{t}}\text{的列秩} = A\text{的行秩}$$

9. 试证明:V上两自伴算子的乘积是自伴的,当且仅当这两个算子可交换.

Proof.

对于自伴的$S, T \in \mathcal{L}(V)$,我们有

$$(ST)^* = ST \Leftrightarrow T^*S^* = ST \Leftrightarrow TS = ST$$

10. 设$\mathbb{F} = \mathbb{C}$且$T \in \mathcal{L}(V)$.试证明:T是自伴的,当且仅当$\langle Tv, v\rangle = \langle T^*v, v\rangle$对任意$v \in V$成立.

Proof.

设$T \in \mathcal{L}(V)$,我们有

$$\begin{aligned}\langle Tv, v\rangle = \langle T^*v, v\rangle, \forall v \in V &\Leftrightarrow \langle (T - T^*)v, v\rangle = 0, \forall v \in V \\ &\Leftrightarrow T - T^* = \mathbf{0} \\ &\Leftrightarrow T\text{是自伴的}\end{aligned}$$

11. 定义算子$S : \mathbb{F}^2 \to \mathbb{F}^2$为$S(w, z) = (-z, w)$.回答下列问题.

(1) 求S^*的表达式.

(2) 试证明:S正规但不自伴.

(3) 求S的所有特征值.

Solution.

(1) 对于任意$v := (w_1, z_1), u := (w_2, z_2) \in \mathbb{F}^2$有

$$\langle Tv, u\rangle = (-z_1, w_1) \cdot (w_2, z_2) = -z_1 w_2 + w_1 z_2 = (w_1, z_1) \cdot (z_2, -w_2) = \langle v, (z_2, -w_2)\rangle$$

于是$S^*(w, z) = (z, -w)$.

(2) 不难验证$S^*S = SS^* = I$而$S \neq S^*$,于是S正规但不自伴.

(3) 当$\mathbb{F} = \mathbb{R}$时,容易验证S没有特征值.

当$\mathbb{F} = \mathbb{C}$时,令$v := (w, z) \in \mathbb{C}^2$,于是$Sv = \lambda v$即$\begin{cases} w = -\lambda z \\ z = \lambda w \end{cases}$.

解上述方程,得$\lambda = \pm \mathrm{i}$,于是S的特征值为$\mathrm{i}, -\mathrm{i}$.

12. 称算子$B \in \mathcal{L}(V)$是**斜的**,如果$B + B^* = \mathbf{0}$.设$T \in \mathcal{L}(V)$,试证明:T是正规算子,当且仅当存在可交换的$A, B \in \mathcal{L}(V)$且A是自伴算子,B是斜算子,使得$T = A + B$.

Proof.

$\Leftarrow$:设$T \in \mathcal{L}(V)$且存在满足题意的A, B使得$T = A + B$.那么$T^* = A^* + B^* = A - B$.于是

$$T^*T = (A - B)(A + B) = A^2 - BA + AB - B^2 = A^2 - B^2$$

$$TT^* = (A + B)(A - B) = A^2 + BA - AB - B^2 = A^2 - B^2$$

于是$TT^* = T^*T$,因而T是自伴的.

$\Rightarrow$:令$A = \dfrac{T + T^*}{2}, B = \dfrac{T - T^*}{2}$.于是

$$AB - BA = \frac{T^*T - TT^*}{2} = \mathbf{0}$$

又$A^* = A, B^* = -B$,因此存在满足条件的A, B使$T = A + B$.

13. 设$\mathbb{F} = \mathbb{R}$,定义$\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$为$\mathcal{A}T = T^*$对所有$T \in \mathcal{L}(V)$成立.回答下列问题.

(1) 求$\mathcal{A}$的所有特征值.

(2) 求$\mathcal{A}$的最小多项式.

Solution.

(1) 考虑$\mathcal{A}$的特征值$\lambda \in \mathbb{R}$,于是$\mathcal{A}T = \lambda T = T^*$.两边取伴随可得$\lambda T^* = T$,从而$(\lambda^2 - 1)T = \mathbf{0}$.

