Linear Algebra Done Right 6B

1. 设$e_1,\cdots,e_n$是V中的向量组,使得

$$||a_1e_1+\cdots+a_ne_n||^2=|a_1|^2+\cdots+|a_m|^2$$

对任意$a_1,\cdots,a_n\in\mathbb{F}$都成立.试证明:$e_1,\cdots,e_n$是规范正交组.

Proof.

对于任意$k\in\{1,\cdots,n\}$,令

$$a_j=\begin{cases}0,j\neq k\\1,j=k\end{cases}$$

可得$||e_k||^2=1$,即$\langle e_k,e_k\rangle=1$.

对于任意$k,j\in\{1,\cdots,n\}$且$j\neq k$,令

$$a_i=\begin{cases}0,i\neq j\text{或}i\neq k\\1,i=j\\t,i=k\end{cases}$$

可得$||e_j+te_k||^2=1+|t|^2\geqslant||e_j||^2$.根据**6A.6**可知$\langle e_j,e_k\rangle=0$.

综上可知$e_1,\cdots,e_n$是规范正交组.

2. 证明下列命题.

(1) 设$\theta\in\mathbb{R}$,试证明:$(\cos\theta,\sin\theta),(-\sin\theta,\cos\theta)$和$(\cos\theta,\sin\theta),(\sin\theta,-\cos\theta)$都是$\mathbb{R}^2$的规范正交基.

(2) 试证明:$\mathbb{R}^2$中的每个规范正交基都具有**(1)**中两个形式之一.

Proof.

(1) 对于任意$a,b\in\mathbb{R}$,都有

$$\begin{aligned}||a(\cos\theta,\sin\theta),b(-\sin\theta,\cos\theta)||^2&=||a\cos\theta-b\sin\theta,a\sin\theta+b\cos\theta||^2\\&=(a\cos\theta-b\sin\theta)^2+(a\sin\theta+b\cos\theta)^2\\&=a^2+b^2\end{aligned}$$

根据**6B.1**可知$(\cos\theta,\sin\theta),(-\sin\theta,\cos\theta)$是$\mathbb{R}^2$中的规范正交组.

又因为$\dim\mathbb{R}^2=2$,于是$(\cos\theta,\sin\theta),(-\sin\theta,\cos\theta)$是$\mathbb{R}^2$中的规范正交基.

另一组向量的证明过程相似,在此不再赘述.

(2) 考虑$\mathbb{R}^2$中所有满足$||u||=1$的向量u,都应当具有$(\cos\alpha,\sin\alpha)$的形式.

考虑向量$u=(\cos\alpha,\sin\alpha),v=(\cos\beta,\sin\beta)$.令$\langle u,v\rangle=0$,则有

$$\cos\alpha\cos\beta+\sin\alpha\sin\beta=0$$

即

$$\cos(\alpha-\beta)=0$$

当且仅当$\beta=\alpha+\dfrac{\pi}{2}+k\pi$,$k\in\mathbb{Z}$时成立.

若k是偶数,则$v=(-\sin\alpha,\cos\alpha)$.

若k是奇数,则$v=(\sin\alpha,-\cos\alpha)$.

于是命题得证.

3. 设$e_1,\cdots,e_m$是V中的一规范正交组,且$v\in V$.试证明:

$$||v||^2=|\langle v,e_1\rangle|^2+\cdots+|\langle v,e_m\rangle|^2\Leftrightarrow v\in\operatorname{span}(e_1,\cdots,e_m)$$

Proof.

根据Bessel不等式,$|\langle v,e_1\rangle|^2+\cdots+|\langle v,e_m\rangle|^2\leqslant||v||^2$,当且仅当

$$v-\langle v,e_1\rangle e_1-\cdots-\langle v,e_m\rangle e_m=\mathbf{0}$$

时等式成立.这等价于$v\in\operatorname{span}(e_1,\cdots,e_m)$.

4. 设$n\in\mathbb{N}^*$,试证明:

$$\frac{1}{\sqrt{2\pi}},\frac{\cos x}{\sqrt{\pi}},\cdots,\frac{\cos nx}{\sqrt{\pi}},\frac{\sin x}{\sqrt{\pi}},\cdots,\frac{\sin nx}{\sqrt{\pi}}$$

是$C[-\pi,\pi]$中的一规范正交组.

