

Recursive Sequences and Fibonacci Sequences

In this discussion we will see how matrices can be used to describe recursive sequences, in particular Fibonacci numbers. Recall that a recursively defined sequence is a sequence where the first one or more values are given along with a formula that relates the n^{th} term to the previous terms. The *Fibonacci sequence* is defined as follows:

$$u_n = u_{n-1} + u_{n-2} \quad u_0 = 1 \quad u_1 = 1$$

We can find the Fibonacci numbers by using the formula

$$u_2 = u_1 + u_0 = 1 + 1 = 2$$

$$u_3 = u_2 + u_1 = 2 + 1 = 3$$

$$u_4 = u_3 + u_2 = 3 + 2 = 5$$

$$u_5 = u_4 + u_3 = 5 + 3 = 8$$

These numbers occur in several disciplines and wherever you don't expect them in mathematics. For example suppose one mating pair of rabbits is introduced into an ecosystem. This breed of rabbits produce a pair of rabbits after their second month of life and all months after that. Assume no rabbits die. Then the zeroth and first month, there is one pair of rabbits. On the second month there will be two pairs: the original pair plus their children. The third month there will three pairs: the two pairs from second month plus the children from the original pair. If we want to find out how many rabbits on the n^{th} month, we add the number of rabbits from the prior month and the number of children. Since only two month old rabbits can have children, we only add the rabbits that were there two months ago. This is the Fibonacci sequence.

We can see how each number is produced, however if we want to find the 1000th Fibonacci number this way, it would be exhausting. We now show how matrices can be used to produce Fibonacci numbers more efficiently. We can write the following two equations:

$$u_n = u_{n-1} + u_{n-2}$$

$$u_{n-1} = u_{n-1}$$

This has matrix equation

$$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_{n-1} \\ u_{n-2} \end{pmatrix}$$

We can write this as

$$\mathbf{w}_k = A\mathbf{w}_{k-1}$$

where

$$\mathbf{w}_k = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Notice that

$$\mathbf{w}_1 = A\mathbf{w}_0$$

$$\mathbf{w}_2 = A\mathbf{w}_1 = AA\mathbf{w}_0 = A^2\mathbf{w}_0$$

In general, we have

$$\mathbf{w}_k = A^k\mathbf{w}_0$$

We now seek a fancy way to find a power of a matrix. The answer comes with diagonalization. We have that if A is diagonalizable with

$$A = PDP^{-1}$$

then

$$A^k = PD^kP^{-1}$$

We can diagonalize this matrix with

$$P = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \quad P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix}$$

The n^{th} Fibonacci number can be found by

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If you carefully multiply this out and take the first component of the result, we get

$$u_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

Hence the 1000th Fibonacci number is

$$u_{1000} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{1001} - \left(\frac{1-\sqrt{5}}{2} \right)^{1001} \right]$$

This number is extremely large.

Other Sequences

We can play this same game with modified rules. For example, if the mature rabbits had six pairs of children instead of one, then the recursion relationship would be

$$u_n = u_{n-1} + 6u_{n-2} \quad u_0 = 1 \quad u_1 = 1$$

and the matrix equation would be

$$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_{n-1} \\ u_{n-2} \end{pmatrix}$$

and we would use a similar technique to find a general formula for u_n . In this case, we have

$$P = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

So that

$$\begin{aligned}
 \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} (-2)^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2(-2)^n & 3^{n+1} \\ -(-2)^n & 3^n \end{pmatrix} \begin{pmatrix} \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2(-2)^n + 3^{n+1} & -6(-2)^n + 2(3)^{n+1} \\ -(-2)^n + 3^n & 3(-2)^n + 2(3)^n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 &= \frac{1}{5} \begin{pmatrix} -4(-2)^n + 3^{n+2} \\ 2(-2)^n + 3^{n+1} \end{pmatrix}
 \end{aligned}$$

Hence

$$u_n = \frac{2(-2)^n + 3^{n+1}}{5}$$

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