Linear Algebra Done Right 6C

1. 设$v_1,\cdots,v_m\in V$.试证明
$$\{v_1,\cdots,v_m\}^{\perp}=\left(\operatorname{span}\left(v_1,\cdots,v_m\right)\right)^{\perp}$$

Proof.

由于$\{v_1,\cdots,v_m\}\subseteq\operatorname{span}\left(v_1,\cdots,v_m\right)$,于是根据正交补的性质有$\left(\operatorname{span}\left(v_1,\cdots,v_m\right)\right)^{\perp}\subseteq\{v_1,\cdots,v_m\}^{\perp}$.

对于任意$u\in\{v_1,\cdots,v_m\}^{\perp}$,都满足$\langle u,v_k\rangle=0$对任意$k\in\{1,\cdots,m\}$成立.

于是对于任意$v:=a_1v_1+\cdots+a_mv_m\in\operatorname{span}\left(v_1,\cdots,v_m\right)$有
$$\langle u,v\rangle=\left\langle u,\sum_{k=1}^{m}a_kv_k\right\rangle=\sum_{k=1}^{m}a_k\left\langle u,v_k\right\rangle=0$$
从而$u\in\left(\operatorname{span}\left(v_1,\cdots,v_m\right)\right)^{\perp}$,因而$\{v_1,\cdots,v_m\}\subseteq\left(\operatorname{span}\left(v_1,\cdots,v_m\right)\right)^{\perp}$.

综上可知$\{v_1,\cdots,v_m\}^{\perp}=\left(\operatorname{span}\left(v_1,\cdots,v_m\right)\right)^{\perp}$.

2. 设U是V的子空间,且有一组基$u_1,\cdots,u_m$.向量组$u_1,\cdots,u_m,v_1,\cdots,v_n$是$V$的一组基.对上述$V$的基运用Gram-Schmidt过程得到$V$的规范正交基$e_1,\cdots,e_m,f_1,\cdots,f_n$.试证明:$e_1,\cdots,e_m$是$U$的规范正交基,$f_1,\cdots,f_n$是$U^{\perp}$的规范正交基.

Proof.

对$u_1,\cdots,u_m$应用Gram-Schmidt过程得到的$e_1,\cdots,e_m$自然是U的规范正交基.

对于任意$k\in\{1,\cdots,n\}$和任意$j\in\{1,\cdots,m\}$,都有$\langle f_k,e_j\rangle=0$,于是$f_k\in U^{\perp}$.

于是$f_1,\cdots,f_n$是$U^{\perp}$中的规范正交组.

又因为$\dim U^{\perp}=\dim V-\dim U=n$,因而$f_1,\cdots,f_n$是$U^{\perp}$的规范正交基.

3. 设U是$\mathbb{R}^4$的子空间,其定义为
$$U=\operatorname{span}\left(\left(1,2,3,-4\right),\left(-5,4,3,2\right)\right)$$
求U的一规范正交基和$U^{\perp}$的一规范正交基.

Proof.

将$\left(1,2,3,-4\right),\left(-5,4,3,2\right)$扩展为$V$的一组基
$$\left(1,2,3,-4\right),\left(-5,4,3,2\right),\left(1,0,0,0\right),\left(0,1,0,0\right)$$

对这组基运用Gram-Schmidt过程,得到

$$e_1=\frac{1}{\sqrt{30}}(1,2,3,-4),e_2=\frac{1}{\sqrt{12030}}(-77,56,39,38)$$

$$e_3=\frac{1}{\sqrt{76190}}(190,117,60,151),e_4=\frac{1}{\sqrt{190}}(0,9,-10,3)$$

根据**6C.2**可知e_1,e_2是U的基,e_3,e_4是$U^{\perp}$的基.

4. 设$e_1,\cdots,e_n$是V中的一组向量,满足

(a) 对任意$k\in\{1,\cdots,n\}$,都有$||e_k||=1$.

(b) 对任意$v\in V$,都有$||v||^2=|\langle v,e_1\rangle|^2+\cdots+|\langle v,e_n\rangle|^2$.

