Entanglement

2-qubit system $\rightarrow$ 4-dimensional state space

Any state in this 4-dimensional state space (bi-partite state) can be classified into two kinds of states.

Product state

$$|\psi\rangle \in \mathbb{C}^4$$

$$|\psi\rangle \longrightarrow |q_1\rangle \otimes |q_2\rangle \qquad |q_1\rangle \in \mathbb{C}^2, \; |q_2\rangle \in \mathbb{C}^2$$

[can be expressed as a tensor product of two single qubit states]

Entangled state :-

$$|\psi\rangle \in \mathbb{C}^4 \not\longrightarrow |q_1\rangle \otimes |q_2\rangle$$

(cannot decompose)

Bell states

Four 2-qubit states that are maximally entangled

$$|\psi_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle + 0|01\rangle + 0|10\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$|\psi_{01}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$|\psi_{11}\rangle = 1/\sqrt{2}\left(|00\rangle - |11\rangle\right)$$

$$|\psi_{10}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

$$|\psi_{00}\rangle = 1/\sqrt{2}\left(|00\rangle + |11\rangle\right)$$

$\Downarrow$ Measurement

collapses to $|00\rangle$ or $|11\rangle$ with equal probability

Measurement two qubits

first qubit $\rightarrow 0$

second qubit has to be $\rightarrow 0$

(or)

first qubit $\rightarrow 1$

second qubit has to be 1

$$|\psi_{01}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

Measurement of first qubit

$\rightarrow 0$

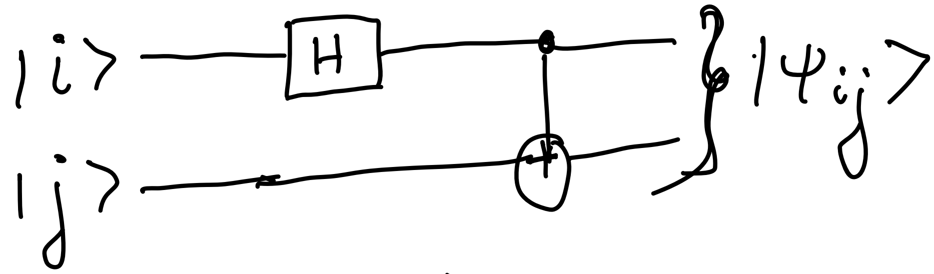
second qubit $\rightarrow 1$

(or)

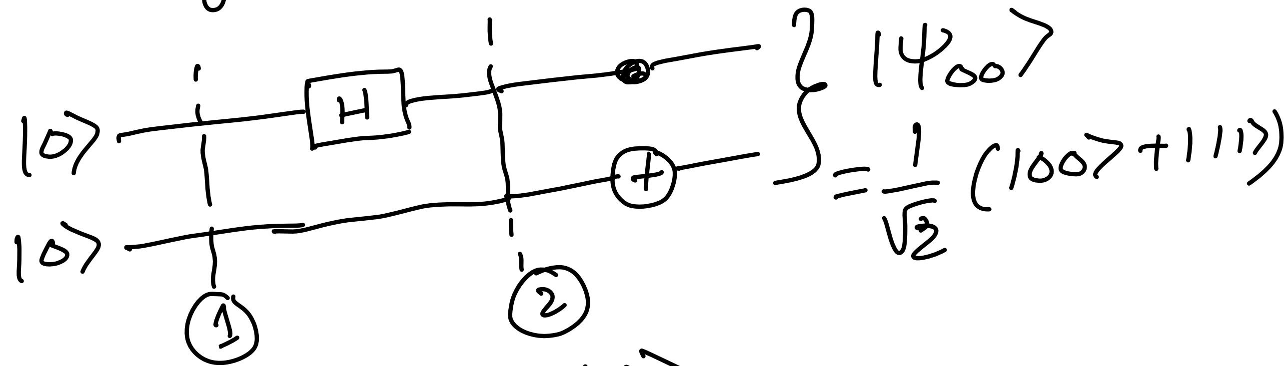
first qubit $\rightarrow 1$

second qubit $\rightarrow 0$

Input	Output	
$\lvert ij\rangle$	$\lvert\psi_{ij}\rangle$	
$\lvert 00\rangle$	$\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$	$\lvert\psi_{00}\rangle$
$\lvert 01\rangle$	$1/\sqrt{2}\,(\lvert 01\rangle + \lvert 10\rangle)$	$\lvert\psi_{01}\rangle$
$\lvert 10\rangle$	$1/\sqrt{2}\,(\lvert 00\rangle - \lvert 11\rangle)$	$\lvert\psi_{10}\rangle$
$\lvert 11\rangle$	$1/\sqrt{2}\,(\lvert 01\rangle - \lvert 10\rangle)$	$\lvert\psi_{11}\rangle$



