Linear Algebra Done Right 7F

1. 设$S,T\in\mathcal{L}(V,W)$.试证明:$|||S||-||T|||\leqslant||S-T||$.

Proof.

由线性映射的范数的定义可知存在$v\in V$且$||v||\leqslant 1$使得$||S-T||=||(S-T)v||$.于是

$$||S-T||=||(S-T)v||=||Sv-Tv||\geqslant|||Sv||-||Tv|||\geqslant|||S||-||T|||$$

以上不等式即**反向三角不等式**.

2. 设$T\in\mathcal{L}(V)$是自伴的(如果$\mathbb{F}=\mathbb{C}$则令T是正规的),试证明:$||T||=\max\{|\lambda|:\lambda\text{是}T\text{的特征值}\}$.

Proof.

由**7.88**和**7E.7**可知

$$||T||=T\text{的最大奇异值}=T\text{的绝对值最大的特征值}$$

3. 设$T\in\mathcal{L}(V,W)$且$v\in V$.试证明:$||Tv||=||T||||v||$当且仅当$T^*Tv=||T||^2v$.

Proof.

$\Leftarrow$:由**7.88(c)**和**7.91**可得

$$||T||^2||v||=||T^*Tv||\leqslant||T^*||||Tv||=||T||||Tv||\Rightarrow||Tv||\geqslant||T||||v||$$

而$||Tv||\leqslant||T||||v||$,于是$||Tv||=||T||||v||$.

$\Rightarrow$:我们有

$$\begin{aligned}||T^*Tv-||T||^2v||^2&=\left\langle T^*Tv-||T||^2v,T^*Tv-||T||^2v\right\rangle\\&=||T^*Tv||^2+||T||^4||v||^2-2\mathrm{Re}\left\langle T^*Tv,||T||^2v\right\rangle\\&\leqslant||T^*||^2||Tv||^2+||T||^4||v||^2-2||T||^2||Tv||^2\\&=0\end{aligned}$$

于是$T^*Tv=||T||^2v$.

4. 设$T \in \mathcal{L}(V,W), v \in V$且$||Tv|| = ||T||||v||$.试证明:如果$u \in V$且$\langle u,v\rangle = 0$,那么$\langle Tu,Tv\rangle = 0$.

Proof.

根据**7F.3**有

$$\langle Tu,Tv\rangle = \langle u,T^*Tv\rangle = \left\langle u,||T||^2 v\right\rangle = ||T||^2 \langle u,v\rangle = 0$$

5. 设U是有限维内积空间,$T \in \mathcal{L}(V,U)$且$S \in \mathcal{L}(U,W)$.试证明:$||ST|| \leqslant ||S||||T||$.

Proof.

由线性映射的范数的定义可知存在$v \in V$且$||v|| \leqslant 1$使得$||ST|| = ||STv||$.于是

$$||ST|| = ||STv|| = ||Tv|| \left|\left|S\left(\frac{Tv}{||Tv||}\right)\right|\right| \leqslant ||Tv||||S|| \leqslant ||S||||T||$$

6. 证明或给出一反例:如果$S,T \in \mathcal{L}(V)$,那么$||ST|| = ||TS||$.

Solution.

令$V = \mathbb{F}^2$,$S(x,y) = (x,0), T(x,y) = (y,0)$.于是$ST = \mathbf{0} \neq TS$.根据**7.87(b)**,即$||ST|| = 0 \neq ||TS||$.

8. 回答下列问题.

(1) 试证明:如果$T \in \mathcal{L}(V)$且$||I - T|| < 1$,那么T可逆.

(2) 设$S \in \mathcal{L}(V)$可逆.试证明:如果$T \in \mathcal{L}(V)$且$||S - T|| < \dfrac{1}{||S^{-1}||}$,那么$T$是可逆的.

Proof.

(1) 如果T不可逆,那么存在$v \in V$使得$Tv = \mathbf{0}$.令$u = \dfrac{v}{||v||}$,于是$||u|| = 1$.于是

$$||I - T|| \geqslant ||(I - T)u|| = ||u - \mathbf{0}|| = ||u|| = 1$$

这与$||I - T|| < 1$矛盾,于是T可逆.

