E2 210 (Jan.–Apr. 2025)

Homework Assignment 3

Submission deadline: Monday, March 3, 11:59pm

1. Show that the set of Pauli matrices $\{X(\mathbf{a})Z(\mathbf{b}) : \mathbf{a}, \mathbf{b} \in \{0,1\}^n\}$ is an orthonormal basis for the vector space, $\mathbb{C}^{N \times N}$, of $N \times N$ complex matrices, under the Hilbert-Schmidt inner product: $(A, B) \stackrel{\text{def}}{=} \frac{1}{N}\,\mathrm{tr}(A^\dagger B)$. (Here, as usual $N = 2^n$.)

2. Let G be a graph on the vertex set $[n] := \{1, 2, \ldots, n\}$ having edge set $E \subseteq \binom{[n]}{2}$. (Here, $\binom{[n]}{2}$ denotes the set of all 2-subsets of $[n]$, so that edges are certain 2-subsets of $[n]$. In particular, the graph has no self-loops, i.e., edges that connect a vertex to itself.)

   Let A be the adjacency matrix of G; this is the $n \times n$ matrix with $0/1$ entries, whose (i,j)-th entry is 1 iff $\{i,j\}$ is an edge of G. Set $H = [I \mid A]$, where I is the $n \times n$ identity matrix. Thus, H is an $n \times 2n$ matrix having rank n.

   (a) Show that the symplectic product between any pair of rows of H is 0.

   (b) If $\mathcal{S}$ is the stabilizer group defined by the check matrix H, what is $\dim \mathcal{Q}_{\mathcal{S}}$?

3. In this exercise, we will prove the following proposition:

   Proposition: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be, respectively, $[n, k_1]$ and $[n, k_2]$ binary linear codes such that $\mathcal{C}_1^\perp \subseteq \mathcal{C}_2$. Let $A_0 := \mathcal{C}_1^\perp, A_1, \ldots, A_{K-1}$ be a listing of the $K = 2^{k_1+k_2-n}$ cosets of $\mathcal{C}_1^\perp$ within $\mathcal{C}_2$. Then, the quantum states

   $$|\phi_j\rangle := \frac{1}{\sqrt{2^{n-k_1}}} \sum_{\mathbf{x} \in A_j} |\mathbf{x}\rangle\,, \quad j = 0, 1, \ldots, K-1,$$

   form an orthonormal basis of the quantum code $\mathcal{Q}$ obtained via the CSS construction from $\mathcal{C}_1$ and $\mathcal{C}_2$.

   (a) Show that $\langle \phi_i | \phi_j \rangle = \delta_{i,j}$.
   [*Hint*: Note that $\langle \mathbf{b} | \mathbf{b}' \rangle = 0$ for any pair of distinct binary n-tuples $\mathbf{b}$ and $\mathbf{b}'$. Now, use the fact that cosets A_i and A_j are disjoint for $i \neq j$.]

   Let H_1 and H_2 be any pair of parity-check matrices for $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, of full row-rank. Thus, H_1 and H_2 are, respectively, $(n-k_1) \times n$ and $(n-k_2) \times n$ binary matrices such that $H_1 H_2^T = \mathbf{0}$ over $\mathbb{F}_2$. By the CSS construction, the stabilizer generators are $X(\mathbf{h})$ and $Z(\mathbf{h}')$, where $\mathbf{h}$ and $\mathbf{h}'$ range over the rows of H_1 and H_2, respectively.

   (b) Argue that, for any binary n-tuples $\mathbf{x}$, $\mathbf{h}$ and $\mathbf{h}'$, we have $X(\mathbf{h})\,|\mathbf{x}\rangle = |\mathbf{x} \oplus \mathbf{h}\rangle$ and $Z(\mathbf{h}')\,|\mathbf{x}\rangle = (-1)^{\mathbf{h}' \cdot \mathbf{x}}\,|\mathbf{x}\rangle$. In other words, the Pauli operator $X(\mathbf{h})$ applied to $|\mathbf{x}\rangle$ yields $|\mathbf{x} \oplus \mathbf{h}\rangle$, and the Pauli operator $Z(\mathbf{h}')$ applied to $|\mathbf{x}\rangle$ yields $(-1)^{\mathbf{h}' \cdot \mathbf{x}}\,|\mathbf{x}\rangle$.

   (c) Show, using (b), that for any row $\mathbf{h}$ of H_1, we have $X(\mathbf{h})\,|\phi_j\rangle = |\phi_j\rangle$, and for any row $\mathbf{h}'$ of H_2, we have $Z(\mathbf{h}')\,|\phi_j\rangle = |\phi_j\rangle$.
   [*Hint*: Write the sum $\sum_{\mathbf{x} \in A_j} |\mathbf{x}\rangle$ as $\sum_{\mathbf{c} \in \mathcal{C}_1^\perp} |\mathbf{a} \oplus \mathbf{c}\rangle$, where $\mathbf{a}$ is a fixed binary vector (a "coset leader") in A_j.]