A study of scalar-tensor theory of gravitation and its application to cosmology

by

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Abstract

The observed late-time accelerated expansion of the universe is motivating researchers to look for an alternative to Einstein's general theory of relativity (GTR) as a description of gravitation. The scalar-tensor theory of gravitation is one of the widely studied alternatives to Einstein's GTR. In scalar-tensor theory, unlike in Einstein's GTR, not just the metric tensor but the value of a non-minimally coupled scalar field is required to describe the gravitational field. A prototype of this class of model is the Brans-Dicke theory, in which the value of Newton's gravitational constant depends on the value of the scalar field and, therefore, it evolves with time. In this dissertation, we plan to study Brans-Dicke's theory of gravitation and other models of scalar-tensor theory and the viability of such models in explaining the observed late-time accelerated expansion of the universe.

Certificate

This is to certify that the dissertation titled `A study of scalar-tensor theory of gravitation and its application to cosmology` *is a bonafide work of* `Nihal Jalal Pullisseri`*, who has done research under my guidance and supervision. This is being submitted to the Department of Physics, St. Stephen's College, University of Delhi, in partial fulfillment, in lieu of a Discipline Specific Elective paper in* `Physics Hons. (567: BSc (Honours) Physics VI semester: Paper Code: 32227627.)` *To the best of my knowledge, the contents of the dissertation have not been previously submitted to this or any other institution, for any other course and degree.*

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Declaration

I certify that.

1. *The work contained in this undergraduate dissertation is original and has been completed by me, under the guidance of my supervisor.*

2. *The work has not been submitted to any other Institute for any degree or diploma.*

3. *Whenever I have used materials (data, plots, theoretical analysis, and text) from other sources, I have given due credit to the original source and author by citing them in the text of the thesis and listing details in the Bibliography.*

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Chapter 1

Introduction

The expansion of the universe has been known for about a century now. The first evidence for expansion came from V. M. Slipher in the 1920s and later extended by Edwin Hubble and Milton Humason [11] which resulted in the famous Hubble's law which relates the velocities of galaxies to their respective distances. This was a monumental discovery in the field of cosmology. The expansion of the universe was explained satisfactorily in the framework of Einstein's General theory of relativity and the so-called Friedmann equations which relate the expansion of the universe to its energy/matter content.

Another important observation that challenged our understanding of cosmology came in 1998 by two independent projects studying Type Ia supernovae [19] [21], it indicated that the universe is not only expanding but doing so at an accelerated rate. This was contrary to what cosmologists had believed, that the rate of expansion should slow down. The accelerated expansion was theorized to be caused by some exotic form of energy named **dark energy**. The exact nature and mechanism of dark energy is still unknown, it is still very much an active area of research.

Numerous theories have been put forward to elucidate the nature of dark energy, encompassing perspectives rooted in both general relativity and particle physics, as well as hybrid approaches that blend the two fields. The existence of dark energy is typically attributed to the cosmological constant Λ in the Einstein field equation. The model for the evolution of the universe incorporating this is known as the ΛCDM model and is the current widely accepted model for the universe.

In this dissertation, we explore the so-called scalar-tensor theory of gravity which is an alternative theory of gravity to that of Einstein's general theory of relativity. Unlike general relativity, the scalar-tensor theory of gravity incorporates a scalar field in addition to the metric tensor to describe gravity. We also look at the Brans-Dicke theory[4] which is a special case of scalar-tensor theory. This theory exhibits a greater resemblance to Mach's principle than general relativity.

Subsequently, we study the applicability of the theory in explaining the observed late-time accelerated expansion of the universe. We will explore the possibility of explaining dark energy and hence the acceleration of the universe without the need for a cosmological constant as such. The motivation for this comes from recent observations in cosmology indicating the possible failure of $GR - \Lambda CDM$ model. One such observational enigma is called the Hubble tension which gives different values of the Hubble's constant from different kinds of observations, one from the cosmic microwave background radiation (from the Planck satellite observations [2]) and the other from a supernova [1]. Recently, many attempts have been made to resolve this in the framework of scalar-tensor gravity. For one such recent work refer [18]. Recent developments like that of Hubble tension have reignited interest in scalar-tensor theory of gravity. Refer [7] for recent developments regarding Hubble tension.

This report is divided into five chapters including the Introduction and Conclusion. **Chapter 2** covers the basics of cosmology in the known general relativistic framework, observational evidence for the accelerated expansion of the universe, and a brief discussion on the ΛCDM model.

In **Chapter 3**, we introduce the scalar-tensor theory of gravity including the Brans-Dicke theory, we obtain the field equations in this framework. Later on, we discuss cosmology within this framework of scalar-tensor theory gravity, including the derivation of Friedmann equations. We end the chapter with some results that will be used in chapter 4.

In **Chapter 4**, we look at the primary goal of this dissertation, we try to explain the acceleration of the universe in the framework of scalar-tensor theory of gravity and draw comparisons from the ΛCDM model

1.1 Conventions

- Following the convention prevalent in literature, we employ the units wherein $\hbar = c = 1$.
- We use the metric signature $(+---)$
- We interchangeably use the terms ΛCDM model and $GR - \Lambda CDM$ model

Chapter 2

Cosmology in the framework of GTR

2.1 Cosmology

Broadly speaking, Cosmology is the study of origin and evolution of the universe, it is a vast field that originated centuries ago. The field of cosmology has significantly changed our understanding of the universe and ourselves. There are many observations and implications of cosmology that are worth mentioning, but in this dissertation, our focus will be on the observed later-time acceleration of the universe.

2.1.1 Expansion of the universe

One of the most important observations in modern cosmology came from observations made by V. M. Slipher, Edwin Hubble, and Milton Humason. Hubble examined the relationship between the radial velocities of the galaxies and their respective distances. The radial velocity is related to the redshift, which is the quantity that we can observe. It was noticed that the relation between the radial velocity/redshift and the distance was linear, and given by the relation:

$$V = cz = H_0 D$$

where, V is the radial velocity, z the redshift, D the distance of the galaxy from Earth and H_0 is known as the Hubble's constant. This relation implies that the mutual separation between galaxies is increasing. As can be seen in the following figure

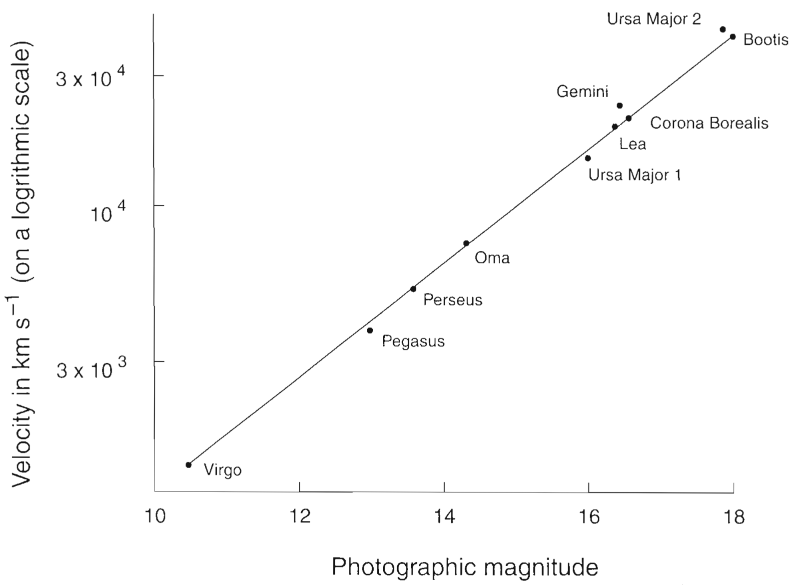

Figure 2.1: Photographic Magnitude(Distance) vs Velocity (From 1.8, An Introduction to Cosmology, J.V Narlikar[16])

An analogy, for this is a balloon that is expanding, suppose that the balloon has dots marked on it, as it expands, the mutual separation between the dots increases, but the dots themselves are fixed in size.

2.2 Basics of Cosmology

There are three main fundamental assumptions to modern cosmology. They are:

- Cosmological Principle: It says that, at sufficiently large scales, the universe is homogeneous and isotropic. Homogeneity implies that the universe "looks the same" or has the same properties everywhere except for local irregularities with regard to the distribution of matter, in other words, the universe has no privileged points, similarly, isotropy implies that the universe has no privileged directions.