(2) 取与**(1)**同样的v,u,我们有$||S - T|| \geqslant ||(S - T)u||| = ||Su||$.
而$1 = ||u|| = ||S^{-1}Su|| \leqslant ||S^{-1}||||Su||$,即$||Su|| \geqslant \dfrac{1}{||S^{-1}||}$.即$||S - T|| \geqslant \dfrac{1}{||S^{-1}||}$.
这与题设不符,于是T可逆.

9. 设$T \in \mathcal{L}(V)$.试证明:对于任意$\varepsilon > 0$,都存在可逆的$S \in \mathcal{L}(V)$使得$0 < ||T - S|| < \varepsilon$.

Proof.
取$\delta \in (0, \varepsilon)$且$\delta$不是$T$的特征值,于是$T - \delta I$可逆.令$S = T - \delta I$,则有

$$||T - S|| = ||\delta I|| = |\delta| \in (0, \varepsilon)$$

于是命题得证.

10. 设$\dim V > 1$,$T \in \mathcal{L}(V)$不可逆.试证明:对于任意$\varepsilon > 0$,都存在$S \in \mathcal{L}(V)$使得$0 < ||T - S|| < \varepsilon$且$S$不可逆.

Proof.
由于T不可逆,于是存在$e_1 \in V$且$||e_1|| = 1$使得$Te_1 = \mathbf{0}$.将e_1扩展为V的规范正交基$e_1, \cdots, e_n$.
令$Se_1 = \mathbf{0}, Se_k = Te_k - \dfrac{\varepsilon}{2} e_k (k = 2, \cdots, n)$,于是$S$不可逆.对于任意$v \in V$且$v \notin \operatorname{span}(e_1)$有

$$0 < ||(T - S)v||^2 = \left|\left| \frac{\varepsilon}{2} \sum_{k=2}^{n} \langle v, e_k \rangle e_k \right|\right|^2 \leqslant \frac{\varepsilon^2}{4}$$

于是$0 < ||T - S|| \leqslant \dfrac{\varepsilon}{2} < \varepsilon$

11. 设$\mathbb{F} = \mathbb{C}$,$T \in \mathcal{L}(V)$.试证明:对于任意$\varepsilon > 0$,都存在可对角化的$S \in \mathcal{L}(V)$使得$0 < ||T - S|| < \varepsilon$.

Proof.
根据Schur定理,存在V的规范正交基$e_1, \cdots, e_n$使得T关于其有上三角矩阵A,其对角线元素为$\lambda_1, \cdots, \lambda_n$.
考虑$D \in \mathcal{L}(V)$满足$De_k = ke_k$,显然$||D|| = n$.
由于$\lambda_1, \cdots, \lambda_n$的数目是有限的,于是存在$\delta \in \left(0, \dfrac{\varepsilon}{n}\right)$使得$\lambda_k + k\delta$互异.
令$S = T + \delta D$,于是

$$0 < ||T - S|| = ||\delta D|| = \delta ||D|| < \frac{\varepsilon}{n} \cdot n = \varepsilon$$

于是命题得证.

12. 设$T \in \mathcal{L}(V)$是正算子,试证明:$\left|\left|\sqrt{T}\right|\right| = \sqrt{||T||}$.

Proof.

不妨设T的最大特征值为λ,于是$\sqrt{T}$的最大特征值为$\sqrt{\lambda}$.

根据**7E.7**可知T和$\sqrt{T}$的最大奇异值分别为λ和$\sqrt{\lambda}$.于是命题得证.

13. 设$S,T\in\mathcal{L}(V)$是正算子.试证明$||S-T||\leqslant\max\{||S||,||T||\}\leqslant||S+T||$.

Lemma.L.13 如果$A\in\mathcal{L}(V)$是自伴的,那么$||A||I-A$是正算子.

Proof.

考虑到A,I均为自伴算子,于是$||A||I-A$也是自伴的.

考虑$||A||I-A$的特征值λ,则存在非零的$v\in V$使得$||A||v-Av=\lambda v$,从而$||A||-\lambda$是A的特征值.

根据**7F.2**可得$|||A||-\lambda|\leqslant||A||$,从而$\lambda\geqslant 0$,即$||A||I-A$是正算子.