$C[\pi,\pi]$是定义在$[-\pi,\pi]$上的连续实值函数构成的向量空间,其内积定义为$\langle f,g\rangle=\displaystyle\int_{-\pi}^{\pi}fg$.

Proof.

首先有

$$\left\langle\frac{1}{\sqrt{2\pi}},\frac{1}{\sqrt{2\pi}}\right\rangle=\int_{-\pi}^{\pi}\frac{1}{2\pi}\mathrm{d}x=1$$

又对于任意$k\in\{1,\cdots,n\}$有

$$\left\langle\frac{\cos kx}{\sqrt{\pi}},\frac{\cos kx}{\sqrt{\pi}}\right\rangle=\int_{-\pi}^{\pi}\frac{\cos^2 kx}{\pi}\mathrm{d}x=\int_{-\pi}^{\pi}\frac{1+\cos 2kx}{2\pi}\mathrm{d}x=\left(\frac{x}{2\pi}+\frac{\sin 2kx}{4k\pi}\right)\Bigg|_{-\pi}^{\pi}=1$$

$$\left\langle \frac{\sin kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\sin^2 kx}{\pi} \mathrm{d}x = \int_{-\pi}^{\pi} \frac{1-\cos 2kx}{2\pi} \mathrm{d}x = \left(\frac{x}{2\pi} - \frac{\sin 2kx}{4k\pi} \right) \Bigg|_{-\pi}^{\pi} 1$$

于是这向量组中各向量的范数均为1.又因为

$$\left\langle \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos kx \mathrm{d}x = \frac{\sin kx}{2k\pi} \Bigg|_{-\pi}^{\pi} = 0$$

$$\left\langle \frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin kx \mathrm{d}x = 0$$

对于任意$j,k \in \{1, \cdots, n\}$且$j \neq k$有

$$\left\langle \frac{\cos jx}{\sqrt{\pi}}, \frac{\cos kx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\cos(j-k)x + \cos(j+k)x}{2} \mathrm{d}x = \left(\frac{\sin(j-k)x}{2(j-k)} + \frac{\sin(j+k)x}{2(j+k)} \right) \Bigg|_{-\pi}^{\pi} = 0$$

$$\left\langle \frac{\sin jx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\cos(j-k)x - \cos(j+k)x}{2} \mathrm{d}x = \left(\frac{\sin(j-k)x}{2(j-k)} - \frac{\sin(j+k)x}{2(j+k)} \right) \Bigg|_{-\pi}^{\pi} = 0$$

$$\left\langle \frac{\sin jx}{\sqrt{\pi}}, \frac{\cos kx}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\sin(j-k)x + \sin(j+k)x}{2} \mathrm{d}x = \left(\frac{\cos(j-k)x}{2(k-j)} - \frac{\cos(j+k)x}{2(j+k)} \right) \Bigg|_{-\pi}^{\pi} = 0$$

于是这向量组满足规范正交组的定义,命题得证.

5. 设$f : [-\pi, \pi] \to \mathbb{R}$是连续的.对于任意$k \in \mathbb{N}$,定义

$$a_k = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \cos(kx) \mathrm{d}x \qquad \text{和} \qquad b_k = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \sin(kx) \mathrm{d}x$$

试证明:

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2 \right) \leqslant \int_{-\pi}^{\pi} \left(f(x) \right)^2 \mathrm{d}x$$

Proof.