$|i\rangle = |0\rangle \quad |j\rangle = |0\rangle$



$$|00\rangle = |0\rangle \otimes |0\rangle$$

$$\Big\Downarrow H$$

$$= H|0\rangle \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$\Big\Downarrow \text{CNOT}$$

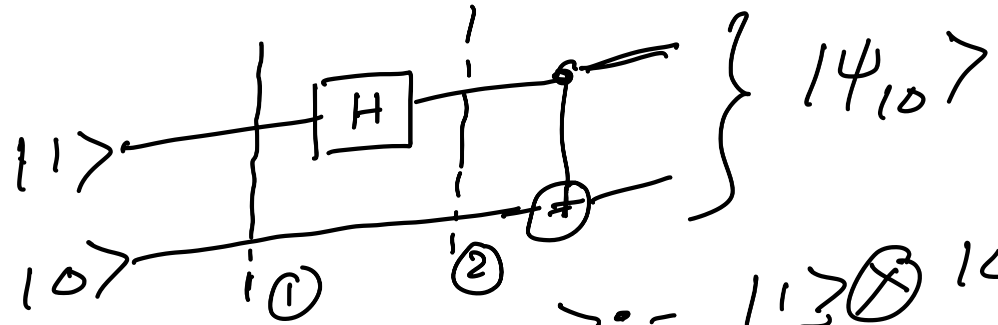
$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

flip 0 to 1

Finally what we have is $|\psi_{00}\rangle$

Second example:-

$|i\rangle = |1\rangle$, $|j\rangle = |0\rangle$



Input state : $|10\rangle := |1\rangle \otimes |0\rangle$

$\Downarrow H$

$H|1\rangle = |-\rangle$

$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) \otimes |0\rangle$

$= \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$

$\Downarrow$ CNOT

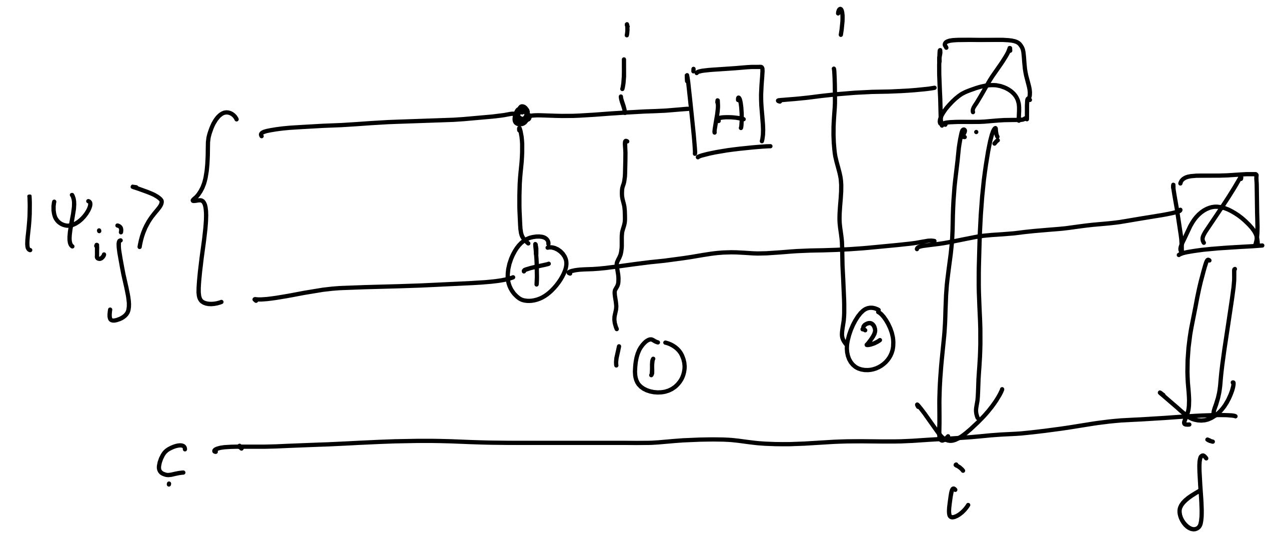
$= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

