Spectrum of a complex quantum system: random matrix theory

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Plan

1. Complex quantum systems
2. Universal distributions
3. Wigner derivation
4. Quantum chaotic systems

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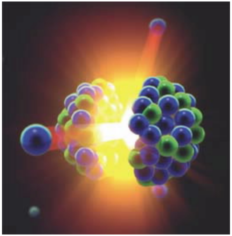
What do these systems have in common?

- $n +^{166} Er$



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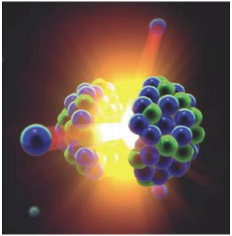
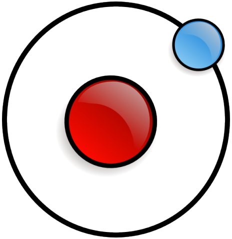


- H in B-field



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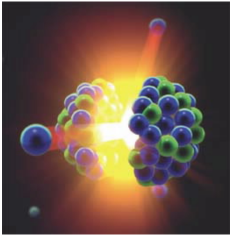

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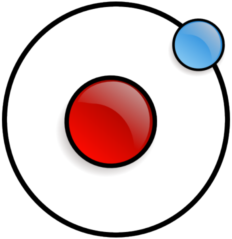
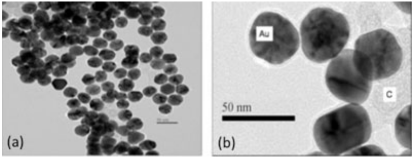
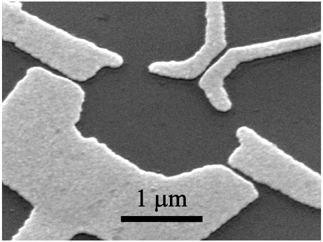


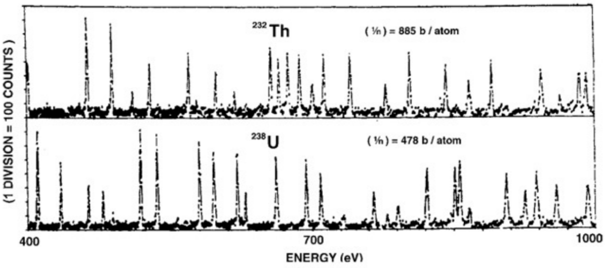
- metal grain

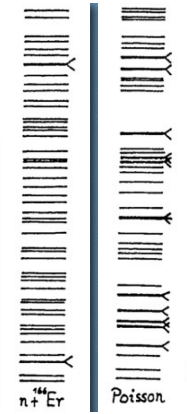


- quantum dot



RMT in nuclear physics





- notion of "level repulsion"
- statistical theory of energy levels
  Wigner, Dyson, Mehta ('60–'70)
- ensamble of Hamiltonians: random $H \longrightarrow$ correleted E_i

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Wigner symmetry classes

Level spacing distribution (Wigner 1957)

$$\mathcal{P}(s) \propto s^{\beta} e^{-s^2}$$

s – spacing: $(E_i - E_{i+1})/\Delta$ normalized with average spacing Δ

- $B = 0$ time reversal symmetry $\longrightarrow \beta = 1$
- $B \neq 0$ no time reversal symmetry $\longrightarrow \beta = 2$
- $B = 0$ time reversal symmetry without spin-rotation symmetry $\longrightarrow \beta = 4$

Geometric origin of level repulsion

we know that $P(\mathbf{r}) = f(r) \longrightarrow P(r) = f(r) r^2$

Geometric origin of level repulsion

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compare with

$$P(H) = f(\{E\}) \longrightarrow P(\{E\}) = f(\{E\}) \prod_{i<j} |E_i - E_j|^\beta$$

- matrix H has got eigenvalues $\{E\} = \{E_1, E_2, \ldots\}$
- all correlations come from the volume element
- volume element is a Jacobian (math)

Gaussian ensamble

useful minimal model $f(\{E\}) = \prod_i f(E_i)$ and let it be $f(E_i) = e^{-cE_i^2}$
we can simplify $f(\{E\}) = \prod_i e^{-cE_i^2} = e^{-c\sum_i E_i^2} = e^{-c\mathrm{Tr}H^2}$ so we get

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$$f(E) = e^{-cE^2} \longrightarrow P(H) = \exp(-c\,\mathrm{Tr}H^2)$$

Gaussian ensamble is a probability distribution over Hamiltonian H matrices, it is choosen for math convenience (not fundamental).