设$V = C[-\pi, \pi]$,于是$f \in C[-\pi, \pi]$.沿用**6B.4**中V上内积的定义,我们有

$$\frac{a_0^2}{2} = \left| \frac{a_0}{\sqrt{2}} \right|^2 = \left| \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} f(x) \mathrm{d}x \right|^2 = \left| \left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle \right|^2$$

$$a_k^2 = |a_k|^2 = \left| \left\langle f, \frac{\cos kx}{\sqrt{\pi}} \right\rangle \right|^2$$

$$b_k^2 = |b_k|^2 = \left| \left\langle f, \frac{\sin kx}{\sqrt{\pi}} \right\rangle \right|^2$$

对于任意$n \in \mathbb{N}$,根据**6B.4**可知

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \cdots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \cdots, \frac{\sin nx}{\sqrt{\pi}}$$

是$C[-\pi,\pi]$上的规范正交组.根据Bessel不等式有

$$\left|\left\langle f,\frac{1}{\sqrt{2\pi}}\right\rangle\right|^2+\left|\left\langle f,\frac{\cos x}{\sqrt{\pi}}\right\rangle\right|^2+\cdots+\left|\left\langle f,\frac{\cos nx}{\sqrt{\pi}}\right\rangle\right|^2+\left|\left\langle f,\frac{\sin x}{\sqrt{\pi}}\right\rangle\right|^2+\cdots+\left|\left\langle f,\frac{\sin nx}{\sqrt{\pi}}\right\rangle\right|^2\leqslant||f||^2$$

即

$$\frac{a_0^2}{2}+\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)\leqslant\int_{-\pi}^{\pi}\left(f(x)\right)^2\mathrm{d}x$$

因为上式对所有正整数n都成立,于是两边对n取极限有

$$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}\left(a_k^2+b_k^2\right)\leqslant\int_{-\pi}^{\pi}\left(f(x)\right)^2\mathrm{d}x$$

6. 设$e_1,\cdots,e_n$是V中的规范正交基.

(1) 试证明:如果$v_1,\cdots,v_n$是V中的向量组,满足

$$||e_k-v_k||<\frac{1}{\sqrt{n}}$$

对于任意$k\in\{1,\cdots,n\}$成立,那么$v_1,\cdots,v_n$是V的基.

(2) 试证明:存在V中的向量组$v_1,\cdots,v_n$,满足

$$||e_k-v_k||\leqslant\frac{1}{\sqrt{n}}$$

对于任意$k\in\{1,\cdots,n\}$成立,且$v_1,\cdots,v_n$不是线性无关组.

Proof.

(1) 设$a_1,\cdots,a_n\in\mathbb{F}$使得

$$a_1v_1+\cdots+a_nv_n=\mathbf{0}$$

则有

$$\sum_{k=1}^{n}|a_k|^2=\left|\left|\sum_{k=1}^{n}a_ke_k\right|\right|^2=\left|\left|\sum_{k=1}^{n}a_k\left(e_k-v_k\right)\right|\right|^2\leqslant\sum_{k=1}^{n}|a_k|^2||e_k-v_k||^2\leqslant\sum_{k=1}^{n}|a_k|^2\cdot\sum_{k=1}^{n}||e_k-v_k||^2$$

又

$$\sum_{k=1}^{n}||e_k-v_k||^2<n\cdot\frac{1}{n}=1$$

于是$\sum\limits_{k=1}^{n}|a_k|^2=0$,即$a_1=\cdots=a_n=0$.

于是$v_1,\cdots,v_n$线性无关.因而$v_1,\cdots,v_n$是V的基.

(2) 令$v_k = e_k - \dfrac{e_1 + \cdots + e_n}{n}$.注意到

$$v_1 + \cdots + v_n = e_1 + \cdots + e_n - n\left(\frac{e_1 + \cdots + e_n}{n}\right) = \mathbf{0}$$

于是$v_1, \cdots, v_n$不是线性无关组.另一方面,有

$$||v_k - e_k|| = \left|\left|\frac{e_1 + \cdots + e_n}{n}\right|\right| = \frac{1}{n}||e_1 + \cdots + e_n|| = \frac{\sqrt{n}}{n} \leqslant \frac{1}{\sqrt{n}}$$

于是这样的$v_1, \cdots, v_n$满足题意.

7. 设$T \in \mathcal{L}(\mathbb{R}^3)$关于基$(1,0,0),(1,1,1),(1,1,2)$有上三角矩阵.求$\mathbb{R}^3$的一规范正交基,使得$T$关于其有上三角矩阵.

Solution.

令$v_1 = (1,0,0), v_2 = (1,1,1), v_3 = (1,1,2)$.