$= |\psi_{10}\rangle$

Try this exercise for constructing $|\psi_{01}\rangle$ and $|\psi_{11}\rangle$

Bell measurement :-

Given $|\psi_{ij}\rangle$ find $|i\rangle$ and $|j\rangle$



Consider my input state to be

$$|\psi_{00}\rangle$$

$$|\psi_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$\Big\Downarrow \text{ CNOT}$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$

1

1 gets flipped to 0

$$\Big\Downarrow \text{ H}$$

$$= \frac{1}{\sqrt{2}}\left(|+0\rangle + |-0\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\otimes|0\rangle + \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\otimes|0\rangle\right]$$

$$= \frac{1}{2}\left(|00\rangle + |10\rangle\right) + \frac{1}{2}\left(|00\rangle - |10\rangle\right)$$

$= |00\rangle$

If my measurement values of two qubits gives me 0 and 0 then my input Bell state is $|\psi_{00}\rangle$

Consider my input state to $|\psi_{10}\rangle$

$$|\psi_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$\Downarrow \text{CNOT}$$

$$= \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$$

$$\Downarrow H$$

$$= \frac{1}{\sqrt{2}}(|+0\rangle - |-0\rangle)$$

$$= \frac{1}{\sqrt{2}}\left[\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle - \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes|0\rangle\right)\right]$$

$$= \frac{1}{2}\left(|00\rangle + |10\rangle\right) - \frac{1}{2}\left(|00\rangle - |10\rangle\right)$$

$$= |10\rangle$$

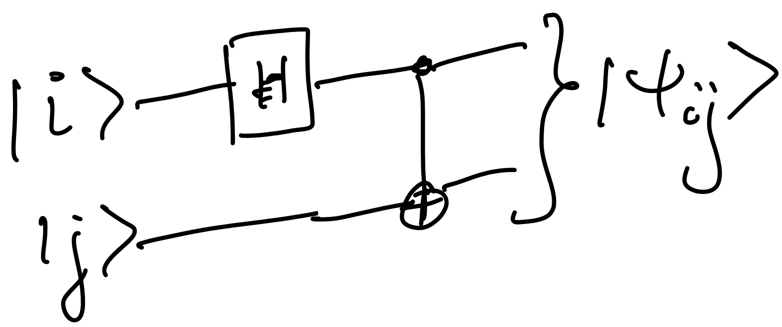
If my measurement values after I execute circuit is 1 and 0 my Bell state $|\psi_{10}\rangle$

Input {Bell states}	Output i	j
$\lvert\psi_{00}\rangle = \frac{1}{\sqrt{2}}\left(\lvert 00\rangle + \lvert 11\rangle\right)$	0	0
$\lvert\psi_{01}\rangle = \frac{1}{\sqrt{2}}\left(\lvert 01\rangle + \lvert 10\rangle\right)$	0	1
$\lvert\psi_{10}\rangle = 1/\sqrt{2}\left(\lvert 00\rangle - \lvert 11\rangle\right)$	1	0
$\lvert\psi_{11}\rangle = \frac{1}{\sqrt{2}}\left(\lvert 01\rangle - \lvert 10\rangle\right)$	1	1

Quantum Teleportation

It is about communicating information over arbitrarily long distances using the power of quantum entanglement.