- Weyl's postulate: It states that the galaxies' worldlines(trajectory in spacetime) are orthogonal to spacelike hypersurface and that these does not intersect with each other(Figure 1.2). Further, we assume that such galaxies form a continuum to achieve a smooth-fluid approximation. Thus, the galaxies behave like dust(pressureless matter)

- General relativity: Einstein argued that the dynamics of the universe is best explained by the theory of general relativity, and hence it constitutes the backbone of cosmology.

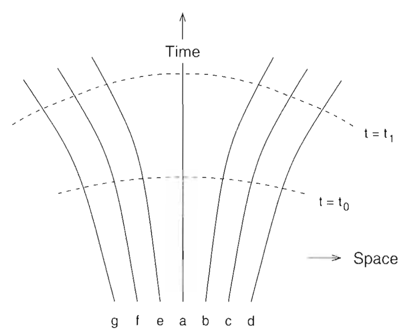

Figure 2.2: World lines of galaxies following Weyl's postulate (From 3.5.1, An Introduction to Cosmology, J.V Narlikar)

In the next section, we cover some basics of the General theory of relativity

2.3 Einstein's Equation

The Einstein Equation is the guiding equation in general relativity. It describes the relationship between matter distribution and the geometry of the spacetime. It is a tensor equation, given by:

$$G_{ab} = 8\pi G T_{ab}$$

where,

G_{ab} is the Einstein tensor ,

G is the universal gravitational constant,

T_{ab} is the energy-momentum (stress-energy) tensor that describes the density and flux of energy and momentum in spacetime.

The fact that Einstein's equation is of tensorial nature follows from the Principle of General Covariance which states that any physical law should be presentable as a tensor equation. We now look closely at each component of Einstein's equation.

2.3.1 The metric tensor

A metric or line element is a fundamental quantity in differential geometry and hence general relativity. It relates the distance in a coordinate system to the coordinate differences in that coordinate system. The metric characterizes a given differential manifold. The metric is given by

$$ds^2 = \mathbf{e}_\mu \cdot \mathbf{e}_\nu dx^\mu dx^\nu$$

where the metric tensor $g_{\mu\nu}$ is defined as $g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu$

2.3.2 Einstein tensor

The Einstein tensor is defined as $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$. This tensor kind of quantifies the curvature of the spacetime. R_{ab} is the Ricci tensor, which is a contraction of a more general tensor called Riemann curvature tensor R_{abcd}.R is the Ricci scalar which is the contraction of the Ricci tensor. Riemann tensor gives us all the information about the curvature of the spacetime. For a given metric Riemann tensor is defined as

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd} \Gamma^a_{ec} - \Gamma_{bc} \Gamma^d_{ed}$$

where Γ^a_{bc} is called the Christoffel symbol of the second kind, defined by,

$$\Gamma^c_{ab} = \frac{1}{2} g^{ad} \left[\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc} \right]$$

2.3.3 Energy momentum tensor

The energy-momentum tensor essentially describes the matter distribution. In case of pressureless matter(dust), it is given by

$$T_{ab} = \rho_0 u_a u_b$$

where ρ_0 is the proper energy density and u_a is the four-velocity. If we allow the possibility of pressure(perfect fluid) it takes the form

$$T_{ab} = (\rho_0 + p) u_a u_b - p g_{ab}$$

2.3.4 The Einstein-Hilbert Action

Like most theories of physics which can be described in terms of the Principle of Least Action, we can also use this principle to obtain the field equations of GR. The action is defined as:

$$I = \int R\sqrt{-g} d^4x + \int \mathscr{L}_m d^4x$$

where the integral is over the spacetime. The first part describes the gravitational field whereas the second describes the matter field. We can obtain the field equation from this by varying the metric g_{ab}. This can be done using the well-known Euler-Lagrange equation, but here we use another method as followed in Ray d' inverno [6] wherein we use the δ notation. This method is preferred because it is computationally much simpler and turns out to be useful when we derive field equations in scalar-tensor theory later. Let

$$I = \int R\sqrt{-g} d^4x$$

be defined over some region Ω of the manifold. According to the principle of stationary action, if we make arbitrary variations of the g_{ab} which vanish on the boundary $\partial\Omega$ of Ω, then action I must be stationary. Equivalently, we have

$$I = \int_{\Omega} \mathfrak{g}^{ab} R_{ab} \, \mathrm{d}\Omega$$

Using the fact that, the Ricci scalar is the contraction of the Ricci tensor. We define $\mathfrak{g}^{ab} = \sqrt{-g} g^{ab}$. Now varying the integral and using the Leibniz rule for products for the δ notation. We obtain:

$$\delta I = \int_{\Omega} \left(\delta\mathfrak{g}^{ab} R_{ab} + \mathfrak{g}^{ab} \delta R_{ab} \right) \mathrm{d}\Omega$$

Here the second term gives, $\int_{\Omega} \mathfrak{g}^{ab} \delta R_{ab} \mathrm{d}\Omega = \int_{\Omega} \partial_c \left(\mathfrak{g}^{ab} \delta \Gamma^c_{ab} - \mathfrak{g}^{ac} \delta \Gamma^b_{ab} \right) \mathrm{d}\Omega$. Using divergence theorem, we get that $\int_{\Omega} \partial_c \left(\mathfrak{g}^{ab} \delta \Gamma^c_{ab} - \mathfrak{g}^{ac} \delta \Gamma^b_{ab} \right) \mathrm{d}\Omega = \int_{\partial\Omega} (\mathfrak{g}^{ab} \delta \Gamma^c_{ab} - \mathfrak{g}^{ac} \delta \Gamma^b_{ab}) dS_c$ but since $\delta \Gamma^c_{ab}$ and $\delta \Gamma^b_{ab})$ is zero on the boundary, this term is zero.
Now, Consider the term $\int_{\Omega} \delta \mathfrak{g}^{ab} R_{ab} d\Omega$ varying this term, $\delta I = \int_{\Omega} \left[-(-g)^{\frac{1}{2}} G^{ab} \right] \delta g_{ab} d\Omega$. Similarly after varying the matter lagrangian, we finally get the field equation

$$G^{ab} = 8\pi G T^{ab}$$

.

2.4 The Robertson-Walker metric

The assumptions of cosmology help us in formulating a metric suitable for describing the universe. Weyl's postulate implies that the metric should have the form

$$ds^2 = cdt^2 + g_{\mu\nu} dx^{\mu} dx^{\nu}$$

Furthermore, it follows from the cosmological principle (homogeneity and isotropy) that the curvature of space at any point should be constant, this gives the metric for 3-space of the form:

$$\mathrm{d}\sigma^2 = a^2 \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right) \right)$$

where k can take values $-1, 0, 1$. Hence the most general metric satisfying both of these conditions is given by:

$$ds^2 = cdt^2 - a(t)^2 \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right) \right)$$

$a(t)$ is called the scale-factor or the expansion-factor. Here r is called the 'comoving coordinate'. For $k = 0$ we have a flat space, $k = -1$ corresponds to hyperbolic space and $k = 1$ to spherical.

2.5 Friedmann's Equations

Now we are in a position to explain the dynamics of the universe. The energy momentum tensor is as defined before $T_{ab} = (\rho_0 + p)u_a u_b - p g_{ab}$. Using this and the metric we just found in the field equation $G^{\mu}_{\nu} = 8\pi G T^{\mu}_{\nu}$ leads to two independent so-called Friedmann's equations, given by:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{2.1}$$

$$\frac{\ddot{a}}{a} = \frac{-4\pi G(\rho + 3p)}{3} \tag{2.2}$$

The contracted Bianchi identity leads to the conservation equation:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$$

We define the equation of state parameter w, $w = \frac{p}{\rho}$.

Also, $H = \frac{\dot{a}}{a}$, H is called the Hubble parameter. In terms of H the first equation becomes:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

2.5.1 Omega Parameter

$\Omega(t)$ is defined as $\Omega(t) = \frac{\rho(t)}{\rho_c(t)}$. Where $\rho_c = \frac{3H^2}{8\pi G}$ is called the critical density (this is the density at which the universe exhibits flat geometry).

From here on, we take $k = 0$, this is in accordance with observations which suggests a flat geometry[24][14].

