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* Linear Algebra [3M]

Topics :-

→ Determinants
→ Inverse of a matrix
→ Rank of matrix
→ Homogeneous & non-homogeneous linear eqⁿ
→ Eigen values & Eigen vectors
→ Cayley - Hamilton theorem

Textbook :-

Matrices by A. R. Vasistha

* Determinants *

* Properties of Determinants :-

Point I : Value of the determinant will not be change if rows and columns are interchange.

i.e., $|A| = |A^T|$

Ex.

$$|A| = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 10 - 4 = 6$$

$$|A^T| = \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = 10 - 4 = 6$$

$\therefore |A| = |A^T|$

Point II : Value of the determinant is multiplied by -1 if two rows & two columns are interchange.

Ex.

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Point III : Value of the determinant can be zero in the following cases :-

i) The elements in two rows or two columns are identical.

Ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

same

ii) The elements in two rows or two columns are proportional to each other.

Ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 3 & 6 & 9 \end{vmatrix} = 0$$

proportional

iii) All elements in a row or column are zeros

Ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 7 & 3 \end{vmatrix} = 0$$

iv) The elements in a determinant are of consicutive order (continuous order)

→ valid for ~~onl~~ 3×3 & high order matrices

Ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

we can start 1st element from any no.

Point IV: The determinant of upper triangular, lower triangular, diagonal, scalar & Identity matrix is the product of its diagonal elements.

1) Upper triangular matrix

Ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{vmatrix} = 1 \times (-4) \times 7 = -28$$

2) Lower triangular matrix

Ex.

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 7 \end{vmatrix} = 1 \times 3 \times 7 = 21$$

3) Diagonal matrix

Ex.

$$|A| = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 3 \times 2 \times 5 = 30$$

A matrix is diagonal iff at least one diagonal element should be non zero & all other non-diagonal elements should be zero.

4) Scalar matrix

Ex.

$$|A| = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 3 \times 3 \times 3 = 27$$

→ Diagonal elements should be same.
→ Non - diagonal elements should be zero.

5) Identity matrix

Ex.

$$|I| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \times 1 \times 1 = 1$$

→ Determinant of Identity matrix is always 1.

Point V: If A is $n \times n$ matrix then

$$|KA| = K^n |A|$$

where n = order of matrix A

Ex.

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$|3A| = 3^2 \cdot |A| = 9 \times (-2) = -18$$

Point VI: If each element of a row or column contains sum of 2 elements then the determinant can be expressed as sum of two determinants of same order.

Ex.

$$|A| = \begin{vmatrix} 1 & 1^2 & 1^3+1 \\ 2 & 2^2 & 2^3+4 \\ 3 & 3^2 & 3^3+5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1^2 & 1^3 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{vmatrix} + \begin{vmatrix} 1 & 1^2 & 1 \\ 2 & 2^2 & 4 \\ 3 & 3^2 & 5 \end{vmatrix}$$

* Note 1: Consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Method I:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Method II:

$$\begin{array}{ccccc} & & - & - & - \\ a_{13} & a_{11} & a_{12} & a_{13} & a_{11} \\ & a_{21} & a_{22} & a_{23} & \\ a_{33} & a_{31} & a_{32} & a_{33} & a_{31} \\ & & + & + & + \end{array}$$

$$|A| = a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} - a_{33}a_{21}a_{12} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11}$$

→ If determinant contains more no. of zeros use method I.

Ex. 1) Find determinant of

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

2 1 2 2 1

2 1 2

1 2 2 1 2

$$= 8 + 1 + 8 - 4 - 4 - 4$$
$$= 5$$

Ex: 2) $A = \begin{pmatrix} 5 & 0 & 2 \\ \bar{0} & 3^{+} & \bar{0} \\ 2 & 0 & 1 \end{pmatrix}$

$$= 3(5 - 4) = 3$$

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Ex. 3) The following represents eqn of straight line

$$\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

The line passes through

a) (0,0) ✓b) (3,4) c) (4,3) d) (4,4)

$$x(8) - 2(y) + 4(y - 8)$$
$$8x - 2y + 4y - 32 = 0$$
$$8x + 2y = 32$$
$$4x + y = 16$$
$$\because 4(3) + 4 = 16$$
$$x = 3 \ \& \ y = 4$$

Note2:— Consider the matrix

$$A = \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}_{4\times 4} \end{array}$$

Method I :— (Complicated)

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Method II :— (Preferable)

~~Co~~

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}_{4\times 4}$$

Procedure:

Step 1 : Among the given element of a matrix, select any non zero element.

Step 2 : Make all elements above & below or left & right of the selected element as zero using row & column operations

Therefore, the matrix is

$$A = \begin{pmatrix} a_{11}^{+} & a_{12} & a_{13} & a_{14} \\ 0^{-} & b_{22} & b_{23} & b_{24} \\ 0^{+} & b_{32} & b_{33} & b_{34} \\ 0^{-} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

→ 1st column has more no. of zeros so [illegible] the determinant along 1st column

$$|A| = a_{11} \begin{vmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{vmatrix}$$

Ex. 1) Find the determinant of

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 9 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$A = \begin{pmatrix} 1^{+} & -1 & 0 & 1 \\ 0^{-} & 3 & 3 & 3 \\ 0^{+} & 5 & 4 & 7 \\ 0^{-} & 1 & 0 & 2 \end{pmatrix}$$

$$|A| = 1 \begin{vmatrix} 3 & 3 & 3 \\ 5 & 4 & 7 \\ 1^{+} & 0^{-} & 2^{+} \end{vmatrix}$$

$$1[21 - 12] + 2[12 - 15]$$

$$9 + 2(-3)$$

$$9 - 6 = 3$$

Problem 1 :- If A has m rows & $m+5$ columns & B has n rows and $11-n$ columns. The orders of A and B if AB and BA are defined ?

$\Rightarrow$

$$(A)_{m \times (m+5)} \cdot (B)_{n \times (11-n)}$$

$$(B)_{n \times (11-n)} \cdot (A)_{m \times (m+5)}$$

$$\therefore \quad m+5 = n \qquad\qquad m - n = -5$$

$$11 - n = m \qquad\qquad m + n = 11$$

$$2m = 6$$

$$m = 3 \qquad \therefore \; n = 8$$

Therefore, orders are $A_{(3,8)}$ & $B_{(8,3)}$ resp.

Problem 2 :- If $A = (a_{ij})_{m \times n}$ such that $a_{ij} = i + j, \forall i, j$ then sum of all element of A is ?

$\Rightarrow$

$$A = \begin{pmatrix} 1+1 & 1+2 & 1+3 & \cdots & 1+n \\ 2+1 & 2+2 & 2+3 & \cdots & 2+n \\ \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ m+1 & m+2 & m+3 & \cdots & m+n \end{pmatrix}$$

according to point VI

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ 3 & 3 & & 3 \\ \vdots & \vdots & & \vdots \\ m & m & \cdots & m \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & & n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

$$\frac{m(m+1)}{2} \quad \underbrace{\frac{m(m+1)}{2} + \dots + \frac{m(m+1)}{2}}$$ | $$\underbrace{\frac{n(n+1)}{2} + \frac{n(n+1)}{2} + \dots + \frac{n(n+1)}{2}}$$

for n columns $= \frac{n \cdot m(m+1)}{2}$ | for m rows $= \frac{m \cdot n(n+1)}{2}$

Sol:

$$\frac{mn(m+1)}{2} + \frac{mn(n+1)}{2}$$

$$\frac{mn}{2}\left[m+1+n+1\right]$$

$$\frac{mn}{2}\left[m+n+2\right]$$

Problem 8 :- If $A = (a_{ij})_{3\times3}$, $B = (b_{ij})_{3\times3}$ such that

$b_{ij} = 2^{i+j} a_{ij} \;\forall\, i, j$; $|A| = 2$; $|B| = ?$

a) 2^{10} b) 2^{11} c) 2^{12} d) 2^{13}

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2$$

$$|B| = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$= 2^2 \cdot 2^3 \cdot 2^4 \begin{vmatrix} 1a_{11} & 2a_{12} & 2^2 a_{13} \\ 1a_{21} & 2a_{22} & 2^2 a_{23} \\ 1a_{31} & 2a_{32} & 2^2 a_{33} \end{vmatrix}$$

$$2^9 \cdot 2 \cdot 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\therefore\ 2^{12} \cdot 2 = \underline{2^{13}}$$

Problem 4:— If $A = (a_{ij})_{n \times n}$ such that

i) $a_{ij} = i^2 - j^2$, $\forall\ i, j$

ii) $a_{ij} = i - j$, $\forall\ i, j$

Find sum of all elements of A

$\Longrightarrow$

i)
$$A = \begin{pmatrix} 0 & -3 & -8 & \cdots & (1^2 - n^2) \\ 3 & 0 & -5 & \cdots & (2^2 - n^2) \\ 8 & 5 & 0 & \cdots & (3^2 - n^2) \\ \vdots & & & & \\ (n^2 - 1^2) & (n^2 - 2^2) & (n^2 - 3^2) & \cdots & 0 \end{pmatrix}$$

A is <u>skew symmetric matrix</u>.

→ In skew symmetric matrix all diagonal elements must be zero & non-diagonal elements should be real no.

<u>Benefit</u> :→ Sum of all elements of skew symmetric matrix is always zero.

Ex.

i) $A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}_{2 \times 2}$ = Sum of all elements of skew sym. matrix = 0

2) $A = \begin{pmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{pmatrix}_{3 \times 3}$ = Sum = 0

→ If i^{th} row, j^{th} column elem. of a matrix is in the form $a_{ij} = i^n - j^n$ $(n > 0)$ then corresponding matrix is always skew symm.

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* Note 3 :-

A matrix A is said to be symmetric if $A^T = A$ or $a_{ij} = a_{ji}$

* Note 4 :-

A matrix A is said to be skew symmetric if $A^T = -A$ or $a_{ij} = -a_{ji}$

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Problem 1 :- The value of $\begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix} = ?$

$C_1 \to C_1 + C_2$

$$\begin{vmatrix} 1+2b & b & 1 \\ 1+2b & 1+b & 1 \\ 1+2b & 2b & 1 \end{vmatrix} = 0$$

proportional

Problem 2 :- Find the determinant of $\begin{vmatrix} 1/a & a & bc \\ 1/b & b & ca \\ 1/c & c & ab \end{vmatrix}$

$$= \begin{vmatrix} bc/abc & a & bc \\ ca/abc & b & ca \\ ab/abc & c & ab \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} bc & a & bc \\ ca & b & ca \\ ab & c & ab \end{vmatrix} = 0$$

same

→ If there are no numbers inside determinant try to make 1 perticular row or column same

<u>Problem 8</u> :- find the value of
$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3 + C_4$

$$= \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix}$$

$$= x+3a \begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$; $R_4 \to R_4 - R_1$

$$= x+3a \begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix}$$

according to pt IV

$$= (x+3a)\{1 \times (x-a) \times (x-a) \times (x-a)\}$$

$$= (x+3a)(x-a)^3$$

* **Short cut method :-**

Procedure :-

i) If the diagonal elements are one category of same elements & non diagonal elements are other category of same elements

2) select the 1^{st} row

3) add all elements of 1^{st} row

4) take the product of $(1^{st} - 2^{nd})(1^{st} - 3^{rd})(1^{st} - 4^{th}) \ldots$

5) (3) x (4)

i.e.,

$$|A| = \begin{vmatrix} x + a + a + a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

Problem 11: The value of $A = \begin{pmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{pmatrix}$

$$|A| = \begin{pmatrix} x+10 & 2 & 3 & 4 \\ x+10 & 2+x & 3 & 4 \\ x+10 & 2 & 3+x & 4 \\ x+10 & 2 & 3 & 4+x \end{pmatrix} = x+10 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{pmatrix}$$

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - R_1 \quad R_4 \to R_4 - R_1$

$$= (x+10) \begin{pmatrix} \cancel{x+}10 & 2 & 3 & 4 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

$$= (x+10)\, 1 \times x \times x \times x$$

$$= x^3 (x+10)$$

Problem 2:

$$A = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3 + C_4$

$$= \begin{vmatrix} 1+a+b+c+d & b & c & d \\ 1+a+b+c+d & 1+b & c & d \\ 1+a+b+c+d & b & 1+c & d \\ 1+a+b+c+d & b & c & 1+d \end{vmatrix}$$

$$= (1+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 1 & 1+b & c & d \\ 1 & b & 1+c & d \\ 1 & b & c & 1+d \end{vmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$; $R_4 \to R_4 - R_1$

$$= (1+a+b+c+d) \begin{vmatrix} 1^{+} & b & c & d \\ 0^{-} & 1 & 0 & 0 \\ 0^{+} & 0 & 1 & 0 \\ 0^{-} & 0 & 0 & 1 \end{vmatrix}$$

$$= 1+a+b+c+d \,\left[\,1\,[\,1\,]\,\right]$$

$$= 1+a+b+c+d$$

Problem 3: Find the value of $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$C_2 \to C_2 - C_1 \qquad C_3 \to C_3 - C_1$

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$$A = \begin{array}{ccc} + & - & + \end{array}$$

$$A = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$= 1\left[(b-a)(c^2-a^2) - (c-a)(b^2-a^2)\right]$$

$$= (b-a)(c+a)(c-a) - (c-a)(b+a)(b-a)$$

$$= (b-a)(c-a)\left[c + a - b - \cancel{a}\right]$$

$$= (b-a)(c-a)(c-b)$$

$$= \{-(a-b)\}(c-a)\{-(b-c)\}$$

$$= \underline{\underline{(a-b)(b-c)(c-a)}}$$

* Short cut method [for (2×2) (3×3) .... (n×n)]

If matrix is in the form $\begin{vmatrix} a & b & c & d & \cdots \\ a^2 & b^2 & c^2 & d^2 & \cdots \\ a^3 & b^3 & c^3 & d^3 & \cdots \\ \vdots \end{vmatrix}$

then select the 2^{nd} row

& take (2^{nd} – 1^{st}) (3^{rd} – 2^{nd}) ... (last – 1^{st})

i.e., $(a-b)(b-c)(c-d)$

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* Inverse of a matrix *

Let $A = (a_{ij})$ be $n \times n$ matrix

i) **Minor:** Minor of an element a_{ij} is denoted by M_{ij} and is defined as

$$M_{ij} = (n-1)^{th} \text{ order determinant.}$$

ii) **Cofactor:** Cofactor of an element a_{ij} is denoted by A_{ij} and is defined as

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Ex. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 7 & -2 \end{pmatrix}$$

Consider the 2nd row 2nd element

$a_{22} = 3$

→ Minor of a_{22} is

$$M_{22} = \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} = -2-8 = -10$$

→ Cofactor of a_{22} is

$$A_{22} = (-1)^{2+2} \times (-10) = -10$$

* **Invertible matrix:**

A matrix A is said to be invertible if we can find some other matrix B such that $AB = BA = I$ then B is called inverse of matrix A.

$$A^{-1} A B = A^{-1} I$$

$$I B = A^{-1} I$$

$$\boxed{B = A^{-1}}$$

* Singular matrix :-

A matrix is said to be singular if $|A| = 0$

* Non Singular matrix :-

A matrix is said to be non singular if $|A| \neq 0$

Inverse of matrix exists if $|A| \neq 0$.

Note 1:

$$A^{-1} = \frac{adj.A}{|A|}$$

$$\Rightarrow adj.A = |A| \cdot A^{-1}$$

$$\Rightarrow A \cdot adj.A = A \cdot |A| \cdot A^{-1}$$

$$\Rightarrow \boxed{A \cdot adj.A = |A| \cdot I = adj.A \cdot A}$$

Note 2:

$$A^{-1} = \frac{adj.A}{|A|}$$

$$\Rightarrow (adj.A)^{-1} \cdot A^{-1} = (adj.A)^{-1} \cdot \frac{adj.A}{|A|}$$

$$\Rightarrow (adj.A)^{-1} \cdot A^{-1} = \frac{I}{|A|} \quad \because A \cdot A^{-1} = I$$

$$\Rightarrow (adj.A)^{-1} \cdot A^{-1} \cdot A = \frac{I \cdot A}{|A|}$$

$$\Rightarrow \boxed{(adj.A)^{-1} = \frac{A}{|A|}}$$

Note 3: Let A be an $n\times n$ matrix

We know that

$$\text{adj}.A = |A|.A^{-1}$$

$$\Rightarrow |\text{adj}.A| = |\underset{K}{\underset{\downarrow}{|A|}}.\underset{A}{\underset{\downarrow}{A^{-1}}}|$$

$\boxed{|KA| = K^n|A|}$

$$\Rightarrow |A|^n.|A^{-1}|$$

$$\Rightarrow |A|^n.|A|^{-1}$$

$$\therefore \boxed{|\text{adj}.A| = |A|^{n-1}}$$

Replacing A by $\text{adj}.A$ in the above relation

$$|\text{adj}.\text{adj}.A| = |\text{adj}.A|^{n-1}$$

$$= \{|A|^{n-1}\}^{n-1}$$

$$\therefore \boxed{|\text{adj}.\text{adj}.A| = |A|^{(n-1)^2}}$$

Similarly,

$$\therefore |\text{adj}.\text{adj}.\text{adj}.A| = |A|^{(n-1)^3}$$ & so on..

Note 4: We know that

$$A.\text{adj}.A = |A|.I$$

replacing A by $\text{adj}.A$

$$\Rightarrow \text{adj}.A(\text{adj}.\cancel{A}\ \text{adj}.A) = |\text{adj}.A|.I$$

$$\Rightarrow \text{adj}.A(\text{adj}.\text{adj}.A) = |A|^{n-1}.I$$

pre multiply both side by A

$$\Rightarrow A.\text{adj}.A(\text{adj}.\text{adj}.A) = A.|A|^{n-1}I$$

$\boxed{A.I = A}$

$$\Rightarrow |A|.I(\text{adj}.\text{adj}.A) = |A|^{n-1}.A$$

$\Rightarrow |A| \cdot (\text{adj. adj. } A) = |A|^{n-1} \cdot A$

$\Rightarrow \text{adj. adj } A = \dfrac{|A|^{n-1} \cdot A}{|A|}$

$\Rightarrow \boxed{\text{adj. adj. } A = |A|^{n-2}\, A}$

Note 5 :- A matrix A is said to be orthogonal if

$\boxed{A\, A^T = A^T \cdot A = I}$

$A \cdot A^T = I$

$A^{-1} \cdot A \cdot A^T = A^{-1} I$

$I \cdot A^T = A^{-1} I$

$\boxed{A^T = A^{-1}}$

Note 6 :- If A is an orthogonal matrix then A^{-1} & A^T are also orthogonal matrices.

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Problem 1 :- For the matrix $M = \begin{pmatrix} 3/5 & 4/5 \\ x & 3/5 \end{pmatrix}$ such that $M^T = M^{-1}$. The value of x is ?

$\frac{3}{5}(x) + \frac{4}{5}\left(\frac{3}{5}\right) = 0$

$\boxed{x = -4/5}$

Note : If A is an orthogonal matrix then its rows & columns are pair wise orthogonal.
But converse of the stmt. may or may not be true.
i.e., If rows & columns of matrix are pair wise orthogonal, then the matrix may or may not be orthogonal.

Problem 2: If $A = (a_{ij})_{5\times5}$ such that

i) $a_{ij} = i-j$, $\forall\, i,j$

ii) $a_{ij} = i^2 - j^2$, $\forall\, i,j$

find A^{-1} in each case.

$\longrightarrow$ i) $a_{ij} = i-j$

$a_{ji} = j-i$

$= -(i-j)$

$= -a_{ij}$

$\therefore \boxed{a_{ij} = -a_{ji}}$ $\therefore A \Rightarrow$ skew symmetric matrix

$\therefore \Rightarrow A^T = -A$

$\Rightarrow |A^T| = |-A|$ as $|KA| = K^n |A|$

$\Rightarrow \quad = (-1)^5 |A|$

$\therefore |A^T| = -1\,|A|$ $\quad |A| = |A^T| \rightarrow$ property No 1

$|A| = -|A|$

$|A| + |A| = 0$

$2|A| = 0$

$2 \neq 0 \quad \therefore |A| = 0$

$\therefore A^{-1}$ does not exist.

Note: ① Inverse of any odd order skew symm matrix does not exist.
Reason : Since every odd order skew symm matrix is singular i.e., $|A| = 0$

② Inverse of even order skew symm. matrix exists.
Reason : Since every even order skew symm. matrix is non singular i.e., $|A| \neq 0$.

ii)

$$A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}_{2\times 2}$$

$$A^T = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$$

$$\boxed{A^T = -A}$$

$$\therefore |A| = \begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} = 0+9 = (3)^2$$

⇒ The determinant of even order skew symm. matrix is a perfect square.

⇒ If ith row, jth column element of a matrix is in the form

$a_{ij} = i^n - j^n$ $(n>0)$, the corresponding matrix is always skew symmetric.

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Problem 3:– If X and Y are two non zero matrices of the same order such that $XY = (0)_{n\times n}$, then

A) $|X| \neq 0$, $|Y| = 0$

B) $|X| = 0$, $|Y| \neq 0$

C) $|X| \neq 0$; $|Y| \neq 0$

D) $|X| = 0$; $|Y| = 0$

$XY = 0$

take determinants on b.s

$|XY| = 0$

$|X||Y| = 0$

then $|X|=0$ or $|Y|=0$ or both $|X|=0$ & $|Y|=0$

∴ C is omitted

Pro

Let $|X| = 0$, $|Y| \neq 0$ $\therefore Y^{-1} \Rightarrow$ exist

$$XY = 0 \Rightarrow XYY^{-1} = 0 \cdot Y^{-1}$$
$$\Rightarrow XI = 0$$
$$\Rightarrow X = 0 \quad \text{(contradiction to hypothesis)}$$

$\therefore \boxed{|Y| = 0}$

If we choose $|Y| = 0$, $|X| \neq 0$ then we get

$\boxed{|X| = 0}$

$\therefore \boxed{|X| = |Y| = 0}$

Problem 4: If A, B, C, D, E, F, G are non-singular matrices of the same order such that CEDBGAF = I then B^{-1} is —

$\Rightarrow$ $CEDBGAF = I$

$$C^{-1}CEDBGAF = C^{-1}I$$
$$IEDBGAF = C^{-1}I$$
$$EDBGAF = C^{-1}$$
$$E^{-1}EDBGAF = E^{-1}C^{-1}$$
$$IDBGAF = E^{-1}C^{-1}$$
$$D^{-1}DBGAF = D^{-1}E^{-1}C^{-1}$$
$$BGAF = D^{-1}E^{-1}C^{-1}$$
$$BGAFF^{-1} = D^{-1}E^{-1}C^{-1}F^{-1}$$
$$BGAI = D^{-1}E^{-1}C^{-1}F^{-1}$$
$$BGAA^{-1} = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}$$
$$BG = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}$$
$$BGG^{-1} = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}G^{-1}$$
$$\boxed{B = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}G^{-1}}$$

$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

$\therefore B^{-1} = \left(D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}G^{-1}\right)^{-1}$

$\boxed{B^{-1} = GAFCED}$

* Short cut method :—

$\underline{CED}\,B\,\underline{GAF} = GAFCED$

B^{-1}

$||^{ly}$ $\underline{CEDBGAF} = FCEDBG$
A^{-1}

$||^{ly}$ $\underline{CEDBGAF} = CEDBGA$
F^{-1}

Prob No. 6 Let k be a +ve real no. & let

$$A = \begin{pmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{pmatrix}_{3\times3} \quad B = \begin{pmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{pmatrix}_{3\times3}$$

find i) det.(adj.B) ii) det.(adj.A)
iii) If det(adj.A) = 10^6, the value of k = ?

$\Longrightarrow$

i) $|adj\ B| = |B|^{n-1} = 0$
$\because$ determinant of odd order skew symm. matrix is zero.

ii) $|adj.\ A| = |A|^{n-1} = |A|^{3-1} = |A|^2$ ——— eq^n(1)

$$|A| = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$= \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 0 & 1+2k & -2k-1 \\ -2\sqrt{k} & 2k & -1 \end{vmatrix} \qquad -(1+2k)$$

$C_2 \to C_2 + C_3$

$$= \begin{vmatrix} 2k-1 & 4\sqrt{k} & 2\sqrt{k} \\ 0 & 0 & -(1+2k) \\ -2\sqrt{k} & 2k-1 & -1 \end{vmatrix}$$

$$\Rightarrow -\left\{-(1+2k)\left((2k-1)^2+8k\right)\right\}$$

$$\cancel{(1+2k)\;\left((2k+1)(2k-1)\right)}$$

$$\Rightarrow (1+2k)\;(4k^2-4k+1+8k)$$

$$\Rightarrow (1+2k)\;(4k^2+4k+1)$$

$$\Rightarrow (2k+1)\,(2k+1)^2 = (2k+1)^3$$

put in eq^n (1)

$$\Rightarrow \left|(2k+1)^3\right|^2 = (2k+1)^6$$

iii) $|adj.\,A| = 10^6$

$(2k+1)^6 = 10^6$

$2k+1 = 10$ $\quad 2k = 9 \quad \therefore \boxed{k = 9/2}$

Problem No. 5: If A, B, C, D are non singular matrices of the same order such that $ABCD = I$ then B^{-1} is ?

$$ABCD = I$$

$$B^{-1}$$

$$\therefore \boxed{B^{-1} = CDA}$$

Prob. No. 7:—

Find the inverse of foll. matrices —

EE-05 2m 1) $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{pmatrix}$

EE-95 2m H.W 3) $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

EE-98 2m H.W 2) $A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

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$\Rightarrow$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{adj.\,A}{|A|}$$

$\Rightarrow$

$$|A| = \begin{vmatrix} 1^{(+)} & 0^{(-)} & -1^{(+)} \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= 1\left[2+3\right] - 1\left[6-2\right] = 5 - 4 = 1$$

$\Rightarrow$ adj. A =

$$\begin{array}{cl} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{pmatrix} & \begin{array}{l} \longrightarrow \mathrm{III} \\ \longrightarrow \mathrm{I} \\ \longrightarrow \mathrm{II} \end{array} \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 1 \\ -1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 2 \\ 1 & 3 & 0 & 1 \end{pmatrix} \qquad \therefore\ adj\,A = \begin{pmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{pmatrix}$$

$$\therefore\ A^{-1} = \begin{pmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{pmatrix}$$

Procedure : (3×3)

$\cdot)$ Select the middle row middle element & move in anticlockwise direction to complete 1 cycle.

