Solutions to E2 210 Midterm Exam

March 4, 2025

TIME: 1½ hours — MAX MARKS: 30

The marks assigned to each part of a question are listed within square brackets at the right margin.

1. `[5 marks]` Let $\mathcal{C}$ be the (classical) binary linear code with parity-check matrix $H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$.

   (a) What is the dimension of $\mathcal{C}$? What is its minimum distance? `[2]`

   Solution: Since H contains the 3×3 identity matrix as a submatrix, it has rank equal to 3. Hence, $\dim(\mathcal{C}) = n - \text{rank}(H) = 5 - 3 = 2$.

   (b) Identify an error vector $\mathbf{e}$ of least weight having the same syndrome as $\mathbf{y} = [1\ 1\ 1\ 1\ 1]$. `[3]`

   Solution: The syndrome of $\mathbf{y}$ is $\mathbf{s} = H\mathbf{y}^T = [1\ 0\ 0]^T$. Since this is identical to the third column of H, if we take $\mathbf{e} = [0\ 0\ 1\ 0\ 0]$, then $H\mathbf{e}^T = [1\ 0\ 0]^T$. Since $w_H(\mathbf{e}) = 1$, we conclude that $\mathbf{e} = [0\ 0\ 1\ 0\ 0]$ is an error vector of least weight having syndrome equal to $[1\ 0\ 0]^T$.

2. `[5 marks]` Consider the depolarizing noise channel:

$$\mathcal{E}(\rho) = (1-p)\rho + p\,\frac{I}{2}.$$

   Compute the fidelity between a single-qubit pure state $|\psi\rangle$ and $\rho := \mathcal{E}(|\psi\rangle\langle\psi|)$.

   Solution: Recall that the fidelity between a pure state $|\psi\rangle$ and a (possibly mixed) state with density matrix ρ is given by $F(|\psi\rangle, \rho) := \sqrt{\langle\psi|\,\rho\,|\psi\rangle}$. Plugging in $\rho = \mathcal{E}(|\psi\rangle\langle\psi|) = (1-p)\,|\psi\rangle\langle\psi| + p\,\frac{I}{2}$ into the expression for fidelity, we obtain

$$F(|\psi\rangle, \rho) \;=\; \sqrt{(1-p)\,\langle\psi|\psi\rangle\,\langle\psi|\psi\rangle \;+\; p\,\langle\psi|\frac{I}{2}|\psi\rangle} \;=\; \sqrt{(1-p) + p/2} \;=\; \sqrt{1-p/2}.$$

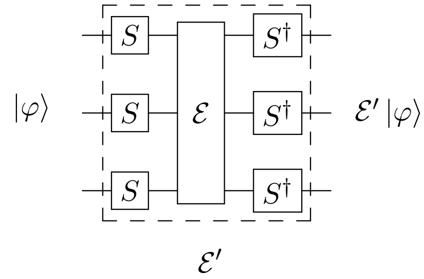
3. `[10 marks]` Recall that phase-flip (Z) errors can be converted to bit-flip (X) errors by padding with Hadamard gates. This idea was used to obtain a 3-qubit single-Z-error correcting code from the 3-qubit single-X-error correcting code $\mathcal{Q} = \text{span}(|000\rangle, |111\rangle)$.

   (a) Describe how Y-errors can be converted to X-errors through the use of S and $S^\dagger$ gates. Here, $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ is the phase gate that applies a $\pi/2$ phase shift on $|1\rangle$ while leaving $|0\rangle$ unaffected. `[4]`

   Solution: We make use of the identity $S \cdot X \cdot S^\dagger = Y$, or equivalently, $S^\dagger \cdot Y \cdot S = X$. In the circuit model, we have

$$-\boxed{S}-\boxed{Y}-\boxed{S^\dagger}- \quad = \quad -\boxed{X}-$$

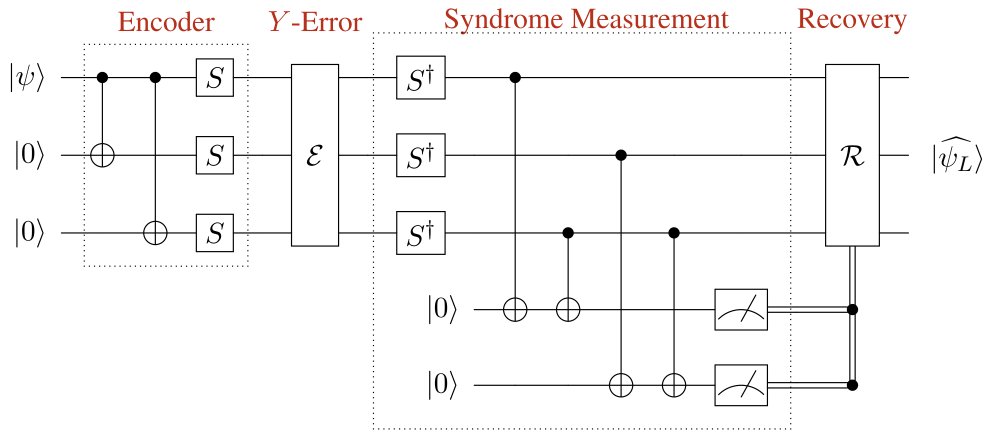
So, to convert Y-errors to X-errors, we pad on the left with S gates and on the right with $S^\dagger$ gates.



