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More on wormholes supported by small amounts of exotic matter

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Recent papers by Fewster and Roman have emphasized that wormholes supported by arbitrarily small amounts of exotic matter will have to be incredibly fine-tuned if they are to be traversable. This paper discusses a wormhole model that strikes a balance between two conflicting requirements, reducing the amount of exotic matter and fine-tuning the metric coefficients, ultimately resulting in an engineering challenge: one requirement can only be met at the expense of the other. The wormhole model is macroscopic and satisfies various traversability criteria.

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I. INTRODUCTION

Two recent papers by Fewster and Roman [1,2] discuss a variety of problems arising from wormholes using arbitrarily small amounts of exotic matter, including wormhole models by Visser [3] and Kuhfittig [4–6]. It was found that two of the Kuhfittig models are seriously flawed by not taking into account the proper distances in estimating the size of certain wormholes. A corrected version of these models resulted in a wormhole that turned out to be traversable but contained an exotic region whose proper thickness was much larger than desired for wormhole construction [2]. Some of the other models discussed in [1] need to be incredibly fine-tuned, even for small throat sizes.

The purpose of this paper is to propose a reasonable balance between reducing the proper thickness of the exotic region (the region in which the weak energy condition is violated) and the amount of fine-tuning required to achieve this reduction. There are no particular restrictions on the throat size. Various constraints are shown to be met, so that the resulting wormhole is traversable for humanoid travelers.

II. THE BASIC MODEL

Our starting point is the line element [6]

$$ds^2 = -e^{2\gamma(r)}dt^2 + e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where $\gamma(r)$ is the redshift function, which must be everywhere finite. Our units are taken to be those in which $G = c = 1$. The function α has a vertical asymptote at the throat $r = r_0$:

$$\lim_{r\to r_0+} \alpha(r) = +\infty.$$

If $e^{2\alpha(r)} = 1/(1 - b(r)/r)$ [Morris and Thorne (MT) [7]], then the "shape function" $b(r)$ can be written

$$b(r) = r(1 - e^{-2\alpha(r)}). \tag{2}$$

In the absence of any other conditions the proper radial distance $\ell(r)$ to the throat from any point outside, given by

$$\ell(r) = \int_{r_0}^{r} e^{\alpha(r')}dr', \tag{3}$$

may diverge as $r \to r_0$. One way to avoid this problem is to start with the function

$$\alpha_1(r) = \ln\frac{K}{(r - r_0)^A}, \qquad 0 < A < 1, \tag{4}$$

in the vicinity of the throat; K is a constant having the same units as $(r - r_0)^A$ and will be determined later. Since this function eventually becomes negative, it will have to be joined smoothly to some $\alpha_2(r) > 0$ which goes to zero as $r \to \infty$. (This construction will be made explicit below.)

Now observe that for $\alpha_1(r)$

$$\ell(r) = \int_{r_0}^{r} e^{\ln K/(r'-r_0)^A}dr' = \int_{r_0}^{r} \frac{K}{(r' - r_0)^A}dr', \tag{5}$$

which is finite for $0 < A < 1$. In fact, $\ell(r_0) = 0$. The redshift function is assumed to have a similar form in the vicinity of the throat:

$$\gamma_1(r) = -\ln\frac{L}{(r - r_2)^B}, \qquad 0 < B < 1, \tag{6}$$

where $0 < r_2 < r_0$ to avoid an event horizon at the throat. This function is also subject to modification.