由于T是$\mathcal{A}$的特征向量,于是$T \neq \mathbf{0}$,于是$\lambda = \pm 1$.

于是$\mathcal{A}$的特征值为1或-1.

(2) 令$p(z) = z^2 - 1$.对于任意$T \in \mathcal{L}(V)$,我们有

$$p(\mathcal{A})T = \mathcal{A}^2 T - T = \mathcal{A}T^* - T = T - T = \mathbf{0}$$

又$\mathcal{A}$的特征值为$1, -1$.于是$p(z) = z^2 - 1$是$\mathcal{A}$的最小多项式.

14. 在$\mathcal{P}_2(\mathbb{R})$上定义内积$\langle \cdot, \cdot \rangle$为$\langle p, q \rangle = \int_0^1 pq$.定义算子$T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$为

$$T(ax^2 + bx + c) = bx$$

回答下列问题.

(1) 试证明:T不是自伴算子.

(2) T关于$\mathcal{P}_2(\mathbb{R})$的标准基$1, x, x^2$的矩阵为

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

这矩阵与它的共轭转置相等,尽管T不是自伴的.试解释这为什么不矛盾.

Proof.

(1) 考虑$p(x) = 1, q(x) = x \in \mathcal{P}_2(\mathbb{R})$.我们有

$$\langle Tp, q \rangle = \langle \mathbf{0}, x \rangle = 0$$

$$\langle p, Tq \rangle = \int_0^1 x \mathrm{d}x = \frac{1}{2}$$

于是$\langle Tp, q \rangle = \langle p, T^* q \rangle \neq \langle p, Tq \rangle$.于是$T \neq T^*$,因而它不自伴.

(2) $1, x, x^2$不是$\mathcal{P}_2(\mathbb{R})$的规范正交基.

15. 设$T \in \mathcal{L}(V)$可逆,试证明下列命题.

(1) T自伴,当且仅当T^{-1}自伴.

(2) T正规,当且仅当T^{-1}正规.

Proof.

(1) 由于$(T^*)^{-1} = (T^{-1})^*$,于是在$T = T^*$两边取逆可得$T^{-1} = (T^{-1})^*$,从而T与T^{-1}的自伴性等价.

(2) 同理,在$T^*T = TT^*$两边取逆可得$T^{-1}(T^{-1})^* = (T^{-1})^* T^{-1}$,于是$T$与$T^{-1}$的正规性等价.

16. 设$\mathbb{F} = \mathbb{R}$,回答下列问题.

(1) 试证明:V上的自伴算子构成的集合是$\mathcal{L}(V)$的子空间.

(2) 求**(1)**中的子空间的维数.

Proof.

(1) 设$U = \{T \in \mathcal{L}(V) : T = T^*\}$.显然$\mathbf{0}^* = \mathbf{0}$,于是$\mathbf{0} \in U$.
对于任意$S, T \in U$,都有$(S + T)^* = S^* + T^* = S + T$,于是$S + T \in U$,即$U$对加法封闭.
对于任意$T \in U$和$\lambda \in \mathbb{R}$,都有$(\lambda T)^* = \lambda T^* = \lambda T$,即$\lambda T \in U$,于是$U$对标量乘法封闭.
于是U是V的子空间.

(2) 令$\dim V = n$.考虑所有$n \times n$的自伴矩阵.对于任意$j, k \in \{1, \cdots, n\}$且$j \leqslant k$,定义$\mathcal{M}_{j,k}$如下

$$\mathcal{M}_{j,k} = \begin{cases} \mathcal{M}_{j,k} = \mathcal{M}_{k,j} = 1\text{且其余元素为}0, j \neq k \\ \mathcal{M}_{k,k} = 1\text{且其余元素为}0, j = k \end{cases}$$

容易验证这样的$\mathcal{M}_{j,k}$线性无关,且所有自伴矩阵都可以写为上述矩阵的线性组合.
具体来说,若$A \in \mathbb{R}^{n,n}$自伴,那么

$$A = \sum_{1 \leqslant j \leqslant k \leqslant n} A_{j,k} \mathcal{M}_{j,k}$$

因而$U = \operatorname{span}(\mathcal{M}_{1,1}, \cdots, \mathcal{M}_{n,n})$.于是$\dim U = \dfrac{n(n+1)}{2}$.