试证明:$e_1,\cdots,e_n$是V的规范正交基.

Proof.

根据题设条件,对于任意$k\in\{1,\cdots,n\}$有

$$||e_k||^2=|\langle e_k,e_1\rangle|^2+\cdots+|\langle e_k,e_n\rangle|^2=1$$

又因为$|\langle e_k,e_k\rangle|=||e_k||=1$,于是

$$\sum_{j=1,j\neq k}^{n}|\langle e_k,e_j\rangle|^2=0$$

这表明对任意$j,k\in\{1,\cdots,n\}$且$j\neq k$都有$\langle e_k,e_j\rangle=0$.

于是$e_1,\cdots,e_n$是V中的规范正交组.

考虑任意的$v\in V$,根据Bessel不等式可知

$$\sum_{k=1}^{n}|\langle v,e_k\rangle|^2\leqslant||v||^2$$

当且仅当$v=\sum\limits_{k=1}^{n}|\langle v,e_k\rangle|e_k$时等号成立.

于是$v\in\operatorname{span}(e_1,\cdots,e_n)$,即$V=\operatorname{span}(e_1,\cdots,e_n)$,从而$e_1,\cdots,e_n$为$V$的规范正交基.

5. 设V是有限维的,且U为V的子空间,试证明:$P_{U^{\perp}}=I-P_U$,其中I是V上的恒等算子.

Proof.

由于$V=U\oplus U^{\perp}$,于是对于任意$v\in V$,其都可以被唯一分解为$v=u+w$,其中$u\in U,w\in U^{\perp}$.

根据正交投影的定义,我们有$P_U v = u, P_{U^\perp} v = w$,于是

$$v = P_U v + P_{U^\perp} v$$

从而

$$I = P_U + P_{U^\perp}$$

移项即可得欲证等式.

6. 设V是有限维的,且$T \in \mathcal{L}(V, W)$.试证明

$$T = TP_{(\text{null } T)^\perp} = P_{\text{range } T} T$$

Proof.

根据**6C.5**有$P_{(\text{null } T)^\perp} = I - P_{\text{null } T}$.对任意$v \in V$,都有$P_{\text{null } T} v \in \text{null } T$,于是

$$TP_{(\text{null } T)^\perp} v = T\left(I - P_{\text{null } T}\right) v = Tv - \mathbf{0} = Tv$$

于是$TP_{(\text{null } T)^\perp} = T$.

对于任意$w \in \text{range } T$,都有$P_{\text{range } T} w = w$.于是$T = P_{\text{range } T} T$.

综上,命题得证.

7. 设X和Y为V的有限维子空间.试证明:$P_X P_Y = \mathbf{0}$当且仅当$\langle x, y\rangle = 0$对所有$x \in X$和所有$y \in Y$都成立.

Proof.

$\Rightarrow$:$P_X P_Y = \mathbf{0}$即对任意$v \in V$有$P_X P_Y v = \mathbf{0}$.又range $P_Y = Y$,于是对于任意$y \in Y$有$P_X y = \mathbf{0}$.

由于$V = X \oplus X^\perp$,于是存在唯一的分解$y = x + x'$使得$x \in X, x' \in X^\perp$.

又因为$P_X y = \mathbf{0}$,即上述分解中$x = \mathbf{0}$,于是$y = x' \in X^\perp$.即对于任意$x \in X$有$\langle x, y\rangle = 0$.

$\Leftarrow$:考虑X的规范正交基$e_1, \cdots, e_n$.对于任意$v \in V$,有$P_Y v \in Y$,于是

$$P_X\left(P_Y v\right) = \sum_{k=1}^{n} \left\langle P_Y v, e_k\right\rangle e_k = \mathbf{0}$$

于是$P_X P_Y = \mathbf{0}$.