The corresponding element will be return in the 1st column seperately.

2) Select the 3rd row middle element & move in anti-clockwise direction to complete 1 cycle.
The corresponding element will be return in the 2nd column seperately.

3) Repeat the same process with 1st row also.

4) Copy 1st column as a last column & find det. of smaller matrices

Procedure : (2 x 2)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \boxed{A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

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2m

Prob. No. 8 :- Given an orthogonal matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \text{ then } (AA^T)^{-1} \text{ is } - ?$$

Orthogonal $\Longrightarrow$ $(AA^T) = I$

$(AA^T)^{-1} = I^{-1}$

but Inverse / adj. of Identity matrix is Identity matrix only.

$\therefore (AA^T)^{-1} = I$

Prob. No. 9 Find the inverse of $A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ Here A is an orthogonal matrix therefore.

$A^{-1} = A^T$

* Rank of Matrix *

* **Sub matrix :** A matrix obtained by deleting some rows or columns or both is called as sub matrix.

Ex. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\underbrace{B_1 = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \quad B_2 = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} \quad B_3 = \begin{pmatrix} 4 & 5 & 6 \\ 1 & -1 & 0 \end{pmatrix}}_{\text{sub matrices of } A}$$

* **Minor :** The determinant of square sub matrix is called its minor.

Ex.

$$\left.\begin{array}{l} |B_1| = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5-8 = -3 \\[2em] |B_2| = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12-15 = -3 \end{array}\right\} \text{Minors of } A$$

* **Rank :** If the determinant of highest possible square matrix is not equal to zero then the order of the determinant is called Rank of matrix.

Ex. Find the rank of

$$A = \begin{pmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 4 & 2 \\ 1 & -11 & 14 & 5 \end{pmatrix}_{3\times 4}$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix}_{3\times 3} = 1[-14+44] - 3[28-4] - 2[-22+1]$$

$$= 30 - 72 + 42$$

$$= 0$$

$$\begin{vmatrix} 3 & -2 & 1 \\ -1 & 4 & 2 \\ -11 & 14 & 5 \end{vmatrix}_{3\times 3} = 3[20-28] + 2[-5+22] + 1[-14+44]$$

$$= -24 + 34 + 30 = 40 \neq 0$$

$\therefore$ Rank of A = $\rho(A) = 3$ i.e., order of matrix

* Row Echelon form :-

A matrix A is said to be in Row Echelon form iff

i) zero rows should occupy the last rows, if any.

ii) the no. of zero's before a non zero element of each row is less than no. of such zeros before a non zero element of the next row.

Ex.

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 7 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{6\times 6}$$

Condition 1 satisfy
—— " —— 2 —— " ——

Note: The rank of Row Echelon form matrix is equal to no. of non zero rows.

$\boxed{\rho(A) = 4}$ ← Linearly Independent Row/Vector

→ These non zero rows are called Linearly Independent rows/vector.

→ To reduce any matrix into row echelon form we should use only row operations.

→ Every upper triangular matrix is in R.E. form but every R.E. form will not be an upper triangular matrix.

Ex. 1

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$

Upper Δ^{lar} matrix
∴ A is in R.E form

Ex. 2

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Not Upper Δ^{lar} matrix
but A is in R.E. Form

Upper Δ^{lar} matrix $\Rightarrow$ R.E form R.E form $\not\Rightarrow$ Upper Δ^{lar} matrix

* Column Echelon form :-

A matrix A is said to be in column echelon form iff

i) zero ~~rows~~ columns should occupy the last columns, if any.

ii) The no. of zeros above a non zero element of each column is less than the no. of zeros above a non zero element of the next column

Ex. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$$

Note :-

Rank of matrix in Column Echelon form is equal to no. of non zero columns.

$$\therefore \boxed{\rho(A) = 4}$$

→ To reduce any matrix into column echelon form, we should use only Column Operations

→ Every lower Δ^{lar} matrix will be in column echelon form.

but every C.E form will not be a lower Δ^{lar} matrix.

Ex. 1

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 9 & 4 & 5 \end{pmatrix}$$

↓

Lower Δ^{lar} matrix

Lower Δ^{lar} M $\Rightarrow$ C.E. form

Ex. 2

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \text{[illegible]} & \text{[illegible]} \\ 2 & \text{[illegible]} & \text{[illegible]} \end{pmatrix}$$

C. E. Form $\not\Longrightarrow$ lower Δ^{lar} matrix

Example: Consider the matrix $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{pmatrix}$

$\Rightarrow$ Row Echelon form

$R_2 \rightarrow R_2 - 2R_1 \qquad R_3 \rightarrow R_3 - R_1$

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{pmatrix} \qquad R_3 \rightarrow R_3 - 2R_1$$

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 7 & 8 \\ 0 & 0 & 0 \end{pmatrix} \qquad \therefore \boxed{\rho(A) = 2}$$

$\Rightarrow$ Column Echelon form

$C_2 \rightarrow C_2 - 3C_1 \qquad C_3 \rightarrow C_3 + 2C_1$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -7 & 8 \\ 1 & -14 & 16 \end{pmatrix} \qquad C_3 \rightarrow 7C_3 + 8C_2$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -7 & 0 \\ 1 & -14 & 0 \end{pmatrix} \qquad \therefore \boxed{\rho(A) = 2}$$

Note: Rank of matrix = no of non zero rows (or) no of non zero columns.

Prob. No. 7

2)

$$A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad A^{-1} = \frac{adj.A}{|A|}$$

$$|A| = \begin{vmatrix} 5 & 0 & 2 \\ 0^{\ominus} & 3^{\oplus} & 0^{\ominus} \\ 2 & 0 & 1 \end{vmatrix} = 3[5-4] = 3$$

$$adj.|A| = \begin{pmatrix} 3 & 0 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 5 & 0 \\ 3 & 0 & 0 & 3 \end{pmatrix}$$

$$adj.A = \begin{pmatrix} 3 & 0 & -6 \\ 0 & 1 & 0 \\ -6 & 0 & 15 \end{pmatrix} \quad \therefore A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & -6 \\ 0 & & 0 \\ -6 & 0 & 15 \end{pmatrix}$$

3)

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad A^{-1} = \frac{Adj.A}{|A|}$$

$$|A| = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0^{\oplus} & 0^{\ominus} & 1^{\oplus} \end{vmatrix} = 1[1+1] \quad 2$$

$$adj.A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

15/7/12

* Information regarding Rank of matrix :

1) $\rho(O_{n\times n}) = 0$
2) $\rho(I_{n\times n}) = n$
3) $\rho\{adj.\ I_{n\times n}\} = n$
4) $\rho(A) = \rho(A^T)$
5) $\rho(A+B) \leq \rho(A) + \rho(B)$
6) $\rho(A-B) \geq \rho(A) - \rho(B)$
7) $\rho(AB) \geq \rho(A) + \rho(B) - n$, if A & B are $n\times n$ matrice
8) $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$
9) If A is an $m\times n$ matrix, then $\rho(A) \leq \min(m,n)$
10) If $\rho(A_{n\times n}) = n$, then $\rho(adj.A) = n$
11) If $\rho(A_{n\times n}) = n-1$, then $\rho(adj.\ A) = 1$
12) If $\rho(A_{n\times n}) = n-2$, then $\rho(adj.\ A) = 0$

2M

Problem 1: If $A = (a_{ij})_{m\times n}$, such that $a_{ij} = i\cdot j$, $\forall i,j$ then $\rho(A) = ?$

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & & & & \vdots \\ m & 2m & 3m & \cdots & mn \end{pmatrix}_{m\times n}$$

to find $\rho(A)$ convert the matrix into Row Echelon form

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_m \to R_m - mR_1$

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$\therefore \rho(A) =$ No. of non zero rows $= 1$

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Problem 2: If $x = (x_1, x_2 \cdots x_n)^T$ is n-tuple non zero vector then

i) $\rho(xx^T)$

ii) $\rho(x^Tx)$

$\Rightarrow$

i) $\rho(xx^T)$

$x = (x_1, x_2 \cdots x_n)^T$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} \qquad x^T = (x_1, x_2 \cdots x_n)_{1\times n}$$

$\rho(xx^T) \le \min\{\rho(x), \rho(x^T)\}$

$\rho(xx^T) \le \min\{1, 1\}$

$\Rightarrow \rho(xx^T) \le 1$ — 1 ✓, 0 ✗ ($x \rightarrow$ Non zero vector)

$\therefore \boxed{\rho(xx^T) = 1}$

Gate

Problem 3: The rank of 5×6 matrix Q is 4 then which of the foll stmt is true.

a) Q will have 4 L.I rows & 4 L.I columns ✓

b) Q will have 4 L.I rows & 5 L.I columns

c) QQ^T is invertible

d) Q^TQ is invertible

$\rho(Q_{5\times 6}) = 4$

$\therefore$ 4 non zero rows or columns

option (a)

option (b) ✗

If det of matrix $\neq 0$ then order of ...

option (c) $(Q)_{5\times6}$ $(Q)^T_{6\times5}$

$(QQ^T)_{5\times5} \rightarrow$ Invertible

$(QQ^T)^{-1}$ exists

$|(QQ^T)_{5\times5}| \neq 0 \Rightarrow \rho(QQ^T) = 5$ (contradiction to stmt)

option (d) $(Q^T)_{6\times5}$ $(Q)_{5\times6}$

$(Q^TQ)_{6\times6} \rightarrow$ Invertible

$(Q^TQ)^{-1}$ exists

$|(Q^TQ)_{6\times6}| \neq 0 \Rightarrow \rho(Q^TQ) = 6$ (contradiction to stmt)

* Linearly Dependent & Independent vectors *

$\Rightarrow$ Two vectors x_1 & x_2 are L.D. if one vector is expressed as multiple of other vector.

x_1 & $x_2 \Rightarrow$ same directional vectors

Example

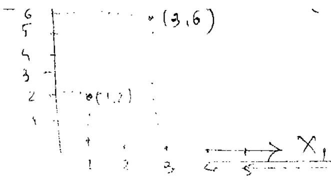
$x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $x_2 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

$x_2 = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\boxed{x_2 = 3x_1}$ or $\boxed{x_1 = \frac{1}{3}x_2}$

$\therefore x_1, x_2 \Rightarrow$ L.D.

collinear.



$\Longrightarrow$ Two vectors in R^2 are L.D. if and only if they are collinear.

$\Longrightarrow$ Three vectors in R^3 are L.D iff they are coplanar.

$\Longrightarrow$ Two vectors x_1, x_2 are L.I. iff it is not possible to express one vector as a multiple of other vector.

Example:

$$x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$x_1 \neq k x_2 \quad \text{OR} \quad x_2 \neq k x_1$$

$$\therefore x_1, x_2 \Longrightarrow \text{L.I.}$$

* Linearly Dependent vectors: (for 2 or more than 2 vectors)

A set of r n-vectors $x_1, x_2, \dots x_r$ are said to be linearly dependent if there exist r scalars $k_1, k_2, \dots, k_r$ such that

$$k_1 x_1 + k_2 x_2 + \dots + k_r x_r = 0$$

where $k_1, k_2, \dots, k_r$ not all zeros.
(atleast one k value is a non zero no.)

* Linearly Independent vectors:

A set of r n-vectors $x_1, x_2, \dots x_r$ are said to be linearly independent if there exist r scalar $k_1, k_2, \dots k_r$ such that

$$k_1 x_1 + k_2 x_2 + \dots + k_r x_r = 0$$

all zeros.

* Criteria for L.I & L.D

1) If $\rho(A)$ = no. of given vectors or $|A| \neq 0$, the given vectors are said to be L.I.

Ex.

Consider the vectors

$x_1 = (1 \quad 2 \quad 2)$, $x_2 = (2 \quad 1 \quad 2)$, $x_3 = (2 \quad 2 \quad 1)$

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 1(1-4) - 2(2-4) + 2(4-2)$$
$$= -3 + 4 + 4$$
$$= 5$$

$\therefore |A| = 5 \neq 0 \implies$ L.I.

2) If $\rho(A) <$ no. of given vectors or $|A| = 0$, the given vectors are said to be L.D.

$n > n$

Ex.

Consider the vectors

$x_1 = (1 \quad 3 \quad -2)$, $x_2 = (2 \quad -1 \quad 4)$, $x_3 = (1 \quad -11 \quad 14)$

$$|A| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix} = 1(-14+44) - 3(28-4) - 2(-22+1)$$
$$= 30 - 72 + 42$$

$|A| = 0 \implies$ L.D.

3) If the given vectors are L.D. then any one of the vector can be expressed as linear combination of other vectors. diff. of

4) If the given vectors are L.I then it is not possible to write any one of the vector as linear combination of other vectors.

at least one elem. must be nonzero

5) Every non zero vector is L.I. vector.

Ex.

Consider the vector

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$kx = 0$$

$$k \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Non zero vector $\leftarrow$ $\boxed{\therefore k = 0}$

$\therefore x \rightarrow$ L.I vector

columns/rows of Identity matrix

6) The set of unit vectors are always L.I.

Ex. Consider the set of vectors

$x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$|A| = 1 \neq 0 \Longrightarrow \text{L.I vectors}$$

7) The set of vectors having atleast one zero vector are L.D.

Ex. Consider the set of vectors

$x_1 = (1, 2, 3)$, $x_2 = (0\ 0\ 0)$, $x_3 = (1\ -1\ 4)$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & -1 & 4 \end{vmatrix} = 0 \quad \therefore x_1 = x_2 = x_3 \Rightarrow \text{L.D.}$$

$$|A| = 0 \Longrightarrow \text{L.D.}$$

No. of vectors = r
No. of elements in each vector = n

8) A set of ~~r~~ r vectors with $r < n$ components (elements) are always L.I., provided the vectors should not be in the same direction.
Ex. Consider the vectors
$(1\ -1\ 2\ 1)$, $(1\ 2\ 3\ 4)$, $(2\ 3\ 4\ 9)$

$r = 3$
$n = 4$
here, $r < n \implies$ L.I

9) A set of r vectors with $r > n$ components then given vectors are L.D.
Ex. Consider the vectors
$x_1 = (1\ 1\ 2)$
$x_2 = (1\ -1\ 0)$ $\quad r = 4$
$x_3 = (1\ -1\ 5)$ $\quad n = 3$
$x_4 = (9\ -5\ 4)$ $\quad$ here, $r > n \implies$ L.D.
$\therefore x_1, x_2, x_3, x_4$ are L.D.

10) A set of r vectors with $r = n$ components may be L.I or L.D

* Dimension & Basis of the vectors

* Dimension :- It is defined as no. of L.I. vectors.
Dimension = No. of L.I. vectors = no. of non zero rows in Row Echelon form = no. of non zero columnns in Column Echelon form.

* Basis :- It is defined as the set of L.I. vectors.
Basis = set of L.I vectors = set of non zero rows in R.E. form = set of non zero columns in C.E. form.

Problem 1: Test whether the following vectors are L.D or L.I. Also find their dimension & basis.

$(1 \; 1 \; -1 \; 0)$, $(4 \; 4 \; -3 \; 1)$, $(-6 \; 2 \; 2 \; 2)$, $(9 \; 9 \; -6 \; 3)$

$n = 4$ here $r = n$ ∴ may be L D or L I

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ -6 & 2 & 2 & 2 \\ 9 & 9 & -6 & 3 \end{vmatrix}$$

$R_2 \to R_2 - 4R_1 \qquad R_3 \to R_3 + 6R_1 \qquad R_4 \to R_4 - 9R_1$

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 3 & 3 \end{vmatrix}$$

zero

[illegible] row only not previous row

$R_2 \longleftrightarrow R_3$

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{vmatrix}$$

$R_4 \to R_4 - 3R_3$

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

3 L.I. vectors

whenever a matrix is reduced to ...
C.F form then Rank of matrix = Dimension ...
Set of L.I vectors are called basis ...

$\rho(A) = 3 <$ No. of given vectors (4)

$\therefore x_1, x_2, x_3, x_4 \Longrightarrow$ L.D

$$\begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & \cdot & 2 \\ 0 & 0 & \cdot & \cdot \end{vmatrix} \neq 0$$

Dimension = 3

Basis = $\{(1\ 1\ {-1}\ 0), (0\ 8\ {-4}\ 2), (0\ 0\ 1\ 1)\}$

Gate (2m)

Problem: If $q_1, q_2, \ldots, q_m$ are n-dimensional vectors with $m < n$. The vectors are L.D. The matrix Q is $q_1, q_2, \ldots q_m$ as columns. The rank of Q = ?

(q_1: 1st col, q_2: 2nd col, q_m: m^{th} col)

a) m b) n c) between m & n d) $< m$ ✓

$$q_1 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$Q = \begin{array}{c} \begin{array}{cccccc} q_1 & q_2 & q_3 & & q_m \end{array} \\ \left(\begin{array}{c|c|c|c|c} \ & \ & \ & \cdots & \ \end{array} \right)_{n\times m} \end{array} \Rightarrow (Q)_{n\times m}$$

$\rho(Q_{n\times m}) \le \min(n, m)$

but $m < n$ — given

$\therefore \rho(Q_{n\times m}) \le m$ — m ✗ / $< m$ ✓ — from criteria (2)

$\therefore \boxed{\rho(Q) < m}$

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* Nullity of a matrix :-

* Nullity :- It is denoted by $N(A)$.
& is defined as the difference betn order of matrix and rank of matrix.

i.e.,

$$\boxed{N(A) = n(A) - \rho(A)}$$

$N(A)$ → Nullity, $n(A)$ → order, $\rho(A)$ → Rank

⟹ Nullity of a non-singular matrix is always zero.

let A be an $n \times n$ non singular matrix.

Then,

$$|A_{n\times n}| \neq 0$$

$$\therefore \rho(A) = n$$

$$N(A) = n(A) - \rho(A)$$

$$= n - n$$

$$\therefore N(A) = 0$$

Problem: The nullity of $A = \begin{pmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{pmatrix} = 1$. The value of $k = ?$

$$N(A) = n(A) - \rho(A)$$

$$1 = 3 - \rho(A)$$

$$\rho(A) = 2$$

for 3×3 matrix

ill $\rho(A) = 3$ then $|A| \neq 0$

$$|A| = 0 \implies$$

$$\begin{vmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = 0$$

$$k = -1$$

$ax + by + cz = 0 \Longrightarrow$ Homogeneous

$ax + by + cz = 2 \Longrightarrow$ Non - Homogeneous

Problem: The rank of matrix A is 5 & nullity of matrix is 3 then order of matrix = ?

$$N(A) = n(A) - \rho(A)$$

$$3 = n(A) - 5$$

$$n(A) = 8$$

* Non-homogeneous system of Linear Equation *

Consider the following non homogeneous system of m linear eq^{ns} in n unknowns.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \text{I}$$

Procedure

1) Write the given system of eq^{ns} in the form
$AX = B$

i.e.,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

coefficient matrix   sol^n matrix   Column matrix of constants.

2) Write the elements of matrix B in the last column of matrix A. The resulting matrix is called Augmented matrix & is denoted by $(A|B)$

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$$(A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

3) Reduce the augmented matrix $(A|B)$ into Row Echelon form & hence find rank of A & rank of $(A|B)$

4) If $\rho(A) < \rho(A|B)$ or $\rho(A|B) \neq \rho(A)$, the given system of equations ~~are said to~~ have no sol^n (inconsistent).

5) If $\rho(A|B) = \rho(A) =$ no. of unknowns, the given system of eq^n have unique solution

6) If $\rho(A|B) = \rho(A) <$ no. of unknowns, the given system of eq^n have infinite no. of solutions.

7) If the given system of equations have a sol^n (unique or infinite sol^n).
The sol^n can be found by reducing the matrix AB into Row Echelon form & by using back code substitution, the variables $x_1, x_2, \ldots x_n$ can be found.

Note :- If the total no. of eq^{ns} < total no. of variables, the given system of eq^n have infinite no. of sol^{ns}.

These infinite no. of sol^{ns} can be found by assigning $(n-r)$ variables as arbitrary constants.
These $(n-r)$ sol^{ns} are Linearly Independent sol^{ns}.

$x+y=3$

$r=1, \quad n=2$

$r<n, \quad n-r=2-1=1$

put $\boxed{y=c} \rightarrow$ L.I sol^n

$x+y=3$

$x+c=3$

$x=3-c$

$x=3-c$ & $y=c$

Note: Consider the system of eq^n.

$ax+by=e$

$cx+dy=f$

The above system of eq^{ns} have

1) No sol^n if $\frac{a}{c}=\frac{b}{d}\neq\frac{e}{f}$

2) Unique sol^n if $\frac{a}{c}\neq\frac{b}{d}$

3) Infinite sol^n if $\frac{a}{c}=\frac{b}{d}=\frac{e}{f}$

ECE 2010 (1m)

Problem 1: The system of eq^{ns} $4x+2y=7$, $2x+y=6$ have

$\frac{4}{2}=\frac{2}{1}\neq\frac{7}{6} \quad \left(\frac{a}{c}=\frac{b}{d}\neq\frac{e}{f}\right)$

$\therefore$ The given system of eq^{ns} have no sol^n.

Problem 2: $x+2y=5$

$2x+3y=9$

$\frac{1}{2}\neq\frac{2}{3}\neq\frac{5}{9}, \quad \frac{1}{2}\neq\frac{2}{1} \quad \left(\frac{a}{c}\neq\frac{b}{d}\right)$

$\therefore$ Unique sol^n

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roblem 3: $x + y = 3$
$3x + 5y = 9$

$\frac{1}{3} \neq \frac{1}{5} = \frac{3}{9}$ ∴ Infinite solⁿ

Ne (S) 2m
roblem 4: How many solutions does the following system of eqⁿˢ have

i) $-x + 5y = -1$
$x - y = 2$
$x + 3y = 3$

a) Infinite b) Exactly 2 c) Unique solⁿ d) No solⁿ

$$(A|B) = \left(\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right)$$

$R_2 \to R_2 + R_1 \quad R_3 \to R_3 + R_1$

$$= \begin{pmatrix} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\left(\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$\rho(A|B) = 2 \qquad \rho(A) = 2$

$\rho(A|B) = \rho(A) =$ No. of unknowns $= 2$

∴ Unique solⁿ

(2m)

Problem 5: Consider following non homogeneous system of Linear eq^n in 3 variables x_1, x_2, x_3.