17. 设$\mathbb{F} = \mathbb{C}$.试证明:V上的自伴算子构成的集合不是$\mathcal{L}(V)$的子空间.

Proof.
对于$\lambda \in \mathbb{C} \backslash \mathbb{R}$有$\lambda \neq \overline{\lambda}$.于是对于某个自伴的$T \in \mathcal{L}(V)$有

$$(\lambda T)^* = \overline{\lambda} T^* = \overline{\lambda} T \neq \lambda T$$

从而λT不是自伴的,于是这集合对标量乘法不封闭,不是$\mathcal{L}(V)$的子空间.

18. 设$\dim V \geqslant 2$,试证明:V上的正规算子构成的集合不是$\mathcal{L}(V)$的子空间.

Proof.

不妨令$A, B \in \mathcal{L}(V)$关于V上的某规范正交基的矩阵为

$$\mathcal{M}(A) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \mathcal{M}(B) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

容易验证$A = A^*, B = B^*$,于是A, B都是自伴的.然而

$$\mathcal{M}((A+B)(A+B)^*) = \begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \mathcal{M}((A+B)^*(A+B))$$

从而$A + B$不是正规算子,于是这集合对加法不封闭.

19. 设$T \in \mathcal{L}(V)$且对任意$v \in V$都有$||T^*v|| \leqslant ||Tv||$.试证明$T$是正规算子.

Proof.

令$e_1, \cdots, e_n$是V的规范正交基,根据**7A.5**有

$$||Te_1||^2 + \cdots + ||Te_n||^2 = ||T^*e_1||^2 + \cdots + ||T^*e_n||^2$$

根据题意可知$||Te_k|| \geqslant ||T^*e_k||$对任意$k \in \{1, \cdots, n\}$成立.于是只能有$||Te_k|| = ||T^*e_k||$对任意$k \in \{1, \cdots, n\}$成立.于是对于任意$e \in V$且$||e|| = 1$都有$||Te|| = ||T^*e||$.于是对于任意$v \in V$有

$$\left|\left|T\left(\frac{v}{||v||}\right)\right|\right| = \left|\left|T^*\left(\frac{v}{||v||}\right)\right|\right|$$

于是$||Tv|| = ||T^*v||$,因而T是正规算子.

20. 设$P \in \mathcal{L}(V)$且$P^2 = P$.试证明下列命题是等价的.

(a) P是自伴的.

(b) P是正规的.

(c) 存在V的子空间U使得$P = P_U$.

Proof.

(a)⇒(b):假设$P = P^*$,显然有$P^*P = PP^* = P^2$.于是P是正规的.

(b)⇒(c):假设P是正规的,于是

$$\text{null } P = \text{null } P^* = (\text{range } P)^\perp$$

根据**6C.9**可知存在V的子空间U使得$P = P_U$.

(c)⇒(a):对于任意$v_1, v_2 \in V$,设$v_1 = u_1 + w_1, v_2 = u_2 + w_2$,其中$u_1, u_2 \in U, w_1, w_2 \in U^\perp$.于是

$$\langle P_U v_1, v_2 \rangle = \langle u_1, u_2 + w_2 \rangle = \langle u_1, u_2 \rangle = \langle u_1 + w_1, u_2 \rangle = \langle v_1, P_U v_2 \rangle$$

于是$P_U^* = P_U$,因而P_U是自伴算子.