8. 设U是V的有限维子空间,且$v \in V$.定义U上的线性泛函$\phi : U \to \mathbb{F}$为 $\phi(u) = \langle u, v\rangle$ 对所有$u \in U$成立.根据Riesz表示定理,存在唯一$w \in U$使得 $\phi(u) = \langle u, w\rangle$ 对所有$u \in U$成立.试证明:$w = P_U v$.

Proof.
因为$v - P_U v \in U^{\perp}$,于是$\langle u, v - P_U v\rangle$.于是

$$\langle u, v\rangle = \langle u, v - P_U v\rangle + \langle u, P_U v\rangle = \langle u, P_U v\rangle$$

因此$\phi(u) = \langle u, v\rangle = \langle u, P_U v\rangle$.而$P_U v \in U$.根据Riesz表示定理,这样的向量是唯一存在的,于是$w = P_U v$.

9. 设V是有限维的,$P \in \mathcal{L}(V)$使得$P^2 = P$且null P中的任意向量都正交与range P中的任意向量.试证明:存在V的子空间U使得$P = P_U$.

Proof.
根据**3B.27**可知$V = \operatorname{range} P \oplus \operatorname{null} P$.又因为$V = \operatorname{range} P \oplus (\operatorname{range} P)^{\perp}$,于是

$$\dim \operatorname{null} P = \dim (\operatorname{range} P)^{\perp}$$

根据题意可知null $P \subseteq (\operatorname{range} P)^{\perp}$.于是null $P = (\operatorname{range} P)^{\perp}$.
令$U = \operatorname{range} P$.对于任意$v := Px + w \in V$,其中$Px \in \operatorname{range} P, w \in \operatorname{null} P$,有

$$P_U v = Px = P\left(Px + w\right) = Pv$$

此时$P_U = P$.

10. 设V是有限维的,$P \in \mathcal{L}(V)$使得$P^2 = P$且$||Pv|| \leqslant ||v||$对任意$v \in V$成立.试证明:存在V的子空间U使得$P = P_U$.

Proof.
考虑$w \in \operatorname{null} P$和$Px \in \operatorname{range} P$,根据题意,对任意$\lambda \in \mathbb{F}$有

$$||Px|| = ||P(Px + \lambda w)|| \leqslant ||Px + \lambda w||$$

根据**6A.6**可知$\langle w, Px\rangle = 0$,即null P中的任意向量都正交与range P中的任意向量.
根据**6C.9**,取$U = \operatorname{range} P$即可使$P = P_U$.

11. 设$T \in \mathcal{L}(V)$且U是V的有限维子空间.试证明:U在T下不变,当且仅当$P_UTP_U = TP_U$.

Proof.

$\Rightarrow$:对于任意$v \in V$,都有$P_Uv \in U$.因为U在T下不变,于是$T(P_Uv) \in U$.

于是$P_U(T(P_Uv)) = T(P_Uv)$.因而$P_UTP_U = TP_U$.

$\Leftarrow$:如果U不在T下不变,那么存在$u \in U$使$Tu \notin U$.于是

$$P_UTP_Uu = P_UTu \neq Tu = TP_Uu$$

于是$P_UTP_U \neq TP_U$.

12. 设V是有限维的,$T \in \mathcal{L}(V)$,且U是V的子空间.试证明U和$U^\perp$在T下不变,当且仅当$P_UT = TP_U$.

Proof.

$\Rightarrow$:对任意$v := u + w \in V$,其中$u \in U, w \in U^\perp$,都有$Tu \in U, Tw \in U^\perp$.于是

$$P_UTv = P_U(Tu + Tw) = Tu = TP_Uv$$

于是$P_UT = TP_U$.

$\Leftarrow$:若存在$u \in U$使得$Tu \notin U$,那么

$$TP_Uu = Tu \notin U, P_U(Tu) \in U$$

于是$TP_U \neq P_UT$,与题设矛盾.从而$Tu \in U$对任意$u \in U$成立.

若存在$w \in U^\perp$使得$Tw \notin U^\perp$,那么

$$TP_Uw = T\mathbf{0} = \mathbf{0}, P_UTw \neq \mathbf{0}$$

同理可推出矛盾,因此$Tw \in U^\perp$对任意$w \in U^\perp$成立.