$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 + 2x_2 + 5x_3 = 2$$
$$-x_1 + 4x_2 + x_3 = 3$$

The above system of eq^n have —
a) No sol^n ✓b) unique sol^n c) More than 1 but finite no of sol^{ns} d) Infinite no. of sol^{ns}

$$(A|B) = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 3 & 2 & 5 & 2 \\ -1 & 4 & 1 & 3 \end{pmatrix}$$

o sol^n

$R_2 \to 2R_2 - 3R_1$ $R_3 \to 2R_3 + R_1$

$$(A|B) = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & 7 & 1 & 1 \\ 0 & 7 & 5 & 7 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$(A|B) = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & 7 & 1 & 1 \\ 0 & 0 & 4 & 6 \end{pmatrix}$$

$\rho(A|B) = 3$ $\rho(A) = 3$

No. of unknowns = 3

$\therefore$ Unique sol^n

2010 (2m)

Problem 6: For the set of eq^{ns} $x_1 + 2x_2 + x_3 + x_4 = 2$
$3x_1 + 6x_2 + 3x_3 + 3x_4 = 6$

which of the foll. stmt is true —
1) There exist only trivial sol^n
2) There are no sol^{ns}
3) Unique ~~non trivial~~ sol^n non zero sol^n
4) Infinite no of non trivial sol^n

$\Longrightarrow$ No of eq^{ns} $(r) = 2$
No of variables $(n) = 4$
here $r < n$
Infinite sol^n (non trivial)

Problem 7: The value of x_3 obtained by solving the foll. system of eq^{ns}.

$$x_1 + 2x_2 - 2x_3 = 4$$
$$2x_1 + x_2 + x_3 = -2$$
$$-x_1 + x_2 - x_3 = 2$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 1 & 1 & -2 \\ -1 & 1 & -1 & 2 \end{array}\right)$$

$R_2 \to R_2 - 2R_1$ $\quad$ $R_3 \to R_3 + R_1$

$$(A|B) = \left(\begin{array}{cccc} 1 & 2 & -2 & 4 \\ 0 & -3 & 5 & -10 \\ 0 & 3 & -3 & 6 \end{array}\right)$$

$R_3 \to R_3 + R_2$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & -3 & 5 & -10 \\ 0 & 0 & 2 & -4 \end{array}\right)$$

$$2x_3 = -4$$
$$\boxed{x_3 = -2}$$

(To find x_2

$$-3x_2 + 5x_3 = -10$$
$$-3x_2 - 10 = -10$$
$$-3x_2 = 0$$
$$\boxed{x_2 = 0}$$)

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2011 (2m)

Problem 8 : find the values of λ, μ for which the following system of eq^{ns}

$$x + y + z = 6$$
$$x + 4y + 6z = 20$$
$$x + 4y + \lambda z = \mu$$ have

1) No sol^n 3) infinite sol^n

2) Unique sol^n

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right)$$

$R_2 \rightarrow R_2 - R_1$ $R_3 \rightarrow R_3 - R_1$

$$(A|B) = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda-1 & \mu-6 \end{pmatrix}$$

$R_3 \longrightarrow R_3 - R_2$

$$(A|B) = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda-6 & \mu-20 \end{pmatrix}$$

$\rho(A) \neq 3$

$\rho(A|B) = 3$

1) No sol^n : If $\rho(A) < \rho(A|B)$

If $\lambda = 6$, $\mu \neq 20$

$\rho(A) = 2$ $\rho(A|B) = 3$

$\Rightarrow \rho(A) < \rho(A|B) \longrightarrow$ No sol^n

2) Unique sol^n : If $\rho(A) = \rho(A|B) =$ No of unknowns

For $\lambda \neq 6$ & $\mu \longrightarrow$ any value $\mu = 20$ or $\mu \neq 20$

3) Infinite no. of sol^n :— If $\rho(A) = \rho(A|B) <$ No. of unknowns

No. of unknowns = 3 | must be 2 bcoz rank can't be 0 & 1

$\rho(A)$ & $\rho(A|B)$ should be less than 3

It is possible if

$\lambda = 6$ & $\mu = 20 \implies$ Infinite sol^n set

HW

Problem 9 : Find the values of A & B for which the following system of eq^{ns}

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b$$

have

1) No sol^n 2) Unique sol^n 3) Infinite sol^{ns}

GATE 2M

Problem 10 : For what values of α & β, the following system of eq^{ns}

$$x + y + z = 5$$
$$x + 3y + 3z = 9$$
$$x + 2y + \alpha z = \beta$$

have infinite no. of sol^n?

a) $\alpha = 2$, $\beta = 7$ c) $\alpha = 3$, $\beta = 4$

b) $\alpha = 7$, $\beta = 2$ d) $\alpha = 4$, $\beta = 3$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right)$$

$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$

$$= \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha - 1 & \beta - 5 \end{pmatrix}$$

$R_3 \to 2R_3 - R_2$

Non-homogeneous System eqⁿ ⟹ draw ...

$$\left(\begin{array}{cccc} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 2\alpha-4 & 2\beta-14 \end{array}\right)$$

$\rho(A) = \rho(A|B) <$ No. of unknowns

$\rho(A) = \rho(A|B) < 3$

$2\alpha - 4 = 0$ $\quad$ $2\beta - 14 = 0$

$2\alpha = 4$ $\quad$ $2\beta = 14$

$\boxed{\alpha = 2}$ $\quad$ $\boxed{\beta = 7}$

Problem 11 : If A is 3×4 matrix & the non homogeneous system of equations $AX = B$ is inconsistent (No sol) The highest possible rank of A is

$\longrightarrow$

$(A|B)_{3\times5}$

$\rho\{(A|B)\}_{3\times5} \leq \min(3, 5)$

$\rho\{(A|B)\} \leq 3$

$= 3$ Highest Possible Rank

But it is given that the given system of eqⁿˢ are inconsistent (No solⁿ)

For inconsistent $\rightarrow \rho(A) < \rho(A|B)$

$\rho(A) < 3$

$\rho(A) \begin{cases} 2 \\ 1 \\ 0 \end{cases}$

Highest possible rank of A $= 2$

* Homogeneous system of Linear Eq^n *

Consider the following homogeneous system of Linear eq^{ns} in m equations of n unknowns.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right\} \quad (I)$$

Procedure to solve problems :-

1) Write the given system of eq^{ns} in the form $AX = 0$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

2) Reduce the matrix A into either row or column echelon form (but always row echelon form is preferable) OR find the determinant of matrix $|A|$.

3) If $\rho(A) =$ No. of unknowns (variables) OR $|A| \neq 0$ ($A \rightarrow$ Non singular matrix), the given system of eq^n have trivial sol^n ($x = y = z = 0$)

4) If $\rho(A) <$ No. of unknowns OR $|A| = 0$, the given system of eq^n have infinite no. of sol^n.

All these infinite no. of sol^n can be found by assigning $(n-r)$ variables as arbitrary constant. These $(n-r)$ sol^{ns} are called L.I. sol^{ns}.

$\boxed{AX = B}$

- $\boxed{B = 0}$
  - $\rho(A) = n$ or $|A| \neq 0$ → Trivial sol^n $(x = y = z = 0)$ → $\cancel{\Downarrow}$ ~~L.D~~
  - $\rho(A) < n$ or $|A| = 0$ → Non-Trivial → Infinite no. of non zero sol^n → $\cancel{\Downarrow}$ ~~L.I~~
- $\boxed{B \neq 0}$
  - $\rho(A) < \rho(A|B)$ → No sol^n
  - $\rho(A|B) = \rho(A) = n$ → Unique sol^n
  - $\rho(A|B) = \rho(A) < n$ → Infinite no. non zero sol^{ns}.

Problem 1: For what values of λ the system of eq^{ns}

$$x + y + z = 0$$
$$(\lambda+1)x + y + (\lambda+1)z = 0$$
$$(\lambda^2 - 1)z = 0$$

have 2 L.I sol^n?

$\Longrightarrow$

No. of L.I sol^n $= n - r = 2$

$= 3 - r = 2$

$\therefore \boxed{r = 1} \rightarrow$ Rank of matrix

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$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ \cancel{\lambda+1}\ 0 & \lambda+1 & \lambda+1 \\ 0 & 0 & \lambda^2-1 \end{vmatrix} = 0 \Rightarrow \text{Upper } \Delta^{lar}$$

$\Rightarrow (\lambda+1)(\lambda^2-1) = 0$

$(\lambda+1)(\lambda+1)(\lambda-1) = 0$

$\lambda = -1, 1$ ... not correct

put in matrix in order to get $\rho(A) = 1$

$\therefore \boxed{\lambda = -1}$

Prob1

Prob

Problem 2 :- The rank of 3x3 matrix A is 1. The homogeneous system of eq^{ns} $AX = 0$ has

a) trivial sol^n b) 1 L.I. sol^n

c) 2 L.I. sol^{ns} d) 3 L.I. sol^{ns}

$(A)_{3\times3} \Rightarrow$ 3 eq^{ns} & 3 variables

$n = 3$ $r = 1 \Longrightarrow r < n$

$\Downarrow$

Infinite sol^n

$\therefore$ L.I sol^n $= n - r$

$= 3 - 1$

$= 2$

2011 (pm)

Problem 3 :- The system of eq^{ns}: $2x_1 + x_2 + x_3 = 0$

$x_2 - x_3 = 0$

$x_1 + x_2 = 0$ have

a) No non trivial sol^n c) 5 Non trivial sol^{ns}

b) Unique non trivial sol^n d) Infinite non trivial sol^{ns}

$$\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 2[1] - 1[0+1] + 1[0-1]$$

$$= 2 - 1 - 1 = 0$$

Infinite sol^n.

Problem 4: The system of eq^n $x + 3y - 2z = 0$

$2x - y + 4z = 0$

$x - 11y + 14z = 0$

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix} = \cancel{1[-14+44] - 2[28-4] - 1[-22+1]}$$

$$= \cancel{30 - 48 + 21}$$

$$1[-14+44] - 3[28-4] + 2[-22+1]$$

$$= 0 \qquad = 30 - 72 + 42 = 0$$

Problem 5: Find the values of k for which the following system of eq^{ns} have infinite no. of non trivial sol^{ns} $\rightarrow |A| = 0$

$(3k-8)x + 3y + 3z = 0$

$3x + (3k-8)y + 3z = 0$

$3x + 3y + (3k-8)z = 0$

$\longrightarrow$ If $|A| = 0$, the given system of eq^{ns} have infinite no. of non zero sol^{ns}.

$$\begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$\Rightarrow (3k - 8 + 3 + 3)(3k - 8 - 3)(3k - 8 - 3) = 0$

$(3k - 2)(3k - 11)(3k - 11) = 0$

$3k = 2 \qquad 3k = 11$

$\therefore k = \frac{2}{3}, \frac{11}{3}$

H.W

Problem 6 : Find the real value of λ for which the foll system of eq^{ns} have non trivial sol^{ns}

infinite no. of non trivial sol^{n}

$x + 2y + 3z = \lambda x$

$3x + y + 2z = \lambda y$

$2x + 3y + z = \lambda z$

$\boxed{\lambda = 6}$

V.imp

* Eigen values & Eigen Vectors *

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

Characteristic matrix

ch. det

ch. polynomial $\rightarrow |A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 4$

$= \lambda^2 - 8\lambda + 16 - 4$

ch. eq^{n} $\leftarrow$ $|A - \lambda I| = 0 \implies \lambda^2 - 8\lambda + 12 = 0$

$\therefore \lambda = 6, 2$

$\therefore$ ch. roots or Eigen values

Null vector can not be Eigen vector

* **Eigen values**: Let A be an $n \times n$ matrix. λ is a scalar (some constant). The matrix $A - \lambda I$ is called as characteristic matrix.

  $|A - \lambda I|$ is called characteristic determinant or characteristic polynomial.

  The roots of this ch. det. are called char. roots OR Eigen values OR Latent roots OR Proper values.

  The set of eigen values of matrix A is called as spectrum of A.

* **Eigen Vector**: If λ is an eigen value of a matrix A then there exist a non zero vector X such that $AX = \lambda X$, then the non zero vector X is called as Eigen vector.

Note:

$$AX = \lambda X$$
$$AX = \lambda X I$$
$$AX - \lambda X I = 0$$
$$A\underline{X} - \lambda I \underline{X} = 0$$
$$\boxed{(A - \lambda I)X = 0}$$

Trivial $sol^n \Rightarrow |A - \lambda I| \neq 0$

$\underbrace{\text{Infinite } sol^n}_{\text{Non trivial } sol^n} \Rightarrow |A - \lambda I| = 0$

$\Downarrow$

Eigen Vectors

Note: Consider the matrix

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

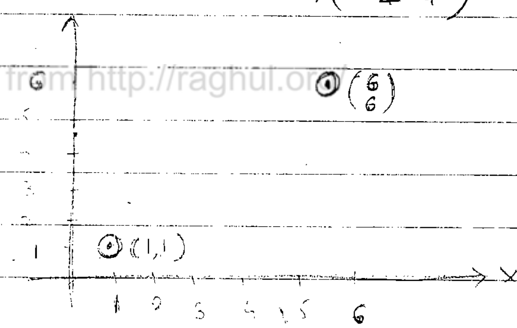
$$\underbrace{\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{X} = \begin{pmatrix} 4+2 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \underbrace{6}_{\lambda} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{X}$$

(Non zero vector)

$X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigen vector corresponding to eigen value $\lambda = 6$ for matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

→ Any pt / vector exist along the same vector will also be an eigen vector corresponding to the same eigen value.

Ex. (2,2) (3,3) ... (7,7) (8,8) ...



Eg $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{X} = \begin{pmatrix} 4+4 \\ 2+8 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix} = 2 \underbrace{\begin{pmatrix} 4 \\ 5 \end{pmatrix}}_{X}$$

Not same

$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is not eigen vector corresponding to eigen value 2

Problem 1: Find the eigen values & eigen vector of

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

Symmetric matrix

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) - 16$$

$$= -9 - \cancel{3\lambda} + \cancel{3\lambda} + \lambda^2 - 16 = 0$$

$$\lambda^2 - 25 = 0$$

$$\lambda^2 = 25$$

$$\lambda = \pm 5$$

$\therefore$ Eigen values $= 5, -5$

case i) $(A - \lambda I) X = 0$

$$\begin{pmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigen vectors

put $\lambda = 5$

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 4x_2 = 0 \quad \text{——— (1)}$$

$$4x_1 - 8x_2 = 0 \quad \text{——— (2)}$$

eq^n(1) $\Longrightarrow$ divide b.s. by (-2)

$$x_1 - 2x_2 = 0 \qquad x_1 = 2x_2$$

$$\frac{x_1}{x_2} = \frac{2}{1} \qquad \therefore X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Case ii) when $\lambda = -5$

$(A - \lambda I) X = 0$

$$\begin{pmatrix} 5-\lambda & 4 \\ 4 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

put $\lambda = -5$

$$\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$8x_1 + 4x_2 = 0$ ——— (1)

$4x_1 + 2x_2 = 0$ ——— (2)

2 vectors are not dep.. only 1 vector is dep

$2x_1 + x_2 = 0$

$2x_1 = -x_2$

$\frac{x_1}{x_2} = \frac{-1}{2}$

$X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$X_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad X_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$X_1^T X_2 = 0$ or $X_1 X_2^T = 0 \Rightarrow X_1$ & X_2 are orthogonal

$X_1^T \cdot X_2 = \begin{pmatrix} 2 & 1 \end{pmatrix}_{1\times 2} \begin{pmatrix} -1 \\ 2 \end{pmatrix}_{2\times 1}$

$(X_1^T \cdot X_2)_{1\times 1} = -2 + 2 = 0$

$\therefore$ X_1 & $X_2 \Rightarrow$ orthogonal vectors

E.V. of Upper Δ, Lower Δ, diag... on
∴ diagonal elem. only

Note 1: The Eigen vectors corresponding to diff. eigen value of a real symmetric matrix are always orthogonal.

Note 2: If the Eigen vectors corresponding to diff. eigen values of any square matrix are always Linearly Independent.

$$X_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\therefore X_1 \neq k X_2 \quad \text{OR} \quad X_2 \neq k X_1$$

$$\therefore \left.\begin{matrix} X_1 \\ X_2 \end{matrix}\right\rangle \text{L.I.} \qquad (\text{acc}^n \text{ to criteria (4)})$$

In the above ex. the eigen values of matrix are 5, -5 the corresponding eigen vectors are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

* Consider the matrix $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$

$\lambda = 2, 2 \longrightarrow$ Repeated twice

when $\lambda = 2$ Max. no. of L.I. vects

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

put $\lambda = 2$

$$\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0x_1 + 3x_2 = 0$$

$$3x_2 = 0 \qquad \therefore x_2 = 0$$

$x_1 \neq 0$

put $x_1 = c$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} \text{ where } c \neq 0$$

Acc^n to criteria (5)
every non zero vector is L.I vector.

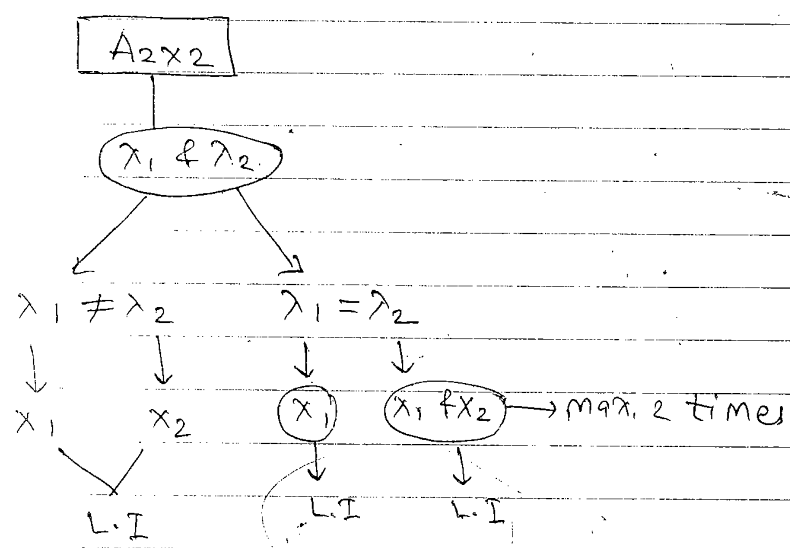
Note:- If some of the eigen value of a matrix are repeated then the eigen vectors corresponding to repeated eigen values may be L.I or L.D.

If an eigen value λ is repeated n times the eigen vectors corresponding to repeated eigen values are always L.I. which are given by

$$p = n - r \quad , \quad 1 \leq p \leq m \quad \{\text{max. } m \text{ times}\}$$

$n \rightarrow$ no. of unknowns or variables

$r \rightarrow \rho(A - \lambda I)$



9.

$A_{3\times 3}$

$\lambda_1, \lambda_2, \lambda_3$

$\lambda_1 \neq \lambda_2 \neq \lambda_3$

$x_1 \quad x_2 \quad x_3$

L.I

$\lambda_1 \neq \lambda_2 \neq \lambda_3$

$x_1 \quad x_2 \quad x_3$

L.I $\quad$ L.I or L.D

L.I

$\lambda_1 = \lambda_2 = \lambda_3$

$x_1 \quad x_1, x_2 \quad x_1, x_2, x_3$

L.I $\quad$ L.I $\quad$ L.I

$$\begin{pmatrix} c_1 + c_2 \\ c_2 \\ c_1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$x_1 \quad x_2$

L.I

2011 Gate Instr. (1M)

Note:-

$$A\,X = \lambda\,X$$

Eigen vector of A

Eigen value of A

$$A^{-1} \cdot A \cdot X = A^{-1} \cdot \lambda \cdot X$$

$$I \cdot X = A^{-1} \cdot \lambda \cdot X$$

$$X = \lambda \cdot A^{-1} X$$

$$\frac{X}{\lambda} = A^{-1} X$$

$$A^{-1} X = \frac{1}{\lambda} X$$

Eigen vector of A^{-1}

Eigen value of A^{-1}

If λ is eigen value of A & X is eigen vector of A but $\frac{1}{\lambda}$ is eigen value of A^{-1} and

classmate
Date
Page

X is eigen vector of A^{-1}. Therefore, A & A^{-1} have same eigen vectors.

A & A^m have same eigen vectors ($m > 0$) corresponding to eigen values.

$AX = \lambda X$

$AAX = A\lambda X$

$A^2 X = \lambda\lambda X$

$A^2 X = \lambda^2 X$

17/07/2012

* Properties of Eigen values & Eigen vectors:—

1) Sum of eigen values is equal to trace of matrix.
(trace of = sum of diagonal elem)

i.e., if $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values of a matrix A then

(i) Trace $A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

2) (ii) $|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots \lambda_n$

Ex. Consider the matrix

$\lambda_1 + \lambda_2 = 7$

$\lambda_1 \cdot \lambda_2 = 6$

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

$$\begin{matrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{matrix}$$

$(5-\lambda)(2-\lambda) - 4$

$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = \lambda^2 - 7\lambda + 6$

$\lambda = 1, 6$

i) → $1 + 6 = 5 + 2$ ∴ Trace of $A = 7$

ii) → $1 \times 6 = 10 - 4 = 6$ $|A| = 6$

3) The eigen values of upper triangular or lower triangular or diagonal or scalar or identity matrix is its diagonal elements.

Not for sale downloaded from http://raghul.org/

1) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$; $\lambda = 1, -4, 7$

Upper Δ^{lar}

2) $A = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 5 & 0 \\ 0 & 8 & 9 \end{pmatrix}$; $\lambda = 3, 5, 9$

Lower Δ^{lar}

3) $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$; $\lambda = 3, 4, 5$

Diagonal matrix

4) $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$; $\lambda = 3, 3, 3$

scalar matrix

5) $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $\lambda = 0, 0, 0$

scalar / Null matrix

$\lambda + 6$

6) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $\lambda = 1, 1, 1$

Identity matrix

4) The eigen values of A & A^T are same

5) The eigen values of A & $P^{-1}AP$ are same where P is a non singular matrix.

6) The eigen values of real symmetric matrix are real.

7) The eigen values of skew symmetric matrix are purely imaginary OR zeros.

8) The eigen values of orthogonal matrix are of unit modulus i.e., ± 1

9) If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also one of its eigen value.

10) If λ is an eigen value of matrix A then

i) $k\lambda$ is an eigen value of kA.

ii) $\frac{1}{\lambda}$ is also an eigen value of A^{-1}.

iii) λ^2 —— " —— A^2.

iv) λ^m —— " —— A^m

v) $\frac{|A|}{\lambda}$ is an eigen value of adj. A

vi) $\lambda \pm K$ —— " —— $A \pm KI$

vii) $(\lambda \pm K)^2$ —— " —— $(A \pm KI)^2$

viii) $\frac{1}{\lambda \pm K}$ —— " —— $(A \pm KI)^{-1}$

Note: If A is a singular matrix i.e., $|A| = 0$ then one of its eigen value should be zero

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A|$$
$$\downarrow$$
$$0 \cdot \lambda_2 \cdot \lambda_3 = |A|$$
$$|A| = 0 \qquad \therefore |A| \longrightarrow \text{singular}$$

2m

Problem 1: $A = \begin{pmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ are . $\qquad j = \sqrt{-1}$

A) $3, 3+5j, 6-j$
B) $-6+5j, 3-j, 3+j$
C) $3-j, 3+j, 5+j$
D) $3, -1+3j, -1-3j$

by prop. 1 :

$$\text{Trace } A = \lambda_1 + \lambda_2 + \lambda_3 \qquad [\text{cross check}]$$
$$-1 - 1 + 3 = 3 - 1 + 3j - 1 - 3j$$
$$1 = 1$$

If (1) or more opt^n satisfy [illegible] (find $|A|$)

Problem 2: The eigen values & eigen vector of a 2×2 matrix are given by

Eigen value	Eigen vector
$\lambda_1 = 8$	$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\lambda_2 = 4$	$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

the matrix is —

a) $\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$ b) $\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$

to find eigen value from eigen vector

formula. $(A - \lambda I)X = 0$

1998 (2m)

Problem 3: The vector $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is an eigen vector of the matrix

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}.$$ The eigen value corresponding to the eigen vector is —

$\Longrightarrow$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

value consider any 1 row & multiply

$$-1 - 4 + \lambda = 0$$

$$\boxed{\lambda = 5}$$

Problem 4: For the matrix $\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$ the eigen value corresponding to the eigen vector $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ is

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2 + 4 - \lambda = 0$$

$$\lambda = 6$$

Problem 5: The min & max eigen values of a matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ are -2 & 6 resp. what would be the 3^{rd} eigen value.

$$\text{Trace } A = \lambda_1 + \lambda_2 + \lambda_3$$
$$= 1+5+1 = -2+6+\lambda_3$$
$$\boxed{\lambda_3 = 3}$$

Problem 6: The matrix $\begin{pmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{pmatrix}$ has an eigen value $= 3$. Sum of other two eigen values is —

$$1+0+p = 3+\lambda_2+\lambda_3$$
$$p+1-3 = \lambda_2+\lambda_3$$
$$\boxed{\lambda_2+\lambda_3 = p-2}$$

2011

Problem 7: Consider the following matrix

$A = \begin{pmatrix} 2 & 3 \\ x & y \end{pmatrix}$ The eigen values of A are 4 & 8. The values of x, y are

prop. 1 $\longrightarrow 2+y = 4+8$
$$y = 12-2 = 10$$

prop. 2 $\longrightarrow$ $\cancel{2 \times 10 = 20}$ $\quad 4 \times 8 = 32$

$$\begin{vmatrix} 2 & 3 \\ x & 10 \end{vmatrix} = 32 \qquad x = -4$$

$$20 - 3x = 32$$
$$20 - 32 = 3x \qquad \therefore x = -10$$
$$3x = -12 \qquad \boxed{x = -4}$$

$2y - 3x = 4 \times 8 \Rightarrow 2 \times 10 - 3x = 32$

$-3x = 32 - 20$

$-3x = 12$

$\boxed{x = -4}$

Pro 8 :- The eigen values of 3×3 matrix are given by $1, -3, 9$. find

i) Trace $(A^2 + A^{-1} - adj.A)$

ii) det $(A^2 + A^{-1} - adj.A)$

$\Rightarrow |A| = 1 \times -3 \times 9 = -27$

$A^2 + A^{-1} - adj.A$

$\rightarrow \lambda^2 + \frac{1}{\lambda} - \frac{|A|}{\lambda}$ by prop. (10)

$\lambda^2 + \frac{1}{\lambda} - \frac{|A|}{\lambda}$

1: $(1)^2 + \frac{1}{1} - \frac{(-27)}{1} = \underline{\underline{29}}$

-3: $(-3)^2 + \frac{1}{(-3)} - \frac{(\cancel{-}27)}{(\cancel{-}3)}$

$9 - \frac{1}{3} - \cancel{9} = \underline{\underline{-1/3}}$

9: $9^2 + \frac{1}{9} - \frac{(-\cancel{27})^3}{\cancel{9}}$

$81 + \frac{1}{9} + 3$

$\frac{729 + 1 + 27}{9} = \underline{\underline{\frac{757}{9}}}$

i) Trace $(A^2 + A^{-1} - adj.A) = 29 - \frac{1}{3} + \frac{757}{9}$

ii) $\det(A^2 + A^{-1} - \text{adj}\,A) = 29 \times \left(\frac{-1}{3}\right) \times \frac{757}{9}$

Prob. 9 :- Given $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$, the eigen values of $3A^3 + 5A^2 - 6A + 2I$ are

Upper Δ^{lar}

$\therefore \lambda = 1, 3, -2$

$3A^3 + 5A^2 - 6A + 2I \Longrightarrow 3\lambda^3 + 5\lambda^2 - 6\lambda + 2$

put $\lambda = 1$

$3(1)^3 + 5(1)^2 - 6(1) + 2$

$3 + 5 - 6 + 2 = 4$

put $\lambda = 3$

$3 \times 27 + 5 \times 9 - 18 + 2$

$81 + 45 - 16 = 126 - 16 = 110$

put $\lambda = -2$

$-(3 \times 8) + 20 + 12 + 2$

$-24 + 24 = 0$

$3 \times (-8) + 5 \times 4 + 12 + 2$

$-24 + 20 + 12 + 2 = 10$

2009 (sem)

Prob. No. 10 :- The eigen values of 2×2 matrix A are given by -2 & -3 resp. the eigen values of $(x+I)^{-1} \cdot (x+5I)$

This is not possible

$\Rightarrow (x+I)^{-1}(x+5I) = (x+I)^{-1}\{(x+I) + 4I\}$

$= (x+I)^{-1} \cdot (x+I) + 4I \cdot (x+I)^{-1}$ $\quad A + kI$

$= I + 4(x+I)^{-1}$

$4(x+I)^{-1} + I \Longrightarrow \frac{4}{1+\lambda} + 1$

$$\Rightarrow \frac{4}{1+\lambda} + 1$$

put $\lambda = -2$ $\qquad \frac{4}{1-2} + 1 = -3$

put $\lambda = -3$ $\qquad \frac{4}{1-3} + 1 = -1$

Prob 11 :- The eigen vector of a 3x3 matrix are $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are orthogonal. What will be the 3rd orthogonal eigen vector.