Next, recall that the weak energy condition (WEC) requires the mass-energy tensor $T_{\alpha\beta}$ to obey

$$T_{\alpha\beta}\mu^\alpha\mu^\beta \geq 0$$

for all timelike vectors and, by continuity, all null vectors. Using the notation in Ref. [6], the violation of the weak energy condition is given by $\rho - \tau < 0$, where

$$\rho - \tau = \frac{1}{8\pi}\left[\frac{2}{r}e^{-2\alpha(r)}[\alpha'(r) + \gamma'(r)]\right]. \tag{7}$$

[Sufficiently close to the asymptote, $\alpha'(r) + \gamma'(r)$ is clearly negative.] To satisfy the Ford-Roman constraints, we would like the WEC to be satisfied outside some small interval $[r_0, r_1]$. To accomplish this, choose r_1 and con-

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struct α and γ so that

$$|\alpha_1'(r_1)| = |\gamma_1'(r_1)|.$$

It will be shown presently that, if

$$B = \frac{r_1 - r_2}{r_1 - r_0} A, \tag{8}$$

then $|\alpha_1'(r)| > |\gamma_1'(r)|$ for $r_0 < r < r_1$, and $|\alpha_1'(r)| < |\gamma_1'(r)|$ for $r > r_1$. More precisely,

$$\alpha_1'(r) = \frac{-A}{r - r_0} < \frac{-A(r_1 - r_2)}{r_1 - r_0} \frac{1}{r - r_2} = -\gamma_1'(r)$$

for $r_0 < r < r_1$; if $r > r_1$, the sense of the inequality is reversed. To see this, observe that equality holds for $r = r_1$. Now multiply both sides by $r - r_2$ and denote the resulting left side by $f(r)$, i.e.,

$$f(r) = \frac{-A(r - r_2)}{r - r_0}.$$

Since $f'(r) > 0$, $f(r)$ is strictly increasing, enough to establish the inequalities.

III. SOME MODIFICATIONS

As noted earlier, α_1 and γ_1 will eventually become negative. Anticipating this, we can cut these functions off at some $r = r_3$ and then connect them smoothly to suitable new functions α_2 and γ_2. (For physical reasons r_3 should be larger than r_1.) Suppose α_2 has the following form:

$$\alpha_2(r) = \frac{C}{r - r_0};$$

then

$$\alpha_2'(r) = -\frac{C}{(r - r_0)^2}.$$

Since we want $\alpha_1'(r_3) = \alpha_2'(r_3)$, we have

$$\alpha_2'(r_3) = -\frac{C}{(r_3 - r_0)^2} = \frac{-A}{r_3 - r_0}.$$

It follows that $C = A(r_3 - r_0)$ and

$$\alpha_2(r) = \frac{A(r_3 - r_0)}{r - r_0}.$$

Similarly,

$$\gamma_2(r) = -\frac{B(r_3 - r_0)}{r - r_0}.$$

Besides having slopes of equal absolute value at r_3, we want the functions to meet, i.e., we want

$$\alpha_1(r_3) = \alpha_2(r_3) \quad \text{and} \quad \gamma_1(r_3) = \gamma_2(r_3).$$

To this end we must determine K and L. For the first case,

$$e^{\ln[K/(r_3 - r_0)^A]} = e^{[A(r_3 - r_0)]/(r_3 - r_0)} = e^A;$$

thus $K = e^A(r_3 - r_0)^A$. In a similar manner, $L = e^B(r_3 - r_0)^B$.

Returning to α_1 and γ_1 at $r = r_1$, A and B were chosen so that α_1 and γ_1 would have the desired properties. For $r < r_1$ the form of γ_1 can safely be altered: retain the value of the function as well as the slope at $r = r_1$ but for $r < r_1$ continue γ_1 as a straight line with slope $\gamma_1(r_1)$ or, better still, as a curve that is approximately linear. (In particular, γ_1 need not have an asymptote close to $r = r_0$.) As a result, $|\gamma_1''|$ is relatively small, while $\gamma' < |\alpha'|$ on $[r_0, r_1]$, as before.

The purpose of this change is twofold: to avoid a large time dilation near the throat (discussed at the end of Sec. VI) and to address the issue raised in Ref. [2]; as long as the asymptote is retained at $r = r_2$, the proper distance across the exotic region is shown to be quite large.