Gaussian ensamble separetes

Written in terms of matrix elements H_{ij}:

$$P(H) = \exp(-c \, \mathrm{Tr} H^2) = \exp(-c \sum_{ij} |H_{ij}|^2)$$

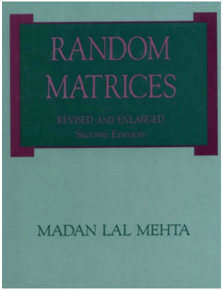
$\longrightarrow$ independend Gaussian distributions for all H_{ij}

$$P(H) = \prod_{i<j} \exp(-2c|H_{ij}|^2) \prod_{i} \exp(-c|H_{ii}|^2)$$

Gaussian ensamble symmetries

Naming convention follows from the property of unitary matrix, which diagonalizes H:

- G. Orhogonal Ensamble $\beta = 1$ ($B = 0$)
- G. Unitary Ensamble $\beta = 2$ ($B \neq 0$)
- G. Symplectic Ensamble $\beta = 4$ ($B = 0$, but with spin-orbit mixing)



Generating Gaussian ensambles

Gaussian is also numerically convenient!

Constrains for a hermitian $H_{N\times N}$

- GOE $\beta = 1$; H is real; $H = H^*$
- GUE $\beta = 2$; H is complex
- GSE $\beta = 4$; H of only real quaternions; $H = \sigma_y H^* \sigma_y$

(real quaternion $a + ib\sigma_x + ic\sigma_y + id\sigma_Z$)

Numerical recipe

Generating Gaussian ensambles

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Numerical recipe

- GOE: draw H_{ij} – independent Gaussian $(0, 1)$
- GUE: draw $H_{ij} = (x + iy)/\sqrt{2}$; x, y Gaussian $(0, 1)$
- GSE; draw $H_{ij} = (x + iy)/\sqrt{2}$; and symmetrize
  $H := (H + \sigma_y H^* \sigma_y)/2$

$\longrightarrow$ in all cases return hermitian $H := (H + H^\dagger)/2$

Plan

Wigner surmise

Wigner got his insight by studing just a 2×2 case

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}$$

real, symmetric, with matrix elements

$$p(H_{11}, H_{12}, H_{22}) \propto e^{-c(H_{11}^2 + H_{22}^2 + 2H_{12}^2)}$$

Eigenvalue decomposition

$$H = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

allow us to change variables $H_{11}, H_{12}, H_{22} \ \rightarrow \ E_1, E_2, \theta$
and calculate $P(|E_1 - E_2|)$ ...

Notes on derivation*

Wigner surmise – derivation

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}$$ – general, real 2×2 matrix

probability distribution

$$p(H_{11}, H_{12}, H_{22}) \propto e^{-c\left(H_{11}^2 + H_{22}^2 + 2H_{12}^2\right)}.$$

with c – an arbitrary constant.

Eigenvalue decomp.

$$H = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Notes on derivation*

with explicit relations

$$H_{11} = E_1 \cos^2\theta + E_2 \sin^2\theta$$
$$H_{12} = (E_1 - E_2)\sin\theta\cos\theta$$
$$H_{22} = E_1 \sin^2\theta + E_2 \cos^2\theta$$

we can calculate the Jacobian:

$$\frac{\partial(H_{11}, H_{12}, H_{22})}{\partial(E_1, E_2, \theta)} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2E_1\cos\theta\sin\theta + 2E_2\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & (E_1-E_2)(\cos^2\theta - \sin^2\theta) \\ \sin^2\theta & \cos^2\theta & 2E_1\sin\theta\cos\theta - 2E_2\cos\theta\sin\theta \end{bmatrix}$$

$$|J| = |\det[\ldots]| = |E_1 - E_2| \cdot \left|\det \begin{bmatrix} \cos^2\theta & \sin^2\theta & -\sin 2\theta \\ \frac{1}{2}\sin 2\theta & -\frac{1}{2}\sin 2\theta & \cos 2\theta \\ \sin^2\theta & \cos^2\theta & \sin 2\theta \end{bmatrix}\right|$$

