Hypergraph Product Codes

Aswanth T

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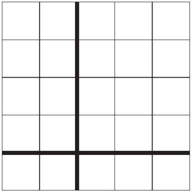

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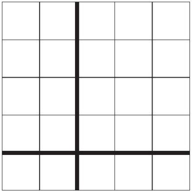

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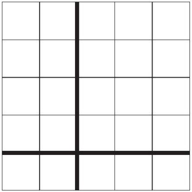

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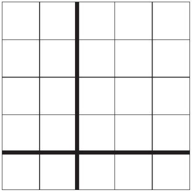

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- Consider a graph $\mathcal{G}$ which is a tiling of the torus as shown in figure
- The vertex set $\mathcal{V} = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$
- Every vertex (x, y) has edges connecting to the four neighbouring vertices $(x \pm 1, y), (x, y \pm 1)$ (additions and subtractions are modulo m).

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- The dual code of the code associated with $\mathbf{H}_Z$, $\mathscr{C}_Z^{\perp}$ is therefore a subspace of the cycle code $\mathscr{C}_X$. $\Rightarrow$ $\mathbf{H}_Z$ and $\mathbf{H}_X$ can be used for CSS construction.

Toric Code

- The number of qubits that the quantum code can encode,

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- The minimum distance of the quantum code,

$$\begin{aligned} d_Q &\stackrel{\text{def}}{=} \min\{d_X, d_Z\}, \text{ where} \\ d_X &\stackrel{\text{def}}{=} \min\left\{|x|, x \in \mathscr{C}_X \backslash {\mathscr{C}_Z}^{\perp}\right\}, \\ d_Z &\stackrel{\text{def}}{=} \min\left\{|x|, x \in \mathscr{C}_Z \backslash {\mathscr{C}_X}^{\perp}\right\}. \end{aligned}$$

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- The coset leaders in $\mathscr{C}_X/\mathscr{C}_Z^\perp$ are horizontal and vertical loops of the form $(a, 0), (a, 1), \ldots, (a, m-1)$ and $(0, a), (1, a), \ldots, (m-1, a)$ for any a.

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- Hence, d_Z is also m.
  $\Rightarrow$ The minimum distance of the quantum code is m.

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A family of $[[n, k, d]]$ qubit stabilizer codes is QLDPC if, as n tends to ∞,

1. the Pauli weight of each stabilizer generator is bounded by a constant,
2. for each physical qubit, the number of stabilizer generators having a non-trivial operator on it is bounded by a constant.

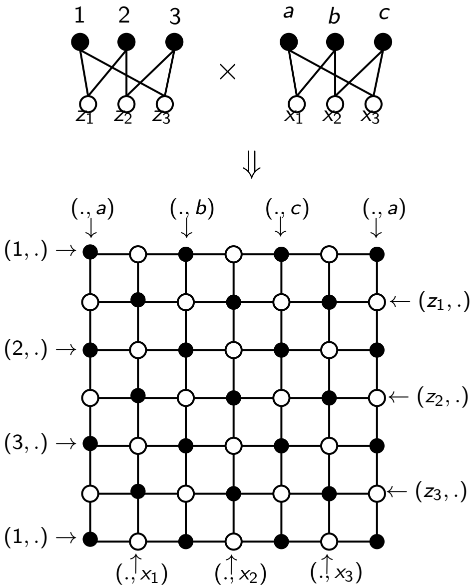
Advantage: The syndrome extraction of the constant-weight check operators can be done using a constant-depth circuit.

Generalizing Toric Code

Definition

Graph product: The product $\mathcal{G}_1 \times \mathcal{G}_2$ of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ has vertex set made up of (x, y), where x and y are vertices of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. The edges of the product graph connect two vertices (x, y) and (x', y') if either $x = x'$ and $\{y, y'\}$ is an edge of $\mathcal{G}_2$ or $y = y'$ and $\{x, x'\}$ is an edge of $\mathcal{G}_1$.

Generalizing Toric Code



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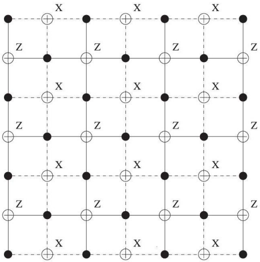
- The product graph is the union of the Tanner graphs of $\mathscr{C}_X$ and $\mathscr{C}_Z$ in toric code.



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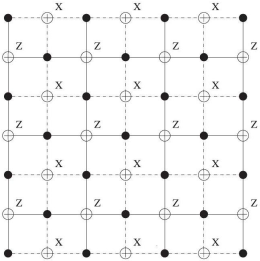

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- This motivates us to generalize the construction using graph product.

