

QUANTUM MECHANICS

Classical physics

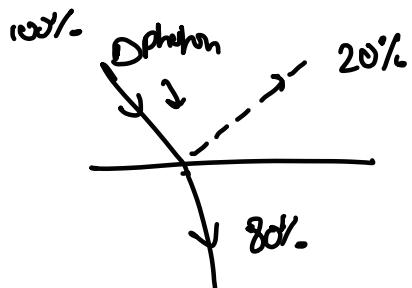
- fails for high speeds → relativity
- fails for small particles → quantum mechanics

In classical physics -

- Identical experiments give identical results.

In quantum physics -

- Identical experiments can give different results.



the same photon hitting in the same condⁿ can sometimes reflect sometimes reflect :P
(different results, identical setup)

But if you were to repeat it multiple times then you could say fixed 20% reflected 80% refracted

Light -

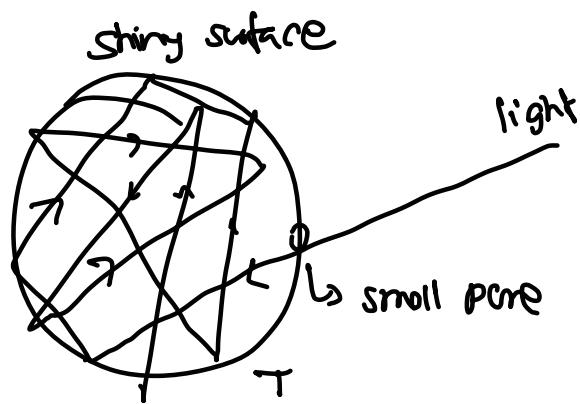
- Travels in a straight line }
- Gets reflected refracted } can be explained thru wave
- Gets diffracted - wave
- Is polarised - wave
- Photoelectric effect - particulate

Black body radiation ~~gives~~ radiation -

All object absorb some radiation and emit some radiation

at a particular temp - what gives them colour

A black body is the best absorber of radiation which absorbs all the radiation that falls on it and \therefore also emits all the radiation (otherwise if temp would \uparrow)



whatever passes through
the pore gets stuck inside
the black-body with no
escape

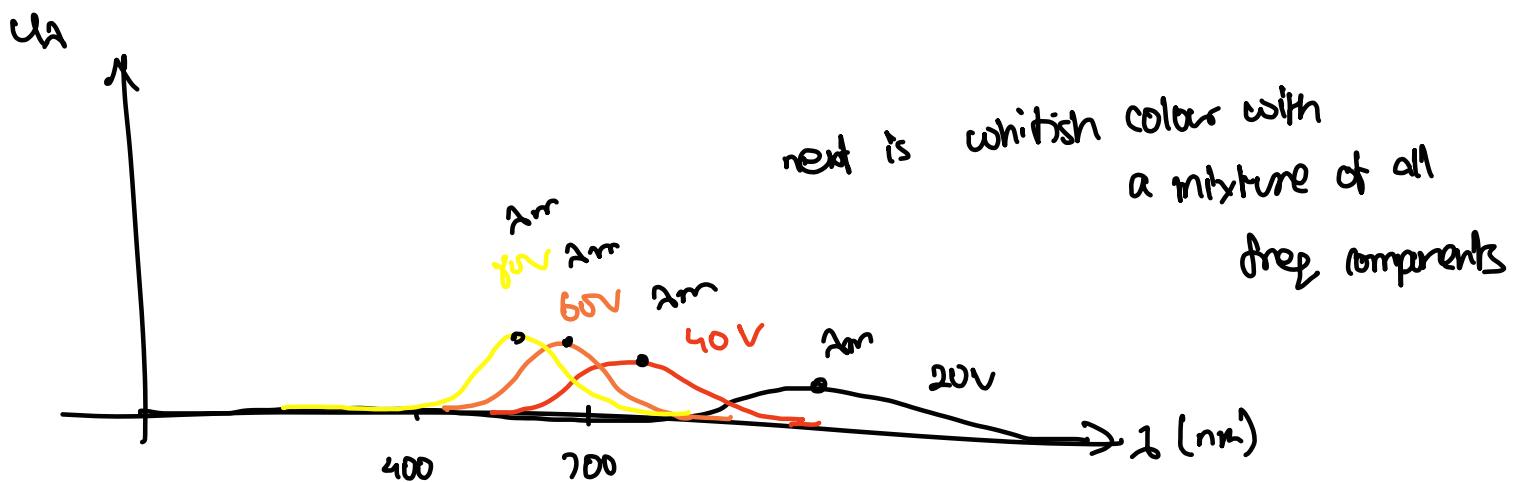
Wein's displacement law -

Bulb at 1000 volt - nada (but it's slightly warm)

slightly high - red
 ↓
 orange
 ↓
 yellow
 ↓
 white

} as you ↑ voltage
or rather
temp. really
(cu² + 2rt)

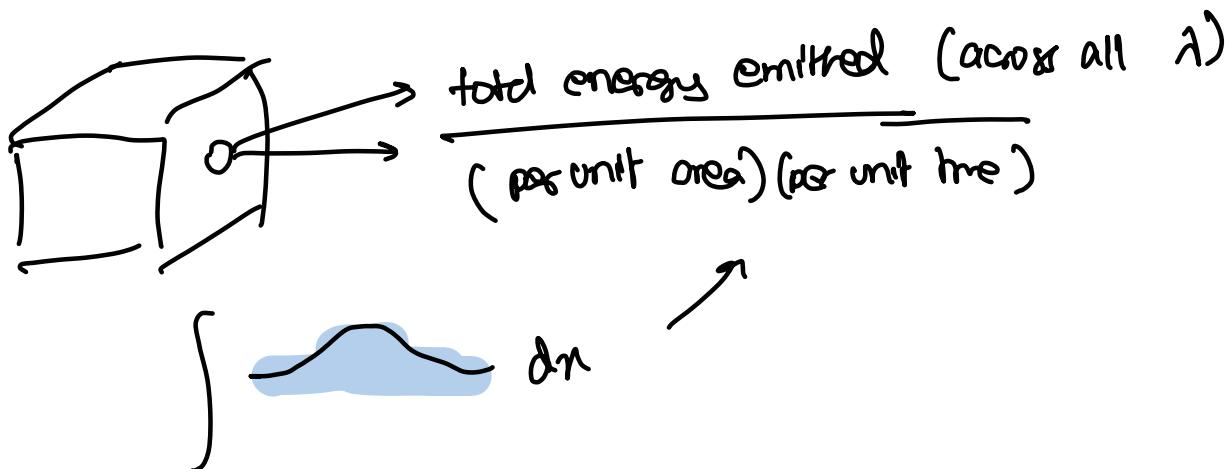
But why?



as $T \uparrow \lambda_m \downarrow$; Experimentally, $\lambda_m T = 2.89 \times 10^{-3}$
fixed value

Stefan's law -

$$U = \sigma T^4 \quad (\text{for a black-body})$$



so we have two expressions for max λ_m and area
of graph

but what about a particular λ ?

$$\frac{(\lambda + d\lambda) \text{ range energy}}{\text{unit volume}} = U_\lambda d\lambda$$

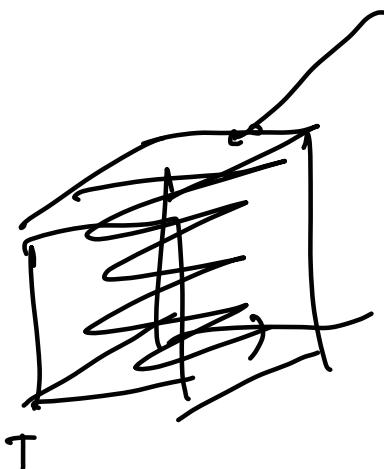
$$\text{Similarly, } v - dv \quad | \quad d\lambda = -\frac{c}{v^2} dv$$

$U_\lambda dv$ = energy per unit volume in
 $v - v + dv$ range

$$U_V = v^3 f(\frac{v}{kT}) = \alpha v^3 e^{-bD/kT} \quad (\text{some experimental fit})$$

matched in some extent.

Rayleigh-Jeans



proposed the radiation energy

inside as standing waves

i.e. a collection of oscillators

$v - v + \delta v$ oscillators / standing wave

If free energy is theoretically calculated, the graph could be theoretically explained.

It came out as ; $v - v + \delta v - \frac{8\pi v^2}{c^3} dv$ if standing waves
unit volume

Every standing wave, (low amplitude C , high amplitude C)

further had a distribution of $e^{-E/kT}$ (exp r.v.)

$$\exp[-E] = kT \left(\frac{1}{2}\right)$$

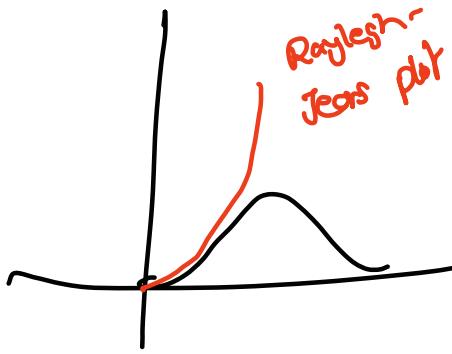
$$f_E(E) = \frac{1}{kT} e^{-E/kT}$$

(I think)

$$\therefore U_V d\omega = \frac{8\pi v^2}{c^3} \times \delta v \times kT$$

$$U_V = \frac{8\pi v^2}{c^3} \times kT$$

this graph
doesn't even
match the
experimental
curve

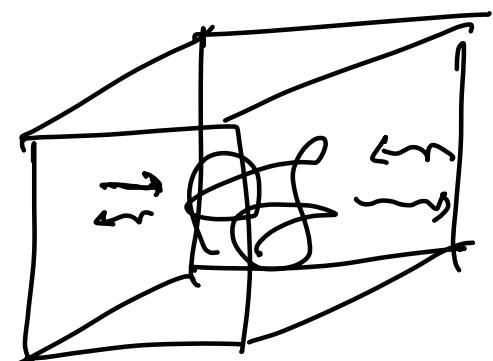


matches only at low freq.

this mismatch at high freq. was called ultraviolet

catastrophe

Max Planck



walls emit oscillations absorb
oscillation emit wall absorbs
at eq. T .

- Made a rule which he didn't believe - energy emitted or absorbed from radiation at freq $\nu = nh\nu$

New expectation, a discrete r.v :-
of happens in seconds

$$\text{Exp}(E) = \sum_{n=0}^{\infty} (nh\nu) e^{-\frac{nh\nu}{kT}} \quad \text{A.G.P formula}$$

$$= (hv) \sum_{n=0}^{\infty} n \left(e^{\frac{hv}{kT}} \right)^n = \frac{hv}{e^{\frac{hv}{kT}} - 1}$$

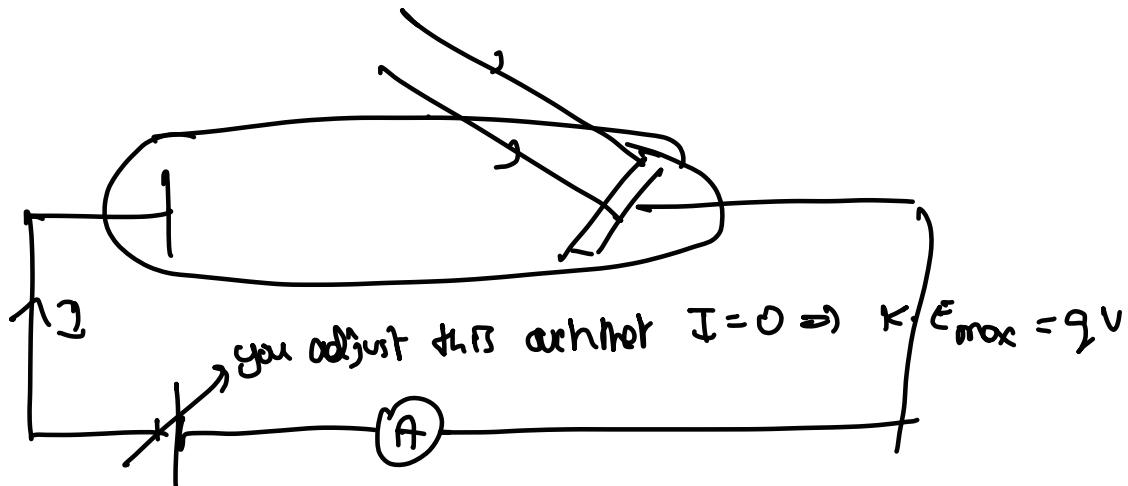
\therefore New expression $U_{v0} dv = \frac{8\pi v^2}{c^3} dv \times \frac{hv}{e^{\frac{hv}{kT}} - 1}$

$$U_{v0} = \frac{8\pi v^3}{c^3} \times \frac{h}{e^{\frac{hv}{kT}} - 1}$$

matched exactly

energy / volume
in v to $v + dv$

II Photoelectric effect



i. Emission of $photo^-$ is almost instantaneous.

classically \therefore Intensity is fixed - it should have taken some time - hours if low intensity

2. If $\lambda < \lambda_0$, even if you ↑ intensity photo's never come out.

$$1 \quad KE_{max} = \frac{hc}{\lambda} - \text{work function}$$

$$k^2_{\text{mes}} = m^2 - \omega^2 ; \text{ k.f depends on freq?}$$

Solⁿ: 2 things can happen

A photon can go collide and give e^- if $h\nu > \omega$
then k.e.

Or multiple photons can go collide with an e- at

enjoy it happened

'it wouldn't account

CBP C-Card then

go ond collide with

go ond collide with

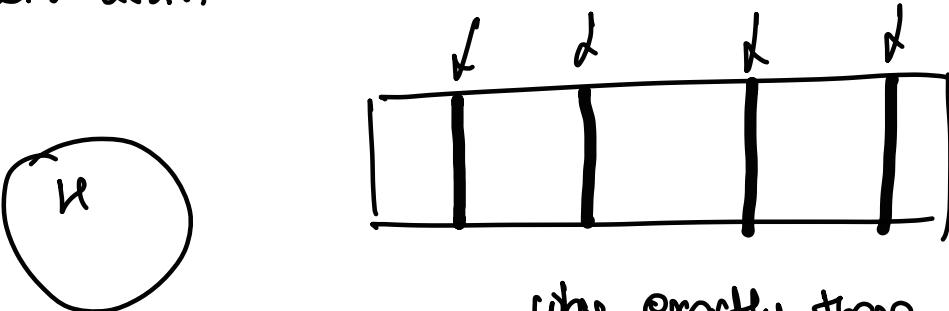
go on and collide with

go and collide with

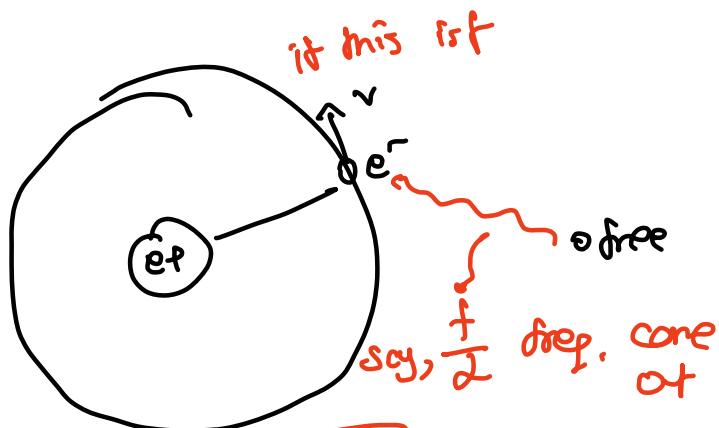
go and collide with
other e^- and base if net

go and collide with
other e^- and lose if net
energy

Bohr atom



why exactly those wavelengths are
absorbed?



$$E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_0 r}$$

$$\frac{e^2}{4\pi\epsilon_0 r^2} = \frac{mv^2}{r}$$

$$\Rightarrow E = \frac{1}{2}mv^2 - mv^2 = -\frac{1}{2}mv^2$$

$$v = \omega r \Rightarrow$$

$$v = 2\pi r \nu$$

assuming this ($f = v b \hbar \nu$)

$$\text{Also, } 0 - E = \frac{n\hbar\nu}{2}$$

After this we Eliminate v, r, ν

Calc:

$$-E = \frac{n\hbar}{2} \times \frac{v}{2\pi r} \Rightarrow \frac{1}{r} = -\frac{4\pi\hbar^2}{n\hbar\nu}$$

$$\frac{1}{r} = \frac{(mv^2) 4\pi \epsilon_0}{e^2}$$

Equating both :-

$$\frac{-4\pi E}{nh^2} = \frac{(mv^2) 4\pi \epsilon_0}{e^2}$$

$$\frac{-4\pi E e^2}{hh} = 4\pi \epsilon_0 m v^3$$

$$\Rightarrow V = - \left[\frac{E e^2}{h h \epsilon_0 m} \right]^{1/3}$$

$$\Rightarrow E = -\frac{1}{2} m \left[\frac{E e^2}{h h \epsilon_0 m} \right]^{2/3} \quad \left(\because E = \frac{1}{2} m v^2 \right)$$

$$\Rightarrow E^{1/3} = -\frac{1}{2} m \left[\frac{e^2}{h h \epsilon_0 m} \right]^{2/3}$$

$$\Rightarrow E = -\frac{1}{8} m^2 \times \frac{e^4}{n^2 h^2 \epsilon_0^2 m^2}$$

$$\Rightarrow E = \frac{-mc^4}{8\epsilon_0^2 h^2 n^2}$$

$$\text{when } n=1 \Rightarrow -13.6 \text{ eV} \quad (\text{min. energy e}^-)$$

$$n=2 = -3.4 \text{ eV}$$

$$n=3 = -1.5 \text{ eV}$$

$$n=4 = -0.85 \text{ eV}$$

De Broglie

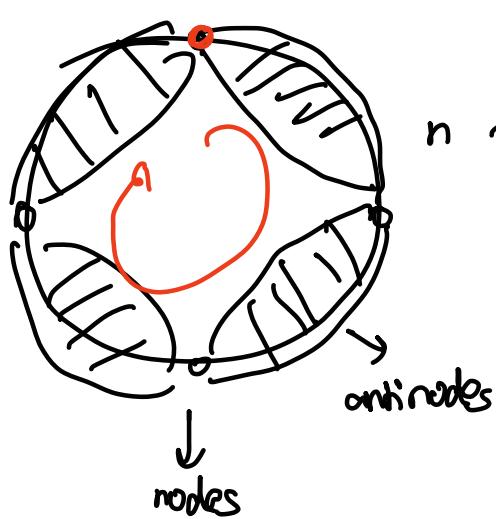
for light, $\lambda = \frac{h}{P}$ → photon momentum
 ↓
 wavelength
 of light