$X = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Let $X_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

here, 3 vectors are orthogonal

$\therefore X_1^T \cdot X_3 = 0 \qquad X_2^T \cdot X_3 = 0$.

$$(1 \;\; 0 \;\; 1) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (1 \;\; 0 \;\; -1) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x + z = 0$ —— (i) $\qquad x - z = 0$ —— (ii)

add (i) & (ii)

$2x = 0$

$\therefore x = 0$ put in (i)

$0 + z = 0$

$z = 0$

$y \neq 0$ $\quad \because$ zero vector can't be eigen vector

$\Rightarrow \therefore y = c \qquad c$ — arbitrary constant & $c \neq 0$

∴ The 3^{rd} eigen vector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

2008 (2m)

Prob.12 :- The eigen vector of 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ are given by $\begin{pmatrix} 1 \\ a \end{pmatrix}, \begin{pmatrix} 1 \\ b \end{pmatrix}$, what is $(a+b)$?

$$\lambda = 1, 2.$$

$$AX = \lambda X \quad \text{OR} \quad (A - \lambda I)x = 0$$

put $\lambda = 1$

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} 1 \\ a \end{pmatrix}}_{2\times 1} = 1 \begin{pmatrix} 1 \\ a \end{pmatrix}$$

$$\cancel{1+2a \quad 2a} \qquad 1 + 2a = 1$$

$$\& \quad 2a = a \qquad \boxed{a = 0}$$

put $\lambda = 2$

$$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ b \end{pmatrix} = 2 \begin{pmatrix} 1 \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 2b \end{pmatrix}$$

$$1 + 2b = 2$$

$$2b = 1$$

$$\boxed{b = 1/2}$$

Method I :—

Case 1 : when $\lambda = 1$

$(A - \lambda I) X = 0$

$$\begin{pmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Prob.

$$\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2x_2 = 0 \qquad \therefore x_2 = 0$

here $x_1 \neq 0$

$x_1 = C \qquad C \rightarrow$ non zero arbitrary const

$$X = \begin{pmatrix} C \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

generally C is replaced by 1

Case 2 : when $\lambda = 2$

$$\begin{pmatrix} 1-2 & 2 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-x_1 + 2x_2 = 0$

$2x_2 = x_1$

$\frac{2}{1} = \frac{x_1}{x_2}$

$X_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$

$x_1 = 2 \qquad x_2 = 1$

$\frac{x_1}{x_2} = \frac{1}{(1/2)}$

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix} \Rightarrow a = 0$$

$$x_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ b \end{pmatrix} \quad b = 1/2$$

Prob. 13: Find eigen values & eigen vectors of foll.

a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

$\Longrightarrow \lambda = 1, 2, 3$

for distinct eigenvalues → consider ... row? → select ... middle elem → ... → put ...

→ put $\lambda = 1$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$x_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

other form of the ... vector

→ put $\lambda = 2$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$x_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

→ put $\lambda = 3$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 0 & -2 & 1 \\ -1 & 2 & 0 & -1 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{2}$$

$$x_3 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$

$\lambda = 1, -4, 7$

put $\lambda = 1$

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 2 & & 3 & & 0 & & 2 \\ 5 & & 2 & & 0 & & -5 \end{array}$$

$$\frac{x_1}{19} = \frac{x_2}{0} = \frac{x_3}{0} \qquad X_1 = \begin{pmatrix} 19 \\ 0 \\ 0 \end{pmatrix}$$

put $\lambda = -4$

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 2 & & 3 & & 5 & & 2 \\ -8 & & 2 & & 0 & & -8 \end{array}$$

$$\frac{x_1}{28} = \frac{x_2}{-10} = \frac{x_3}{-40} \qquad X_2 = \begin{pmatrix} 28 \\ -10 \\ -40 \end{pmatrix}$$

put $\lambda = 7$

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 2 & & 3 & & -6 & & 2 \\ -11 & & 2 & & 0 & & -11 \end{array}$$

$$\frac{x_1}{37} = \frac{x_2}{12} = \frac{x_3}{66} \qquad X_3 = \begin{pmatrix} 37 \\ 12 \\ 66 \end{pmatrix}$$

Note :– To find the eigen vectors corresponding to nonrepeated eigen value of a matrix, we proceed as follows :-

1) Select the 1^{st} two rows only
2) Start from the 1^{st} row middle no. & move in anticlockwise direction to complete 1 cycle. If any element exist in the main diagonal while we are moving in anticlockwise direction, then the eigen value should be subtracted from the corresponding diagonal elements.
These elements will be return seperately in row wise
3) Repeate the same procedure with 2^{nd} row also. & find eigen vectors

* Cayley - Hamiltom theorem *

Statement :

Every square matrix satisfies its own characteristic eqn.

Ex. Consider the matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

Its char. eqn. is $\lambda^2 - 8\lambda + 12 = 0$

by cayley - Hamiltom theorem, every square matrix satisfies its own char. eqn.

i.e., $A^2 - 8A + 12I = 0$

Note : (2×2)

Consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

$$\Longrightarrow \lambda^2 - \lambda[\text{trace of } A] + |A| = 0$$

by cayley-hamilton theorem, it is

$$A^2 - A[\text{trace of } A] + |A| \cdot I = 0$$

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Note: (3×3)

Consider the 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$, its characteristic eq^n. is—

General procedure:—

$$\lambda^3 - \lambda^2(\text{trace of } A) + \lambda\{|\ | + |\ | + |\ |\} - |A| = 0$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Using Cayley Hamilton theorem, we can find

i) Inverse of a matrix.
2) Powers of a matrix.

* Positive powers of matrix

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

char. eq^n $\longrightarrow$ $\lambda^2 - 8\lambda + 12 = 0$

Cayley Hamilton theorem $\longrightarrow$ $A^2 - 8A + 12I = 0$ ——— (1)

$A^2 = 8A - 12I$ ——— (2)

$A^3 = 8A^2 - 12A \longrightarrow (3)$

$A^4 = 8A^3 - 12A^2 \longrightarrow (4)$

$A^5 = 8A^4 - 12A^3 \longrightarrow (5)$

⋮

is —

* Negative powers of matrix

$$A^2 - 8A + 12I = 0$$

$$A^2 \cdot A^{-1} - 8 \cdot A \cdot A^{-1} + 12 \cdot I \cdot A^{-1} = 0$$

$$A - 8I + 12A^{-1} = 0$$

$$12A^{-1} = 8I - A$$

$$A^{-1} = \frac{1}{12}\left[-A + 8I\right]$$

$$A^{-1} \cdot A^{-1} = \frac{1}{12}\left[-A^{-1} \cdot A + 8IA^{-1}\right]$$

$$A^{-2} = \frac{1}{12}\left[-I + 8A^{-1}\right]$$

$$A^{-1} \cdot A^{-2} = \frac{1}{12}\left[-A^{-1} \cdot I + 8 A^{-1} \cdot A^{-1} \right]$$

$$A^{-3} = \frac{1}{12}\left[-A^{-1} + 8 A^{-2} \right]$$

Problem 1:— Find A^8 using Kaylay hamiltom theorem for

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$|A| = -1 - 4 = -5$

trace of $A = 1 \cancel{+} (-1) = 0$

$$A^2 - A\,(\text{trace of } A) + |A| \cdot I = 0$$

$$A^2 - A(0) + (-5) \cdot I = 0$$

$\cancel{A^2 + A - 5I}$

$$A^2 - 5I = 0$$

$$A^2 = 5I$$

$$(A^2)^4 = (5I)^4 = 5^4 \cdot I^4$$

$$A^8 = 625\, I$$

$$= 625 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^8 = \begin{pmatrix} 625 & 0 \\ 0 & 625 \end{pmatrix}$$

(4M)

Problem 2:

Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic eq^n.

Consider the matrix

$$A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}.$$

once we get char. eq^n after replacing λ by A 1st time we get char eq^n we should not multiply it by A^{-1}.

1) A satisfies the relation.

(a) $A + 3I + 2A^{-1} = 0$ (b) $A^2 + 2A + 2I = 0$

(c) $(A+I)(A+2I) = 0$ (d) $\exp(A)$

$$|A| = 0 + 2 = 2$$

trace of $A = -3 + 0 = -3$

$$\Rightarrow A^2 - A(\text{trace of } A) + |A| \cdot I = 0$$

$$A^2 + 3A + 2I = 0$$

$$\cancel{A^2 + 2A}$$

$$A^2 + 2AI + A + 2I = 0$$

$$A(A+2I) + I(A+2I) = 0$$

$$(A+2I)(A+I) = 0$$

2) A^9 equals $511 > 510$

a) $511 A + 510 I$ c) $154A + 155I$

b) $309 A + 104 I$ d) $\exp(9A)$

$$\Rightarrow A^2 + 3A + 2I = 0$$

$$A^2 = -3A - 2I \quad \text{——— (1)}$$

$$A^3 = -3A^2 - 2A \quad \text{——— (2)}$$

$(3 > 2)$

$A^{(\text{even})} \rightarrow$ all $-$ve elem

$A^{\text{odd}} \rightarrow$ all $+$ve elem

classmate Date ______ Page ______

$$= -3(-3A - 2I) - 2A$$

$$\Rightarrow A^3 = 7A + 6I \longrightarrow (2)$$

$(7 > 6)$

$$\Rightarrow A^4 = 7A^2 + 6A$$

$$= 7[-3A - 2I] + 6A$$

$$= -21A - 14I + 6A$$

$$A^4 = -15A - 14I \longrightarrow (3)$$

$(15 > 14)$

$$\Rightarrow A^5 = -15A^2 - 14A$$

$$= -15[-3A - 2I] - 14A$$

$$= 45A + 30I - 14A$$

$$A^5 = 31A + 30I \longrightarrow (4)$$

$(31 > 30)$

Imp * $A^{(\text{Higher } +\text{ve Number})} \Rightarrow A^{5000}, A^{10000}$

* Method :-

char eq^n $\Rightarrow \lambda^2 - (-3+0)\lambda + 2 = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

Pro

$\Rightarrow$ If λ values are not repeated then

$$\lambda^n = a\lambda + b \longrightarrow (1)$$

for $\lambda = -1$ $\quad (-1)^n = -a + b \longrightarrow (2)$

for $\lambda = -2$ $\quad (-2)^n = -2a + b \longrightarrow (3)$

$$(-1)^n - (-2)^n = a$$

$$\therefore \boxed{a = (-1)^n - (-2)^n}$$

$$\therefore \boxed{b = (-1)^n + a}$$

$$b = (-1)^n + a$$
$$= (-1)^n + (-1)^n - (-2)^n$$
$$= 2(-1)^n - (-2)^n$$

put a & b in eq^n (1)

$$\lambda^n = \left[(-1)^n - (-2)^n\right]\lambda + \left[2(-1)^n - (-2)^n\right]$$

by c-h theorem

$$\boxed{A^n = \left[(-1)^n - (-2)^n\right]A + \left[2(-1)^n - (-2)^n\right]I}$$

For ex. put $n = 9$

$$A^n = \left[(-1)^9 - (-2)^9\right]A + \left[2(-1)^9 - (-2)^9\right]I$$
$$= (-1 + 512)A + (-2 + 512)I$$
$$A^9 = 511A + 510 \cdot I$$

Problem 3: The char eq^n of 3x3 matrix P is given by

$$\alpha(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1$$

where I denoted the Identity matrix. The inverse of the matrix P will be —

a) $P^2 + P + 2I$
b) $P^2 + P + I$
c) $-(P^2 + P + I)$
d) $-(P^2 + 2P + 2I)$ ✓

$$\lambda^3 + \lambda^2 + 2\lambda + 1$$
$$\Longrightarrow P^3 + P^2 + 2P + 1 \cdot I = 0$$
$$\cancel{2P = -I - P^2 - P^3}$$
$$P^{-1} \cdot P^3 + P^{-1} \cdot P^2 + 2P^{-1}P + P^{-1} \cdot I = 0$$
$$P^2 + P + 2I + P^{-1} = 0$$
$$P^{-1} = -P^2 - P - 2I = -(P^2 + P + 2I)$$

~~Ircate~~

Problem (E. value & E. vectors)

How many L.I. eigen vectors of $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$\lambda = 2, 2$

No. of L.I e.vectors $\Rightarrow 1 \le p \le m$ ← No. of times e.value repeated

$m = 2$

$p = n - r$

n = no of unknowns, $r = \rho(A - \lambda I)$

$$(A - \lambda I) = \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} \qquad \lambda = 2$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\rho(A - \lambda I) = 1$

$p = n - r$

$= 2 - 1 = \boxed{1}$

——— x ———

30

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Numericals Methods (2 marks)

Topics :-

1. Solutions to Algebric and Transcandantal Eqn
   - Bisection method
   - ***** - Newton-Raphson Method
   - Regula-false method
   - Secant method

2. Solutions to Systems of Linear Equation
   - * - Gauss Elimination Method
   - * - LU Decomposition

3. Solution to Integration of function
   - ** - Trapezoidal Rule
   - Simpson $\frac{1}{3}$ Rule
   - Simpson $\frac{3}{8}$th Rule

4. Solution to differential Equations
   - * - Euler's method
     - forward
     - Backward
   - Runge Kutta method

1987 - 2012

36 Question

26 Question (Newton-Raphson method)

4 Question (Simpson Rule)

2 Que LU Decom

2 Ques Gauss

2 Ques. Euler

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(I) Mathematical methods are of two types.

(1) Analytical method

(2) Numerical method

(1) Analytical method:-

Eg. 1 Ex. find roots of $x^2-5x+6=0$ using analytical method

→ Analytical solⁿ is $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\therefore x = 3, 2$

Eg. (2) find $\int_1^2 x\,dx$ using analytical method

→ $\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2$

$= \left[\frac{4}{2} - \frac{1}{2}\right]$

$= \frac{3}{2}$

Eg. (3) Solve $\frac{dy}{dx} = x$, Using analytical method.

→ $\frac{dy}{dx} = x$ $\therefore dy = x\,dx$

Integrate $\int \frac{dy}{dx} dx$ $\int dy = \int x\,dx$

$y = \frac{x^2}{2} + c$

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Note :- - Drawback of analytical method is, it is not applicable for higher degree equations & also not applicable for non-linear Eq's.

TO overcome this, we use NUMERICAL METHODS

- NUMERICAL METHODS provides Approximation value

[I] Solution to Algebric and Transcandantal Equation

* **Intermediate Mean Value theorem :-**

f(x) is a continuous function defined on [a, b] f(a) & f(b) having opposite signs. In such case there exist at least one Root of f(x) in [a, b]

Let,

$f(x) = x^3 - 4x - 9$, in [2, 3]

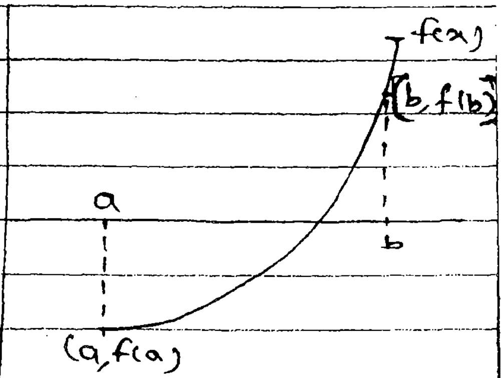
$\therefore f(2) = 2^3 - 4(2) - 9$

$= -9$

$\therefore -9 < 0$

& $f(3) = 3^3 - 4(3) - 9$

$= 6 > 0$



Since, $f(2) < 0$ & $f(3) > 0$

So, there exist atleast one root in [2, 3]

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$\exists$ - there exists

* Bisection Method :-

Step I :- Let, $f(x)$ is a continuous on $[a, b]$

Step 2 :- $f(a)$ & $f(b)$ having opposite signs

Say $f(a) < 0$ & $f(b) > 0$

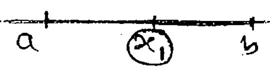
Using intermediate mean value theorem

there exist ($\exists$) atleast one root in $[a, b]$

Step 3 :- Let,

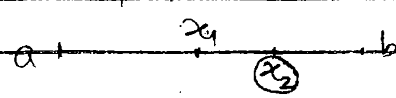
Approximation root is x_1 and

$$x_1 = \frac{a+b}{2}$$



Case I :-

If $f(x_1) = 0 \Longrightarrow$ 'x_1' is root then Stop process.



Case II :-

If $f(x_1) < 0$ and $f(b) > 0$

then $x_2 = \frac{x_1 + b}{2}$

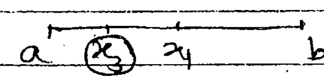
Continue this process until desired root is found

Case III :-

If $f(x_1) > 0$ and $f(a) < 0$

So $\exists$ atleast one root between $[a, x_1]$

Say, $x_3 = \frac{a + x_1}{2}$



Continue this process until desired root is found

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Q find x_2 and x_3 using Bisection method where

$f(x) = x^3 - 4x - 9$, $[2, 3]$

→ Put intervals in $f(x)$

$f(2) = 2^3 - 4 \times 2 - 9$

$= -9$

$\therefore -9 < 0$ $\qquad \therefore f(2) < 0$

$f(3) = 3^3 - 4 \times 3 - 9$

$= 6$

$\therefore 6 > 0$ $\qquad \therefore f(3) > 0$.

$\exists$ atleast one root between $[2, 3]$

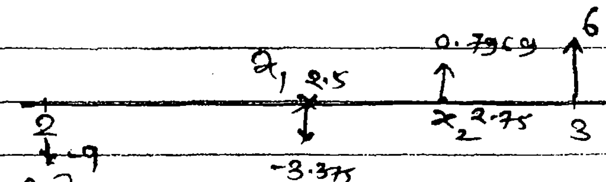
$x_1 = \frac{2+3}{2} = 2.5$

$f(2.5) = (2.5)^3 - 4(2.5) - 9$

$= -3.375$

$\therefore -3.375 < 0$ $\qquad \therefore f(2.5) < 0$.

Since $f(2.5) < 0$ and $f(3) > 0$



$\exists$ root b/w $[2.5, 3]$

$\therefore$ Let $x_2 = \frac{2.5+3}{2} = \frac{5.5}{2} = 2.75$

$f(x_2) = f(2.75) = (2.75)^3 - 4(2.75) - 9 = 0.7969$

$\therefore 0.7969 > 0$

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Since $f(2.75) > 0$ & $f(2.5) < 0$

So $\exists$ atleast one root in $[2.5, 2.75]$

Say,

$$x_3 = \frac{2.5 + 2.75}{2}$$

$$\boxed{x_3 = 2.62}$$

* Newton Raphson Method :-

Step 1 :- Let, $f(x)$ is continuous function $[a, b]$

Step 2 :- Newton Raphson iteration formula for finding root of $f(x) = 0$ is,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Q. find Newton raphson iteration formula for square root of +ve real number 'c'.

GATE 99

$\rightarrow$ Let, $x = \sqrt{c}$

Squaring b.s.

$x^2 = c$

$\therefore x^2 - c = 0$

$\therefore f(x) = x^2 - c$ $\quad \therefore f(x_n) = x_n^2 - c$

$f'(x) = 2x$ $\quad f'(x_n) = 2x_n$

$\therefore$ By Newton Raphson method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

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$$x_{n+1} = x_n - \frac{x_n^2 - c}{2x_n}$$

$$\boxed{x_{n+1} = \frac{x_n^2 + c}{2x_n}}$$

Q. find N-R iteration formula for $f(x) = x^2 - 117 = 0$

GATE 2009 →

$f(x) = x^2 - 117$; $f(x_n) = (x_n)^2 - 117$

$f'(x) = 2x$; $f'(x_n) = 2x_n$

$$x_{n+1} = x_n - \frac{x_n^2 - 117}{2x_n}$$

$$\boxed{x_{n+1} = \frac{x_n^2 + 117}{2x_n}}$$

Q. If $f(x) = x^2 - 13$ and $x_0 = 3.5$, then value of x_1 using N-R iteration formula

GATE 2010 →

$f(x) = x^2 - 13$ $\quad$ $f(x_n) = x_n^2 - 13$

$f'(x) = 2x$ $\quad$ $f'(x_n) = 2x_n$

$$\therefore x_{0+1} = x_0 - \frac{x_0^2 - 13}{2x_0}$$

$$x_1 = \frac{x_0^2 + 13}{2x_0}$$

$$x_1 = \frac{(3.5)^2 + 13}{2 \times 3.5} = \underline{3.6071}$$

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Q. Newton Raphson iteration formula for finding $\sqrt[3]{c}$

GATE 1996

→ Let $x = \sqrt[3]{c}$

Cubing

$x^3 = c$

$x^3 - c = 0$

$f(x) = x^3 - c$

$f(x_n) = x_n^3 - c$

$f'(x) = 3x^2 - 0$

$f'(x_n) = 3x_n^2$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - c}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + c}{3x_n^2}$$

$$\boxed{x_{n+1} = \frac{2x_n^3 + c}{3x_n^2}}$$

Q. The N-R method is used to find the root of $x^2 - 2 = 0$ & starting value is $x_0 = -1$; The iteration formula will be

GATE 1997

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Fastness of Convergence is Rate of Convergence

(a) Converges to -1 (b) Converges to $-\sqrt{2}$
(c) Converges to $\sqrt{2}$ (d) Not convergent

$f(x) = x^2 - 2$ $\qquad$ $f'(x_n) = \cancel{2}x_n^2 - 2$

$f(x_n) = x_n^2 - 2$ $\qquad$ $f'(x_n) = 2x_n$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = -1 - \frac{(-1)^2 - 2}{2(-1)}$$

$$= -1 - \frac{1-2}{-2}$$

$$= -1 - \frac{+1}{2}$$

$$= -1 + 0.5$$

$$x_1 = -1.5$$

$$\therefore x_2 = \frac{x_1^2 + 2}{2x_1} = \frac{(-1.5)^2 + 2}{2(-1.5)} = -1.416$$

$$\therefore x_3 = \frac{x_2^2 + 2}{2x_2} = \frac{(-1.416)^2 + 2}{2(-1.416)} = -1.414$$

||ly $x_4 = -1.414$ i.e. $-\sqrt{2}$

Recess 10:30 am

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Q find Newton Raphson iteration formula, ~~for~~ reciprocal of a where $a>0$

GATE 2005

Let, $x = \frac{1}{a}$

$$\frac{1}{x} = a$$

$$\therefore \frac{1}{x} - a = 0$$

$$\therefore f(x) = 1/x - a$$

$$f'(x) = \frac{-1}{x^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(1/x_n - a)}{(-1/x_n^2)}$$

$$= x_n + x_n^2 (1/x_n - a)$$

$$= x_n + x_n - a x_n^2$$

$$\boxed{x_{n+1} = 2x_n - a x_n^2}$$

Q Given $a>0$, we wish to compute N-R iteration formula for reciprocal of a for $a=7$ and $x_0 = 0.2$, then first two iteration will be.