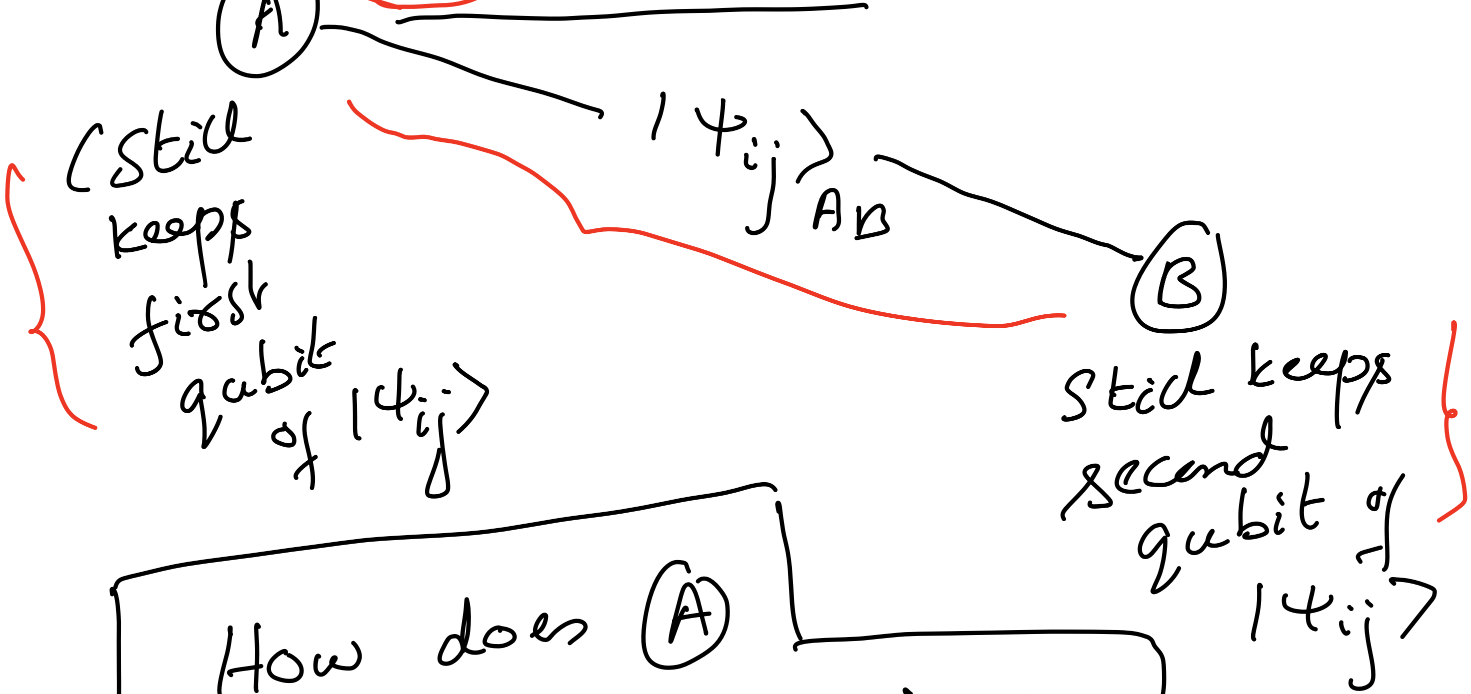
Ⓐ Alice, Ⓑ Bob } together ran an experiment and produced an entangled state $|\psi_{ij}\rangle$