De'Broglie proclaimed the same exp was due for particles
 as well

$$\lambda = \frac{h}{P} = \frac{h}{\frac{m}{\gamma} v} = \frac{h}{mv} \sqrt{1 - \frac{v^2}{c^2}}$$



A standing wave:



Bohr said "I ask why but $mv\lambda = \frac{nh}{2\pi}$,

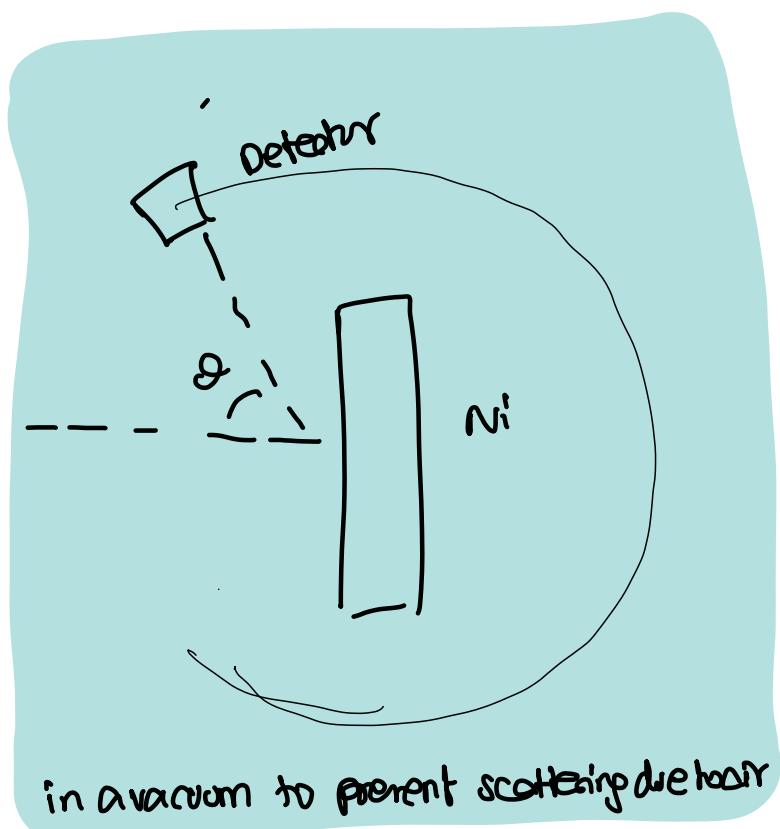
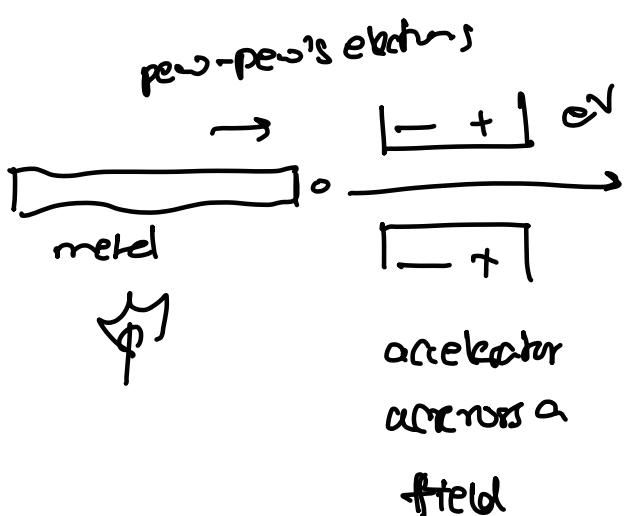
the total red must be a multiple of λ else at the red point, the left-hand limit and right-hand limit won't be inphase

$$n\lambda = 2\pi \Rightarrow n \frac{h}{mv} = 2\pi$$

$$\Rightarrow n \frac{h}{2\pi} = mv$$

#Dorison and Germer -

wanted to see the roughness/wiggles on Ni surface,
they wanted to see this by the diffused reflection of e^-
on Ni-surface. But did that happen?



Remember: Goal was to see किसी angle पर कितने e^- .

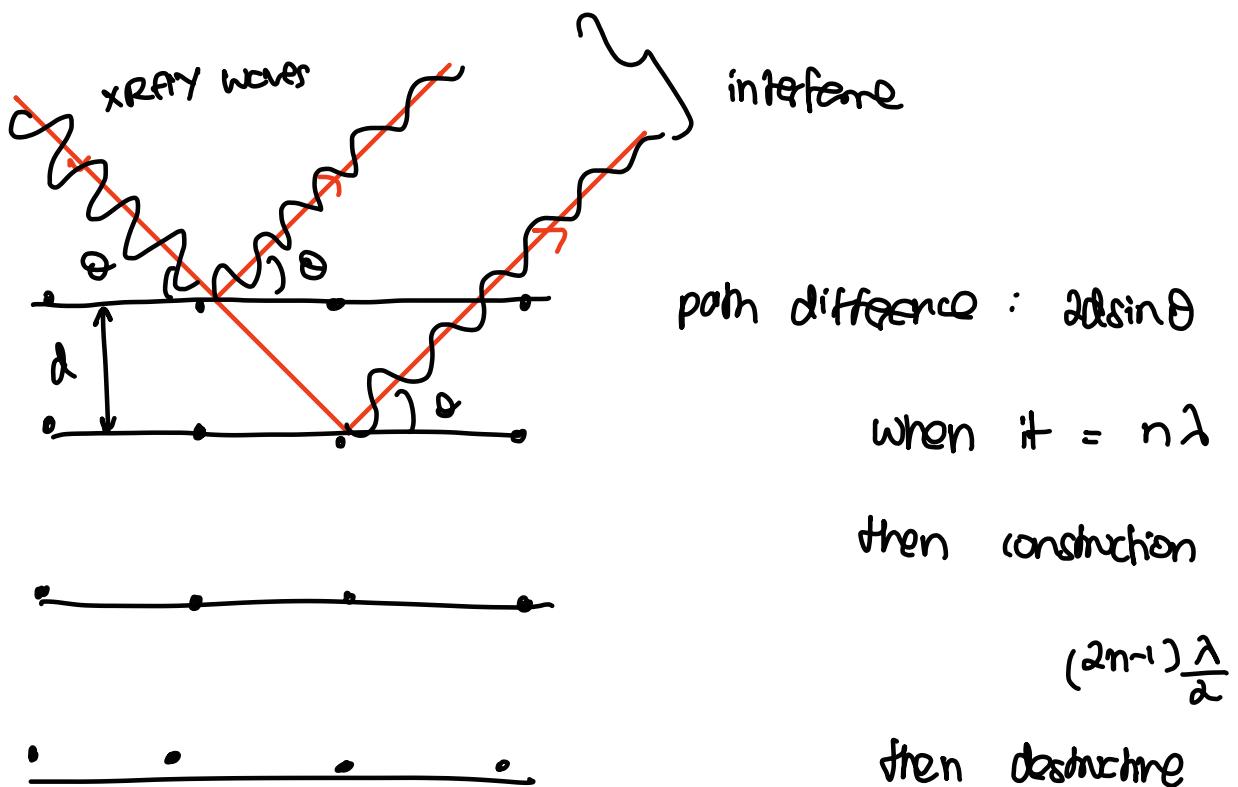
What happened was the vacuum kinda failed, so NiO got created, so to make Ni again they heated $\text{NiO} \rightarrow \text{Ni} + \text{O}_2$

However heating had produced a side-effect! The previously polycrystalline Ni has now aligned into a single crystal structure

When they repeated the experiment again, they ended up seeing peaks!

To understand that, let's first learn X-ray diffraction technique

- Used to understand structure of a material



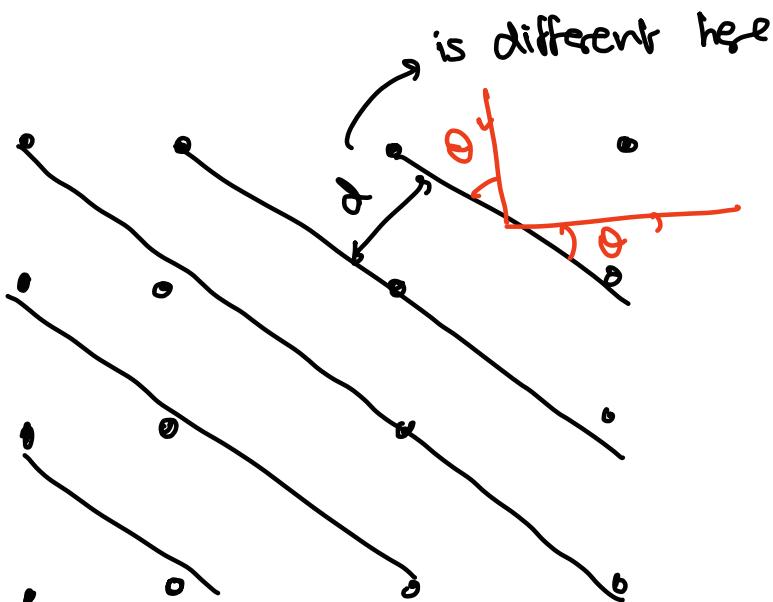
\therefore based on diff. diff &

diff diff intensities

was seen on the detector

If you remember your material sciences planes can be made arbitrarily $[110]$ $[001]$ so on!

so what if we considered some other plane



नहीं कि अस्ति नहीं

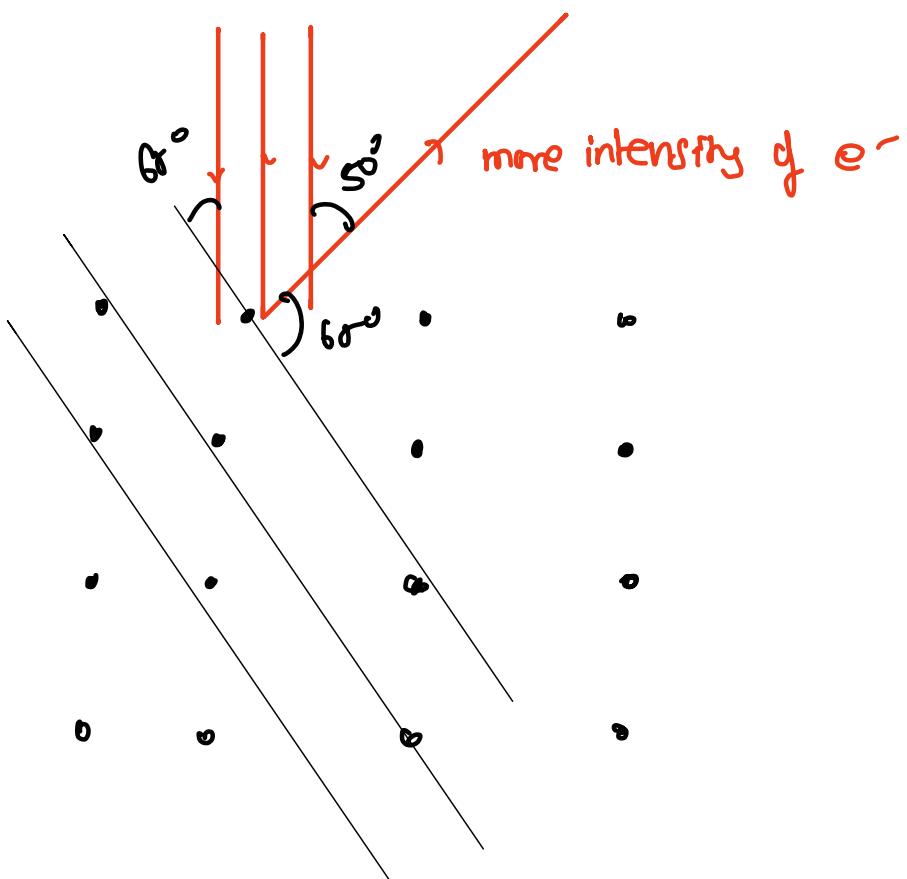
तो यहाँ कि नहीं

so based on (d, θ) pair there could be multiple 'centre lobe peaks' (unlike the usual Young's experiment with only 1 main central band)

With these peaks they

can figure out if it is PCC, BCC or -

In Davisson Germer - the accidentally heated Ni



(TLDR : a wave hits metal scatters in all directions ,

now every family of planes form a path difference

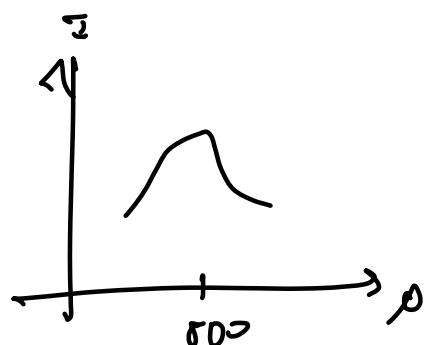
set-up so every family produces a main lobish thingy)

(maybe its like a multi-slit (multi-families) experiment?)

For example, in the Davisson - Germer experiment a

detector at 50° saw a peak at 54° or

(changing K.E = changing p = changing
 λ = changing which angle constructive
might occur)



With X-ray : $d = 0.0909\text{nm}$

Now

$$2ds \sin \theta = \lambda_e$$

65°

$$\lambda_e = 0.165\text{nm} \sim \frac{h}{p} !$$

Wave and wavepackets -

for an electric-field -

$$E = E_0 \cos(kx - \omega t)$$

A physical quantity is changing, with a direction of propagation with a wavelength λ , propagation constant $k = \frac{2\pi}{\lambda}$.

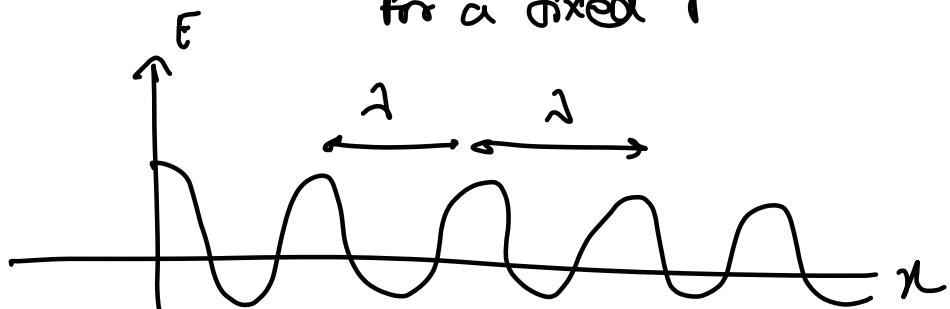
$\vec{k} = \left(\frac{2\pi}{\lambda}\right) \hat{e} \rightarrow$ in the direction of propagation,

propagation vector / wave vector

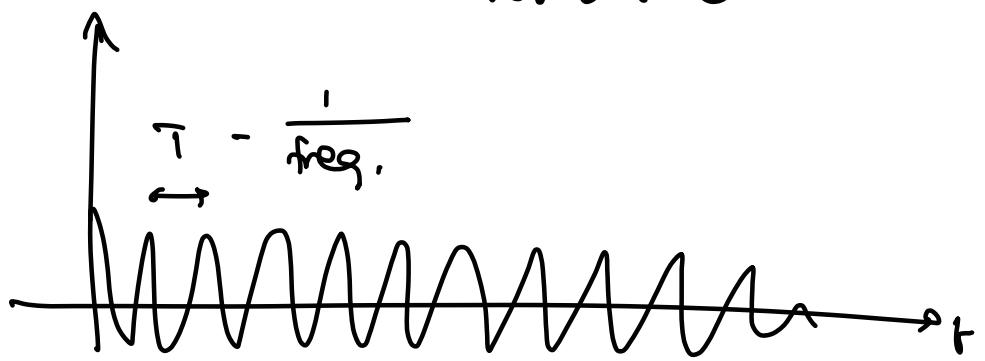
frequency $v = \frac{\omega}{2\pi}$; $\omega = 2\pi v$

Monochromatic
wave

for a fixed t

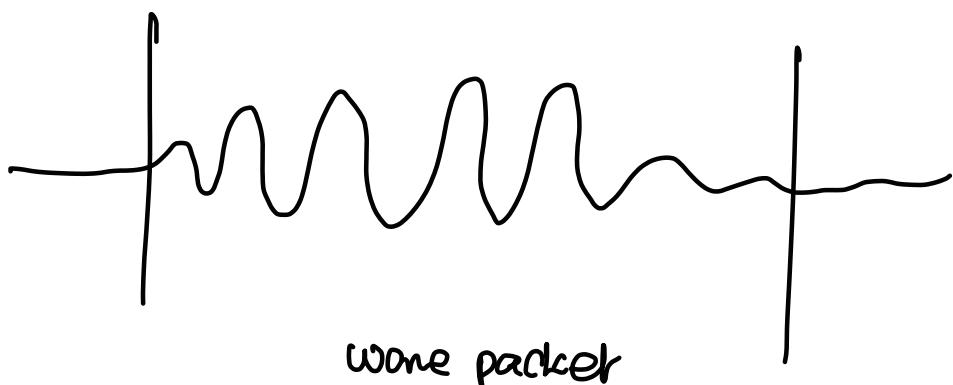


For a fixed λ



Reality -

In reality however, the wave isn't spread across all of space, but is limited \Rightarrow it's a mixture of many wavelengths



\hbar notation > for an EM wave

$$p = \frac{h}{\lambda} = \frac{h}{2\pi} \times \frac{2\pi}{\lambda} = \hbar \times k$$

$$E = h\nu = \frac{h}{2\pi} \times 2\pi\nu = \hbar\omega$$

Speed of a wave

$$\frac{\omega}{k} = c$$

| for a wave

(use your signal's brain)

$$\frac{\hbar\omega}{\hbar k} = c \Rightarrow \frac{E}{P} = c \Rightarrow E = PC \quad (\text{for an EM wave})$$

Description of a wave packet -

instead of mentioning all the components that make up the wave and their amplitudes, we rather describe it using a differential equation - superposition of whose soln can re-create the wave.

