Ans-6 – Definition of bilinear interpolation function

$$f(u,v) = \theta_1 + \theta_2 u + \theta_3 v + \theta_4 uv$$

$f(P_{ij}) = F_{ij}$ where

$P_{ij} = (x_i, y_j)$ of a given $M \cdot N$ grid points of a grid in $\mathbb{R}^2$

$i = 1, 2, 3 \ldots M$ $\quad x_1 < x_2 < \ldots x_M$

$j = 1, 2, 3 \ldots N$ $\quad y_1 < y_2 < \ldots y_N$

(a) consider

$$A = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ \vdots \\ 1 & x_1 & y_N & x_1 y_N \\ 1 & x_2 & y_1 & x_2 y_1 \\ \vdots \\ 1 & x_M & y_N & x_M y_N \end{bmatrix} \in \mathbb{R}^{MN \times 4}$$

$\therefore$ interpolation conditions can be represented as $\boxed{A\theta = b}$.

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \in \mathbb{R}^{4 \times 1} \qquad b = \begin{bmatrix} F_{11} \\ F_{12} \\ \vdots \\ F_{1N} \\ \vdots \\ F_{MN} \end{bmatrix} \in \mathbb{R}^{MN \times 1}$$

(b) For unique solution to $A\theta = b$ columns of A must be linearly independent which means $\boxed{MN \geq 4}$

(otherwise $\operatorname{rank}(A) < 4$, making columns of A not linearly independent)

case 1: $M = 1,\ N = 4$

then
$$A = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ 1 & x_1 & y_3 & x_1 y_3 \\ 1 & x_1 & y_4 & x_1 y_4 \end{bmatrix}$$

1st and 2nd columns are linearly dependent

so unique solution wouldn't exist

case 2): $M = 4 \quad N = 1$

similar to case 1, columns 1 and columns 3 are linearly dependent.

case 3): $M = 2 \quad N = 2$

this case might lead to have independent columns

$$A = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ 1 & x_2 & y_1 & x_2 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \end{bmatrix}$$

consider $x_1 = 0$, $x_2 = 2$

$y_1 = 0$ $\quad y_2 = 2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 2 & 2 \end{bmatrix}$$ has all columns linearly independent

$\therefore$ $M \geq 2$ and $N \geq 2$ is the condition that enables unique solution to $A\theta = b$.

thus minimum values of M and N is both 2.