Linear Algebra Done Right 7A

1. 设$n \in \mathbb{N}^*$,定义$T \in \mathcal{L}(\mathbb{F}^n)$为

$$T\left(z_1, \cdots, z_n\right) = \left(0, z_1, \cdots, z_{n-1}\right)$$

求$T^*\left(z_1, \cdots, z_n\right)$的表达式.

Solution.

令$v = \left(a_1, \cdots, a_n\right), w = \left(b_1, \cdots, b_n\right)$,于是

$$\begin{aligned}
\langle Tv, w\rangle &= \langle (0, a_1, \cdots, a_{n-1}), (b_1, \cdots, b_n)\rangle \\
&= \sum_{k=1}^{n-1} a_k b_{k+1} \\
&= \langle (a_1, \cdots, a_n), (b_2, \cdots, b_n, 0)\rangle \\
&= \langle v, T^* w\rangle
\end{aligned}$$

于是

$$T^*\left(z_1, \cdots, z_n\right) = \left(z_2, \cdots, z_n, 0\right)$$

2. 设$T \in \mathcal{L}(V, W)$.试证明

$$T = \mathbf{0} \Leftrightarrow T^* = \mathbf{0} \Leftrightarrow TT^* = \mathbf{0} \Leftrightarrow T^*T = \mathbf{0}$$

Proof.

我们有

$$\begin{aligned}
T = \mathbf{0} &\Leftrightarrow \operatorname{null} T = V, \operatorname{range} T = \{\mathbf{0}\} \\
&\Leftrightarrow \left(\operatorname{range} T^*\right)^{\perp} = V, \left(\operatorname{null} T^*\right)^{\perp} = \{\mathbf{0}\} \\
&\Leftrightarrow \operatorname{range} T^* = \{\mathbf{0}\}, \operatorname{null} T = V \\
&\Leftrightarrow T^* = \mathbf{0}
\end{aligned}$$

从$T = \mathbf{0}$出发推出$TT^* = T^*T = \mathbf{0}$是容易的.现在假设$T^*T = \mathbf{0}$,于是对任意$v \in V$有

$$TT^* = \mathbf{0} \Rightarrow TT^*v = \mathbf{0} \Rightarrow \langle T^*Tv, v\rangle = 0 \Rightarrow \langle Tv, Tv\rangle = 0 \Rightarrow Tv = \mathbf{0}$$

这表明$T = \mathbf{0}$.

现在假设$TT^* = \mathbf{0}$,同理可推出$T^* = \mathbf{0}$,从而$T = \mathbf{0}$.

于是上述四个命题等价.

3. 设$T \in \mathcal{L}(V)$和$\lambda \in \mathbb{F}$.试证明:λ是T的特征值,当且仅当$\overline{\lambda}$是T^*的特征值.

Proof.

我们有

$$\begin{aligned}
\lambda\text{是}T\text{的特征值} &\Leftrightarrow T-\lambda I\text{不是满射} \\
&\Leftrightarrow \text{range } (T-\lambda I) \neq V \\
&\Leftrightarrow \left(\text{null } \left(T-\lambda I\right)^*\right)^{\perp} \neq V \\
&\Leftrightarrow \left(\text{null } T^*-\overline{\lambda} I\right)^{\perp} \neq V \\
&\Leftrightarrow \text{null } \left(T^*-\overline{\lambda} I\right) \neq \{\mathbf{0}\} \\
&\Leftrightarrow T^*-\overline{\lambda} I\text{不是单射} \\
&\Leftrightarrow \overline{\lambda}\text{是}T^*\text{的特征值}
\end{aligned}$$

4. 设$T \in \mathcal{L}(V)$且U是T的子空间.试证明U在T下不变,当且仅当$U^{\perp}$在T^*下不变.

Proof.

首先假设U在T下不变.对于任意$u \in U$和$w \in U^{\perp}$,我们有

$$\langle Tu, w\rangle = \langle u, T^*w\rangle$$

由于U在T下不变,于是$Tu \in U$.因此$\langle Tu, w\rangle = 0$,于是$\langle u, T^*w\rangle = 0$.

由于上式对任意$u \in U$成立,于是$T^*w \in U^{\perp}$,即$U^{\perp}$在T^*下不变.

由于$(T^*)^* = T, \left(U^{\perp}\right)^{\perp} = U$,于是将上述条件中的$U$和$T$分别替换为$U^{\perp}$和$T^*$即可证得另一方向.

5. 设$T \in \mathcal{L}(V, W)$,$e_1, \cdots, e_n$为V的规范正交基,$f_1, \cdots, f_m$为W的规范正交基.试证明:

$$||Te_1||^2 + \cdots + ||Te_n||^2 = ||T^*f_1||^2 + \cdots + ||T^*f_m||^2$$

Proof.