Lemma.L.14 如果$A,B\in\mathcal{L}(V)$且$A,B-A$均为正算子,那么$||A||\leqslant||B||$.

Proof.

根据**7C.6**可知$B=A+(B-A)$是正算子,于是根据**Lemma.L.13**可知$||B||I-B$是正算子.于是

$$||B||I-A=(||B||I-B)+(B-A)$$

是正算子.考虑A的任意特征值$\lambda\geqslant 0$和对应的特征向量v,我们有

$$(||B||I-A)\,v=(||B||-\lambda)\,v$$

由于$B-A$是正算子,于是$||B||-\lambda\geqslant 0$,从而$||B||\geqslant\lambda$.

于是$||A||=\max\{\lambda:\lambda\text{为}A\text{的特征值}\}\leqslant||B||$,命题得证.

Proof.

回到我们的命题.设λ为自伴算子$S-T$的特征值,其特征向量为v.于是

$$(||S||I-(S-T))\,v=(||S||-\lambda)\,v$$

根据**Lemma.L.13**和**7C.6**可知$||S||I-(S-T)$是正算子,于是$||S||-\lambda\geqslant 0$.

同理可知$||T||+\lambda\geqslant 0$.由于上述两条等式对于所有$S-T$的特征值λ都成立,于是$||S-T||\leqslant\max\{||S||,||T||\}$.

根据**Lemma.L.14**,令$A=S,B=S+T$可知$||S||\leqslant||S+T||$,同理有$||T||\leqslant||S+T||$.

于是$\max\{||S||,||T||\}\leqslant||S+T||$.

14. 设U,W是V的子空间且$||P_U - P_W|| < 1$.试证明$\dim U = \dim W$.

Proof.

注意到$P_U = I - P_{U^\perp}, P_W = I - P_{W^\perp}$.根据**7F.8(1)**可知$P_U + P_{W^\perp}$和$P_W + P_{U^\perp}$都是可逆的.于是

$$U \cap W^\perp = (\text{null } P_{U^\perp}) \cap (\text{null } P_W) \subseteq \text{null } (P_{U^\perp} + P_W) = \{\mathbf{0}\} \Rightarrow U \cap W^\perp = \{\mathbf{0}\}$$

$$V = \text{range } (P_U + P_{W^\perp}) \supseteq (\text{range } P_U) + (\text{range } P_{W^\perp}) = U + W^\perp \Rightarrow U + W^\perp = V$$

于是

$$\dim V = \dim(U + W^\perp) = \dim U + \dim W^\perp - \dim(U \cap W^\perp) = \dim U + \dim V - \dim W$$

于是$\dim U = \dim W$.

15. 定义$T \in \mathcal{L}(\mathbb{F}^3)$为$T(z_1, z_2, z_3) = (z_3, 2z_1, 3z_2)$.求使得$T = S\sqrt{T^*T}$的$S \in \mathcal{L}(\mathbb{F}^3)$的表达式.

Solution.

考虑T关于$\mathbb{F}^3$的标准基的矩阵

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

于是

$$\mathcal{M}(T^*T) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

于是$\sqrt{T^*T}e_1 = 2e_1, \sqrt{T^*T}e_2 = 3e_2, \sqrt{T^*T}e_3 = e_3$.于是$S$应满足

$$Te_1 = S\sqrt{T^*T}e_1 = 2Se_1 = 2e_2 \Rightarrow Se_1 = e_2$$

$$Te_2 = S\sqrt{T^*T}e_2 = 3Se_2 = 3e_3 \Rightarrow Se_2 = e_3$$

$$Te_3 = S\sqrt{T^*T}e_3 = Se_3 = e_1 \Rightarrow Se_3 = e_1$$

于是

$$S(z_1, z_2, z_3) = (z_3, z_1, z_2)$$

16. 设$S \in \mathcal{L}(V)$是可逆正算子.试证明:存在$\delta > 0$使得任意满足$||S-T|| < \delta$的自伴算子T都是正算子.

Proof.

设μ是S的最小奇异值.由于S可逆,于是$\mu > 0$.