From equation 2.1 we get that $a(t) \propto t^{\frac{2}{3(1+w)}}$ for constant value of the equation of state parameter $w = \frac{p}{\rho}$. The conservation equation implies that $\rho \propto a^{-3(1+w)}$. In the next section, we look a the so-called ΛCDM model of the universe.

2.6 Accelerated expansion of the universe

The energy density is not just due to matter, it's also due to radiation(and vacuum energy which will discuss later on). The equation of state for matter corresponds to $w = 0$ which is to say it is pressureless and for radiation, it is $w = \frac{1}{3}$. In these cases, we have

Radiation : $a(t) \propto t^{1/2}, \quad \rho_r \propto a^{-4}$

Dust : $\quad a(t) \propto t^{2/3}, \quad \rho_m \propto a^{-3}$

but both of these cases correspond to decelerated expansion of the universe when put into 2.2. So, how do we explain the accelerated expansion of the universe? We see from 2.2 that for the equation of state $w < \frac{-1}{3}$ corresponds to accelerated expansion, $\ddot{a} > 0$. So we need a form of energy which satisfies this condition. This form of energy is referred to as "dark energy".

What is the origin of dark energy? Well, we're not exactly sure, but it is often attributed to the so called cosmological constant Λ which arises from a slight modification to the Einstein field equation in the form

$$G_{ab} = 8\pi G T_{ab} - \Lambda g_{ab}$$

So this adds one more contribution to the energy density apart from matter and radiation with $w = -1$. Let us see how, Expressing 2.2 in terms of Omega parameters, we get

(neglecting contribution from radiation)

$$\left(\frac{1}{H_0^2}\right)\left(\frac{\ddot{a}}{a}\right) = \frac{-1}{2}\left[\Omega_m a^{-3} + (1+3w_\Lambda)\Omega_\Lambda a^{-3(1+w_\Lambda)}\right]$$

Here we take w_Λ to be very close to -1 but greater than -1. This implies that $(1 + 3w_\Lambda)$ is negative. The first term in the square brackets is trivially positive always, it's straightforward to see that at small a the first term will dominate but for later a the second term will start dominating which ends up making the whole sum in the square brackets to be net negative and results in $\ddot{a} > 0$ which implies acceleration.

Below we look at some observational evidence for dark energy and hence the acceleration of the universe.

2.6.1 Observational evidence

There is numerous observational evidence [5] indicating the accelerated expansion of the universe at the present epoch. Before we get into the evidence, I introduce some relevant quantities in astronomy. One is the redshift(z) which is the increase in the wavelength of light. In this case, due to the expansion of the universe. It is defined as

$$1 + z = \frac{\lambda_0}{\lambda} = \frac{a_0}{a}$$

where the subscript zero is to indicate the present epoch.

Another important quantity is called the luminosity distance, which defines distance in through the luminosity of a stellar object which is an observationally measurable quantity. It is defined as

$$d_L^2 = \frac{L_s}{4\pi\mathcal{F}}$$

where L_s is the luminosity of the source and $\mathcal{F}$ is the energy flux. In terms of redshift

$$d_L = (1+z)\int_0^z \frac{dz'}{H(z')}$$

Including contribution from all types of energy, the Hubble parameter becomes

$$H^2 = H_0^2 \sum_i \Omega_i^{(0)} (1+z)^{3(1+w_i)} \tag{2.3}$$

Where the superscript (0) denotes present epoch.

Using this, we get the luminosity distance to be

$$d_L = \frac{(1+z)}{H_0} \int_0^z \frac{\mathrm{d}z'}{\sqrt{\sum_i \Omega_i^{(0)} (1+z')^{3(1+w_i)}}} \tag{2.4}$$

For small values of z, $d_L \approx \frac{z}{H_0}$.

Supernovae Ia

Type Ia supernovae occur when white dwarf stars exceed the Chandrashekar mass limit and explode. Luminosity distance d_L is related to quantities apparent magnitude m and absolute magnitude M by

$$m - M = 5 \log_{10}(d_L) + 25$$

Here m is defined in terms of the logarithm of energy flux and M in terms of the logarithm of luminosity distance. These quantities are observationally measurable. M is independent of z. The equation above provides us a method to measure experimentally whereas 2.4 is the theoretical prediction for d_L.

Consider two different sets of observations[19]

1. 1992P: z = 0.026 with m = 16.08

2. 1997ap: z = 0.83 with m = 24.32

Using the first set and the approximation for d_L at small z, we find that $M = -19.9$. Using this the luminosity distance of 1997ap is found to be $H_0 d_L \approx 1.16$. Using 2.4 with $\Omega_M = 1$, we find that $H_0 d_L \approx 0.95$. Now, if we were to include a contribution from dark energy such that $\Omega_\Lambda^{(0)} \approx 0.7$ and $\Omega_m^{(0)} \approx 0.3$, we get $H_0 d_L \approx 1.23$ which is in good agreement with the observational value. Numerous other observations from SN Ia supernovae are in agreement with this result[22] [25].

Age of the universe

Another set of evidence comes from the comparison of the age of the universe(t_0) and the age of the oldest stellar populations (t_s). Trivially, the condition $t_0 > t_s$ should be satisfied, but we will see that only the presence of normal matter violates this.

The age of Globular clusters in the Milky Way is estimated to be $12.7 \pm 0.7 Gyr$[12] [20] [26], this can be used as a lower bound for the age of the universe such that $t_0 > 11 - 12 Gys$. From the Friedmann equation 2.1, the age of the universe can be shown to have the form

$$t_0 = \int_0^\infty \frac{\mathrm{d}z}{H_0 x \left[\Omega_r^{(0)} x^4 + \Omega_m^{(0)} x^3 + \Omega_\Lambda^{(0)} - \Omega_K^{(0)} x^2\right]^{1/2}} \tag{2.5}$$

where $x(z) = 1 + z$. Since the radiation-dominated period is much shorter than the age of the universe, we can neglect the contribution from radiation.

First, let there be no contribution from $\Omega_\Lambda^{(0)}$. Using the fact that $\Omega_K^{(0)} = \Omega_m^{(0)} - 1$. We get from (1.5)

$$t_0 = \frac{2}{3H_0}$$

Using the Hubble parameter value $H_0^{-1} = 9.776 h^{-1} Gyr$ with $0.64 < h < 0.80$ [9]. We get for a flat universe ($K = 0$), $t_0 = 8 - 10 Gyr$. This does not satisfy the bound.

We can find a cure for this, by using a flat model with a cosmological constant. In this case 3.2 gives

$$\begin{aligned} H_0 t_0 &= \int_0^\infty \frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_m^{(0)}(1+z)^3 + \Omega_\Lambda^{(0)}}} \\ &= \frac{2}{3\sqrt{\Omega_\Lambda^{(0)}}} \ln\left(\frac{1+\sqrt{\Omega_\Lambda^{(0)}}}{\sqrt{\Omega_m^{(0)}}}\right) \end{aligned}$$

when $\Omega_\Lambda^{(0)} = 0.7$ and $\Omega_m^{(0)} = 0.3$ we get $t_0 = 13.1 Gyr$ which satisfies the bound.