- $y = A f(x-vt)$

$$\frac{\partial y}{\partial x} = A \times \frac{\partial f}{\partial(x-vt)} \times \cancel{\frac{\partial(x-vt)}{\partial x}}$$

$$\frac{\partial^2 y}{\partial x^2} = A \left[\frac{\partial^2 f}{\partial(x-vt)^2} \right] \times \cancel{\frac{\partial(x-vt)}{\partial x}}$$

$$\frac{\partial y}{\partial t} = A \frac{\partial f}{\partial(x-vt)} \times \cancel{\frac{\partial(x-vt)}{\partial t}}^{-v}$$

$$\frac{\partial^2 y}{\partial t^2} = A \frac{\partial^2 f}{\partial(x-vt)^2} \times (-v) (-v) = v^2 A \left[\frac{\partial^2 f}{\partial(x-vt)^2} \right]$$

Combining the two equations

$$\frac{1}{A} \frac{\partial^2 y}{\partial x^2} = \frac{1}{A v^2} \frac{\partial^2 y}{\partial t^2} \quad (\text{Provided } v \text{ is a constant})$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Schrödinger

- free particle

Has particulate properties - E, p

Wave properties - k, ω

$$\begin{aligned}\psi &= A e^{j(kx - \omega t)} \\ \Rightarrow \psi &= A e^{j\left(\frac{p}{\hbar}x - \frac{E}{\hbar}t\right)} \\ &= A e^{\frac{j}{\hbar}(px - Et)}\end{aligned}$$

$$\Rightarrow \frac{\partial \psi}{\partial x} = A e^{\frac{j}{\hbar}(px - Et)} \times \frac{j}{\hbar} \frac{\partial (px - Et)}{\partial x}$$

$$= A e^{\frac{j}{\hbar} (px - Et)} \times \frac{j}{\hbar} \times p$$

$$= (\psi) \times \frac{j}{\hbar} \times p \Rightarrow \boxed{-j\hbar \frac{\partial \psi}{\partial x} = px\psi}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = A e^{\frac{j}{\hbar} (px - Et)} \times \left(\frac{j}{\hbar} \times p \right)^2$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -A \times \frac{p^2}{\hbar^2} e^{\frac{j}{\hbar} (px - Et)}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -\frac{p^2}{\hbar^2} \times \psi \Rightarrow \boxed{-\hbar^2 \frac{\partial^2 \psi}{\partial x^2} = p^2 \psi}$$

Now in DMR,

$$\psi = A e^{j \left(\frac{p}{\hbar} x - \frac{E}{\hbar} t \right)}$$

$$\Rightarrow \frac{\partial \psi}{\partial t} = -\psi \times \frac{j \times E}{\hbar}$$

$$\Rightarrow j \times \hbar \frac{\partial \psi}{\partial t} = E \psi$$

- for a free particle $E = K \cdot t^2$ (\because no P.B) $= \frac{P^2}{2m}$
(for non-relativistic speeds)

$$\therefore E\psi = \frac{P^2}{2m} \psi$$

$$\Rightarrow jx \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \quad // \text{differential equation!}$$

with no p.E terms

- for a bounded particle

$$E = \frac{P^2}{2m} + V(x) \quad | \text{ K.E + P.E}$$

$$\Rightarrow E\psi = \left[\frac{P^2}{2m} + V(x) \right] \psi$$

$$\Rightarrow \boxed{j\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi}$$

Schrödinger's wave equation

Reviewing a few prior equations we just derived

$$P\psi = -j\hbar \frac{\partial \psi}{\partial x}$$

for a sec remove ψ from both sides, just don't.

$$p = -j\hbar \frac{\partial}{\partial x}$$

T. Fuck is this, one side has momentum, other side doesn't even make sense $\frac{\partial}{\partial x}$ of what?

it is an operator, it operates on a function and gives new function.

and equality doesn't really make sense

$$p \Rightarrow -j\hbar \frac{\partial}{\partial x} \quad | \quad P = -j\hbar \frac{\partial}{\partial x}$$

(capital p)

operator corresponding
to linear momentum

Similarly

$$\frac{p^2 \psi}{2m} = -\hbar^2 \frac{\partial^2 \psi}{\partial x^2} \times \frac{1}{2m}$$

(magically remove ψ)

$$K.E \Rightarrow \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \text{ an operator}$$

$$\frac{p^2}{2m} \psi + V(x) \psi = j\hbar \frac{\partial \psi}{\partial t}$$

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi = j\hbar \frac{\partial \psi}{\partial t}$$

potential energy an operator, multiplication
is also an operator if takes 1

function, gives another

Potential energy operator is thus $V(x)$.

Position's operator is x , again just a multiplication

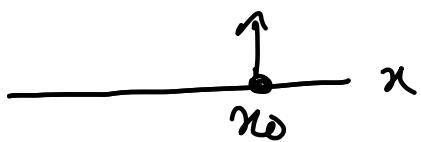
Ex. An observable is given by operator \hat{x} है,

† Heisenberg's uncertainty principle -

ψ can be a complex wave-function as well.

Say a particle is known to be exactly here

$\psi(x)$:



Then it opposes the common sense that the 'wave' exists only at that point and nowhere else. (impulse)

i.e. $\psi(x) = f(x-x_0)$ (∞ at x_0 ; 0 elsewhere)

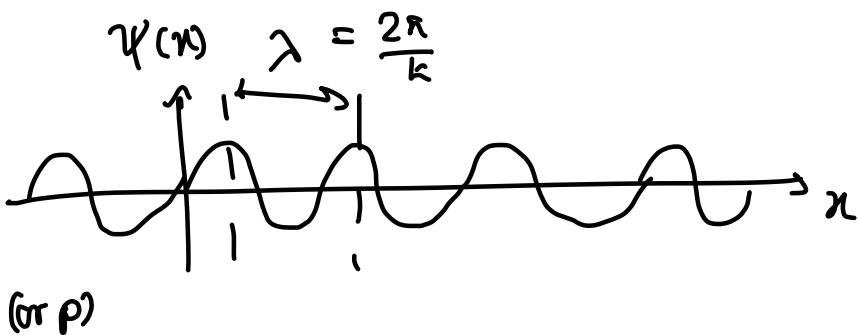
however, just like an impulse $\int \psi(x) dx = 1$

OK, now question, what is its wavelength? ?? It's not periodic

OK then what about $p = \frac{h}{\lambda}$; how shall this make sense to you? λ is gibberish so is p .

let's look at some other functions whose p and hence λ we know

$$\psi(n) = \sin(kn)$$



Okay, we know λ nicely now, but what about its position?

The wave is spread everywhere... so where is our particle?

- ~~But~~ 'p' is definitely known, ~~then~~ x is spread everywhere
(\because wave has equal amplitude everywhere)

- Let us now apply some signals knowledge, we know a f is made up of all possible sinusoids ($x(n)$)
 \therefore when we claim ' x ' is definitely known \Leftrightarrow These are general (∞)

To restate,

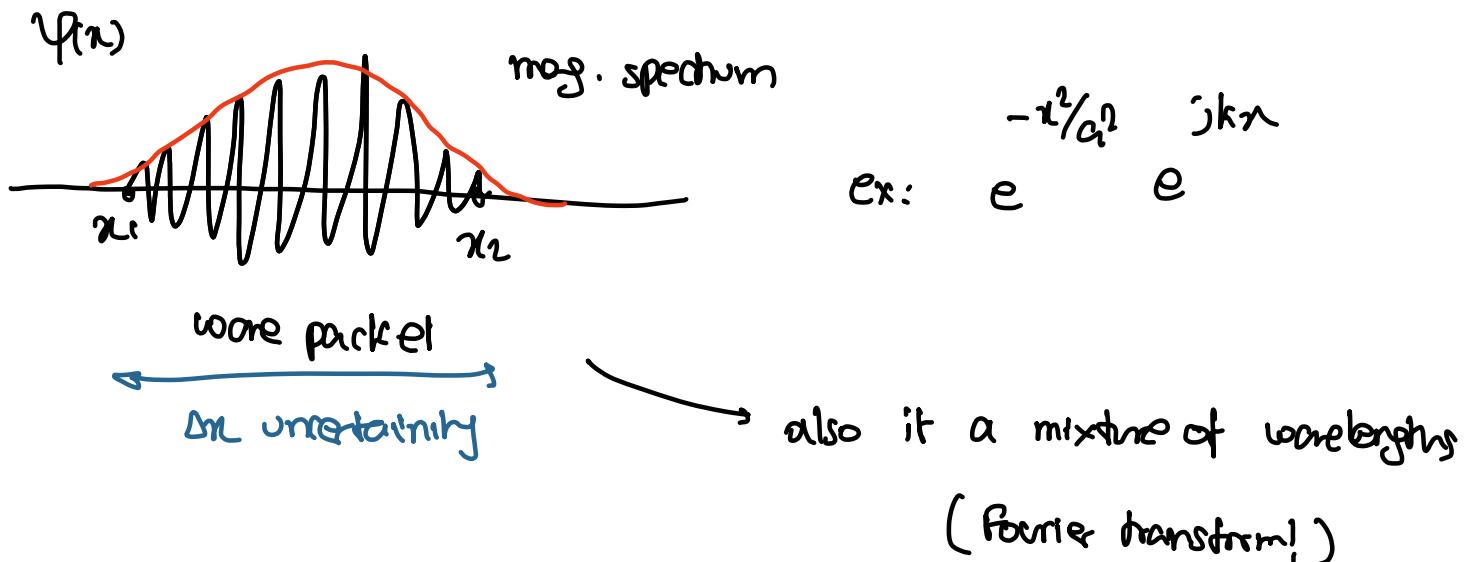
if p is known with certainty (ψ), x is equally likely to be across all of space. \Rightarrow i.e. we don't know n .

If x is known with certainty ($\underline{\psi}$), p is equally likely to be all possible values. \Rightarrow i.e. we don't know p .

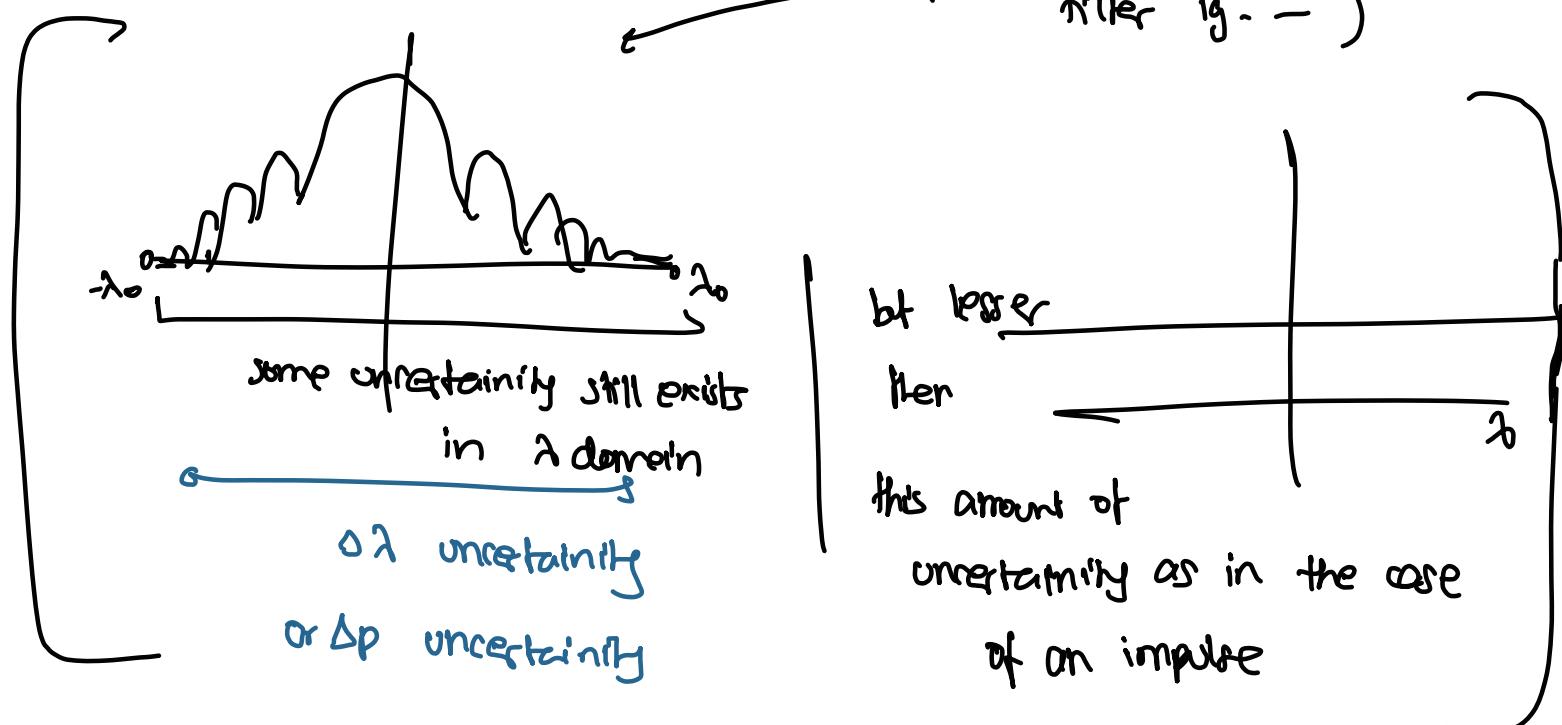
Realistically :-

We know x as a range - it is within this box

p as a range - it is $p \pm 4 \text{ kgm/s}$



(λ should be like a blackman filter if --)



$\therefore \Delta x \propto \frac{1}{\Delta p}$ is (OK OK stay with me so even in a domain is not strictly bounded if slightly looking till ∞ , with lesser

$$(\Delta x)(\Delta p) > \frac{\hbar}{2}$$

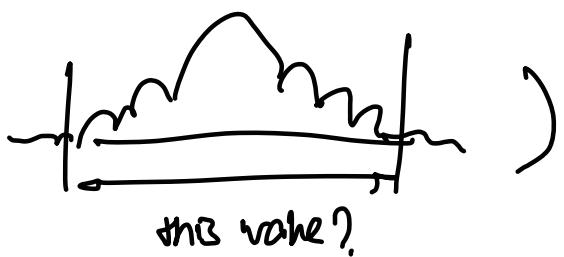
HEISENBERG

UNCERTAINTY

PRINCIPLE

mag of course but yes -'. I think it's justified here we can't use an equality but have to use an inequality to state a lower bound! (THIS IS WHAT I THINK)

(like given a Δx ; $\Delta p >$ then



Max Born

(In a 1-D world)

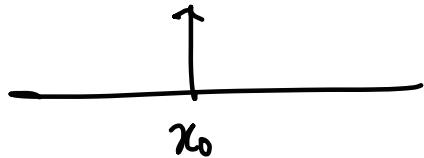
$\Psi(x) \overline{\Psi(x)} dx = \text{Prob of finding } e^- \text{ in } (x - n, x + n)$

(i.e. $f_x(x) = \Psi(x) \overline{\Psi(x)}$)

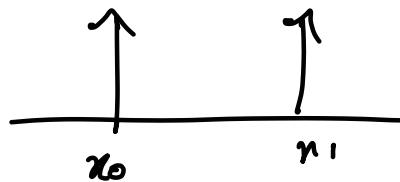
* 2 Scenarios -

- At this instant, particle has a definite position, but we do not know, हमें कैरल उसके probabilities पता कौ?

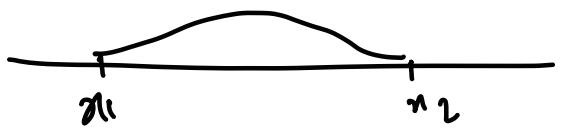
2. At this instant, the particle exists in a mixed state of different positions. — it's not that I do not know, it that it itself isn't in a decided state yet.



pure state,
at an instant of time
it is at x_0
(whether or not we know it)



mixed state,
at an instant of time
it doesn't have a single
position



it's not a matter of
whether we know it
or not it doesn't even
exist in just one position
at an instant.

- Similarly pure state / mixed state for momentum -

which
 in time here
 x is

$$\frac{e^{j k_0 x}}{\sqrt{2\pi\hbar}} = \psi(x)$$

← to normalize such that
it integrates to 1.

$$\psi(n) = \frac{e^{\frac{j p_0}{\hbar} n}}{\sqrt{2\pi\hbar}}$$

$$|\psi(n)|^2 = \psi(n) \overline{\psi(n)} = \left[\frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{j}{\hbar} p_0 n} \right] \left[\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{j}{\hbar} p_0 n} \right]$$

$$|\psi(n)|^2 = \frac{1}{2\pi\hbar} (= f_x(n))$$

\therefore it is a uniform random variable, all n are equally probable.

This is \therefore it is a pure momentum state but a mixed position state

For a realistic wave function -

(i) Should be continuous

(ii) Must be finite everywhere (fig d, is an unrealistic function)

(iii) Should be square integrable

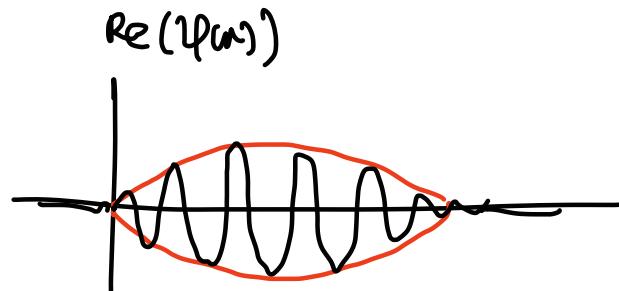
$$\int |\psi_m|^2 dx \text{ must exist}$$

with

$$\int_{-\infty}^{\infty} |\psi_m|^2 dx = 1$$

Example :

$$\psi(x) = e^{-x^2/a^2} e^{jk_0 x}$$



Okay now tell me if λ -transform

or rather p -transform, tell me what all pure momentum states make up this $\psi(x)$.

$$|\psi_0\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{j}{\hbar} p_0 x} \quad (\text{one } p \text{ component of many})$$

(Ket ψ_0)

(p_0 state के corresponding)
wave function

(Inverse F.T.)