根据**7E.12(2)**可知$\sqrt{\mu}$是$\sqrt{T}$的最小奇异值,于是由**7E.14**可得

$$\langle Sv, v\rangle = \left\langle \sqrt{S}v, \sqrt{S}v \right\rangle = \left|\left|\sqrt{S}v\right|\right|^2 \geqslant \mu||v||^2$$

设自伴算子$T \in \mathcal{L}(V)$满足$||S-T|| < \mu$.由Cauchy不等式可得

$$\langle (S-T)v, v\rangle \leqslant ||(S-T)v||||v|| \leqslant ||S-T||||v||^2 \leqslant \mu||v||^2 \leqslant \langle Sv, v\rangle$$

于是$\langle Tv, v\rangle \geqslant 0$对任意$v \in V$都成立,因而$T$是正算子.

17. 试证明:如果$u \in V$而φ_u是V上定义为$\varphi_u(v) = \langle v, u\rangle$的线性泛函,那么$||\varphi_u|| = ||u||$.

Proof.

对于任意$v \in V$,由Cauchy不等式可得

$$|\varphi_u(v)| = |\langle u, v\rangle| \leqslant ||u||||v||$$

当且仅当$v = \lambda u$时等号成立.对于所有满足$||v|| = 1$的$v \in V$则有

$$|\varphi_u(v)| \leqslant ||u||$$

根据线性映射的范数的定义可知$||\varphi_u|| = ||u||$.

18. 设$e_1, \cdots, e_n$是V的规范正交基,$T \in \mathcal{L}(V, W)$.试证明下列命题.

(1) $\max\limits_{1\leqslant k\leqslant n}\{||Te_k||\} \leqslant ||T|| \leqslant \sqrt{\sum\limits_{k=1}^{n}||Te_k||^2}$.

(2) $||T|| = \sqrt{\sum\limits_{k=1}^{n}||Te_k||^2}$当且仅当$\dim \operatorname{range} T \leqslant 1$.

Proof.

(1) 根据$||T||$的定义,对任意$v \in V$且$||v|| = 1$都有$||T|| \geqslant ||Tv||$,于是

$$\max_{1\leqslant k\leqslant n}\{||Te_k||\} \leqslant ||T||$$

令$s_1, \cdots, s_n$为T的(按降序排列的)奇异值,根据**7E.11(1)**有

$$\sum_{k=1}^{n} ||Te_k||^2 = \sum_{k=1}^{n} s_k^2$$

由**7.88(a)**可知$||T|| = s_1$,于是

$$||T|| = s_1 = \sqrt{s_1^2} \leqslant \sqrt{\sum_{k=1}^{n} s_k^2} = \sqrt{\sum_{k=1}^{n} ||Te_k||^2}$$

于是命题得证.

(2) 这要求T除了其最大奇异值外其余所有奇异值为0.于是$\dim \operatorname{range} T \leqslant 1$.

19. 试证明:如果$T \in \mathcal{L}(V, W)$,那么$||T^*T|| = ||T||^2$.

Proof.

考虑$||T||$的最大奇异值s_1,根据**7.88(a)**有$||T|| = s_1$.

而T^*T是正算子,其关于V的某规范正交基$e_1, \cdots, e_n$由对角矩阵.不妨设对角线元素为$\lambda_1, \cdots, \lambda_n$.于是

$$||T^*Tv||^2 = \left|\left|\sum_{k=1}^{n} \lambda_k \langle v, e_k \rangle e_k\right|\right| = \sum_{k=1}^{n} \lambda_k^2 |\langle v, e_k \rangle|^2 \leqslant \left(\max_{1 \leqslant k \leqslant n} \{\lambda_k\}\right)^2 \sum_{k=1}^{n} |\langle v, e_k \rangle| \leqslant \left(\max_{1 \leqslant k \leqslant n} \{\lambda_k\}\right)^2 ||v||^2$$

于是$||T^*Tv|| \leqslant \max_{1 \leqslant k \leqslant n} \{\lambda_k\} ||v||$,取$v = e_j$使得$\lambda_k$最大即可取等.

因而$||T^*T|| = \max\limits_{1 \leqslant k \leqslant n} \{\lambda_k\} = s_1^2 = ||T||^2$.

Proof.

