Chapter 2

Convolution arithmetic

The analysis of the relationship between convolutional layer properties is eased by the fact that they don't interact across axes, i.e., the choice of kernel size, stride and zero padding along axis j only affects the output size of axis j. Because of that, this chapter will focus on the following simplified setting:

- 2-D discrete convolutions ($N = 2$),
- square inputs ($i_1 = i_2 = i$),
- square kernel size ($k_1 = k_2 = k$),
- same strides along both axes ($s_1 = s_2 = s$),
- same zero padding along both axes ($p_1 = p_2 = p$).

This facilitates the analysis and the visualization, but keep in mind that the results outlined here also generalize to the N-D and non-square cases.

2.1 No zero padding, unit strides

The simplest case to analyze is when the kernel just slides across every position of the input (i.e., $s = 1$ and $p = 0$). Figure 2.1 provides an example for $i = 4$ and $k = 3$.

One way of defining the output size in this case is by the number of possible placements of the kernel on the input. Let's consider the width axis: the kernel starts on the leftmost part of the input feature map and slides by steps of one until it touches the right side of the input. The size of the output will be equal to the number of steps made, plus one, accounting for the initial position of the kernel (Figure 2.8a). The same logic applies for the height axis.

More formally, the following relationship can be inferred:

Relationship 1. *For any i and k, and for $s = 1$ and $p = 0$,*

$$o = (i - k) + 1.$$

2.2 Zero padding, unit strides

To factor in zero padding (i.e., only restricting to $s = 1$), let's consider its effect on the effective input size: padding with p zeros changes the effective input size from i to $i + 2p$. In the general case, Relationship 1 can then be used to infer the following relationship:

Relationship 2. *For any i, k and p, and for $s = 1$,*

$$o = (i - k) + 2p + 1.$$

Figure 2.2 provides an example for $i = 5$, $k = 4$ and $p = 2$.

In practice, two specific instances of zero padding are used quite extensively because of their respective properties. Let's discuss them in more detail.

2.2.1 Half (same) padding

Having the output size be the same as the input size (i.e., $o = i$) can be a desirable property:

Relationship 3. *For any i and for k odd ($k = 2n + 1, \quad n \in \mathbb{N}$), $s = 1$ and $p = \lfloor k/2 \rfloor = n$,*

$$\begin{aligned} o &= i + 2\lfloor k/2 \rfloor - (k - 1) \\ &= i + 2n - 2n \\ &= i. \end{aligned}$$

This is sometimes referred to as *half* (or *same*) padding. Figure 2.3 provides an example for $i = 5$, $k = 3$ and (therefore) $p = 1$.

2.2.2 Full padding

While convolving a kernel generally *decreases* the output size with respect to the input size, sometimes the opposite is required. This can be achieved with proper zero padding:

Relationship 4. *For any i and k, and for $p = k - 1$ and $s = 1$,*

$$\begin{aligned} o &= i + 2(k - 1) - (k - 1) \\ &= i + (k - 1). \end{aligned}$$

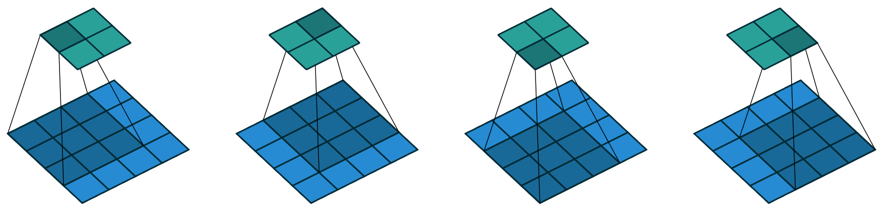

Figure 2.1: (No padding, unit strides) Convolving a 3×3 kernel over a 4×4 input using unit strides (i.e., $i = 4$, $k = 3$, $s = 1$ and $p = 0$).