IV. THE CONSTRAINTS

Wormhole solutions allowed by general relativity may be subject to severe constraints from quantum field theory, particularly the quantum inequalities [8,9], also discussed in [1]. Of particular interest to us is Eq. (95) in Ref. [9]:

$$\frac{r_m}{r_0} \leq \left(\frac{1}{v^2 - b_0'}\right)^{1/4} \frac{\sqrt{\gamma}}{f} \left(\frac{l_p}{r_0}\right)^{1/2}, \tag{9}$$

where r_m is the smallest of several length scales, $\gamma = 1/\sqrt{1 - (v/c)^2}$, l_p is the Planck length, f is a small scale factor, and $b_0' = b'(r_0)$. Returning to Eq. (2), the shape function, we have

$$b'(r) = \frac{d}{dr}[r(1 - e^{-2\alpha(r)})] = \frac{d}{dr}\left[r\left(1 - \frac{1}{K^2}(r - r_0)^{2A}\right)\right]$$
$$= 1$$

when $r = r_0$, provided that $A > \frac{1}{2}$, required to meet the radial tidal constraint, discussed below. (Observe that b_0' does not even exist unless $A \geq \frac{1}{2}$.) For the right-hand side of inequality (9) to be defined and real, we must have $v^2 > b_0'$. Since $b_0' = 1$ and $v \leq 1$, the inequality is trivially satisfied, thereby removing any concerns about the size r_0 of the radius of the throat. Of course, the quantum inequality (9) cannot simply be circumvented in this manner—there exist other constraints which may be equally severe: (i) as noted in Ref. [1], a wormhole for which $b_0' = 1$ may be unstable and (ii) the exponent B in $\gamma_1(r)$ is subject to severe fine-tuning resulting in part from the tidal constraints (MT [7]).

To check these constraints, we need some of the components of the Riemann curvature tensor. From Ref. [5]

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = e^{-2\alpha_1(r)}(\gamma_1''(r) - \alpha_1'(r)\gamma_1'(r) + [\gamma_1'(r)]^2), \tag{10}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = \frac{1}{r} e^{-2\alpha(r)} \gamma_1'(r), \tag{11}$$

and

$$R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} = \frac{1}{r} e^{-2\alpha(r)} \alpha_1'(r). \tag{12}$$

For the radial tidal constraint we have

$$|R_{\hat{r}\hat{t}\hat{r}\hat{t}}| = |e^{-2\alpha_1(r)}(\gamma_1''(r) - \alpha_1'(r)\gamma_1'(r) + [\gamma_1'(r)]^2)|$$
$$= \left| e^{-2\ln K/(r-r_0)^A}\left(\frac{-B}{(r-r_2)^2} - \frac{-A}{r-r_0}\frac{B}{r-r_2} + \frac{B^2}{(r-r_2)^2}\right)\right|$$
$$= \left| \frac{(r-r_0)^{2A}}{K^2}\left(-\frac{B}{(r-r_2)^2} + \frac{AB}{(r-r_0)(r-r_2)} + \frac{B^2}{(r-r_2)^2}\right)\right|$$
$$\leq (10^8\ \mathrm{m})^{-2}. \tag{13}$$

As long as $A > \frac{1}{2}$, as before, $|R_{\hat{r}\hat{t}\hat{r}\hat{t}}|$ is close to zero whenever r is close to r_0. Of particular interest is the constraint at $r = r_1$, discussed next.