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \alpha(p) e^{\frac{j}{\hbar} px} dp$$

$(x(u) = \int n(t) e^{-j\omega t} dt)$

$$\alpha(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{\frac{-j}{\hbar} px} dx \quad (\text{like Fourier transform!})$$

$\frac{i}{\sqrt{2\pi\hbar}} e^{\frac{j}{\hbar} px}$; ~~for normalizing!~~ (ix wrong)

\Rightarrow comes because pure state is

Interestingly, (By Parseval)

probability of lin. momentum in range $p - p + dp$ is

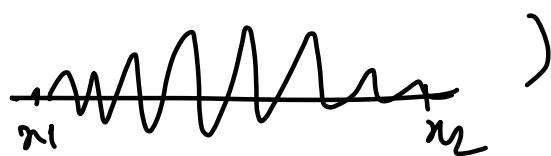
$$|\alpha(p)|^2 dp \quad (\because \int |\psi(n)|^2 dn = \int |\alpha(p)|^2 dp \text{ by Parseval})$$

(This is the reason I think)

Review -

1. The state of the particle at an instant is represented by a wave function.
2. At a given instant, the wave function may or may not be a pure state of an observable quantity say 'A'.

(like this



(instead of classical



3. If the wave function is not a pure state, one can only tell the probabilities of different values of the observable

4. If the wavefunction is

$\psi(n)$ corresponding
to pure state

$$\psi(n) = \sum_i c_i \phi_i(n) \quad (\phi_i(n) = |a\rangle)$$

$$|n\rangle = \delta(n) \quad |p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$$

then probability of
finding the observable a

(pure state of observable
a)

to be a_i is $|c_i|^2$ (Parseval's for Fourier series I think)

{ also PMF so $|c_i|^2$
just one term}

5. If the wavefunction is

$$\psi(n) = \int_{-\infty}^{\infty} c(a) \phi_a(n) da$$

then probability of finding the observable

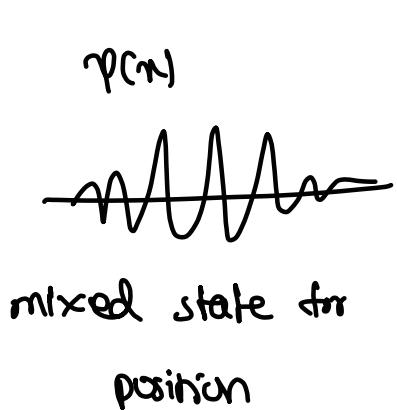
a between a - $a+\Delta a$ is $|c(a)|^2 da$ (Parseval's for f.o.t)
(c : continuous
R.V.)

6. WHAT MEASURING DOES TO A WAVE FUNCTION

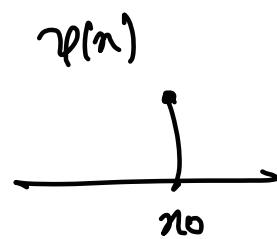
Suppose the wave function is a mixed state of
observable A. Let a precise measurement of A is done
and the value obtained is a. (like n is observable
 n_0 is measured)

Then, the wave function just after the measurement will be the pure state of A corresponding to the value a .

i.e say



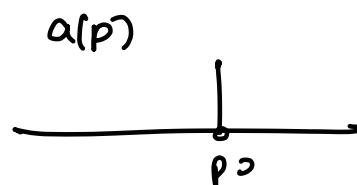
\Rightarrow
Measurement
of position



pure state for position

mixed state for momentum

\Rightarrow
Measurement
of momentum



$$\psi(n) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p_0 x}$$



↳ this is only $\Re\psi(n)$

'cos I can't plot

complex $\psi(n)$

Heisenberg's uncertainty concretised -

$$\text{Var}(x) = E(x^2) - (E(x))^2 \quad (\text{has } m^2 \text{ units})$$

$$\text{S.D.} = \sqrt{\text{Var}(x)} \quad (\text{so we take root})$$

the uncertainty is defined as the standard deviation

$$\Delta p = \sqrt{E(p^2) - (E(p))^2}$$

$$\Delta x = \sqrt{E(x^2) - (E(x))^2}$$

$$E(p^2) = \int p^2 |\alpha(p)|^2 dp$$

$$E(x^2) = \int x^2 |\psi(x)|^2 dx$$

$$E(p) = \int p |p(p)|^2 dp$$

$$E(x) = \int x |\psi(x)|^2 dx$$

if you do math, you will get

$$(\Delta x)(\Delta p) \geq \frac{\hbar}{2}$$

in the same state!

when you collapse $\psi(n) \Rightarrow \alpha(p)$

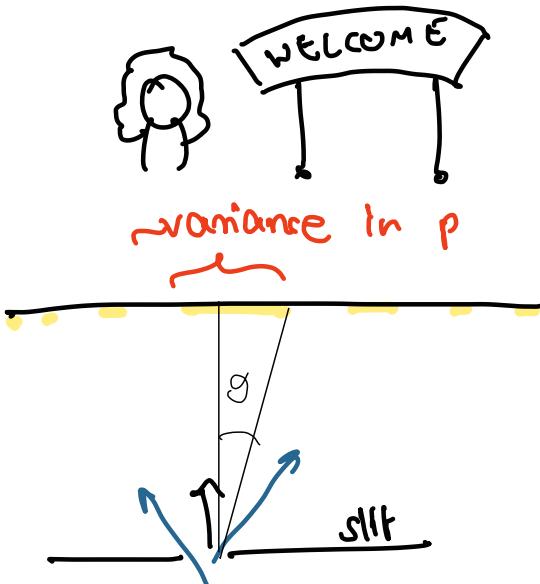
everything you collapse,

but even after collapse; $(\Delta x)(\Delta p) \geq \frac{\hbar}{2}$



@ collapsed state

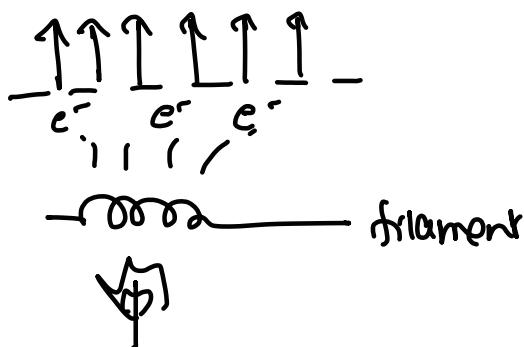
Allow me to indulge you on a thought experiment



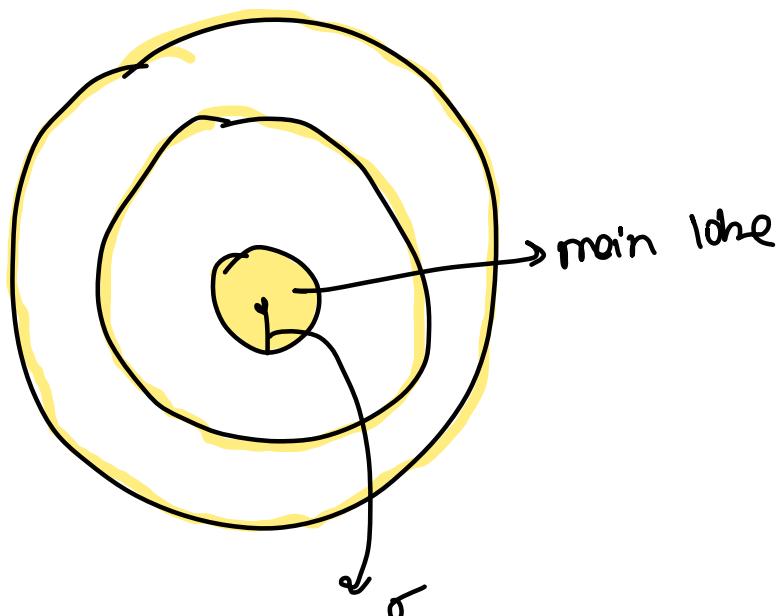
diffraction pattern

$\Delta p \sim 0$ so now what? DIFFRACTION!

$$p \approx 0 \Rightarrow \Delta p \approx 0$$



You shall get an annular ring diffraction pattern!



By diffraction principle,

$$\sin\theta = \frac{1.22\lambda}{d}$$

$d \downarrow$ (small width)

$\sin\theta \uparrow$ (more uncertainty
in p !!!)

(p in a way tells you
direction of e^-)

$$\Delta p \sim p \sin\theta - (-p \sin\theta) = 2p \sin\theta$$

$$O(p \sin\theta) = O\left(p \times \frac{\lambda}{d}\right) = O\left(\frac{h}{\lambda} \times \frac{\lambda}{d}\right)$$

↓

$$(\sin\theta = \frac{1.1222}{d}) = O\left(\frac{h}{d}\right)$$

$\Delta n \sim O(d)$; They are inverted!

II Which $\psi(n)$ has minimum $(\Delta n)(\Delta p) = \frac{\pi}{2}$?

Gaussian!

$$f(n) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{n-\mu}{\sigma}\right)^2}$$

$$\text{Now, } |\psi(n)|^2 = f(n)$$

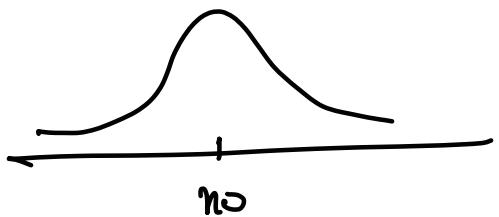
$$\Rightarrow \psi(n) = \sqrt{f(n)} ; \text{ technically this should be } e^{-\frac{1}{2} \left(\frac{n-\mu}{\sigma}\right)^2}$$

case but still defined gaussian a little weirds

instead of half; so -

$$\psi(n) = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-\frac{(n-n_0)^2}{a^2}} e^{\frac{j}{\hbar}pn}$$

$$|\psi(x)|^2 = A^2 e^{-2\left(\frac{x-x_0}{a}\right)^2}$$



$$E(x) = \mu = x_0$$

$$\text{Var}(x) = \sigma^2 = \left(\frac{a}{2}\right)^2 ; \text{ S.D} = \frac{a}{2}$$

it because of

this factor everything is screwed up

otherwise you can just do $\sqrt{\text{fcn}}$

you will get $\left(\frac{\hbar}{a}\right)^2$ if find it by transform and $= \sqrt{\text{fcn}}.$

$$\text{find } \text{Var}(p) = \frac{\hbar^2}{a^2} \quad | \quad \therefore \text{S.D} = \frac{\hbar}{a}$$

$$\therefore \Delta x \times \Delta p = \frac{\hbar}{a} \times \frac{a}{2} = \frac{\hbar}{2} !$$

More broadly ; $f(n) = |\text{f}(n)| e^{jn\phi}$
 a phase shift
 is also allowed Gaussian linear phase in n

(\because variance independent
 of time (here x) shift?)

II State-space

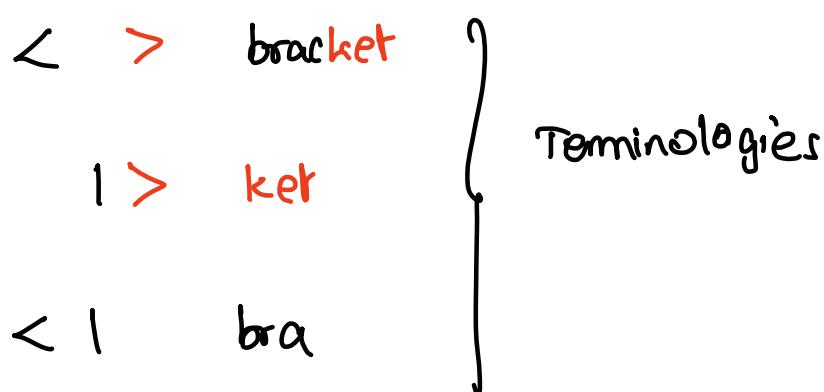
<u>state</u>	<u>State space</u>	
represented with $\Psi(n)$	represented with : all possible states set of $\Psi(n)$	for the particle
	$\Psi_1(n), \Psi_2(n) \dots$	(all possible mixed states I think, 1 mixed state 1 $\Psi(n)$)
$\{\Psi_1(n), \Psi_2(n), \Psi_3(n) \dots\}$		
makes up a vector-space apparently		

★ Final answer:

The set of all wavefunctions (square-integrable complex functions) forms a vector space because:

- "experimentally, quantum states obey superposition,"
- "mathematically, square-integrable functions automatically satisfy vector space axioms,"
- "quantum evolution (Schrödinger equation) is linear,"
- "probabilities depend on inner products,"
- "interference requires vector addition."

So the structure of quantum mechanics *forces* wavefunctions to live in a complex vector space (a Hilbert space).



Unit vector analogue - Normalised wave function

those $\psi(n)$ for whom $\text{norm} = 1$; $\|\psi_i\| = \sqrt{\langle \psi_i | \psi_i \rangle}$

$$\|\psi\| = \left(\int_{-\infty}^{\infty} \psi(x) \overline{\psi(x)} dx \right)^{1/2} = 1$$

More strictly a dot product is not a part of vector space.

Vector space + dot product is a Hilbert space

$\{\psi(n)\}$ forms a Hilbert space.

- The dot product of $\psi_1(x)$ and $\psi_2(x)$ is defined as

$$\int_{-\infty}^{\infty} \overline{\psi_1(x)} \psi_2(x) dx \quad \left| \begin{array}{l} \text{Take conjugate of the} \\ \text{first term! Do not forget} \end{array} \right.$$

$$\langle \psi_1 | \psi_2 \rangle$$

NOTE : $\langle \psi_1 | \psi_2 \rangle = \overline{\langle \psi_2 | \psi_1 \rangle}$ \therefore it has conjugate symmetry

$$\Rightarrow \langle \psi_1 | \psi_1 \rangle = \int_{-\infty}^{\infty} |\psi_1|^2 dx$$

but is NOT commutative!

- If $\langle \psi_1 | \psi_2 \rangle = 0$ then ψ_1 and ψ_2 are orthogonal to each other
- Basis is a "thing! That set of basis in which $\|b_i\| = 1$ and also $\langle b_i | b_j \rangle = 0$ is called an orthonormal basis set

$$|\psi\rangle = \sum_i c_i |\phi_i\rangle ; \text{ where } |\phi_i\rangle \text{ are basis elements}$$

if ϕ_i are an orthonormal set, then

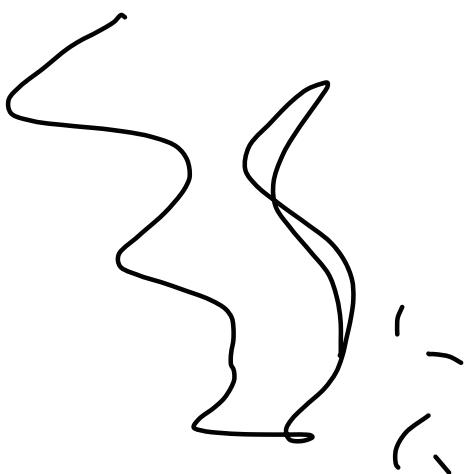
$$c_i = \langle \phi_i | \psi \rangle$$

$$\begin{array}{c} \psi(n) \\ \downarrow \quad \nearrow \\ \dots \end{array}$$

If Pure states as a basis-

$$1. |p_0\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{j}{\hbar} p_0 x}$$

$$\psi(n) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} a(p_0) e^{\frac{j}{\hbar} p_0 x} dp_0$$



$$\text{we know } c_i = \langle \phi_i | \psi \rangle$$

$$\therefore a(p_0) = \langle p_0 | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{j}{\hbar} p_0 x} \psi(n) dx = 0$$

Some remarks tho:

) (then how is this allowed?)