由于$f_1, \cdots, f_m$是W的规范正交基,于是对于任意$k \in \{1, \cdots, n\}$有

$$||Te_k||^2 = |\langle Te_k, f_1\rangle|^2 + \cdots + |\langle Te_k, f_m\rangle|^2$$

同理,对于任意$j \in \{1, \cdots, m\}$有

$$||T^* f_j||^2 = |\langle T^* f_j, e_1 \rangle|^2 + \cdots + |\langle T^* f_j, e_n \rangle|^2$$

又因为$\langle Te_k, f_j \rangle = \langle e_k, T^* f_j \rangle$,于是$|\langle Te_k, f_j \rangle|^2 = |\langle T^* f_j, e_k \rangle|^2$.于是

$$\sum_{k=1}^{n} ||Te_k||^2 = \sum_{k=1}^{n} \sum_{j=1}^{m} |\langle Te_k, f_j \rangle|^2 = \sum_{j=1}^{m} \sum_{k=1}^{n} |\langle T^* f_j, e_k \rangle|^2 = \sum_{j=1}^{m} ||T^* f_j||^2$$

6. 设$T \in \mathcal{L}(V, W)$,试证明下列命题.

(1) T是单射,当且仅当T^*是满射.

(2) T是满射,当且仅当T^*是单射.

Proof.

(1) 由于$\text{null } T = (\text{range } T^*)^\perp$,我们有

$$T\text{是单射} \Leftrightarrow \text{null } T = \mathbf{0} \Leftrightarrow \text{range } T^* = V \Leftrightarrow T^*\text{是满射}$$

(2) 由于$(T^*)^* = T$,于是交换**(1)**中的T和T^*即可.

7. 设$T \in \mathcal{L}(V, W)$,试证明下列命题.

(1) $\dim \text{null } T^* = \dim \text{null } T + \dim W - \dim V$.

(2) $\dim \text{range } T^* = \dim \text{range } T$.

Proof.

(1) 我们有

$$\dim \text{null } T^* = \dim (\text{range } T)^\perp = \dim W - \dim \text{range } T = \dim W - \dim V + \dim \text{null } T$$

(2) 我们有

$$\dim \text{range } T^* = \dim (\text{null } T)^\perp = \dim V - \dim \text{null } T = \dim \text{range } T$$

8. 设A为$m \times n$矩阵,试证明其行秩等于列秩.

Proof.

记$\overline{v}$为列向量v的复共轭.考虑A的列向量的张成空间的基$v_1, \cdots, v_l$,我们有

$$a_1\overline{v_1} + \cdots + a_l\overline{v_l} = \mathbf{0} \Rightarrow \overline{a_1}v_1 + \cdots + \overline{a_l}v_l = \mathbf{0} \Leftrightarrow \overline{a_1} = \cdots = \overline{a_l} = 0 \Leftrightarrow a_1 = \cdots = a_l = 0$$

从而$\overline{v_1}, \cdots, \overline{v_l}$线性无关.于是$\overline{A}$的列秩不小于$A$的列秩.交换两者可知$\overline{A}$与$A$的列秩相等.

现在我们假设A是线性映射$T \in \mathcal{L}(\mathbb{F}^m, \mathbb{F}^n)$对应于标准基的矩阵.于是

$$A\text{的列秩} = \dim \operatorname{range} T = \dim \operatorname{range} T^* = A^*\text{的列秩} = A^{\mathrm{t}}\text{的列秩} = A\text{的行秩}$$

9. 试证明:V上两自伴算子的乘积是自伴的,当且仅当这两个算子可交换.

Proof.

对于自伴的$S, T \in \mathcal{L}(V)$,我们有

$$(ST)^* = ST \Leftrightarrow T^*S^* = ST \Leftrightarrow TS = ST$$

10. 设$\mathbb{F} = \mathbb{C}$且$T \in \mathcal{L}(V)$.试证明:T是自伴的,当且仅当$\langle Tv, v\rangle = \langle T^*v, v\rangle$对任意$v \in V$成立.

Proof.

设$T \in \mathcal{L}(V)$,我们有

$$\begin{aligned} \langle Tv, v\rangle = \langle T^*v, v\rangle, \forall v \in V &\Leftrightarrow \langle (T - T^*)v, v\rangle = 0, \forall v \in V \\ &\Leftrightarrow T - T^* = \mathbf{0} \\ &\Leftrightarrow T\text{是自伴的} \end{aligned}$$

11. 定义算子$S : \mathbb{F}^2 \to \mathbb{F}^2$为$S(w, z) = (-z, w)$.回答下列问题.