$$A \xleftrightarrow{|\psi_{ij}\rangle} B$$

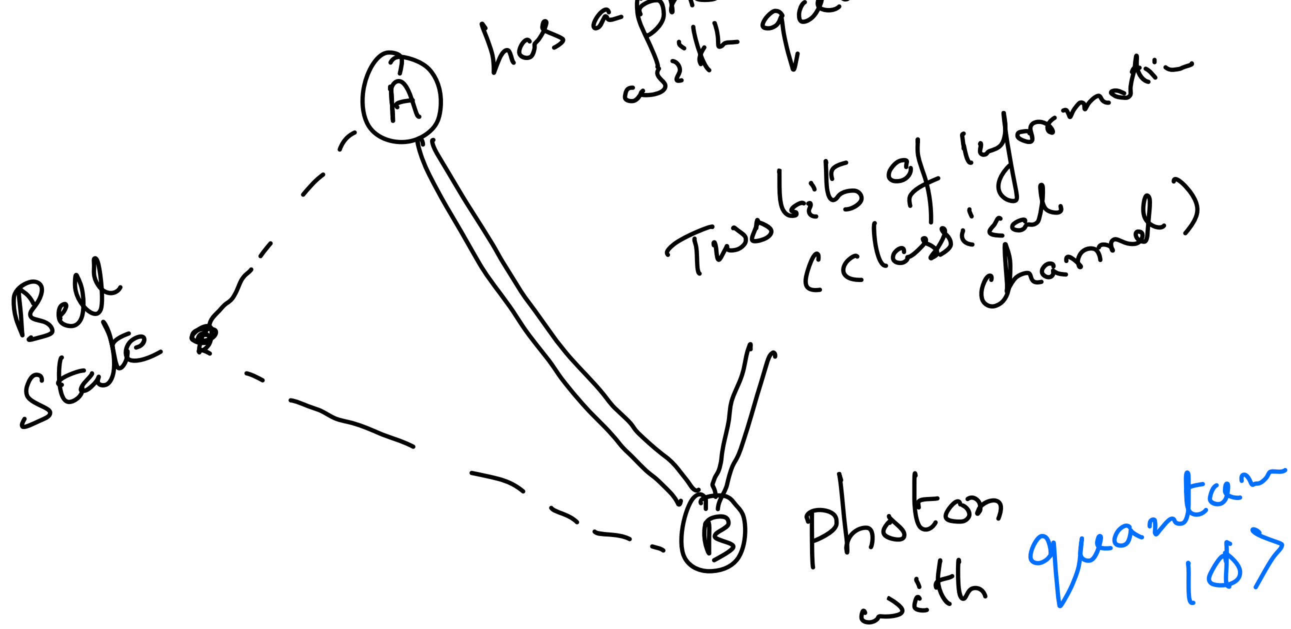
Ⓐ keeps the first qubit and Ⓑ keeps the second qubit

$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$



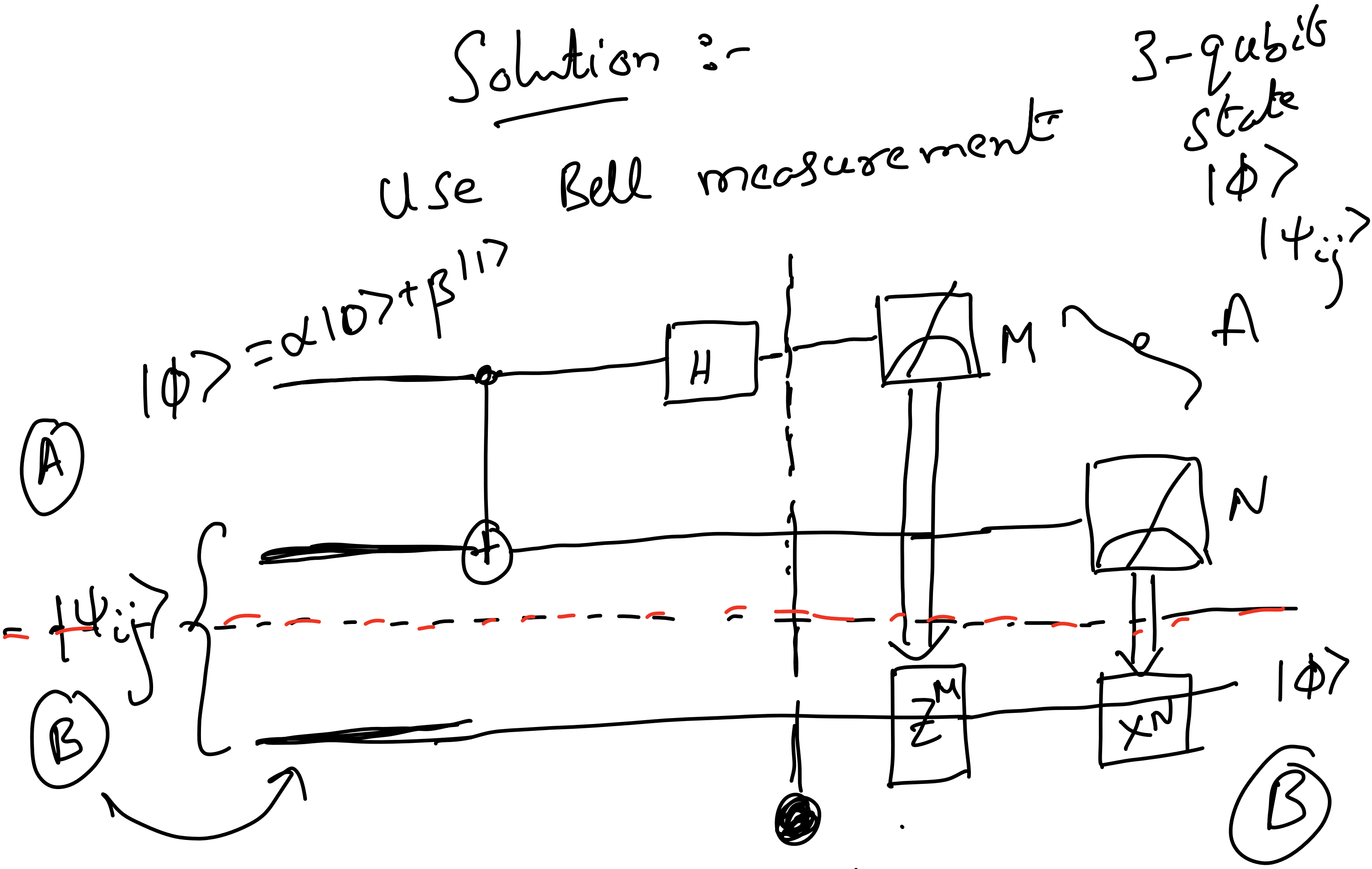
How does A communicate $|\phi\rangle$ to B without traversing the distance?

