计算方法第二次作业　　学号 20011109 汪耀

P101.

5.

Doolittle 分解：

$$L=\begin{pmatrix}1&0&0&0\\ \frac{6}{5}&1&0&0\\ \frac{7}{5}&-\frac{1}{2}&1&0\\ 1&0&\frac{3}{5}&1\end{pmatrix}\quad U=\begin{pmatrix}5&7&9&10\\ 0&-\frac{2}{5}&-\frac{4}{5}&-3\\ 0&0&-5&-\frac{17}{2}\\ 0&0&0&\frac{1}{10}\end{pmatrix}$$

则 $Ax=b$ 可化为 $LUx=b$，令 $\begin{cases}Ux=y\\ Ly=b\end{cases}$

先解 $Ly=b$，可解得 $y=(1,-\frac{1}{5},-\frac{1}{2},\frac{3}{10})^T$

再解 $Ux=y$，解得 $x=(20,-12,-5,3)^T$

Crout 分解：

$$\hat{L}=\begin{pmatrix}5&0&0&0\\ 6&-\frac{2}{5}&0&0\\ 7&\frac{1}{5}&-5&0\\ 5&0&-3&\frac{1}{10}\end{pmatrix}\quad \hat{U}=\begin{pmatrix}1&\frac{7}{5}&\frac{9}{5}&2\\ 0&1&2&\frac{15}{2}\\ 0&0&1&\frac{17}{10}\\ 0&0&0&1\end{pmatrix}$$

先解 $\hat{L}y=b$，得 $y=(\frac{1}{5},\frac{1}{2},\frac{1}{10},3)^T$

再解 $\hat{U}x=y$，得 $x=(20,-12,-5,3)^T$

6. 公式 $l_{kk}=\sqrt{a_{kk}-\sum_{j=1}^{k-1}l_{kj}^2}$　　$l_{ik}=\frac{1}{l_{kk}}(a_{ik}-\sum_{j=1}^{k-1}l_{ij}l_{kj})$

其中 $k=1,2,\cdots n$　　$i=k+1,k+2,\cdots n$

直接计算得

$$L=\begin{pmatrix}1&0&0\\ \frac{1}{2}&\frac{\sqrt{3}}{6}&0\\ \frac{1}{3}&\frac{\sqrt{3}}{6}&\frac{\sqrt{5}}{30}\end{pmatrix}$$

8. (1) 对A作 Crout 分解

$$\hat{L}=\begin{pmatrix}2&0&0&0\\ 1&\frac{7}{2}&0&0\\ 0&1&\frac{26}{7}&0\\ 0&0&1&\frac{45}{26}\end{pmatrix}\quad \hat{U}=\begin{pmatrix}1&\frac{1}{2}&0&0\\ 0&1&\frac{2}{7}&0\\ 0&0&1&\frac{7}{26}\\ 0&0&0&1\end{pmatrix}$$

$\hat{L}\hat{U}x=b$，令 $\begin{cases}\hat{L}y=b\\ \hat{U}x=y\end{cases}$

解 $\hat{L}y=b$，可得 $y=(\frac{1}{2},-\frac{5}{7},1,-\frac{26}{45})^T$

解 $\hat{U}x=y$，可得 $x=(\frac{46}{45},-\frac{41}{45},\frac{52}{45},-\frac{26}{45})^T$

(2) 对A作 Crout 分解

$$\hat{L}=\begin{pmatrix}4&0&0&0&0\\ -1&\frac{15}{4}&0&0&0\\ 0&-1&\frac{56}{15}&0&0\\ 0&0&-1&\frac{209}{56}&0\\ 0&0&0&-1&\frac{780}{209}\end{pmatrix}$$

$$\tilde{U}=\begin{pmatrix} 1 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 1 & -\frac{4}{15} & 0 & 0 \\ 0 & 0 & 1 & -\frac{15}{56} & 0 \\ 0 & 0 & 0 & 1 & -\frac{56}{209} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

先解 $\tilde{L}y=b$ 解得 $y=(25, \frac{20}{3}, \frac{25}{14}, \frac{100}{209}, \frac{2095}{39})^T$

