Sums of Consecutive Numbers

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Date: 06/20/2002 at 06:42:20
From: Laura Tyers
Subject: consecutive numbers

In what way can 1000 be expressed as the sum of consecutive numbers?
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Date: 06/20/2002 at 16:27:44
From: Doctor Ian
Subject: Re: consecutive numbers

Hi Laura,

Well, it can't be the sum of two consecutive numbers, since that
would have to be an odd number.  But what about three?  If the
middle number is n, then

  (n-1) + n + (n+1) = 1000

                 3n = 1000

So that won't work.  How about four?

  (n-1) + n + (n+1) + (n+2) = 1000

                     4n + 2 = 1000

That's not going to work, either.  (Do you see why?)

What didn't work for 3 will work for 5, however:

  (n-2) + (n-1) + n + (n+1) + (n+2) = 1000

                                 5n = 1000

                                  n = 200

So we've found one way.  What you need to do now is figure out
whether what worked for 5 will work for any other numbers!  The
key was to have an odd number that evenly divided 1000, so you
could get a bunch of differences from the middle number to cancel
out.  Are there odd numbers other than 5 that evenly divide 1000?
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And then you're half done, because you still have to worry about
the even numbers.

Let's say that you want to have k consecutive numbers, where k is
even.  (We've tried k=2 and k=4 so far.)  If we try to distribute
the resulting consecutive numbers around some 'center' number,
it's never quite balanced, the way it is when we use odd numbers.

Let's look at some examples:

  k = 2                    n + (n+1)

  k = 4            (n-1) + n + (n+1) + (n+2)

  k = 6    (n-2) + (n-1) + n + (n+1) + (n+2) + (n+3)

Already we can see a pattern here.  All the differences except
the last one will cancel out, leaving us with

   k*n + k/2

So we can have an even number of consecutive numbers only if we
can find some value of k such that

  k*n + k/2 = 1000

If you can find integer values of k and n that make the equation
true, you'll have found more ways to add consecutive numbers to
get 1000.  If you can show that there are _no_ integer values of
k and n that make the equation true, then you've shown that you
can ignore even values of k.

Can you take it from here?

- Doctor Ian, The Math Forum
  http://mathforum.org/dr.math/
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Date: 06/20/2002 at 17:21:08
From: Doctor Greenie
Subject: Re: consecutive numbers

Whenever you add an ODD number of consecutive integers, the total is
equal to the number of numbers times the number "in the middle".  For
example, the sum of the three numbers 73, 74, and 75, is just 3 times
74; or the sum of the seven numbers 124, 125, 126, 127, 128, 129, and
130 is just 7 times 127.

And whenever you add an EVEN number of consecutive integers, the
total is equal to the number of numbers times the number halfway
between the two numbers "closest to the middle".  For example, the
sum of the four numbers 11, 12, 13, and 14, is just 4 times 12.5; or
the sum of the eight numbers 87, 88, 89, 90, 91, 92, 93, and 94 is
just 8 times 90.5.

So the sum of ANY sequence of consecutive integers is either

(1) an ODD number times ANY INTEGER; or
(2) an EVEN number times the number halfway between two integers
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You want to write 1000 as the sum of consecutive numbers.  Let's look
at the divisors of 1000:

     1  1000
     2   500
     4   250
     5   200
     8   125
    10   100
    20    50
    25    40

Let's first look at the ways we can write 1000 as the sum of an ODD
number of consecutive integers.  For these we need a factorization of
1000 in which one of the factors is ODD.

Ignoring the divisor 1 (we aren't interested in the "sequence"
consisting of the single number "1000"), there are three divisors of
1000 which are odd.  We have

  (1)    5 x 200 = 1000
  (2)   25 x  40 = 1000
  (3)  125 x   8 = 1000

These factorizations tell us that we can write 1000 as

  (1)  5 consecutive integers with middle number 200
  (2)  25 consecutive integers with middle number 40
  (3)  125 consecutive integers with middle number 8

The first solution has middle number 200, with (5-1)/2=2 numbers each
side of 200: 198, 199, 200, 201, 202.

The second solution has middle number 40, with (25-1)/2=12 numbers
each side of 40: 28, ..., 40, ... 52

The third solution has middle number 8, with (125-1)/2=62 numbers
each side of 8:  -54, ..., 8, ..., 70

Now let's look for the ways we can write 1000 as the sum of an EVEN
number of consecutive integers.  To find these solutions, we need to
be able to write 1000 as the product of an even integer "a" and some
number of the form "b and 1/2" where b is an integer.  Any number of
the form "b and 1/2" is half of an odd integer.  Because the only odd
divisors of 1000 are 5, 25, and 125,  we have the following for the
possibilities for making 1000 as the sum of an EVEN number of
consecutive integers:

  (1) 400 x  2.5  (400 consecutive integers with middle numbers 2,3)
  (2)  80 x 12.5  (80 consecutive integers with middle numbers 12,13)
  (3)  16 x 62.5  (16 consecutive integers with middle numbers 62,63)

The first solution here has middle numbers 2 and 3 with (400-2)/2=199
numbers each side of those two:  -197, ..., 2, 3, ..., 202

The second solution here has middle numbers 12 and 13 with (80-
2)/2=39 numbers each side of those two:  -27, ..., 12, 13, ..., 52

The third solution here has middle numbers 62 and 63 with (16-2)/2=7
numbers each side of those two:  55, ..., 62, 63, ..., 70
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Notice that some of these solutions include negative integers.  You
didn't specify whether that was allowed.

For any solution you have containing only positive integers, you will
always have a corresponding solution involving negative integers,
because, for example, if you have found 3+4+5=12 as one solution for
a sum of consecutive integers equalling 12, then the sequence

  (-2)+(-1)+0+1+2+3+4+5

will also equal 12, because the (-2) and the 2 add to zero, as do the
(-1) and the 1, and so the part of the sequence

   (-2)+(-1)+0+1+2

has the value 0.

So we have three pairs of solutions to your problem (each pair
consisting of a sequence of positive integers and the corresponding
sequence which involves negative integers also), as follows:

  (1a)  198, 199, 200, 201, 202; or
  (1b)  -197, -196, ..., 196, 197, 198, 199, 200, 201, 202

  (2a)  55, 56, ..., 69, 70; or
  (2b)  -54, -53, ..., 53, 54, 55, 56, ..., 69, 70

  (3a)  28, 29, ..., 51, 52; or
  (3b)  -27, -26, ..., 26, 27, 28, 29, ..., 51, 52

I hope you were able to follow all of this.  Please write back if you
have any further questions on any of this.

- Doctor Greenie, The Math Forum
  http://mathforum.org/dr.math/
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