V. THE EXOTIC REGION

To help determine the size of the exotic region, we examine the above constraint at $r = r_1$. Since $|\alpha_1'(r_1)| = |\gamma_1'(r_1)|$, we have

$$\frac{A}{r_1 - r_0} = \frac{B}{r_1 - r_2}.$$

Substituting in Eq. (13) and using

$$r_1 - r_2 = \frac{B}{A}(r_1 - r_0)$$

from Eq. (8), we get

$$|R_{\hat{r}\hat{t}\hat{r}\hat{t}}| = \frac{(r_1-r_0)^{2A}}{K^2}\left(-\frac{A}{(r_1-r_0)}\frac{1}{(B/A)(r_1-r_0)} + \frac{2A^2}{(r_1-r_0)^2}\right)$$
$$= \frac{(r_1-r_0)^{2A}}{K^2}\frac{A^2(2-1/B)}{(r_1-r_0)^2} = (10^8\ \mathrm{m})^{-2}, \tag{14}$$

assuming now that the constraint is just met.

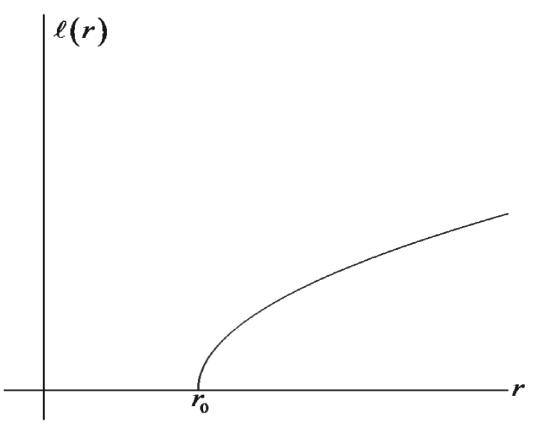

FIG. 1. The graph of $\ell(r) = \frac{K}{1-A}(r - r_0)^{1-A}$.

$$|R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}| = |R_{\hat{r}\hat{t}\hat{r}\hat{t}}| \leq (10^8\ \mathrm{m})^{-2}.$$

So from Eq. (10),

Taking the square root of the reciprocals, we get

$$K(r_1 - r_0)^{1-A}\frac{1}{A\sqrt{2-1/B}} = 10^8$$

or

$$K(r_1 - r_0)^{1-A} = 10^8 A\sqrt{2 - \frac{1}{B}}. \tag{15}$$

Returning to Eq. (3),

$$\ell(r) = \int_{r_0}^{r} K(r' - r_0)^{-A} dr' = \frac{K}{1-A}(r - r_0)^{1-A}. \tag{16}$$

(See Fig. 1; the graph is plotted using $A = \frac{1}{2}$, $K = 5$, and $r_0 = 200$ m.) In particular,

$$\ell(r_1) = \frac{K}{1-A}(r_1 - r_0)^{1-A} = \frac{A}{1-A}10^8\sqrt{2 - \frac{1}{B}}. \tag{17}$$

Even though $\ell(r_1) \to 0$ as $B \to \frac{1}{2}$, considerable fine-tuning is required, as can be seen from Fig. 2 (plotted using $A = \frac{1}{2}$): $\ell(r_1)$ rises so rapidly that even a small increase in B can result in a large increase in $\ell(r_1)$.

Next, let us obtain some numerical estimates: let $A = \frac{1}{2}$ again and recall that $K = e^A(r_3 - r_0)^A$. Then from

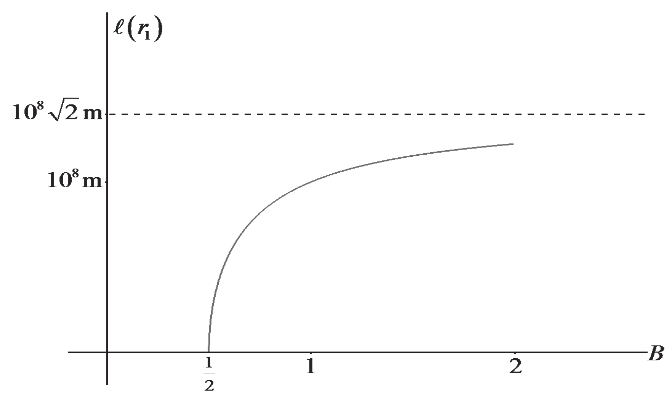

FIG. 2. The size of the exotic region as a function of B.

