From deterministic chaos to quantum chaos part II

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Plan

1. Short summary
2. Experimental realizations
3. Quantum kinematics, quantum dynamics
4. Phase space portraits
5. Exponential sensitivity

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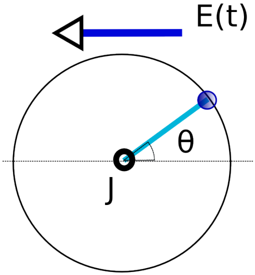
Kicked rotator model

Prototype model for studying deterministic chaos and quantum chaos.

$$H(t) = \frac{J^2}{2I_0} + K\frac{I_0}{\tau_0}\cos\theta \sum_n \delta(t - n\tau_0)$$



- leads to classical standard map
- tunable effective Plank constant $\hbar_{\text{eff}} = \hbar\tau_0/I_0$
- quantum map at $\hbar_{\text{eff}} = 2\pi/M$

Standard map

Dimensionless Hamiltonian

$$H = \frac{p^2}{2} + K\cos(x)\sum_n \delta(t-n)$$

Equations of motion

$$\begin{cases} p_{n+1} &= p_n + K\sin(x_n) \\ x_{n+1} &= x_n + p_{n+1} \end{cases}$$

for (x_n, p_n) – just before n-th kick, with the periodic $x \in [0, 2\pi)$.

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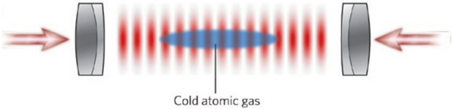
Physical systems

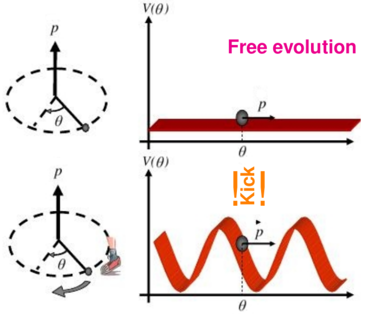
- particle dynamics in accelerators (plasma physics)
- microwave ionization of Rydberg atoms
- cold atoms in magneto-optical traps (most successful)
- superconducting nanodevices (new development)

Note: relatively easy to imagine classical system, harder to have a good controll over a quantum one (with reduced noise).

Experiments with cold atoms

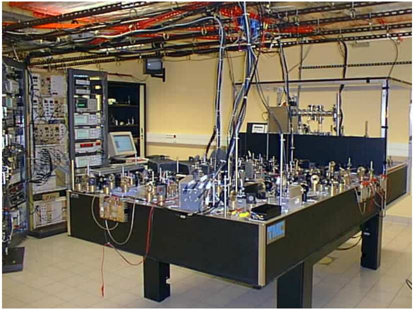


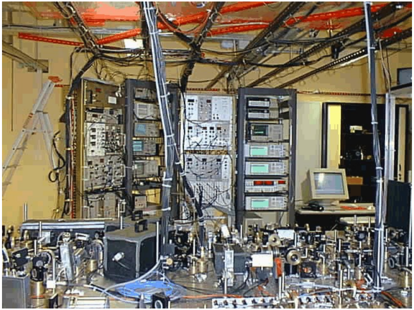
- effective optical lattice with periodically swiched potential
- atoms propagate freely between the kicks
- dilute gas, interactions can be neglected (Raizen PRL ’95)



Laboratory in Lille

Quantum chaos group



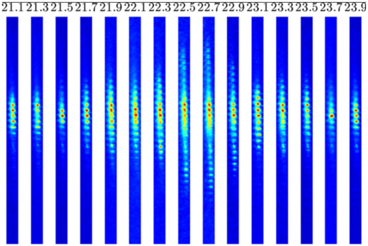


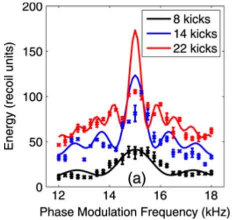
Typical measurement: time of flight after releasing from the trap:

$$|\psi(p)|^2 \text{ and } \langle p^2 \rangle.$$

Auckland experiment

color map $|\psi(p)|^2$ (exp.)





