$$z(r) = \sqrt{1-r^2} + \ln \frac{r}{1+\sqrt{1-r^2}}$$

$$ds^2 = \underbrace{dx^2 + dy^2}_{dr^2 + r^2 d\varphi} + dz^2$$

$$dz = \frac{1-r^2}{r\sqrt{1-r^2}}\, dr$$

$$ds^2 = \left(1 + \left(\frac{1-r^2}{r\sqrt{1-r^2}}\right)^2\right) dr^2 + r^2 d\varphi^2 =$$

$$= \frac{1}{r^2} dr^2 + r^2 d\varphi^2$$

$$g_{ij} = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\Gamma_{k,ij} = \frac{1}{2}\left\{ \frac{\partial g_{ij}}{\partial x_k} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{kj}}{\partial x_i} \right\}$$

$$\Gamma_{r,rr} = \frac{1}{2}\frac{\partial g_{rr}}{\partial r} = -\frac{1}{r^3}$$

$$\Gamma_{r,\varphi r} = 0 = \Gamma_{r,r\varphi}$$

$$\Gamma_{\varphi,rr} = 0$$

$$\Gamma_{\varphi,\varphi\varphi} = 0$$

$$\Gamma_{r,\varphi\varphi} = -\frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial r} = -r$$

$$\Gamma_{\varphi,r\varphi} = \frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial r} = r$$

$$\Gamma^{k}_{ij} = g^{k\lambda}\,\Gamma_{\lambda,ij}$$

$$\Gamma^r_{rr} = g^{rr}\Gamma_{r,rr} = -\frac{1}{r}, \qquad g^{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

$$\Gamma^r_{\varphi\varphi} = g^{rr}\Gamma_{r,\varphi\varphi} = -r^3$$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = g^{\varphi\varphi}\Gamma_{\varphi,r\varphi} = \frac{1}{r}$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \Gamma^\varphi_{rr} = \Gamma^\varphi_{\varphi\varphi} = 0$$

$$\begin{cases} \frac{d^2r}{ds^2} - \frac{1}{r}\left(\frac{dr}{ds}\right)^2 - r^3\left(\frac{d\varphi}{ds}\right)^2 = 0 \\ \frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} = 0 \end{cases}$$

Пусть $\dot{r} = \frac{dr}{ds}$, $\dot{\varphi} = \frac{d\varphi}{ds}$

$$\begin{cases} \ddot{r} - \frac{1}{r}(\dot{r})^2 - r^3(\dot{\varphi})^2 = 0 \\ \ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \end{cases}$$

$$\frac{d}{ds}(\ln\dot{\varphi} + 2\ln r) = 0$$

$$\ln\dot{\varphi} + 2\ln r = \text{const}$$

$$\dot{\varphi} = \frac{e^{\text{const}}}{r^2} = \frac{\alpha}{r^2}$$

$$\ddot{r} - \frac{1}{r}(\dot{r})^2 - r^3\cdot\frac{\alpha^2}{r^4} = 0$$

$$\ddot{r} - \frac{1}{r}(\dot{r})^2 - \frac{\alpha^2}{r} = 0$$

Решим однородное ДУ:

$$\ddot{r} - \frac{1}{r}(\dot{r})^2 = 0$$

$$\frac{\ddot{r}}{\dot{r}} = \frac{\dot{r}}{r} \Rightarrow \ln \dot{r} - \ln r = \mathrm{const}$$

$$\frac{\dot{r}}{r} = C \Rightarrow \dot{r} = C(s)\, r(s)$$

$$\dot{C} r + C\dot{r} - \frac{1}{r}(Cr)^2 - \frac{\alpha^2}{r} = 0$$

$$\dot{C} r - \frac{\alpha^2}{r} = 0$$

$$\dot{C} = \frac{\alpha^2}{r^2}$$

$$\frac{dC}{ds} = \frac{dC}{dr}\frac{dr}{ds} = \frac{dC}{dr} Cr$$

$$\frac{dC}{dr} Cr = \frac{\alpha^2}{r^2} \Rightarrow \int C\,dC = \alpha^2 \int \frac{dr}{r^3}$$

$$\frac{C^2}{2} = -\frac{\alpha^2}{2r^2} + \Phi \Rightarrow C^2 + \frac{\alpha^2}{r^2} = \Phi \Rightarrow$$

$$\Rightarrow C = \sqrt{\Phi - \frac{\alpha^2}{r^2}} = \frac{\sqrt{\Phi r^2 - \alpha^2}}{r}$$

Подставим:

$$\dot{r} = \frac{dr}{d\varphi} \cdot \frac{d\varphi}{ds} = \frac{dr}{d\varphi} \cdot \frac{\alpha}{r^2}$$

$$\frac{dr}{d\varphi} \cdot \frac{\alpha}{r^2} = \frac{\sqrt{\Phi r^2 - \alpha^2}}{r} \cdot r \Rightarrow \frac{dr}{d\varphi} = \frac{r^2}{\alpha}\sqrt{\Phi r^2 - \alpha^2}$$

$$\int \frac{dr}{r^2}\frac{\alpha}{\sqrt{\Phi r^2-\alpha^2}} = \int d\varphi$$

$$\int \frac{\alpha\, dr}{r^3\sqrt{\Phi-\frac{\alpha^2}{r^2}}} = \int -\frac{1}{2}\frac{\alpha\, d\left(\frac{1}{r^2}\right)}{\sqrt{\Phi-\frac{\alpha^2}{r^2}}} = -\frac{\alpha}{2}\int \frac{d\left(\frac{\alpha^2}{r^2}\right)}{\sqrt{\Phi-\frac{\alpha^2}{r^2}}} =$$

$$= \frac{1}{\alpha}\sqrt{\Phi-\frac{\alpha^2}{r^2}} = \sqrt{\frac{\Phi}{\alpha^2}-\frac{1}{r^2}}$$

Тогда $\varphi-\varphi_0 = \sqrt{\frac{\Phi}{\alpha^2}-\frac{1}{r^2}}$

$$(\varphi-\varphi_0)^2 = \frac{\Phi}{\alpha^2}-\frac{1}{r^2} \Rightarrow r^2 = \frac{1}{\frac{\Phi}{\alpha^2}-(\varphi-\varphi_0)^2}$$

$$r(\varphi) = \frac{1}{\sqrt{\frac{\Phi}{\alpha^2}-(\varphi-\varphi_0)^2}}$$