LAAIML - Test 2

Q.1) Let $Ax = b$ be given system of linear equations where $A \in \mathbb{R}^{n \times n}$ is an invertible matrix and $b \in \mathbb{R}^n$ is a given vector. Consider the perturbed system of equations

$$A\hat{x} = b + \delta b$$

What is the choice for δb so that $\hat{x} - x$ is the largest perturbation (in magnitude) to the exact solution x of the system $Ax = b$.

Q.2) Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and let $b_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Compute the least squares solution $\hat{x}_1$ and $\hat{x}_2$ respectively to the problem $Ax = b_1$ and $Ax = b_2$ with geometrical argument. (Do not compute Gram matrix, pseudo inverse etc.)

Q.3) Construct a 2×2 matrix A with the following properties.

i) $e_1 \longmapsto Ae_1$: direction of maximum magnification

ii) $e_2 \longmapsto Ae_2$: direction of minimum magnification

iii) condition number of $A = 10 = \|A\|_2$

iv) $\langle Ae_1, Ae_2 \rangle = (Ae_1)^T (Ae_2) = 0$

v) $\langle Ae_2, e_1 \rangle = (Ae_2)^T e_1 = \frac{1}{\sqrt{2}}$

Q.4) Consider the following 'scalar' multiobjective problem.

$$\min_{x \in \mathbb{R}} \sum_{i=1}^{N} \lambda_i (x - b_i)^2$$

where $\lambda_i > 0$ for $i = 1, 2, \dots, N$ and $b_i \in \mathbb{R}, i = 1, \dots, N$ are given.

Show that $\hat{x}$ that minimizes the above problem is in fact a convex combination of b_i's. i.e.

$$\hat{x} = \sum_{i=1}^{N} w_i b_i$$

where $w_i \geq 0$ for $i = 1, \dots, N$ and $\sum_{i=1}^{N} w_i = 1$.

How are w_i's related to λ_i's ??

Q.5) Let $a \in \mathbb{R}^n$ be a given vector and $\beta \in \mathbb{R}$ be a given scalar.

Find $x \in \mathbb{R}^n$ which is closet to the given vector $a \in \mathbb{R}^n$ among all n-vectors with the property that have average value β ?

Q.6) Consider the data fitting problem with the first basis function $\phi_1(x) = 1$ and the data set $x^{(1)}, \ldots, x^{(N)}$ and $y^{(1)}, \ldots, y^{(N)}$. Assume that the matrix A has linearly independent columns. Let $\hat{\alpha}$ be the parameter values that minimize the mean square prediction error over the data set. Let $\hat{r}^d \in \mathbb{R}^N$ denote the prediction errors using the optimal model parameter $\hat{\alpha}$.

Then prove $\operatorname{avg}(\hat{r}^d) = 0$.

(All the notations are as discussed in the class.)

Q.7) Suppose that the n-vector x and the m-vector y are such that their relationship can be approximated linearly as $y \approx Ax$ where $A \in \mathbb{R}^{m \times n}$.

We do not know A, but we have data $x^{(1)}, x^{(2)}, \dots, x^{(N)}$ and $y^{(1)}, y^{(2)}, \dots, y^{(N)}$.

Estimate A using this data by minimizing

$$\sum_{i=1}^{N} \|Ax^{(i)} - y^{(i)}\|_2^2 .$$

show that $\hat{A}$ which minimizes above cost function is given as $\hat{A} = YX^+$ where $X = [x^{(1)}, \dots, x^{(N)}] \in \mathbb{R}^{n \times N}$ and $Y = [y^{(1)}, \dots, y^{(N)}] \in \mathbb{R}^{m \times N}$.

[Assume: X has linearly independent rows and X^+ denotes pesudo-inverse of X].