1. $|p_0\rangle$ is not normalised

$$\langle p_0 | p_0 \rangle = \int_{-\infty}^{\infty} \frac{1}{2\pi\hbar} \times 1 \times dx = \infty \quad \vdots$$

2. $|p_1\rangle, |p_2\rangle$ are orthogonal

$$\langle p_1 | p_2 \rangle = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\hbar} \right) e^{\frac{-i}{\hbar} p_1 x} \times e^{\frac{i}{\hbar} p_2 x} dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\hbar} \right) e^{\frac{i}{\hbar} (p_2 - p_1)x} dx$$

$$= \frac{1}{2\pi\hbar} \times 2\pi\hbar \delta(p_2 - p_1) = \delta(p_2 - p_1)$$

($\neq 0$ for all values
where $p_2 \neq p_1$)

2. $|x_0\rangle$ as a basis = $\delta(x - x_0)$

$$\psi(x) = \int \psi(l) \delta(x - l) dl \quad (\text{looks like convolution})$$

$$\psi(x_0) = \langle x_0 | \psi \rangle = \int \delta(x - x_0) \psi(x) dx = \psi(x_0)$$

Operators

$$1. \Psi(x) = |p_0\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{j\frac{p_0}{\hbar}x}$$

$$\frac{\partial \psi}{\partial x} = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{j}{\hbar} p_0 \right) e^{j\frac{p_0}{\hbar}x}$$

$$-j\hbar \frac{\partial \psi}{\partial x} = \frac{1}{\sqrt{2\pi\hbar}} p_0 e^{j\frac{p_0}{\hbar}x}$$

$$\Rightarrow -j\hbar \frac{\partial \psi}{\partial x} = p_0 \psi \quad | \quad \boxed{(-j\hbar \frac{\partial}{\partial x}) \psi = p_0 \psi}$$

$$\boxed{-j\hbar \frac{\partial}{\partial x} \leq p_0 \text{ operator}}$$

$$IP = -j\hbar \frac{\partial}{\partial x}$$

$$P\psi = p_0 \psi$$

2. Position operator -

\times = multiply by n

$$\hat{X} \delta(x - x_0) = x \delta(x - x_0)$$

↓ ↓

operator ψ

$$= x_0 \delta(x - x_0)$$

↓ ↓

x_0 observable value ψ

$$3. \ K.E = \frac{P^2}{2m} \quad \therefore \text{it is } P \text{ applied twice} \times \frac{1}{2m}$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \quad (\text{I mean you makes sense})$$

4. Potential energy -

r. Total energy -

Multiply by \sqrt{n}

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \Psi = (E - V(x)) \Psi$$

3. If ψ is operator on pure state & $\psi \propto$ actual value
 ↓
 3. If ψ is operator on pure state & $\psi \propto$ actual value
 ↓
 same function

Eigenvalues and eigen functions

- $\hat{X}\psi = x_0\psi$ i.e. operation on ψ produces a scalar multiple of same ψ

Equations like these are

called eigenvalue equations

that ψ is called the eigen functions of that particular operator ($\hat{f}(x-\omega)$)

x_0 is the corresponding eigen value

- $\hat{P}\psi = p_0\psi$

$$\psi = \frac{1}{\sqrt{2\pi\hbar}} e^{i \frac{p_0}{\hbar} x} \quad \leftarrow \text{eigen function}$$

p_0 ← eigen value
corresponding

- A set of all the eigenfunctions for an operator form a

basis for the state space. *

Ex: $\Psi = \int \psi(x_0) \delta(x - x_0) dx_0$ Basis 1

or

$$\Psi = \int a(p_0) |p_0\rangle dp_0 \quad \text{Basis 2}$$

or

$$\Psi = \int (\text{function of observable}) (\text{eigenfunction of observable corresponding to shot eigenvalue})$$

Algebra of operators -

1. $(A + B)\Psi = A\Psi + B\Psi$

2. $(AB)\Psi = A(B\Psi) \neq B(A\Psi)$ (not always)

(read as B of Ψ then A of that)

Example -

$$\begin{aligned} \bullet (ax + bP)\Psi &= (ax)\Psi + (bP)\Psi \\ &= ax\Psi + bP\Psi \end{aligned}$$

$$\bullet (XP)\Psi = x \left(-j\hbar \frac{\partial \Psi}{\partial x} \right)$$

$$= -j\hbar x \frac{\partial \Psi}{\partial x}$$

$$\bullet (Px)\Psi = -j\hbar \frac{\partial(x\Psi)}{\partial x} = -j\hbar \left\{ x \frac{\partial \Psi}{\partial x} + \Psi \right\}$$

i.e. $(XP)\Psi \neq (Px)\Psi$ (Not commutative)

XP do not commute with each other

* New operator

$$(XP - Px)\Psi = +j\hbar \Psi$$

i.e., doing $(XP - Px)$ on Ψ is equivalent to multiplication

by $j\hbar$

$$(XP - Px) \equiv [x, p] \quad \text{Commutator of } x \text{ with } p$$

$$[P, x] = -[x, P] \quad (\text{order matters})$$

Non commutators are those observables whom you can't definitely know together at the same instant.

further if $[A, B] = j\hbar$ then $\Delta a \Delta b \geq \frac{\hbar}{2}$

Example -

p - momentum K - Kinetic energy

$$-\mathrm{j}\hbar\frac{\partial}{\partial x} \quad \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

∴ If

$$\Psi = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{j}{\hbar} p_0 x} ; \text{ this } \therefore \text{ is an eigenfunction}$$

of both momentum and k_0

(knowing one is knowing the other)

Non-degenerate eigenvalues

n-degenerate eigenvalues

When one eigen value corresponds to 1 independent eigenfunctions (upto scaling) और कोई नहीं।

Dably-degenerate / 2-fold degenerate eigenvalues

When one eigen value corresponds to 2 linearly independent eigenfunctions

$$\text{ex } \underset{(\text{K.E})}{\text{IK}} (\sin kx) = \frac{\hbar^2 k^2}{2m} (\sin kx)$$

$$\underset{(\text{K.E})}{\text{IK}} (\cos kx) = \frac{\hbar^2 k^2}{2m} (\cos kx)$$

Where $\sin kx$, $\cos kx$ are linearly independent eigenfunctions corresponding to the same eigenvalue.

$$\text{i.e. } A |\psi_1\rangle = a |\psi_1\rangle \quad \text{with } \psi_1, \psi_2 \text{ independent}$$
$$A |\psi_2\rangle = a |\psi_2\rangle$$

* Similarly you can have n-fold degenerate

Hermitian Operator

A operator, $|\psi_1\rangle$, $|\psi_2\rangle$

Consider the scalar product of $|\psi_1\rangle$ with $A|\psi_2\rangle$

$$\langle \psi_1 | A\psi_2 \rangle \quad \text{and also} \quad \langle A\psi_1 | \psi_2 \rangle$$

i.e. $\overbrace{A, \psi_1, \psi_2}$ and $\overbrace{A, \overbrace{\psi_1}, \psi_2}$

$$\text{If } \langle \psi_1 | A\psi_2 \rangle = \langle A\psi_1 | \psi_2 \rangle \neq \langle \psi_1 | \psi_2 \rangle$$

Then A is called hermitian operator

$$\text{Also } \langle \psi_1 | A\psi_2 \rangle = \langle A\psi_1 | \psi_2 \rangle = \langle \psi_1 | A^\dagger \psi_2 \rangle$$

All operators corresponding to an observable are necessarily hermitian

* Hermitian adjoint

$$\text{If } \langle \psi_1 | A\psi_2 \rangle \neq \langle A\psi_1 | \psi_2 \rangle$$

$$\text{but } \overbrace{\langle \psi_1 | A\psi_2 \rangle} = \langle B\psi_1 | \psi_2 \rangle$$

then B is the hermitian adjoint of A

$$B = A^\dagger \quad (\text{A dagger})$$

by maths $A = B^\dagger$ (i.e. $\langle \psi_1 | B \psi_2 \rangle = \langle A \psi_1 | \psi_2 \rangle$
 if $\langle \phi_1 | A \phi_2 \rangle = \langle B \phi_1 | \phi_2 \rangle$
 $B = A^\dagger$)

• further,

$$1. \quad (A + B)^\dagger = A^\dagger + B^\dagger$$

$$2. \quad (AB)^\dagger = B^\dagger A^\dagger$$

Expectation of an observable

$$\langle \psi | A | \psi \rangle \quad (\text{assuming discrete})$$

Now,

$$A|\psi\rangle = A \times \sum_i c_i |\phi_i\rangle \quad (\text{all possible eigenfunctions of } A)$$

$$= \sum_i c_i \times A |\phi_i\rangle$$

$$= \sum_i c_i a_i |\phi_i\rangle \quad (\because \text{eigenfunction})$$

$$\text{Now } \langle \psi | A | \psi \rangle$$

$$= \sum_i c_i a_i \langle \psi_i | \phi_i \rangle$$

$$= \sum_i c_i q_i \langle \phi_i | \psi_i \rangle$$

$$= \sum_i c_i q_i \bar{x_i} \Rightarrow \sum_i c_i |c_i|^2$$

$$\Rightarrow \text{Exp}(a)$$

Quick reminder

$$(Aa)^2 = E(a^2) - (E(a))^2 \quad (\text{uncertainty is just the standard deviation})$$

* Schrödinger's equation , with time included

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + v(x) \right] \psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t)$$



Total energy operator

(Hamiltonian)

$$H \psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t)$$

Consider the eigenvalue equation -

$\kappa \phi(\kappa) = E \phi(\kappa)$; for all possible E you find

{ $\phi(\kappa)$ } \rightarrow a really good

basis

If the basis turns out to be
discrete -

$$|\Phi\rangle, |\Phi_2\rangle, \dots \text{ (pure states of energy)}$$

$$|\Psi\rangle = \sum_i c_i |\Phi_i\rangle$$

Now what is thus $|\Psi(t)\rangle$?

$$|\Psi(t)\rangle = \sum_i c_i |\Phi_i\rangle \xrightarrow{\text{this is the time part}}$$

$$\Rightarrow H|\Psi(t)\rangle = \sum_i c_i H|\Phi_i\rangle$$

$$= \sum_i c_i E_i |\Phi_i\rangle \quad (\because |\Phi_i\rangle \text{ are pure states of energy})$$

from Schrödinger's equation:

$$H|\Psi\rangle = j\hbar \frac{\partial}{\partial t} |\Psi\rangle$$

$$\sum_i c_i E_i |\Phi_i\rangle = j\hbar \frac{\partial}{\partial t} \left[\sum_i c_i |\Phi_i\rangle \right]$$

$$= j\hbar \left[\sum_i \left(\frac{\partial c_i}{\partial t} \right) |\Phi_i\rangle \right]$$

$\frac{\partial c_i}{\partial t}$ $\because c_i$ is not a function of space

if $\alpha \neq 0$.

$$\text{ie } c_1 E_1 |\psi_1\rangle + c_2 E_2 |\psi_2\rangle - -$$

$$= j\hbar \frac{d}{dt} c_1 |\psi_1\rangle + j\hbar \frac{d}{dt} c_2 |\psi_2\rangle - -$$

$$\therefore c_1 E_1 = j\hbar \frac{d(c_1)}{dt}$$

$$c_2 E_2 = j\hbar \frac{d(c_2)}{dt}$$

⋮

⋮

for each i ,

$$c_i E_i = j\hbar \frac{d(c_i)}{dt}$$

$$\Rightarrow \frac{1}{j\hbar} \int_0^t E_i dt = \int \frac{dc_i}{c_i}$$

$$\Rightarrow \frac{E_i t}{j\hbar} = \ln c_i \Big| \frac{c_i(t)}{c_i(0)}$$

$$\Rightarrow -j \frac{E_i}{\hbar} = \left[\ln c_i \right]_{c_i(0)}^{c_i(t)}$$

$$\Rightarrow c_i(t) = c_i(0) e^{-\frac{j}{\hbar} E_i t}$$

$$\text{Now } \sim : |\psi(t)\rangle = \sum c_i |\psi_i\rangle$$

$$|\psi(t)\rangle = \sum c_i(0) e^{-\frac{j}{\hbar} E_i t} |\psi_i\rangle$$

\therefore if you knew $|\psi\rangle$ at any one time, it's easy to predict its entire behavior with time

Quick recap

$$H|\psi\rangle = E|\psi\rangle$$

$$H\phi(n) = E\phi(n)$$

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \phi(n) = E \phi(n)$$

$\{\phi_i(x)\}$ → orthonormal basis

$$\Psi(x, 0) = \sum c_i(0) \phi_i(x) \quad \text{or} \quad \int G(0) \phi_\lambda(x) d\lambda$$

$$\Psi(x, t) = \sum c_i(0) e^{-\frac{-i}{\hbar} E_i t} \phi_i(x) \quad | E_i \text{ is the eigenvalue corresponding to the eigenfunction } \phi_i(x)$$

or

$$\int C(\lambda, 0) e^{-\frac{-i}{\hbar} E(\lambda) t} \phi(\lambda, x) d\lambda$$

* THE REAL QUANTUM MECHANICS

* free-particle

Step 1: \hat{H} - ?

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + 0$$

Step 2: find the eigenfunctions

$$H|\psi\rangle = E|\psi\rangle$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}(\psi(x)) = E \psi(x) \rightarrow \textcircled{1}$$

$$\textcircled{2} \quad \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2} \psi = 0$$

$$\textcircled{3} \quad \psi'' + k^2\psi = 0 \quad \text{where } k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\textcircled{4} \quad \psi'' = -k^2\psi$$

$$jkx$$

$$\text{you will get } \psi(x) = e^{jkx}$$

$$\phi_p(x) = \left(e^{j\frac{p}{\hbar}x} \right) \frac{1}{\sqrt{2\pi/\hbar}}$$

for all p_s it will satisfy!

You can verify, if you put it in ①, you will get

$$\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} (\Phi_{p_0}(x)) = \frac{p_0^2}{2m} \Phi_{p_0}(x)$$

Now for time,

$$\Psi(x, 0) = \int \alpha(p, 0) \frac{e^{i \frac{p}{\hbar} x}}{\sqrt{2\pi\hbar}} dp$$

where $\alpha(p, 0) = \int \chi_p(x_0) \frac{1}{\sqrt{2\pi\hbar}} e^{-i \frac{p}{\hbar} x_0} dx$

$$\Psi(x, t) = \int \alpha(p, 0) e^{i \frac{j}{\hbar} \frac{p^2}{2m} t} \frac{e^{i \frac{p}{\hbar} x}}{\sqrt{2\pi\hbar}} dp$$

$$= \int \underbrace{\alpha(p, 0)}_{\sqrt{2\pi\hbar}} e^{i \frac{j}{\hbar} \left(px - \frac{p^2}{2m} t \right)} dp$$

$$j(kx - \omega t)$$

e

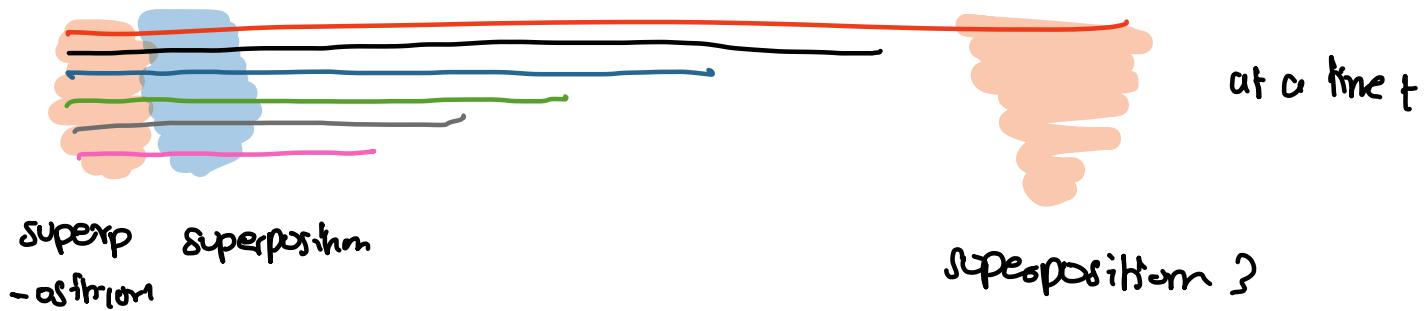
$$k = \frac{p}{\hbar}$$

$$\frac{\omega}{k} = \frac{p^2}{2m\hbar} \times \frac{\pi}{p} = \frac{p}{2m}$$

$$\omega = \frac{E}{\hbar}$$

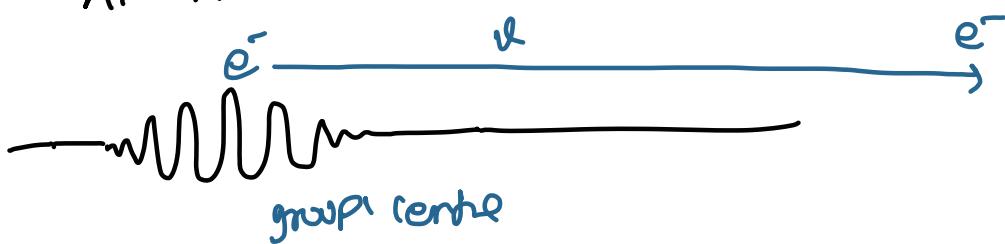
for each p component

$\Psi(k, t)$ is the superposition of all those waves
each with different speeds

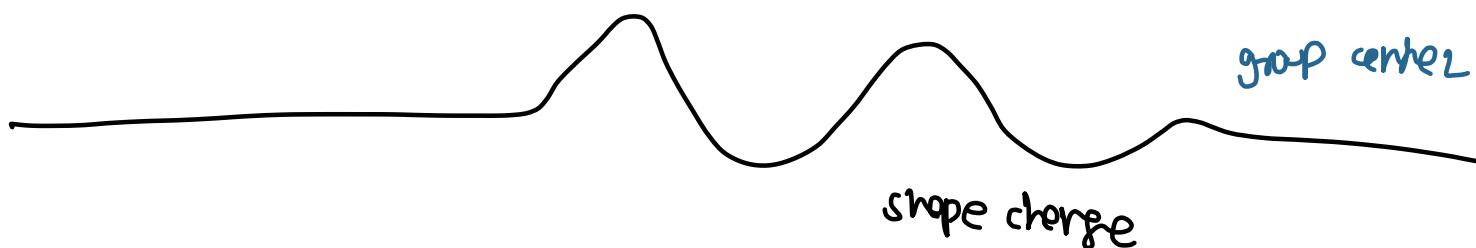


The components of superposition
or shape of Ψ thus changes
as it travels forward.

At t_1



At t_2



$$v = \text{group's velocity} = \frac{d\omega}{dk} = \frac{\frac{d\omega}{dp}}{\frac{\partial k}{\partial p}} = \frac{p}{m} \quad (\frac{\hbar}{m} = v)$$

* रीचन गत

$$-\frac{j}{\hbar} \frac{p^2}{2m} t$$

$$\alpha(p,t) = \alpha(p,0) e$$

$$(|\alpha(p,t)|)^2 = (|\alpha(p,0)|)^2$$

so the probability of momentum being in the range $p-dp$ at any time t is the same!

(because if you remember the derivation was done for

a state $|\psi(t)\rangle$

not $\psi(x,t)$

could also

be $\psi(p,t)$ which here is $\alpha(p,t)$)

linear momentum's probability is thus a constant of motion.

* What if my $\psi(x)$ is a pure state of Hamiltonian at some point in time?

$$\psi(x,0) = \sum c_i \phi_i(x)$$

$$= (x | \Phi_k(x) + 0 - - -$$

$\Psi(x,t)$ from Schrödinger's equation is thus,

$$\Psi(x,t) = \Psi(x,0) e^{-j\frac{E_k}{\hbar}t} \left(\Phi_k(x) e^{-j\frac{E_k}{\hbar}t} \right)$$

$$H \Psi(x,t) = e^{-j\frac{E_k}{\hbar}t} H \Psi(x,0)$$

$$= e^{-j\frac{E_k}{\hbar}t} E_k \Psi(x,0) \quad \left(\because \Psi(x,0) = \Phi_k(x) \right)$$

$\Rightarrow H \Psi(x,t) = E_k \Psi(x,t) \rightarrow$ which happens
to mean $\Psi(x,t)$
i.e if $\Psi(x,0)$ is an eigenfunction
then $\Psi(x,t)$ will always remain an eigenfunction of H
corresponding to that eigenvalue

\therefore This state has a fixed definite (only 1) and constant.