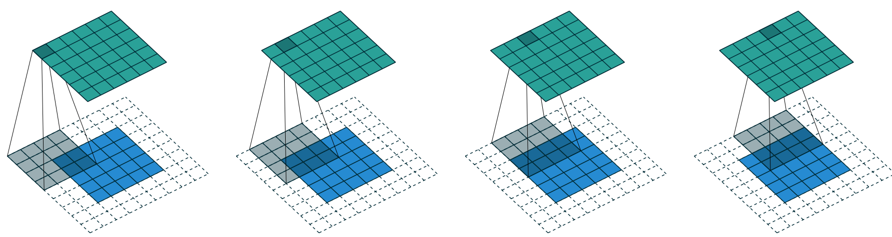

Figure 2.2: (Arbitrary padding, unit strides) Convolving a 4×4 kernel over a 5×5 input padded with a 2×2 border of zeros using unit strides (i.e., $i = 5$, $k = 4$, $s = 1$ and $p = 2$).

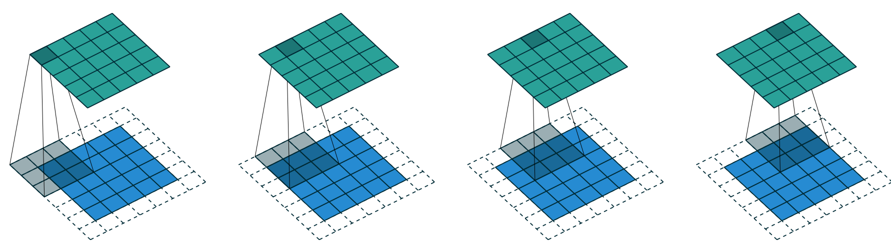

Figure 2.3: (Half padding, unit strides) Convolving a 3×3 kernel over a 5×5 input using half padding and unit strides (i.e., $i = 5$, $k = 3$, $s = 1$ and $p = 1$).

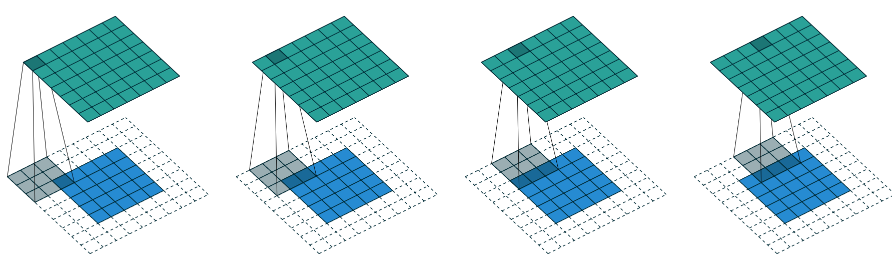

Figure 2.4: (Full padding, unit strides) Convolving a 3×3 kernel over a 5×5 input using full padding and unit strides (i.e., $i = 5$, $k = 3$, $s = 1$ and $p = 2$).

This is sometimes referred to as *full* padding, because in this setting every possible partial or complete superimposition of the kernel on the input feature map is taken into account. Figure 2.4 provides an example for $i = 5$, $k = 3$ and (therefore) $p = 2$.

2.3 No zero padding, non-unit strides

All relationships derived so far only apply for unit-strided convolutions. Incorporating non unitary strides requires another inference leap. To facilitate the analysis, let's momentarily ignore zero padding (i.e., $s > 1$ and $p = 0$). Figure 2.5 provides an example for $i = 5$, $k = 3$ and $s = 2$.

Once again, the output size can be defined in terms of the number of possible placements of the kernel on the input. Let's consider the width axis: the kernel starts as usual on the leftmost part of the input, but this time it slides by steps of size s until it touches the right side of the input. The size of the output is again equal to the number of steps made, plus one, accounting for the initial position of the kernel (Figure 2.8b). The same logic applies for the height axis.

From this, the following relationship can be inferred:

Relationship 5. *For any i, k and s, and for $p = 0$,*

$$o = \left\lfloor \frac{i - k}{s} \right\rfloor + 1.$$

The floor function accounts for the fact that sometimes the last possible step does *not* coincide with the kernel reaching the end of the input, i.e., some input units are left out (see Figure 2.7 for an example of such a case).

