Homework 2 Solutions

E2-210, Jan–Apr 2025

1. Fidelity is defined as $F(\rho,\sigma) := \operatorname{tr}\sqrt{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}$.

   (a) $|\psi\rangle$ is a pure state. Then $(|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| \implies |\psi\rangle\langle\psi| = (|\psi\rangle\langle\psi|)^{\frac{1}{2}}$. Therefore,

   $$\begin{aligned} F(|\psi\rangle,\rho) &= \operatorname{tr}\sqrt{(|\psi\rangle\langle\psi|)^{\frac{1}{2}}\rho(|\psi\rangle\langle\psi|)^{\frac{1}{2}}} \\ &= \operatorname{tr}\sqrt{|\psi\rangle\langle\psi|\,\rho\,|\psi\rangle\langle\psi|} = \sqrt{\langle\psi|\,\rho\,|\psi\rangle}\operatorname{tr}\sqrt{|\psi\rangle\langle\psi|} = \sqrt{\langle\psi|\,\rho\,|\psi\rangle}\underbrace{\operatorname{tr}(|\psi\rangle\langle\psi|)}_{=\langle\psi|\psi\rangle=1} = \sqrt{\langle\psi|\,\rho\,|\psi\rangle}. \end{aligned}$$

   (b) If ρ and σ commute, they are simultaneously diagonalizable in the same ON basis. Let $\{|i\rangle\}$ be the ON basis. Then, $\rho = \sum_i \lambda_i |i\rangle\langle i|$ and $\sigma = \sum_i \mu_i |i\rangle\langle i|$. With this, we have

   $$\begin{aligned} F(\rho,\sigma) = \operatorname{tr}\sqrt{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}} &= \operatorname{tr}\sqrt{(\sum_i \lambda_i^{\frac{1}{2}}|i\rangle\langle i|)(\sum_j \mu_j |j\rangle\langle j|)(\sum_k \lambda_k^{\frac{1}{2}}|k\rangle\langle k|)} \\ &= \operatorname{tr}\sqrt{\sum_{i,j,k}(\lambda_i\cdot\lambda_k)^{\frac{1}{2}}\mu_j |i\rangle\langle i|j\rangle\langle j|k\rangle\langle k|} = \operatorname{tr}\sqrt{\sum_i \lambda_i\mu_i |i\rangle\langle i|} \\ &= \operatorname{tr}\left(\sum_i \sqrt{\lambda_i\mu_i}\,|i\rangle\langle i|\right) = \sum_i \sqrt{\lambda_i\mu_i}\operatorname{tr}(|i\rangle\langle i|) = \sum_i \sqrt{\lambda_i\mu_i} \end{aligned}$$

2. (a) $\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$. We want to show that, for $\sigma = \mathcal{E}(|\psi\rangle\langle\psi|)$, we have $F(|\psi\rangle,\sigma) = \sqrt{1-2p/3}$. There are at least two ways that one can show this.

   (i) The first way is a direct calculation, where we plug in $\sigma = (1-p)|\psi\rangle\langle\psi| + \frac{p}{3}(X|\psi\rangle\langle\psi|X + Y|\psi\rangle\langle\psi|Y + Z|\psi\rangle\langle\psi|Z)$ into $F(|\psi\rangle,\sigma) = \sqrt{\langle\psi|\,\sigma\,|\psi\rangle}$ to obtain

   $$F(|\psi\rangle,\sigma) = \sqrt{1-p+\tfrac{p}{3}(\langle\psi|X|\psi\rangle^2 + \langle\psi|Y|\psi\rangle^2 + \langle\psi|Z|\psi\rangle^2)} \tag{1}$$

   We claim that for any state $\psi = a|0\rangle + b|1\rangle$, with $|a|^2+|b|^2=1$, we have $\langle\psi|X|\psi\rangle^2 + \langle\psi|Y|\psi\rangle^2 + \langle\psi|Z|\psi\rangle^2 = 1$. Indeed, we have

   $$\begin{aligned} \langle\psi|X|\psi\rangle &= (a^*\langle 0| + b^*\langle 1|)(a|1\rangle + b|0\rangle) = a^*b + ab^* \\ \langle\psi|Y|\psi\rangle &= (a^*\langle 0| + b^*\langle 1|)(ia|1\rangle - ib|0\rangle) = i(ab^* - a^*b) \\ \langle\psi|Z|\psi\rangle &= (a^*\langle 0| + b^*\langle 1|)(a|0\rangle - b|1\rangle) = |a|^2 - |b|^2 \end{aligned}$$

Hence,

$$\begin{aligned}
\langle\psi| X |\psi\rangle^2 + \langle\psi| Y |\psi\rangle^2 + \langle\psi| Z |\psi\rangle^2 &= \underbrace{(a^*b + ab^*)^2 - (ab^* - a^*b)^2}_{=4|a|^2|b|^2} + (|a|^2 - |b|^2)^2 \\
&= (|a|^2 + |b|^2)^2 \\
&= 1.
\end{aligned}$$

Plugging this back into (1), we obtain $F(|\psi\rangle, \sigma) = \sqrt{1 - 2p/3}$ as desired.