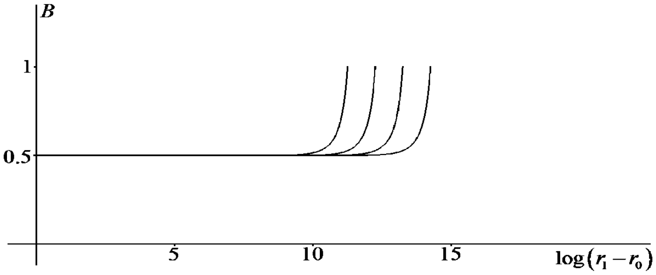

FIG. 3. A plot illustrating the dependence of the exponent B on $\log(r_1 - r_0)$ for several values of $r_3 - r_0$.

Eq. (14)

$$\frac{1}{4}\frac{r_1 - r_0}{e(r_3 - r_0)}\frac{2 - 1/B}{(r_1 - r_0)^2} = \frac{2 - 1/B}{4e(r_3 - r_0)(r_1 - r_0)} = (10^8\ \text{m})^{-2}$$

whence

$$B = [2 - 10^{-16}(4e)(r_3 - r_0)(r_1 - r_0)]^{-1}.$$

(Since $\frac{1}{2} < A < B$, A is also severely restricted.) In Fig. 3, B is plotted against $\log(r_1 - r_0)$ for four values of $r_3 - r_0$ (right to left): 5 m, 50 m, 500 m, and 5000 m. Observe that the exponent B remains very close to $1/2$ for any reasonable value of $r_1 - r_0$ and is virtually independent of $r_3 - r_0$.

Now consider the following coordinate distances: $r_1 - r_0 = 0.000\,001$ m and $r_3 - r_0 = 10$ m. Then

$$B = (2 - 1.09 \times 10^{-20})^{-1}.$$

For the proper distance $\ell(r_1)$ we get

$$\ell(0.000\,001) = \sqrt{1.09 \times 10^{-20}}(10^8\ \text{m}) \approx 1\ \text{cm}.$$

Since Eq. (14) is really an inequality, the result is, strictly speaking, $\ell(0.000\,001) \gtrsim 1$ cm. However, the proper distance can also be obtained by direct integration, using the same values for K and A:

$$\ell(0.000\,001) = \int_{r_0}^{r_0 + 0.000\,001} \frac{K}{(r - r_0)^A} dr \approx 1\ \text{cm}.$$

Remark.—If γ_1 is modified for $r < r_1$, as described in Sec. III, then the constraint, Eq. (13), is met *a fortiori:* since $(r - r_0)^{2A-1} < (r_1 - r_0)^{2A-1}$ for any $r < r_1$,

$$|R_{\hat{r}\hat{t}\hat{r}\hat{t}}| \approx \frac{1}{K^2}(0 + A(r - r_0)^{2A-1}\gamma_1'(r_1) + (r - r_0)^{2A} \times [\gamma_1'(r_1)]^2)$$

is less than the value of $|R_{\hat{r}\hat{t}\hat{r}\hat{t}}|$ at $r = r_1$.

While the distances chosen are rather arbitrary, the outcome, a proper distance of 1 cm, seems promising. It is also clear that reducing this distance any further would require even more fine-tuning. On the positive side, without any special restriction on r_0, the wormhole can be macroscopic.

