Linear Algebra Done Right 6C

1. 设$v_1,\cdots,v_m\in V$.试证明

$$\{v_1,\cdots,v_m\}^{\perp}=(\operatorname{span}(v_1,\cdots,v_m))^{\perp}$$

Proof.

由于$\{v_1,\cdots,v_m\}\subseteq\operatorname{span}(v_1,\cdots,v_m)$,于是根据正交补的性质有$(\operatorname{span}(v_1,\cdots,v_m))^{\perp}\subseteq\{v_1,\cdots,v_m\}^{\perp}$.

对于任意$u\in\{v_1,\cdots,v_m\}^{\perp}$,都满足$\langle u,v_k\rangle=0$对任意$k\in\{1,\cdots,m\}$成立.

于是对于任意$v:=a_1v_1+\cdots+a_mv_m\in\operatorname{span}(v_1,\cdots,v_m)$有

$$\langle u,v\rangle=\left\langle u,\sum_{k=1}^{m}a_kv_k\right\rangle=\sum_{k=1}^{m}a_k\langle u,v_k\rangle=0$$

从而$u\in(\operatorname{span}(v_1,\cdots,v_m))^{\perp}$,因而$\{v_1,\cdots,v_m\}\subseteq(\operatorname{span}(v_1,\cdots,v_m))^{\perp}$.

综上可知$\{v_1,\cdots,v_m\}^{\perp}=(\operatorname{span}(v_1,\cdots,v_m))^{\perp}$.

2. 设U是V的子空间,且有一组基$u_1,\cdots,u_m$.向量组$u_1,\cdots,u_m,v_1,\cdots,v_n$是$V$的一组基.对上述$V$的基运用Gram-Schmidt过程得到$V$的规范正交基$e_1,\cdots,e_m,f_1,\cdots,f_n$.试证明:$e_1,\cdots,e_m$是$U$的规范正交基,$f_1,\cdots,f_n$是$U^{\perp}$的规范正交基.

Proof.

对$u_1,\cdots,u_m$应用Gram-Schmidt过程得到的$e_1,\cdots,e_m$自然是U的规范正交基.

对于任意$k\in\{1,\cdots,n\}$和任意$j\in\{1,\cdots,m\}$,都有$\langle f_k,e_j\rangle=0$,于是$f_k\in U^{\perp}$.

于是$f_1,\cdots,f_n$是$U^{\perp}$中的规范正交组.

又因为$\dim U^{\perp}=\dim V-\dim U=n$,因而$f_1,\cdots,f_n$是$U^{\perp}$的规范正交基.

3. 设U是$\mathbb{R}^4$的子空间,其定义为

$$U=\operatorname{span}((1,2,3,-4),(-5,4,3,2))$$

求U的一规范正交基和$U^{\perp}$的一规范正交基.

Proof.

将$(1,2,3,-4),(-5,4,3,2)$扩展为V的一组基

$$(1,2,3,-4),(-5,4,3,2),(1,0,0,0),(0,1,0,0)$$

对这组基运用Gram-Schmidt过程,得到

$$e_1=\frac{1}{\sqrt{30}}(1,2,3,-4),e_2=\frac{1}{\sqrt{12030}}(-77,56,39,38)$$

$$e_3=\frac{1}{\sqrt{76190}}(190,117,60,151),e_4=\frac{1}{\sqrt{190}}(0,9,-10,3)$$

根据**6C.2**可知e_1,e_2是U的基,e_3,e_4是$U^{\perp}$的基.

4. 设$e_1,\cdots,e_n$是V中的一组向量,满足

(a) 对任意$k\in\{1,\cdots,n\}$,都有$||e_k||=1$.

(b) 对任意$v\in V$,都有$||v||^2=|\langle v,e_1\rangle|^2+\cdots+|\langle v,e_n\rangle|^2$.

试证明:$e_1,\cdots,e_n$是V的规范正交基.

Proof.

根据题设条件,对于任意$k\in\{1,\cdots,n\}$有

$$||e_k||^2=|\langle e_k,e_1\rangle|^2+\cdots+|\langle e_k,e_n\rangle|^2=1$$

又因为$|\langle e_k,e_k\rangle|=||e_k||=1$,于是

$$\sum_{j=1,j\neq k}^{n}|\langle e_k,e_j\rangle|^2=0$$

这表明对任意$j,k\in\{1,\cdots,n\}$且$j\neq k$都有$\langle e_k,e_j\rangle=0$.

于是$e_1,\cdots,e_n$是V中的规范正交组.

考虑任意的$v\in V$,根据Bessel不等式可知

$$\sum_{k=1}^{n}|\langle v,e_k\rangle|^2\leqslant||v||^2$$

当且仅当$v=\sum\limits_{k=1}^{n}|\langle v,e_k\rangle|e_k$时等号成立.