If $\mathcal{E}$ is an error operator of the form $Y^{e_1} \otimes Y^{e_2} \otimes Y^{e_3}$, with $e_1, e_2, e_3 \in \{0, 1\}$, then $\mathcal{E}'$ is of the form $X^{e_1} \otimes X^{e_2} \otimes X^{e_3}$.

(b) Explain how a 3-qubit single-Y-error correcting code can be obtained from the single-X-error correcting code $\mathcal{Q}$. Specifically, give an encoding circuit for a 3-qubit single-Y-error correcting code. [6]

Solution:



4. [10 marks] Consider the stabilizer group $\mathcal{S}$ in $\mathcal{P}_5$ generated by

$$g_1 = XZZXI, \quad g_2 = IXZZX, \quad g_3 = XIXZZ, \quad g_4 = ZXIXZ.$$

The corresponding stabilizer code $\mathcal{Q}_\mathcal{S}$ is known to be a $[[5, 1, 3]]_2$ code.

(a) Explain why $\mathcal{E} = \{E \in \mathcal{P}_5 : \mathrm{wt_s}(E) \le 1\}$ is a set of correctable errors for $\mathcal{Q}_\mathcal{S}$. [2]

Solution: Since $\dim \mathcal{Q}_\mathcal{S} = 2^1 > 1$, its minimum distance is given by $d_{\min}(\mathcal{Q}_\mathcal{S}) = \min\{\mathrm{wt_s}(E) : E \in C(\mathcal{S}) \setminus \mathcal{S}^\Phi\}$. For any $E_1, E_2 \in \mathcal{E}$, we have $\mathrm{wt_s}(E_1^\dagger E_2) \le \mathrm{wt_s}(E_1) + \mathrm{wt_s}(E_2) \le 2$, which is strictly less than $d_{\min}(\mathcal{Q}_\mathcal{S})$. Hence, it must be the case that $E_1^\dagger E_2 \notin C(\mathcal{S}) \setminus \mathcal{S}^\Phi$. Therefore, by the Knill-Laflamme condition for stabilizer codes and Pauli errors, the set $\mathcal{E}$ is a correctable set of errors for $\mathcal{Q}_\mathcal{S}$.

(b) For the $\mathcal{E}$ in part (a), identify, if possible, an $E \in \mathcal{E}$ having syndrome $\mathrm{s} = [1\ 1\ 1\ 1]$. [4]

Solution: We want an operator $E \in \mathcal{E}$ that anti-commutes with each of the stabilizer generators g_1, g_2, g_3, g_4. Note that, ignoring the phase factor of i^ℓ, operators in $\mathcal{E}$ are of the form $M_1 \otimes \cdots \otimes M_5$, with at most one of the M_j's being something other than the identity operator I.

Now, every operator of the form $MIIII$ commutes with g_2, since g_2 has an I in the first position. Similarly, every operator of the form $IMIII$, $IIMII$ and $IIIIM$ commutes with g_3, g_4 and g_1, respectively. Thus, for an operator in $\mathcal{E}$ to anti-commute with *all* the stabilizer generators, it must be of the form $IIIMI$, with $M \in \{X, Y, Z\}$. Each of the generators g_i has either an X or a Z in the fourth position, and Y anti-commutes with both. Thus, $IIIYI$ anti-commutes with all the stabilizer generators, and hence, is the required $E \in \mathcal{E}$ with syndrome $\mathbf{s} = [1\ 1\ 1\ 1]$.

(c) Identify any one operator in $C(\mathcal{S}) \setminus \mathcal{S}^{\Phi}$. (Recall that, in our notation, $\mathcal{S}^{\Phi} = \langle \mathcal{S} \cup \{i \cdot I\} \rangle$.) [4]

Solution: We want an operator that commutes with all the generators g_i, but is not itself a product of the g_i's up to a phase factor. A little experimentation will quickly reveal that $XXXXX$ and $ZZZZZ$ are both in $C(\mathcal{S})$, i.e., both these operators commute with all the g_i's. It remains to demonstrate that neither of these is in $\mathcal{S}^{\Phi}$.

The check matrix corresponding to the generators g_1, g_2, g_3, g_4 is

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & | & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & | & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & | & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

One can readily check that one cannot get $[0\ 0\ 0\ 0\ 0\ \ |\ * \ * \ * \ * \ *]$ or $[* \ * \ * \ * \ * \ \ |\ 0\ 0\ 0\ 0\ 0]$ as a non-trivial linear combination (over the binary field $\mathbb{F}_2$) of the rows of H. This is because the submatrix H_X consisting of the first five columns of H is of full row rank over $\mathbb{F}_2$, and the same is true of the H_Z submatrix consisting of the last five columns of H. From this, one concludes that $\mathcal{S}^{\Phi}$ does not contain Pauli operators consisting of X's and I's only nor does it contain Pauli operators consisting of Z's and I's only. In particular, $\mathcal{S}^{\Phi}$ does not contain $XXXXX$ or $ZZZZZ$. Thus, these operators are in $C(\mathcal{S}) \setminus \mathcal{S}^{\Phi}$.