Let us continue to work with this state, consider an observable A ;

$$\begin{aligned} \text{Exp}(A) &= \langle \Psi | A | \Psi \rangle = \int \bar{\Psi}(x,t) A \Psi(x,t) dx \\ &= \int \bar{\Psi}(x,t) A \left(e^{-j\frac{E_k}{\hbar}t} \times \Psi(x,0) \right) dx \end{aligned}$$

$$= \int \overline{\psi(x,t)} e^{-j\frac{E}{\hbar}t} A(\psi(x,0)) dx$$

$$= \int e^{j\frac{E}{\hbar}t} \times \overline{\psi(x,0)} A(\psi(x,0)) \times e^{-j\frac{E}{\hbar}t} dx$$

$$= \int \overline{\psi(x,0)} A(\psi(x,0)) dx ; \text{ which is time-independent}$$

$\therefore \text{Exp(any observable)}$

remains independent of
time.

- 1. E_i is const.
- 2. $\text{Exp(any observable)}$ is a const.
- 3. Even the probability of each particle stays the same

Even though the wave function is changing with time.

↓
∴ $|\psi(x,t)|^2$ is a const.

↓
only phase of $\psi(x,t)$ is changing with time
(like a normal wave \Rightarrow)

* Such states (where the distribution remains same with time
not the wavefunction itself)

are called stationary states

- all observables remain same (like an e^- in a hydrogen atom :))

* Side note, the eigenvalue equation -

$H\psi = E\psi$ is sometimes also called
time independent

Schrödinger's equation.

* Time-independent Schrödinger's equation -

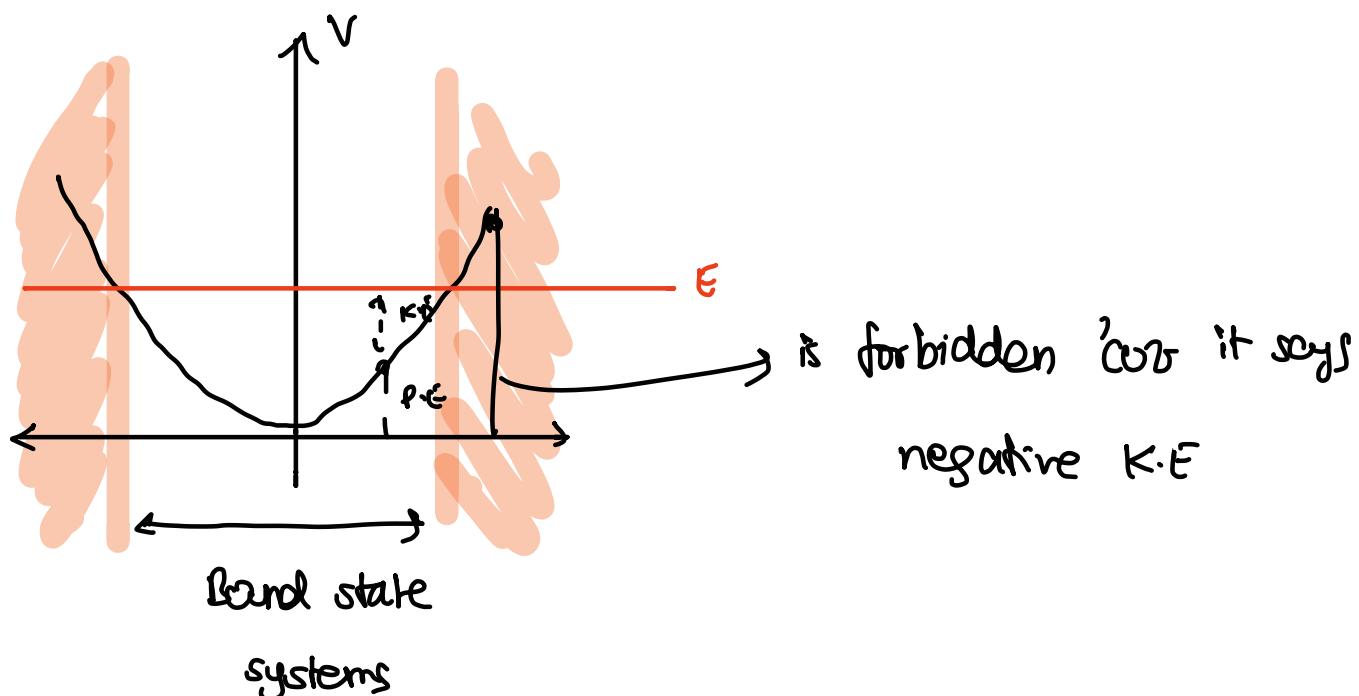
$$H\psi = E\psi$$

- When 10M people say potential, they mean potential energy

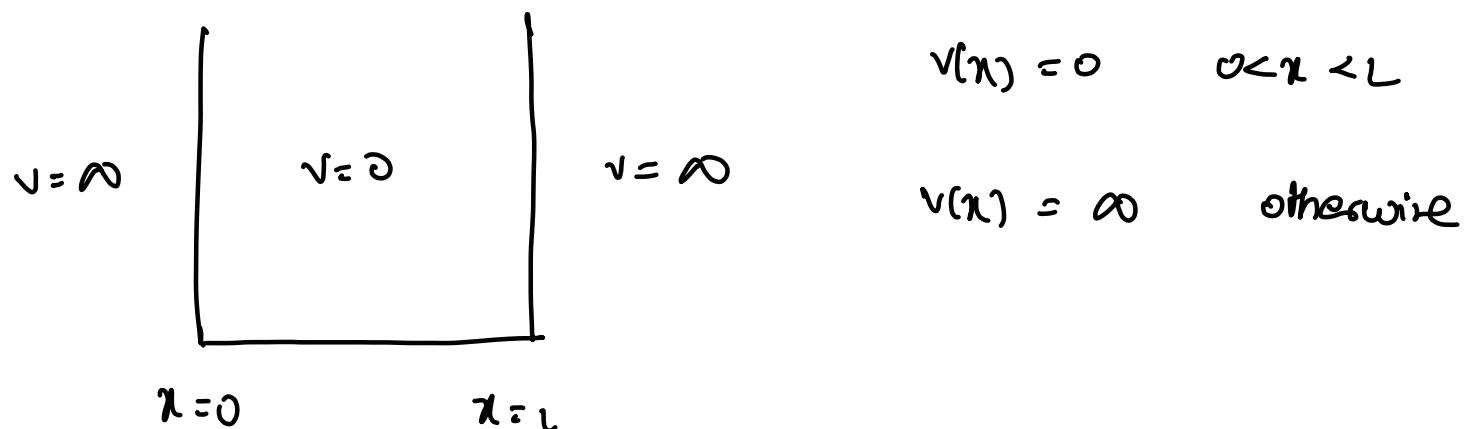
(this is more like part potential

energy if you remember Balaji sir, itz just how
much energy it takes do bring only THIS particular
charge from ∞ , not everyone)

* forbidden regions for bound states -



* 1D box - deep square well potential



Reminder : A wave function must be , cont'n [otherwise Schrö won't]

- finite (impulse ok :))
- square integrable \rightarrow else prob. won't make sense

Independent time :-

$$\mu |\psi\rangle = E |\psi\rangle$$

$$\left[-\frac{\partial^2}{\partial x^2} \frac{n^2 \hbar^2}{2m} + V(x) \right] \phi(x) = E \phi(x)$$

if $V(x) = \infty$ (outside D_L) $\phi(x)$ has to be zero

\because RHS is finite, LHS must be finite $\Rightarrow \phi(x) = 0$

$$x < 0, x > L$$

$$\Rightarrow \text{all } \phi_k(x) = 0$$

$$\Rightarrow \psi(x) = 0$$

$$\Rightarrow \rho(e^-) = 0$$

$$0 < x < L$$

$$\left(-\frac{\partial^2}{\partial x^2} \frac{\hbar^2}{2m} + 0 \right) \phi(x) = E \phi(x)$$

$$\therefore \frac{d^2}{dx^2} \phi(x) = -\frac{2mE}{\hbar^2} \phi(x)$$

$$\therefore \frac{d^2}{dx^2} \phi(x) = -\kappa^2 \phi(x) \quad \left| \begin{array}{l} \kappa = \sqrt{\frac{2mE}{\hbar^2}} \\ (\text{why } \kappa \text{ 'cuz } P = \hbar) \\ \text{when } V=0 \end{array} \right.$$

$$\Rightarrow \psi(x) = A \sin kx + B \cos kx ; \text{ for a } k$$

$\therefore \psi(x)$ is contin

$$\therefore \psi(0) = 0 ; \psi(L) = 0$$

$$\psi(0) = B \Rightarrow B = 0$$

$$\psi(L) = A \sin kL \Rightarrow A \sin kL = 0$$

$A = 0$ makes $\psi(x)$ always 0

we know particle exist

$$\therefore A \neq 0$$

$$\sin kL = 0 \Leftrightarrow kL = n\pi$$

$$\Rightarrow k = \frac{n\pi}{L}$$

(other than)
0

inside

some

But not 0 \because if $k=0$

$\sin(0x)$ will always
be 0

(n can be negative but

that doesn't really make a new eigenfunction

it just adds a scalar factor \therefore n positive is enough)

$$\therefore \psi_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3, \dots$$

(it's a discrete basis :))

- What are its corresponding eigenvalues?

$$\frac{2mE}{\hbar^2} = k^2 \Rightarrow E = \frac{\hbar^2 k^2}{2m}$$

$$\Rightarrow E_n = \frac{\hbar^2 \times n^2 \times \pi^2}{2m L^2} \quad (\because kL = n\pi)$$

- How to find A_n ?

Well remember we work with normalised basis \therefore .

$$\|\psi_n\| = 1 \Rightarrow \sqrt{\int_{-\infty}^{\infty} \psi_n(x) \psi_n(x) dx} = 1$$

$$\|\psi_1\| \Rightarrow \sqrt{|A_1|^2 \int_0^L \sin^2 \frac{n\pi x}{L} dx} = 1 \Rightarrow \text{S.O.B.S} \\ (\text{sg on both sides})$$

$$\Rightarrow |A_1|^2 \int_0^L \sin^2 \frac{n\pi x}{L} dx = 1$$

$$\Rightarrow |A_1|^2 = \frac{2}{L} \Rightarrow A_1 = \sqrt{\frac{2}{L}}$$

(To make simple life you only take +ve)

↑
+ doesn't make much
difference bcs $|\psi|^2$

List!

Eigenvalue

$$\frac{\pi^2 \hbar^2}{2m L^2}$$

Eigenfunction

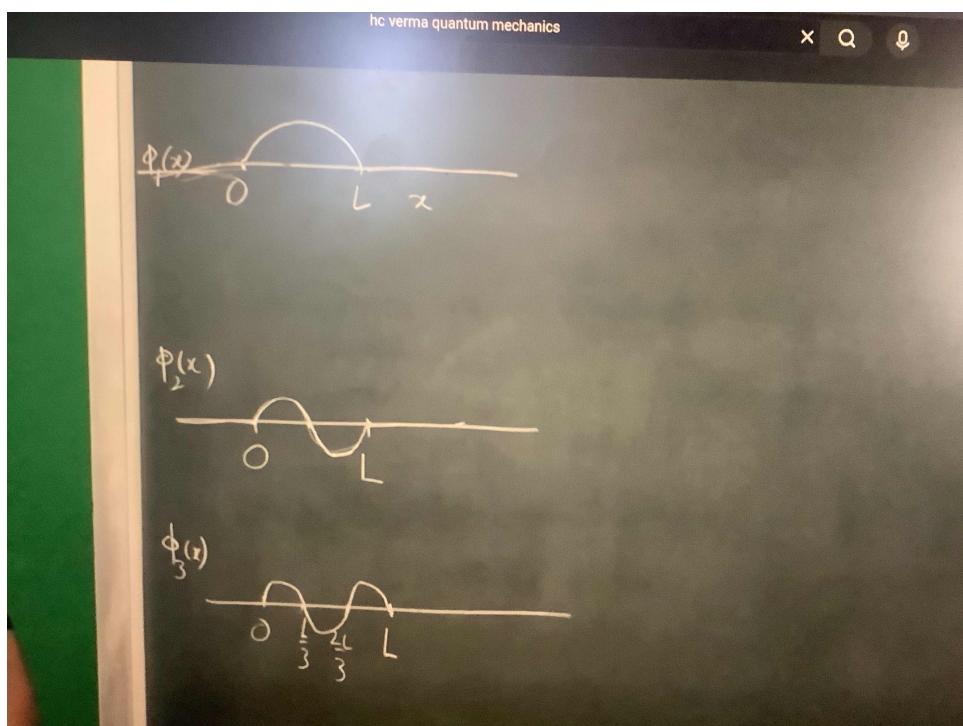
$$\sqrt{\frac{2}{L}} \sin \frac{\pi x}{L}$$

$$\frac{4\pi^2 \hbar^2}{2m L^2}$$

$$\sqrt{\frac{2}{L}} \sin \frac{2\pi n}{L}$$

$$\frac{9\pi^2 \hbar^2}{2m L^2}$$

$$\sqrt{\frac{2}{L}} \sin \frac{3\pi n}{L}$$

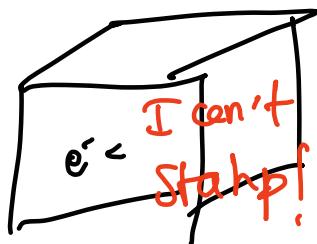


- $E \propto \frac{1}{L^2}$

- short well more
Energy

- Energy is in discrete eigenstates

$E = 0$ has 0 probability \Rightarrow the particle must be in motion



Q/ A particle of mass m in an infinite square well

$$\text{[Diagram of a square well]} \quad \text{has the wavefunction } \psi(x) = \frac{\sqrt{30} \sin(\frac{n\pi}{L}x)}{L^{5/2}}$$

Find the probability of finding the energy to be $\frac{\hbar^2 n^2}{2mL^2}$

Ans.

$$|\Psi\rangle = \sum c_i |\psi_i\rangle$$

Probability of finding $|\psi_i\rangle$ $|c_i|^2$ pure state of energy
(eigenfunction of Hamiltonian)

$$c_i = \langle \psi_i | \Psi \rangle$$

$$= \int_0^L \overline{\psi_i(x)} \psi(x) dx = \int_0^L \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \times \frac{\sqrt{30} \sin(\frac{n\pi}{L}x)}{L^{5/2}} dx$$

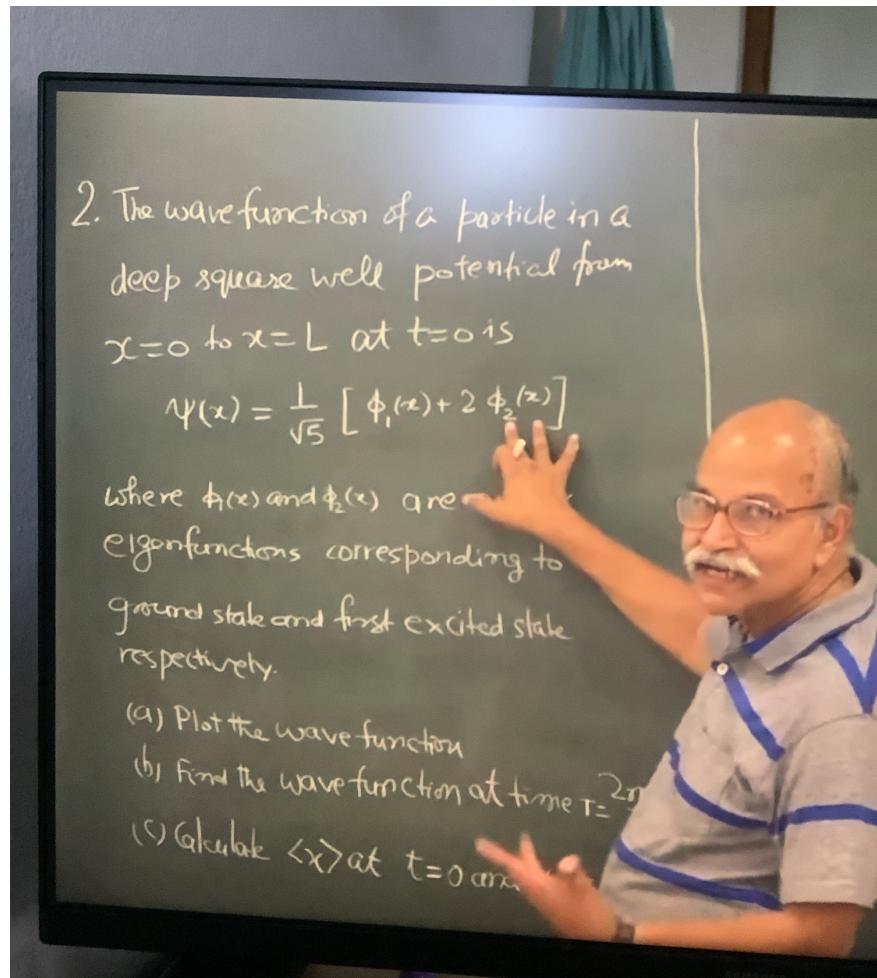
$$= \frac{\sqrt{60}}{L^4} \int_0^L \sin \frac{\pi n}{L} x (L-x) dx$$