CSS code $\mathcal{Q}(\mathcal{G}_1 \times \mathcal{G}_2)$ associated to a graph product

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- The union of $\mathcal{G}_1 \times_X \mathcal{G}_2$ and $\mathcal{G}_1 \times_Z \mathcal{G}_2$ equals $\mathcal{G}_1 \times \mathcal{G}_2$.

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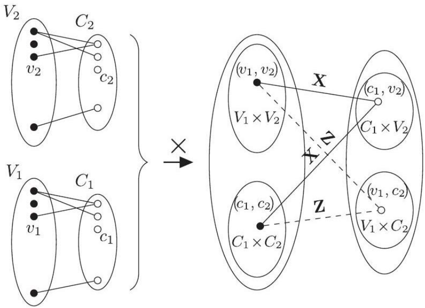

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- Define $\mathscr{C}_X$ and $\mathscr{C}_Z$ as the codes associated to the Tanner graphs $\mathcal{G}_1 \times {}_X\mathcal{G}_2$ and $\mathcal{G}_1 \times {}_Z\mathcal{G}_2$ respectively.

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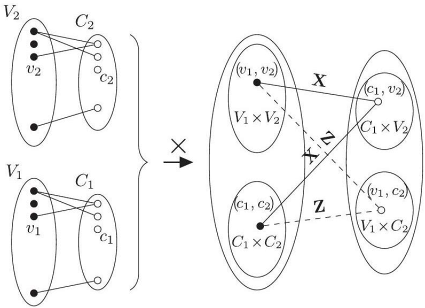

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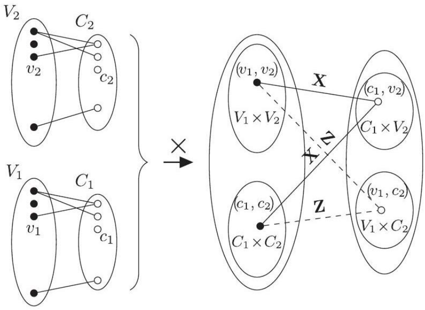

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- Construct a CSS code using $\mathscr{C}_X$ and $\mathscr{C}_Z$.
- It is still unclear whether $\mathscr{C}_Z^\perp \subseteq \mathscr{C}_X$.

Validity of the Construction

Proposition

$$\mathscr{C}_X^{\perp} \subseteq \mathscr{C}_Z$$

Proof:

- Let $\mathbf{H}_X$ and $\mathbf{H}_Z$ be the parity-check matrices associated to the Tanner graphs $\mathcal{G}_1 \times_X \mathcal{G}_2$ and $\mathcal{G}_1 \times_Z \mathcal{G}_2$ respectively.

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- Identify $\mathbf{h}_X(c_1, v_2)$ with the set of neighbors of (c_1, v_2) in $\mathcal{G}_1 \times \mathcal{G}_2$.
  (Similarly, for $\mathbf{H}_Z$ as well)

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- It is sufficient to prove that for any $v_1 \in V_i, c_i \in C_i, i \in \{1, 2\}$, $\mathbf{h}_X(c_1, v_2)$ of $\mathbf{H}_X$ is orthogonal to $\mathbf{h}_Z(v_1, c_2)$ of $\mathbf{H}_Z$.

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- If n_1 and r_1 (n_2 and r_2) are the number of variable nodes and check nodes in $\mathcal{G}_1$ ($\mathcal{G}_2$),

$$\begin{aligned} dim(\mathscr{C}_Z) &= n_1 n_2 + r_1 r_2 - rank(\mathbf{H}_Z) \\ &\geq n_1 n_2 + r_1 r_2 - n_1 r_2 \\ dim(\mathscr{C}_X^{\perp}) &= rank(\mathbf{H}_X) \\ &\leq n_2 r_1 \end{aligned}$$

Validity of the Construction

- For S to be nonempty, we need v_1 connected to c_1 and v_2 connected to c_2.
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- If we choose $\mathcal{G}_1$ and $\mathcal{G}_2$ properly, we can have $\mathscr{C}_X^{\perp} \subset \mathscr{C}_Z$ and get a non-trivial quantum code by CSS construction.

Hypergraphs and Their Product

- A **hypergraph** $\mathcal{H} = (V, \mathcal{E})$ is a set V together with a collection $\mathcal{E}$ of subsets called hyperedges.

