Subsystem stabilizer codes

Start again with ~~two~~ two sets of n hyperbolic pairs in $\mathcal{P}_n$:

$$
\begin{array}{lccccccc}
 & Z_1 & \longleftrightarrow & X_1 & & \bar{Z}_1 & \longleftrightarrow & \bar{X}_1 \\
 & Z_2 & \longleftrightarrow & X_2 & & \bar{Z}_2 & \longleftrightarrow & \bar{X}_2 \\
 & \vdots & & \vdots & & \vdots & & \\
 & Z_k & \longleftrightarrow & X_k & & \bar{Z}_k & \longrightarrow & \bar{X}_k \\
\text{n-k stabilizer generators for } Q=\{|\phi\rangle|0\rangle^{\otimes(n-k)} : |\phi\rangle \in \mathbb{C}^{2^k}\} & \left\{\begin{array}{c} Z_{k+1} \\ \vdots \\ Z_n \end{array}\right. & \begin{array}{c}\longleftrightarrow \\ \\ \longleftrightarrow\end{array} & \begin{array}{c} X_{k+1} \\ \vdots \\ X_n \end{array} & \text{n-k stabilizer generators for some other } Q_S & \left\{\begin{array}{c} g_1 \\ \vdots \\ g_{n-k} \end{array}\right. & \begin{array}{c}\longrightarrow \\ \\ \longleftarrow\end{array} & \begin{array}{c} h_1 \\ \vdots \\ h_{n-k} \end{array}
\end{array}
$$

There is a unitary U s.t. $U(\text{operator in first set})U^\dagger = \text{operator in } 2^{\text{nd}} \text{ set}$

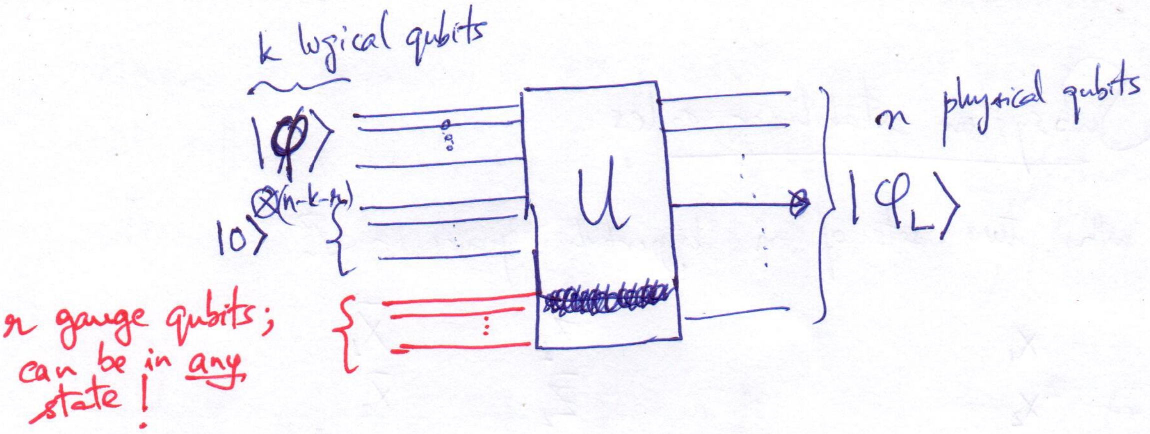
$$
\underset{\in Q}{|\phi\rangle|0\rangle^{\otimes(n-k)}} \xrightarrow{\quad U \quad} \underset{\in Q_S}{|\phi_L\rangle}
$$

Then, for an arbitrary $E \in \mathcal{P}_n$, we can write it as ($\bmod \langle\omega\rangle$)

$$
E = g(E)\, h(E)\, \mathcal{L}(E)
$$

We convert some of the stabilizers on either side into <u>gauge operators</u>.

$$
\begin{array}{lcc c lcc}
 & Z_1 & X_1 & & & \bar{Z}_1 & \bar{X}_1 \\
 & \vdots & \vdots & & & \vdots & \\
 & Z_k & X_k & \xleftrightarrow{\text{Unitary } U} & & \bar{Z}_k & \bar{X}_k \\
\text{n-k-r stabilizer generators} & \left\{\begin{array}{c} Z_{k+1} \\ \vdots \\ Z_{n-r} \end{array}\right. & \begin{array}{c} X_{k+1} \\ \vdots \\ X_{n-r} \end{array} & & \text{n-k-r stabilizer generators} & \left\{\begin{array}{c} g_1 \\ \vdots \\ g_{n-k-r} \end{array}\right. & \begin{array}{c} h_1 \\ \vdots \\ h_{n-k-r} \end{array} \\
2r \text{ gauge operators} & \boxed{\begin{array}{cc} Z_{n-r+1} & X_{n-r+1} \\ \vdots & \vdots \\ Z_n & X_n \end{array}} & & & 2r \text{ gauge operators} & \boxed{\begin{array}{cc} g_{n-k-r+1} =: G^z_1 & h_{n-k-r+1} =: G^x_1 \\ \vdots & \vdots \\ g_{n-k} =: G^z_r & h_{n-k} =: G^x_r \end{array}} & \text{2r gauge operators}
\end{array}
$$



The ~~sub~~ subspace on the left (input end) of the encoder U is of the form $\underbrace{A}_{|\varphi\rangle|0\rangle^{\otimes(n-k-r)}} \otimes \underbrace{B}_{r\text{-qubit space.}}$

The logical qubits come from subsystem A; hence the term "subsystem codes".

Now, any Pauli error E can be decomposed as

$$E = \underbrace{g(E)\,h(E)}_{\substack{\text{these come from} \\ g_1 \cdots g_{n-k-r} \\ h_1 \cdots h_{n-k-r}}} \ \underbrace{\mathcal{L}(E)}_{\substack{\text{from the} \\ \text{logical operators} \\ \bar{X}_1 \cdots \bar{X}_k \\ \bar{Z}_1 \cdots \bar{Z}_k}} \ \underbrace{\mathcal{G}(E)}_{\substack{\text{from the gauge operators} \\ G_1^x \cdots G_r^x \\ G_1^z \cdots G_r^z}}$$

The syndrome $\underline{s} = (s_1, \dots, s_{n-k-r})$ of E is obtained by measuring the stabilizer generators $g_1 \cdots g_{n-k-r}$.

This, as before, determines $h(E) = \prod_{i=1}^{n-k-r} h_i^{s_i}$

Again, as before, $g(E)$ need not be recovered exactly, as this part of E has no effect on the encoded state

But now, even $\mathcal{G}(E)$ need not be recovered exactly, as this too has no effect on the logical qubits encoded!

To see why:

Applying, say, G_i^x to $|\varphi_L\rangle$, an encoding of the k logical qubits $|\varphi\rangle$:

For any $1 \le i \le r$

$$G_i^x |\varphi_L\rangle = U X_{n+1-i} U^\dagger \, U\Big(|\varphi\rangle|0\rangle^{\otimes(n-k-r)} \underbrace{|\psi\rangle}_{\text{gauge qubits}}\Big)$$

$$= U\Big(|\varphi\rangle|0\rangle^{\otimes(n-k-r)} \underbrace{X_{n+1-i}|\psi\rangle}_{=:\,|\psi'\rangle}\Big)$$

which is also an encoding of $|\varphi\rangle$.

Removing some of the stabilizers implies that less information about the error E is available to the decoder.

However, this loss is compensated by an increase in the code's degeneracy, meaning ~~errors~~ there are more errors that do not affect the encoded (logical) qubits. Also, the min. dist. is unaffected as the logical operators do not change!