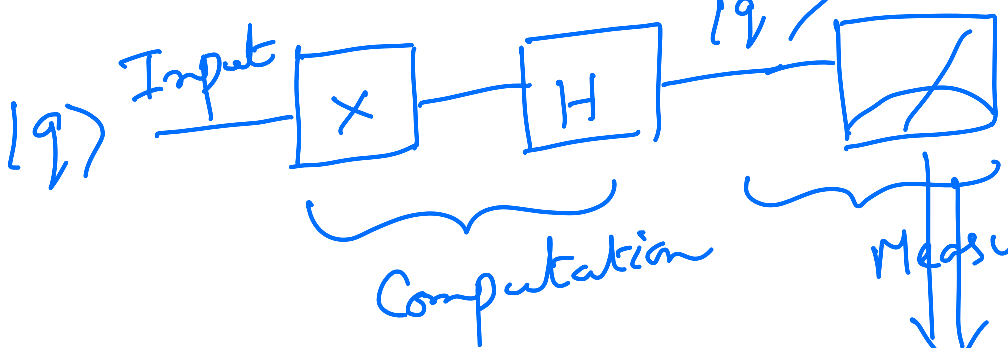
Quantum circuits

$|q'\rangle = \alpha'|0\rangle + \beta'|1\rangle$



$C = |0\rangle$ with probability $|\alpha'|^2$

$C = |1\rangle$ with probability of $|\beta'|^2$

→ Input
→ Computation
→ Measurement

Scale up state space

(By combining single qubits)

With 1 qubit → 2-dimensional state space

2 qubit state

$\underset{2\times1}{|q_1\rangle} \otimes \underset{2\times1}{|q_2\rangle}$

$= 4\times1$ vector

$$\boxed{\underset{m\times p}{A} \otimes \underset{q\times r}{B} = mq \times pr}$$

$|0\rangle \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |1\rangle \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$|q_1\rangle = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \quad ; \quad |q_2\rangle = \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix}$$

$$|q_1\rangle \otimes |q_2\rangle = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \otimes \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix}$$

$$= \begin{bmatrix} q_{11} q_{21} \\ q_{11} q_{22} \\ q_{12} q_{21} \\ q_{12} q_{22} \end{bmatrix}_{4\times 1}$$

4-dimensional vector

3 qubit system

$$\underbrace{|q_1\rangle \otimes |q_2\rangle}_{4\times 1} \otimes \underbrace{|q_3\rangle}_{2\times 1}$$

$\Downarrow$

8×1 vector

$\Downarrow$

8 dimensional vector space

1 qubit $\rightarrow$ 2-D state space

2 qubit $\rightarrow$ 4-D state space

3 qubits $\rightarrow$ 8-D state space

$\vdots$

n-qubit $\rightarrow$ 2^n dimensional state space

1-qubit : $|0\rangle, |1\rangle$ 2 basis states

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2-qubit : 4 basis states

$\{|0\rangle, |1\rangle\}$ q_1

$\{|0\rangle, |1\rangle\}$ q_2

$$|00\rangle = |0\rangle \otimes |0\rangle$$
$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|01\rangle = |0\rangle \otimes |1\rangle$$
$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = |1\rangle \otimes |0\rangle$$
$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|11\rangle = |1\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$$

4 basis states with 2-qubit system

$$|q_1\rangle \otimes |q_2\rangle = \begin{bmatrix} q_{11}\, q_{21} \\ q_{11}\, q_{22} \\ q_{12}\, q_{21} \\ q_{12}\, q_{22} \end{bmatrix}$$

$$|q_1\rangle = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix}$$

$$|q_2\rangle = \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix}$$

$$\boxed{\begin{aligned} |q_1\, q_2\rangle = {} & q_{11}\, q_{21}\, |00\rangle + q_{11}\, q_{22}\, |01\rangle \\ & + q_{12}\, q_{21}\, |10\rangle + q_{12}\, q_{22}\, |11\rangle \end{aligned}}$$