再解 $\tilde{U}x=y$，解得 $x=\begin{pmatrix} 27.05128205128205 \\ 8.205128205128205 \\ 5.769230769230770 \\ 14.871794871794872 \\ 53.717948717948720 \end{pmatrix}$

7. $\|A\|_1=\max\{0.6+0.1, 0.5+0.3\}=0.8$

$\|A\|_\infty=\max\{0.6+0.5, 0.1+0.3\}=1.1$

$\|A\|_F=\sqrt{0.6^2+0.5^2+0.1^2+0.3^2}=\sqrt{0.71}=0.842615$

$AA^T=\begin{pmatrix} 0.3 & 0.33 \\ 0.33 & 0.34 \end{pmatrix}$

$|\lambda I-AA^T|=\begin{vmatrix} \lambda-0.3 & -0.33 \\ -0.33 & \lambda-0.34 \end{vmatrix}=\lambda^2-0.71\lambda+0.0169$

令 $|\lambda I-AA^T|=0$ 解得：

$\lambda_1=0.685341$ $\quad\lambda_2=0.024659$

故 $\|A\|_2=\sqrt{\rho(A^TA)}=\sqrt{0.685341}=0.827853$

11. (1) $Cond(A)=\|A\|\cdot\|A^{-1}\|\geq\|A\cdot A^{-1}\|=1$

(2) $Cond(AB)=\|AB\|\cdot\|(AB)^{-1}\|=\|AB\|\cdot\|B^{-1}A^{-1}\|$

$\leq\|A\|\|B\|\|B^{-1}\|\|A^{-1}\|=\|A\|\|A^{-1}\|\|B\|\|B^{-1}\|$

$=Cond(A)\cdot Cond(B)$

(3) 对 $\forall\, c\neq0$

$Cond(cA)=\|cA\|\|(cA)^{-1}\|=c\cdot c^{-1}\|A\|\|A^{-1}\|$

$=Cond(A)$

12. Jacobi 迭代法：

$$\begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{pmatrix}=\begin{pmatrix} 0, & -\frac{a_{12}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 \end{pmatrix}\begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix}+\begin{pmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \end{pmatrix}$$

Gauss-Seidel 迭代法

$$\begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{pmatrix}=\begin{pmatrix} 0, & -\frac{a_{12}}{a_{11}} \\ 0, & \frac{a_{12}a_{21}}{a_{11}a_{22}} \end{pmatrix}\begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix}+\begin{pmatrix} \frac{b_1}{a_{11}} \\ -\frac{a_{21}b_1}{a_{11}a_{22}}+\frac{b_2}{a_{22}} \end{pmatrix}$$

则 $\rho(B_J)=\sqrt{\left|\frac{a_{12}a_{21}}{a_{11}a_{22}}\right|}$，$\rho(B_G)=\left|\frac{a_{12}a_{21}}{a_{11}a_{22}}\right|$

两种迭代法求解此方程组时，收敛的充要条件均为 $\left|\frac{a_{12}a_{21}}{a_{11}a_{22}}\right|<1$

(2) 由(1)可知：$\left|\frac{a_{12}a_{21}}{a_{11}a_{22}}\right|<1$ 时，两种方法同时收敛；

$\left|\frac{a_{12}a_{21}}{a_{11}a_{22}}\right|\geq1$ 时 两种方法同时发散

13、方程组的系数阵 $A=\begin{pmatrix}1 & 0.4 & 0.4\\0.4 & 1 & 0.8\\0.4 & 0.8 & 1\end{pmatrix}$

因为A对称正定，故Gauss-Seidel迭代法收敛

又 $|2D-A|=\begin{vmatrix}1 & -0.4 & -0.4\\-0.4 & 1 & -0.8\\-0.4 & -0.8 & 1\end{vmatrix}<0$，故 $2D-A$ 不是正定

