From deterministic chaos to quantum chaos

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Plan

1. Modeling and simulating quantum mechanical systems
2. Chaotic classical dynamics
3. Standard map
4. Quantum dynamics

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Computer modeling ”cornerstones”

- Deterministic chaos
- Inhomogeneous systems
  (e.g. fractals, oil in rocks, pattern formation)
- Molecular Dynamics simulations and (equilibrium) Monte Carlo

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take a simple model $\rightarrow$ use QM formulation $\hbar \neq 0$

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- Random wave function interference, random matrix theory
- Path integrals and quantum MC simulation

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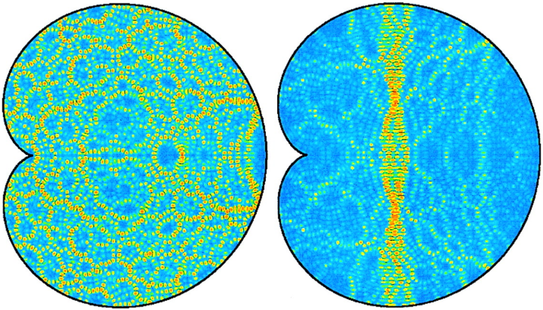
Michael Berry: from a student response to my quantum mechanics lectures
”... didn’t explain what use quantum mechanics is ... alright, so it explains things, but they would happen anyway”

Quantum system with a chaotic dynamics in the classical limit.



Quantum chaos

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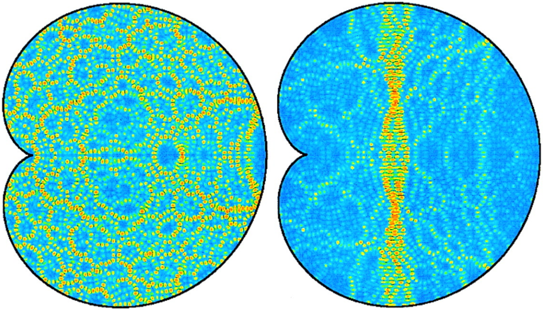

Is there a simple model? Quantum billards, quantum map (our lab).

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Classical mechanics

Analytical solutions

- special, integrable case (e.g. 2-body problem)
- essentially periodic motion

Computer modeling

- easy to illustrate chaotic trajectories
- ... just for a few bodies

Chaos theory inspirations

Jules Henri Poincare 1854-1912
"The Last Universalist"

3-body problem

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Jules Henri Poincare 1854-1912
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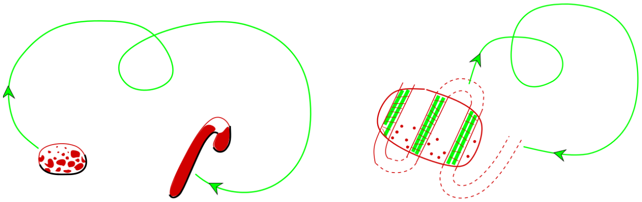
3-body problem

- mathematical fundations of chaos theory
- prize from King of Sweden Oscar II

Chaos essentials

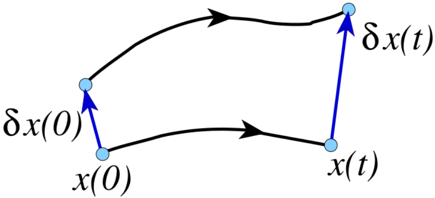
Necessary requirements for CHAOS emergence:

- exponential sensitivity to a change of initial conditions
- limited available space (or ”mixing” property)



Predrag Cvetanovic “Chaos Book”

Laypunov exponent and Lyapunov time



Quantitative description of sensitivity to the initial condition

$$\delta \boldsymbol{x}(t) \approx e^{\lambda t} |\delta \boldsymbol{x}(0)|$$

λ – *Lyapunov exponent*

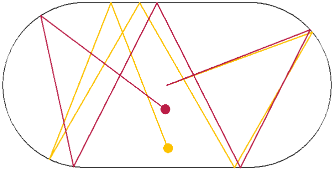
Dynamics is predictable until *Lyapunov time*:

$$T \approx -\frac{1}{\lambda} \ln |\delta x / L|$$

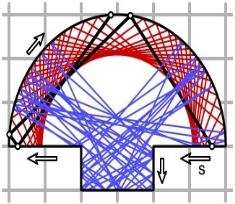
estimates for the Solar System $T \approx 50\mathrm{My}$ (2009)

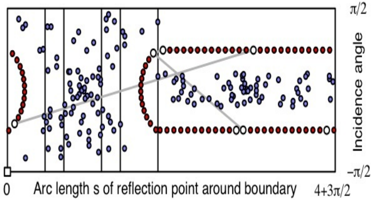
Chaotic billards

Bunimovich stadium (with almost everywere positive Lyapunov exp.)



mixed case – chaotic and regular trajectories





$\Longrightarrow$ phase space portrait: **angle-perimeter map**

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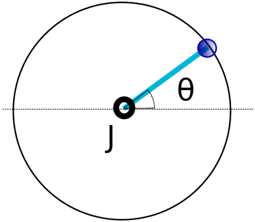
Kicked rotator model

Prototype model for studying chaos and quantum chaos.



