Ans-4. $A \in \mathbb{R}^{m \times n}$ has Linearly Independent columns

Given equation $Ax = b$

a) existence: if $b \notin \text{col-span}(A)$

$\Rightarrow$ there does not exist any x s.t. $Ax = b$

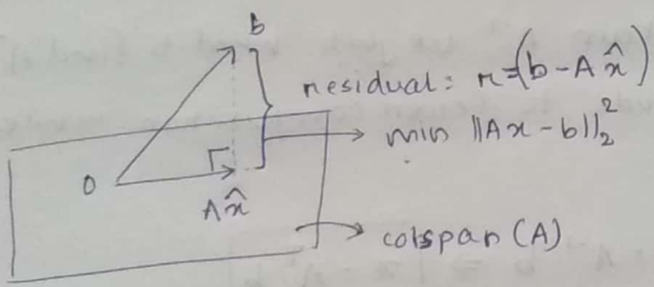
In such a case we try to find $\hat{x}$ (called least square solution).

s.t. $\hat{x} = \underset{x \in \mathbb{R}^n}{\arg\min} \|Ax - b\|_2$

Let $A_i \rightarrow i^{th}$ column of A

then $\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 = \min_{x \in \mathbb{R}^n} \|x_1 A_1 + x_2 A_2 + \dots + x_n A_n - b\|_2^2$

Geometrically,



To minimize r

Now r is the direction of normal to col-span(A), thus it is perpendicular to all vectors in col-span(A).

$\Rightarrow A^T r = 0$

$\Rightarrow A^T(b - A\hat{x}) = 0$

$\Rightarrow \boxed{A^T A \hat{x} = A^T b}$

(hence Normal equations)

$A^T A \hat{x} = A^T b$

$\Rightarrow \boxed{\hat{x} = (A^T A)^{-1} A^T b}$

we know, if A has linearly independent columns then

$A^T A$ is invertible, $\Rightarrow \hat{x}$ will have a unique value. (for least squares problem)

If A doesnot have linearly independent columns then solution for least squares problem may not be unique and there will exist infinitely many solutions.