Notes on derivation*

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$$= |E_1 - E_2| \cdot \Big[-\tfrac{1}{2}\cos^2\theta \sin^2 2\theta + \cos 2\theta \sin^4\theta - \tfrac{1}{2}\sin^2 2\theta \cos^2\theta$$
$$- \tfrac{1}{2}\sin^2 2\theta \sin^2\theta - \cos^4\theta \cos 2\theta - \tfrac{1}{2}\sin^2 2\theta \sin^2\theta \Big]$$

$$= |E_1 - E_2| \left| \left[-\sin^2 2\theta + \cos 2\theta \underbrace{(\sin^4\theta - \cos^4\theta)}_{-\cos 2\theta} \right] \right| = |E_1 - E_2| \cdot 1$$

Notes on derivation*

we need the distribution, from Gaussian

$$p(H_{11}, H_{12}, H_{22}) \cdot |J(E_1, E_2, \theta)| = A \exp\Big[-c\Big((E_1 \cos^2\theta + E_2 \sin^2\theta)^2$$

$$+ (E_1 \sin^2\theta + E_2 \cos^2\theta)^2 + 2(E_1 - E_2)^2 \sin^2\theta \cos^2\theta\Big)\Big] \, |E_1 - E_2|$$

$$= A\,|E_1 - E_2| \exp\left[-c(E_1^2 + E_2^2)\right]$$

level spacing distribution

$$P(s) = A \int dE_1\, dE_2\, d\theta\, |E_1 - E_2|\, e^{-c(E_1^2+E_2^2)}\, \delta(s - |E_1 - E_2|)$$

$$= \left\{ \text{changing to } \delta = E_1 - E_2;\ \bar{E} = \frac{E_1+E_2}{2} \right\}$$

$$= \tilde{A} \int d\delta\, |\delta|\, e^{-c\frac{1}{2}\delta^2}\, \delta(s-\delta) = \tilde{A}\, s\, e^{-\frac{c}{2}s^2}$$

Notes on derivation*

fix $\tilde{A}$ by normalization $\int ds\, P(s) = 1$

and c fixed unit mean level spacing $\int ds\, s\, P(s) = 1$

$$\int_0^\infty ds\, s\, e^{-\frac{c}{2}s^2} = \frac{1}{c} \quad ; \text{ so } \tilde{A} = c$$

$$c \cdot \int_0^\infty ds\, s^2 e^{-\frac{c}{2}s^2} = \{\text{by parts}\} = \int_0^\infty e^{-\frac{c}{2}s^2} ds = \sqrt{\frac{\pi}{2c}} = 1 \Rightarrow \boxed{c = \frac{\pi}{2}}$$

Finally $P(s) = \frac{\pi s}{2} e^{-\frac{\pi}{4}s^2}$

similar for two-level GUE : $P(s) = \frac{32}{\pi^2} s^2 e^{-\frac{4}{\pi}s^2}$

for four-level GSE : $P(s) = \frac{2^{18}}{3^6 \pi^3} s^4 e^{-\frac{64}{9\pi}s^2}$

Wigner surmise – summary

Wigner's prediction is very accurate also for large N (roughly 2% agreement with the exact RMT formulas)

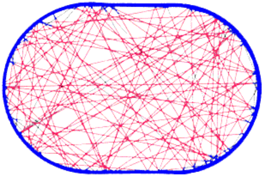
$$P(s) = \begin{cases} \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2} & \text{for GOE} \\ \frac{32}{\pi^2} s^2 e^{-\frac{4}{\pi}s^2} & \text{for GUE} \\ \frac{2^{18}}{3^6\pi^3} s^4 e^{-\frac{64}{9\pi}s^2} & \text{for GSE} \end{cases}$$

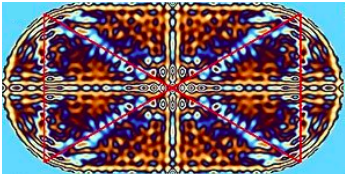
where s is normalized such that $\int sP(s)\,ds = 1$ (in units of average spacing).