对v_1, v_2, v_3应用Gram-Schmidt过程得到$\mathbb{R}^3$的一规范正交基

$$e_1 = (1,0,0), e_2 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), e_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

且满足$\operatorname{span}(v_1, \cdots, v_k) = \operatorname{span}(e_1, \cdots, e_k)$.于是$T$在$\operatorname{span}(e_1, \cdots, e_k)$下不变,即$T$关于$e_1, \cdots, e_3$有上三角矩阵.

8. 定义$\mathcal{P}_2(\mathbb{R})$的内积为$\langle p, q\rangle = \displaystyle\int_0^1 pq$,使得$\mathcal{P}_2(\mathbb{R})$成为内积空间.回答下列问题.

(1) 对$\mathcal{P}_2(\mathbb{R})$的标准基应用Gram-Schmidt过程,得到$\mathcal{P}_2(\mathbb{R})$的一组规范正交基.

(2) $\mathcal{P}_2(\mathbb{R})$上的微分算子$D$关于基$1, x, x^2$有上三角矩阵.求$\mathcal{P}_2(\mathbb{R})$关于**(1)**中所求规范正交基的矩阵,并验证这矩阵是上三角矩阵.

Proof.

(1) 令$v_1 = 1, v_2 = x, v_3 = x^2$.我们有

$$f_1 = v_1 = 1$$
$$f_2 = v_2 - \frac{\langle v_2, f_1\rangle}{||f_1||^2}f_1 = x - \frac{1}{2}$$
$$f_3 = v_3 - \frac{\langle v_3, f_1\rangle}{||f_1||^2}f_1 - \frac{\langle v_3, f_2\rangle}{||f_2||^2}f_2 = x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right) = x^2 - x + \frac{1}{6}$$

于是

$$e_1 = \frac{f_1}{||f_1||} = 1, e_2 = \frac{f_2}{||f_2||} = 2\sqrt{3}\left(x - \frac{1}{2}\right), e_3 = \frac{f_3}{||f_3||} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)$$

即为所求的规范正交基.

(2) 不难得出,所求矩阵为

$$\mathcal{M}(D)=\begin{pmatrix}0 & 2\sqrt{3} & 0\\ 0 & 0 & 2\sqrt{15}\\ 0 & 0 & 0\end{pmatrix}$$

这显然是一个上三角矩阵.

9. 设$e_1,\cdots,e_m$是对V中的线性无关组$v_1,\cdots,v_m$运用Gram-Schmidt过程得到的规范正交组.试证明:对于任意$k\in\{1,\cdots,m\}$,都有$\langle v_k,e_k\rangle>0$.

Proof.

我们有

$$\begin{aligned}\langle v_k,e_k\rangle&=\frac{1}{||f_k||}\langle v_k,f_k\rangle\\&=\frac{1}{||f_k||}\left\langle v_k,v_k-\sum_{j=1}^{k-1}\frac{\langle v_k,f_j\rangle}{||f_j||^2}f_j\right\rangle\\&=\frac{1}{||f_k||}\left(||v_k||^2-\sum_{j=1}^{k-1}\frac{|\langle v_k,f_j\rangle|^2}{||f_j||^2}\right)\\&=\frac{1}{||f_k||}\left(||v_k||^2-\sum_{j=1}^{k-1}|\langle v_k,e_j\rangle|^2\right)\end{aligned}$$

又因为$v_k\notin\operatorname{span}(e_1,\cdots,e_{k-1})$,于是根据Bessel不等式有

$$||v_k||^2>\sum_{j=1}^{k-1}|\langle v_k,e_j\rangle|^2$$

于是$\langle v_k,e_k\rangle>0$.

10. 设$v_1,\cdots,v_m$是V中的线性无关组.试证明:通过Gram-Schmidt过程得到的规范正交组$e_1,\cdots,e_m$是仅有的对任意$k\in\{1,\cdots,m\}$都有$\langle v_k,e_k\rangle>0$且$\operatorname{span}(v_1,\cdots,v_k)=\operatorname{span}(e_1,\cdots,e_k)$的规范正交组.