Probability of my state $|q_1\, q_2\rangle$ collapsing to $|00\rangle = |q_{11}\, q_{21}|^2$

$$P(C \to |01\rangle) = |q_{11}\, q_{22}|^2$$

$$P(C \to |10\rangle) = |q_{12}\, q_{21}|^2$$

$$P(C \to |11\rangle) = |q_{12}\, q_{22}|^2$$

n-qubit state

$$|q_1\rangle \otimes |q_2\rangle \otimes |q_3\rangle \otimes \cdots |q_n\rangle$$

$$\begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \otimes \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} \otimes \begin{bmatrix} q_{31} \\ q_{32} \end{bmatrix} \otimes \cdots \begin{bmatrix} q_{n1} \\ q_{n2} \end{bmatrix}$$

$$\begin{bmatrix} q_{11} q_{21} q_{31} \cdots q_{n1} \\ \vdots \\ q_{12} q_{22} \cdots q_{n2} \end{bmatrix} \rightarrow \underline{2^n \text{ rows}}$$

$$2^n \times 1$$

1^{st} basis state

$$\underbrace{|00\cdots 0\rangle} = |0\rangle \otimes |0\rangle \otimes \cdots |0\rangle$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} 2^n$$

$$||$$

$$|0\rangle^{\otimes n} = |00\cdots 0\rangle$$

$$q_{11} \cdots q_{n1} |0\rangle^{\otimes n} + \cdots + q_{1n} \cdots q_{n2} |1\rangle^{\otimes n}$$

We will have 2^n probabilities

2 qubit states

Bipartite state (All states which are lying in 2-qubit state space)

$$\underbrace{|01\rangle} = |0\rangle \otimes |1\rangle$$

$$\underbrace{\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)}$$

Bipartite states

$$= \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)} \otimes \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)}$$

$\Downarrow$

Tensor product of 2 single qubit states

Can we write states in my 2-qubit state space as a tensor product of 2 single qubit states?

Not all bipartite states can be expressed as tensor product of 2 single qubit states eg: $|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$