对于任意$v \in V$有

$$\left|\left|\sqrt{T^*T}v\right|\right|^2 = \langle T^*Tv, v \rangle = \langle Tv, Tv \rangle = ||Tv||^2 \leqslant ||T||^2 ||v||^2$$

于是$\left|\left|\sqrt{T^*T}\right|\right| \leqslant ||T||$.考虑$T$的极分解$T = S\sqrt{T^*T}$,由于$S$是酉算子,于是$||S|| = 1$.于是根据**7F.5**有

$$||T|| = \left|\left|S\sqrt{T^*T}\right|\right| \leqslant ||S|| \left|\left|\sqrt{T^*T}\right|\right| = \left|\left|\sqrt{T^*T}\right|\right|$$

于是$\left|\left|\sqrt{T^*T}\right|\right| = ||T||$.根据**7F.12**有$||T^*T|| = ||T||^2$.

20. 设$T \in \mathcal{L}(V)$是正规的,试证明:对于任意$k \in \mathbb{N}^*$都有$||T||^k = ||T^k||$.

Proof.

设$s_1, \cdots, s_n$为T的奇异值,那么$||T|| = s_1$.

根据**7E.12(2)**,考虑V的规范正交基$e_1, \cdots, e_n$使得$T^*Te_j = s_j^2 e_j (j = 1, \cdots, n)$.于是根据$T$的正规性可得

$$\left(T^k\right)^* T^k e_j = (T^*T)^k e_j = s_j^{2k} e_j$$

从而T^k的奇异值为$s_1^k, \cdots, s_n^k$,于是$||T^k|| = s_1^k = ||T||^k$.

21. 设$\dim V > 1$且$\dim W > 1$.试证明:$\mathcal{L}(V, W)$上的范数并不来自内积.换言之,试证明:不存在$\mathcal{L}(V, W)$上的内积使得

$$\max\{||Tv|| : v \in V\text{且}||v|| = 1\} = \sqrt{\langle T, T\rangle}$$

对所有$T \in \mathcal{L}(V, W)$都成立.

Proof.

考虑V的规范正交基$e_1, \cdots, e_n$和W的规范正交基$f_1, \cdots, f_m$,其中$m, n \geqslant 2$.

令$T, S \in \mathcal{L}(V, W)$为

$$Se_1 = f_1, Se_2 = f_2, Se_k = \mathbf{0} (k > 2)$$

$$Te_1 = -f_1, Te_2 = f_2, Te_k = \mathbf{0} (k > 2)$$

于是$||S + T|| = ||S - T|| = 2$且$||S|| = ||T|| = 1$,则有

$$||S + T||^2 + ||S - T||^2 = 8 \neq 4 = 2\left(||S||^2 + ||T||^2\right)$$

这不符合平行四边形等式,于是不存在这样的内积.

22. 设$T \in \mathcal{L}(V, W)$.令$n = \dim V$,$s_1, \cdots, s_n$为T的按降序排列的奇异值.试证明:如果$1 \leqslant k \leqslant n$,那么$\min\{||T|_U|| : U\text{是}V\text{的子空间且}\dim U = k\} = s_{n-k+1}$.

当你做到这里的时候,是否感到了绝望?

Proof.

对于给定的$k \in \{1, \cdots, n\}$,令$E_k = \min\{||T|_U|| : U\text{是}V\text{的子空间且}\dim U = k\}$.

如果T没有正奇异值,也即$T = \mathbf{0}$,我们有$T|_U = \mathbf{0}$,于是$E_k = 0 = s_{n-k+1}$.

当$k = n$时,唯一可取的子空间为V,于是$E_n = ||T|| = s_1$.

现在,假定$1 \leqslant k < n$,并假设$s_1, \cdots, s_m$为T的正奇异值.

考虑T的奇异值分解

$$Tv = \sum_{j=1}^{m} s_j \langle v, e_j \rangle f_j$$

将$e_1, \cdots, e_m$扩展为V的规范正交基$e_1, \cdots, e_n$,于是

$$Te_j = \begin{cases} s_j f_j, 1 \leqslant j \leqslant m \\ \mathbf{0}, m < j \leqslant n \end{cases}$$

令$X = \operatorname{span}(e_{n-k+1}, \cdots, e_k)$,那么$\dim X = k$.考虑以下两种情况.