Plan

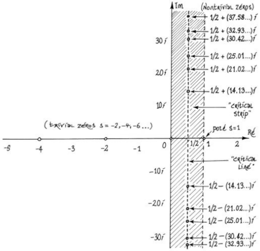
Other chaotic systems

- chaotic billiards



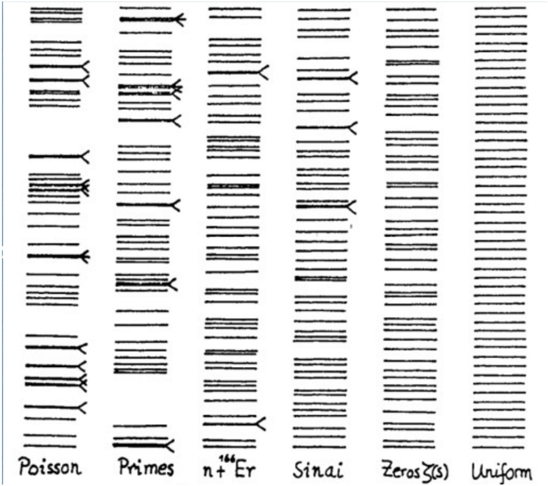


- Riemann's hypothesis $\zeta(\frac{1}{2} + i\epsilon)$



Universal level spacing

Bohigas & Giannoni (1984): chaotic dynamical system (conjecture) also described by statistical theory of spectra (RMT)



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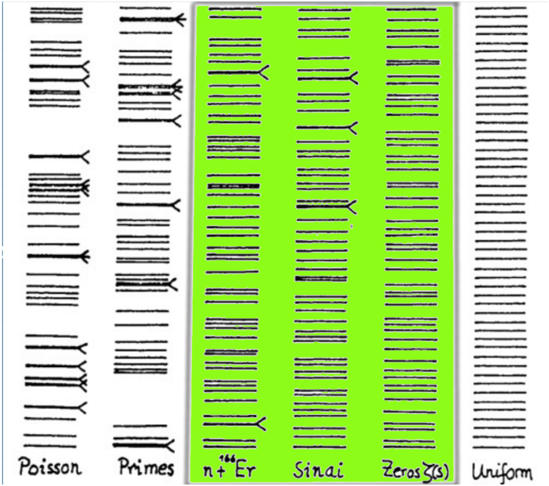
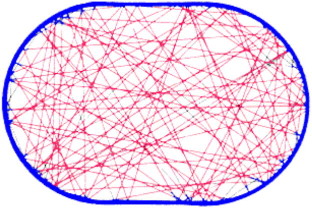


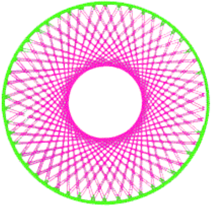
What do we know about zeros of Riemman's zeta function?

- accurate statistics, tested beyond Wigner surmise
- semiclassical expansions share formal resemblence to zeta function
- there might be a dynamical system, with eigenvalues corresponding to Riemman's zeros

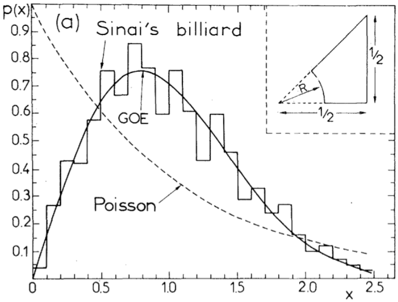
Quantum chaos

quantization of chaotic motion only leaves
geometric level correlations





stadium (chaotic)	circle (regular)
Wigner ($\beta = 1$):	Poisson:
level repulsion	no repulsion



level repulsion is a quantum signature of chaotic motion

from Bohigas, Giannoni (1984); also Berry & Tabor (1977)

Quantum chaos – to – regular crossover

- ergodic time: $\tau = L/v$
- correlation energy $E_c = \hbar/\tau$
- levels further apart than E_c are uncorrelated
- $\delta \sim E_c$ Wigner-to-Poisson crossover with level spacing δ