Ans-1 - (a) $P_n(\mathbb{R})$: set of all polynomials in indeterminate x with real coefficients.

t.p.
(a) $P_n(\mathbb{R})$ is a vector space

Let $f(x)$ and $g(x) \in P_n(\mathbb{R})$, $h(x) \in P_n(\mathbb{R})$
$\alpha, \beta \in \mathbb{R}$

i) $f(x) + g(x) = g(x) + f(x)$ [polynomial addition] coeff. on both LHS & RHS are same

ii) $f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x)$

[polynomial addition] coeff. on both LHS & RHS are same

iii) $0 \in P_n(\mathbb{R})$ (when all real coeff. are 0)

$f(x) + 0 = 0 + f(x) = f(x)$ ← existence of additive identity

iv) $f(x) + p(x) = 0$

$\boxed{p(x) = -f(x)}$ is additive inverse of $f(x)$.

v) $\alpha(\beta f(x)) = (\alpha\beta) f(x)$ [scalar multiplication in polynomials]

vi) $1 \in P_n(\mathbb{R})$ $(1 \cdot x^0 + 0 \cdot x^1 + \dots + 0 \cdot x^n = 1)$
thus $1 \in P_n(\mathbb{R})$

$1 \cdot f(x) = f(x)$ ← existence of multiplicative identity

vii) $(\alpha + \beta) f(x) = \alpha f(x) + \beta f(x)$

viii) $\alpha(f(x) + g(x)) = \alpha f(x) + \alpha g(x)$

Since all the properties of a vector space are satisfied, thus $P_n(\mathbb{R})$ is a vector space.

(b) $F(p(x)) = \left.\frac{d}{dx} p(x)\right|_{x=0}$ $\quad F : P_n(\mathbb{R}) \longrightarrow \mathbb{R}$

consider a basis for $p(x)$: $\{1, x, x^2, \dots\}$

(b) $F : P_n(\mathbb{R}) \longrightarrow \mathbb{R}$

$$F(p(x)) = \frac{d}{dx} p(x) \Big|_{x=0}$$

$$F(p(x).\alpha + q(x).\beta)$$

$$= \frac{d}{dx} (\alpha \cdot p(x) + \beta \cdot q(x)) \Big|_{x=0}$$

$$= \frac{d}{dx} (\alpha \cdot p(x)) \Big|_{x=0} + \frac{d}{dx} (\beta \cdot q(x)) \Big|_{x=0}$$

$$= \alpha \cdot \frac{d}{dx}(p(x)) \Big|_{x=0} + \beta \cdot \frac{d}{dx} q(x) \Big|_{x=0}$$

Thus F is a linear functional

$$= \alpha \cdot F(p(x)) + \beta \cdot F(q(x))$$

(c) Consider a basis for $P_n(\mathbb{R})$: $\{1, x, x^2, \ldots\}$

Let $f(x) \in P_n(\mathbb{R})$; $f(x) = 1 . a_0 + x . a_1 + \ldots$

$$F(f(x)) = \frac{d}{dx} f(x) \Big|_{x=0} = a_i$$

Consider $a = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$; $\mathbf{x} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \end{pmatrix}$ $\boxed{a^T \mathbf{x} = a_1}$

for $\mathbf{x}$, F forms a map to $a^T \mathbf{x}$

$$F : \mathbf{x} \longrightarrow a^T \mathbf{x}$$

$a^T x$ is the inner product representation for linear functional F.