故 Jacobi 迭代法不收敛

14. (1) 方程系数阵 $A=\begin{pmatrix}3 & -10\\9 & -4\end{pmatrix}$

对于Jacobi迭代法

$$B_J=I-D^{-1}A=\begin{pmatrix}0 & \frac{10}{3}\\\frac{9}{4} & 0\end{pmatrix}$$

$$\rho(B_J)=\sqrt{\frac{90}{12}}=\frac{\sqrt{30}}{2}>1$$

故 Jacobi 法发散

对于Gauss-Seidel迭代法

$$B_G=-(D+L)^{-1}U=\begin{pmatrix}0 & \frac{10}{3}\\0 & \frac{15}{2}\end{pmatrix}$$

$$\rho(B_G)=\frac{15}{2}>1$$

故 Gauss-Seidel 法发散

(2) 方程组系数阵 $A=\begin{pmatrix}9 & -4\\3 & -10\end{pmatrix}$

对Jacobi法：$B_J=I-D^{-1}A=\begin{pmatrix}0 & \frac{4}{9}\\\frac{3}{10} & 0\end{pmatrix}$

$$\rho(B_J)=\sqrt{\frac{12}{90}}=\frac{\sqrt{30}}{15}<1$$

故Jacobi迭代法收敛

对Gauss-Seidel法：

$$B_G=-(D+L)^{-1}U=\begin{pmatrix}0 & \frac{4}{9}\\0 & \frac{2}{15}\end{pmatrix}$$

$$\rho(B_G)=\frac{2}{15}<1$$

故 Gauss-Seidel 迭代法收敛

16. (1) 若A正定，则 $|1|>0$，$\begin{vmatrix}1 & a\\a & 1\end{vmatrix}>0$，$\begin{vmatrix}1 & a & a\\a & 1 & a\\a & a & 1\end{vmatrix}>0$

即 $\begin{cases}1>0\\1-a^2>0\\1+a^3+a^3-a^2-a^2-a^2>0\end{cases}$

解得 $-\frac{1}{2}<a<1$

(2) Jacobi收敛的充要条件为 A和 $2D-A$ 均对称正定且对角线元素为正

$2D-A=\begin{pmatrix}1 & -a & -a\\-a & 1 & -a\\-a & -a & 1\end{pmatrix}$，即 $\begin{cases}1>0\\1-a^2>0\\1-a^3-a^3-a^2-a^2-a^2>0\end{cases}$

解得 $-1<a<\frac{1}{2}$

又由(1)：$-\frac{1}{2}<a<1$

故 $-\frac{1}{2}<a<\frac{1}{2}$ 时 Jacobi 迭代法收敛

(5) Gauss-seidel 迭代矩阵

$$B_G=\begin{pmatrix}1&0&0\\a&1&0\\a&a&1\end{pmatrix}^{-1}\begin{pmatrix}0&a&a\\0&0&a\\0&0&0\end{pmatrix}$$

$$=\begin{pmatrix}0&-a&-a\\0&a^2&a^2-a\\0&a^2-a^3&2a^2-a^3\end{pmatrix}$$

其特征多项式 $p(\lambda)=\begin{vmatrix}\lambda&a&a\\0&\lambda-a^2&a^2-a\\0&a^3-a^2&\lambda-2a^2+a^3\end{vmatrix}$

$$=\lambda\left(\lambda^2+(a^3-3a^2)\lambda+a^3\right)$$

则要使 Gauss-Seidel 法收敛 求 $p(\lambda)=0$ 的根为 $\lambda_1, \lambda_2, \lambda_3$

则 $|\lambda_1|<1$, $|\lambda_2|<1$, $|\lambda_3|<1$

又 $\lambda_1=0$, λ_2, λ_3 为 $\lambda^2+(a^3-3a^2)\lambda+a^3=0$ 的根

则 $|\lambda_2|<1$ $|\lambda_3|<1$ 的充要条件为:

$$|a^3-3a^2|<1+a^3<2$$

解得 $-\frac{1}{2}<a<1$

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2. 线性插值: 取 $x_0=0.2$, $x_1=0.3$

$$L_1(x)=\frac{x-0.2}{0.3-0.2}\times1.3499+\frac{x-0.3}{0.2-0.3}\times1.2214$$

$$f(0.23)\approx L_1(0.23)=1.2700$$

二次插值: $x_0=0.1$, $x_1=0.2$, $x_2=0.3$

$$L_2(x)=1.1052\cdot\frac{(x-0.2)(x-0.3)}{(0.1-0.2)(0.1-0.3)}+1.2214\cdot\frac{(x-0.1)(x-0.3)}{(0.2-0.1)(0.2-0.3)}$$
$$+1.3499\times\frac{(x-0.1)(x-0.2)}{(0.3-0.1)(0.3-0.2)}$$

