Linear Algebra Done Right 7A

1. 设$n \in \mathbb{N}^*$,定义$T \in \mathcal{L}(\mathbb{F}^n)$为

$$T\left(z_1, \cdots, z_n\right) = (0, z_1, \cdots, z_{n-1})$$

求$T^*\left(z_1, \cdots, z_n\right)$的表达式.

Solution.

令$v = \left(a_1, \cdots, a_n\right), w = \left(b_1, \cdots, b_n\right)$,于是

$$\begin{aligned}
\langle Tv, w \rangle &= \langle (0, a_1, \cdots, a_{n-1}), (b_1, \cdots, b_n) \rangle \\
&= \sum_{k=1}^{n-1} a_k b_{k+1} \\
&= \langle (a_1, \cdots, a_n), (b_2, \cdots, b_n, 0) \rangle \\
&= \langle v, T^* w \rangle
\end{aligned}$$

于是

$$T^*\left(z_1, \cdots, z_n\right) = \left(z_2, \cdots, z_n, 0\right)$$

2. 设$T \in \mathcal{L}(V, W)$.试证明

$$T = \mathbf{0} \Leftrightarrow T^* = \mathbf{0} \Leftrightarrow TT^* = \mathbf{0} \Leftrightarrow T^*T = \mathbf{0}$$

Proof.

我们有

$$\begin{aligned}
T = \mathbf{0} &\Leftrightarrow \operatorname{null} T = V, \operatorname{range} T = \{\mathbf{0}\} \\
&\Leftrightarrow \left(\operatorname{range} T^*\right)^{\perp} = V, \left(\operatorname{null} T^*\right)^{\perp} = \{\mathbf{0}\} \\
&\Leftrightarrow \operatorname{range} T^* = \{\mathbf{0}\}, \operatorname{null} T = V \\
&\Leftrightarrow T^* = \mathbf{0}
\end{aligned}$$

从$T = \mathbf{0}$出发推出$TT^* = T^*T = \mathbf{0}$是容易的.现在假设$T^*T = \mathbf{0}$,于是对任意$v \in V$有

$$TT^* = \mathbf{0} \Rightarrow TT^*v = \mathbf{0} \Rightarrow \langle T^*Tv, v \rangle = 0 \Rightarrow \langle Tv, Tv \rangle = 0 \Rightarrow Tv = \mathbf{0}$$

这表明$T = \mathbf{0}$.

现在假设$TT^* = \mathbf{0}$,同理可推出$T^* = \mathbf{0}$,从而$T = \mathbf{0}$.

于是上述四个命题等价.

3. 设$T \in \mathcal{L}(V)$和$\lambda \in \mathbb{F}$.试证明:λ是T的特征值,当且仅当$\overline{\lambda}$是T^*的特征值.

Proof.

我们有

$$\begin{aligned}
\lambda\text{是}T\text{的特征值} &\Leftrightarrow T-\lambda I\text{不是满射} \\
&\Leftrightarrow \operatorname{range}\ (T-\lambda I) \neq V \\
&\Leftrightarrow \left(\operatorname{null}\ (T-\lambda I)^*\right)^{\perp} \neq V \\
&\Leftrightarrow \left(\operatorname{null} T^*-\overline{\lambda} I\right)^{\perp} \neq V \\
&\Leftrightarrow \operatorname{null}\ \left(T^*-\overline{\lambda} I\right) \neq \{\mathbf{0}\} \\
&\Leftrightarrow T^*-\overline{\lambda} I\text{不是单射} \\
&\Leftrightarrow \overline{\lambda}\text{是}T^*\text{的特征值}
\end{aligned}$$

4. 设$T \in \mathcal{L}(V)$且U是T的子空间.试证明U在T下不变,当且仅当$U^{\perp}$在T^*下不变.

Proof.