2.4 Zero padding, non-unit strides

The most general case (convolving over a zero padded input using non-unit strides) can be derived by applying Relationship 5 on an effective input of size $i + 2p$, in analogy to what was done for Relationship 2:

Relationship 6. *For any i, k, p and s,*

$$o = \left\lfloor \frac{i + 2p - k}{s} \right\rfloor + 1.$$

As before, the floor function means that in some cases a convolution will produce the same output size for multiple input sizes. More specifically, if $i + 2p - k$ is a multiple of s, then any input size $j = i + a, \quad a \in \{0, \ldots, s - 1\}$ will produce the same output size. Note that this ambiguity applies only for $s > 1$.

Figure 2.6 shows an example with $i = 5$, $k = 3$, $s = 2$ and $p = 1$, while Figure 2.7 provides an example for $i = 6$, $k = 3$, $s = 2$ and $p = 1$. Interestingly, despite having different input sizes these convolutions share the same output size. While this doesn't affect the analysis for *convolutions*, this will complicate the analysis in the case of *transposed convolutions*.

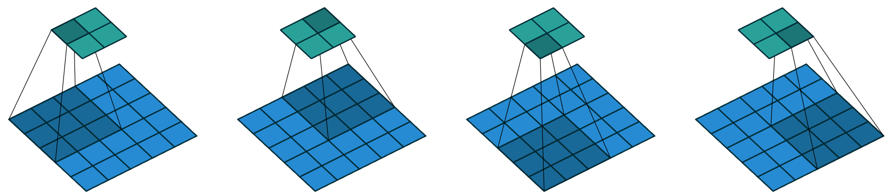

Figure 2.5: (No zero padding, arbitrary strides) Convolving a 3×3 kernel over a 5×5 input using 2×2 strides (i.e., $i = 5$, $k = 3$, $s = 2$ and $p = 0$).

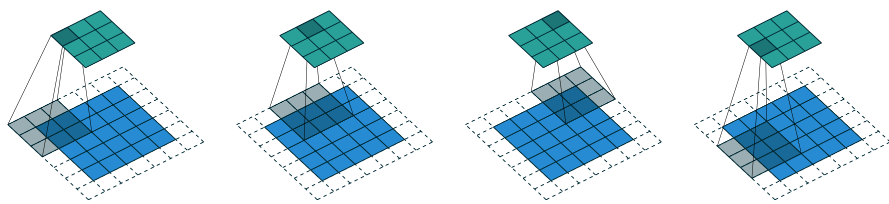

Figure 2.6: (Arbitrary padding and strides) Convolving a 3×3 kernel over a 5×5 input padded with a 1×1 border of zeros using 2×2 strides (i.e., $i = 5$, $k = 3$, $s = 2$ and $p = 1$).

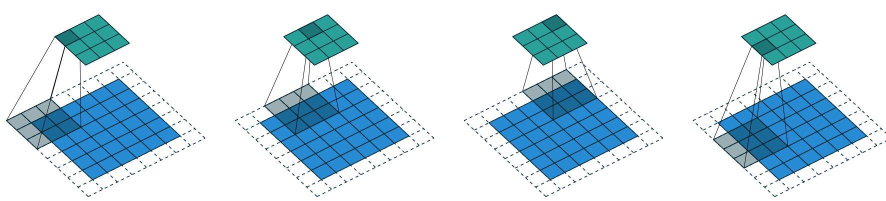

Figure 2.7: (Arbitrary padding and strides) Convolving a 3×3 kernel over a 6×6 input padded with a 1×1 border of zeros using 2×2 strides (i.e., $i = 6$, $k = 3$, $s = 2$ and $p = 1$). In this case, the bottom row and right column of the zero padded input are not covered by the kernel.

(a) The kernel has to slide two steps to the right to touch the right side of the input (and equivalently downwards). Adding one to account for the initial kernel position, the output size is 3×3.

(b) The kernel has to slide one step of size two to the right to touch the right side of the input (and equivalently downwards). Adding one to account for the initial kernel position, the output size is 2×2.

Figure 2.8: Counting kernel positions.