GATE 2005

(a) 0.11, 0.1299 (b) 0.12, 0.1392

$0.4 - 7 \times 0.04$ अश्लील, गंदे विचारवाली पुस्तक पढ़ना जहर पीने से भी ज्यादा खतरनाक है।

$= 0.4 - .28 = \frac{.28}{0.12}$

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Given $x = \frac{1}{a}$

$\therefore \frac{1}{x} - a = f(x)$

$\therefore x_{n+1} = 2x_n - a x_n^2$

$x_{0+1} = 2x_0 - a x_0^2$

$= 2 \times 0.2 - 7(0.2)^2$

$x_1 = 0.12$

$x_{1+1} = x_2 = 2x_1 - a x_1^2$

$= 2(0.12) - 7(0.12)^2$

$= 0.24 - 7 \times 0.0144$

$x_2 = 0.1392$

I.W

Q $f(x) = x - \cos x$, then $x_{n+1} = $ ______ ?

395 $\Rightarrow$ Ans. $\Rightarrow x_n - \frac{(x_n - \cos x_n)}{1 + \sin x_n}$

I.W Q $f(x) = x.e^x - 2$, $x_0 = 0.8679$ then $x_1 = $ ______ ?

$\Rightarrow$ Ans. $\Rightarrow x_1 = 0.853$

I.W Q $f(x) = x^3 - x^2 + 4x - 4 = 0$, $x_0 = 2$, then $x_1 = $ ______ ?

$\Rightarrow$ Ans $\Rightarrow x_1 = 4/3$

I.W Q $f(x) = e^x - 1$, $x_0 = -1$, then $x_1 = $ ______ ?

$\Rightarrow$ Ans $\Rightarrow x_1 = 0.71828$

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$$2x + y + z = 10$$
$$y + 3z = 6$$
$$-2z = -10$$

$\therefore$ By Using Back Substitution,
we get $z = 5$, $y = -9$, $x = 7$

Case II :-

If $\rho(A) = \rho(AB) = r$
but $r < n$

* * (i) No. of linearly independent solutions $= n - r$.

(ii) No. of linearly dependent solutions $= r$

In this case system has infinitely many solutions

Ex:- Same example only 3rd row elements are made zeros.

$$\therefore \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here,

$$\rho(A) = 2 = \rho(AB)$$

(i) No. of linearly independent soln $= n - r$
$= 3 - 2$
$= 1$.

(ii) No. of linearly dependent $= r$
$= 2$.

अश्लील, गंदे विचारवाली पुस्तक पढ़ना जहर पीने से भी ज्यादा खतरनाक है।

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$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix}$$

$2x + y + z = 10$ ——— (i)

$y + 3z = 6$ ——— (ii)

$\therefore y = 6 - 3z$ —— (i) —— y is dependent, z is independent

& from (i)

$x = 2 + z$ —— (ii) x is dependent, z is independent

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+z \\ 6-3z \\ z \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

for different values of z we get different sol^n. So system has infinite sol^n.

Bisection $f(x)$ $[a,b]$ $\frac{d}{dx}\sqrt{x} = \frac{1}{2}$
N·R $f(x)$ x_0.

पृष्ठ सं.: 30
दि.: / /

Q $f(x) = x - e^{-x}$, then $x_{n+1} =$ ______ ?

2008 ⟹ Ans. ⟹ $x_{n+1} = \dfrac{e^{-x_n}(1+x_n)}{1+e^{-x_n}}$

2001 Q $f(x) = x + \sqrt{x-3}$ and $x_0 = 2$, then $x_1 =$ ______ ?

⟹ Ans ⟹ $x_1 = 1.8124$

1999 Q The newton raphson method used to find the root of the equation and $f'(x)$ is derivative of f then the method is converges

(a) Always (b) Only if 'f' is polynomial
(c) Only if $f(x_0) < 0$
(d) None of the above ⟹ Ans. = (d).

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Newton Raphson method is useful for finding roots of Eq^n whether curve is ler to x axis, i.e the curves which are generating high slopes we can get better results using N-R method

If slope is less then N-R method is not providing accurate results

The N-R method converging to the root if it satisfy the following eq^n

$$|f(x) \cdot f''(x)| \le |f'(x)|^2$$

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* Regula false Method :-

Step 1 :- Let, $f(x)$ is continuous function in $[a, b]$

Step 2 :- Let us Assume that x_0 and x_1 are initial approximation values for the required root such that $f(x_0)$ and $f(x_1)$ having opposite signs
Say $f(x_0) < 0$, $f(x_1) > 0$

Step 3 :- Regula false Iteration formula for finding root of $f(x)$ in $[x_0, x_1]$ is

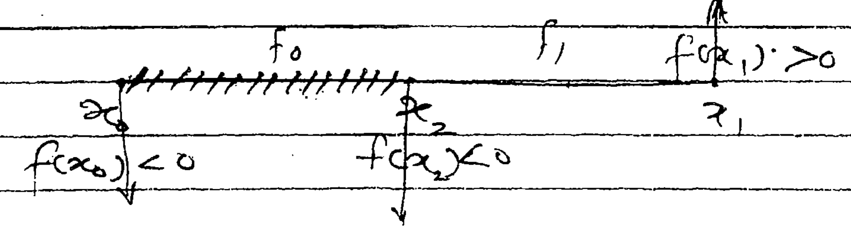
if $$\boxed{x_n = \frac{f_n \cdot x_{n-1} - f_{n-1} \cdot x_n}{f_n - f_{n-1}}}$$

In particular $$\boxed{x_2 = \frac{f_1 \cdot x_0 - f_0\, x_1}{f_1 - f_0}} \quad n=1 \qquad \text{—— (i)}$$

Case I :-
If $f(x_2) = 0 \implies x_2$ is root, then Stop process

Case II :-
If $f(x_2) < 0$ and $f(x_1) > 0$

i.e.



To ~~find~~ compute 'x_3', replace 'x_0' by x_2

$$\therefore x_3 = \frac{f_1 x_2 - f_2 x_1}{f_1 - f_2}$$

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Case III :-

If $f(x_2) > 0$ and $f(x_0) < 0$

So to compute 'x_3', we have to replace 'x_1' by 'x_2' 'f_1' by f_2 in Eq^n (i)

$$\therefore \boxed{x_3 = \frac{f_2 \cdot x_0 - f_0 x_2}{f_2 - f_0}}$$

Continue the process until desired accuracy is found

Q. $f(x) = x^3 + x - 1$ and $[0.5 \;\; 1]$ then find x_2, x_3 using Regula false method

→ $[0.5, 1] = [x_0, x_1]$

$f(x) = x^3 + x - 1$

$f(0.5) = (0.5)^3 + 0.5 - 1 = -0.375$ i.e < 0

$f(1) = 1^3 + 1 - 1 = 1$ i.e. > 0

Let, $x_2 = \dfrac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$

$$= \frac{1 \times 0.5 - (-0.375) \times 1}{1 - (-0.375)}$$

$x_2 = 0.64$

Now, $f_2 = f(x_2) = f(0.64) = (0.64)^3 + 0.64 - 1$

$= -0.0979$

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Re fala $f(x)$ Assume $[x_0, x_1]$ $\rightarrow f(x_0) < 0$

Secant $f(x)$ $[x_0, x_1]$, $f(x_0), f(x_1) < 0$. $f(x_1) > 0$

$$\therefore x_3 = \frac{f_1 \cdot x_2 - f_2 \cdot x_1}{f_1 - f_2}$$

formula

$$x_3 = \frac{f_1 x_2 - f_2 x_1}{f_1 - f_2}$$

$$= \frac{-0.0979 \times 0.5 - (-0.375) \times (0.64)}{-0.0979 - (-0.375)}$$

wrong.

$$= \frac{-0.0979 \times 0.5 + 0.375 \times 0.64}{-0.979 + 0.375}$$

$$\boxed{Ans. = 0.672}$$

* Secant Method :-

The difference between Regula false & Secant method is, in Secant method the initial Guess values x_0, x_1 need not satisfy the condition.

Let, $[f(x_0) \times f(x_1)] < 0$

i.e Secant method does not provide 100% Gaurentee that the root is existing in the initial guess interval (x_0, x_1)

Iteration formula for finding roots of Given Eq^n using Secant method is

$$\boxed{x_{n+1} = \frac{f_n \cdot x_{n-1} - f_{n-1} \cdot x_n}{f_n - f_{n-1}}}$$

$x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4$

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$$\frac{f_0 \quad x_2 \quad f_1}{x_0 \qquad\quad x_1}$$

In particular,

$$\boxed{x_2 = \frac{f_1 \cdot x_0 - f_0 \cdot x_1}{f_1 - f_0}} \quad \text{——— (i)}$$

To compute x_3, x_0 is replaced by x_1, x_1 replaced by x_2 in eq^n (i)

Continue process until desired accuracy of root is found.

Q Using Secant Method, find 1st & 2nd approximation of the real root for the equation $x^3 - 2x - 5 = 0$, with $[2, 3]$

→

$f(x) = x^3 - 2x - 5$, ~~f(2) =~~

$f(2) = 2^3 - 2(2) - 5 = -1 < 0$ ——— f_0

$f(3) = 3^3 - 2(3) - 5 = 16 > 0$ ——— f_1

$$x_2 = \frac{f_1 \cdot x_0 - f_0 x_1}{f_1 - f_0} = \frac{16 \times 2 - (-1) \times 3}{16 (-) (-1)}$$

$$= \frac{32 + 3}{17}$$

$$= \frac{35}{17} = 2.058$$

$\therefore f(x_2) = (2.058)^3 - 2(2.058) - 5$

$= -0.3996 \approx -0.3907$

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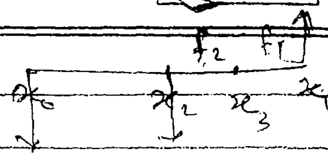

$$x_3 = \frac{f_2 \cdot x_1 - f_1 \cdot x_2}{f_2 - f_1}$$

$$= \frac{-0.3907 \times 3 - 16 \times 2.058}{-0.3907 - 16}$$

$$x_3 = 2.0812$$

* Method	Order of Convergence
① Bisection	linear Convergence $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^1$ means of order 1
② Regula falsi	linear Convergence $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^1$ — " — " — 1
③ Secant Method	Quadratic Convergence $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^{1.62}$ — " — — " — 1.62
④ Newton Raphson	Quadratic Convergen. $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^2$ — " — — " — 2

$\alpha = 2$

if $x_n = 2.02$,

$x_{n+1} = 2.004$

Error = Exact − Approxi

$\varepsilon_n = 2 - 2.02$

$= -0.02$

$\varepsilon_{n+1} = -0.004$

$$\boxed{\varepsilon_{n+1} = \varepsilon_n^2}$$

Bisec 1
Reg 1.62
Sec
Newto 2

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magnitude

and

$$x_n = 2.03$$
$$x_{n+1} = 2.06$$

$$\varepsilon_{n+1} = 0.06$$
$$\varepsilon_n = 0.03$$

$$\boxed{\varepsilon_{n+1} = 2\varepsilon_n^1}$$

Note:- Let us consider n^{th} degree polynomial $f(x) = 0$

(a) The number of +ve real roots for $f(x) \leq$ The number of sign changes in $f(x) = 0$

(b) The number of -ve real roots for $f(x) \leq$ The number of sign changes in $f(-x) = 0$

(c) The number of imaginary roots =
n − (No. of +ve roots + No. of -ve roots)

Q Polynomial $P(x) = x^5 + x + 2$ has

GATE 2003

(a) All real roots (c) Three real roots & 2 complex
✓(b) 1 real & 4 complex roots (d) All complex roots

→ from Note (a)

no. of +ve real roots $\leq$ '0' No. of sign changes in $P(x)$

∴ No. of +ve real roots = 0

from Note (b) No. of -ve real roots $\leq$ No. of sign changes in $P(-x)$

$$P(-x) = -x^5 - x + 2$$

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$P(-x) = 1$ Change

$\therefore$ '1' (-ve) real roots.

Imaginary roots $= n - [(+ve) + (-ve)$ roots$]$

$= 5 - [0+1]$

$= 4$

$\therefore$ Ans. 1 Real root & 4 complex root

Q. If $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_n$ are n roots of Eq^n.

$a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1} x + a_n = 0$

(a) $\sum_{i=1}^{n} \alpha_i = \_\_\_\_$ (b) $\sum \alpha_1 \alpha_2 = \_\_\_\_$

(c) $\sum \alpha_1 \alpha_2 \alpha_3 = \_\_\_\_$ (d) $\alpha_1, \alpha_2, \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$

$\rightarrow$

$ax^2 + bx + c = 0$

$\therefore \alpha_1 + \alpha_2 = -\frac{b}{a}$ $\quad \alpha_1 \cdot \alpha_2 = \frac{c}{a} = \frac{\text{Const.}}{\text{Coeff of } x^2}$

$\therefore \alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \frac{\text{Const}}{\text{Coeff} \cdot \text{of } x^3}$

$\vdots$

$\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \ldots \alpha_n = \frac{\text{Const}}{\text{Coeff.} \cdot x^n}$

$\therefore \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \ldots \alpha_n = \frac{a_n}{a_0}$

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$\therefore ax^3 + bx^2 + cx + d = 0.$

$$\therefore \alpha_1 + \alpha_2 + \alpha_3 = \frac{-b}{a} = \frac{-a_1}{a_0}$$

$$\sum \alpha_1 \cdot \alpha_2 = \frac{a_2}{a_0}$$

$$\therefore \sum \alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \frac{a_3}{a_0}$$

$$\therefore \sum_{i=1}^{n} \alpha_i = \frac{-a_1}{a_0}$$

$$\sum \alpha_1 \cdot \alpha_2 = \frac{a_2}{a_0}$$

$$\sum \alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \frac{-a_3}{a_0}$$

$$\alpha_1 \cdot \alpha_2 \cdots \alpha_n = (-1)^n \cdot \frac{a_n}{a_0}$$

alternate +ve, −ve signs.

GATE 2008

Q. It is known that the roots of the non-linear eq^n $x^3 - 6x^2 + 11x - 6 = 0$ are 1 & 3 then 3rd root will be

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{-b}{a} = \frac{-(-6)}{1} = 6$$

$$\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \frac{c}{a} = \frac{-6}{1} = -6$$

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Here, $x^3 - 6x^2 + 11x - 6 = 0$

$\therefore a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0$

$$\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \frac{c}{a} = \frac{(-1)^3(-6)}{1}$$

$$1 \times 3 \times \alpha_3 = (-1)(-6)$$

$$\alpha_3 = \frac{6}{3}$$

$$\boxed{\alpha_3 = 2}$$

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24/08/12

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II Solutions to System of linear Equation

(A) Gauss Elimination :-

⟹ "MATRIX METHOD"

Q. Solve
$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

⟶

Step I :- Construct Augmented matrix

i.e
$$[A:B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

— A = Coefficient matrix
B = Constant matrix.

Step II :- Convert augmented matrix into an upper triangular matrix using elementary row operations

Here, in above prob we have to do Row operation as, $R_2 - \frac{3R_1}{2}$ and $R_3 - \frac{R_1}{2}$

$$\therefore [A:B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/2 & 11 \end{array}\right]$$

अश्लील, गंदे विचारवाली पुस्तक पढ़ना जहर पीने से भी ज्यादा खतरनाक है।

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$$R_3 - \frac{(7/2)R_2}{1/2} = R_3 - 7R_2$$

$$\therefore [redacted] \approx \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

Imp
****
Expected
Ques.

$\rho(A)$ = No. of Non Zero rows in an uppertriangular matrix of A.

Case I :-

If $\rho(A) = \rho(AB) = r = n$

where, n is no. of unknowns

i.e. $(x, y, z \ldots \text{etc})$

Then the System is said to CONSISTENT and it has a UNIQUE SOLUTION.

$\Rightarrow$ Continue to prob.

$$\therefore \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

Here $\rho(A) = \rho(AB) = 3 = n$

So, it has a Unique sol^n.

$$\therefore \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ -10 \end{bmatrix}$$

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Case (iii) :-

If $\rho(A) \neq \rho(AB)$ then System is said to be "INCONSISTENT", then it has "NO SOLUTION"

Ex:- Same example, but in constant matrix in third row a const is present i.e -5.

$$\therefore \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

$\therefore \rho(A) = 2$

$\rho(AB) = 3$

$\therefore \rho(A) \neq \rho(AB)$ —— "NO SOLUTION"

Gauss Elimination :-

$\Longrightarrow$ "PIVOTAL SOLUTION"

Q. Solve :-

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

Step I :- Construct augmented matrix

i.e. $[A : B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$

अश्लील, गंदे विचारवाली पुस्तक पढ़ना जहर पीने से भी ज्यादा खतरनाक है ।

from (2, 3, -10, 5, 1, -7) the largest absolute value is -10

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absolute = |modulus|

Since, $a_{11} = 2$

Now, scan entire 1st column and select largest absolute value and make it as "Pivot". Exchange pivot element row with 1st row and then eliminate x from row 2 & row 3.

$R_2 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$R_2 - \frac{2R_1}{3}$ & $R_3 - \frac{R_1}{3}$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 18 \\ 0 & -1/3 & -1 & -2 \\ 0 & 10/3 & 8 & 10 \end{array}\right]$$

Since, $a_{22} = -1/3$

Now, scan entire 2nd column from a_{22} and select largest absolute value i.e. $\left|\frac{-1}{3}\right| < \left|\frac{10}{3}\right|$ and make it as pivot

Exchange pivot element row with 2nd row and then eliminate y from row 3

a_{11}

a_{22}

a_{33}

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$R_2 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 3 & 2 & 3 & 18 \\ 0 & 10/3 & 8 & 10 \\ 0 & -1/3 & -1 & -2 \end{array}\right]$

$$R_3 + \frac{\left(\frac{1}{3}\right)R_2}{\left(\frac{10}{3}\right)} = R_3 + \frac{R_2}{10}$$

$$\approx \left[\begin{array}{ccc|c} 3 & 2 & 3 & 1 \\ 0 & 10/3 & 8 & 10 \\ 0 & 0 & -1/5 & -1 \end{array}\right]$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 0 & 10/3 & 8 \\ 0 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 10 \\ -1 \end{bmatrix}.$$

By solving, $z = 5$
$y = -9$
$x = 7$

Q. In the solutions of the following set of linear equations by Gauss Elimination using Pivotal soln the pivots for eliminating "x" & "y" respt.

GATE 2009

$5x + y + 2z = 34$
$4y - 3z = 12$
$10x - 2y + z = -4$

(a) 10 & 4
(b) 10 & 2
(c) 5 & 4
(d) 5 & -4

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$$\left[\begin{array}{ccc|c} 5 & 1 & 2 & 34 \\ 0 & 4 & -3 & 12 \\ 10 & -2 & 1 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 10 & -2 & 1 & -4 \\ 0 & 4 & -3 & 12 \\ 5 & 1 & 2 & 34 \end{array}\right]$$

$\cancel{34}$ 68 +4

$R_3 - R_1/2$

$$\left[\begin{array}{ccc|c} 10 & -2 & 1 & -4 \\ 0 & 4 & -3 & 12 \\ 0 & \cancel{2} & 3/2 & 36 \end{array}\right]$$

Ans. $a_{11} = 10$

$a_{22} = 4$

$\therefore$ Ans. 10 & 4.

(B) LU Decomposition (Method of factorisation) or Do-little method.

Step I :- Let us consider

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Step II :- Matrix representation of given system of equation is

$$AX = B \quad \text{——— (i)}$$

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Let, $A = LU$ ——— (ii)

——— where L = lower Unit Δ'ular matrix

i.e. $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

U = Upper Δular matrix

$= \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$

$\therefore$ (i) $\Rightarrow$ $LUX = B$ ——— (iii)

Let, $UX = Y$ ——— (iv)

where. $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\therefore$ $LY = B$.

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

By solving, we get, y_1, y_2, y_3 in terms of elements of lower Unit Δ'ular matrix.

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$$\therefore \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

By Solving, we get x_1, x_2, x_3 in terms of the elements of Upper Δ'ular and lower Unit Δular matrix.

The order computing elements of L & U is $U_{11}, U_{12}, U_{13}, l_{21}, U_{22}, U_{23}, l_{31}, l_{32}, U_{33}$.

Note:- CROUT'S method is similar to Do-little method except that in Crout's method 'A' is decomposed with lower triangular matrix & Unit upper triangular matrix.

i.e. $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

$U = \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$

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GATE 2011

Q In matrix A is $\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ is decomposed into product of Lower and upper triangular matrices Using Crout's method The properly decomposed L & U matrices respectively

(a) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & U_{12} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11} U_{12} \\ l_{21} & l_{21} \cdot U_{12} + l_{22} \end{bmatrix}$$

$\therefore \; l_{11} = 2$

$l_{11} U_{12} = 1 \qquad \therefore U_{12} = 1/2$

$l_{21} = 4$

$l_{21} \cdot U_{12} + l_{22} = -1 \qquad 4 \times \frac{1}{2} + l_{22} = -1$

$l_{22} = -3$

$$\therefore \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & U_{12} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

III Solution to Integration of function :-

Let us consider the given curve is $f(x)$ and ordinates on x axis is $x=a$ & $x=b$

The area bounded by the given curve and the ordinates is denoted by

$$\int_a^b f(x)\,dx \quad \text{———} \quad (*)$$

Divide $[a,b]$ into "n" equal subintervals where, length of each interval is "h" (Step Size)

$a = x_0$

$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + h + h = x_0 + 2h$

$x_3 = x_2 + h = x_0 + 2h + h = x_0 + 3h$

$\vdots$

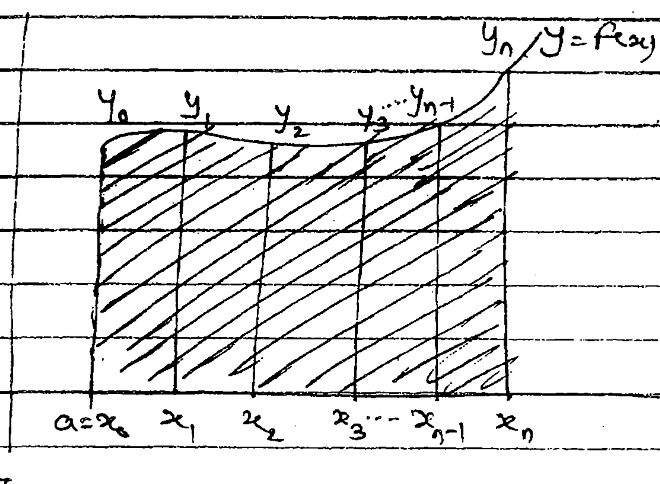
$x_n = x_0 + nh$

i.e $b = x_0 + nh$

$\therefore x_n = x_0 + nh$

$b = a + nh$

$$\therefore \boxed{n = \frac{b-a}{h}}$$



In Q., Only Simpson rule is mentioned then take Simpson $\frac{1}{3}^{rd}$ rule

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Equation (*) Can be evaluated by Using

1) Trapezoidal Rule

$$= \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\cdots\cdots y_{n-1})\right]$$

2) Simpson $\frac{1}{3}$ Rule

$$= \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+y_6+\cdots\cdots y_{n-2}) + 4(y_1+y_3+y_5+\cdots\cdots y_{n-1})\right]$$

3) Simpson $\frac{3}{8}$th Rule

$$= \frac{3h}{8}\left[(y_0+y_n) + 2(y_3+y_6+y_9+\cdots\cdots y_{n-3}) + 3(y_1+y_2+y_4+y_5+y_7+y_8+y_{10}+y_{11}\cdots y_{n-1})\right]$$

Recess. 10:40 am

Continue 11:15 am.

Gate Q 2010

h

x	0 (x_0)	0.25 (x_1)	0.5	0.75	1.0
$f(x)$	1	0.9412	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

The value of the integrated betn the limits 0 & 1 Using Simpson Rule (if not mentioned then take $\frac{1}{3}^{rd}$ rule)

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Simpson $\frac{1}{3}$ rule $= \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4\cdots)+4(y_1+y_3\cdots)\right]$

$= \frac{0.25}{3}\left[(y_0+y_4)+2(y_2)+4(y_1+y_3)\right]$

$= \frac{0.25}{3}\left[(1+0.5)+2(0.8)+4(0.9412+0.64)\right]$

$= 0.7854$

Note:- ① Simpson's Rule is applicable if the number of intervals are "EVEN"

② Simpson's $\frac{3}{8}$th Rule is applicable if the number of intervals are multiples of 3 i.e ($n = 3, 6, 9, 12,$ etc)

③ Trapezoidal Rule is applicable for any number of intervals

Q. A 2nd degree polynomial $f(x)$ takes a following values

GATE 2006

x	0	1	2
$f(x)$	1	4	15

~~The integration 0 to 2~~ $\int_0^2 f(x)\,dx$ is Evaluated Using trapezoidal Rule then the error estimation is

(a) $4/3$ (b) $-4/3$ (c) $2/3$ (d) $-2/3$

Error = Exact value − Approximate value

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→ Here,

It is mentioned that 2^{nd} degree polynomial.

$\therefore f(x) = a_0 + a_1 x + a_2 x^2$.

and it takes a followings values —— given.

$\therefore f(0) = 1$ —— i.e. $x = 0,\ f(x) = 1$.