21. 设$D : \mathcal{P}_8(\mathbb{R}) \to \mathcal{P}_8(\mathbb{R})$是定义为$Dp = p'$的微分算子.试证明:$\mathcal{P}_8(\mathbb{R})$上不存在使得$D$为正规算子的内积.

Proof.

根据**7A.27**可知如果D是正规算子,那么null $D^2 =$ null D.

然而$D^2(x) = \mathbf{0}, D(x) = 1$,于是null $D \neq$ null D^2,从而D无论如何不能是正规算子.

22. 给出一例$T \in \mathcal{L}(\mathbb{R}^3)$正规但不自伴.

Solution.

令T关于$\mathbb{R}^3$的标准基的矩阵为

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

于是$TT^* = T^*T = \mathbf{0}$,而$T \neq T^*$.

23. 设T是V上的正规算子,设$v, w \in V$满足

$$||v|| = ||w|| = 2, Tv = 3v, Tw = 4w$$

试证明:$||T(v + w)|| = 10$.

Proof.

由于v, w对应于T的不同特征值,又因为T是正规算子,于是v, w正交.于是

$$||T(v + w)|| = \sqrt{\langle 3v + 4w, 3v + 4w \rangle} = \sqrt{9||v||^2 + 16||w||^2} = \sqrt{100} = 10$$

24. 设$T \in \mathcal{L}(V)$且T的最小多项式为

$$a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m$$

试证明:T^*的最小多项式为

$$\overline{a_0} + \overline{a_1} z + \cdots + \overline{a_{m-1}} z^{m-1} + z^m$$

Proof.

设T的最小多项式为$p(z)$,其系数取复共轭得到的多项式为$\overline{p}(z)$.根据**7A.2**,我们有

$$p(T) = \mathbf{0} \Rightarrow (p(T))^* = \mathbf{0} \Rightarrow \overline{p}(T^*) = \mathbf{0}$$

假定$s \in \mathcal{P}(\mathbb{F})$且$\deg s < \deg p$,那么$\overline{s}$一定不是$T$的最小多项式.我们有

$$\overline{s}(T) \neq \mathbf{0} \Rightarrow (\overline{s}(T))^* \neq \mathbf{0} \Rightarrow s(T^*) \neq \mathbf{0}$$

于是T^*的最小多项式的次数不低于p的次数.

又$\overline{p}(T^*) = \mathbf{0}$且$\deg \overline{p} = \deg p$且$\overline{p}$是首一的,于是$\overline{p}$是$T^*$的最小多项式.

25. 设$T \in \mathcal{L}(V)$.试证明:T可对角化,当且仅当T^*可对角化.

Proof.

假设T可对角化,那么不妨设T的最小多项式为$p(z) = (z - \lambda_1) \cdots (z - \lambda_n)$.于是对于任意$k \in \{1, \cdots, n\}$有

$$p(\lambda_k) = 0 \Leftrightarrow \overline{p(\lambda_k)} = 0 \Leftrightarrow \overline{p}(\overline{\lambda_k}) = 0$$

于是$\overline{p}(z) = (z - \overline{\lambda_1}) \cdots (z - \overline{\lambda_n})$.由**7A.24**可知$\overline{p}(z)$是$T^*$的最小多项式,于是$T^*$可对角化.

交换T^*与T即可证得另一方向.于是命题得证.

26. 给定$u, x \in V$.定义$T \in \mathcal{L}(V)$为$Tv = \langle v, u \rangle x$对任意$v \in V$成立.回答下列问题.

(1) 试证明:如果V是实向量空间,那么T自伴当且仅当u, x线性相关.

(2) 试证明:T正规当且仅当u, x线性相关.

Proof.

注意到$T^*v = \langle v, x\rangle u$.

(1) $\Rightarrow$:若$u = \mathbf{0}$,那么显然u, x线性相关.否则,取任意$v \in V$且$v \neq 0$,则有

$$Tv = \langle v, u\rangle x = \langle v, x\rangle u = T^*v$$

于是$x = \dfrac{\langle v, x\rangle}{\langle v, u\rangle}u$,即$x, u$线性相关.

