Ans - 2 - $A \in \mathbb{R}^{n \times n}$ is inventible

Definition of $\text{maxmag}(A) = \max\limits_{\substack{x \neq 0 \\ x \in \mathbb{R}^n}} \frac{\|Ax\|_2}{\|x\|_2} = \max\limits_{\|x\|_2 = 1} \|Ax\|_2$

$\text{minmag}(A) = \min\limits_{\substack{x \neq 0 \\ x \in \mathbb{R}^n}} \frac{\|Ax\|_2}{\|x\|_2} = \min\limits_{\|x\|_2=1} \|Ax\|_2$

(a) to prove $\text{maxmag}(A) = \frac{1}{\text{minmag}(A^{-1})}$

$$Ax = y \Rightarrow x = A^{-1}y$$

$$\text{maxmag}(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{y \neq 0} \frac{\|y\|_2}{\|A^{-1}y\|_2}$$

$$= \max_{y \neq 0} \frac{1}{\left(\frac{\|A^{-1}y\|_2}{\|y\|_2}\right)} = \frac{1}{\min\limits_{y \neq 0} \frac{\|A^{-1}y\|_2}{\|y\|_2}} = \frac{1}{\text{minmag}(A^{-1})}$$

(b) to prove condition number

$$\text{cond}(A) = \frac{\text{maxmag}(A)}{\text{minmag}(A)}$$

$\text{cond}(A) = \|A\|_2 \|A^{-1}\|_2$ (definition)

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \text{maxmag}(A)$$

$$\text{cond}(A) = \text{maxmag}(A) \cdot \text{maxmag}(A^{-1})$$

$$= \text{maxmag}(A) \cdot \frac{1}{\text{minmag}(A)} \quad \text{[from part (a)]}$$

$$= \frac{\text{maxmag}(A)}{\text{minmag}(A)}$$