$\therefore a_0 = 1$

$\therefore f(1) = 4$

$\therefore a_0 + a_1 + a_2 \cancel{x}^2 = 4$

$1 + a_1 + a_2 = 4$

$a_1 + a_2 = 3.$ —— (i)

$f(2) = 15$

$a_0 + a_1 \times 2 + a_2 \times 2^2 = 15$

$2a_1 + 2a_2 = 14$

$a_1 + 2a_2 = 7$ —— (ii)

$\therefore$ Solve (i) & (ii)

$a_2 = 4$

$a_1 = -1$

$\therefore a_0 = 1,\ a_1 = -1,\ a_2 = 4$

$\therefore f(x) = 1 - x + 4x^2.$

$\therefore$ Exact value $= \int_0^2 f(x)\,dx = \int_0^2 (1 - x + 4x^2)\,dx$

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$$= \left[x - \frac{x^2}{2} + \frac{4x^3}{3} \right]_0^2$$

Exact $= \frac{32}{3}$

Approximate value = Trapezoidal Rule value

$\therefore$ T.R. value $= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \cdots) \right]$

$$= \frac{1}{2}\left[(1+15) + 2(4) \right]$$

$$= 12$$

$\therefore$ Error = Exact − Approximate

$$= \frac{32}{3} - 12$$

$$= \frac{-4}{3}.$$

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$\frac{2\pi}{8} .. \frac{\pi}{4}$ $\quad 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, 2\pi$ $\quad \frac{h}{2}[(\;+\;)$

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5 significant digit = upto 5^{th} decimal pt.

ME Q. GATE 2007

$\int_0^{2\pi} \sin x \, dx$ is evaluated by T.R. Rule with Eight Equal intervals, with **5** Significant digits

(a) 0.00000 (b) 1.00000 (c) 0.00500 (d) 0.00025

→ Here, $n = 8$, $h = \frac{b-a}{n} = \frac{2\pi - 0}{8} = \frac{\pi}{4}$.

$h = \frac{\pi}{4}$

x		$\sin x$	
$\sin x$	$x = 0$	$\sin 0 = 0$	-- y_0
	$x + h = 0 + \pi/4$	0.70710	y_1
	$\pi/2$	1	y_2
	$3\pi/4$	0.70710	y_3
	π	0	y_4
	$5\pi/4$	-0.70710	y_5
	$6\pi/4$	-1	y_6
	$7\pi/4$	-0.70710	y_7
	2π	0	y_8

$$\text{T.R. Rule} = \frac{h}{2}\left[(y_8 + y_0) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{\pi}{4} \cdot \frac{1}{2}\left[(0+0) + 2(0.70710 + 1 + 0.70710 + 0 - 0.70710 - 1 - 0.70710)\right]$$

$$= 0.00000$$

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ME GATE 2010

x	0	60	120	180	240	300	360
y	0	1068	-323	0	323	-355	0

Evaluate $\int_0^{2\pi} y\,dx$ Using Simpson's Rule

(a) 542 (b) 995 (c) 1444 (d) 1986

$\rightarrow = \frac{h}{3}\left[(y_0+y_6)+2(y_3)+4(y_1+y_2+y_4+y_5)\right]$

$= \frac{60}{3}\left[(0+\right.$

Here $h = 60$.

But the Integration limit is in π term

$\therefore$ ~~take~~ $h = \frac{\pi}{3}$

$\frac{1}{3}$ rd $= \frac{\pi}{3}\left[(y_0+y_6)+2(y_3)+4(y_1+y_2+y_4+y_5)\right]$

$= 995$

x	1	2	3
$\frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$
	y_0	y_2	

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MEQ. 2011 GATE

The integral $\int_1^3 \frac{1}{x}\,dx$ when evaluated using $\frac{1}{3}$rd rule on two equal intervals each of length 1.

(a) 1.000 (b) 1.111

$\longrightarrow$ $\int_1^3 \frac{1}{x}\,dx$ $\quad h = \frac{b-a}{n}$ —— $n = 2$

$= \frac{3-1}{2}$

$h = 1$

	x	$1/x$
y_0	1	$1/1 = 1$
y_1	2	$1/2$
y_2	3	$1/3$

$\frac{1}{3}$rd $= \frac{h}{3}\left[(y_0 + y_2) + \cancel{\ldots}\right]$

$\cancel{= \frac{1}{3}\left[0 + 0.5\right]}$

$\cancel{= \frac{0.5}{3}}$

$\cancel{= 0.}$

$\frac{1}{3}$rd $= \frac{1}{3}\left[\left(1 + \frac{1}{3}\right) + 2(0) + 4(y_1)\right]$

$= \frac{1}{3}\left[\frac{4}{3} + 4 \times \frac{1}{2}\right]$

$= \frac{1}{3}\left[\frac{4}{3} + 2\right]$

$= \frac{1}{3}\left[\frac{10}{3}\right]$

$= \frac{10}{9}$

$= 1.111$

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Error order T·R rule = h^2
Error order Simpson $\frac{1}{3}$ = h^4
Error order Simpson $\frac{3}{8}$ = h^5.

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Q GATE 2008

The minimum number of equal length of subinterval needed to approximate

$$\int_1^2 x \cdot e^x \, dx$$ to an accuracy of at least $\frac{1}{3} \times 10^{-6}$ Using T·R rule

(a) $1000 \cdot e$ (b) 1000 (c) $100 \cdot e$ (d) 100

→ $n = ?$

$f(x) = x \cdot e^x$

Note :- (a) $\boxed{\text{Error in T.R. Rule} = -\left(\frac{b-a}{12}\right) \cdot h^2 \cdot \max\left[f''(x)\right]}$

(b) $\boxed{\text{Truncation Error in T.R. Rule} = \frac{(b-a)}{12} h^2 \cdot \max\left[f''(x)\right]}$

—— Error order h^2

At least means $\geq$

Here, Error ↑, Accuracy ↓

Given; Accuracy $\geq \frac{1}{3} \times 10^{-6}$

Truncation Error $\leq \frac{1}{3} \times 10^{-6}$

$$\frac{(b-a)}{12} \cdot h^2 \cdot \max\left[f''(x)\right] \leq \frac{1}{3} \times 10^{-6}$$

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Here,

$f(x) = xe^x$

$f'(x) = xe^x + e^x$

$f''(x) = xe^x + e^x + e^x$

$\therefore f''(x) = 2e^x + xe^x$

— Here, in $f''(x)$, e^x is Increasing function as x increases.

& xe^x is also.

$\therefore f''(x)$ is Increasing function in $(1,2)$

$\therefore f''(2) = 2\times e^2 + 2\times e^2$

$= 4e^2.$

Now, $h = \frac{b-a}{n}.$

$\therefore \frac{(2-1)}{12} \cdot \left(\frac{2-1}{n}\right)^2 \max f''(x) \leq \frac{1}{3}\times 10^{-6}.$

$\frac{1}{\cancel{12}_{\cancel{3}}} \times \frac{1}{n^2} \times \cancel{4} e^2 \leq \frac{1}{\cancel{3}} \times 10^{-6}$

$\frac{e^2}{n^2} \leq 10^{-6}$

$e^2 \times 10^{6} \leq n^2$

$n^2 \geq e^2 (10^3)^2$

$n \geq e \cdot 10^3$

$n \geq 1000 \cdot e$

$\therefore$ Ans. $1000 \cdot e$.

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Note :-

① Error in $S\text{-}\frac{1}{3}$ rule $= \frac{-(b-a)}{180} \cdot h^4 \cdot \max \left[f^{IV}(x) \right]$

— Error order h^4

② Error in $S\text{-}\frac{3}{8}$ Rule $= \frac{-3}{80} \cdot h^5 \cdot \max \left[f^{IV}(x) \right]$

— Error order h^5.

IV Solutions to differential Equation

Let us consider differential Eq^n.

$$\frac{dy}{dx} = f(x, y) \quad \text{where} \quad y(x_0) = y_0 \quad \text{——} \circledast$$

Eq^n ⊛ Can be solved by using :-

1) Eulers method
   - forward Eulers method
   - Backward Eulers method.
2) Runge - kutta method
   - Runge Kutta of 1^{st} order - [Eulers method]
   - Runge Kutta of 2^{nd} order - [modified Eulers method]
   - Runge Kutta of 3^{rd} order
   - Runge Kutta of 4^{th} order

Not asked in GATE yet.

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* Eulers Method :- (forward)

Eulers Iterative formula for finding solution curve to the eq^n (*) is

$$\boxed{y_{n+1} = y_n + h \cdot f(x_n, y_n)}$$

In particular for $n=0$

$$\boxed{y_1 = y_0 + h \cdot f(x_n, y_n)}$$

Q. find an approximate value of 'y' corresponding to $x = 0.2$ & $\frac{dy}{dx} = x + y$, $y = 1$, ~~when~~ $x = 0$. Using Eulers method where $h = 0.1$

make it $f(x,y)$

	x	y	comment
①	$x_0 = 0$	$y_0 = 1$	Initial value.
②	$x_1 = x_0 + h$ $= 0 + 0.1$ $= 0.1$	$y_1 = ?$ $\therefore y_1 = 0.1$	$y_1 = y_0 + h \cdot f(x_0, y_0)$ $= 1 + 0.1\,(x_0 + y_0)$ $= 1 + 0.1\,(0+1)$ $y_1 = 1.1$
③	$x_2 = x_1 + h$ $= 0.1 + 0.1$ $= 0.2$	$y_2 = ?$ $\therefore y_2 = 1.22$	$y_2 = y_1 + h \cdot f(x_1, y_1)$ $= 1.1 + 0.1\,(0.1 + 1.1)$ $= 1.1 + 0.1\,(0.2)$ $= 1.22$

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Q. GATE 2010 — $\frac{dy}{dx} - y = x$ when $y(0) = 0$, where $h = 0.1$

Compute $y(0.3)$ using Eulers 1st order method.
$\Longrightarrow$ i.e Eulers forward method.

$\rightarrow$ Ans. $\underline{\underline{0.031}}$

x	y	Comment
$x_0 = 0$	0	$y_1 = y_0 + h\, f(x_0, y_0)$
$x_1 = 0.1$	$y_1 = 0$	$y_1 = 0 + 0.1\,(0+0)$ $y_1 = 0$ $y_2 = y_1 + h\, f(x_1, y_1)$
$x_2 = 0.2$	$y_2 = 0.01$	$y_2 = 0 + 0.1\,(0.1 + 0)$ $= 0.01$ $y_3 = y_2 + h\, f(x_2, y_2)$
$x_3 = 0.3$	$\boxed{y_3 = 0.031}$	$y_3 = 0.01 + 0.1\,(0.2 + 0.01)$ $= 0.01 + 0.1 \times 0.21$ $= 0.01 + 0.021$ $= 0.031$

03/09/

* Eulers Backward Method :-

Let, $\frac{dy}{dx} = f(x, y)$ ——— (*)

where, $y(x_0) = y_0$

$\therefore$ Eulers Backward iterative formula for solving Eq^n (*) i.e $\frac{dy}{dx} = f(x,y)$

$$\boxed{y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})}$$

Here, y_{i+1} is in both L.H.S & R.H.S.
Since, y_{i+1} is defined in function
Therefore, this method is called as Implicit ~~Eq~~^{c}.
Euler's method.

Q.1 find an appropriate value for $x = 0.2$ using Implicit Eulers method where, $\frac{dy}{dx} = x+y$, $y(0) = 1$ where step size $h = 0.1$

$\rightarrow$ Given, $\frac{dy}{dx} = x+y$

$$\therefore y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})$$

$$y_{i+1} = y_i + h \cdot (x_{i+1} + y_{i+1})$$

$$(1-h) \cdot y_{i+1} = y_i + h \cdot x_{i+1}$$

$$y_{i+1} = \frac{y_i + h \cdot x_{i+1}}{(1-h)}$$

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In perticular $x=0$ then

$$\therefore \boxed{y_1 = \frac{y_0 + h \cdot x_1}{(1-h)}}$$

$\therefore$ Construct the table for y with respective to values of x.

x	y	Comment.
$x_0 = 0$	$y_0 = 1$	Initial Condition
$y_1 = 0.1$	$\boxed{y_1 = 1.222}$	$y_1 = \frac{y_0 + h \cdot x_1}{(1-h)} = \frac{1 + 0.1 \times 0.1}{(1-0.1)}$, $y_1 = 1.222$
$x_2 = 0.2$	$\boxed{y_2 = 1.38}$	$y_2 = \frac{y_1 + h \cdot x_2}{(1-h)} = \frac{1.222 + 0.1 \times 0.2}{(1-0.1)} = 1.38.$

Note:- Eulers Backward method is more stable than forward method.

The exact Solution for differential eq^n $\frac{dy}{dx} = x+y$ with $y(0) = 1$ is $y = 2e^x - x - 1$

at $x=1$, $y = 3.44$.

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By observing forward & backward Eulers method you can say that Backward method is converging to required value very quickly

GATE Q 2006 The diff. eqⁿ $\frac{dy}{dx} = 0.25y^2$ is to be solved using backward Eulers method with boundary conditions $y=1$ at $x=0$ and $h=1$ what would be the value of y at $x=1$.

(A) 1.33 (B) 1.67 (C) 2.0 (D) 2.33

→ Here, $\frac{dy}{dx} = 0.25y^2$

$$y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1})$$

$$y_{i+1} - 1\left[0.25\, y_{i+1}^2\right] = y_i$$

$$1 \times 0.25\, y_{i+1}^2 - y_{i+1} + y_i = 0.$$

— Here, value of $h=1$, ∴ No need to construct table

$$0.25\, y_{i+1}^2 - y_{i+1} + y_i = 0.$$

Comparing with $ax^2 + bx + c = 0$

∴ roots of Eqⁿ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2 \cdot a}$$

$$\therefore y_{i+1} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(0.25)(y_i)}}{2 \times 0.25}$$

$$y_{i+1} = \frac{1 \pm \sqrt{1 - y_i}}{0.5}$$

put $i=0$,

$$y_1 = \frac{1 \pm \sqrt{1-y_0}}{0.5} \quad \text{—— } y_0 = 1.$$

i.e. $y(0) = 1$ — given

$$\therefore y_1 = \frac{1 \pm \sqrt{1-1}}{0.5}$$

$$y_1 = \frac{1}{0.5}$$

$$\boxed{y_1 = 2}$$

Runge Kutta Method :-

GATE 1996 2nd order R-K → Given diff. eq^n $\frac{dy}{dx} = x - y$ with $y(0) = 0$ then value of $y(0.1)$ using 2^{nd} order Runge Kutta method, with step size $h = 0.1$

Runge kutta method of 2^{nd} order iterative formula for finding sol^n curve to eq^n i)

$$\boxed{y_1 = y_0 + \frac{1}{2}(k_1 + k_2)}$$

where $\boxed{k_1 = h\, f(x_0, y_0)}$

$k_1 = 0.1 \times (x_0 - y_0)$

$= 0.1 \times (0 - 0)$

$\boxed{k_1 = 0}$

$\boxed{k_2 = h\, f(x_0 + h, y_0 + k_1)}$

$k_2 = 0.1[(x_0 + h) - (y_0 + k_1)]$

$= 0.1[(0 + 0.1) - (0 + 0)]$

$\boxed{k_2 = 0.01}$

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$\therefore y_1 = y(x_0+h) = y(0+0.1) = y(0.1)$

$$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$$

$$= 0 + \frac{1}{2}(0+0.01)$$

$$= \frac{0.01}{2}$$

$$\boxed{y_1 = 0.005}$$

Q.2. 4th Order R-K → Apply Runge-Kutta method of 4th order where $\frac{dy}{dx} = x+y$, $y=1$ when $x=0$, $h=0.2$

Compute $y(0.2)$

Runge Kutta method of 4th order formula

$$\boxed{y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)}$$

where

$k_1 = h \cdot f(x_0, y_0)$	$k_2 = h \cdot f(x_0+\frac{h}{2}, y_0+\frac{k_1}{2})$	$k_3 = h\, f(x_0+\frac{h}{2}, y_0+\frac{k_2}{2})$
$= 0.2(x_0+y_0)$	$= 0.2\left[0+\frac{0.2}{2}+1+\frac{0.2}{2}\right]$	$= 0.2\left[0+\frac{0.2}{2}+1+\frac{0.24}{2}\right]$
$= 0.2(0+1)$		$= 0.2[0.1+1+0.12]$
$k_1 = 0.2$	$k_2 = 0.24$	$k_3 = 0.244$

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$$\boxed{k_4 = h\cdot f(x_0+h\,,\,y_0+k_3)}$$

$$= 0.2\left[x_0+h+y_0+k_3\right]$$

$$= 0.2\left[0+0.2+1+0.2+4\right]$$

$$k_4 = 0.2888$$

$$\therefore y_1 = y_0+\frac{1}{6}\left[k_1+2k_2+2k_3+k_4\right]$$

$$= 1+\frac{1}{6}\left(0.2+2\times 0.24+2\times 0.244+0.288\right)$$

$$\boxed{y_1 = 1.2428}$$

Q.3. 3rd order R-K

Apply Runge kutta method of 3^{rd} order with $\frac{dy}{dx} = x+y$, when $y=1$, $x=0$ & $h=0.2$ then compute $y(0.2)$

→ Runge kutta method of 3^{rd} order iterative formula for finding solution curve to the Eq^n is.

$$\boxed{y_1 = y_0+\frac{1}{6}\left(k_1+4k_2+k_3\right)}$$

where,

$$\boxed{k_1 = h\cdot f(x_0,y_0)}$$

$$= 0.2\,(x_0+y_0)$$

$$= 0.2\,(0+1)$$

$$k_1 = 0.2$$

$$\boxed{k_2 = h\cdot f\left(x_0+\frac{h}{2}\,,\,y_0+\frac{k_1}{2}\right)}$$

$$= 0.2\left[0+\frac{0.2}{2}+1+\frac{0.2}{2}\right]$$

$$k_2 = 0.24$$

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$$\boxed{k_3 = h \cdot f(x_0 + h,\ y_0 + k')}$$

where, $\cancel{k' = 0}$

$$\boxed{k' = h \cdot f(x_0 + h,\ y_0 + k_1)}$$

$\therefore k' = 0.2\,[0 + 0.2 + 1 + 0.2]$

$= 0.2\,(1.4)$

$k' = 0.28$

$k_3 = 0.2\,[0 + 0.2 + 1 + 0.28]$

$k_3 = 0.296$

$y_1 = y_0 + \frac{1}{6}\,[k_1 + 4k_2 + k_3]$

$\therefore y_0 = 1 + \frac{1}{6}\,[0.2 + 4 \times 0.24 + 0.296]$

$$\boxed{y_0 = 1.2428}$$

Q.4. GATE 2007 Euler's method

The diff. eq^n $\frac{dx}{dt} = \frac{1 - x}{\tau}$ is evaluated using Eulers method with step size $h = \Delta T$, where $\Delta T > 0$. What is the maximum value of ΔT, To Ensure Stability in sol^n

(A) 1 (B) $\frac{\tau}{2}$ (C) τ ✓(D) 2τ

अश्लील, गंदे विचारवाली पुस्तक पढ़ना जहर पीने से भी ज्यादा खतरनाक है।

→ Stable :-

An iterative method is said to be stable if the round off error is remains bounded as $n \to \infty$, where n is no. of Iterations.

The Eulers method formula

$$\boxed{y_{i+1} = y_i + h \cdot f(x_i, y_i)}$$ can be written as

$$\boxed{y_{i+1} = E \cdot y_i + K} \quad \text{———} (*)$$

where,

K = terms which are involved in x or Consts.

Eq^n (*) is said to be stable, if $|E| < 1$

i.e $-1 < |E| < 1$

$$\frac{dx}{dt} = \frac{1-x}{\tau}$$

Similarly $\frac{dy}{dx} = \frac{1-y}{\tau}$

$$\therefore \frac{dy}{dx} = \frac{1-y}{\tau} = f(x, y)$$

$$\therefore y_{i+1} = y_i + h \cdot f(x_i, y_i)$$

$$= y_i + h \cdot \left(\frac{1-y_i}{\tau} \right)$$

$$y_{i+1} = \underbrace{\left(1 - \frac{h}{\tau}\right)}_{E} \cdot y_i + \underbrace{\frac{h}{\tau}}_{K} \quad \text{————} (I)$$

G

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Eq^n (I) is stable if $\left|1-\frac{h}{\tau}\right| < 1$

$\therefore \left|1-\frac{h}{\tau}\right| < 1$

$\left|1-\frac{\Delta T}{\tau}\right| < 1$

i.e. $-1 < \left|1-\frac{\Delta T}{\tau}\right| < 1$

Substract 1 throughout to reduce 1 or Cancell 1 from middle term

$\therefore -1-1 < \left|1-1-\frac{\Delta T}{\tau}\right| < 1-1$

$\therefore -2 < \left|-\frac{\Delta T}{\tau}\right| < 0$

$\therefore -2\tau < |-\Delta T| < 0$

$\therefore 2\tau > \Delta T > 0$ —— Removing Signs and changing directions of Equality Signs.

i.e $0 < \Delta T < 2\tau$

$\therefore \Delta \tau < 2\tau$

Ans. 2τ

Q.1. Simpson Rule — The min. no. of Equal length subintervals needed to approximate $\int_0^2 e^{2x}dx$ to an accuracy of atleast $\frac{8}{45}\times 10^{-8}$ using Simpson's Rule

(A) 200 e (B) 200 (C) 2000 e (D) 2000

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$\rightarrow$ $f(x) = e^{2x}$ —— $[a,b] = [0,2]$

Accuracy atleast means

$$\text{accuracy} \geq \frac{8}{45} \times 10^{-8}$$

Here, if Accuracy ↑ then Error ↓

$$\therefore \text{Error} \leq \frac{8}{45} \times 10^{-8}$$

In Numerical method error is considered as ~~Simpson's~~ Truncation method

$$\left| \left[\text{Truncation Error} \right]_{\text{Simpson rule}} \right| \leq \frac{8}{45} \times 10^{-8}$$

$$\left| \frac{b-a}{180} \times h^4 \times \text{Max}\left(f^{IV}(x)\right) \right| \leq \frac{8}{45} \times 10^{-8}$$

But $h = \dfrac{b-a}{n}$

$$\therefore \left| \frac{2-0}{180} \times \left(\frac{2-0}{n}\right)^4 \times \text{Max}.\, f^{IV}(x) \right| \leq \frac{8}{45} \times 10^{-8}$$

Now, $f(x) = e^{2x}$

$\therefore f'(x) = 2 \cdot e^{2x}$

$f''(x) = 4 \cdot e^{2x}$

$f'''(x) = 8 \cdot e^{2x}$

$f^{IV}(x) = 16 \cdot e^{2x} = 16 \times e^{2\times 2} = 16 \cdot e^4$ — last limit ~~for~~ $x = 2$

$$\therefore \left| \frac{\cancel{2}-0}{\cancelto{4}{180}} \times \left(\frac{2-0}{n}\right)^4 \times \cancelto{4}{16} \cdot e^4 \right| \leq \frac{8}{\cancel{45}} \times 10^{-8}$$

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$$\therefore \left| \left(\frac{2}{n}\right)^4 \cdot e^4 \right| \le \frac{8}{1} \times 10^{-8}$$

$$\left| \frac{16}{n^4} \cdot e^4 \right| \le 8 \times 10^{-8}$$

$$\left| \frac{16 \cdot e^4}{1\,8 \times 10^{-8}} \right| \le n^4$$

$$\left| 16 \times e^4 \times 10^{+8} \right| \le n^4$$

$$\therefore n^4 \geq \left| 16 \times e^4 \times 10^{+8} \right|$$

$$\therefore n^4 \geq \left| 2^4 \times e^4 \times (10^2)^4 \right|$$

$$\therefore n \geq \left| 2 \times e^4 \times 10^2 \right|$$

$$n \geq 200\, e^4 \quad \text{———— } \therefore \text{Ans.}$$

Q. 2.
GATE

If we use Newton raphson method to find roots $f(x)=0$, Using x_0, x_1 & x_2 respectively As initial guess values & then the roots obtained would be

(A) 1.3, 0.6, 0.6

(B) 0.6, 0.6, 1.3

(C) 1.3, 1.3, 0.6

(D) 1.3, 0.6, 1.3

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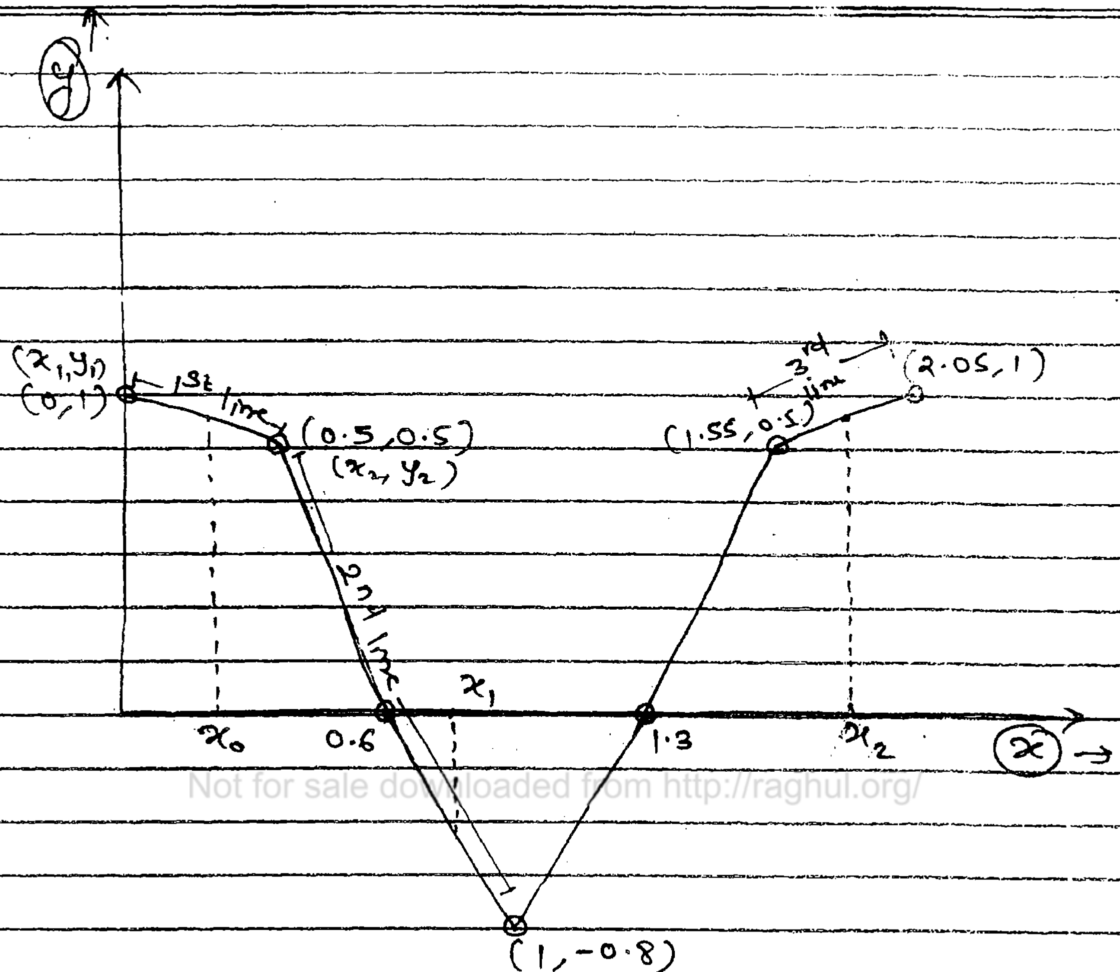

→ 1st line for x_0

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

(0, 1) → (0.5, 0.5)

$$m = \frac{0.5 - 1}{0.5 - 0}$$

(0.1, 1) — (0.5, 0.5) ← original

$$m = -1$$

Eqn of 1st line $\Rightarrow (y - y_1) = m(x - x_1)$

$$(y - 1) = -1(x - 0)$$

$$\underline{\underline{x + y = 1}} \Rightarrow y = -x + 1$$

$$\Rightarrow y = mx + c$$

प्रेम सबसे करो, बुरा किसीका न करो।

$$\Longrightarrow c \to 1$$

∴ converges to 1.3

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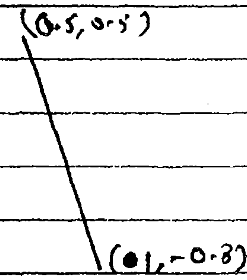
On x axis, $y=0$,
So, $x=1$ which is near to 1.3 than 0.6 on x axis $\therefore$ x_0 is converging to 1.3.