$\Leftarrow$:令$u = \lambda x$,其中$\lambda \in \mathbb{R}$.那么对于任意$v \in V$有

$$Tv = \langle v, u\rangle x = \langle v, \lambda x\rangle x = \lambda \langle v, x\rangle x = \langle v, x\rangle (\lambda x) = \langle v, x\rangle u = T^*v$$

从而$T^* = T$,于是T自伴.

(2) 注意到

$$(T^*T - TT^*)v = \langle v, x\rangle \langle u, u\rangle x - \langle v, u\rangle \langle x, x\rangle u$$

$\Leftarrow$若u, x线性相关,不妨令$u = \lambda x$,其中$\lambda \in \mathbb{F}$.于是

$$\begin{aligned}(T^*T - TT^*)v &= \langle v, x\rangle \langle \lambda x, \lambda x\rangle x - \langle v, \lambda x\rangle \langle x, x\rangle (\lambda x)\\ &= \langle v, x\rangle |\lambda|^2 ||x||^2 x - \overline{\lambda}\lambda \langle v, x\rangle ||x||^2 x\\ &= \mathbf{0}\end{aligned}$$

于是$T^*T - TT^* = \mathbf{0}$,即T正规.

$\Rightarrow$:若$u = \mathbf{0}$或$x = \mathbf{0}$,那么显然u, x线性相关.否则令$v = u$可得

$$\mathbf{0}u = \langle u, x\rangle \langle u, u\rangle x - \langle u, u\rangle \langle x, x\rangle u = \mathbf{0}$$

于是可得$x = \dfrac{\langle u, x\rangle}{||x||^2}u$,因而$u, x$线性相关.

27. 设$T \in \mathcal{L}(V)$正规.试证明:对于任意$k \in \mathbb{N}^*$,都有

$$\operatorname{null} T^k = \operatorname{null} T, \operatorname{range} T^k = \operatorname{range} T$$

Proof.

我们使用归纳法证明$\operatorname{null} T^k = \operatorname{null} T$.$k = 1$的情形是显然成立的.

现在设$k > 1$,并且命题对所有小于k的正整数都成立.

null $T^{k-1} \subseteq$ null T是显然的.现在,对于任意$v \in$ null T^k有

$$||T^k v|| = 0 \Leftrightarrow ||T(T^{k-1}v)|| = 0 \Leftrightarrow ||T^*(T^{k-1}v)|| = 0 \Leftrightarrow T^*T^{k-1}v = \mathbf{0}$$

于是我们有

$$\left\langle T^*T^{k-1}v, T^{k-2}v \right\rangle = 0 \Leftrightarrow \left\langle T^{k-1}v, T^{k-1}v \right\rangle = 0 \Leftrightarrow T^{k-1}v = \mathbf{0}$$

于是null $T^k \subseteq$ null T^{k-1},进而null $T^k =$ null $T^{k-1} =$ null T.

由于T是正规算子,于是

$$\text{range } T = (\text{null } T^*)^\perp = (\text{null } (T^*)^k)^\perp = (\text{null } (T^k)^*)^\perp = \text{range } T^k$$

综上,命题得证.

28. 设$T \in \mathcal{L}(V)$是正规的.试证明:如果$\lambda \in \mathbb{F}$,那么T的最小多项式不是$(x-\lambda)^2$的多项式倍.

Proof.

设$p \in \mathcal{P}(\mathbb{F})$是$T$的最小多项式.假定存在$r \in \mathcal{P}(\mathbb{F})$和$\lambda \in \mathbb{F}$使得$p(z) = (z-\lambda)^2 r(z)$.

由于$\deg(z-\lambda)r(z) < \deg p$,于是$(z-\lambda)r(z)$不是$T$的最小多项式,也即$r(T)v \notin$ null $(T-\lambda I)$.

