Let's take another look at the picture with n hyperbolic pairs within $\mathcal{P}_n$:

$$\begin{array}{ccc}
g_1 & \xleftrightarrow{\text{anticommute}} & h_1 \\
g_2 & \xleftrightarrow{\quad "\quad} & h_2 \\
\vdots & & \vdots \\
g_{n-k} & \xleftrightarrow{\quad "\quad} & h_{n-k} \\
\bar{Z}_1 & \xleftrightarrow{\quad "\quad} & \bar{X}_1 \\
\bar{Z}_2 & \xleftrightarrow{\quad "\quad} & \bar{X}_2 \\
\vdots & & \vdots \\
\bar{Z}_k & \xleftrightarrow{\quad "\quad} & \bar{X}_k
\end{array}$$

$$S = \langle g_1, \dots, g_{n-k} \rangle$$

$$C(S)/\langle \omega \rangle = \langle g_1, \dots, g_{n-k}, \bar{Z}_1, \dots, \bar{Z}_k, \bar{X}_1, \dots, \bar{X}_k \rangle$$

In an abuse of notation ~~just~~ c we just call this $C(S)$

$$\mathcal{P}_n/\langle \omega \rangle = \underbrace{\langle g_1, \dots, g_{n-k}, h_1, \dots, h_{n-k}, \bar{Z}_1, \dots, \bar{Z}_k, \bar{X}_1, \dots, \bar{X}_k \rangle}$$

Just call this $\mathcal{P}_n$.

By virtue of forming hyperbolic pairs, these $2n$ ~~gen~~ Pauli operators are indep., hence they generate $\mathcal{P}_n/\langle\omega\rangle$

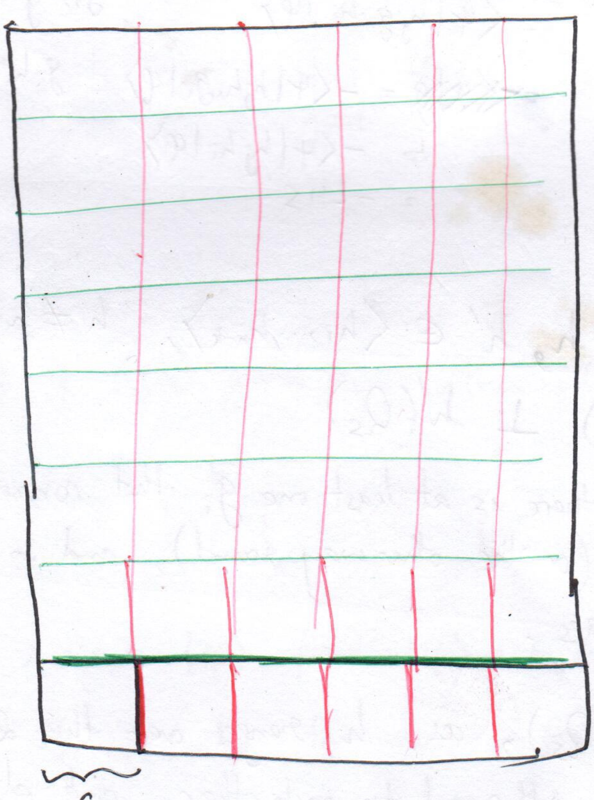
We clearly have $S \leq C(S) \leq \mathcal{P}_n$

$$C(S)/S \cong \langle \bar{Z}_1, \dots, \bar{Z}_k, \bar{X}_1, \dots, \bar{X}_k \rangle$$

$$\mathcal{P}_n/C(S) \cong \langle h_1, \dots, h_{n-k} \rangle$$

[Recall that $|C(S)| = 2^{n+k}$ and $|\mathcal{P}_n| = 2^{2n}$. So, $|\mathcal{P}_n/C(S)| = \frac{2^{2n}}{2^{n+k}} = 2^{n-k}$]

$\mathcal{P}_n$



$\} \, C(S)$

S

The green horizontal rectangles are cosets of $C(S)$ within $\mathcal{P}_n$

The ~~red boxes~~ boxes are cosets of S within cosets of $C(S)$ in $\mathcal{P}_n$.

We have seen that for all $|\varphi\rangle \in Q_s$,

$$g_i|\varphi\rangle = |\varphi\rangle \qquad \text{for } i=1,2,\dots,n-k$$

$$\bar{Z}_i|\varphi\rangle,\ \bar{X}_i|\varphi\rangle \in Q_s \qquad \text{for } i=1,2,\dots,k$$

So, what about $h_i|\varphi\rangle$?

① $h_i|\varphi\rangle \in Q_s^\perp$ for $i=1,2,\dots,n-k$

Proof. For any $|\psi\rangle \in Q_s$, we have

$$\begin{aligned}\langle\psi|h_i|\varphi\rangle &= \langle\psi|g_i h_i|\varphi\rangle \\ &= -\langle\psi|h_i g_i|\varphi\rangle \qquad \text{as } g_i, h_i \text{ anticommute} \\ &= -\langle\psi|h_i|\varphi\rangle\end{aligned}$$

$$\Rightarrow \langle\psi|h_i\varphi\rangle = 0 \qquad \cancel{\Rightarrow h_i|\varphi\rangle \in Q_s^\perp}$$

Thus, $h_i|\varphi\rangle \in Q_s^\perp$

② For $i \neq j$ and any $|\varphi\rangle, |\psi\rangle \in Q_s$, including the case $|\varphi\rangle = |\psi\rangle$, we have

$$h_i|\varphi\rangle \perp h_j|\psi\rangle$$

i.e., $\langle\psi|h_j h_i|\varphi\rangle = 0$

In other words, $h_i(Q_s) \perp h_j(Q_s)$

(Again this is because
$$\begin{aligned}\text{LHS} &= \langle\psi|g_i h_j h_i|\varphi\rangle \\ &= \langle\psi|h_j g_i h_i|\varphi\rangle \qquad g_i, h_j \text{ commute} \\ &= -\langle\psi|h_j h_i g_i|\varphi\rangle \qquad g_i, h_i \text{ anticommute} \\ &= -\langle\psi|h_j h_i|\varphi\rangle \\ &= -\text{LHS}\end{aligned}$$
)

③ More generally for any $h, h' \in \langle h_1,\dots,h_{n-k}\rangle$, $h \neq h'$, we have $h(Q_s) \perp h'(Q_s)$

Indeed if $h \neq h'$, then there is at least one g_i that commutes with h but anticommutes with h' (or the other way round), and so the same argument as in ② applies.

