E2 210 (Jan.–Apr. 2025)

Homework Assignment 2

Submission deadline: Monday, Feb. 10, 11:59pm

1. Fidelity is a distance measure between two density matrices, defined as $F(\rho,\sigma) := \operatorname{tr}\sqrt{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}$.

   (a) Let $F(|\psi\rangle,\rho)$ denote the fidelity between the density matrix, $|\psi\rangle\langle\psi|$, of a pure state $|\psi\rangle$ and an arbitrary density ρ. Show that $F(|\psi\rangle,\rho) = \sqrt{\langle\psi|\rho|\psi\rangle}$. [*Hint*: Note that $(|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|$.]

   (b) If ρ and σ commute (i.e., $\rho\sigma = \sigma\rho$), then show that $F(\rho,\sigma) = \sum_i \sqrt{\lambda_i\mu_i}$, where the λ_i's and μ_i's are the eigenvalues of ρ and σ, respectively.

   [*Hint*: Since ρ and σ commute, they are simultaneously diagonalizable, i.e., they are diagonalizable in the same ON basis.]

2. Quantum noise and fidelity.

   (a) Consider the depolarizing noise channel:

   $$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}\big(X\rho X + Y\rho Y + Z\rho Z\big).$$

   Show that if a qubit in state $|\psi\rangle$ passes through the depolarizing noise channel, then the fidelity between $|\psi\rangle$ and $\sigma := \mathcal{E}(|\psi\rangle\langle\psi|)$ is $F(|\psi\rangle,\sigma) = \sqrt{1-2p/3}$.

   (b) Consider the amplitude damping channel with parameter γ:

   $$\mathcal{E}_{\text{AD}}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger,$$

   where $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}$ and $E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$. Let $\sigma := \mathcal{E}_{\text{AD}}(|\psi\rangle\langle\psi|)$ be the density matrix of a qubit $|\psi\rangle$ after it passes through the amplitude damping channel. Show that

   $$F(|\psi\rangle,\sigma) \geq \sqrt{1-\gamma},$$

   with equality if and only if $|\psi\rangle = |1\rangle$.

3. The S gate is a single-qubit gate that applies a $\pi/2$ phase shift to the $|1\rangle$ state while leaving $|0\rangle$ unaffected: $S|0\rangle = |0\rangle$ and $S|1\rangle = i|1\rangle$. Consider the 2-qubit quantum code $\mathcal{Q} = \text{span}(|{++}\rangle, |{--}\rangle)$. Can this code recover from a single $\pi/2$ phase shift error? In other words, is there a recovery operation that allows any quantum state $|\varphi\rangle \in \mathcal{Q}$ to be recovered from any error in the set $\mathcal{E} = \{I\otimes I, S\otimes I, I\otimes S\}$?