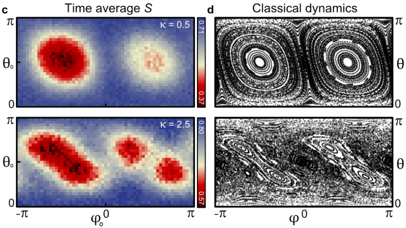
lines – QKR simulation

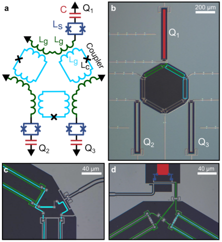
D.H. White, S.K. Ruddell, and M.D. Hoogerland, New J. Phys. (2014)

$\longrightarrow$ detecting resonances in the presence of noise

Superconducting qubits

J.M. Martinis in Google labs, Nature Physics (2016)





Evolution of quantum kicked top (QKT)

$$\mathcal{F} = \exp\left(-i\frac{\kappa}{2j\hbar}J_z^2\right)\exp\left(-i\frac{\pi}{2\hbar}J_y\right)$$

simulated with 3 qubits $J_z = \sum_{j=1,2,3} s_z^j$, each initialized in a state

$$\cos\theta_0|+z\rangle + e^{i\phi_0}\sin\theta_0|-z\rangle.$$

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Quantum evolution

Unitary evolution in the Schrödinger picture – formal solution:

$$\psi(t) = \mathcal{F}(t)\psi(0),$$

where $\mathcal{F}(t) = \mathrm{T}\left\{\exp[-i\int^t H(t')\,dt'/\hbar_{\mathrm{eff}}]\right\}$.

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For our stroboscopic model

$$\mathcal{F}(t) = \mathcal{F}^t = \left[\exp\left(-\frac{i\hat{p}^2}{2\hbar_{\mathrm{eff}}}\right)\exp\left(-\frac{iK\cos(x)}{\hbar_{\mathrm{eff}}}\right)\right]^t.$$

Quantum map

Momentum dependent evolution in (angular) momentum basis (with $p = \hbar_{\text{eff}} m$):

$$\exp\left(-\frac{i(p+2\pi)^2}{2\hbar_{\text{eff}}}\right) = \exp\left(-i\frac{\hbar_{\text{eff}}}{2}m^2 - i2\pi m - i\frac{(2\pi)^2}{2\hbar_{\text{eff}}}\right).$$

... is periodic for $\hbar_{\text{eff}} = \frac{2\pi}{M}$ with M-even.

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... is periodic for $\hbar_{\rm eff} = \frac{2\pi}{M}$ with M-even.

One obtains a consistent description with finite momenta

$$p = \tfrac{2\pi}{M} m, \quad m = 0, \ldots, M-1,$$

and with discrete positions

$$x = \frac{2\pi}{M} n, \quad n = 0, \ldots, M-1.$$

formal quantization of area-preserving invertable maps: Balazs, Voros ’87

Discrete Fourier transform

Changing representation from position to momenta:

$$\tilde{\psi}(p) = \int dx\, e^{-ipx/\hbar} \psi(x)$$

and substituting $x = \frac{2\pi n}{M}$, $p = \frac{2\pi m}{M}$, we get

$$\tilde{\psi}(p_m) = \frac{1}{\sqrt{M}} \sum_n e^{-2\pi i m n/M} \psi(x_n).$$

Now $\psi_n = \psi(x_n)$ is a discrete vector in position basis,
$\tilde{\psi}_m = \tilde{\psi}(p_m)$ is a discrete vector in momentum basis.

Quantum map calculation

One step of quantum evolution is encoded in:

$$\bar{\psi} = \exp\left(-\frac{i}{2\hbar}p^2\right)\exp\left(-\frac{i}{\hbar}K\cos x\right)\psi,$$

where $\bar{\psi}/\psi$ is defined just after/before the kick.

The evolution explicitely written for discrete variables

$$\bar{\psi}_{n'} = \frac{1}{M}\sum_{m,n} e^{i\frac{2\pi}{M}n'm} e^{-\frac{i}{2\hbar}p_m^2} e^{-i\frac{2\pi}{M}mn} e^{-\frac{i}{\hbar}K\cos x_n}\psi_n$$

encodes one step of our quantum map (with $\hbar = \frac{2\pi}{M}$, $p_m = \frac{2\pi m}{M}$, $x_n = \frac{2\pi n}{M}$).

Quantum map – implementation

Let's look at some technicalities:

$$V_n = \exp\left(-\tfrac{i}{\hbar} K \cos x_n\right), \; P_m = \exp\left(-\tfrac{i}{2\hbar} p_m^2\right)$$

can be easily implemented with NumPy arrays.