$\alpha = \frac{1}{\sqrt{2}}$

$\beta = \frac{1}{\sqrt{2}}$

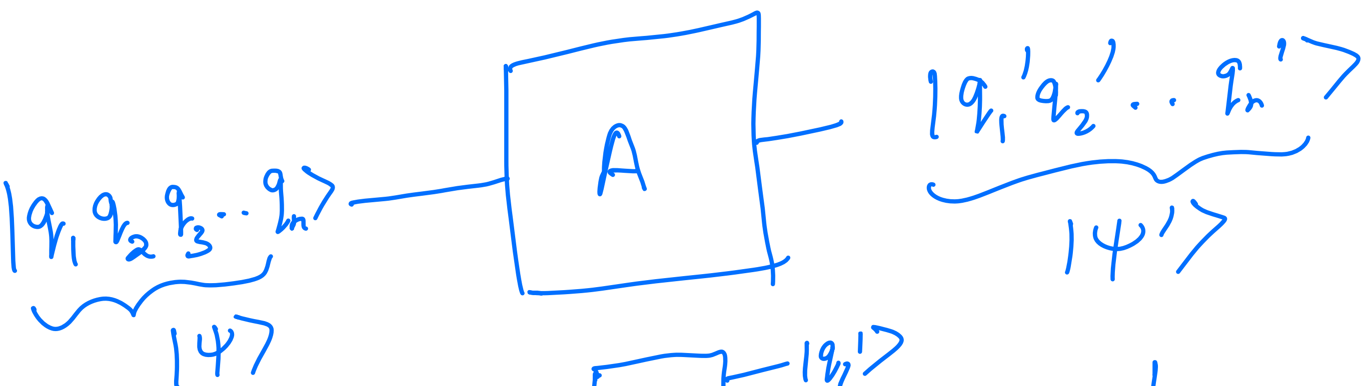
$\gamma = 0$

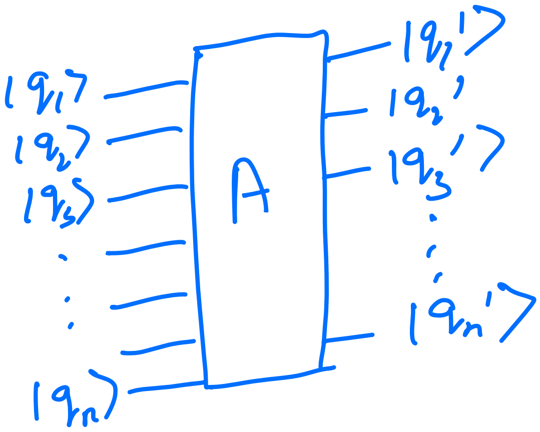
$\delta = 0$

$$|\tilde{\psi}\rangle = \underbrace{1/\sqrt{2}\,(|00\rangle + |11\rangle)}_{\text{2-qubit state}}$$

We cannot find 2 single qubit states whose tensor product gives me $|\tilde{\psi}\rangle$

Multi-qubit Quantum gates





$$A^\dagger A = AA^\dagger = I$$

Given input state and A

$$A|\psi\rangle = |\psi'\rangle$$

Given the output state and A

can we find input state ?

$$A^{\dagger} = A^{-1}$$

$$A|\psi\rangle = |\psi'\rangle$$

$$A^{\dagger}|\psi'\rangle = |\psi\rangle$$

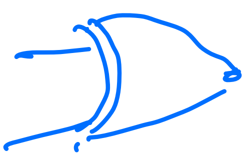
$$\boxed{\text{CNOT Gate}}$$

classical

XOR Gate

A

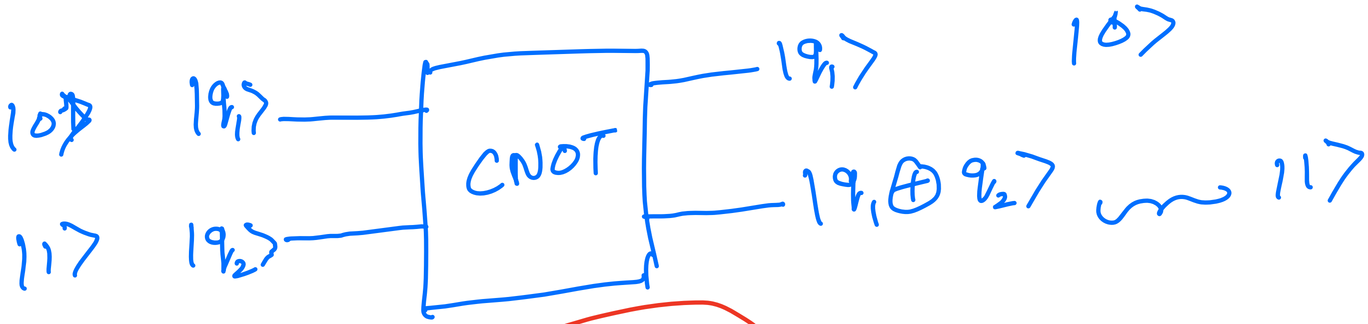
B



$A \oplus B$

If two bits A and B are different, output is 1

otherwise output is 0.



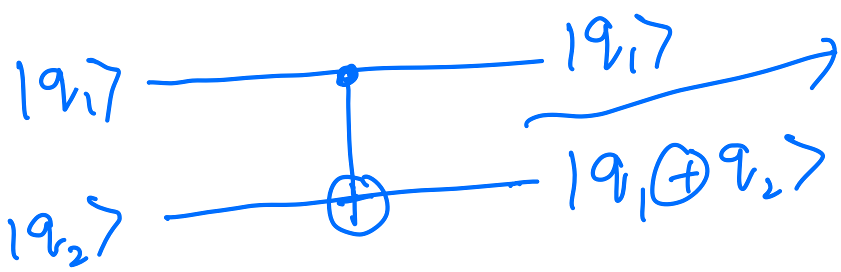
$$\left\{\begin{aligned} CNOT\,|00\rangle &= |00\rangle \\ CNOT\,|01\rangle &= |01\rangle \\ CNOT\,|10\rangle &= |11\rangle \\ CNOT\,|11\rangle &= |10\rangle \end{aligned}\right\}$$