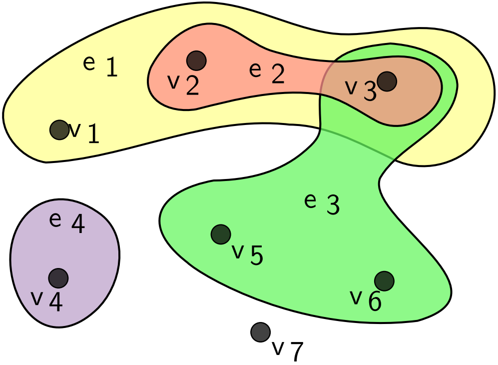

Figure: A hypergraph of vertex set $\{v_1, v_2, \ldots, v_6\}$ and hyperedges $\{e_1, e_2, e_3, e_4\}$

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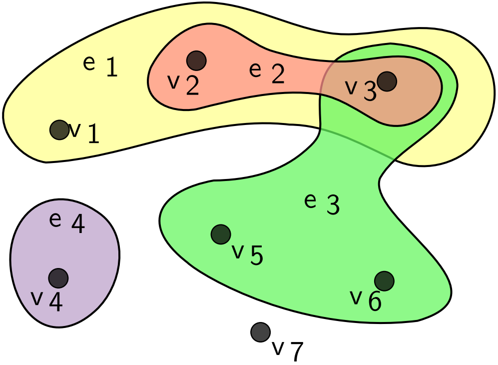

Figure: A hypergraph of vertex set $\{v_1, v_2, \ldots, v_6\}$ and hyperedges $\{e_1, e_2, e_3, e_4\}$

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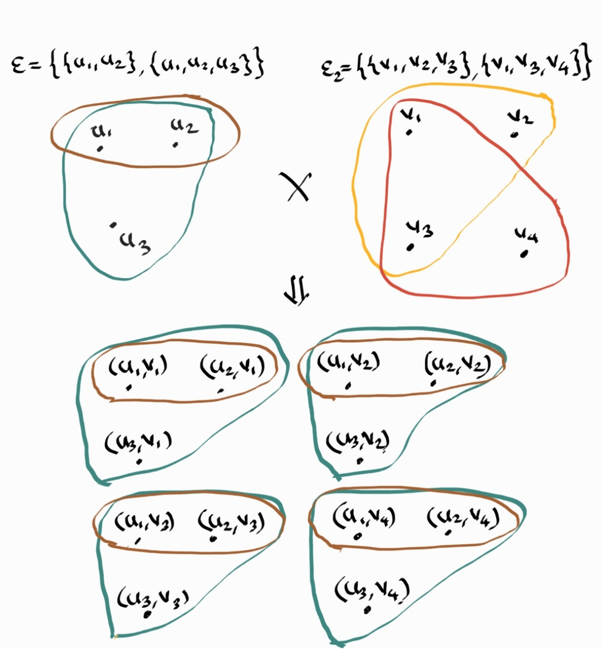
- A hypergraph is a graph when every edge has cardinality 2.

Hypergraphs and Their Product

- Let $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. The product hypergraph $\mathcal{H}_1 \times \mathcal{H}_2$ is defined as $\mathcal{H} = (V, \mathcal{E})$ such that $V = V_1 \times V_2$ and $\mathcal{E}$ is the collection of subsets of V
  - either of the form $\{v_1\} \times e_2$ with $v_1 \in V_1$ and $e_2 \in \mathcal{E}_2$,
  - or of the form $e_1 \times \{v_2\}$ with $e_1 \in \mathcal{E}_1$ and $v_2 \in V_2$.

Hypergraphs and Their Product - Example



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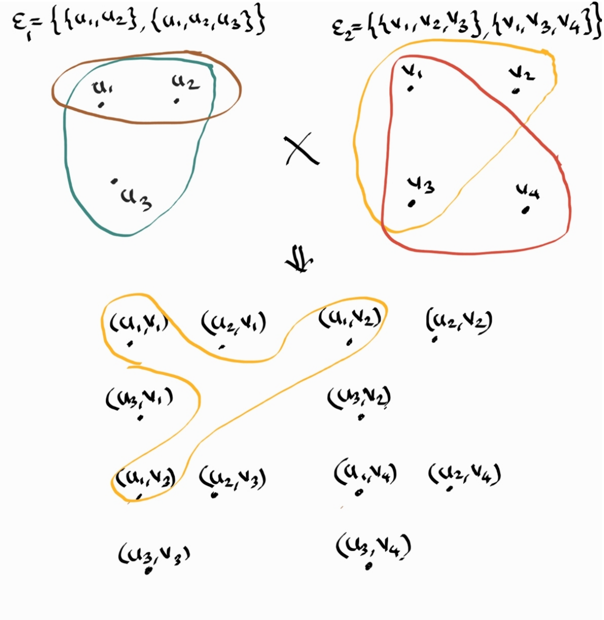