又因为$p(T) = \mathbf{0}$,于是$r(T)v \in$ null $(T-\lambda I)^2$,因此null $(T-\lambda I) \neq$ null $(T-\lambda I)^2$.

因为T是正规算子,所以$T-\lambda I$是正规算子.根据**7A.27**可知null $(T-\lambda I)^2 =$ null $(T-\lambda I)$.

这与前面的推导矛盾,于是不存在$\lambda \in \mathbb{F}$使得$p(z)$是$(z-\lambda)^2$的多项式倍.

29. 证明或给出反例:如果$T \in \mathcal{L}(V)$且存在V的规范正交基$e_1, \cdots, e_n$使得$||Te_k|| = ||T^*e_k||$对任意$k \in \{1, \cdots, n\}$成立,那么T是正规的.

Proof.

令T关于$\mathbb{F}^2$的标准基e_1, e_2的矩阵为

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

我们在**7A.18**中已经证明T不是正规的,然而

$$||Te_1|| = ||T^*e_1|| = \sqrt{2}, ||Te_2|| = ||T^*e_2|| = 1$$

于是就给出了一个反例.

30. 设$T \in \mathcal{L}(\mathbb{F}^3)$是正规的且$T(1,1,1)=(2,2,2)$.设$(z_1,z_2,z_3) \in \operatorname{null} T$,试证明:$z_1+z_2+z_3=0$.

Proof.
令$u:=(z_1,z_2,z_3)$.若$u=\mathbf{0}$,那么命题显然成立.
若$u \neq \mathbf{0}$,那么u是对应于0的T的特征向量.又因为$(1,1,1)$是对应于2的T的特征向量,于是

$$u \cdot (1,1,1)=0$$

即$z_1+z_2+z_3=0$.

31. 给定$n \in \mathbb{N}^*$.在由$[-\pi,\pi]$上全体连续实值函数构成的,具有内积$\langle f,g\rangle=\displaystyle\int_{-\pi}^{\pi} fg$的内积空间中,令

$$V=\operatorname{span}(1,\cos x,\cos 2x,\cdots,\cos nx,\sin x,\sin 2x,\cdots,\sin nx)$$

回答下列问题.

(1) 定义$D \in \mathcal{L}(V)$为$Df=f'$.试证明$D^*=-D$.

(2) 定义$T \in \mathcal{L}(V)$为$Tf=f''$.试证明T是自伴的.

Proof.
根据**6B.4**可知

$$\frac{1}{\sqrt{2\pi}},\frac{\cos x}{\sqrt{\pi}},\cdots,\frac{\cos nx}{\sqrt{\pi}},\frac{\sin x}{\sqrt{\pi}},\cdots,\frac{\sin nx}{\sqrt{\pi}}$$

是$C[-\pi,\pi]$上的规范正交组.易知这就是V的一组规范正交基.不妨令$v=\dfrac{1}{\sqrt{2\pi}},e_k=\dfrac{\cos kx}{\sqrt{\pi}},f_k=\dfrac{\sin kx}{\sqrt{\pi}}$.

(1) 我们有

$$D^*v=\mathbf{0}=-Dv$$

$$D^*e_k=kf_k=-De_k$$

$$D^*f_k=-ke_k=-Df_k$$

于是$D=-D^*$.

(2) 注意到$T=D^2$.于是

$$T^*=\left(D^2\right)^*=\left(D^*\right)^2=(-D)^2=D^2=T$$

于是T是自伴的.

32. 设$T \in \mathcal{L}(V, W)$.试证明

$$T'(\varphi_w) = \varphi_{T^*w}$$

对所有$w \in W$成立.

Proof.

对于任意$v \in V$和任意$w \in W$有

$$T'(\varphi_w)v = (\varphi_w \circ T)v = \varphi_w(Tv) = \langle Tv, w\rangle = \langle v, T^*w\rangle = \varphi_{T^*w}v$$

从而$T'(\varphi_w) = \varphi_{T^*w}$.