Calculating the residue of power towers

I want to calculate the residue of a power tower. How do I do that?

For example, I want to know the answer to this:

$$2 \uparrow\uparrow 10 \pmod{10^9}$$

(modular-arithmetic) (exponentiation) (tetration) (power-towers)

edited Jul 1 '13 at 14:13
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asked Aug 18 '12 at 7:38
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1 Answer

When dealing with power towers with bases not relatively prime to the modulus, it's useful to employ the Chinese Remainder Theorem. And then repeatedly apply the Euler's Theorem.

$2 \uparrow\uparrow 10 \pmod{2^9} = 0$, so we only need to calculate $2 \uparrow\uparrow 10 \pmod{5^9}$.

By Euler's Theorem, we need to first study $2 \uparrow\uparrow 9 \pmod{\phi(5^9)} = 2 \uparrow\uparrow 9 \pmod{4 \cdot 5^8}$. So, as $2 \uparrow\uparrow 9 \equiv 0 \pmod{4}$, so by Chinese Remainder Theorem, we only need to settle the case when $2 \uparrow\uparrow 9 \pmod{5^8}$

Similarly proceeding at every step, we go a few more levels deeper, to get that we need to settle the congruence $2 \uparrow\uparrow 4 \pmod{4 \cdot 5^3}$. As $2 \uparrow\uparrow 4 = 2^{16} = 256^2 \equiv 6^2 \equiv 36 \pmod{125}$, so we get that $2 \uparrow\uparrow 4 \equiv 36 \pmod{4 \cdot 5^3}$, as $4|36$.

Now, we unwrap the calculations. $2 \uparrow\uparrow 5 = 36^2 \equiv 1296 \pmod{5^4}$, hence $2 \uparrow\uparrow 5 \equiv 1296 \pmod{4 \cdot 5^4}$ as $4|1296$.

At this stage the calculations become too tedious to perform, but I hope you get the idea.

answered Aug 18 '12 at 8:01
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