— 2^{nd} line for x_1 $\underline{(0.5, 0.5)}$ $(1, -0.8)$



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-0.8 - 0.5}{1 - 0.5}$$

$$m = -1.8$$

Eq^n of 2^nd line $\Rightarrow$ $(y - y_1) = m(x - x_1)$

$$(y - 0.5) = -1.8(x - 0.5)$$

$$y - 0.5 = -1.8x + 0.9$$

$$y + 1.8x = 0.9 + 0.5$$

$$\underline{\underline{y + 1.8x = 1.3}}$$

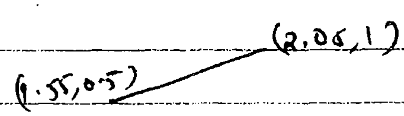
$$\Rightarrow y = -1.8x + 1.3$$

$$\Rightarrow y = mx + c$$

$$\Rightarrow c \rightarrow 1.3.$$

$\therefore$ Converges to 1.3.

— 3^{rd} line for x_2 $\underline{(1.55, 0.5)}$ $(2.05, 1)$



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0.5}{2.05 - 1.55}$$

$$m = 0$$

Eq^n

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Eqn of 3rd line

$$(y - y_1) = m(x - x_1)$$

~~y = 1.55~~)

$$(y - 0.5) = 0(x - \cancel{1}.55)$$

$$y - 0.5 = 0$$

$$y = 0.5$$

$$\Rightarrow \therefore y = mx + c$$

$$\therefore c \Rightarrow 0.5$$

Converges to $\underline{0.6}$

PROBABILITY AND STATISTICS

(Probability, r.v's
r. processes
Veeraraj an
McGraw Hills.)

(i) Basics.

(ii) Probability

(iii) Random variable / Expectation.

(iv) Distribution → Discrete
→ Continues.

(v) Mathematical

STATISTICS

→ collection of data.

→ Analysis of data

→ Interpretation of data.

Definition : According to prof. R.A Fisher, Statistics is defined as a collection of data, analysis of data and interpretation of data.

TYPES OF DATA

(i) Grouped and Ungrouped.

(ii) closed and open data.

GROUPED DATA

If data is in the form of class intervals and frequency then the data is known as grouped data, or distributing the frequencies to their corresponding class intervals, then the data is known as frequency distribution.

UNGROUPED DATA : If the data contains only observations, without any class intervals, then the data is known as ungrouped data or Raw Data.

CLOSED DATA : If the class intervals are in a continous form without any discontinuity, then the data is known as closed data otherwise open data.

MEAN (AVERAGE)

$$\boxed{\bar{X}_{UGD} = \frac{\sum_{i=1}^{n} x_i}{n}}$$

$$\boxed{\bar{X}_{GD} = \frac{\sum_{i=1}^{n} f_i x_i}{N}}$$

n :→ no of observations.

x : midpoint , $\frac{UL+LL}{2}$

N : sum of frequencies.

f : frequencies.

MEDIAN

- If n is odd, the middle observation itself is the median.
- If n is even, average between the middle observations provided

i) Data is rearranged either in increasing or decreasing order.

ii) No: of observations above the middle is equal to the number of observations below.

$$\boxed{M_d = l + \frac{\left(\frac{N}{2} - m\right) \times c}{f}}$$

l :- Lower limit for the ideal class

f :- frequency for the ideal class.

Cumulative frequency for the ideal class.

Q) Find the median for the following frequency data.

C.I	FREQUENCY	CUMULATIVE FREQUENCY
0-5	3	3
5-10	7	10
10-15	11	21
15-20	8	29
20-25	2	31

→ Ideal class.

N = 31

$$\frac{N}{2} = \frac{31}{2} = 15.5$$

$$M_d = 10 + \left(\frac{15.5 - 10}{11}\right) 5$$

$$= 10 + \frac{5.5 \times 5}{11} = 12.5$$

Note: If the first class itself is ideal, the cumulative frequency and frequency are ideal $(m = f) \Rightarrow M_d = l$

MODE

The most frequently repeated observation is known as Mode.

1, 2, 3, 4, 5, 2, 3, 11, 14, 2, 3, 21, 2, 16, 21, 3, 19

$M_o = 2, 3 \longrightarrow$ Bimodal.

For grouped data

$$\boxed{M_o = 3\,Md - 2\,Mean.}$$

$$M_o = l + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) c$$

$\Delta_1 = f - f_{-1}$

$\Delta_2 = f - f_{+1}$

Q Find the mode for the following frequency distribution.

C·I	Frequency
0-10	11
10-20	14
$\boxed{20\text{-}30}$	$\boxed{17}$ ideal class.
30-40	8
40-50	5
50-60	3

Frequency 17 can be treated as ideal class [Highest frequency]

$\Delta_1 = 17 - 14 = 3$

$\Delta_2 = 17 - 8 = 9$

$$M_0 = 20 + \left(\frac{3}{12}\right) \times 10 = \frac{45}{2} = \underline{\underline{22.5}}$$

* If the maximum frequencies are repeated ~~itself,~~ first, last and in between, select 'in between' as the ideal class.

* If the maximum frequencies are repeated in between, select randomly (bimodal).

* If all the frequencies are equal, mode is undefined $\left[\frac{0}{0} \text{ form}\right]$

* If the maximum frequencies are repeated first and last select randomly (bimodal).

MEASURES OF CENTRAL TENDENCIES

Among the 3 measures, mean, mode and median,
Mean is the best measure.

MEASURES OF DISPERSION

→ Range
→ Mean Deviation.
→ Quartile Deviation (Q.D)
→ Standard Deviation. (SD)
→ Coefficient of Variation. (C.V)

Measures of dispersion helps us to identify the deviation within the data.

RANGE : [Max — Min.
Greatest value — Least value]

STANDARD DEVIATION

$$\sqrt{\text{Variance}} = (S.D)$$

$$\text{Variance} = (S.D)^2 = \sigma_x^2$$

$$\boxed{\sigma_x^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\bar{x})^2}$$

Variance is the sum of the squares of deviation from mean. The differences or deviations within the data, is known as Variance.

Note:

i Lesser variance is more consistent or more uniform.

ii Variance will never be negative.

iii Variance of constant is 0.

iv Sum of the differences from the mean is always zero.

$$\boxed{\left[\sum_{i=1}^{n}(x_i - \bar{x})\right] = 0}$$

* If the variances are equal for the different groups, greater mean is more consistent.
* Sum of the squares of the deviation from the mean should be minimum.

For grouped data

$$\boxed{\sigma_x^2 = \frac{1}{N}\sum f_i x_i^2 - \left(\frac{1}{N}\sum f_i x_i\right)^2}$$

GROUPED DATA VARIANCE

$$\boxed{\sigma_x^2 = \frac{1}{N}\sum f_i (x_i-\bar{x})^2}$$

RELATION B/W QD/MD/SD

$$\boxed{6\,QD = 5\,MD = 4\,SD}$$

$QD = \frac{2}{3}\sigma$, $MD = \frac{4}{5}\sigma$

OEFFICIENT OF VARIATION (C.V)

$$\boxed{C.V = \frac{\text{Standard Deviation}}{\text{Mean}} \times 100}$$

$$\boxed{C.V = \frac{\sigma}{\bar{x}} \times 100}$$

Lesser σ implies lesser C.V ⇒ Data is more consistent or uniform.

or identifying the consistency within the data, which can e measurable by standard deviation ...

Q, Find mean and Variance for the first n natural numbers.

$$\bar{X} = \frac{[1+2+3+\cdots+n]}{n}$$

$$= \frac{\cancel{n}(n+1)}{2\cancel{n}}$$

$$\bar{X} = \frac{n+1}{2}$$

$$\sigma_x^2 = \frac{1}{n}\sum x_i^2 - (\bar{x})^2$$

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = \frac{1}{n}\left[1^2+2^2+3^2+\cdots+n^2\right]$$

$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$\sigma_x^2 = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right] = \frac{n+1}{2}\left[\frac{4n+2-3n-3}{6}\right] = \frac{n+1}{2}\times\frac{n-1}{6} = \frac{n^2-1}{12}$$

$$\boxed{\bar{X} = \frac{n+1}{2}} \qquad \boxed{\sigma_x^2 = \frac{n^2-1}{12}}$$

Mean of n natural numbers

Variance of n natural numbers.

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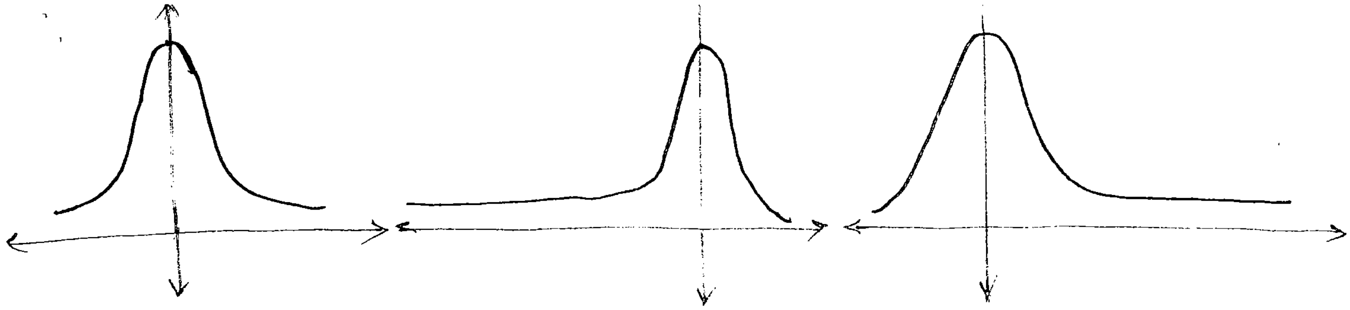
n statistics, geometrical representations or graph representations purely helps us to determine the behaviour of grouped data.)

SKEWNESS : Opposite of Symmetry.

"Lack of symmetry".



Symmetry.　　Negatively skewed.　　Positively skewed.

→ To check the skewness of a distribution.

→ **PEARSON'S COEFFICIENT OF SKEWNESS**

$$\boxed{S_{KP} = \frac{\text{Mean} - \text{Mode}}{\sigma}}$$

$$\boxed{S_{KP} = \frac{3(\text{Mean} - \text{Median})}{\sigma}}$$

→ practical limit of S_{KP}

$$\boxed{-3 \le S_{KP} \le 3}$$

→ For Symmetry $S_{KP} = 0$

Symmetry condition	Negative skewness	positive skewness.
Mode = Median = Mean	Mode > Median > Mean	Mode < Median < Mean.

PROBABILITY

RANDOM EXPERIMENT: Unpredictable outcomes of an experiment is known as a Radom experiment.

eg: Tossing a unbiased coin.

Rolling a Die.

Drawing a card from the pack of 52.

AMPLE SPACE (S): The collection of all possible outcomes of an experiment is known as a sample space. It is denoted by S.

EVENT: The outcomes of an experiment is known as a event. Mathamatically, event is the subset of Sample space.

ition of OBABILITY: The probability of an event is defined as the ratio ~~of~~ b/w the favourable cases to the event and the number of outcomes of an experiment. (The outcomes are mutually exclusive, exhaustive events)

$$\therefore \boxed{P(E) = \frac{m}{n}} \quad \text{where } m \leqslant n$$

AXIOMATIC APROACH / PROBABILITY FUNCTION

le 1 $\quad P(S) = 1$

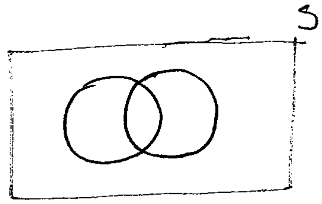
le 2 $\quad 0 < P(E) \leqslant 1$

$P(E) = 0 \quad ; \quad P(\phi) = 0 \longrightarrow$ Impossible Event

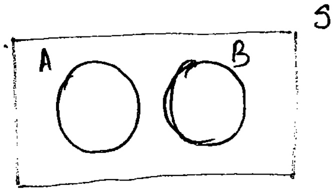
$P(E) = 1 \quad ;$ — Certain Event / Sure Event

Rule 3 : $\boxed{P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)}$

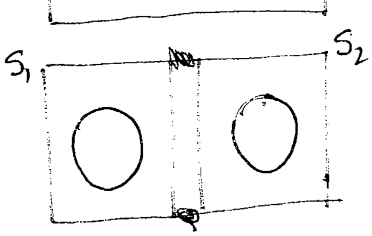
where E_i's are disjoint/Mutually Exclusive.



Dependent.



Mutually Exclusive Events



Independent

Note: Occurance of an event doesnot depends ~~on +~~ upon the occurance of the events in the same sample space, then such events are called Mutually Exclusive event.

→ Let A and B are mutually exclusive events.

$$A \cap B = \phi \quad \& \quad P(A \cap B) = 0$$

→ Occurance of an event doesnot depends ~~on~~ upon the occurance of same event in a different sample space, then those events are called Independent events.

→ Mutually Exclusive events never be independent, Independent events never be ~~not~~ equal to Mutually Exclusive.

RESULTS

1. Compliment theorem.

$$P(A^c) = 1 - P(A)$$

$$P(A) = 1 - P(A^c)$$

$\frac{1}{2} \times \frac{1}{6}$

2. Addition Theorem.

If A, & B are two events (If nothing specified, take it as ~~ind~~ dependent)

~~P(A∩~~

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

→ If A and B are mutually exclusive events

$$P(A \cup B) = P(A) + P(B)$$

$$P(A + B) = P(A) + P(B)$$

∴ Here it doesn't mean '+' = '∪' ~~In unio~~ If '+' Sign is given it is indirectly shown or it is sure that A & B are mutually exclusive. only mutually exclusive events can be added.

$$P(A+B+C) = P(A) + P(B) + P(C)$$

3. Multiplication Theorem for Dependent Events.

If A & B are two events,

$$P(A \cap B) = P(A)\ P(B/A)$$ → Here A must be happened already.

conditional probability

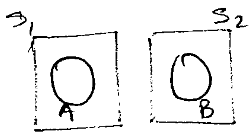
$$= P(B)\ P(A/B)$$ → Here B must be happened already.

if A, B and C are three events.

$$\boxed{P(A\cap B\cap C) = \underbrace{P(A)\,P(B/A)}_{P(A\cap B)}\;P(C/A\cap B)}$$

3. Multiplication Theorem for independent events.

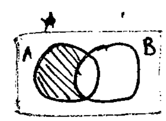
$$P(A\cap B) = P(A)\cdot P(B)$$



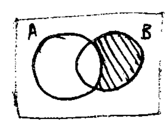
$$P(A\cap B\cap C) = P(A)\,P(B)\,P(C)$$

4. If A & B are 2 events.

$$\boxed{P(A\cap B^c) = P(A) - P(A\cap B)}$$ only A

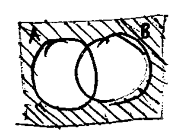


$$\boxed{P(A^c\cap B) = P(B) - P(A\cap B)}$$ only B.

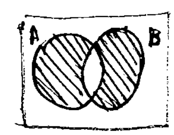


Neither A nor B

$$\boxed{P(A^c\cap B^c) = P(\overline{A\cup B}) = 1 - P(A\cup B)}$$ not in A B



$$\boxed{P(A\,\Delta\,B) = P(A\cap B^c) + P(A^c\cap B)}$$ only once



5. $$P(A^c/B) = \frac{P(A^c\cap B)}{P(B)}$$

$$= \frac{P(B) - P(A\cap B)}{P(B)} = 1 - \frac{P(A\cap B)}{P(B)} = 1 - P(A/B)$$

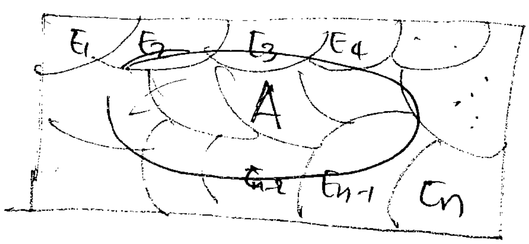
$$P\left(A/B^c\right) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{P(A) - P(A \cap B)}{1 - P(B)} \quad (\because P(B) \neq 1)$$

$$P\left(A^c/B^c\right) = \frac{P(A^c \cap B^c)}{P(B^c)} = \frac{1 - P(A \cup B)}{1 - P(B)} \quad (\because P(B) \neq 1)$$

Note: If A and B are independent events, the probability of $P(A \cap B^c)$, $P(A^c \cap B)$ and $P(A^c \cap B^c)$ are also independent.

6. <u>BAYE'S THEOREM</u>

If $E_1, E_2, E_3, \ldots E_n$ are the mutually exclusive events $(P(E_i) \neq 0)$ such that A is an arbitrary events which is a subset of "$\bigcup_{i=1}^{n} E_i$", then $P(A)$ is



$$P(A) = P(E_1 \cap A) + P(E_2 \cap A) + \cdots + P(E_n \cap A)$$

$$P(A) = P(E_1)\,P\left(A/E_1\right) + P(E_2)\,P\left(A/E_2\right) + P(E_3)\,P\left(A/E_3\right) + \cdots + P(E_n)\,P\left(A/E_n\right)$$

$$\boxed{P(A) = \sum_{i=1}^{n} P(E_i)\,P\left(A/E_i\right)}$$ Total probability of unknown event.

Part (ii) of Bayer's theorem: Reverse probability.

$$\boxed{P\left(E_i/A\right) = \frac{P\left(E_i \cap A\right)}{P(A)}}$$

$$\boxed{P\left(E_i/A\right) = \frac{P(E_i)\, P\left(A/E_i\right)}{\sum_{i=1}^{n} P(E_i)\, P\left(A/E_i\right)}}$$

Indirectly A is known.

steps in Bayer's theorem.

S1 Identify the known events in the data (Mutually exclusive).

S2 Select the unknown event (It is a part of known events).

S3 write the probability of unknown in terms of known.

S4 Find the total probability of unknown events.

S5 Compute Reverse probability for known events.

'Atleast'	Minimum	$\geq$
'Atmost'	Maximum	$\leq$
'And'	Product	$\cap$
'OR'	Sum	$\cup$

Application of addition theory

Cases with terminologies

→ 'either - or'

→ 'atleast once'

→ 'OR'

Application of Multiplication theory

Cases with terminologies

→ 'Simontaneously'

→ 'one after other'

→ 'as well as'

→ 'Successively'

→ 'One by one'.

→ 'alternatively'.

→ 'and'.

52 CARD CASE

Total 52 cards

13 Hearts + 13 Diamond + 13 club ~~spade~~ + 13 Spade .

Each 13 contains 1, 2, 3, 4, ... 10, J (11), Q (12), K (13)

ie, 10 number cards + 3 Face cards

Total ~~of~~ no: of Face cards = $4 \times 3 = 12$

Here 1 sample space =

Q. 3 coins are tossed at a time. Find the probability of yielding atmost one head for

$$S = \begin{Bmatrix} HHH \\ HHT \\ HTH \\ HTT \\ THH \\ THT \\ TTH \\ TTT \end{Bmatrix}$$

$\cancel{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}$

$$P(X \leq 1) = P(X < 1) + P(X = 1)$$

$$= \frac{1}{8} + \frac{3}{8}$$

$$= \frac{4}{8} = \frac{1}{2}$$

Q. Above data same, Find the probability that atleast one tail.

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$

Q. Find the probability that atleast one head and one tails

P(atleast 1 head $\boxed{\text{and}}$ atleast 1 tail min).

→ whenever "and" is used as the conjugation b/w any number of events, the all the events must occurs simontaneously in all ~~cases~~ Favourable cases.

$$P = \frac{6}{8} = \frac{3}{4}$$

Q, Same data. Find the probability that atleast one head and atmost one tail.

HHH, HHT, HTH, THH

$P = 4/8 = \underline{1/2}$

Q, A player tosses 4 coins Find the probability that atleast 2 heads and atleast 2 tails.

$$\frac{4P_4}{2!\,2!} =$$

$$\frac{4!}{0!} \qquad \frac{4\times3\times2\times1}{4} = 6$$

HHTT
HTTH
TTHH
THHT
THTH
HTHT

$= 6 = 4C_2$ — repeated, favourable.

$${}^{n}C_r = \frac{n!}{(n-r)!\,r!}$$

H	No of cases	T
$^4C_0$ =	1	= $^4C_4$
$^4C_1$ =	4	= $^4C_3$
$^4C_2$ =	6	= $^4C_2$
$^4C_3$ =	4	= $^4C_1$
$^4C_4$ =	1	= $^4C_0$

Favourable case.

12

~~Alm~~ Almost 2 head, Almost 2 tails = same = $\frac{6}{16}$

Q, A coin is ~~xapeo~~ tossed 6 times. Find the probability that the number of heads are more than the number of tails.

$$2^6 = 64$$

$$\underbrace{{}^6C_0 \quad {}^6C_1 \quad {}^6C_2 \quad {}^6C_3 \quad {}^6C_4 \quad {}^6C_5 \quad {}^6C_6}$$

Favourable cases of Head.

$$P = \frac{{}^6C_6 + {}^6C_5 + {}^6C_4}{64}$$

$$= \frac{1 + \frac{6!}{1 \times 5!} + \frac{6!}{2!\,4!}}{64}$$

$$= \frac{1 + 6 + \frac{\cancel{6}^{3} \times 5}{\cancel{2}}}{64}$$

$$= \frac{1+6+15}{64}$$

$$= \frac{22}{64}$$

Q, A coin is repeated n times. Find the probability that the head appears in the odd terms.

$${}^nC_1 + {}^nC_3 + {}^nC_5 + \dots + {}^nC_{n-1} = 2^{n-1}$$

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots + {}^nC_n = 2^{n-1}$$

$$\text{Req prob} = \frac{2^{n-1}}{2^n} = \frac{1}{2}$$

1, Two dies are rolled two times. Find the probability that for getting a sum 7

(i) atleast once

(ii) only once

(iii) twice.