This simplifies the notation:

$$\bar{\psi}_{n'} = \frac{1}{M} \sum_{m,n} e^{i\frac{2\pi}{M} n' m} P_m e^{-i\frac{2\pi}{M} mn} V_n \psi_n .$$

Furthermore, the command `u=fft(w)` just calulates:

$$u_{n'} = \sum_n e^{-i\frac{2\pi}{M} n' n} w_n .$$

To implement one step we need: two element-wise vector multiplications and two Fast Fourier Transforms.

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Gaussian wave packet

- Packet centered at (x_0, p_0) with a minimal uncertainty $\Delta x \Delta p = \hbar/2$,
- take the continuum expression

$$\psi_0(x) = N \exp\left(i\frac{p_0 x}{\hbar}\right) \exp\left(-\frac{(x - x_0)^2}{2\hbar}\right)$$

and substitute our discrete $x_n = \frac{2\pi n}{M}$.

Problem: not periodic in $x \in [0, 2\pi)$. We use "method of images"

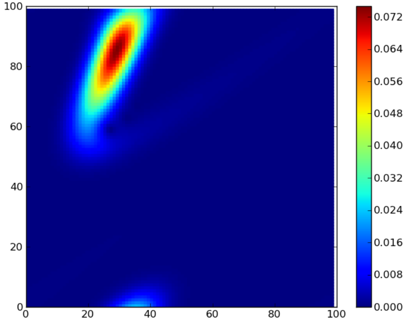
$$\psi_G(x) = \tilde{N} \exp\left(i\frac{p_0 x}{\hbar}\right) \sum_d \exp\left(-\frac{(x - x_0 + 2\pi d)^2}{2\hbar}\right),$$

where in practice $|d| \le 4$ is enough.

We can represent the wave function ψ in the phase space using

$$Q(x_0, p_0) = \sum_n \psi_G^*(x_n)\psi(x_n),$$

where ψ_G is the Gaussian packet centered at (x_0, p_0).



$t = 4$ kicks, $K = 2.1$, $M = 100$, color map of $|Q|$

Calculating Q-distribution

Husimi distribution

$$Q \equiv \sum_n \psi_0^*(n)\, \psi(n)$$

Gaussian our wave func.

denote $g_{n_0}(n) \equiv \psi_0(n)$

└ centered at $(n_0, m_0 = 0)$

in general $e^{i\frac{2\pi}{M} m_0 n} g_{n_0}(n)$ is at (n_0, m_0)

Husimi $Q(n_0, m_0)$ is then

$$Q(n_0, m_0) = \sum_n e^{-i\frac{2\pi}{M} m_0 n} g_{n_0}(n)\, \psi(n),$$

which can be calculated for all $m_0 = 0, \ldots, M-1$ with a single use of FFT (n_0 – fixed).

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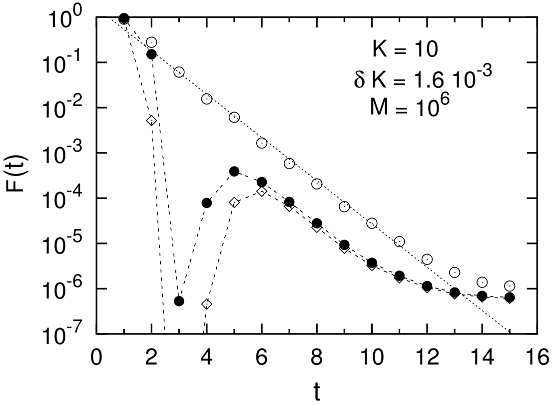
Proper quantity

- Fidelity F may be used as a measure of system sensitivity to a small perturbation
- Decay $F(t) = |\langle\psi_{\text{pert.}}(t)|\psi_0(t)\rangle|^2 \propto e^{-\lambda t}$
  independent of perturbation in a large parameter range for $\hbar \to 0$
  (Jalabert, Pastawski Phys. Rev. Lett. '01)

How to calculate

- Initial Gaussian wave packet at (x_0, p_0) is evolved for t steps
- Overlap integral
  $F(t) = | < \mathcal{F}^t(K + \delta K)\psi_0 | \mathcal{F}^t(K)\psi_0 > |^2$,
  δK – parametric perturbation
- Average F over different initial (x_0, p_0) (let $M = 10^4, 10^5, 10^6$)

Simulation results



○ — $\overline{F}$

◇ — $\exp(\overline{\ln F})$

Straight line: decay with the classical Lyapunov exponent $\lambda = 1.1$

P.G. Silvestrov, J. Tworzydło, and C.W.J. Beenakker Phys. Rev. E (2003)