于是U和$U^\perp$在T下不变.

13. 设$\mathbb{F} = \mathbb{R}$,V是有限维向量空间.对任意$v \in V$,令ϕ_v为V上的线性泛函,定义为$\phi_v(u) = \langle u, v\rangle$对所有$u \in V$成立.

(1) 试证明:$v \mapsto \phi_v$是V到V'的单的线性映射.

(2) 试证明:$v \mapsto \phi_v$是V到V'的同构.

Proof.

令映射$T: V \rightarrow V'$为$Tv = \phi_v$对所有$v \in V$成立.

(1) 对于任意$v, w \in V$,对于任意$u \in V$有

$$\phi_v(u) + \phi_w(u) = \langle u, v \rangle + \langle u, w \rangle = \langle u, v + w \rangle = \phi_{v+w}(u)$$

于是$Tv + Tw = T(v + w)$.

对于任意$v \in V$和$\lambda \in \mathbb{F}$,对于任意$u \in V$有

$$\lambda \phi_v(u) = \lambda \langle u, v \rangle = \langle u, \lambda v \rangle = \phi_{\lambda v}(u)$$

于是$\lambda Tv = T(\lambda v)$.

因此T满足可加性和齐次性,是V到V'的线性映射.

现在,假定存在$v, w \in V$使得$Tv = Tw$.于是对于任意$u \in V$有$\phi_v(u) = \phi_w(u)$.而

$$\phi_v(u) = \phi_w(u) \Leftrightarrow \langle u, v \rangle = \langle u, w \rangle \Leftrightarrow \langle u, v - w \rangle = 0 \Leftrightarrow v - w = \mathbf{0} \Leftrightarrow v = w$$

于是T是单射.

(2) 注意到$\dim V = \dim V'$,又因为T是单射,于是T是V到V'的同构.

14. 设$e_1, \cdots, e_n$是V的规范正交基.对于$v \in V$,令$\phi_v(u) = \langle u, v \rangle$对所有$u \in U$成立.试证明:$e_1, \cdots, e_n$的对偶基为$\phi_{e_1}, \cdots, \phi_{e_n}$.

Proof.

对于任意$j, k \in \{1, \cdots, n\}$且$j \neq k$有

$$\phi_{e_k}(e_j) = \langle e_j, e_k \rangle = 0$$

又$\phi_{e_k}(e_k) = \langle e_k, e_k \rangle = 1$.于是

$$\phi_{e_k}(e_j) = \begin{cases} 1, j = k \\ 0, j \neq k \end{cases}$$

这符合e_k的对偶ϕ_k的定义.于是$\phi_{e_1}, \cdots, \phi_{e_n}$为$e_1, \cdots, e_n$的对偶基.

15. 在$\mathbb{R}^4$中,令$U = \operatorname{span}((1,1,0,0),(1,1,1,2))$.求$u \in U$使得$||u - (1,2,3,4)||$尽可能小.

Proof.

对U的基使用Gran-Schmidt过程得到其规范正交基$e_1 = \frac{1}{\sqrt{2}}(1,1,0,0), e_2 = \frac{1}{\sqrt{5}}(0,0,1,2)$.

令$v=(1,2,3,4)$,则

$$P_Uv=\langle v,e_1\rangle e_1+\langle v,e_2\rangle e_2=\frac{3}{2}(1,1,0,0)+\frac{11}{5}(0,0,1,2)=\left(\frac{3}{2},\frac{3}{2},\frac{11}{5},\frac{22}{5}\right)$$

于是$u=P_Uv$即为所求向量.

16. 设$C[-1,1]$是闭区间$[-1,1]$上所有连续实值函数构成的向量空间,其内积定义为$\langle f,g\rangle=\int_{-1}^{1}fg$对所有$f,g\in C[-1,1]$成立.令$U$是$C[-1,1]$的子空间,定义为$U=\{f\in C[-1,1]:f(0)=0\}$.