$$f(0.23)\approx L_2(0.23)=1.2659$$

5. (1) 线性插值: 取 $x_0=0$, $x_1=\frac{\pi}{6}$

$$L_1(x)=\frac{x-\frac{\pi}{6}}{0-\frac{\pi}{6}}\times0+\frac{x-0}{\frac{\pi}{6}-0}\times0.5=\frac{3}{\pi}x$$

$$\sin\left(\frac{\pi}{12}\right)\approx L_1\left(\frac{\pi}{12}\right)=0.25$$

$$\left|\sin\frac{\pi}{12}-L_1\left(\frac{\pi}{12}\right)\right|=|0.258819045-0.25|=0.008819045$$

(2) 二次插值: 取 $x_0=0$, $x_1=\frac{\pi}{6}$, $x_2=\frac{\pi}{4}$

$$L_2(x)=1\times\frac{(x-\frac{\pi}{6})(x-\frac{\pi}{4})}{(0-\frac{\pi}{6})(0-\frac{\pi}{4})}+\frac{\sqrt{3}}{2}\frac{(x-0)(x-\frac{\pi}{4})}{(\frac{\pi}{6}-0)(\frac{\pi}{6}-\frac{\pi}{4})}+\frac{\sqrt{2}}{2}\frac{(x-0)(x-\frac{\pi}{6})}{(\frac{\pi}{4}-0)(\frac{\pi}{4}-\frac{\pi}{6})}$$

$$\cos\frac{\pi}{5}\approx L_2\left(\frac{\pi}{5}\right)=0.80981246$$

$$\left|\cos\frac{\pi}{5}-L_2\left(\frac{\pi}{5}\right)\right|=|0.809016994-0.80981246|=0.00795466$$

线性插值的余项估计：

$$|f(x)-p_1(x)| \le \frac{1}{2}\max_{x_0\le\xi\le x_1}|f''(\xi)||(x-x_0)(x-x_1)|$$

则 $\left|\sin\frac{\pi}{12}-L_1(\frac{\pi}{12})\right| \le \frac{1}{2}\left|\frac{\pi}{12}-0\right|\left|\frac{\pi}{12}-\frac{\pi}{6}\right|\times\frac{1}{2} = \frac{\pi^2}{576}$

二次插值的余项估计

$$|f(x)-p_2(x)| \le \frac{1}{6}\max_{x_0\le\xi\le x_2}|f'''(\xi)||(x-x_0)(x-x_1)(x-x_2)|$$

故 $\left|\cos\frac{\pi}{5}-L_2(\frac{\pi}{5})\right| \le \frac{1}{6}\times\frac{\sqrt{2}}{2}\times\frac{\pi}{5}\times\frac{\pi}{30}\times\frac{\pi}{20} = \frac{\sqrt{2}\pi^3}{36000}$

8.(1) 令 $f(x)\equiv 1$

利用 Lagrange 插值余项定理

$$f(x)-p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

显然 $p_n(x)\equiv 1$

由 Lagrange 基函数插值公式

$$p_n(x) = \sum_{j=0}^{n} f(x_j)l_j(x) = \sum_{j=0}^{n} l_j(x)$$

故 $\sum_{j=0}^{n} l_j(x) = 1$

(2) 令 $f(x)\equiv x^k$, $k=1,2,\cdots,n$

$$f(x)-p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

显然 $p_n(x)=f(x)\equiv x^k$

故 $p_n(x) = \sum_{j=0}^{n} f(x_j)l_j(x) = \sum_{j=0}^{n} x_j^k l_j(x) = x^k$

(3)

$$\sum_{j=0}^{n}(x_j-x)^k l_j(x)$$

$$= \sum_{j=0}^{n}\sum_{i=0}^{k} C_k^i x_j^i(-x)^{k-i} l_j(x) = \sum_{i=0}^{k} C_k^i(-x)^{k-i}\sum_{j=0}^{n} x_j^i l_j(x)$$

代入(2)得 $= \sum_{i=0}^{k} C_k^i(-x)^{k-i}x^i = (x-x)^k = 0$

故试得证