2.6.2 Graphical analysis of Cosmological parameters

As in 2.3, we can write the Hubble parameter as
$H^2 = H_0^2 \left[\Omega_m^{(0)} a^{-3} + \Omega_\Lambda^{(0)} a^{-3(1+w_\Lambda)}\right]$
Using which we obtain the following relations $\Omega_m(a)$ and $\Omega_\Lambda(a)$ and

$$\Omega_m(a) = \frac{\Omega_m^{(0)} a^{-3}}{\Omega_m^{(0)} a^{-3} + \Omega_\Lambda^{(0)} a^{-3(1+w_\Lambda)}}$$

$$\Omega_\Lambda(a) = \frac{\Omega_\Lambda^{(0)} a^{-3(1+w_\Lambda)}}{\Omega_m^{(0)} a^{-3} + \Omega_\Lambda^{(0)} a^{-3(1+w_\Lambda)}}$$

Plotting these against a for $w_\Lambda = -1$, $\Omega_\Lambda^{(0)} = 0.7$ and $\Omega_m^{(0)} = 0.3$ we get

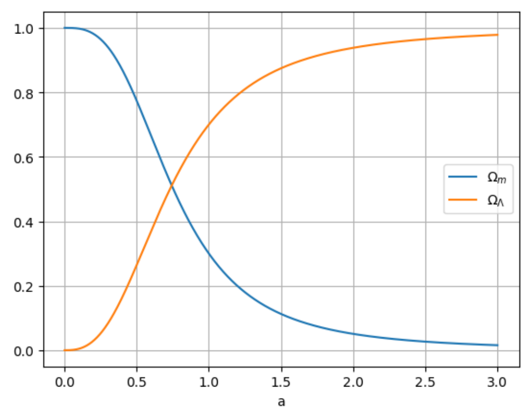

Figure 2.3: $\Omega_m(a)$ and $\Omega_\Lambda(a)$ vs a

From the figure, it is clear that initially, matter dominates, whereas dark energy is negligible. But, with increasing a matter density decrease and dark energy becomes dominant. The present epoch is at $a = 1$ where we have $\Omega_\Lambda^{(0)} = 0.7$ and $\Omega_m^{(0)} = 0.3$. Notice that these are the same orders of magnitude, this is called the Cosmic coincidence problem.

Deceleration parameter

The deceleration parameter, q is defined as

$$q(a) = \frac{-a\ddot{a}}{a^2}$$

This gives us a dimensionless measure of the acceleration of the universe.

Since $\frac{a}{a^2}$ is a positive quantity, $q < 0$ if $\ddot{a} > 0$ and $q > 0$ if $\ddot{a} < 0$.

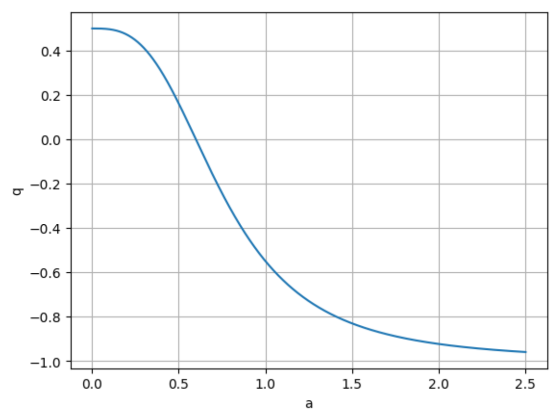

Figure 2.4: q vs a

This graph can be interpreted as follows, initially, during the matter-dominant epoch, the universe's expansion rate was decelerating, but as dark energy become more and more prevalent, the universe's expansion is accelerating. From the graph, we see that at the present epoch($a = 1$), the universe is accelerating.

Chapter 3

Cosmology in Scalar-Tensor theory of gravity

3.1 Mach's Principle

An important principle that helped Einstein formulate general relativity is the **Mach's Principle**, proposed by Ernst Mach in 1883. He pointed out some logical and philosophical inconsistencies in Newton's formulations. Newton believed that all motion in the universe, uniform and accelerated are all with respect to an absolute space which exists 'without relation to anything else, always similar and immovable' in his own words. According to him, Inertial forces were a result of acceleration with respect to this absolute space.

Mach questioned the validity of these ideas, why should the so-called inertial reference frame have the special status that it does not require inertial forces? He found the argument to be circular, one needs the second law to identify an inertial reference frame, but the second law is in turn dependent on the existence of inertial frames. Mach suggested that motion only makes sense when it is relative, that is, in an otherwise empty universe, a body cannot be said to be in motion, it only makes sense with reference to other matter.

He postulated that inertial effects are the result of interaction between all matter in the universe.
In this spirit, it basically means that what happens locally (inertial frame), is determined by the stars spread out through the universe through their masses, distribution, and motion. Einstein was greatly influenced by this principle and incorporated some of the ideas from Mach into general relativity, but not completely so.
Refer [3] to see different versions of Mach's principle which appear in literature.

3.2 Brans-Dicke theory

1n 1961, C. Brans and R.H Dicke came up with a theory of gravity which tries to fully incorporate Mach's principle[4]. Although in general relativity spatial geometry are affected by mass distribution, it is not uniquely specified by the distribution. Brans and Dicke tried to include this aspect also in their theory, but only partially.

According to Mach's principle, as discussed above, inertial forces observed locally in an accelerated laboratory may be interpreted as gravitational effects, having their origin in distant matter accelerated relative to the laboratory. If this is the case, then that means that the locally observed values of inertial masses will depend upon the point in consideration. However, to compare the masses at different points in spacetime we need a consistent unit of mass since a usual unit of masses are unreliable as they are measured in terms of masses of elementary particles, which are themselves subject to change. Therefore we need an independent unit of mass against which an increase or decrease in the mass of a particle can be measured. One such characteristic mass is provided by gravity in the form of

$$\sqrt{\frac{\hbar c}{G}} \approx 2.16 \times 10^{-5}$$

so a dimensionless quantity $m\sqrt{\frac{G}{\hbar c}}$ measured at different points in spacetime can tell us whether the masses m are changing.

So, if there is a change of this quantity as predicted by Mach's principle, This would mean that either m or G is changing (we consider $\hbar$ and c to be fixed in accordance with the principle of least modification of existing theories. Thus special relativity and quantum theory are unaffected). They pointed out that a varying m and G are equivalent physically but the formal structure of theory would be different. But the theoretical structure is simpler if we let the masses of elementary particles be constant and G to vary.

In 1953, D. W. Sciama[23] had given some argument which would explain G in terms of the large scale structure of the universe [10] [17]. This can be derived from Friedmann cosmology, where we have $\rho_0 = \frac{3H^2}{4\pi G} q_0$. If we interpret $\frac{4\pi\rho_0 R_0^3}{3}$ as characteristic mass of the universe, where $R_0 = \frac{c}{H}$ is the characteristic radius. We obtain the following relation

$$\frac{1}{G} = \frac{M_0}{R_0 c^2} q_0^{-1} \sim \frac{M_0}{R_0 c^2} \sim \sum \frac{m}{rc^2}$$

This suggests that the reciprocal of the gravitational constant which is determined as a linear superposition of contributions from the matter m at distance r from the point where G is measured is causally connected to the point in consideration. Since $\frac{m}{r}$ is a solution of a wave equation with a point source m. They postulated that G behaves as the reciprocal of a scalar field ϕ, $G^{-1} \sim \phi$.

3.2.1 Brans-Dicke Action

Generalizing the usual GR action or the Einstein-Hilbert action by dividing it by G which can be identified with ϕ^{-1}. The simple wave equation for ϕ with a scalar matter density as the source would give

$$S = \frac{1}{16\pi} \int \left(-\phi R + \frac{\omega_{BD}}{\phi} \partial_a \phi \partial^a \phi \right) \sqrt{-g} d^4x$$

This is the generalized action for the gravitational field. Comparing with Einsteinian version $I = \frac{1}{16\pi G}\int R\sqrt{-g}dx^{\mu}$. We see that the scalar field ϕ is now coupled to the Ricci scalar and the second term which is the Lagrangian density of ϕ ensures that ϕ will satisfy a wave equation. The ϕ in the denominator of the second term is to ensure that ω_{BD} is constant and ω_{BD} itself is the coupling constant referred to as the Brans-Dicke(BD) parameter.

3.3 Scalar-tensor theories of gravity

Scalar-tensor theories of gravity are the generalization of Brans-Dicke theory. These theories are commonly referred to as such because they go beyond the traditional framework of Einstein's gravity, which solely incorporates the metric tensor g_{ik} to describe gravitational interactions. In contrast, these theories introduce additional scalar fields in addition to the metric tensor to describe the gravitational field. The action in a general scalar-tensor theory is given by (here ψ is the scalar field),

$$S = \frac{1}{16\pi}\int(-f(\psi)R + \frac{1}{2}\partial_a\psi\partial^a\psi - U(\psi))\sqrt{-g}d\Omega \tag{3.1}$$

Where $U(\psi)$ is the potential.