(1) 试证明:$U^{\perp}=\{\mathbf{0}\}$.

(2) 试证明:不假设有限维的情况下,$V=U\oplus U^{\perp}$和$U=\left(U^{\perp}\right)^{\perp}$不再成立.

Proof.

(1) 显然$\mathbf{0}\in U^{\perp}$.考虑$g\in U^{\perp}$,令$f(x)=x^2g(x)\in U$,于是

$$\langle f,g\rangle=\int_{-1}^{1}\left(xg(x)\right)^2=0$$

这就要求$xg(x)=0$对任意$x\in[-1,1]$成立,于是$g=\mathbf{0}$,则$U^{\perp}=\{\mathbf{0}\}$.

(2) 在**(1)**中有$U^{\perp}\subset U\subset C[-1,1]$,于是$U\oplus U^{\perp}=U\neq C[-1,1]$.
又有$\left(U^{\perp}\right)^{\perp}=\{\mathbf{0}\}^{\perp}=C[-1,1]\neq U$.于是题设的两式不再成立.

17. 求$p\in\mathcal{P}_3(\mathbb{R})$使得$p(0)=p'(0)=0$,且$\int_0^1\left|2+3x-p(x)\right|^2\mathrm{d}x$尽可能小.

Solution.
考虑$\mathcal{P}_3(\mathbb{R})$的内积$\langle p,q\rangle=\int_0^1 pq$对所有$p,q\in\mathcal{P}_3(\mathbb{R})$成立.
令$U=\{p\in\mathcal{P}_3(\mathbb{R}):p(0)=p'(0)=0\}$为$\mathcal{P}_3(\mathbb{R})$的子空间,$U$的一组基为$x^2,x^3$.对这组基运用Gram-Schmidt过程得到$U$的一组规范正交基

$$e_1=\sqrt{5}x^2,e_2=6\sqrt{7}\left(x^3-\frac{5}{6}x^2\right)$$

令$q=2+3x$.我们只需求使得$||q-p||^2$最小的$p(x)$,于是

$$p(x)=P_Uq=\langle q,e_1\rangle e_1+\langle q,e_2\rangle e_2=24x^2-\frac{203}{10}x^3$$

18. 求$p \in \mathcal{P}_5(\mathbb{R})$使得$\displaystyle\int_{-\pi}^{\pi} |\sin x - p(x)|^2 \, \mathrm{d}x$尽可能小.

Proof.

本题的求解涉及到复杂的积分计算,因此过程不再赘述.最终的解为

$$p(x) = \frac{105\left(1465 - 153\pi^2 + \pi^4\right)}{8\pi^6}x - \frac{315\left(1155 - 125\pi^2 + \pi^4\right)}{4\pi^8}x^3 + \frac{693\left(945 - 105\pi^2 + \pi^4\right)}{8\pi^{10}}x^5$$

19. 设V是有限维的,$P \in \mathcal{L}(V)$是将V映成其某个子空间的正交投影.试证明:$P^\dagger = P$.

Proof.

设$P = P_U$,其中U是V的子空间.那么

$$P^\dagger = \left(P|_{(\text{null } P)^\perp}\right)^{-1} P_{\text{range } P} = (P|_U)^{-1} P_U = I_U P_U = P_U = P$$

于是命题成立.

20. 设V是有限维的,且$T \in \mathcal{L}(V, W)$.试证明:$\text{null } T^\dagger = (\text{range } T)^\perp$且$\text{range } T^\dagger = (\text{null } T)^\perp$.

Proof.

考虑T的伪逆$T^\dagger = \left(T|_{(\text{null } T)^\perp}\right)^{-1} P_{\text{range } T}$.

由于$T|_{(\text{null } T)^\perp}$是双射,于是$\left(T|_{(\text{null } T)^\perp}\right)^{-1} w = \mathbf{0}$当且仅当$w = \mathbf{0}$.

而$P_{\text{range } T} w = \mathbf{0}$当且仅当$w \in (\text{range } T)^\perp$,于是$\text{null } T^\dagger = (\text{range } T)^\perp$.

