Ans-7. $A \in \mathbb{R}^{n \times n}$

L_{ij} matrix is a lower triangular matrix which is multiplied with A to convert a_{ij} th element into using a_{jj} th element $(i \neq j)$. $(i > j)$

$$L_{ij} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ & 1 & 0 & \cdots & 0 \\ & & \ddots & & \vdots \\ & \underset{ij}{\lambda} & \underset{jj}{\cdots} & & 0 \\ & & & & 1 \end{bmatrix}$$

ij th position of L_{ij} has λ where $\boxed{\lambda = -\dfrac{a_{ij}}{a_{jj}}}$

$A = [a_{ij}]_{n \times n}$.

Let's say

$1 . a_{ij} + \lambda a_{jj} = 0$

$\boxed{\lambda = -\dfrac{a_{ij}}{a_{jj}}}$

$L_{ij} A$ transform a_{ij} to 0.

$$\boxed{L_{ij} = \begin{cases} 1, & i = j \\ -\dfrac{a_{ij}}{a_{jj}}, & i = i, j = j \\ 0, & \text{elsewhere} \end{cases}}$$

To compute LU decomposition:

for j from 1 to n:
    for i from (j+1) to n:
        construct L_{ij}
        $A := L_{ij} \cdot A$ (multiply L_{ij} with A).

the output 'A' matrix after the completion of j-for loops (and i-for loops in each such loop). is equivalent to U.

To compute L, we store all L_{ij}^{-1} (inverse of L_{ij} matrix)

$$L = \prod_{j=1,\, i=j+1}^{j=n,\, i=n} L_{ij}^{-1} \quad \text{(multiplication of all } L_{ij}^{-1} \text{ matrices)}$$

Thus $\boxed{A = LU}$.

To find L_{ij}^{-1}

Clearly $L_{ij} = I - l_{ij} e_j^T \qquad (l_{ij} = a_{ij}/a_{jj})$

$$\Rightarrow (I - l_{ij} e_j^T)(I + l_{ij} e_j^T)$$

$$= I + \cancel{l_{ij} e_j^T} - \cancel{l_{ij} e_j^T} - l_{ij} \underbrace{(e_j^T l_{ij})}_{0} e_j^T$$

$$= I \qquad [\because e_j^T l_{ij} = 0]$$

$$\boxed{L_{ij}^{-1} = I + l_{ij} e_j^T}$$

where $l_{ij} = -\lambda = + \dfrac{a_{ij}}{a_{jj}}$