VI. OTHER CONSTRAINTS

It remains to check some of the other constraints in MT [7]. The gradient of the redshift function must satisfy the condition

$$|\gamma'(r)| \leq g_\oplus/(c^2\sqrt{1 - b(r)/r}) = 1.09 \times 10^{-16}\ \text{m}^{-1} \tag{18}$$

at the stations [7]. If r, the distance to the stations, is chosen to be 225 000 km and if $r_0 \ll 225\,000$ km, then (since $B \approx 1/2$)

$$|\gamma_2'(r)| = \frac{B(r_3 - r_0)}{(r - r_0)^2} = \frac{\frac{1}{2}(10\ \text{m})}{(225\,000\,000\ \text{m})^2} = 1 \times 10^{-16}\ \text{m}^{-1}.$$

Also, the stations should be far enough away from the throat so that $b(r)/r \approx 0$, making the space nearly flat. A similar formulation is that $1 - b(r)/r = e^{-2\alpha(r)}$ is close to unity. This constraint is easily met given that $\alpha_2(r) = A(r_3 - r_0)/(r - r_0)$.

Using Eqs. (11) and (12), the remaining tidal constraints can be written [7]

$$|R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}| = |R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'}| = \gamma^2|R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}| + \gamma^2\left(\frac{v}{c}\right)^2|R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}}| = \gamma^2\left(\frac{1}{r}e^{-2\alpha(r)}\gamma_1'(r)\right) + \gamma^2\left(\frac{v}{c}\right)^2\left(\frac{1}{r}\right)e^{-2\alpha(r)}\alpha_1'(r) \leq (10^8\ \text{m})^{-2};$$

(here $\gamma^2 = 1/[1 - (v/c)^2]$). The first term is close to zero near the throat while the second is merely a constraint on the velocity of the traveler.

A final consideration is the time dilation near the throat. Let $v = d\ell/d\tau$, so that $d\tau = d\ell/v$ (assuming that $\gamma \approx 1$.) Since $d\ell = e^{\alpha(r)}dr$ and $d\tau = e^{\gamma(r)}dt$, we have for any coordinate interval Δt:

$$\Delta t = \int_{t_a}^{t_b} dt = \int_{\ell_a}^{\ell_b} e^{-\gamma(r)}\frac{d\ell}{v} = \int_{r_a}^{r_b} \frac{1}{v}e^{-\gamma(r)}e^{\alpha(r)}dr.$$

Since we are concerned mainly with the vicinity of the throat, let us consider the interval $[r_0, r_3]$. If the form of $\gamma_1(r)$ in Eq. (6) were retained, the result would be

$$\Delta t = \int_{r_0}^{r_3} \frac{1}{v}\frac{L}{(r - r_2)^B}\frac{K}{(r - r_0)^A}dr.$$

Since both A and B exceed $1/2$, $\Delta t \to \infty$ as r_2 approaches r_0. It was noted in Sec. III, however, that to the left of r_1, $\gamma_1(r)$ is "straightened out," that is, $\gamma_1(r)$ no longer has an asymptote close to $r = r_2$. As a result, letting $A = \frac{1}{2}$ again,

$$\Delta t = \int_{r_0}^{r_3} \frac{1}{v}e^{-\gamma_1(r)}\frac{K}{(r - r_0)^{1/2}}dr,$$

which is now finite. In addition to the proper velocity v, the time interval Δt evidently depends on $\gamma_1(r)$, as well as the choice of K. But since we are dealing with a relatively small distance and hence a small proper traversal time, the time dilation is of little significance. This conclusion also holds for $r > r_3$ since $\gamma_2(r)$ and $\alpha_2(r)$ fall off so rapidly.

VII. SUMMARY

This paper discusses a wormhole model with the following characteristics: since the derivative of the shape function is unity at the throat, the quantum inequalities are trivially satisfied. In place of this requirement one is faced with a restriction on the spacetime geometry that may be equally severe. The severity of this restriction notwithstanding, the model allows a compromise between shrinking the exotic region and fine-tuning the metric coefficients: reducing the amount of exotic matter can only be accomplished by further fine-tuning. There is no way to determine *a priori* which requirement is easier to meet, or whether, being interdependent, either one can be met. One can only hope that in the end this problem reduces to an engineering challenge.

No particular restriction is placed on the throat size. Moreover, various traversability criteria are met, resulting in a wormhole that is traversable for humanoid travelers.

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