$\sigma_x|0\rangle = |1\rangle$

$\sigma_x|1\rangle = |0\rangle$

X-gate

Controlled X-Gate



controlled by 1st qubit

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

4×4

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{CNOT}\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{|00\rangle} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle$$

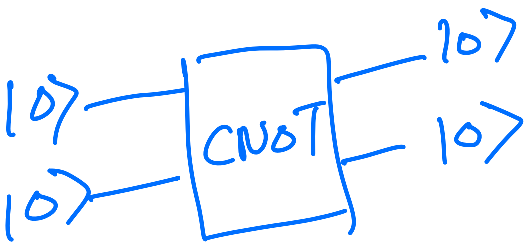
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

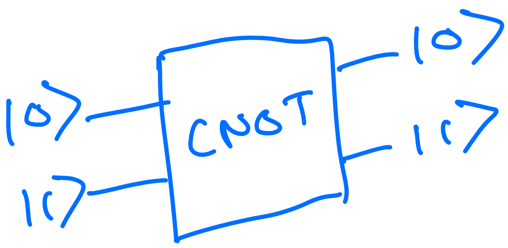
$$\text{CNOT} \quad |01\rangle = |01\rangle$$

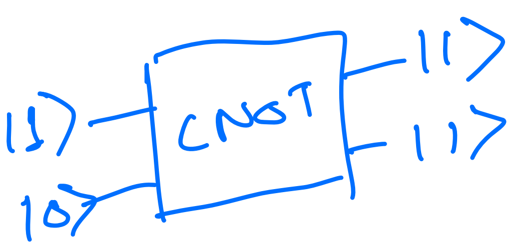
$\|q_1 q_2\rangle$	output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

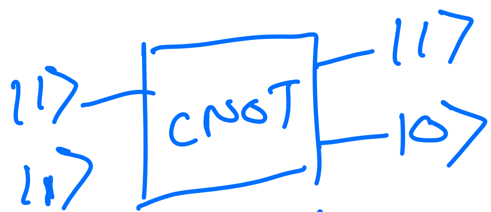
Controlled X-Gate

It is like acting σ_x on $|q_2\rangle$ but controlled by what is $|q_1\rangle$

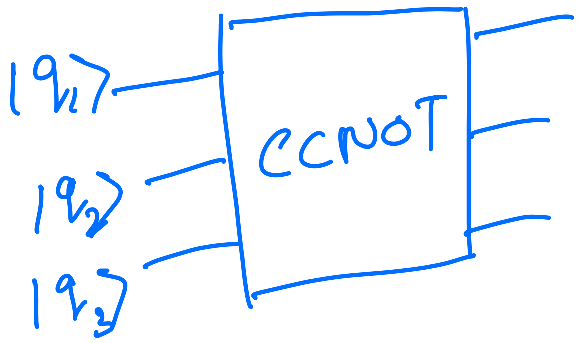








Controlled X-Gate



output of $|q_3\rangle$ is controlled by $|q_1\rangle$ and $|q_2\rangle$

Toffoli gate

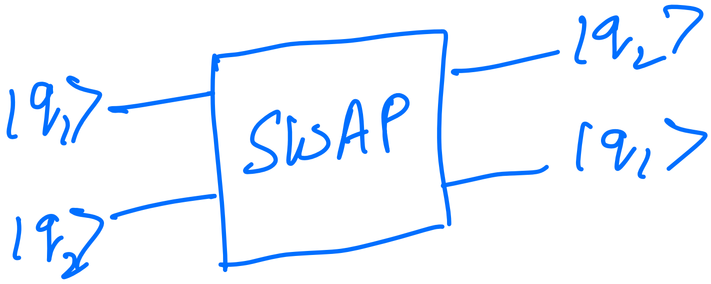
Controlled Y-gate

controlled by $|q_1\rangle$ and $|q_2\rangle$ output is $\sigma_y |q_2\rangle$

Controlled Z-Gate

controlled by $|q_1\rangle$ and $|q_2\rangle$ output is $\sigma_z |q_2\rangle$

SWAP Gate



$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$SWAP\,|00\rangle = |00\rangle$$
$$SWAP\,|01\rangle = |10\rangle$$
$$SWAP\,|10\rangle = |01\rangle$$
$$SWAP\,|11\rangle = |11\rangle$$