By part

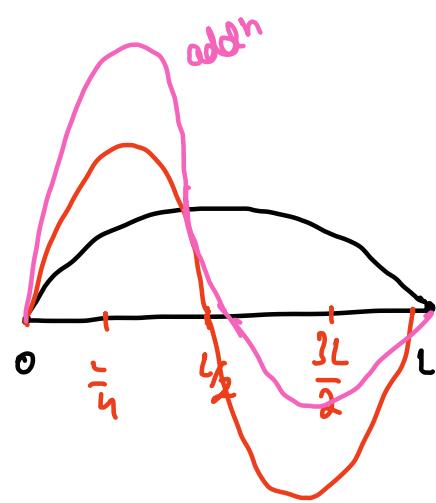
$$= \frac{\sqrt{960}}{\pi^2}$$

$$\therefore \text{Prob} : |C|^2 = \frac{960}{\pi^6} = 0.94$$

Q1



(a)



$$\Phi_1(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L}$$

$$2\Phi_2(x) = \left(\sqrt{\frac{2}{\pi}} \sin \frac{2\pi x}{L} \right) \times 2$$

(b)

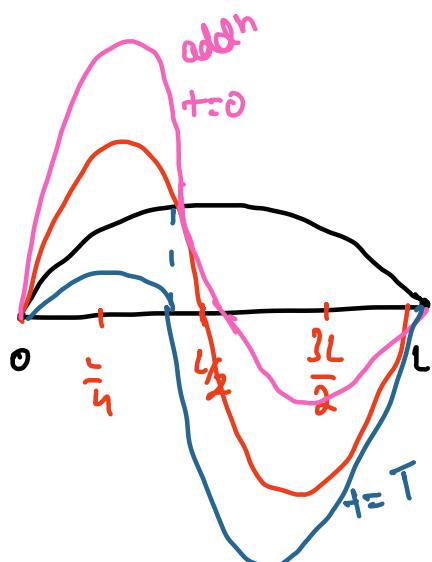
$$\Psi(x, t) = \left(\Phi_1(x) e^{-j \frac{E_1}{\hbar} t} + 2\Phi_2(x) e^{-j \frac{E_2}{\hbar} t} \right) \frac{1}{\sqrt{2}}$$

$$E_1 = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2m L^2}$$

$$E_2 = \frac{4\hbar^2 \pi^2}{2m L^2}$$

substituting the + asked in the question, we get

$$\Psi(x, t) = \left[-\Phi_1(x) + 2\Phi_2(x) \right] \frac{1}{\sqrt{2}}$$



(it changes with time)

$$\textcircled{C} \quad E(x) = \langle \Psi | x | \Psi \rangle$$

$$= \int \bar{\psi} x \psi dx \Rightarrow \int x |\psi(x, t)|^2 dx$$

Consider a better way

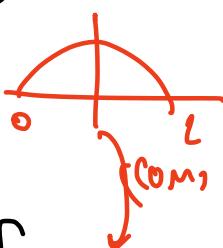
$$|\Psi\rangle = \frac{1}{\sqrt{5}} [|\Phi_1\rangle + 2 |\Phi_2\rangle]$$

$$\langle \Psi | = \frac{1}{\sqrt{5}} [\langle \Phi_1 | + 2 \langle \Phi_2 |] \quad (\text{Basically it's complex conjugate})$$

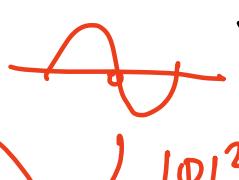
$$\langle \Psi | x | \Psi \rangle$$

$$= \frac{1}{\sqrt{5}} [\langle \Phi_1 | + 2 \langle \Phi_2 |] [x |\Phi_1\rangle + 2x |\Phi_2\rangle]$$

$$= \frac{1}{\sqrt{5}} [\langle \Phi_1 | x |\Phi_1\rangle + 2 \langle \Phi_1 | x |\Phi_2\rangle + 2 \langle \Phi_2 | x |\Phi_1\rangle + 4 \langle \Phi_2 | x |\Phi_2\rangle]$$

" ∵ red $\bar{\Phi}_1 = \Phi_1$ $\bar{\Phi}_2 = \Phi_2$


$$= \frac{1}{\sqrt{5}} \left[\frac{L}{2} + \left(\frac{-16L}{9\pi^2} \right) 4 + \frac{+4L}{2} \right]$$



$$= \frac{L}{2} - \frac{64}{9\pi} \frac{L}{\pi^2} \quad (Ai + 0)$$



$C_0 - C_2$ के लिए, makes sense from graph

$$At t = T$$

$$= C_2 + \frac{6u}{45} C_{12}$$

• x does not commute with $h \Rightarrow$ For a mixed state

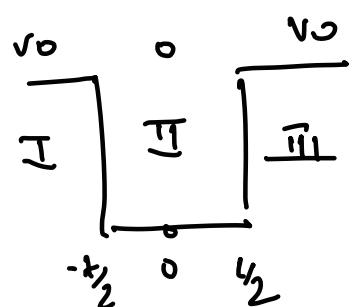


 E(x) changes with time
 E(energy) stays same though

* finite Square well

I:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V_0 \psi = E \psi$$



$$\Rightarrow \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2} (V_0 - E) \psi$$

for any bound state $V_0 > E$:-

$$\Rightarrow \frac{d^2\psi}{dx^2} = \gamma^2 \psi ; \gamma = \sqrt{\frac{2m}{\hbar^2} (V_0 - E)}$$

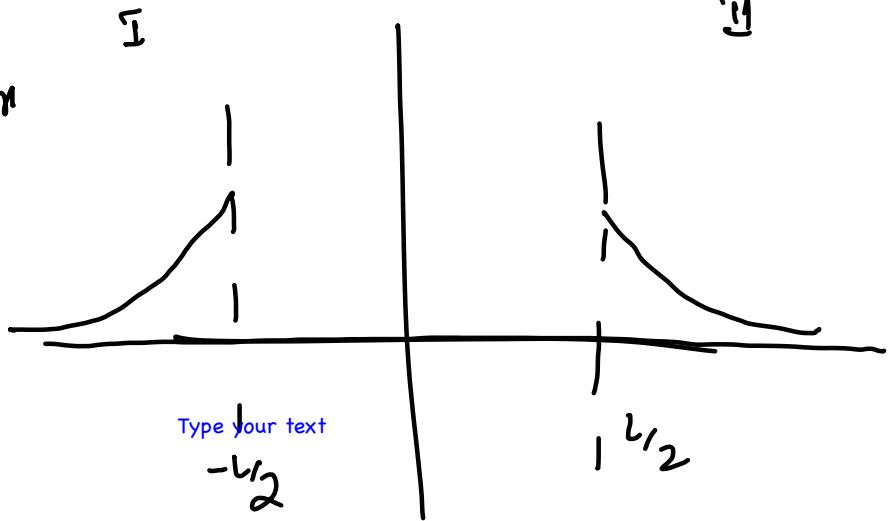
$$\Rightarrow \psi(x) = A e^{\gamma x} + B e^{-\gamma x}$$

At $x \rightarrow \infty$ ψ will blow up due to $B e^{-\gamma x}$

$$\therefore B=0$$

$$\Rightarrow \phi(x) = Ae^{ix}$$

In III



$$\Rightarrow \phi(x) = Be^{-ix}$$

In II -

$$\Rightarrow \frac{d^2\phi}{dx^2} = \frac{2m}{\hbar^2} (\omega_0^2 - E) \phi$$

$$\Rightarrow \frac{d^2\phi}{dx^2} = -\frac{2mE}{\hbar^2} \phi$$

$$\Rightarrow \phi = A \sin kx + B \cos kx \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

For me some E : per E you get 1 eigenfunction across regions
 $\phi(x)$ must be contin: $\phi'(x)$ must be contin:

$$\phi_I(x = -\frac{L}{2}) = \phi_{II}(x = -L_1)$$

$$\phi'_I(x = -L_1) = \phi''_{II}(x = -L_1)$$

$$\phi_{II}(x = L_1) = \phi_{III}(x = L_1)$$

$$\phi'_{II}(x = L_1) = \phi'''_{III}(x = L_1)$$

4 equations 4 variables :)

★ Correct logic (clean version)

The time-independent Schrödinger equation is:

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

Rearranged:

$$\psi''(x) = \frac{2m}{\hbar^2}(V(x) - E)\psi(x)$$

Now look at the right-hand side:

- If $V(x)$ is finite,
- and $\psi(x)$ is finite (wavefunction always is),

then the right-hand side is finite.

Therefore:

🔥 $\psi''(x)$ must be finite everywhere.

And that forces:

✓ $\psi(x)$ must be continuous

✓ $\psi'(x)$ must be continuous

+ Ask anything

LHS $V(n) - E$

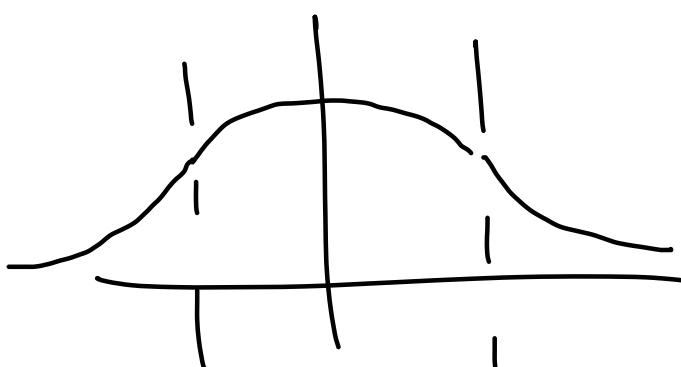
is finite for a finite well, $\therefore \psi''(n)$ must be finite (i.e. exist not as impulses)

$\therefore \psi(n), \psi'(n)$

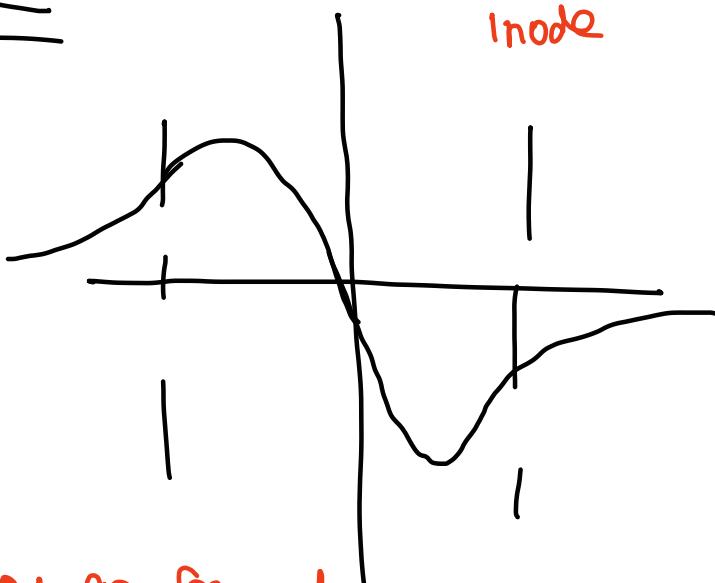
must be contn

(In ∞ well ψ'' would be impulsive so ψ' needn't (and isn't) contn)

ground state E_1



E_2



nodes will keep ↑ as you go forward

ϵ can keep increasing as long as it is less than ν_0 (to remain in bound state) thus unlike in ∞ case, there are a finite number of eigenfunctions.

* Linear Harmonic oscillator

$$\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \psi + \frac{1}{2}m\omega^2x^2\psi = \epsilon\psi$$

ψ must be contin, ψ' must be contin

(no ∞ jump)

exist in the ψ

anywhere)

$$v(x) = \frac{1}{2}kx^2$$

$$= \frac{1}{2}m\omega^2x^2$$

$$\left(\because \omega = \sqrt{\frac{k}{m}} \right)$$

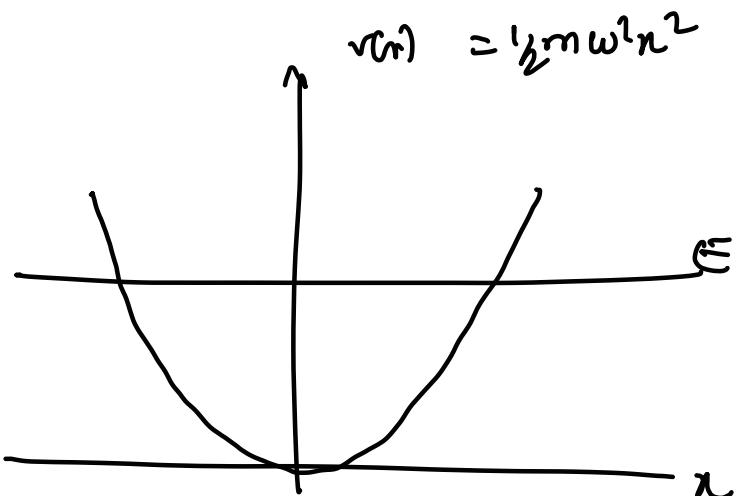
Energy eigenfunctions -

$$E_n = (n + \frac{1}{2})\hbar\omega, n=0,1,2, \dots$$

$$E_0 = \frac{1}{2}\hbar\omega$$

$$E_1 = \frac{3}{2}\hbar\omega$$

$$E_2 = \frac{5}{2}\hbar\omega$$



$$\psi_n(x) = C_n H_n(\alpha x) e^{-\frac{1}{2} \alpha^2 x^2}$$

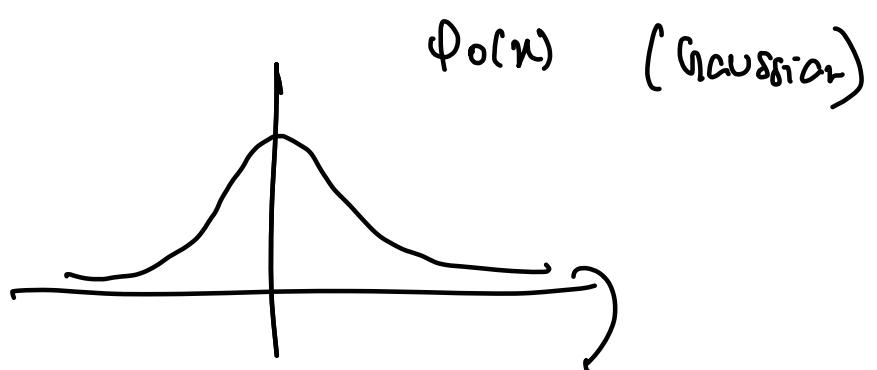
↳ Hermite polynomial

$$\alpha = \sqrt{\frac{m\omega}{\hbar}}$$

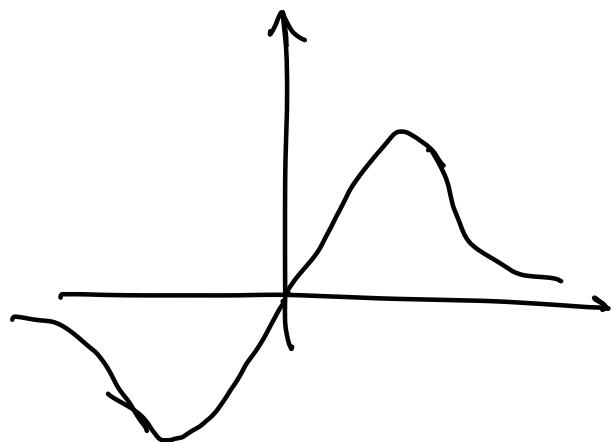
$$H_0(y) = 1$$

$$H_1(y) = 2y$$

$$H_2(y) = 4y^2 - 2$$

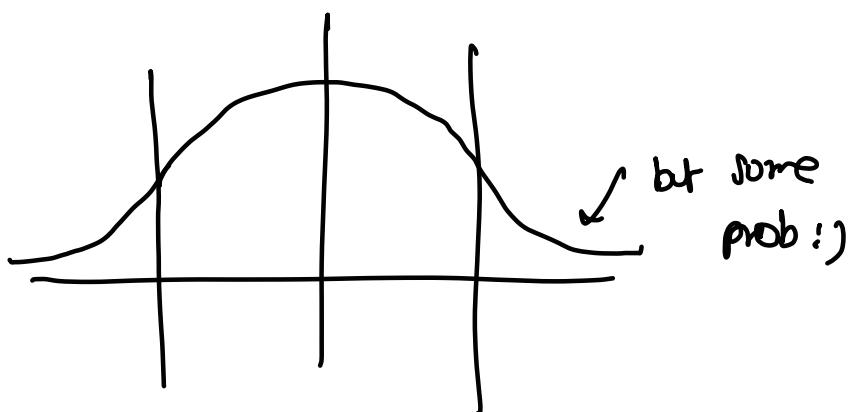
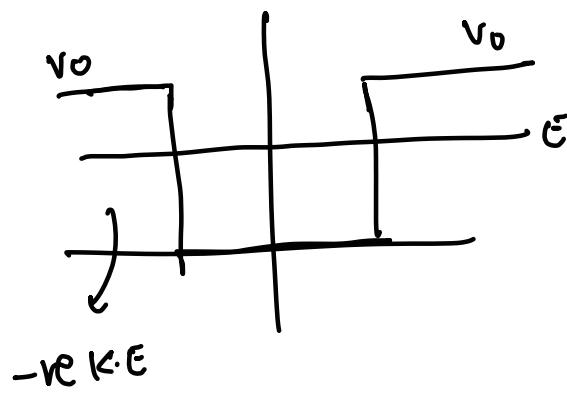


hmm... is it not
becoming 0?