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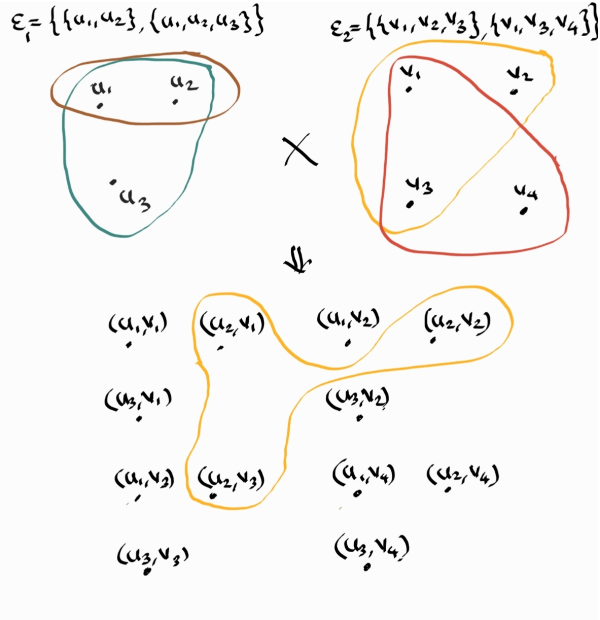

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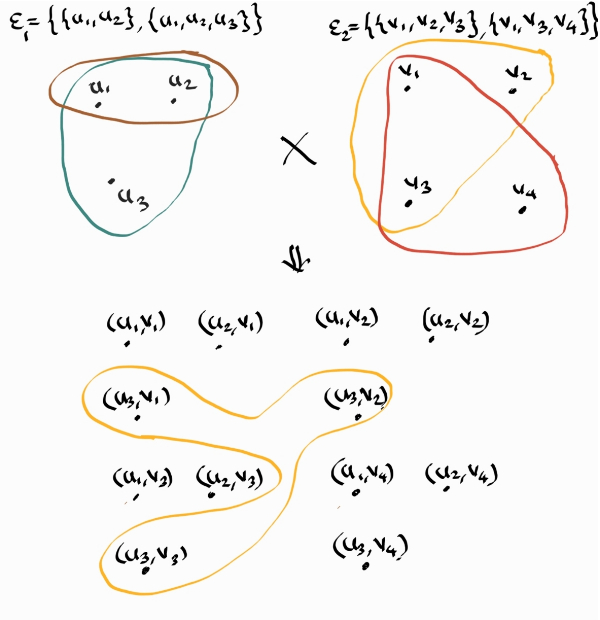
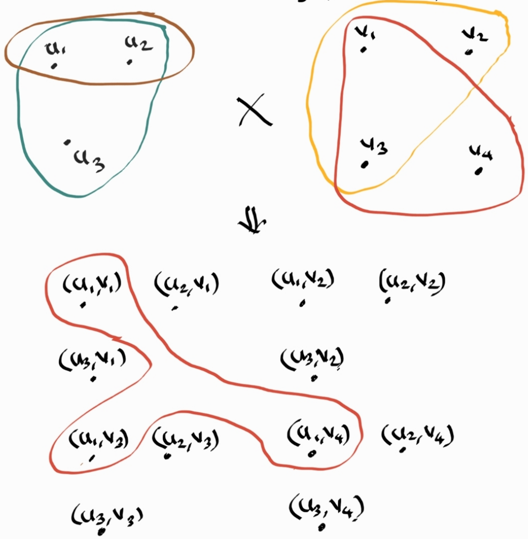


Hypergraphs and Their Product - Example

$\mathcal{E}_1 = \{\{u_1, u_2\}, \{u_1, u_2, u_3\}\}$ $\qquad$ $\mathcal{E}_2 = \{\{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}\}$



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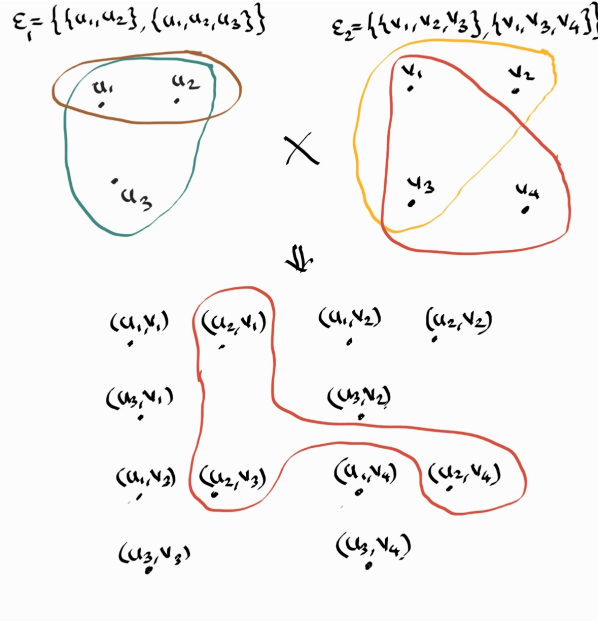