Classical Gates

{AND, OR, NOT}

{NAND}

{NOR}

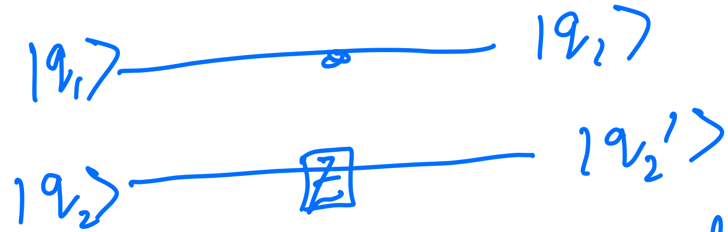
Quantum computing

$\{H, T, CNOT, \sigma_x, \sigma_y, \sigma_z\}$

{CCNOT}

universal Gates

Controlled Z - gate

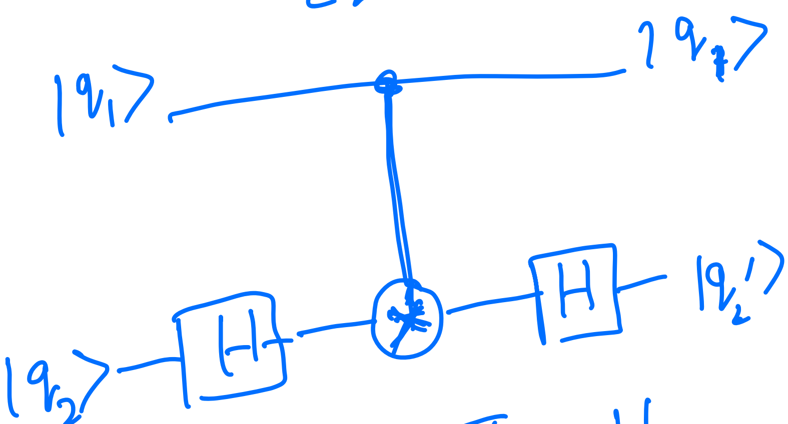


We can derive controlled Z - gate from CX gate

$$\boxed{HXH = Z}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

CZ



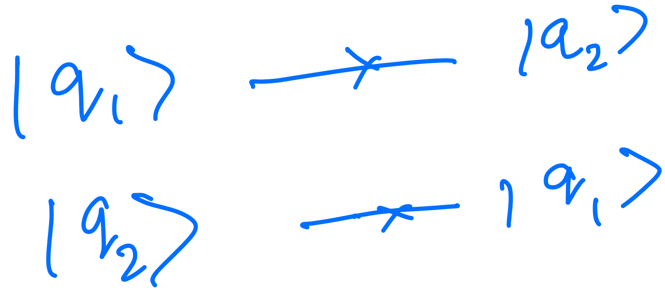
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