于是$v\in\operatorname{span}(e_1,\cdots,e_n)$,即$V=\operatorname{span}(e_1,\cdots,e_n)$,从而$e_1,\cdots,e_n$为$V$的规范正交基.

5. 设V是有限维的,且U为V的子空间,试证明:$P_{U^{\perp}}=I-P_U$,其中I是V上的恒等算子.

Proof.

由于$V=U\oplus U^{\perp}$,于是对于任意$v\in V$,其都可以被唯一分解为$v=u+w$,其中$u\in U,w\in U^{\perp}$.

根据正交投影的定义,我们有$P_U v = u, P_{U^\perp} v = w$,于是

$$v = P_U v + P_{U^\perp} v$$

从而

$$I = P_U + P_{U^\perp}$$

移项即可得欲证等式.

6. 设V是有限维的,且$T \in \mathcal{L}(V, W)$.试证明

$$T = T P_{(\text{null } T)^\perp} = P_{\text{range } T} T$$

Proof.

根据**6C.5**有$P_{(\text{null } T)^\perp} = I - P_{\text{null } T}$.对任意$v \in V$,都有$P_{\text{null } T} v \in \text{null } T$,于是

$$T P_{(\text{null } T)^\perp} v = T\left(I - P_{\text{null } T}\right) v = Tv - \mathbf{0} = Tv$$

于是$T P_{(\text{null } T)^\perp} = T$.

对于任意$w \in \text{range } T$,都有$P_{\text{range } T} w = w$.于是$T = P_{\text{range } T} T$.

综上,命题得证.

7. 设X和Y为V的有限维子空间.试证明:$P_X P_Y = \mathbf{0}$当且仅当$\langle x, y\rangle = 0$对所有$x \in X$和所有$y \in Y$都成立.

Proof.

$\Rightarrow$:$P_X P_Y = \mathbf{0}$即对任意$v \in V$有$P_X P_Y v = \mathbf{0}$.又range $P_Y = Y$,于是对于任意$y \in Y$有$P_X y = \mathbf{0}$.

由于$V = X \oplus X^\perp$,于是存在唯一的分解$y = x + x'$使得$x \in X, x' \in X^\perp$.

又因为$P_X y = \mathbf{0}$,即上述分解中$x = \mathbf{0}$,于是$y = x' \in X^\perp$.即对于任意$x \in X$有$\langle x, y\rangle = 0$.

$\Leftarrow$:考虑X的规范正交基$e_1, \cdots, e_n$.对于任意$v \in V$,有$P_Y v \in Y$,于是

$$P_X\left(P_Y v\right) = \sum_{k=1}^{n} \left\langle P_Y v, e_k\right\rangle e_k = \mathbf{0}$$

于是$P_X P_Y = \mathbf{0}$.

8. 设U是V的有限维子空间,且$v \in V$.定义U上的线性泛函$\phi : U \to \mathbb{F}$为 $\phi(u) = \langle u, v \rangle$ 对所有$u \in U$成立.根据Riesz表示定理,存在唯一$w \in U$使得 $\phi(u) = \langle u, w \rangle$ 对所有$u \in U$成立.试证明:$w = P_U v$.

Proof.
因为$v - P_U v \in U^{\perp}$,于是$\langle u, v - P_U v \rangle$.于是

$$\langle u, v \rangle = \langle u, v - P_U v \rangle + \langle u, P_U v \rangle = \langle u, P_U v \rangle$$

因此$\phi(u) = \langle u, v \rangle = \langle u, P_U v \rangle$.而$P_U v \in U$.根据Riesz表示定理,这样的向量是唯一存在的,于是$w = P_U v$.

9. 设V是有限维的,$P \in \mathcal{L}(V)$使得$P^2 = P$且null P中的任意向量都正交与range P中的任意向量.试证明:存在V的子空间U使得$P = P_U$.

Proof.
根据**3B.27**可知$V = \text{range } P \oplus \text{null } P$.又因为$V = \text{range } P \oplus (\text{range } P)^{\perp}$,于是

$$\dim \text{null } P = \dim (\text{range } P)^{\perp}$$

根据题意可知null $P \subseteq (\text{range } P)^{\perp}$.于是null $P = (\text{range } P)^{\perp}$.
令$U = \text{range } P$.对于任意$v := Px + w \in V$,其中$Px \in \text{range } P, w \in \text{null } P$,有

$$P_U v = Px = P\,(Px + w) = Pv$$

此时$P_U = P$.

10. 设V是有限维的,$P \in \mathcal{L}(V)$使得$P^2 = P$且$||Pv|| \leqslant ||v||$对任意$v \in V$成立.试证明:存在V的子空间U使得$P = P_U$.