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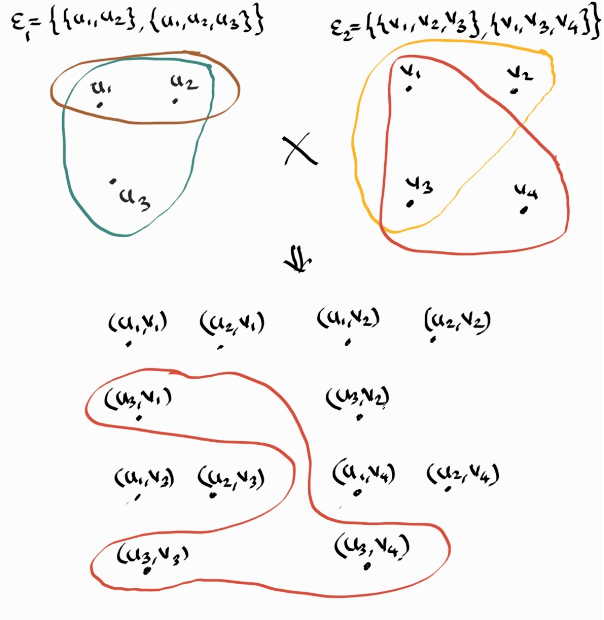

Hypergraphs and Their Product

- A hypergraph can be identified with a Tanner graph: for a hypergraph $\mathcal{H} = (V, \mathcal{E})$, the associated Tanner graph has vertex set $V \cup \mathcal{E}$ and $v \in V$ and $e \in \mathcal{E}$ are connected whenever $v \in e$ in $\mathcal{H}$.

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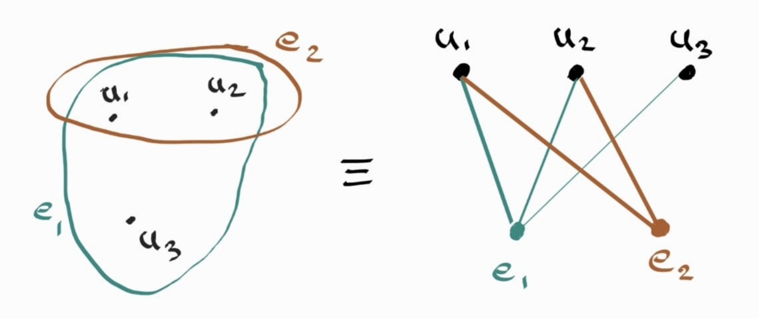
- Conversely, any Tanner graph $\mathcal{T}(V, C, E)$, generates a hypergraph on vertex set V with the hyperedges as the neighborhoods of the check vertices c, for all $c \in C$.



Hypergraph Product of Tanner Graphs

- Let $\mathcal{G}_1 = \mathcal{T}\left(V_1, C_1, E_1\right)$ and $\mathcal{G}_2 = \mathcal{T}\left(V_2, C_2, E_2\right)$ be two Tanner graphs.

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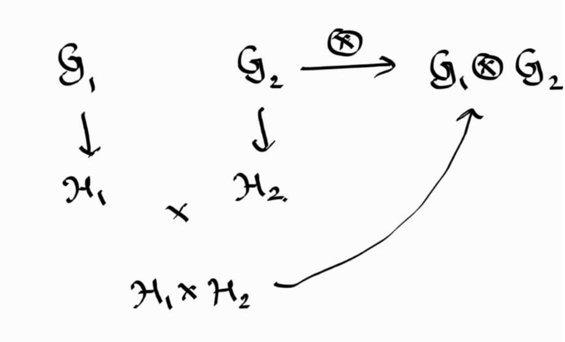

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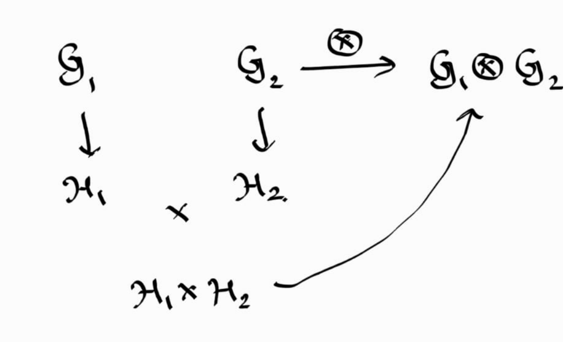

- $\mathcal{G}_1 \otimes \mathcal{G}_2$ can be connected to the notion of product codes.