$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$



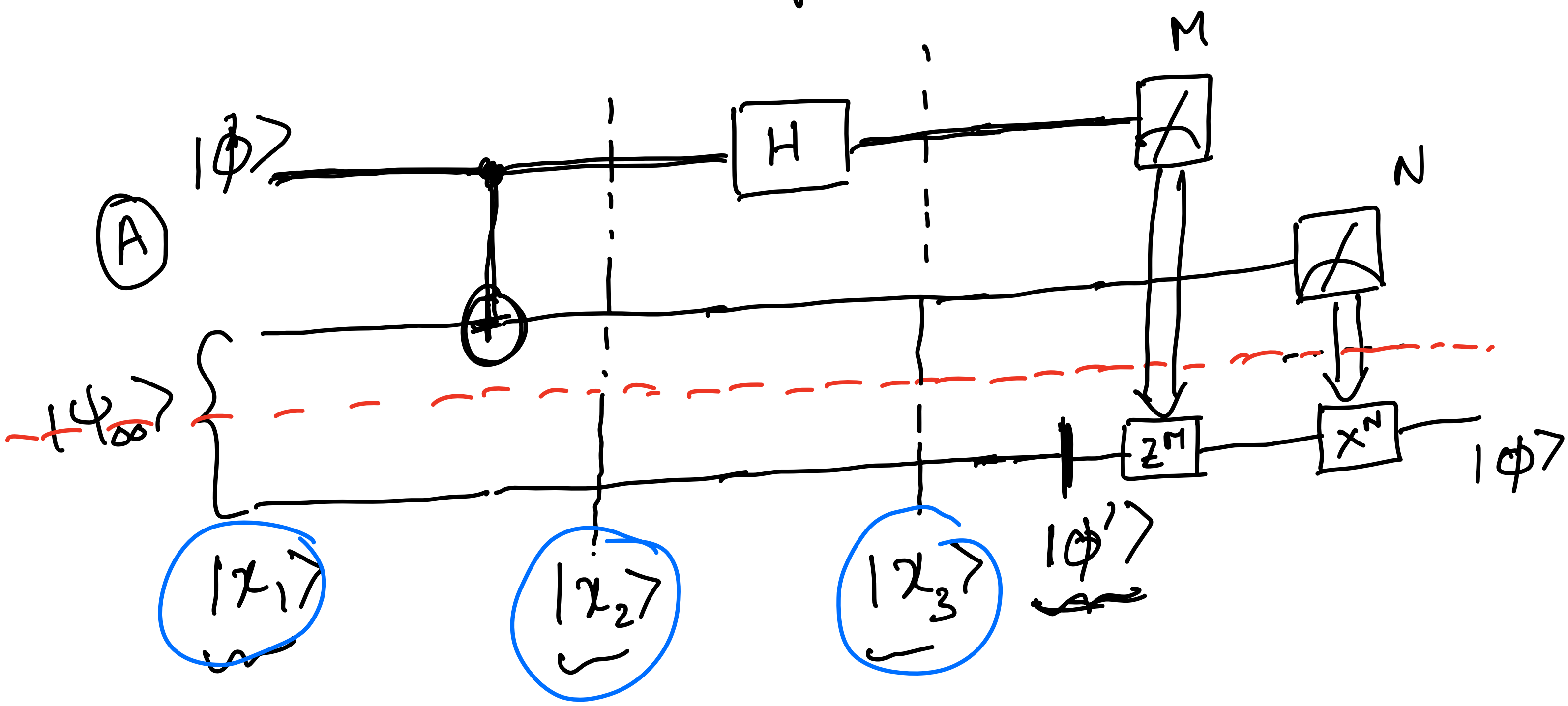
Solution :-

Use Bell measurement



Let us begin with

$$|\psi_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$



Start state

$$|\chi_1\rangle = |\phi\rangle \otimes |\psi_{00}\rangle$$

$$= (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}\left\{\alpha|0\rangle \otimes (|00\rangle + |11\rangle) + \beta|1\rangle \otimes (|00\rangle + |11\rangle)\right\}$$

$\Downarrow$ CNOT

$$|\chi_2\rangle = \frac{1}{\sqrt{2}}\left\{\alpha|0\rangle \otimes (|00\rangle + |11\rangle) + \beta|1\rangle \otimes (|10\rangle + |01\rangle)\right\}$$

$\Downarrow$ H

$$|\chi_3\rangle = \frac{1}{\sqrt{2}}\left\{\alpha|+\rangle \otimes (|00\rangle + |11\rangle) + \beta|-\rangle \otimes (|10\rangle + |01\rangle)\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\alpha\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes (|00\rangle + |11\rangle) + \beta\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes (|10\rangle + |01\rangle)\right\}$$

$$= \frac{1}{2}\left\{\alpha(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \beta(|010\rangle + |001\rangle - |110\rangle - |101\rangle)\right\}$$

$$= \frac{1}{2}\Big\{ |00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle[\alpha|1\rangle + \beta|0\rangle] + |10\rangle[\alpha|0\rangle - \beta|1\rangle] + |11\rangle[\alpha|1\rangle - \beta|0\rangle]\Big\}$$