Proof.
考虑$w \in \text{null } P$和$Px \in \text{range } P$,根据题意,对任意$\lambda \in \mathbb{F}$有

$$||Px|| = ||P(Px + \lambda w)|| \leqslant ||Px + \lambda w||$$

根据**6A.6**可知$\langle w, Px \rangle = 0$,即null P中的任意向量都正交与range P中的任意向量.
根据**6C.9**,取$U = \text{range } P$即可使$P = P_U$.

11. 设$T \in \mathcal{L}(V)$且U是V的有限维子空间.试证明:U在T下不变,当且仅当$P_U T P_U = T P_U$.

Proof.
$\Rightarrow$:对于任意$v \in V$,都有$P_U v \in U$.因为U在T下不变,于是$T\left(P_U v\right) \in U$.
于是$P_U\left(T\left(P_U v\right)\right) = T\left(P_U v\right)$.因而$P_U T P_U = T P_U$.
$\Leftarrow$:如果U不在T下不变,那么存在$u \in U$使$Tu \notin U$.于是

$$P_U T P_U u = P_U T u \neq T u = T P_U u$$

于是$P_U T P_U \neq T P_U$.

12. 设V是有限维的,$T \in \mathcal{L}(V)$,且U是V的子空间.试证明U和$U^{\perp}$在T下不变,当且仅当$P_U T = T P_U$.

Proof.
$\Rightarrow$:对任意$v := u + w \in V$,其中$u \in U, w \in U^{\perp}$,都有$Tu \in U, Tw \in U^{\perp}$.于是

$$P_U T v = P_U\left(Tu + Tw\right) = Tu = T P_U v$$

于是$P_U T = T P_U$.
$\Leftarrow$:若存在$u \in U$使得$Tu \notin U$,那么

$$T P_U u = T u \notin U, P_U(Tu) \in U$$

于是$T P_U \neq P_U T$,与题设矛盾.从而$Tu \in U$对任意$u \in U$成立.
若存在$w \in U^{\perp}$使得$Tw \notin U^{\perp}$,那么

$$T P_U w = T\mathbf{0} = \mathbf{0}, P_U T w \neq \mathbf{0}$$

同理可推出矛盾,因此$Tw \in U^{\perp}$对任意$w \in U^{\perp}$成立.
于是U和$U^{\perp}$在T下不变.

13. 设$\mathbb{F} = \mathbb{R}$,V是有限维向量空间.对任意$v \in V$,令ϕ_v为V上的线性泛函,定义为$\phi_v(u) = \langle u, v\rangle$对所有$u \in V$成立.

(1) 试证明:$v \mapsto \phi_v$是V到V'的单的线性映射.

(2) 试证明:$v \mapsto \phi_v$是V到V'的同构.

Proof.

令映射$T : V \to V'$为$Tv = \phi_v$对所有$v \in V$成立.

(1) 对于任意$v, w \in V$,对于任意$u \in V$有

$$\phi_v(u) + \phi_w(u) = \langle u, v\rangle + \langle u, w\rangle = \langle u, v + w\rangle = \phi_{v+w}(u)$$

于是$Tv + Tw = T(v + w)$.

对于任意$v \in V$和$\lambda \in \mathbb{F}$,对于任意$u \in V$有

$$\lambda\phi_v(u) = \lambda \langle u, v\rangle = \langle u, \lambda v\rangle = \phi_{\lambda v}(u)$$

于是$\lambda Tv = T(\lambda v)$.

因此T满足可加性和齐次性,是V到V'的线性映射.

现在,假定存在$v, w \in V$使得$Tv = Tw$.于是对于任意$u \in V$有$\phi_v(u) = \phi_w(u)$.而

$$\phi_v(u) = \phi_w(u) \Leftrightarrow \langle u, v\rangle = \langle u, w\rangle \Leftrightarrow \langle u, v - w\rangle = 0 \Leftrightarrow v - w = \mathbf{0} \Leftrightarrow v = w$$

于是T是单射.

(2) 注意到$\dim V = \dim V'$,又因为T是单射,于是T是V到V'的同构.

14. 设$e_1, \cdots, e_n$是V的规范正交基.对于$v \in V$,令$\phi_v(u) = \langle u, v\rangle$对所有$u \in U$成立.试证明:$e_1, \cdots, e_n$的对偶基为$\phi_{e_1}, \cdots, \phi_{e_n}$.

Proof.

对于任意$j, k \in \{1, \cdots, n\}$且$j \neq k$有

$$\phi_{e_k}(e_j) = \langle e_j, e_k\rangle = 0$$

又$\phi_{e_k}(e_k) = \langle e_k, e_k\rangle = 1$.于是

$$\phi_{e_k}(e_j) = \begin{cases} 1, j = k \\ 0, j \neq k \end{cases}$$

这符合e_k的对偶ϕ_k的定义.于是$\phi_{e_1}, \cdots, \phi_{e_n}$为$e_1, \cdots, e_n$的对偶基.