Hypergraph Products and Product Codes

- Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two binary codes of length n_1 and n_2 respectively. The product code $\mathscr{C}_1 \otimes \mathscr{C}_2$ is the binary code of length $n_1 n_2$ whose codewords may be viewed as binary matrices of size $n_1 \times n_2$ such that all of its columns belong to $\mathscr{C}_1$ and all its rows to $\mathscr{C}_2$.

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- If $\mathcal{G}_1 = \mathcal{T}(V_1, C_1, E_1)$ and $\mathcal{G}_2 = \mathcal{T}(V_2, C_2, E_2)$ are the Tanner graphs of $\mathscr{C}_1$ and $\mathscr{C}_2$ respectively, then $\mathcal{G}_1 \otimes \mathcal{G}_2$ is a Tanner graph of $\mathscr{C}_1 \otimes \mathscr{C}_2$.

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- The dimension of the product code,

$$dim\left(\mathscr{C}_1 \otimes \mathscr{C}_2\right) = dim\left(\mathscr{C}_1\right) \times dim\left(\mathscr{C}_2\right)$$

Transpose Tanner Graph

- The transpose of a Tanner graph $\mathcal{G} = \mathcal{T}(V, C, E)$ is the Tanner graph $\mathcal{G}^T := \mathcal{T}(C, V, E)$.

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- The transpose of the parity check matrix of $\mathscr{C}$ is the parity check matrix of $\mathscr{C}^T$.
- Let $\mathbf{H}$ be the parity check matrix of $\mathscr{C}$. Then

$$\begin{aligned} dim(\mathscr{C}) &= |V| - rank(\mathbf{H}) \\ &= |V| - rank(\mathbf{H}^T) \\ &= |V| - (|C| - dim(\mathscr{C}^T)) \\ &= |V| - |C| + dim(\mathscr{C}^T) \end{aligned}$$

Connecting the Construction to Hypergraph Product

Proposition

As defined eariler, let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the Tanner graphs of $\mathscr{C}_1$ and $\mathscr{C}_2$ respectively. Then,

$$\mathscr{C}_X = \mathcal{G}_1 \times_X \mathcal{G}_2 = \left(\mathcal{G}_1^T \otimes \mathcal{G}_2\right)^T$$
$$\mathscr{C}_Z = \mathcal{G}_1 \times_Z \mathcal{G}_2 = \left(\mathcal{G}_1 \otimes \mathcal{G}_2^T\right)^T.$$

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- Equivalently,

$$\mathscr{C}_X^T = \mathscr{C}_1^T \otimes \mathscr{C}_2$$
$$\mathscr{C}_Z^T = \mathscr{C}_1 \otimes \mathscr{C}_2^T.$$

Connecting the Construction to Hypergraph Product

Proof.

- From the definition of hypergraph product,

$$
\begin{aligned}
\left(\mathcal{G}_1^T \otimes \mathcal{G}_2\right)^T &= \left(\mathcal{T}_1\left(V_1, C_1\right)^T \otimes \mathcal{T}_2\left(V_2, C_2\right)\right)^T \\
&= \left(\mathcal{T}_1\left(C_1, V_1\right) \otimes \mathcal{T}_2\left(V_2, C_2\right)\right)^T \\
&= \left(\mathcal{T}_3\left(C_1 \times V_2, V_1 \times V_2 \cup C_1 \times C_2\right)\right)^T \\
&= \mathcal{T}_3\left(V_1 \times V_2 \cup C_1 \times C_2, C_1 \times V_2\right) \\
&= \mathcal{G}_1 \times_X \mathcal{G}_2.
\end{aligned}
$$

- Similarly, for $\mathcal{G}_1 \times_Z \mathcal{G}_2$ as well.