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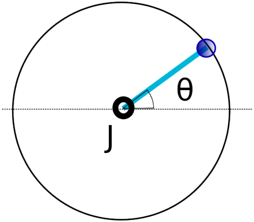
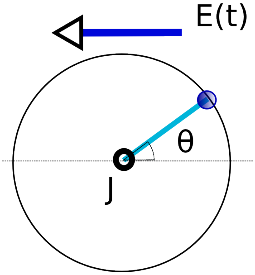


Kicked rotator model

Prototype model for studying chaos and quantum chaos.

Dipole fixed on one end point in a uniform electric field;
field is applied periodically in short pulses.



$\boldsymbol{E}(t)$ – pulsed electric field
J – angular momentum
θ – angle position

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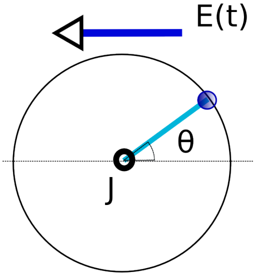


$\boldsymbol{E}(t)$ – pulsed electric field
J – angular momentum
θ – angle position
I_0 – moment of inertia
τ_0 – kicking period
K – dimensionless coupling const.

$$H(t) = \frac{J^2}{2I_0} + K\frac{I_0}{\tau_0}\cos\theta\sum_n \delta(t - n\tau_0)$$

Physical systems

- particle dynamics in accelerators (plasma physics)
- microwave ionization of Rydberg atoms
- electron magneto-transport in a resonant tunneling diode
- cold atoms in magneto-optical traps (most successful)

Equation of motion

Evolution of the generalized momentum and position:

$$\begin{cases} \dot{J} &= K\frac{I_0}{\tau_0}\sin(\theta)\sum_n \delta_n(t) \\ \dot{\theta} &= \frac{1}{I_0}J \end{cases}$$

We denote
(θ, J) – values just before a kick, (θ', J') – just before the next kick

$$\begin{cases} J' &= J + K\frac{I_0}{\tau_0}\sin(\theta) \\ \theta' &= \theta + \frac{\tau_0}{I_0}J' \end{cases}$$

It is easier to use dimensionless variables:

$$x \equiv \theta,\ p = \frac{\tau_0}{I_0}J,\ \tilde{t} = t/\tau_0,\ \tilde{H} = H\frac{\tau_0^2}{I_0}$$

Standard map (named after Boris Chirikov)

Dimensionless equations of motion

$$\begin{cases} p_{n+1} & = & p_n + K \sin(x_n) \\ x_{n+1} & = & (x_n + p_{n+1}) \bmod 2\pi \end{cases}$$

for (x_n, p_n) – just before n-th kick

Dimensionless Hamiltonian

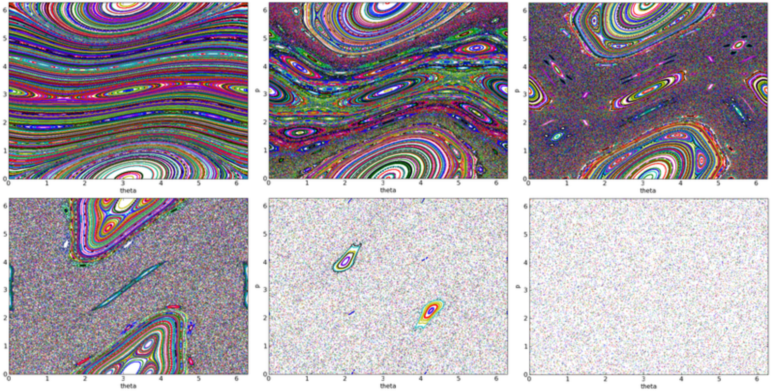
$$\tilde{H} = \frac{p^2}{2} + K \cos(x) \sum_n \delta(\tilde{t} - n)$$

With the (angular) position: $x \in [0, 2\pi)$

Hamiltonian formulation ensures that the resulting map preserves the phase-space volume.

Road to chaos in the standard map

Colors mark evolution from different initial conditions. Subsequent phase space pictures for $K = 0.6, 0.97, 1.2, 2, 5, 7$. System changes from integrable, through mixed phase space to fully chaotic.



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Towards quantum description

Canonical quantization:

$$[\theta, J] = i\hbar,$$

after the rescaling $x = \theta$, $p = J\tau_0/I_0$ we get

$$[x, p] = i\hbar_{\text{eff}}.$$

- We can tune $\hbar_{\text{eff}} = \hbar\tau_0/I_0$ in the quantum kicked rotator!
- The limit $\hbar_{\text{eff}} \to 0$ becomes physically accessible.

Quantum evolution

Unitary evolution in the Schrödinger picture – formal solution:

$$\psi(t) = \mathcal{F}(t)\psi(0),$$

where $\mathcal{F}(t) = \mathrm{T}\left\{\exp[-i\int H(t)\,dt/\hbar_{\mathrm{eff}}]\right\}$.

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For our stroboscopic model (with $\hat{p}$ – operator):

$$H(t) = \frac{\hat{p}^2}{2} + K\cos(x)\sum_n \delta(t-n)$$

we can take adventage of instantaneous kick and compute

$$\mathcal{F}(t) = \mathcal{F}^t = \left[\exp\left(-\frac{i\hat{p}^2}{2\hbar_{\mathrm{eff}}}\right)\exp\left(-\frac{iK\cos(x)}{\hbar_{\mathrm{eff}}}\right)\right]^t.$$