(1) 求S^*的表达式.

(2) 试证明:S正规但不自伴.

(3) 求S的所有特征值.

Solution.

(1) 对于任意$v := (w_1, z_1), u := (w_2, z_2) \in \mathbb{F}^2$有

$$\langle Tv, u\rangle = (-z_1, w_1)\cdot(w_2, z_2) = -z_1w_2 + w_1z_2 = (w_1, z_1)\cdot(z_2, -w_2) = \langle v, (z_2, -w_2)\rangle$$

于是$S^*(w, z) = (z, -w)$.

(2) 不难验证$S^*S = SS^* = I$而$S \neq S^*$,于是S正规但不自伴.

(3) 当$\mathbb{F} = \mathbb{R}$时,容易验证S没有特征值.

当$\mathbb{F} = \mathbb{C}$时,令$v := (w, z) \in \mathbb{C}^2$,于是$Sv = \lambda v$即$\begin{cases} w = -\lambda z \\ z = \lambda w \end{cases}$.

解上述方程,得$\lambda = \pm\mathrm{i}$,于是S的特征值为$\mathrm{i}, -\mathrm{i}$.

12. 称算子$B \in \mathcal{L}(V)$是**斜的**,如果$B + B^* = \mathbf{0}$.设$T \in \mathcal{L}(V)$,试证明:T是正规算子,当且仅当存在可交换的$A, B \in \mathcal{L}(V)$且A是自伴算子,B是斜算子,使得$T = A + B$.

Proof.

$\Leftarrow$:设$T \in \mathcal{L}(V)$且存在满足题意的A, B使得$T = A + B$.那么$T^* = A^* + B^* = A - B$.于是

$$T^*T = (A - B)(A + B) = A^2 - BA + AB - B^2 = A^2 - B^2$$

$$TT^* = (A + B)(A - B) = A^2 + BA - AB - B^2 = A^2 - B^2$$

于是$TT^* = T^*T$,因而T是自伴的.

$\Rightarrow$:令$A = \dfrac{T + T^*}{2}, B = \dfrac{T - T^*}{2}$.于是

$$AB - BA = \frac{T^*T - TT^*}{2} = \mathbf{0}$$

又$A^* = A, B^* = -B$,因此存在满足条件的A, B使$T = A + B$.

13. 设$\mathbb{F} = \mathbb{R}$,定义$\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$为$\mathcal{A}T = T^*$对所有$T \in \mathcal{L}(V)$成立.回答下列问题.

(1) 求$\mathcal{A}$的所有特征值.

(2) 求$\mathcal{A}$的最小多项式.

Solution.

(1) 考虑$\mathcal{A}$的特征值$\lambda \in \mathbb{R}$,于是$\mathcal{A}T = \lambda T = T^*$.两边取伴随可得$\lambda T^* = T$,从而$(\lambda^2 - 1)T = \mathbf{0}$.

由于T是$\mathcal{A}$的特征向量,于是$T \neq \mathbf{0}$,于是$\lambda = \pm 1$.

于是$\mathcal{A}$的特征值为1或-1.

(2) 令$p(z) = z^2 - 1$.对于任意$T \in \mathcal{L}(V)$,我们有

$$p(\mathcal{A})T = \mathcal{A}^2 T - T = \mathcal{A}T^* - T = T - T = \mathbf{0}$$

又$\mathcal{A}$的特征值为$1, -1$.于是$p(z) = z^2 - 1$是$\mathcal{A}$的最小多项式.

14. 在$\mathcal{P}_2(\mathbb{R})$上定义内积$\langle \cdot, \cdot \rangle$为$\langle p, q \rangle = \displaystyle\int_0^1 pq$.定义算子$T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$为

$$T(ax^2 + bx + c) = bx$$

回答下列问题.

(1) 试证明:T不是自伴算子.

(2) T关于$\mathcal{P}_2(\mathbb{R})$的标准基$1, x, x^2$的矩阵为

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

这矩阵与它的共轭转置相等,尽管T不是自伴的.试解释这为什么不矛盾.

Proof.

(1) 考虑$p(x) = 1, q(x) = x \in \mathcal{P}_2(\mathbb{R})$.我们有

$$\langle Tp, q \rangle = \langle \mathbf{0}, x \rangle = 0$$

$$\langle p, Tq \rangle = \int_0^1 x \mathrm{d}x = \frac{1}{2}$$

于是$\langle Tp, q \rangle = \langle p, T^* q \rangle \neq \langle p, Tq \rangle$.于是$T \neq T^*$,因而它不自伴.