Dimension of the Quantum Code

- Assume $\mathscr{C}_i$ has block length n_i, dimension k_i and r_i rows in its parity check matrix. Then,

$$\begin{aligned}
\dim\left(\mathscr{C}_X\right) &= n_1 n_2 + r_1 r_2 - r_1 n_2 + \dim\left(\mathscr{C}_X^T\right) \\
&= n_1 n_2 + r_1 r_2 - r_1 n_2 + \dim\left(\mathscr{C}_1^T \otimes \mathscr{C}_2\right) \\
&= n_1 n_2 + r_1 r_2 - r_1 n_2 + \dim\left(\mathscr{C}_1^T\right) \dim\left(\mathscr{C}_2\right) \\
&= n_1 n_2 + r_1 r_2 - r_1 n_2 + k_1^T k_2,
\end{aligned}$$

and

$$\dim\left(\mathscr{C}_Z\right) = n_1 n_2 + r_1 r_2 - r_2 n_1 + k_2^T k_1,$$

where k_1^T and k_2^T are the dimensions of the transpose codes $\mathscr{C}_1^T$ and $\mathscr{C}_2^T$.

Dimension of the Quantum Code

- This gives the quantum dimension of the hypergraph product code,

$$
\begin{aligned}
k_Q &= dim\left(\mathscr{C}_X\right) - \left(n_1 n_2 + r_1 r_2 - dim\left(\mathscr{C}_Z\right)\right) \\
&= n_1 n_2 + r_1 r_2 - r_1 n_2 - n_1 r_2 + k_1 k_2^T + k_1^T k_2 \\
&= \left(n_1 - r_1\right) \cdot \left(n_2 - r_2\right) + k_1 k_2^T + k_1^T k_2 \\
&= \left(k_1 - k_1^T\right) \cdot \left(k_2 - k_2^T\right) + k_1 k_2^T + k_1^T k_2 \\
&= k_1 k_2 + k_1^T k_2^T
\end{aligned}
$$

Minimum Distance

(The minimum distance of a trivial code is defined to be ∞.)

Proposition

For $i \in \{1, 2\}$, let d_i be the minimum distance of a code with Tanner graph $\mathcal{G}_i$ and let d_i^T denote the minimum distance of the code specified by the transpose Tanner graph $\mathcal{G}_i^T$. The minimum distance d_Q of the quantum code $\mathcal{Q}(\mathcal{G}_1 \times \mathcal{G}_2)$ satisfies

$$d_Q \geq \min\left(d_1, d_2, d_1^T, d_2^T\right).$$

Minimum Distance

Detailed Proof:

- $d_X := \min\left(|\mathbf{y}| : \mathbf{y} \in \mathscr{C}_X\left(\mathcal{G}_1 \times \mathcal{G}_2\right) \setminus \mathscr{C}_Z\left(\mathcal{G}_1 \times \mathcal{G}_2\right)^{\perp}\right)$
- $d_Z := \min\left(|\mathbf{y}| : \mathbf{y} \in \mathscr{C}_Z\left(\mathcal{G}_1 \times \mathcal{G}_2\right) \setminus \mathscr{C}_X\left(\mathcal{G}_1 \times \mathcal{G}_2\right)^{\perp}\right)$
- Let $\mathbf{x} \in \mathscr{C}_X\left(\mathcal{G}_1 \times \mathcal{G}_2\right)$.

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- Let $\mathbf{x} \in \mathscr{C}_X \left(\mathcal{G}_1 \times \mathcal{G}_2 \right)$.
- $\sup(\mathbf{x})$ denotes the support of $\mathbf{x}$ which is a subset of $V_1 \times V_2 \cup C_1 \times C_2$.

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- Let $V_1' := \{v' \in V_1 : \exists v \in \ V_2, (v', v) \in \mathrm{supp}(\mathbf{x})\}$ and $C_2' := \{c' \in C_2 : \exists c \in C_1, (c, c') \in \ \mathrm{supp}(\mathbf{x})\}$.

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- Let $\mathscr{C}_i$, $\mathscr{C}_i'$ be the binary code defined by the Tanner graph $\mathcal{G}_i$ and $\mathcal{G}_i'$.

Minimum Distance

If $w_H(\mathbf{x}) < \min(d_1, d_2^T)$:

- $|V_1'| < d_1 \Rightarrow |V_1 \setminus V_1'| > n_1 - d_1 \qquad (|V_1| =: n_1)$

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- A codeword of $\mathscr{C}_2'^T$ can be viewed as a codeword of $\mathscr{C}_2^T$ by extending it with zeros on the positions of $C_2 \setminus C_2'$.

Minimum Distance

If $w_H(\mathbf{x}) < \min(d_1, d_2^T)$:

- $|V_1'| < d_1 \Rightarrow |V_1 \setminus V_1'| > n_1 - d_1 \qquad (|V_1| =: n_1)$
- A codeword of $\mathscr{C}_1'$ can be viewed as a codeword of $\mathscr{C}_1$ by extending it with zeros on the positions of $V_1 \setminus V_1'$.
- A non-zero codeword in $\mathscr{C}_1'$ would give a non-zero codeword in $\mathscr{C}_1$ with weight less than $d_1 \Rightarrow \dim(\mathscr{C}_1') = 0$.