3.3.1 Field equations

We can vary the action with respect to the metric tensor to obtain the Einstein tensor and also with respect to the scalar field to obtain the equation for ψ. Varying with g_{ab} gives

$$\delta I_m = \frac{1}{16\pi}\int(-f(\psi)\delta(\sqrt{-g}g^{ab}R_{ab}) + \frac{1}{2}\partial_\mu\psi\partial^\mu\psi\delta(\sqrt{-g}) - U(\psi)\delta(\sqrt{-g}))d\Omega \tag{3.2}$$

Where we have used the fact that $R = g^{ab}R_{ab}$. Let us look at the first of the three terms, $\delta I_{m_1} = \int(-f(\psi)\delta(\sqrt{-g}g^{ab}R_{ab})d\Omega$. Applying the product rule of variations, this can be turned into the following two terms

$$\delta I_{m_1} = \int (f(\psi)G^{ab}\delta g_{ab} - f(\psi)\left[\nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac}\right])\sqrt{-g}g^{ab}d\Omega$$

Notice that, unlike the Einsein-Hilbert action, the second term cannot be zero by virtue of the divergence theorem. Hence by varying the Christoffel symbol, after a long calculation, we obtain

$$\int f(\psi)\left[\nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac}\right]\sqrt{-g}g^{ab}d\Omega = \int (\nabla^a\nabla^b f(\psi) - g^{ab}\Box f(\psi))\delta g_{ab}\sqrt{-g}d\Omega$$

The variation of the other two terms in are straightforward, so we finally get,

$$\delta I_m = \frac{1}{16\pi}\int (f(\psi)G^{ab}\delta g_{ab} - (\nabla^a\nabla^b f(\psi) - g^{ab}\Box f(\psi)) + \frac{1}{4}\partial_c\psi\partial^c\psi g^{ab} - \frac{1}{2}\partial^a\psi\partial^b\psi - \frac{1}{2}g^{ab}U(\psi))\delta g_{ab}\sqrt{-g}d\Omega$$

which yields the field equation

$$G^{ab} = \frac{8\pi T^{ab}}{f(\psi)} + \frac{1}{f(\psi)}(\nabla^a\nabla^b f(\psi) - g^{ab}\Box f(\psi)) + \frac{1}{2f(\psi)}(\partial^a\psi\partial^b\psi - \frac{1}{2}\partial_c\psi\partial^c\psi g^{ab}) + \frac{1}{2f(\psi)}g^{ab}U(\psi) \tag{3.3}$$

Similarly, varying the action with respect to ψ yields the field equation

$$\Box\psi = -(f_\psi R + U_\psi)$$

Notice that this indeed looks likes a scalar-wave equation.

3.4 Cosmology in Scalar tensor theory

Now that we have derived the field equations, the Friedmann equations are straightforward to obtain. We use the same Robertson-Walker metric here. Therefore, the Friedmann equations become

$$H^2 = \frac{1}{3f(\psi)}\left[8\pi\rho + \frac{\dot{\psi}^2}{4} - 3\dot{f}H + \frac{U(\psi)}{2}\right] \tag{3.4}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6f(\psi)}\left[8\pi(3p+\rho) + 3\ddot{f} + 3\dot{f}H + \dot{\psi}^2 - U(\psi)\right] \tag{3.5}$$

Here, ρ and p are functions of time

3.4.1 Equivalence to Generalised Brans-Dicke theory

There are several other ways to present scalar-tensor theories, one common alternative often found in literature is one which resembles the usual Brans-Dicke action. This can be obtained from 3.1 under the following transformations:

$$\phi = f(\psi)$$

then we obtain the following relation

$$\frac{\partial_\mu\psi\partial^\mu\psi}{2} = \frac{\partial_\mu\phi\partial^\mu\phi}{2f'(\phi)^2}$$

Define $2f'(\psi)^2 = \frac{\phi}{\omega_{BD}(\phi)}$ and $U(\psi) = V[f(\psi)]$.

Under these transformations, the action takes the form

$$S = \frac{1}{16\pi}\int(-\phi R + \frac{\omega_{BD}(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi))\sqrt{-g}d\Omega \tag{3.6}$$

This is called the Generalized Brans-Dicke theory. For $\omega_{BD}(\phi) = const$ and $V(\phi) = 0$ this reduces to the usual Brans-Dicke theory.

Using these transformations in equations 3.4 and 3.5 gives

$$H^2 = \frac{1}{3\phi}\left[8\pi\rho + \frac{\omega_{BD}}{2}\frac{\dot{\phi}^2}{\phi} - 3H\dot{\phi} + \frac{V(\phi)}{2}\right] \tag{3.7}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6\phi}\left[8\pi(\rho+3p) + 3\ddot{\phi} + 2\omega_{BD}\frac{\dot{\phi}^2}{\phi} + 3H\dot{\phi} - V(\phi)\right] \tag{3.8}$$

For the scalar equation, it is easier to vary the action directly and this gives

$$\Box\phi = -\frac{\phi}{2\omega_{BD}}\left(R + V_\phi\right) + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\left(\frac{1}{\phi} - \frac{\frac{d\omega_{BD}}{d\phi}}{\omega_{BD}}\right) \tag{3.9}$$

We can find the Ricci Scalar by contracting the field equation 3.3, we obtain

$$R = \frac{1}{\phi}\left(-8\pi T + 3\Box\phi + \frac{\omega_{BD}\nabla^\nu\phi\nabla_\nu}{\phi} - 2V(\phi)\right)$$

Where T is the trace of the energy-momentum tensor. Substituting this in 3.9 and noting that ϕ is only a function of time, we obtain

$$\ddot{\phi} + 3H\dot{\phi} = \frac{1}{2\omega_{BD}+3}\left[8\pi T - \frac{d\omega_{BD}}{d\phi}\dot{\phi}^2 - \frac{dV}{d\phi}\phi + 2V(\phi)\right] \tag{3.10}$$

The trace T has the value, $T = \rho - 3p$. From here on, we assume that matter is assumed to be pressureless so $T = \rho$.

3.5 Some results

3.5.1 Constancy of ω_{BD} for power law solutions

One of the results we obtained for the generalized Brans-Dicke theory in the simplest case of power law solutions for a and ϕ is that it implies that ω_{BD} is necessarily constant, this is straightforward to see and we will see how. Similar results have been derived in other works, for example, refer [15]

Let $a(t) = a_0(\frac{t}{t_0})^n$ and $\phi(t) = \phi_0(\frac{t}{t_0})^m$. We can obtain $V(a, \dot{a}, \ddot{a}, \phi, \dot{\phi}, \ddot{\phi})$ and $\omega_{BD}(a, \dot{a}, \ddot{a}, \phi, \dot{\phi}, \ddot{\phi})$ by simple algebraic manipulation of equations 3.7 and 3.8. We get

$$V = \ddot{\phi} + 5H\dot{\phi} + 4\phi H^2 + 2\phi\frac{\ddot{a}}{a} \tag{3.11}$$

$$\omega_{BD} = \frac{\phi}{\dot{\phi}^2}\left[2\phi H^2 + H\dot{\phi} - \ddot{\phi} - 2\phi\frac{\ddot{a}}{a}\right] \tag{3.12}$$

Substituting for $a(t)$ and $\phi(t)$ gives

$$\omega_{BD} = \frac{m - m^2 + mn + 2n}{m^2}$$

This implies that under power law, the BD parameter is necessarily a constant, hence, in this case the Generalized Brans-Dicke theory reduces to the usual Brans-Dicke theory with an added potential term. In our treatment of the Generalized Brans-Dicke later to explain the accelerated expansion of the universe, we assume that ω_{BD} be constant.

3.5.2 Power law solution for Brans-Dicke theory

In the case for which $\omega_{BD} = const$ and $V(\phi) = 0$, 3.6 reduces to the usual Brans-Dicke theory. We attempt to find power law solution of the form $a(t) = a_0(\frac{t}{t_0})^n$ and $\phi(t) = \phi_0(\frac{t}{t_0})^m$ for this.

Using 3.7, 3.8 and 3.10 we obtain

$$n = \frac{2\omega_{BD} + 2}{3\omega_{BD} + 4}, m = \frac{2}{3\omega_{BD} + 4} \tag{3.13}$$

For detailed derivation of this result, refer [16].

Chapter 4

Accelerated expansion of the universe in scalar-tensor gravity

In the previous chapter, we have built up all the prerequisites(mainly the Friedmann equations) to study cosmology in scalar-tensor theory. In this chapter, we consider Brans-Dicke theory with potential $V(\phi) \propto \phi^n$ and with $\omega_{BD} = const.$ In such a model we obtain the numerical solutions for $a(t)$ and $\phi(t)$, and investigate whether such model can be used to explain the accelerated expansions of the universe. Subsequently, we will plot the Omega parameters and deceleration parameters and compare them with the Einstanian/$GR - \Lambda CDM$ case.