(2) $1, x, x^2$不是$\mathcal{P}_2(\mathbb{R})$的规范正交基.

15. 设$T \in \mathcal{L}(V)$可逆,试证明下列命题.

(1) T自伴,当且仅当T^{-1}自伴.

(2) T正规,当且仅当T^{-1}正规.

Proof.

(1) 由于$(T^*)^{-1}=(T^{-1})^*$,于是在$T=T^*$两边取逆可得$T^{-1}=(T^{-1})^*$,从而T与T^{-1}的自伴性等价.

(2) 同理,在$T^*T=TT^*$两边取逆可得$T^{-1}\left(T^{-1}\right)^*=\left(T^{-1}\right)^*T^{-1}$,于是$T$与$T^{-1}$的正规性等价.

16. 设$\mathbb{F}=\mathbb{R}$,回答下列问题.

(1) 试证明:V上的自伴算子构成的集合是$\mathcal{L}(V)$的子空间.

(2) 求**(1)**中的子空间的维数.

Proof.

(1) 设$U=\{T\in\mathcal{L}(V):T=T^*\}$.显然$\mathbf{0}^*=\mathbf{0}$,于是$\mathbf{0}\in U$.
对于任意$S,T\in U$,都有$(S+T)^*=S^*+T^*=S+T$,于是$S+T\in U$,即U对加法封闭.
对于任意$T\in U$和$\lambda\in\mathbb{R}$,都有$(\lambda T)^*=\lambda T^*=\lambda T$,即$\lambda T\in U$,于是$U$对标量乘法封闭.
于是U是V的子空间.

(2) 令$\dim V=n$.考虑所有$n\times n$的自伴矩阵.对于任意$j,k\in\{1,\cdots,n\}$且$j\leqslant k$,定义$\mathcal{M}_{j,k}$如下

$$\mathcal{M}_{j,k}=\begin{cases}\mathcal{M}_{j,k}=\mathcal{M}_{k,j}=1\text{且其余元素为}0,j\neq k\\ \mathcal{M}_{k,k}=1\text{且其余元素为}0,j=k\end{cases}$$

容易验证这样的$\mathcal{M}_{j,k}$线性无关,且所有自伴矩阵都可以写为上述矩阵的线性组合.
具体来说,若$A\in\mathbb{R}^{n,n}$自伴,那么

$$A=\sum_{1\leqslant j\leqslant k\leqslant n}A_{j,k}\mathcal{M}_{j,k}$$

因而$U=\operatorname{span}(\mathcal{M}_{1,1},\cdots,\mathcal{M}_{n,n})$.于是$\dim U=\dfrac{n(n+1)}{2}$.

17. 设$\mathbb{F}=\mathbb{C}$.试证明:V上的自伴算子构成的集合不是$\mathcal{L}(V)$的子空间.

Proof.

对于$\lambda\in\mathbb{C}\backslash\mathbb{R}$有$\lambda\neq\overline{\lambda}$.于是对于某个自伴的$T\in\mathcal{L}(V)$有

$$(\lambda T)^*=\overline{\lambda}T^*=\overline{\lambda}T\neq\lambda T$$

从而λT不是自伴的,于是这集合对标量乘法不封闭,不是$\mathcal{L}(V)$的子空间.

18. 设$\dim V \geqslant 2$,试证明:V上的正规算子构成的集合不是$\mathcal{L}(V)$的子空间.

Proof.

不妨令$A, B \in \mathcal{L}(V)$关于V上的某规范正交基的矩阵为

$$\mathcal{M}(A) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \mathcal{M}(B) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

容易验证$A = A^*, B = B^*$,于是A, B都是自伴的.然而

$$\mathcal{M}((A+B)(A+B)^*) = \begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \mathcal{M}((A+B)^*(A+B))$$

从而$A+B$不是正规算子,于是这集合对加法不封闭.

19. 设$T \in \mathcal{L}(V)$且对任意$v \in V$都有$||T^*v|| \leqslant ||Tv||$.试证明$T$是正规算子.

Proof.

令$e_1, \cdots, e_n$是V的规范正交基,根据**7A.5**有

$$||Te_1||^2 + \cdots + ||Te_n||^2 = ||T^*e_1||^2 + \cdots + ||T^*e_n||^2$$

根据题意可知$||Te_k|| \geqslant ||T^*e_k||$对任意$k \in \{1, \cdots, n\}$成立.于是只能有$||Te_k|| = ||T^*e_k||$对任意$k \in \{1, \cdots, n\}$成立.

对于任意$v := a_1e_1 + \cdots + a_ne_n$,我们有

$$||Tv||^2 = \langle , \rangle$$