E2 210 (Jan.–Apr. 2025)

Homework Assignment 1

Submission deadline: Wednesday, Jan. 29, 11:59pm

1. Let $\mathcal{C}$ be a linear code with minimum distance d, and let s, t be non-negative integers such that $2t + s \leq d - 1$. Describe a decoder for $\mathcal{C}$ that can correct every error-and-erasure pattern that contains at most t errors along with at most s erasures.

2. Let $\mathbb{F}_{11}$ be the field of integers modulo-11, i.e., the set of integers $\{0, 1, 2, \ldots, 9, 10\}$ with mod-11 arithmetic. Consider the ISBN code over $\mathbb{F}_{11}$:

$$\mathcal{C}_{\text{ISBN}} = \left\{ (c_1, c_2, \ldots, c_{10}) \in (\mathbb{F}_{11})^{10} : \sum_{i=1}^{10} (11 - i) \cdot c_i \equiv 0 \pmod{11} \right\}.$$

   (a) Verify that $\mathcal{C}_{\text{ISBN}}$ is a linear code by providing a parity-check matrix for it.

   (b) Determine the dimension and minimum distance of $\mathcal{C}_{\text{ISBN}}$.

   (c) Derive a generator matrix for $\mathcal{C}_{\text{ISBN}}$.

3. Consider a new and improved ISBN code $\mathcal{C}^{\#}_{\text{ISBN}}$ that consists of all codewords $(c_1, c_2, \ldots, c_{10})$ of the ISBN code $\mathcal{C}_{\text{ISBN}}$ that additionally satisfy the requirement that $c_1 + c_2 + \cdots + c_{10} \equiv 0 \pmod{11}$.

   (a) Determine the dimension and minimum distance of $\mathcal{C}^{\#}_{\text{ISBN}}$.

   (b) How many errors can this code correct?

4. An international publishing house is seeking to design a linear code of length 10 over $\mathbb{F}_{11}$ to be used as an enhanced International Standard Book Number (ISBN) scheme. Their requirement is that the code, with a suitable choice of decoder, should be able to correct single transposition errors. These are errors in which *exactly one* pair (c_i, c_j) of coordinates in a codeword $\mathbf{c} = (c_1, c_2, \ldots, c_{10})$ gets interchanged; for example, the codeword $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ may get transformed to $(1, 2, \mathbf{6}, 4, 5, \mathbf{3}, 7, 8, 9, 10)$. Other kinds of errors may be ignored.

   Does the code $\mathcal{C}^{\#}_{\text{ISBN}}$ of Problem 3, under syndrome decoding, meet the requirements of the enhanced ISBN scheme?

   [*Hint*: All the error vectors that are to be corrected must lie in distinct cosets, so that they can be chosen to be coset leaders for the purposes of syndrome decoding. Identify the error vectors $\mathbf{e}$ obtainable as $\mathbf{e} = \mathbf{y} - \mathbf{c}$, where $\mathbf{c}$ is a codeword, and $\mathbf{y}$ is obtained by interchanging some two coordinates of $\mathbf{c}$. How many such error vectors are there? How many cosets does $\mathcal{C}^{\#}_{\text{ISBN}}$ have?]