再次根据$T|_{(\text{null } T)^\perp}$是双射,可知$\left(T|_{(\text{null } T)^\perp}\right)^{-1}$是将$\text{range } T$映成$(\text{null } T)^\perp$的双射.

又因为$P_{\text{range } T}$是W映到$\text{range } T$的满射,于是$T^\dagger$将W映到$(\text{null } T)^\perp$,即$\text{range } T^\dagger = (\text{null } T)^\perp$.

21. 设$T \in \mathcal{L}(\mathbb{F}^3, \mathbb{F}^2)$定义为$T(a, b, c) = (a + b + c, 2b + 3c)$.回答下列问题.

(1) 对于$(x, y) \in \mathbb{F}^2$,求$T^\dagger(x, y)$的表达式.

(2) 用本题的例子验证$TT^\dagger = P_{\text{range } T}$.

(3) 用本题的例子验证$T^\dagger T = P_{(\text{null } T)^\perp}$.

Solution.

(1) 由题意可知null $T = \operatorname{span}(1,-3,2)$.于是$T^\dagger(x,y)=(a,b,c)$需满足

$$\begin{cases} x = a+b+c \\ y = 3b+2c \\ a-3b+2c=0 \end{cases}$$

解得$a=\dfrac{1}{14}(13x-5y), b=\dfrac{1}{14}(3x+y), c=-\dfrac{1}{7}(-x+2y)$.于是

$$T^\dagger(x,y)=\frac{1}{14}(13x-5y,3x+y,-2x+4y)$$

(2) 我们有

$$TT^\dagger(x,y)=\frac{1}{14}T(13x-5y,3x+y,-2x+4y)=\frac{1}{14}T(14x,14y)=(x,y)$$

于是$TT^\dagger = P_{\text{range } T}$.

(3) 这也是不难验证的,在此不再赘述.

22. 设V是有限维的,且$T\in\mathcal{L}(V,W)$.试证明:$TT^\dagger T=T$且$T^\dagger TT^\dagger=T^\dagger$.

Proof.
我们有

$$TT^\dagger T = P_{rangeT}T=T$$

又有

$$T^\dagger TT^\dagger = T^\dagger P_{\text{range } T}=\left(T|_{(\text{null } T)^\perp}\right)^{-1}P_{\text{range } T}P_{\text{range } T}=\left(T|_{(\text{null } T)^\perp}\right)^{-1}P_{\text{range } T}=T^\dagger$$

于是命题成立.

23. 设V是有限维的,且$T\in\mathcal{L}(V,W)$.试证明:$\left(T^\dagger\right)^\dagger=T$.

Proof.
根据**6C.20**可知

$$\left(T^\dagger\right)^\dagger=\left(T^\dagger|_{(\text{null } T^\dagger)^\perp}\right)^{-1}P_{\text{range } T^\dagger}=\left(T^\dagger|_{\text{range } T}\right)^{-1}P_{(\text{null } T)^\perp}$$

考虑到$T^\dagger=\left(T|_{(\text{null } T)^\perp}\right)^{-1}P_{\text{range } T}$,于是$T^\dagger|_{\text{range } T}=\left(T|_{(\text{null } T)^\perp}\right)^{-1}$.于是

$$\left(T^\dagger\right)^\dagger=\left(\left(T|_{(\text{null } T)^\perp}\right)^{-1}\right)^{-1}P_{(\text{null } T)^\perp}=T|_{(\text{null } T)^\perp}P_{(\text{null } T)^\perp}=TP_{(\text{null } T)^\perp}$$

对于任意$v \in V$,存在唯一的分解方式$v = u + w$使得$u \in \operatorname{null} T, w \in (\operatorname{null} T)^{\perp}$.此时

$$TP_{(\operatorname{null} T)^{\perp}} v = Tw = Tw + Tu = T(w + u) = Tv$$

于是$T = TP_{(\operatorname{null} T)^{\perp}}$,因而$\left(T^{\dagger}\right)^{\dagger} = T$.