$$
\begin{array}{rl|l}
 & \text{(A)} & \text{(B)} \\
|\chi_3\rangle = \frac{1}{2}\{ & |00\rangle & (\alpha|0\rangle + \beta|1\rangle \\
 & + |01\rangle & (\alpha|1\rangle + \beta|0\rangle \\
 & + |10\rangle & (\alpha|0\rangle - \beta|1\rangle \\
 & + |11\rangle & (\alpha|1\rangle - \beta|0\rangle)\}.
\end{array}
$$

Alice's measurement	Bob's state $|\phi'\rangle$
0 0 $\longrightarrow$ Do nothing	$\alpha|0\rangle + \beta|1\rangle$
0 1 $\longrightarrow$ Bit flip (X-Gate)	$\alpha|1\rangle + \beta|0\rangle$
1 0 $\longrightarrow$ Phase flip (Z-Gate)	$\alpha|0\rangle - \beta|1\rangle$
1 1 $\longrightarrow$ phase flip and Bit flip	$\alpha|1\rangle - \beta|0\rangle$

$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

Bob

$\underset{M\ N}{|0\ 0\rangle} \longrightarrow |\phi'\rangle - \underbrace{\boxed{Z^0} - \boxed{X^0}}_{\text{Nothing} \times} - |\phi\rangle$ (Do nothing)

$\underset{M\ N}{|0\ 1\rangle} \longrightarrow |\phi'\rangle - \boxed{Z^0} - \boxed{X^1} - |\phi\rangle$ (Bit flip)

$\underset{M\ N}{|1\ 0\rangle} \longrightarrow |\phi'\rangle - \boxed{Z^1} - \boxed{X^0} - |\phi\rangle$ (Phase flip)

$\underset{M\ N}{|1\ 1\rangle} \longrightarrow |\phi'\rangle - \boxed{Z^1} - \boxed{X^1} - |\phi\rangle$ Phase flip and Bit flip

$|\psi_{01}\rangle$, $|\psi_{10}\rangle$ and $|\psi_{11}\rangle$

Exercise ?