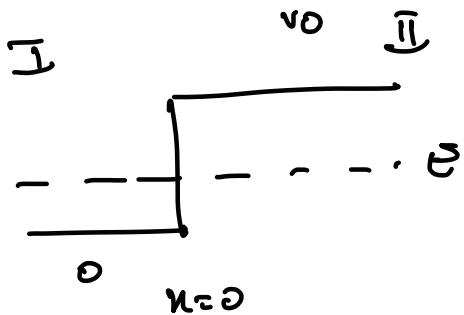
* Quantum mechanics and forbidden regions (Tunneling)

Consider the finite well case -



it was ~~2D~~ outside only for
 ∞ square well

Potential step



I:

$$H\psi = E\psi$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2} = E\phi$$

$$\Rightarrow \frac{d^2\phi}{dx^2} = -\frac{2mE}{\hbar^2}\phi$$

$$\Rightarrow \phi = A e^{jkx} + B e^{-jkx}$$

II:

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2} + V_0\phi = E\phi$$

$$\Rightarrow \frac{d^2\phi}{dx^2} = \frac{2m(V_0 - E)}{\hbar^2}\phi$$

+ve

$$\Rightarrow \phi = \cancel{C e^{j\gamma x}} + D e^{-j\gamma x}$$

not possible

$$\gamma = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

(will blow)

$$\therefore \phi = D e^{-j\gamma x}$$

$$\varphi_I(\sigma) = \varphi_{II}(\sigma) \Rightarrow A + B = D$$

$$\varphi_I'(\sigma) = \varphi_{II}'(\sigma) \Rightarrow jkA - jkB = -\gamma D$$

$$A = \frac{\begin{vmatrix} D & 1 \\ -\gamma D & -jk \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ jk & -jk \end{vmatrix}} = \frac{-jkD + \gamma D}{-jk - jk} = \frac{D(\gamma - jk)}{-2jk}$$

$$\Rightarrow D = \frac{2jkA}{jk - \gamma}$$

$$B = \frac{jk + \gamma}{jk - \gamma} A$$

What all E's are allowed?

(Everything is in A, you get
remove A by |||)

All \therefore until v_0 ,

these are not restrictions, whatever k you fix, γ will be

fixed accordingly -

Probability current -

$$|\psi(x)|^2 : \text{probability density} = f$$

$$\frac{\partial f}{\partial t} = \bar{\psi} \frac{\partial \psi}{\partial t} + \psi \frac{\partial \bar{\psi}}{\partial t}$$

$$\text{But } j\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + v(x)\psi \quad (\text{Schrödinger's Eq.})$$

$$\Rightarrow \bar{\psi} \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + v(x)\psi \right] \frac{1}{j\hbar} \bar{\psi}$$

Compliment of this equation

$$\Rightarrow \psi \frac{\partial \bar{\psi}}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2 \bar{\psi}}{\partial x^2} + v(x)\bar{\psi} \right] - \frac{1}{j\hbar} \psi$$

$$\therefore \frac{\partial f}{\partial t} = \frac{1}{j\hbar} \left\{ \left(-\frac{\hbar^2}{2m} \right) \left(\bar{\psi} \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \bar{\psi}}{\partial x^2} \right) \right\}$$

$$= -\frac{\hbar}{2mj} \left[\frac{\partial}{\partial x} \left(\bar{\psi} \frac{\partial \psi}{\partial x} - \psi \frac{\partial \bar{\psi}}{\partial x} \right) \right]$$

$$= -\frac{\partial}{\partial x} \left[\frac{\hbar}{2mj} \left(\bar{\psi} \frac{\partial \psi}{\partial x} - \psi \frac{\partial \bar{\psi}}{\partial x} \right) \right]$$

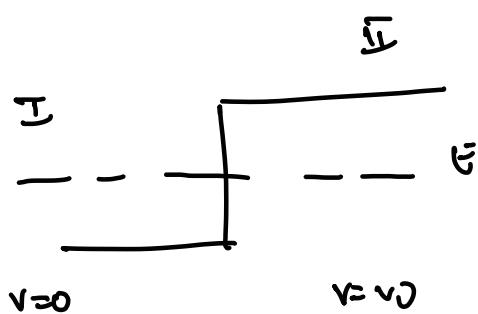
$$\frac{\partial \phi}{\partial t} = -\frac{\partial J}{\partial x} \quad (\text{continuity eq})$$



tells rate of flow of prob density

$$J = \frac{i}{2jm} \left[\bar{\psi} \frac{\partial \psi}{\partial x} - \psi \frac{\partial \bar{\psi}}{\partial x} \right] \quad \text{Prob current density}$$

* Going into the step



$$\phi_I = A e^{j k x} + B e^{-j k x}$$

$$\phi_{II} = C e^{-k x}$$

Consider - $A e^{j k x}; \quad J_1 = |A|^2 \frac{e^{j k x}}{m}$

$B e^{-j k x}; \quad J_2 = -|B|^2 \frac{e^{-j k x}}{m}$

per area per time
→ this much prob
moving →
→ this much prob
moving ←

$$\left| \frac{J_2}{J_1} \right| = \left| \frac{B}{A} \right|^2 \quad \frac{\text{prob away}}{\text{prob towards}}$$

$$\left| \frac{j k + \gamma}{j k - \gamma} \frac{A}{B} \right|^2 \Rightarrow 1$$

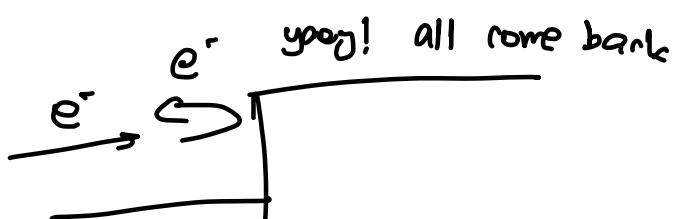
$$\therefore \left| \frac{J_2}{J_1} \right| = 1$$

मितने left जा रहे हैं, उतने ही right जा रहे हैं,

$\psi_I = [Ae^{jkn} + Be^{-jkn}]$ → is an Eigenfunction of energy
(only 1)

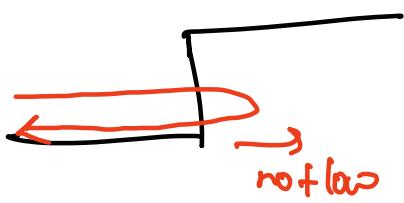
so all qly. are time invariant

including $\left| \frac{J_2}{J_1} \right| = 1$



$$\phi_{II} \rightarrow J_{II} = 0 : 0$$

But prob there exist! यह सकते हैं मज़ा उदाहरण दें कहाँ नहीं सकते



👉 How these two facts coexist

Think of the evanescent wave like putting your hand against a soft cushion:

- Your hand **indents** the cushion (penetration)
- But your hand does **NOT pass through** the cushion (no movement to the other side)

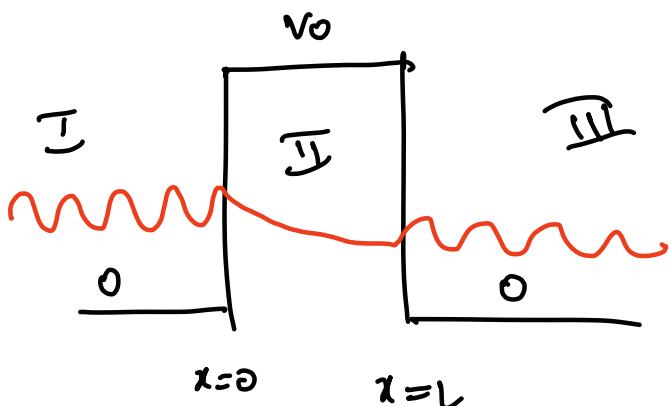
The wavefunction can "indent" into the forbidden region.
But there's no flow taking the particle deeper.

This is because:

$$\psi(x) \underset{\downarrow}{=} Ae^{-\kappa x}$$

THE PROB CURRENT OPERATOR IS 0 FOR REAL ψ

Square Barriers -



$$\text{I} - k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\text{II} - \gamma = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$\text{III} - k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\text{I} - Ae^{jkn} + Be^{-jkn}$$

$$\text{II} - Ce^{jkn} + De^{-jkn} \quad (\text{region is finite so both exist})$$

$$\text{III} - Fe^{jkn} + \cancel{G}e^{-jkn}$$

\rightarrow there is no way a reflected wave

and come, there isn't any barrier there

$$R - \left| \frac{J_2}{J_1} \right|$$

(any swiggly oscillating \Rightarrow)

non-zero J , real $\Rightarrow J=0$)

$$\Psi_I(0) = \Psi_{\text{II}}(0) \rightarrow \textcircled{1}$$

$$\Psi_{\text{II}}(l) = \Psi_{\text{III}}(l) \rightarrow \textcircled{2}$$

$$\Psi_{\text{II}}'(0) = \Psi_{\text{III}}'(l) \rightarrow \textcircled{3}$$

$$+ || || = 1 \rightarrow \textcircled{5}$$

$$\Psi_{\text{II}}'(l) = \Psi_{\text{III}}'(l) \rightarrow \textcircled{4}$$

Σc_p , Σ variables :)

You can find A, B, C, D, f (Or even finding 4 of those
 If you were to sit and solve is enough for 3 fraction
 discussions)

$$T = \frac{4E(v_0 - E)}{4E(v_0 - E) + v_0^2 \sinh^2 \sqrt{\frac{2m(v_0 - E)L^2}{\hbar^2}}}$$

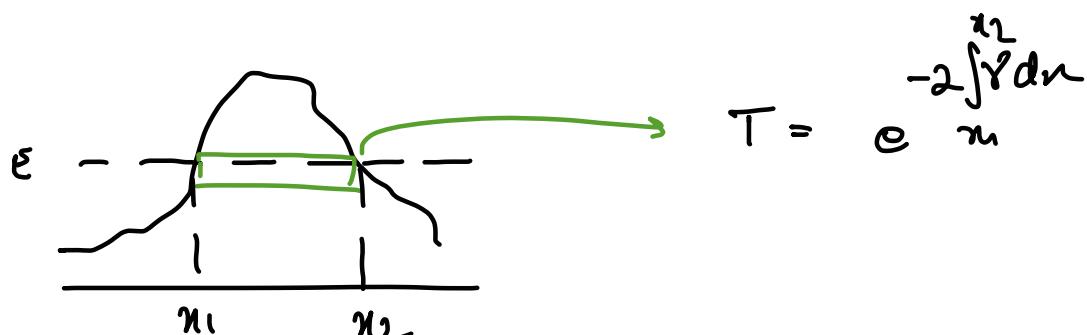
if $T \ll 1$

then $T \sim e^{-2\gamma L}$

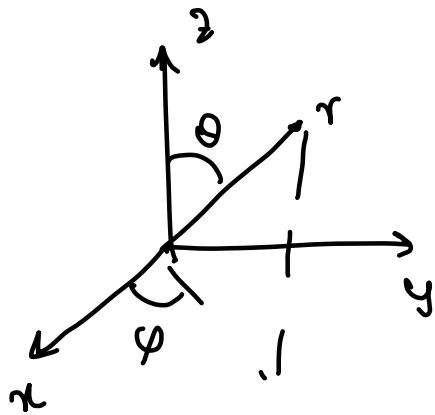
$$\ln(v_0 - E) T \quad \downarrow T$$

$$\gamma = \sqrt{\frac{2m(v_0 - E)}{\hbar^2}}$$

If the barrier is some other shape -



3 Dimension



$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

A really important observable : angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$\Rightarrow (y p_z - z p_y) \hat{i} + (z p_x - x p_z) \hat{j} + (x p_y - y p_x) \hat{k}$$

वास्तव में यहाँ की बात "orbital angular momentum"

(has nothing to do with orbit)

- $L_x \text{ operator} = y p_z - z p_y$

$$\left. \begin{array}{c} \downarrow \\ -j\hbar \frac{\partial}{\partial z} \end{array} \right] \quad \left. \begin{array}{c} \downarrow \\ -j\hbar \frac{\partial}{\partial y} \end{array} \right]$$

- $L_z \text{ operator} = x p_y - y p_x$

- $L^2 \text{ operator} = Lx^2 + Ly^2 + Lz^2$

$$Lx(Lx(\phi)) + Ly(Ly(\phi)) + Lz(Lz(\phi))$$

- Further L^2 and L_z commute with each other

(so they can have common eigenfunctions)

Their eigenfunctions look like:

$$\psi(r, \theta, \phi) = f(r) Y_l^m(\theta, \phi)$$

↓

this can be any function but this is quite specific

$$l = 0, 1, 2, 3$$

$$m = \pm l$$

- $L^2(\psi(r, \theta, \phi)) = \underbrace{l(l+1)\hbar^2}_{\text{eigenvalue}} \psi(r, \theta, \phi)$

- $L_z(\psi(r, \theta, \phi)) = \underbrace{m\hbar}_{\text{eigenvalue}} \psi(r, \theta, \phi)$

Spin angular momentum

Apart from the usual orbital angular momentum by virtue of its motion in space ($r \times p$)

The quantum particles also have a spin angular momentum -

This is an intrinsic momentum not corresponding to any spin or motion

$$S^2 \quad S_z \quad \left| \begin{array}{l} \text{स्पिन} \\ \text{spin} = \pm \frac{1}{2} \end{array} \right.$$

$Y_l^m(\theta, \phi)$

l	m	$Y_l^m(\theta, \phi)$
0	0	$\frac{1}{\sqrt{4\pi}}$
1	0	$\sqrt{\frac{3}{4\pi}} \cos \theta$
1	1	$-\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$
1	-1	$\sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi}$
2	0	$\sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)$
2	2	$\sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{2i\phi}$

m comes out as $e^{im\phi}$
so whenever $m=0$,
no ϕ dependence

* Solving the Schrödinger's equation for an atom-

$$H \Psi(r, \theta, \phi) = E \bar{\Psi}(n\theta, \phi)$$

$$\left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) + \frac{-e^2}{4\pi\epsilon_0 r} \right] \bar{\Psi}(r, \theta, \phi)$$

hmm?

maybe we
spherical

$$= E \bar{\Psi}(r, \theta, \phi)$$

In spherical coordinates-

$$\left[-\frac{\hbar^2}{2m} \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) - \frac{e^2}{4\pi\epsilon_0 r} \right] \bar{\Psi}(r, \theta, \phi)$$

$$\text{Also, } = E \bar{\Psi}(r, \theta, \phi)$$

$$[H L^2] = 0 \quad [H L_z] = 0$$

(*: they both share eigenfunctions)

$\therefore Y_L^m(\theta, \phi)$ is fixed for H's eigenfunction, the question is

$$f(r) - ?$$

For solving they took the substitution $u(r) = r\psi(r)$

With constraint

- ψ : finite everywhere, $\psi \rightarrow 0$ as $r \rightarrow \infty$
- ψ : square integrable

Then

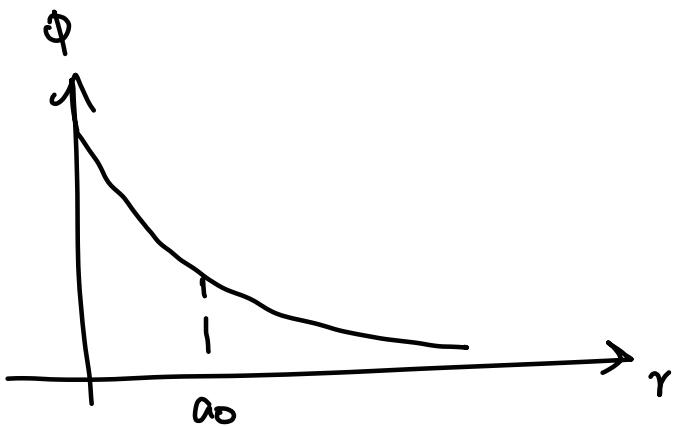
$$E_n = \frac{-m_e e^4}{2(4\pi\epsilon_0)^2 \hbar^2} \times \frac{1}{n^2} \quad ; \quad n=1, 2, 3, \dots$$

$\xrightarrow{\quad}$ $\frac{-13.6 \text{eV}}{n^2}$ (same as Bohr!)

THE HYDROGEN ATOM -

- Ground state - $\psi(n=1, l=0, m=0) = \frac{2}{a_0^{3/2}} e^{-r/a_0}$

$$E_1 = -13.6 \text{eV} \quad (\text{same as Bohr!})$$



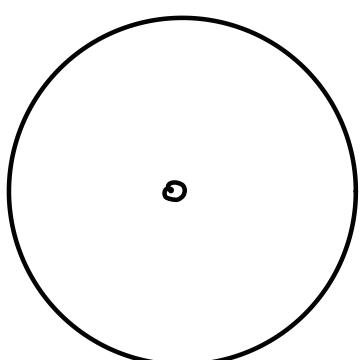
$$a_0 = \frac{4\pi \epsilon_0 \hbar^2}{m e^2}$$

(Bohr radius)

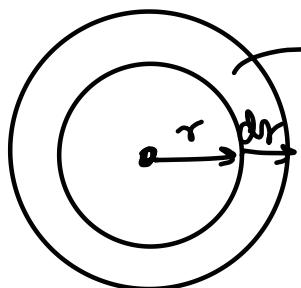
NOT PROB! volume = 0
so PROB = 0

- graph says max in prob density
finding σ^2 is at the origin
- Spherically symmetric

But density is the highest



This is prob/volume ; what above radical prob

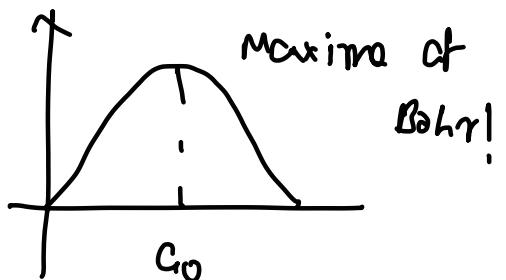


prob of being b/w
r, r+dr

$$(4\pi r^2) | \Psi |^2$$

$$\propto (4\pi r^2) e^{-2r/a_0}$$

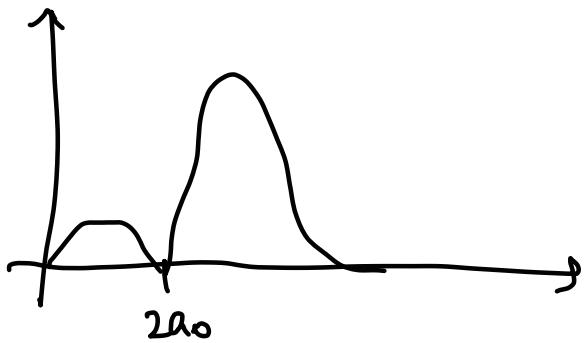
This says r=0 Prob b=0



- n=2 E = -3.4 eV

$$(n=2, l=0, m=0) = \left(\frac{1}{2a_0}\right)^{3/2} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$$

Radial probability -



no relation with Bohr's theory