Ans-7 - (a) Iterative LS definition

$$x^{(k+1)} = x^{(k)} - \frac{1}{\|A\|^2} A^T (Ax^{(k)} - b)$$

for $k = 0, 1, 2, \ldots$

$$\Rightarrow x^{(k+1)} = x^{(k)} - \frac{1}{\|A\|^2} A^T A x^{(k)} + \frac{A^T b}{\|A\|^2}$$

Let $P = I - \frac{1}{\|A\|^2} A^T A$, $Q = \frac{A^T b}{\|A\|^2}$

$$\boxed{x^{(k+1)} = P x^{(k)} + Q}$$

Spectral radius of $P < 1$ thus it will converge (i.e. $\{x^{(k)}\}$)

$\therefore$ expanding it more

$$x^{(k+1)} = P^{k+1} x^{(0)} + (P^k + P^{k-1} + \cdots + P^2 + P + I) Q$$

$$= \underbrace{(I + P + P^2 + \cdots + P^k)}_{S_k} Q$$

$$S = \lim_{k \to \infty} S_k = I + P + P^2 + \cdots + P^k$$

$$(I - P) S = (I + P + P^2 + \cdots) - (P + P^2 + \cdots) = I$$

$$\boxed{S = (I - P)^{-1}}$$

$$\Rightarrow \lim_{k \to \infty} x^{(k+1)} = SQ$$

$$= (I - P)^{-1} Q$$

$$= \left(I - \left(I - \frac{1}{\|A\|^2} A^T A\right)\right)^{-1} \cdot \frac{1}{\|A\|^2} A^T b$$

$$= \frac{(A^T A)^{-1}}{\|A\|^2} \cdot \frac{1}{\|A\|^2} A^T b = (A^T A)^{-1} \cdot A^T b.$$

Thus as $k \to \infty$ $\{x^{(k)}\}$ sequence converges to $\hat{x}$.

(b) Computations

$A \in \mathbb{R}^{m \times n}$

$\|A\|_F^2 \longrightarrow$ computed in mn time (assuming we consider frobenius norm)

$A^T \longrightarrow$ computed in mn time (access each element)

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order notations

Now for $Ax^{(k)} \longrightarrow mn$ steps (i.e. $O(mn)$ or $\approx mn$ steps)

$Ax^{(k)} - b \longrightarrow m$ steps

$A^T(Ax^{(k)} - b) \longrightarrow nm$ steps

$\frac{1}{\|A\|^2} A^T(Ax^{(k)} - b) \longrightarrow n$ steps

$x^{(k)} - \frac{1}{\|A\|^2} A^T(Ax^{(k)} - b) \longrightarrow n$ steps

sum ↓

$x^{(k+1)} \longrightarrow 2mn + m + 2n$ steps

To compute $\{x^{(k)}\}$

$\longrightarrow k(2mn + m + 2n)$ steps

$\longrightarrow O(kmn + km + kn)$ steps (Ans).