How to find this LCM sum function? $\text{lcm}(1,n)+\text{lcm}(2,n)+\cdots+\text{lcm}(n,n)$

Given an integer n, How do we find

$$S=\text{lcm}(1,n)+\text{lcm}(2,n)+\ldots+\text{lcm}(n,n)$$

I know how find the gcd sum

$$\gcd(1,n)+\gcd(2,n)+\cdots+\gcd(n,n)$$

Because there is

$$\sum \phi(n/i)*i$$

where $i|n$. But how do I use it to calculate lcm ?

I found a formula googling here . But there is no proof there, and the research paper journal is unreadable for me (i.e I can't understand the hard mathy notations). So it would be helpful if someone could explain how this formula came.

(elementary-number-theory)

edited Aug 30 at 11:14
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asked Apr 20 at 12:26
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You could perhaps try $lcm(a,b)=\frac{ab}{gcd(a,b)}$. – rah4927 Apr 20 at 13:15

I know that, but for this I have to iterate through all the integers 1...n . – Tamim Ad Dari Apr 20 at 13:18

Well,it perhaps gives us a start,for there are $\phi(n)$ numbers with the denominator 1. – rah4927 Apr 20 at 13:26

ϕ(n) numbers with denominator 1, but to find the sum , I have to know all the a's , right? – Tamim Ad Dari Apr 20 at 13:33

1 Answer

Lemma 1: $\text{lcm}(a,n)+\text{lcm}(n-a,n)=\frac{an}{\gcd(a,n)}+\frac{(n-a)n}{\gcd(n-a,n)}=\frac{n\times n}{\gcd(a,n)}$.

Lemma 2: $\sum\frac{n}{\gcd(a,n)}=\sum_{f|n}\frac{n}{f}\times\phi(\frac{n}{f})=\sum_{d|n}d\phi(d)$,

Proof: consider what happens if $\gcd(a,n)=f\mid n$. It appears $\phi(\frac{n}{f})$ times on the LHS, and each time it has value of $\frac{n}{f}$. Now substitute $d=\frac{n}{f}$, which is also a divisor of n.

Now, to your problem, pull out $\text{lcm}(n,n)=n$.

We have $2\sum_{a=1}^{n-1}\text{lcm}(a,n)=\sum\left[\text{lcm}(a,n)+\text{lcm}(n-a,n)\right]=n\sum\frac{n}{\gcd(a,n)}=n\times\sum_{d|n}d\phi(d)$.

Add back $\text{lcm}(n,n)=n$, and you get the formula in OEIS.

edited Apr 20 at 16:00
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answered Apr 20 at 15:24
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