4.1 Solving for $a(t)$ and $\phi(t)$

We can obtain a system of two differential equations from the Friedmann equation 3.8 and the scalar equation 3.10

$$\frac{d^2a}{dt^2} = -\frac{a}{6\phi}\left[8\pi\rho_m^{(0)}a^{-3} + 3\frac{d^2\phi}{dt^2} + \frac{2\omega_{BD}}{\phi}\left(\frac{d\phi}{dt}\right)^2 + \frac{3}{a}\frac{da}{dt}\frac{d\phi}{dt} - V(\phi)\right] \tag{4.1}$$

$$\frac{d^2\phi}{dt^2} = \frac{1}{2\omega_{BD}+3}\left[8\pi\rho_m^{(0)}a^{-3} - \frac{dV}{d\phi}\phi + 2V(\phi)\right] - \frac{3}{a}\frac{d\phi}{dt}\frac{da}{dt} \tag{4.2}$$

where we have used the fact that $\rho(t) = \rho_m^{(0)} a^{-3}$ from the conservation equation for matter. It is clear to see that this is a second-order non-linear differential equation, the solution is analytically intractable. We will solve this numerically, we use the solve ivp package in Python to tackle this system. We define natural scales a_i, ϕ_i and H_i for a, ϕ and t respectively. a_i and ϕ_i can be seen as the initial value of the corresponding quantities and H_i as the initial value of the Hubble parameter. Therefore the dimensionless quantities become $\tilde{a} = \frac{a}{a_i}$, $\tilde{\phi} = \frac{\phi}{\phi_i}$ and $\tau = H_i t$

We assume that the potential has a power law form so that $V(\phi) = V_0\phi^p$. Therefore, the dimensionless differential equations become

$$\frac{d^2\tilde{a}}{d\tau^2} = -\frac{\tilde{a}}{6\tilde{\phi}}\left[3\Omega_m^{(0)}\tilde{a}^{-3} + 3\frac{d^2\tilde{\phi}}{d\tau^2} + \frac{2\omega_{BD}}{\tilde{\phi}}\left(\frac{d\tilde{\phi}}{d\tau}\right)^2 + \frac{3}{\tilde{a}}\frac{d\tilde{a}}{d\tau}\frac{d\tilde{\phi}}{d\tau} - 6\beta\tilde{\phi}^p\right] \tag{4.3}$$

$$\frac{d^2\tilde{\phi}}{d\tau^2} = \frac{1}{2\omega_{BD}+3}\left[3\Omega_m^{(0)}\tilde{a}^{-3} + 6\beta(2-m)\tilde{\phi}^p\right] - \frac{3}{\tilde{a}}\frac{d\tilde{\phi}}{d\tau}\frac{d\tilde{a}}{d\tau} \tag{4.4}$$

where, $\Omega_m^{(0)} = \frac{8\pi\rho_m^{(0)}}{3\phi_0 H_0^2}$ and $\beta = \frac{V_0\phi_0^{p-1}}{6H_0^2}$, where H_0 is the present value of the Hubble constant. These quantities arise from the non-dimensionalizing.

Notice that 4.3 has a $\ddot{\tilde{\phi}}$ term, so substituting for this from 4.4 we get

$$\frac{d^2\tilde{a}}{d\tau^2} = -\tilde{a}\left[-\frac{\Omega_m^{(0)}}{2\tilde{\phi}}\left(\frac{6+2\omega_{BD}}{3+2\omega_{BD}}\right)\tilde{a}^{-3} - \frac{\omega_{BD}}{3\tilde{\phi}^2}\left(\frac{d\tilde{\phi}}{d\tau}\right)^2 + \frac{1}{\tilde{a}\tilde{\phi}}\frac{d\tilde{a}}{d\tau}\frac{d\tilde{\phi}}{d\tau} + \beta\left[\frac{3(m-2)}{3+2\omega_{BD}}+1\right]\tilde{\phi}^{p-1}\right] \tag{4.5}$$

4.1.1 Initial Conditions

To solve this system of differential equations, we need initial conditions on $\tilde{a}, \dot{\tilde{a}}, \tilde{\phi}$ and $\dot{\tilde{\phi}}$. We also need to know the initial values of $\Omega_m^{(0)}$ and β. Since $\tilde{a} = \frac{a}{a_i}$, $\tilde{\phi} = \frac{\phi}{\phi_i}$. Initial conditions for $\tilde{a}$ and $\tilde{\phi}$ are $\tilde{a} = 1$ and $\tilde{\phi} = 1$. Further, since $\tau = H_i t$, $\frac{d\tilde{a}}{d\tau} = 1$ initially. The initial condition for $\frac{d\tilde{\phi}}{d\tau}$ will come clearer later in this discussion. Using these initial conditions, consider the dimensionless equation for H^2

$$\tilde{H}^2 = \frac{\Omega_m^{(0)} a^{-3}}{\tilde{\phi}} + \frac{\omega_{BD}}{6}\frac{\dot{\tilde{\phi}}^2}{\tilde{\phi}^2} - \frac{H\dot{\tilde{\phi}}}{\tilde{\phi}} + \beta\tilde{\phi}^{m-1} \tag{4.6}$$

where, $\tilde{H} = \frac{H}{H_i}$

For the initial time t_i(note that $H_i = 1$) we get

$$1 = \Omega_m^{(0)} - \alpha + \frac{\alpha^2 \omega_{BD}}{6} + \beta$$

where $\alpha = \left(\frac{d\tilde{\phi}}{d\tau}\right)_{t=t_i}$

Rearranging,

$$\Omega_m^{(0)} = 1 + \alpha\left(1 - \frac{\alpha\omega_{BD}}{6}\right) - \beta \tag{4.7}$$

For $\Omega_m^{(0)}$ we turn to the Einstenian case and look for the initial value for $\Omega_m(t)$ (Note that we use $\Omega_\Lambda^{(0)} = 0.7$ and $\Omega_m^{(0)} = 0.3$), this will be very close to 1 as initially matter is completely dominant, we take this to be

$$\Omega_m^{(0)} = 1 - 10^{-8}, \text{at } a = 10^{-3}$$

As for α, for the power law case, $V(\phi) = V_0\phi_0(\frac{t}{t_0})^{pm}$.

$V(\phi)$ goes as $\frac{1}{t^{m(1-p)}}$ in equation 4.9 (as we shall see, for values of m and p we will consider, $m(1-p) < 2$) whereas, the other terms go as $\frac{1}{t^2}$, so, for t close to 0, these other terms dominate compared to the potential term and hence we can neglect the potential for initial times, in which case we can use the result obtained in section 3.5.2. For this power law case we get that

$$\alpha = \left(\frac{d\tilde{\phi}}{d\tau}\right)_{t=t_i} = \frac{m}{n}$$

where, $n = \frac{2\omega_{BD}+2}{3\omega_{BD}+4}, m = \frac{2}{3\omega_{BD}+4}$ as in 3.13

Now that we know $\Omega_m^{(0)}$ and α. Equation 4.7 fixes β.

4.1.2 Omega parameters

To study the evolution of matter and dark energy, we need to find out the Ω parameters. In this case, instead of the Ω_Λ term which explains dark energy as in the $GR - \Lambda CDM$ model, we will have a Ω_ϕ term. The H^2 equation is

$$\tilde{H}^2 = \frac{\Omega_m^{(0)} a^{-3}}{\tilde{\phi}} + \frac{\omega_{BD}}{6} \frac{\dot{\tilde{\phi}}^2}{\tilde{\phi}^2} - \frac{\tilde{H}\dot{\tilde{\phi}}}{\tilde{\phi}} + \beta \tilde{\phi}^{m-1} \tag{4.8}$$

Comparing with the corresponding H^2 equation in $GR = \Lambda CDM$

$$H^2 = H_0^2 \left[\Omega_m^{(0)} a^{-3} + \Omega_\Lambda^{(0)} a^{-3(1+w_\Lambda)} \right]$$

We define, $\Omega_m(t) = \frac{\Omega_m^{(0)} a^{-3}}{\tilde{\phi}\tilde{H}^2}$ and $\Omega_\phi(t) = \frac{1}{\tilde{H}^2} \left[\frac{\omega_{BD}}{6} \frac{\dot{\tilde{\phi}}^2}{\tilde{\phi}} - \frac{H\dot{\tilde{\phi}}}{\tilde{\phi}} + \beta \tilde{\phi}^{m-1} \right]$

Its clear to see from 4.8 that $\Omega_m(t) + \Omega_\phi(t) = 1$, so, this kind of serves as a conservation equation. Computationally, these relations can be used to check the validity of the solutions we obtain.