如果$1 \leqslant k \leqslant n-m$,那么$s_{n-k+1} = 0$.另外,$T|_X = \mathbf{0}$,从而$E_k = ||\mathbf{0}|| = 0 = s_{n-k+1}$.

如果$n - m < k < n$,那么$1 \leqslant n - k < m$.考虑V的任意一个k维子空间U.

对于任意$v \in V$且$||v|| \leqslant 1$都有$P_U v \in U$且$||P_U v|| \leqslant ||v|| \leqslant 1$.

于是根据$||T_U||$的定义可知$||TP_U v|| \leqslant ||T|_U||$,即$||TP_U|| \leqslant ||T|_U||$. 现在,注意到

$$\dim \operatorname{range} \, (TP_{U^\perp}) \leqslant \dim \operatorname{range} \, P_{U^\perp} = \dim U^\perp = n - k$$

根据**7.92**可知$||T - TP_{U^\perp}|| \geqslant s_{n-k+1}$.另一方面,有$P_U = I - P_{U^\perp}$,于是$||TP_U|| \geqslant s_{n-k+1}$.

于是$||T|_U|| \geqslant ||TP_U|| \geqslant s_{n-k+1}$,我们得到了$||T|_U||$的下界.

现在,对任意$x \in X$有

$$||T|_X x||^2 = \sum_{j=n-k+1}^{m} s_j^2 \left| \langle x, e_j \rangle \right|^2 \leqslant s_{n-k+1}^2 ||x||^2$$

从而$||T|_X|| \leqslant s_{n-k+1}$.结合$T|_X$的下界可知$||T|_X|| = s_{n-k+1}$,于是$E_k = s_{n-k+1}$.

综上可知命题得证.

24. 设$T \in \mathcal{L}(V)$可逆.试证明:$||T^{-1}|| = \dfrac{1}{||T||}$当且仅当$\dfrac{T}{||T||}$是酉算子.

Proof.

设T的奇异值为$s_1, \cdots, s_n$.由于T可逆,于是$s_1, \cdots, s_n > 0$.

根据伪逆的奇异值分解和伪逆的性质可知T^{-1}的奇异值(按降序排列)为$\dfrac{1}{s_n}, \cdots, \dfrac{1}{s_1}$.

于是$||T^{-1}|| = \dfrac{1}{s_n}$.于是

$$||T^{-1}|| = \frac{1}{||T||} \Leftrightarrow \frac{1}{s_n} = \frac{1}{s_1} \Leftrightarrow s_1 = \cdots = s_n \Leftrightarrow \frac{T}{||T||}\text{的所有奇异值均为}1 \Leftrightarrow \frac{T}{||T||}\text{是酉算子}$$

25. 取定$u, x \in V$,其中$u \neq \mathbf{0}$.定义$T \in \mathcal{L}(V)$为对任意$v \in V$有$Tv = \langle v, u \rangle x$.试证明:

$$\sqrt{T^*T}v = \frac{||x||}{||u||} \langle v, u \rangle u$$

对任意$v \in V$成立.

Proof.

我们有

$$\langle Tv, v \rangle = \langle v, u \rangle \cdot \langle x, v \rangle = \langle v, \langle v, x \rangle u \rangle = \langle v, T^* v \rangle$$

于是$T^* v = \langle v, x \rangle u$.令$R = \dfrac{||x||}{||u||} \langle v, u \rangle u$,我们有

$$T^*Tv = \langle v, u \rangle T^* x = ||x||^2 \langle v, u \rangle u$$

而

$$R^2 v = \frac{||x||^2}{||u||^2} \langle v, u \rangle ||u||^2 u = ||x||^2 \langle v, u \rangle u = T^*Tv$$

于是$R^2 = T^*T$.对于任意$v \in V$有

$$\langle Rv, v \rangle = \frac{||x||}{||u||} \langle v, u \rangle \langle u, v \rangle = \frac{||x||}{||u||} |\langle v, u \rangle|^2 \geqslant 0$$

于是R是正算子,因而$R = \sqrt{T^*T}$.

26. 设$T \in \mathcal{L}(V)$.试证明:T可逆,当且仅当存在唯一的酉算子$S \in \mathcal{L}(V)$使得$T = S\sqrt{T^*T}$.