Lemma.10 设$v_1,\cdots,v_n$是V中的线性无关组,对其应用Gram-Schmidt过程得到V上的规范正交组$e_1,\cdots,e_n$.令$S=\{\lambda\in\mathbb{F}:|\lambda|=1\}$和$m\in\mathbb{N}^*$,设$S^m$为所有将$\{1,\cdots,m\}$映射到$S$的函数的集合.那么对于$V$的规范正交组$u_1,\cdots,u_m$使得对于任意$k\in\{1,\cdots,n\}$都有$\operatorname{span}(u_1,\cdots,u_k)=\operatorname{span}(v_1,\cdots,v_k)$,一定存在$f\in S^m$使得对于任意$k\in\{1,\cdots,n\}$有$u_k=f(k)e_k$.

Proof.

对于给定的$f \in S^m$,可知对于任意$j,k \in \{1,\cdots,m\}$且$j \neq k$有

$$||f(j)e_j|| = |f(j)|||e_j|| = 1 \qquad \langle f(j)e_j, f(k)e_k \rangle = f(j)\overline{f(k)} \langle e_j, e_k \rangle = 0$$

又因为$0 \notin S$,于是

$$\operatorname{span}(f(1)e_1,\cdots,f(k)e_k) = \operatorname{span}(e_1,\cdots,e_k) = \operatorname{span}(v_1,\cdots,v_k)$$

于是$f(1)e_1,\cdots,f(m)e_m$是满足

$$\operatorname{span}(f(1)e_1,\cdots,f(k)e_k) = \operatorname{span}(v_1,\cdots,v_k), \forall k \in \{1,\cdots,m\}$$

的规范正交组.现在考虑题设中的规范正交组$u_1,\cdots,u_m$,则有

$$\operatorname{span}(u_1,\cdots,u_k) = \operatorname{span}(v_1,\cdots,v_k) = \operatorname{span}(e_1,\cdots,e_k), \forall k \in \{1,\cdots,m\}$$

于是$\operatorname{span}(u_1) = \operatorname{span}(e_1)$.令$u_1 = \lambda_1 e_1$.因为$||u_1|| = ||e_1|| = 1$,于是$|\lambda_1| = 1$,即$\lambda_1 \in S$.
由于$\operatorname{span}(u_1,u_2) = \operatorname{span}(e_1,e_2)$,于是$u_2 \in \operatorname{span}(e_1,e_2)$.于是

$$u_2 = \langle u_2, e_1 \rangle e_1 + \langle u_2, e_2 \rangle e_2$$

又因为u_1,u_2是规范正交组,于是

$$0 = \langle u_1, u_2 \rangle = \langle \lambda_1 e_1, u_2 \rangle = \lambda_1 \langle e_1, u_2 \rangle$$

因为$\lambda_1 \neq 0$,于是$\langle e_1, u_2 \rangle = 0$,于是

$$u_2 = \langle u_2, e_2 \rangle e_2$$

令$\lambda_2 = \langle u_2, e_2 \rangle$,同理可知$|\lambda_2| = 1$,即$\lambda_2 \in S$.
依此过程构造$\lambda_1,\cdots,\lambda_m \in S$,令$f \in S^m$满足$f(k) = \lambda_k$对任意$k \in \{1,\cdots,m\}$成立.
于是$u_1,\cdots,u_m$就被写成$f(1)e_1,\cdots,f(m)e_m$的形式,命题得证.

Solution.

回到**6B.10**,假定存在$u_1,\cdots,u_k$亦满足题设条件,则存在$f \in S^m$使得$u_k = f(k)e_k$对任意$k \in \{1,\cdots,m\}$成立.
此时有

$$\langle v_k, u_k \rangle = \langle v_k, f(k)u_k \rangle = \overline{f(k)} \langle v_k, e_k \rangle > 0$$

这要求$\overline{f(k)} \in \mathbb{R}$且$\overline{f(k)} > 0$.又因为$f(k) \in S$,于是$f(k) = 1$,从而表明$u_k = e_k$.
于是只有$e_1,\cdots,e_m$满足题意.

11. 求多项式$q \in \mathcal{P}_2(\mathbb{R})$使得$p\left(\frac{1}{2}\right) = \int_0^1 pq$对任意$p \in \mathcal{P}_2(\mathbb{R})$都成立.