LAB III
Practical RMT: level repulsion

Jakub Tworzydło

Institute of Theoretical Physics
Jakub.Tworzydlo@fuw.edu.pl

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Density of states

We first need to check the level density for some test models. We use only Gaussian ensambles for this lab. Matrix entries of a test matrix are random, uncorrelated. We want to discover a semi-cirlce law, also attributed to Wigner, which describes the density of states of a large, random matrix. This semicirlce law acts as a central limit theorem for large (symmetric) random matrices.

With Gaussian ensemble one can derive many results analytically, the semicirle law is a limit for $N \to \infty$. It is known, that there deviations for smaller values of N, which we should also see in our lab.

Task 1

Use subplots, colors and labels to illustrate the results.

Calculate eigenvalues of n_{sample} random matrices drawn from **GOE ensamble** ($N \times N$ matrix size). Make a histogram (normalized) of eigenvalues and compare with the analytical Wigner semicircle law. For this task we accumulate all eigenvalues from all generated matrices.

Wigner semicircle law:

$$\rho(E) = \frac{2}{\pi R^2}\sqrt{R^2 - E^2},$$

where in **GOE**: $R = \sqrt{2N}$.

Example test values:
(A) $N = 6$, $n_{\text{sample}} = 20000$;
(B) $N = 20$, $n_{\text{sample}} = 10000$;
(C) $N = 200$, $n_{\text{sample}} = 500$

Hints

We can plot the histogram by using `hist` function, e.g. with the following parameters

```
n, bins, _ = plt.hist(eigen, 50, density=True,\
                          facecolor='cyan', alpha=0.75)
```

We can also plot an exact function `wigner` on top of that

```
plt.plot(bins, wigner(bins), 'r-', linewidth=2)
```

Simplest way to get a random Hamiltonian from GOE:

```
h = np.random.randn(N,N)
return (h+h.T)/2
```

Task 2

Calculate the histogram of **energy spacings** for **GOE** and **GUE** ensembles. Normalize the accumulated energy spacings to unit mean level spacing (divide by the average).

Compare with the Wigner surmise prediction:

$$P(s) = \begin{cases} \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2} & \text{for } GOE \\ \frac{32}{\pi^2} s^2 e^{-\frac{4}{\pi}s^2} & \text{for } GUE \end{cases}$$

Spectrum unfolding:
We have to get levels with approximately uniform density, so we will only probe the middle part of the whole spectrum. For matrix sizes $N \geq 10$ use $1/4$ of the spectrum around the eigenvalue $N/2$, for smaller matrices use only the two middle eigen-energies.

Present the results:
Illustrate the results for the marix sizes $N = 8$ and $N = 200$. *Try also to use bigger matrix sizes: how far can we reasonably increse* N*?*

Hints

Sorting the eigenvalues

```
eigen = np.sort(eigen)
```

and calculating the differences (spacings):

```
diff  = np.diff(eigen)
```

Extra task

Calculate eigen-phases of the one-step kicked rotator unitary matrix $\mathcal{F}$. Here we have to explicitly store the whole $M \times M$ matrix $\mathcal{F}$.

We can prepare an ensemble by substituting $K\cos(x_n) \to K\cos(x_n+\theta)$ and using some 20 random values for θ. Plot a histogram of the eigen-phases to ensure it is flat and then prepare the histogram of energy spacings. Show the results e.g. for $K = 10$ and $K = 2.1$ (you can also experiment with other values).

Try to compare with the Wigner and Poisson distributions.

Hint: eigenphases of a unitary matrix F:

$$\mathcal{F}u = e^{i\phi}u$$

calculate (all) eigenphases $\phi \in [0, 2\pi)$ of $F(\theta)$, collect the ensamble by taking different values of θ