Proof.

$\Rightarrow$:设T可逆,那么根据T的极分解可知存在酉算子S使得$T = S\sqrt{T^*T}$.

又因为T可逆,于是null $\sqrt{T^*T}$ = null T^*T = null $T = \{\mathbf{0}\}$,进而$\sqrt{T^*T}$可逆.于是

$$S = T\left(\sqrt{T^*T}\right)^{-1}$$

是唯一的.

$\Leftarrow$:如果T不可逆,那么考虑T的奇异值分解

$$Tv = \sum_{k=1}^{m} s_k \langle v, e_k \rangle f_k$$

将$e_1, \cdots, e_m$和$f_1, \cdots, f_m$扩展为V的规范正交基$e_1, \cdots, e_n$和$f_1, \cdots, f_n$.

由于T不可逆,那么其存在为0的奇异值,则有$n > m$.

令$S, R \in \mathcal{L}(V)$,满足$Se_k = Re_k = f_k (k = 1, \cdots, n-1)$和$Se_n = -Re_n = f_n$.

显然$R \neq S$,并且它们都是酉算子.而

$$S\sqrt{T^*T}v = S\left(\sum_{k=1}^{m} s_k \langle v, e_k\rangle e_k\right) = Tv$$

$$R\sqrt{T^*T}v = R\left(\sum_{k=1}^{m} s_k \langle v, e_k\rangle e_k\right) = Tv$$

于是存在互异的S, R使得$T = S\sqrt{T^*T} = R\sqrt{T^*T}$,这与假设矛盾,于是$T$可逆.

27. 设$T \in \mathcal{L}(V)$,T的奇异值为$s_1, \cdots, s_n$.令$e_1, \cdots, e_n$和$f_1, \cdots, f_n$为V的规范正交基,使得

$$Tv = \sum_{k=1}^{n} s_k \langle v, e_k\rangle f_k$$

对任意$v \in V$成立.定义$S \in \mathcal{L}(V)$为

$$Sv = \sum_{k=1}^{n} \langle v, e_k\rangle f_k$$

试证明下列命题.

(1) S是酉算子且$||T - S|| = \max_{k=1,\cdots,n}\{|s_k - 1|\}$.

(2) 如果$E \in \mathcal{L}(V)$是酉算子,那么$||T - E|| \geqslant ||T - S||$.

Proof.

(1) 注意到$Se_k = f_k$对任意$k = 1, \cdots, n$都成立,于是S是酉算子.我们有

$$(T - S)v = \sum_{k=1}^{n} (s_k - 1)\langle v, e_k\rangle f_k = \sum_{k=1}^{n} |s_k - 1|\langle v, e_k\rangle \frac{f_k}{\operatorname{sgn}(s_k - 1)}$$

上式即为$T - S$的奇异值分解.于是$T - S$的奇异值为$|s_1 - 1|, \cdots, |s_n - 1|$.

于是$||T - S|| = \max_{k=1,\cdots,n}\{|s_k - 1|\}$.

(2) 对于任意$k = 1, \cdots, n$都有$||Ee_k|| = ||e_k|| = 1$,于是

$$||T - E|| \geqslant ||(T - E)e_k|| \geqslant |||Te_k|| - ||Ee_k||| \geqslant |||s_k f_k|| - 1| = |s_k - 1|$$

于是$||T - E|| \geqslant \max_{k=1,\cdots,n}\{|s_k - 1|\} = ||T - S||$.

28. 设$T \in \mathcal{L}(V)$.试证明:存在酉算子$S \in \mathcal{L}(V)$使得$T = \sqrt{TT^*}S$.

Proof.

考虑T的奇异值分解

$$Tv = \sum_{k=1}^{m} s_k \langle v, e_k \rangle f_k$$

根据T^*的奇异值分解,代入可得

$$\sqrt{TT^*}v = \sum_{k=1}^{m} s_k \langle v, f_k \rangle f_k$$

将$e_1, \cdots, e_m$和$f_1, \cdots, f_m$扩展为V的规范正交基$e_1, \cdots, e_n$和$f_1, \cdots, f_n$.