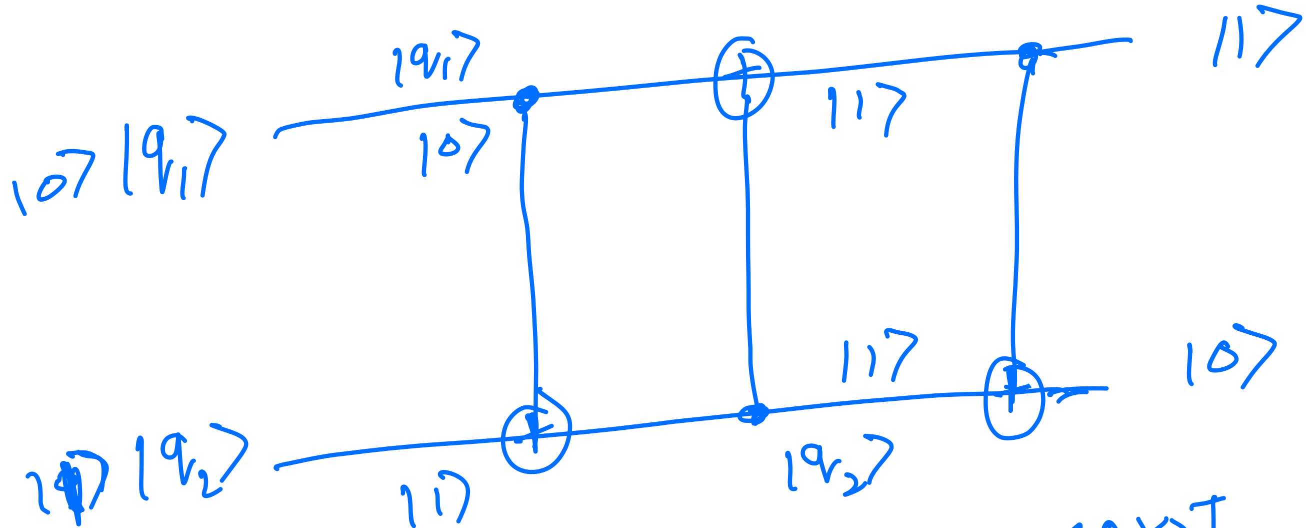
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

H - CNOT - H

C - Z gate

SWAP Gate





CNOT – CNOT – CNOT

= SWAP

Hadamard Gate

on a multi-qubit system

Single qubit

$$H|0\rangle = |+\rangle$$

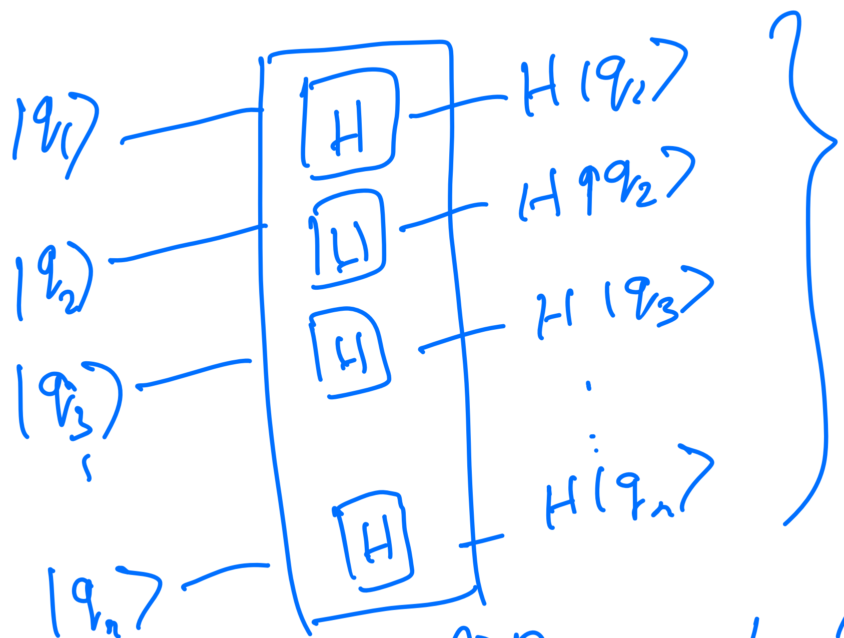
$$H\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$|00\rangle \quad H^{\otimes 2}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{1}{\sqrt{2^2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$\vdots$

$|00\cdots0\rangle \quad H^{\otimes n}\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = \frac{1}{\sqrt{2^n}}\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}$



$|q_1 q_2 \cdots q_n\rangle \quad H^{\otimes n} \quad |q_1' q_2' \cdots q_n'\rangle$