4.2 Exploring Cosmological Solutions

4.2.1 Plotting $a(t)$ and $\phi(t)$

Using the initial conditions we found above, we try to solve the system of differential equations 4.3 and 4.4. For the BD parameter, we take $w_{BD} = 3000$ which is consistent with the bound of $\omega_{BD} > 2230$ set by [13]. We plot $a(t)$ and $\phi(t)$ for p values (power of the potential term) ranging from -3 to 4. Why we choose these particular values will become clear later. All the related Python code can be found in the github repository [1]

[1] https://github.com/Nihal/Cosmology-in-Scalar–tensor-gravity.git

$\underline{a(t)}$

Plotting $a(t)$ for an initial time $t = 0$ and final time $t = 100$, we get for different values of p.

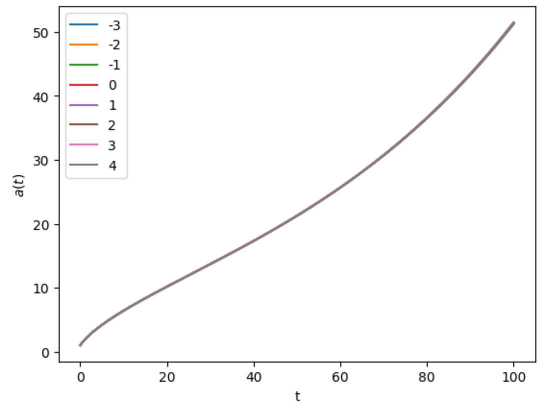

Figure 4.1: $a(t)$ vs t, here the legend indicates values of p

For a longer time interval, we see that $a(t)$ increases very rapidly.

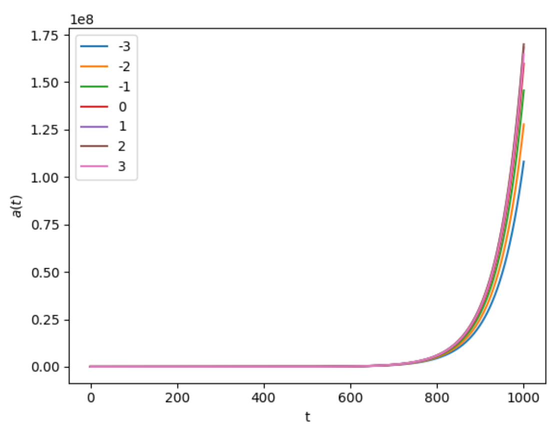

Figure 4.2: $a(t)$ vs t for longer time, note that the y axis is being multiplied by 10^8

$\underline{\phi(t)}$

Plotting $\phi(t)$ for an initial time $t = 0$ and final time $t = 100$ for different values of p, we get

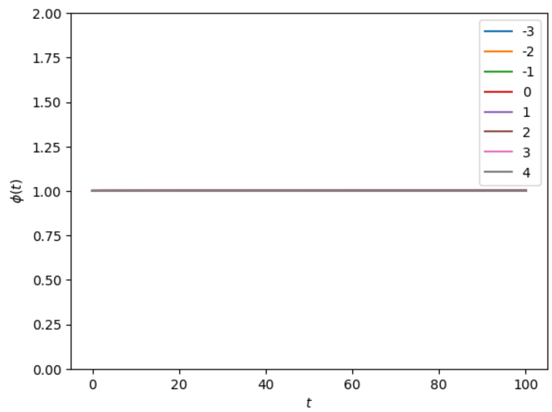

Figure 4.3: $\phi(t)$ vs t

We notice that the scalar field $\phi(t)$ seems to behave like a constant. As we discussed in earlier sections, $\phi(t)$ can be identified with the reciprocal of Newton's gravitational constant and we don't expect it to vary much, but, to see more a complete picture, we should plot for a longer time interval as below

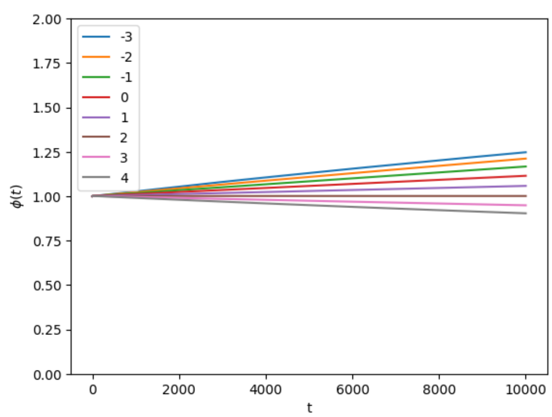

Figure 4.4: $\phi(t)$ vs t for longer time

Interestingly enough, we see that for a longer time, $\phi(t)$ is not exactly constant, but varies very slowly as expected.

The conservation equation

In 4.1.2 we mentioned how the relation $\Omega_m(t) + \Omega_\phi(t) = 1$ can serve as a check to see if the numerical solutions we obtain are consistent, below we plot it for different values of p

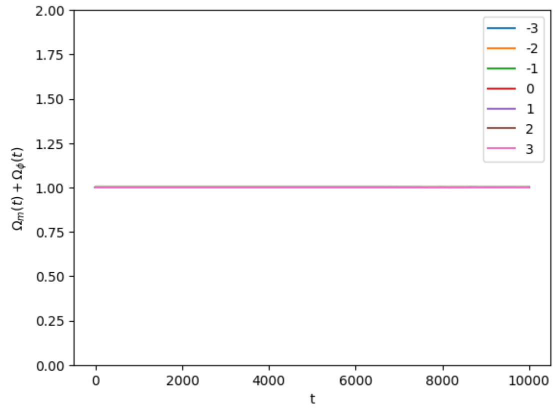

Figure 4.5: $\Omega_m(t) + \Omega_\phi(t)$ vs t

It is clear that the relation indeed holds for all values of p thereby confirming the validity of our solutions.

4.2.2 Ω vs a

Using the definitions derived above, we try to plot the variation of the density parameters with the scale factor a and compare it with the Einsteinian case. Note that, here we take values of p ranging from -3 to 4, below is the plot for this range of p values.

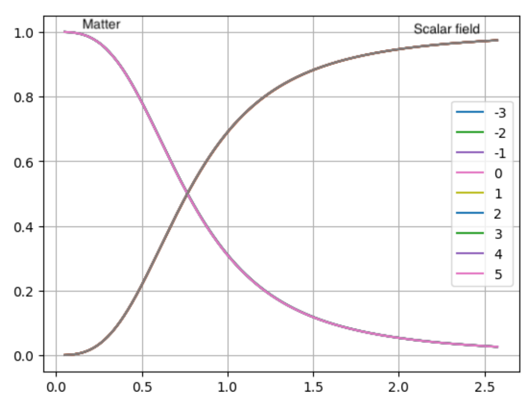

Figure 4.6: $\Omega_m(a) + \Omega_\phi(a)$ vs a

Here the legends indicate the different values of p. The Omega parameters corresponding to matter($\Omega_m(a)$) and scalar field ($\Omega_\phi(a)$) are as indicated in the plot. It is evident that this is almost identical to the $GR-\Lambda CDM$ model case(Figure 2.3) we discussed in section 2.6.2.

We see that $\Omega_m^{(0)}$ is in good agreement with the $GR-\Lambda CDM$ case with $\Omega_m^{(0)} = 0.3$ (as mentioned in Chapter 2) and $\Omega_\phi^{(0)}$ with the $\Omega_\Lambda^{(0)} = 0.7$ case in $GR-\Lambda CDM$.

In conclusion, comparing this result with the $GR-\Lambda CDM$ model, it's clear to see that the scalar field mimics dark energy very well.