(ii) The second approach is to use the identity

$$\frac{I}{2} = \frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z), \tag{2}$$

which holds for any single-qubit density matrix ρ. A proof of this identity will be given later. Using the identity, we get

$$\mathcal{E}(\rho) = (1-p)\rho + \tfrac{p}{3}(2I - \rho) = (1 - \tfrac{4p}{3})\rho + \tfrac{2p}{3}I$$

so that $\sigma = \mathcal{E}(|\psi\rangle\langle\psi|) = (1 - \frac{4p}{3})|\psi\rangle\langle\psi| + \frac{2p}{3}I$. From this, we obtain

$$\langle\psi| \sigma |\psi\rangle = (1 - \tfrac{4p}{3}) + \tfrac{2p}{3} = 1 - \tfrac{2p}{3}$$

and hence, $F(|\psi\rangle, \sigma) = \sqrt{\langle\psi| \sigma |\psi\rangle} = \sqrt{1 - \frac{2p}{3}}$.

Proof of (2) : Denote the RHS of (2) by $\mathcal{L}(\rho)$. By direct calculations, using $XY = -YX$, $XZ = -ZX$ and $YZ = -ZY$, and also $X^2 = Y^2 = Z^2 = I$, we get

$$\begin{aligned}
\mathcal{L}(\tfrac{I}{2}) &= \tfrac{I}{2} \\
\mathcal{L}(X) &= \frac{1}{4}(X + X + YXY + ZXZ) \\
&= \frac{1}{4}(X + X - XYY - XZZ) = \frac{1}{4}(X + X - X - X) = 0 \\
\mathcal{L}(Y) &= \frac{1}{4}(Y + XYX + Y + ZYZ) \\
&= \frac{1}{4}(Y - YXX + Y - YZZ) = \frac{1}{4}(Y - Y + Y - Y) = 0 \\
\mathcal{L}(Z) &= \frac{1}{4}(Z + XZX + YZY + Z) \\
&= \frac{1}{4}(Z - ZXX - ZYY + Z) = \frac{1}{4}(Z - Z - Z + Z) = 0
\end{aligned}$$

So, if we write ρ using its Bloch vector representation, $\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$, then

$$\mathcal{L}(\rho) = \mathcal{L}(\tfrac{I}{2}) + r_x\mathcal{L}(X) + r_y\mathcal{L}(Y) + r_z\mathcal{L}(Z) = \tfrac{I}{2},$$

which proves the identity. □

(b) $\mathcal{E}_{\mathrm{AD}}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$. First note that

$$\langle\psi|\, \mathcal{E}_{\mathrm{AD}}(|\psi\rangle\langle\psi|)\, |\psi\rangle \;=\; |\, \langle\psi|\, E_0\, |\psi\rangle\, |^2 + |\, \langle\psi|\, E_1\, |\psi\rangle\, |^2.$$

Plugging in $\psi = a\,|0\rangle + b\,|1\rangle$, we get after some calculations,

$$\langle\psi|\, \mathcal{E}_{\mathrm{AD}}(|\psi\rangle\langle\psi|)\, |\psi\rangle \;=\; 1-\gamma + |a|^2 \left[\gamma + 2|b|^2\sqrt{1-\gamma}\,(1-\sqrt{1-\gamma})\right] \;\geq\; 1-\gamma$$

with equality iff $a = 0$, i.e., $|\psi\rangle = |1\rangle$. Thus,

$$F\big(|\psi\rangle\,,\, \mathcal{E}_{\mathrm{AD}}(|\psi\rangle\langle\psi|)\big) \;=\; \sqrt{\langle\psi|\, \mathcal{E}_{\mathrm{AD}}(|\psi\rangle\langle\psi|)\, |\psi\rangle} \;\geq \sqrt{1-\gamma}$$

with equality iff $|\psi\rangle = |1\rangle$.

3. We simply check if the Knill-Laflamme condition holds for $\mathcal{E}$ and the ON basis $|\phi_1\rangle = |{+}{+}\rangle$, $|\phi_2\rangle = |{-}{-}\rangle$. Take $E_1 = S \otimes I$ and $E_2 = I \otimes S$. Then, $E_1^\dagger E_2 = S^\dagger \otimes S$, and $\langle\phi_1|E|\phi_2\rangle = \langle + + |S^\dagger \otimes S| - -\rangle = \langle +|S^\dagger|-\rangle\ \langle +|S|-\rangle$. We then compute

$$\begin{aligned}\langle +|S^\dagger|-\rangle &= \frac{1}{2}(\langle 0| + \langle 1|)(|0\rangle + i\,|1\rangle) = \frac{1}{2}(1+i)\\ \langle +|S|-\rangle &= \frac{1}{2}(\langle 0| + \langle 1|)(|0\rangle - i\,|1\rangle) = \frac{1}{2}(1-i)\end{aligned}$$

Hence, $\langle\phi_1|E|\phi_2\rangle = \frac{1}{4}(1+i)(1-i) = \frac{1}{2}$, which is not equal to $C_{12}\,\langle\phi_1|\phi_2\rangle$, for any constant C_{12}, since $\langle\phi_1|\phi_2\rangle = 0$. Therefore, the Knill-Laflamme condition does not hold, so that there is no recovery operation that allows an arbitrary $|\varphi\rangle \in \mathcal{Q}$ to be recovered from any error in the set $\mathcal{E}$.