Consequently, the 2^k-dim'l subspaces $h(Q_s)$, as h ranges over the 2^{n-k} operators in $\langle h_1,\dots,h_{n-k}\rangle$, are all orthogonal to each other, and altogether they span $\mathbb{C}^{2^n}$ [The ~~ON bases~~ union of the ON bases for the subspaces $h(Q_s)$ is an ON basis for $\mathbb{C}^{2^n}$]

Syndromes revisited.

Going back to $\mathcal{P}_n/\langle\omega\rangle = \langle g_1,\dots,g_{n-k}, h_1,\dots,h_{n-k}, \bar{Z}_1,\dots,\bar{Z}_k, \bar{X}_1,\dots,\bar{X}_k\rangle$.

Any Pauli error operator E (ignoring the ω^l factor) is uniquely expressible as a product of ~~the~~ some of the g_i's, h_i's, $\bar{Z}_i$'s and $\bar{X}_i$'s.

Thus,

$$E = \underbrace{g(E)}_{\text{product of some } g_i\text{'s}} \underbrace{h(E)}_{\text{product of some } h_i\text{'s}} \underbrace{\mathcal{L}(E)}_{\text{product of some of the logical operators } \bar{Z}_i \text{ and } \bar{X}_i}.$$

Recall that the syndrome of E, $s(E) = (s_1,\dots,s_{n-k}) \in \{0,1\}^{n-k}$, simply ~~tells us~~ records whether or not each generator g_j commutes with E:

$$E\, g_j = (-1)^{s_j} g_j\, E,$$

and ~~for~~ given $|\psi\rangle = E|\varphi\rangle$ for some $|\varphi\rangle \in Q_S$, the syndrome of E can be determined by measuring $|\psi\rangle$ wrt each (observable) g_j.

Now, ~~whether~~

$s_j = 0 \Rightarrow g_j$ commutes with E

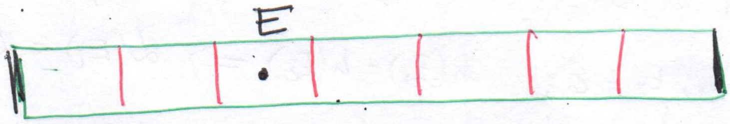
$\Rightarrow h_j$ is not a part of the $h(E)$ component of E

and $s_j = 1 \Rightarrow h_j$ is a part of the $h(E)$ component of E.

Thus, $h(E) = \prod_{j=1}^{n-k} h_j^{s_j}$, ie, the syndrome of E uniquely determines $h(E)$.

(this is called the "pure error" component of E)

In other words, the syndrome of E uniquely determines the coset of $C(S)$ in $\mathcal{P}_n$ within which E lies.



← coset of $C(S)$ in $\mathcal{P}_n$ within which E lies.

However, to recover $|\varphi\rangle$ from $|\psi\rangle = E|\varphi\rangle$, one needs to further determine which coset of S (within the above coset of $C(S)$ in $\mathcal{P}_n$) the error ~~operator~~ E lies. This suffices because any operator F within that ~~coset~~ S-coset has the property that $F^\dagger E \in S$, and hence applying any F from that S-coset will recover $|\varphi\rangle$ from $|\psi\rangle$:

$$F^\dagger|\psi\rangle = F^\dagger E|\varphi\rangle = |\varphi\rangle, \text{ since } F^\dagger E \in S.$$

This is equivalent to identifying $\mathcal{L}(E)$ — the "logical error" of E

because the coset we want is precisely

$$S \cdot h(E)\, \mathcal{L}(E).$$

To identify $\mathcal{L}(E)$, one requires more information, eg., knowledge of the quantum operation (channel) that introduces errors.

[This discussion leads to the idea behind subsystem codes and also to decoding of toric codes]

Remark: The Knill-Laflamme condition says that a set of Pauli errors $\mathcal{E}$ is correctable by Q_S iff $E_1^\dagger E_2 \notin C(S)\setminus S$ for all E_1, E_2.

In terms of syndrome decodings

Either ① E_1 and E_2 do not have the same syndrome

i.e., $h(E_1) \neq h(E_2) \iff E_1$ & E_2 lie in different cosets of $C(S)$ in $\mathcal{P}_n$

$\iff E_1^\dagger E_2 \notin C(S)$

OR ② E_1 and E_2 have the same syndrome, ie, $h(E_1) = h(E_2)$

For such errors, KL is the same as $\mathcal{L}(E_1) = \mathcal{L}(E_2)$ so that $E_1^\dagger E_2 \in S$.

$$\boxed{\text{In summary, a set of Pauli errors } \mathcal{E} \text{ is correctable by } Q_S \text{ iff } \forall E_1, E_2 \in \mathcal{E},\ h(E_1) = h(E_2) \Rightarrow \mathcal{L}(E_1) = \mathcal{L}(E_2).}$$