定义$S \in \mathcal{L}(V)$为

$$Sv = \sum_{k=1}^{n} \langle v, e_k \rangle f_k$$

则有$\langle Sv, f_k \rangle = \langle v, e_k \rangle$对于任意$k = 1, \cdots, m$成立.于是

$$\sqrt{TT^*}Sv = \sum_{k=1}^{m} s_k \langle Sv, f_k \rangle f_k = \sum_{k=1}^{m} s_k \langle v, f_k \rangle f_k = Tv$$

于是$T = \sqrt{TT^*}S$.

29. 设$T \in \mathcal{L}(V)$.试证明下列命题.

(1) 存在酉算子$S \in \mathcal{L}(V)$使得$TT^* = ST^*TS^*$.

(2) 上一问的结论蕴含T和T^*的奇异值相同.

Proof.

(1) 根据**7.93**和**7F.28**可知存在酉算子S使得$T = S\sqrt{T^*T} = \sqrt{TT^*}S$.

由于S是酉算子,于是S可逆,并且$S = S^*$.在上式右乘S^*可得

$$\sqrt{TT^*} = S\sqrt{T^*T}S^*$$

于是

$$TT^* = S\sqrt{T^*T}S^*S\sqrt{T^*T}S^* = S\sqrt{T^*T}\sqrt{T^*T}S^* = ST^*TS^*$$

(2) 由**5A.13**可得T^*T和TT^*的特征值相同,于是T和T^*的奇异值相同.

30. 设$T \in \mathcal{L}(V)$,酉算子$S \in \mathcal{L}(V)$,正算子$R \in \mathcal{L}(V)$使得$T = SR$.试证明:$R = \sqrt{T^*T}$.

Proof.

注意到

$$T^*T=(SR)^*(SR)=R^*S^*SR=R^*R=R^2$$

由于R是正算子,于是$R=\sqrt{T^*T}$.

31. 设$\mathbb{F}=\mathbb{C}$且$T\in\mathcal{L}(V)$是正规的.试证明:存在酉算子$S\in\mathcal{L}(V)$使得$T=S\sqrt{T^*T}$,并且S和$\sqrt{T^*T}$关于V的同一个规范正交基有对角矩阵.

Proof.

根据**7.93**和**7F.28**可知存在酉算子S使得$T=S\sqrt{T^*T}=\sqrt{TT^*}S$.

由于T正规,于是$T^*T=TT^*$,因此$S\sqrt{T^*T}=\sqrt{T^*T}S$,即$S$和$\sqrt{T^*T}$可交换.

根据**7B.16**可知S和$\sqrt{T^*T}$关于V的同一个规范正交基有对角矩阵.

32. 设$T\in\mathcal{L}(V,W)$且$T\neq\mathbf{0}$.令$s_1,\cdots,s_m$为T的正奇异值,试证明:存在$(\text{null }T)^\perp$的规范正交基$e_1,\cdots,e_m$使得

$$T\left(E\left(\frac{e_1}{s_1},\cdots,\frac{e_m}{s_m}\right)\right)$$

为range T中以$\mathbf{0}$为中心,半径为1的球.

Proof.

考虑T的奇异值分解

$$T=\sum_{k=1}^m s_k\left\langle v,e_k\right\rangle f_k$$

其中$e_1,\cdots,e_m$和$f_1,\cdots,f_m$分别为V和W中的规范正交组.

又因为$Tv\neq\mathbf{0}$当且仅当$v\in\operatorname{span}(e_1,\cdots,e_m)$,于是$e_1,\cdots,e_m$是$(\text{null }T)^\perp$的规范正交基.

对于任意$v\in E\left(\frac{e_1}{s_1},\cdots,\frac{e_m}{s_m}\right)$有

$$s_1^2\left|\left\langle v,e_1\right\rangle\right|^2+\cdots+s_m^2\left|\left\langle v,e_m\right\rangle\right|^2<1\Leftrightarrow\left|\left\langle Tv,f_1\right\rangle\right|^2+\cdots+\left|\left\langle Tv,f_m\right\rangle\right|^2<1\Leftrightarrow||Tv||<1$$

由于$f_1,\cdots,f_m$为range T的基,于是

$$T\left(E\left(\frac{e_1}{s_1},\cdots,\frac{e_m}{s_m}\right)\right)$$

为range T中的单位球.