In the next section, we look at the deceleration parameter to study the accelerated expansion in more detail and see which value of p is in best agreement with the $GR-\Lambda CDM$ model.

4.2.3 Deceleration parameter

We can do a deeper analysis on how close our results are by comparing the corresponding deceleration parameters to the $GR-\Lambda CDM$ case.

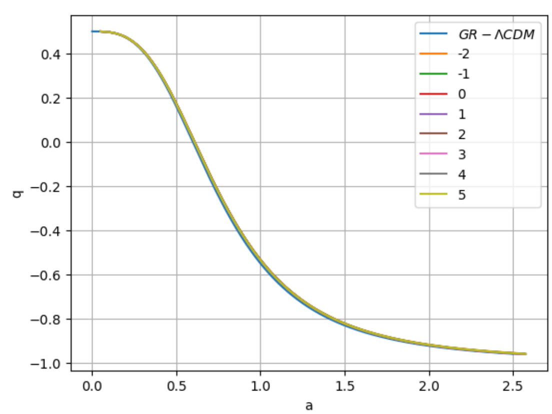

Figure 4.7: $q(a)$ vs a

We see that our results are in very good agreement with the $GR-\Lambda CDM$ model. The plot confirms that the universe does undergo acceleration at late times as expected.

For greater or lesser values of p than in the range we took, the deviation is seen to be substantial enough to disregard them, hence justifying why we took this range in the first place. An example here is shown for $p = 15$ (there is no particular reason for this value, it is just to demonstrate the deviation for higher values of p)

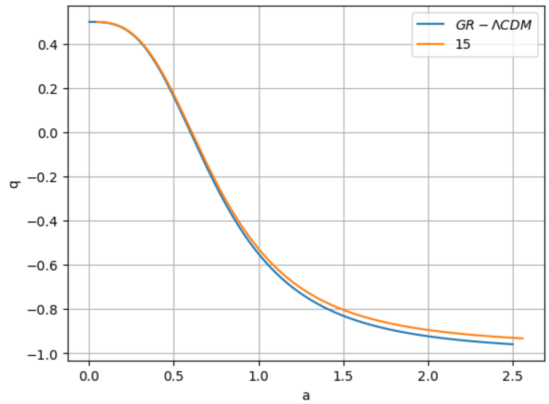

Figure 4.8: $q(a)$ vs a for $p = 15$

We see that for late times, it deviates from the $GR - \Lambda CDM$ model case.

Additionally, we can numerically try to find out for which value p is our q curve the closest to the $GR - \Lambda CDM$ model. We do this by calculating the distance between the curves for different ranges of p and the $GR - \Lambda CDM$ curve.

We find that the closest approximation to the $GR - \Lambda CDM$ curve is when $p \approx 4.615$. The corresponding plot is

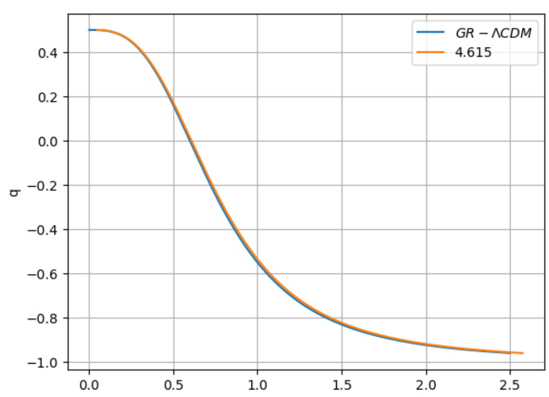

Figure 4.9: $q(a)$ vs a for $p = 4.6156$

We can also see what value q takes at $z = 0 (a = 1)$, summarized by the following table(rounded to four decimal places)

Table 4.1: $q(0)$ for different p

p	$q(0)$
-3	-0.5314
-2	-0.5325
-1	-0.5334
0	-0.5341
1	-0.5347
2	-0.5351
3	-0.5354
4	-0.5355
5	-0.5355
6	-0.5354

Comparing with the $GR - \Lambda CDM$ model value $q(0) = -0.55$, we see that the $p = 4$ in the given table is the closest. We can get a more precise p by trial method(computationally) and it comes out to be $p \approx 4.4264$. This in fact matches well with our previous approximation of 4.615. Hence, in our case, we can conclude that the power of potential p takes a value approximately in the range of $4.2 - 4.7$.

These plots corroborate the main subject of our dissertation, that late time acceleration of the universe can be explained well within in this alternative formulation of general relativity, namely, scalar-tensor theory.

4.3 Equation of state parameter for dark energy

We use the $a(t)$ we found in Brans-Dicke theory for different values of p in the potential $V(\phi) = V_0\phi^n$ to find what equation of state parameter for dark energy, $w_{de} = \frac{p_{de}}{\rho_{de}}$ in GR would lead to the same $a(t)$.

From the Friedmann equations 2.1 and 2.2 we obtain,

$$\rho_{de} = \frac{3H^2}{8\pi G}\left(1 - \frac{\Omega_m^{(0)} a^{-3}}{\frac{H^2}{H_0^2}}\right)$$

$$p_{de} = \frac{-H^2}{4\pi G}\left(\frac{1}{2} - q\right)$$

Consequently, the equation of the state parameter w_{de} in GR can be expressed as

$$w_{de} = -\frac{1}{3}\left(\frac{H^2(1-2q)}{H^2 - \Omega_m a^{-3} H_0^2}\right)$$

From the numerical solutions of $a(t)$ in Brans-Dicke theory with $V(\phi) = V_0\phi^n$, we find that at $z = 0$, for different values of p

p	$1 + w_{de}$
-2	2.6045×10^{-4}
-1	-1.0125×10^{-4}
0	-3.6376×10^{-4}
1	-5.2691×10^{-4}
2	-5.9061×10^{-4}
3	-5.5475×10^{-4}
4	-4.1927×10^{-4}
5	-1.8430×10^{-4}

We see that $1 + w_{de}$ is very close to the $GR - \Lambda CDM$ case of $1 + w_{de}(z) = 0$ for $z = 0$.

Chapter 5

Conclusion

In this dissertation, I studied the scalar-tensor theory of gravity and investigated the possibility of scalar fields inherent in the theory as a candidate for explaining dark energy and the observed late-time accelerated expansion of the universe.

We covered the basics of scalar-tensor theory(including Brans-Dicke theory) and the motivation for such a formulation of gravity as a potential contender to Einstein's general relativity in explaining the universe at the largest scales. We derived the Friedmann equations in the framework of this formulation hence helping us study the dynamics of the universe. An interesting consequence of the Brans-Dicke parameter (ω_{BD}) being constant under power law solutions was shown.

Using the Friedmann equation so derived, we delved deeper into the dynamics. For a given set of initial conditions that are grounded in observations and supported by theoretical predictions, we solved for the scale factor $a(t)$ and $\phi(t)$ and consequently the density parameters corresponding to both these. We see that these predictions closely resemble the $GR - \Lambda CDM$ model for a wide range of parameter p in the potential $V(\phi) = V_0\phi^p$ and that the scalar field density mimics dark energy, thereby presenting itself as a compelling candidate for dark energy.

Further, we studied the deceleration parameters and saw that our results match very well with the $GR - \Lambda CDM$ model for a given range of p, i.e the value of the exponent for the power law potential ansatz we had chosen. We tried numerically find out the 'best-fit' p value, by calculating the value of p for which our model is the closest to the $GR - \Lambda CDM$ model and also by comparing the present epoch value of the deceleration parameter. We found out that this p is approximately within the range 4.2 to 4.7 for our model.

In conclusion, we found that the scalar-tensor theory of gravity is a very promising candidate for explaining the late-time accelerated expansion of the universe.

5.1 Future work

I would like to continue my work in this field and explore the scalar-tensor theory of gravity and its applications in more detail. Below are some examples of the work ahead

1. To include non-canonical scalar fields: We would like to explore generalizations of scalar-tensor theory with non-canonical scalar fields and study the conditions which would give a power law solution for the scale factor.

2. Hubble tension: As mentioned in the introduction, scalar-tensor theories could provide a potential solution to the puzzling Hubble tension, and we would like to explore this avenue more.

3. Stability analysis: We would like to examine in detail the stability of the models we obtained.

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