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- Let $\mathbf{H}_Z$ and $\mathbf{H}_Z'$ respectively be the parity-check matrices associated to the Tanner graphs $\mathcal{G}_1 \times_Z \mathcal{G}_2$ and $\mathcal{G}_1' \times_Z \mathcal{G}_2'$.

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- Denote by $\mathbf{h}_Z\left(v_1, c_2\right)$ the row of $\mathbf{H}_Z$ and by $\mathbf{h}_Z'\left(v_1', c_2'\right)$ the row of $\mathbf{H}_Z'$.

Minimum Distance

- We may write $\mathbf{x}'$ as a (not necessarily unique) linear combination of rows of $\mathbf{H}'_Z$,

$$\mathbf{x}' = \sum_{(v'_1, c'_2) \in I} \mathbf{h}'_Z \left(v'_1, c'_2 \right),$$

where I is some subset of $V'_1 \times C'_2$.

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- The set of neighbors of any (v'_1, c'_2) in the Tanner graph of $\mathscr{C}_Z\left(\mathcal{G}'_1 \times \mathcal{G}'_2\right)$ is the same as the set of neighbors of (v'_1, c'_2) in the Tanner graph of $\mathscr{C}_Z\left(\mathcal{G}_1 \times \mathcal{G}_2\right)$.

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- $\Rightarrow$ The extension of $\mathbf{x}'$

$$\mathbf{x} = \bigoplus_{(v'_1, c'_2) \in I} \mathbf{h}_Z \left(v'_1, c'_2\right)$$

which implies that $\mathbf{x}$ belongs to $\mathscr{C}_Z\left(\mathcal{G}_1 \times \mathcal{G}_2\right)^{\perp}$

Minimum Distance

- We have proved that any $\mathbf{x} \in \mathscr{C}_X(\mathcal{G}_1 \times \mathcal{G}_2)$ having weight less than $\min(d_1, d_2^T)$ also belongs to $\mathscr{C}_Z(\mathcal{G}_1 \times \mathcal{G}_2)^\perp$.

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- $\Rightarrow$ Anything in $\mathscr{C}_X\left(\mathcal{G}_1 \times \mathcal{G}_2\right) \setminus \mathscr{C}_Z\left(\mathcal{G}_1 \times \mathcal{G}_2\right)^\perp$ should have weight greater than or equal to $\min(d_1, d_2^T)$.

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- A similar analysis gives anything in $\mathscr{C}_X(\mathcal{G}_1 \times \mathcal{G}_2) \setminus \mathscr{C}_Z(\mathcal{G}_1 \times \mathcal{G}_2)^\perp$ should have weight also greater than or equal to $\min(d_2, d_1^T)$.

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- $$d_X \geq \min(d_1, d_2, d_1^T, d_2^T).$$
- Similarly, one can prove
$$d_Z \geq \min(d_1, d_2, d_1^T, d_2^T)$$
- Hence,
$$d_Q = \min(d_X, d_Z) \geq \min\left(d_1, d_2, d_1^T, d_2^T\right)$$

Summary of Properties

- If the starting codes $\mathscr{C}_1$ and $\mathscr{C}_2$ have dimensions linear on blocklength, the resulting quantum code also have **dimension linear on blocklength**.
- If the starting codes have minimum distance linear on blocklength, the resulting quantum code has **minimum distance proportional to the square root of blocklength**.

Overview of Currently Known QLDPC Codes

k	d	Code
2	$\sqrt{n}$	Kitaev toric
2	$\sqrt{n\sqrt{\log n}}$	Freedman-Meyer-Luo
$\Theta(n)$	$\sqrt{n}$	hypergraph product
$\sqrt{n}/\log n$	$\sqrt{n}\log n$	high-dimensional expander (HDX)
$\sqrt{n}$	$\sqrt{n}\log^c n$	tensor-product HDX
$n^{3/5}/\mathrm{polylog}(n)$	$n^{3/5}/\mathrm{polylog}(n)$	fiber-bundle
$\log n$	$n/\log n$	lifted-product (LP)
$\Theta(n)$	$\Theta(n)$	expander LP
$\Theta(n)$	$\Theta(n)$	quantum Tanner
$\Theta(n)$	$\Theta(n)$	Dinur-Hsieh-Lin-Vidick

Figure: https://errorcorrectionzoo.org/c/qldpc

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