(i) $\underbrace{1+6,\ 5+2,\ 4+3}_{3 \times 2 = 6}$

~~$6 \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{6}$~~

sum 7 in first : $P(A) = 6/36 = 1/6$ $\quad P(A^c) = 5/6$.

sum 7 in second : $P(B) = 6/36 = 1/6$ $\quad P(B^c) = 5/6$.

$$P(\text{atleast once}) = P(A \cup B) = 1 - P(A^c \cap B^c) = 1 - P(A^c)P(B^c)$$

$$= 1 - \frac{5}{6} \times \frac{5}{6}$$

$$= 1 - \frac{25}{36}$$

p(only once)

$= P(A \cap B^c) + P(B \cap A^c)$

$= P(A)\,P(B^c) + P(B)\,P(A^c)$

$= 1/6 \times 5/6 + 5/6 \times 1/6$

$= \frac{10}{36}$

$P(\text{twice}) = P(A \cap B) = P(A)\,P(B)$

$= 1/6 \times 1/6 = 1/36$

Q, Two dices are rolled. Find the probability that the first die should contain a prime number or a total of eight.

 ~~(6,2)(2,6),(5,3)(3,5),(4,4)~~

prime 2, 3, 5,

$2 \leftarrow (2,1)\ (2,2)\ \ldots\ (2,6)$

$3 \leftarrow (3,1)\ (3,2)\ \ldots\ (3,6)$

$5 \leftarrow (5,1)\ \ldots\ (5,6)$

$6 \times 3 = 18 \qquad P = \frac{18}{36}.$

Total 8

(6,2) (2,6) (5,3) (3,5) (4,4)

But A & B are dependant.

Hence to find $P(A \cap B)$

ie, Prime in first & sum 8

$(5,3)\ (3,5),\ (2,6) \rightarrow$ 3 cases $\quad P(A \cap B) = \frac{3}{36}$.

we need to find

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{18}{36} + \frac{5}{36} - \frac{3}{36}$$

$$= \frac{20}{36}$$

Two dice are rolled. Find the probability neither sum 9 nor sum 11

P(neither sum 9, nor 11)

$= 1 - P$(sum 9, and sum 11)

sum 9 $\rightarrow (5,4)\ (4,5)\ (6,3)\ (3,6) \rightarrow \frac{4}{36}$

sum 11 $\rightarrow (5,6)\ (6,5) \rightarrow \frac{2}{36}$

$$P(9^c \cap 11^c) = 1 - P(9 \cup 11)$$

$$= 1 - \left(\frac{4}{36} + \frac{2}{36}\right) = \cancel{\frac{6}{36}}\ \frac{30}{36}$$

Q. A determinant of order 2. with the elements 0,(and) or 1. Find the probability that the choosen det is non zero

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad S = 2^4 = 16$$

$\Delta \neq ad - bc$

$\therefore$ case $\Delta = 1$ [$a = d = 1$ atleast one of b & c is zero]

$$\Delta = 1 \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$= 3$ cases

Case II $\Delta = -1$

$ad = 0 \quad bc = 1$

[$b = c = 1$, atleast one of a & d is zero]

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \Rightarrow 3 \text{ cases}.$$

$P(\text{non zero } \Delta) = 3/16 + 3/16 = 6/16$

$P(\text{non neg } \Delta) = 1 - 3/16 = 13/16$

~~$P(\text{zero } \Delta) = 1 - 6/16 = 10/16$~~

Q, 4 cards are drawn ~~from~~ at random from a pack of 52 cards. Find the probability that

(i) All the 4 cards are drawn from same suit.

(ii) No two cards are drawn from the same suit.

(i) $S = {}^{52}C_4$

$$\frac{4 \times {}^{13}C_4}{{}^{52}C_4} \quad \text{(at a time)}$$

(ii) $$\frac{{}^{13}C_1 \, {}^{13}C_1 \, {}^{13}C_1 \, {}^{13}C_1}{{}^{52}C_4} = \frac{(13)^2}{{}^{52}C_4} \quad \text{(one by one).}$$

Q, A card is drawn from a pack of 52 cards. Find the probability that ~~neither a diamond nor a face~~ car

(i) neither a diamond nor a face.

(ii) neither a 10 nor a king

$P(D) = 13/52$

$P(F) = 12/52$

(i) $P(D^c \cap F^c) = 1 - P(D \cup F)$

$= 1 - \left[\frac{13}{52} + \frac{12}{52} - \frac{3}{52}\right] = \frac{30}{..}$

(ii) $P(10) = \frac{4}{52}$

$P(K) = \frac{4}{52}$

$$P(10^c \cap K^c) = 1 - P(10 \cup K)$$

10 & King are mutually exclusive.

$$= 1 - \frac{4}{52} - \frac{4}{52}$$

$$= 1 - \frac{8}{52}$$

$$= \frac{44}{52}$$

Q, A and B are the two players rolling a die on the condition that one who gets the two first wining the game. If A start the game, what are the winning chances of player A, B.

$$\frac{1}{6} + \frac{5}{6} \times \frac{1}{6} \quad \frac{5}{6} \times \frac{1}{6}.$$

$P(2) = 1/6 \qquad P(2^c) = 5/6$

Let getting 2 is p not getting q.

$$P(\text{win } B) = \cancel{q+p}\ qp + q^3p + q^5p + \dots = qp + q^2 qp +$$

$$= p\left[q + q^3 + q^5 + \dots\right] = pq\left[1 + q^2 + q^4 + \dots\right]$$

$$= (qp) \times \frac{1}{(1-q^2)}$$

$$= \frac{1}{6} \times \frac{5}{6} \times \frac{1}{1-\frac{25}{36}}$$

$$= 5/11$$

$$P(\text{win } A) = \cancel{0}\, p + q^2 p + q^4 p + \dots$$

$$= p\left[1 + q^2 + q^4 + \dots\right]$$

$$= p \times \frac{1}{1-q^2} = \frac{1}{6} \times \frac{1}{1-\frac{25}{36}}$$

$$= \frac{1}{\cancel{6}} \times \frac{\cancel{36}\,6}{11}$$

$$= \frac{6}{11}$$

A, B, C are the 3 players ~~as~~ in the order. Tossing the same coin on the condition that one who gets the head first winning game. If A starts the game, what are the ~~a~~ winning chances of player C in 3rd trial.

$$P(\text{win } C \text{ in } 3^{rd} \text{ trial}) = \cancel{\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)} \times q^3 q^3 q^2 p$$

$P(H) = 1/2$ $\quad P(T) = 1/2$

'p' $\quad$ (q)

$$= \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2}$$

64

4

$$\frac{1}{8} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{512}$$

Q. A number is choosen at random from 100 numbers,

$\{(00), 01, 02, 03, \ldots 99\}$

Let x denote the sum of the digits on the number, and y denotes product of the digits on the number. Find the probability that

$$P\left(\frac{x=9}{y=0}\right) = \frac{P(x=9 \cap y=0)}{P(y=0)}$$

0 0 – 9

$$P(y=0) = \frac{\cancel{19}\,2/100}{19/100} = \frac{2}{19}$$

9 ~~10~~ + 10 – 0

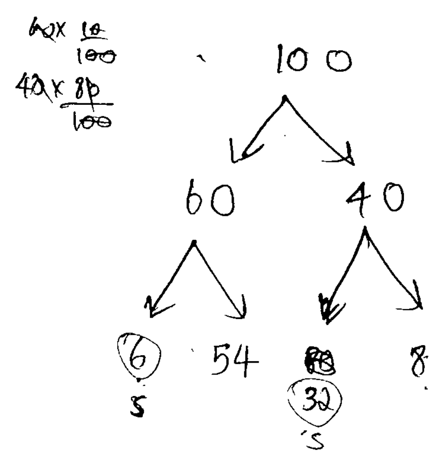
Q. 60% of the employees of the company are college graduates. Of these, 10% are in the sales department. Of the employees who did not graduate from the college are 80% in the sales department. A person is selected at random. Find the probability that

(i) The person is in the sales department.

(ii) Neither in the sales department nor a college ~~departm~~ graduate.

(i) $\frac{38}{100}$ ~~$P(CG \cap SD)$~~

(ii) $\frac{8}{100}$



(i) $P(SD) = P(CG \cap SD) + P(CG^c \cap SD)$

$= P(CG)\; P(SD/CG) + P(CG^c) \cdot P(SD/CG^c)$

$= 0.6 \times 0.1 + 0.4 \times 0.8$

$P(SD) = \underline{\underline{0.38}}.$

(ii) $P(CG^c \cap SD^c) = 1 - P(CG \cup SD)$

$= 1 - \left[P(CG) + P(SD) - P(CG \cap SD)\right]$

$= 1 - \left[P(CG) + P(SD) - P(CG)\, P\left(\dfrac{SD}{CG}\right)\right]$

$= 1 - \left[0.6 + 0.38 - 0.6 \times 0.1\right]$

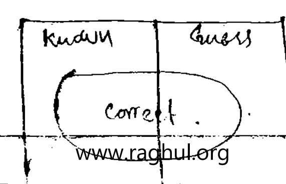
$= 1 - 0.92$

$= \underline{\underline{0.08}}$ $\cancel{0}$ $\cancel{0}$

Q. In answering a qn on multiple choise qn, a student either knows the answer or guess the answer. Let P be the probability that student knowing the answer to the qn and (1-P) be the probability that guessing the ans to a qn. Assume that if the student guess the answer to ~~the~~ a question will be correct, with probability 1/5. what is the conditional probability that if the student knew the ans to a qn, given that, he answered correctly?

$P(K) = P$

$P(G) = 1 - P$



E: answering correctly.

$P(K) = p$

$P(G) = 1-p$

$P(E) = P(E \cap K) + P(E \cap G)$

$P(E) = P(E \cap K) + \cancel{P(G)\, P(E/G)}\; P(E \cap G)$

$P(E) = P(K)\, P\left(E/_K\right) + P(G)\, P(E/G)$

$= p\, P\left(E/_K\right) + (1-p)\, 1/5$

$P\left(E/_K\right) = 1$

if Knowing correct then he will correct it for sure.

$= p \times 1 + (1-p)\, 1/5$

$= p + 1/5 - p/5$

$P(E) = \dfrac{4p+1}{5}$

rve probability

$P\left(K/_E\right) = \dfrac{P(K \cap E)}{P(E)}$

$= \dfrac{p \times 1}{\frac{4p+1}{5}} = \dfrac{5p}{4p+1}$

In qn. final

known even $\rightarrow$ correct ans.

unknown event $\rightarrow$ knowing ans.

Hence go for Reverse probability.

Q, There are 3 coins. Of these two are unbiased. One is a biased coin with 2 heads. A coin is drawn at random and tossed two times. It appears head on both the ~~side~~ times, Find the probability that it is from the biased coin.

$\cancel{\frac{1}{3}\times\frac{1}{2}}$ $\cancel{\frac{1}{3}\times 1}$

UB B
2 1 = 3

$P(UB) = 2/3$ $P(B) = 1/3$.

E: Getting a head two times. is an unknown event.

$$P(E) = P(E \cap UB) + P(E \cap B)$$

$$\cancel{= P(UE)\, P(UB/E) + P(}$$

$$= P(UB)\, P(E/UB) + P(B)\, P(E/B)$$

$P(E/B) = 1$

$$= 2/3 \times \frac{1}{4} + 1/3 \times 1$$

$P(E/UB) = \frac{1}{2}\times\frac{1}{2} = 1/4$

$$= \frac{1}{6} + \frac{1}{3}$$

$$P(E) = \frac{3}{6}$$

Ans: $P(B/E) = \dfrac{P(E \cap B)}{P(E)}$

$$= \frac{P(B)\, P(E/B)}{3/6} = \frac{1/3 \times 1}{3/6} = \frac{\cancel{6}^{2}}{\cancel{9}_{3}} = \frac{2}{3}$$

Ans: 2/3

1. player A speaking truth 4 out of 7 times, A card is drawn from the pack of 52 cards. He reports that there is a diamond what is the probability that actually there was a diamond.

$\frac{4}{7}$

T: A Player telling truth

~~$P(A) = 4/7$~~

$P(T) = (4/7)$

Lie probability

$P(L) = (3/7)$

D: Reporting a diamond.

$$P(D) = P(T \cap D) + P(L \cap D)$$

$$= P(T)\,P(D/T) + P(L)\,P(D/L)$$

$P(D/T) = 13/52$

$P(D/L) = 39/52$

$$= \frac{4}{7} \times \cancel{13}\,\frac{1}{4} + \frac{3}{7} \times \frac{3}{4}$$

$$= \frac{4+9}{28}$$

$$= \frac{13}{28}$$

~~$P(D/T) = P($~~

$$P(T/D) = \frac{P(T \cap D)}{P(D)}$$

$$= \frac{\frac{4}{7} \times \frac{1}{4}}{\frac{13}{28}} = \frac{4}{13}$$

Q, A letter is known to have come from either Tatanagar or CALCUTTA ~~Calcutta~~. On the envelope, the just two consecutive letters 'TA' are visible. Find the probability that the letter has come from Tatanagar.

$$P(T) = 1/2 \quad : \quad P(C) = 1/2.$$

E : Getting a TA consider as single letter.

Given only 1 letter is visible (which is a combination of 2 consecutive) Not more than 1 is visible.

1 2 3 4 5 6 7 8
T A T A N A G A R

CALCUTTA
1 2 3 4 5 6 7

$$P(E/T) = 2/8$$

$$P(E/C) = 1/7$$

Hence can take either one of 'TA' in TATANAGAR.

No need of confusion.

$$P(E) = P(E \cap T) + P(C \cap E)$$

$$= P(T)\,P(E/T) + P(C)\,P(E/C)$$

$$= 1/2 \times 2/8 + 1/2 \times 1/7$$

$$= 11/56$$

$$Q_n \quad P(T/E) = \frac{P(T \cap E)}{P(E)} = \frac{1/2 \times 2/8}{11/56} = \frac{7}{11}$$

There are 3 bags A, B, C, with Balls Blue, Red, and Green in the form ~~of 1, 2, 3~~ ~~B R G~~

		colour	
	B	R	G
Bags. A	1	2	3
B	2	3	1
C	3	1	2

A bag is drawn at random, and two balls are taken from it. They are found to be one blue and one red. Find the probability that the selected balls are from bag C.

$$P(A) = P(B) = P(C) = 1/3 .$$

E : Getting one blue and one red.

$$P(E) = P(E \cap A) + P(E \cap B) + P(E \cap C)$$

$$= P(A)\,P(E/A) + P(B)\,P(E/B) + P(C)\,P(E/C)$$

$$= \frac{1}{3}\left[\frac{2}{15} + \frac{6}{15} + \frac{3}{15}\right]$$

$$= \frac{1}{3}\left[\frac{11}{15}\right]$$

$$= \frac{11}{45}$$

~~P(E/A)=~~

$$P(E/A) = \frac{{}^1C_1 \times {}^2C_1}{{}^6C_2} = \frac{2}{15}$$

$$P(E/B) = \frac{{}^2C_1 \times {}^3C_1}{{}^6C_2} = 6/15$$

$$P(E/C) = \frac{{}^3C_1 \times {}^1C_1}{{}^6C_2} = \frac{3}{15}$$

Qn: $P(C/E) = \cancel{P(CA} \; \frac{P(C \cap E)}{P(E)}$

$= \frac{1}{3} \times \frac{3}{15}$

Also others can be found out.

$$P(B/E) = \frac{\frac{1}{3} \times \frac{6}{15}}{\frac{11}{45}} = \frac{6}{11}$$

$$P(A/E) = \frac{\frac{1}{3} \times \frac{2}{15}}{\frac{11}{45}} = \frac{2}{11}$$

LEAP YEAR CONCEPT

LEAP YEAR

366 Days

52 weeks + 2 days

S-M
M-T
T-W
W-T
T-F
F-S
S-S

$$P(53 \text{ sunday}) = \frac{2}{7}$$

NON LEAP YEAR.

365 Days.

52 weeks + 1 day

$$P(53 \text{ sundays}) = \frac{1}{7}$$

Two dice.

$P(\text{diff zero}) = P(\text{Doublets}) = \frac{6}{36} = \frac{1}{6}$

$P(1,1)$

$P(2,2)$

$\vdots$

$P(6,6)$

Three dice

$P(\text{Triplet}) = 6/6^3 = \frac{1}{36}$

$(1,1,1)$
$(2,2,2)$
$\vdots$
$(6,6,6)$

Consider now

$\{1, 2, 3, \ldots, 200\}$

$P(\text{div } 6 \text{ OR div by } 8)$

$P(\text{div } 6) = \frac{33}{200}$

$P(\text{div } 8) = \frac{25}{200}$

$P(\text{div } 6 \text{ AND div } 8) = \text{LCM}(6,8)$

$= P(\text{div by } 24) = \frac{8}{200}$

Hence $P(\text{div by } 6 \text{ OR div by } 8) = P(6) + P(8) - P(24)$

$$= \cancel{25}\ \frac{33}{200} + \frac{25}{200} - \frac{8}{200}$$

$$= \frac{50}{200} = \frac{1}{4}$$

RANDOM VARIABLE AND EXPECTATION

(r.v)

RANDOM VARIABLE : Connecting the outcomes of an experiment with real values is known as Random Variables. (It is a rule to assign Real number to the outcome is known as 1D Random Variable)

The corresponding data is known as univariate data.

2-D RANDOM VARIABLE : Connecting 2 outcomes at a times to the one real value provided those two outcomes are drawn from same Sample space.

The corresponding data is known as Bivariate data.

⇒ Similarly the concept of n-D Random variable which corresponds to an n-tuple.

TYPES OF RANDOM VARIABLE

Random Variable

- Discrete Random Variable
  - Probability MASS FUNCTION $P(x)$
  - DISCRETE PROBABILITY DISTRIBUTION
    - Binomial
    - Poisson
- Continous Random Variable
  - Probability DENSITY FUNCTION $f(x)$
  - CONTINUOUS PROBABILITY DISTRIBUTION
    - Normal
    - Uniform.

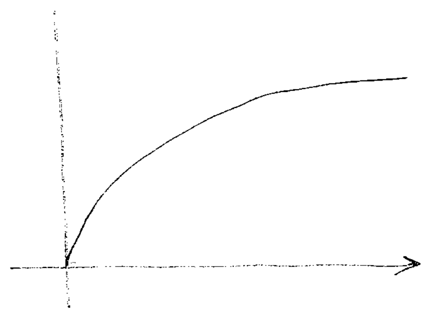
Probability Mass Function $\longrightarrow P(x)$

Probability Density Function $\longrightarrow f(x)$

Distributive Function/Cumulative Function $\longrightarrow F(x)$

$$\frac{dF(x)}{dx} = f(x), \qquad F(x) = \int^{x} f(x)\, dx.$$

Distribution function graph will always be a non decreasing function.



RANDOM PROCESS : Random variable along with time domain.

EXPECTATION

It is actually the mean in the probability ~~function~~ distribution.

$$E(X) = \sum_{x=0}^{n} x \cdot P(x) \quad \text{where, } x \text{ is Discrete r.v}$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \quad \text{where } x \text{ is continous r.v}$$

The above relations are derived from Frequency distribution where freq is replaced by probability

$$\bar{X} = \frac{\sum f \cdot x}{\sum f}$$

$$\bar{X} = \frac{\sum P(x) \cdot x}{\sum P(x)}$$

But $\sum P(x) = 1$

$$\therefore \bar{X} = \sum x\, P(x) = E(x)$$

(ii) VARIANCE

From frequency distribution, the variance is given by .

$$\sigma^2 = \frac{1}{N}\sum f_i x_i^2 - \left(\frac{1}{N}\sum f_i x_i\right)^2$$

$$\frac{1}{N}\sum f_i x_i = E(x)$$

$$\frac{1}{N}\sum f_i x_i^2 = E(x^2)$$

In general $\boxed{\frac{1}{N}\sum f_i x^r = E(x^2)}$

In probability distribution,

$$\boxed{V(x) = E(x^2) - (E(x))^2}$$

$$\boxed{V(x) = E\left(x - (E(x))^2\right)}$$

$$\boxed{V(x) = \sum x^2 p(x) - \left(\sum x \cdot p(x)\right)^2}$$ where x is a discrete r.V

$$\boxed{V(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int x f(x)\,dx\right)^2}$$ where x is a continuous r.V .

PROPERTIES OF EXPECTATION

(i) If X is a r.v and 'a' is a constant,

then $E(aX) = aE(X)$

If X and Y are r.v's.

(ii) Then $E(X+Y) = E(X) + E(Y)$

$$E(X-Y) = E(X) - E(Y)$$

(iii) If X & Y are r.v's

$$E(X.Y) = E(X) . E(Y/X)$$

$$= E(Y) . E(X/Y)$$

(iv) If X & Y are Independent random variables,

$$E(X \cdot Y) = E(X) . E(Y)$$

(v) If $Y = aX + b$, where a & b are constants,

Then $E(Y) = aE(X) + b$

(vi) ie, $\boxed{E[\text{constant}] = \text{constant}}$ ie, Mean of constant = That constant itself.

(vii) $\boxed{E[E[E(X)]] = \text{constant} = E(X)}$

PROPERTIES OF VARIANCE

i) If X is a r.v and 'a' is a constant,

$$V(aX) = a^2 V(x)$$

$$V(-Y) = (-1)^2 V(y) = V(y)$$

If X and Y are independent r.v's.

$$V(X+Y) = V(X) + V(Y)$$

$$V(X-Y) = V(X) + V(-Y)$$

$$\Rightarrow V(X-Y) = V(X) + V(Y)$$

$$\Rightarrow \boxed{V(X \pm Y) = V(X) + V(Y)}$$

If a & b are constants, X & Y are independent r.v's,

$$V(aX - bY) = a^2 V(X) + b^2 V(Y)$$

$$V\left(X/a - Y/b\right) = 1/a^2\, V(X) + 1/b^2\, V(Y)$$

f $Y = aX + b$, where a & b are constants,

$$V(Y) = V(aX + b)$$

$$= V(aX) + V(b)$$

$$V(y) = a^2 V(X) + 0$$

i.e. $V[\text{Constant}] = 0$

If X and Y are two random variables (Dependent r.v's).

$$\boxed{V(X+Y) = V(X) + V(Y) + 2\,CoV(X,Y)}$$

where $CoV(X,Y) \rightarrow$ Covariance of X, Y

where

$$\boxed{CoV(X,Y) = E(X \cdot Y) - E(X) \cdot E(Y)}$$

$\boxed{V(X,Y) \text{ is meaningless}}$

$\rightarrow \boxed{CoV(X,X) = V(X)}$

$\rightarrow$ $CoV(a,b) = E(a \cdot b) - E(a)E(b)$

$= ab - a \cdot b$

$= \underline{\underline{0}}$

$\boxed{CoV[a,b] = 0}$ where a and b are constants.

1 If x and y are independent r.v, then covariance of X, Y = 0

$CoV(X,Y) = 0$ X, Y are independent

But Converse of the statement is not true.

2. Variance and covariance are independent of change of origin dependent of change of scale.

$$V[aX+b] = a^2 V[x]$$

∴ Mean [Expectation] dependent of origin as well as Dependent of change of scale.

$$E[aX+b] = a^2 V[x] + b$$

SKEWNESS

$$\boxed{\beta_1 = \frac{\mu_3^2}{\mu_2^3}}$$

$\mu_3 \rightarrow 3^{rd}$ central moment.

$\mu_2 \rightarrow$ Variance.

Skewness is defined in terms of 'μ_3'

$$\boxed{\gamma = \sqrt{\beta_1}}$$

γ is also a measure of skewness.

Note:

If $\mu_3 = 0 \implies \beta_1 = 0$ Then the curve is SYMMETRY

If $\mu_3 \rightarrow -ve \implies$ Then the curve is NEGATIVELY SKEWED

If $\mu_3 \rightarrow +ve \implies$ Then the curve is POSITIVELY SKEWED

Q, Find the expectation of the number on a die when it is ~~throug~~ thrown.

$$E(X) = \sum_{0}^{n} x\, p(x)$$

$$= 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$$= \frac{1}{6}\left[1+2+3+4+5+6\right]$$

$$= \frac{1}{\cancel{6}}\left[\frac{\cancel{6}\times 7}{2}\right]$$

$$= \cancel{2.5}\ \ \underline{\underline{3.5}}$$

x r.v	1	2	3	4	5	6
$P(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

Q, Find the variance for the single die

$$V(X) = \cancel{E(X)}\ E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{0}^{n} x^2\, p(x)$$

$$= \frac{1}{6}\left[1^2+2^2+3^2+4^2+5^2+6^2\right]$$

$$= \frac{\cancel{6}(7)(13)}{6\ \cancel{6}}$$

$$= \frac{91}{6}$$

$$\frac{13}{7}$$

$$V(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12}$$

Note:

The mean and variance for the sum of the numbers on the ~~dies~~ dice is

$$\boxed{E(X) = \frac{7n}{2}} \qquad \boxed{V(X) = \frac{35}{12}n}$$

where 'n' is the number dice rolled

Q 3 unbiased dice are thrown Find the mean ~~of~~ and variance for the sum of the numbers on them.

X : sum of number on 3 dies.

X : 2, 3, 4, . . . , 18

$$E(X) = \frac{7n}{2} = \frac{7 \times 3}{2} = \frac{21}{2}$$

$$V(X) = \frac{35}{12}n = \frac{35 \times 3}{12} = \frac{35}{4}$$

Two unbiased dies are rolled. Find the expectation for sum 7 on them.

X : sum for the number obtained on 2 dies.

Here X is assuming only 7.

Hence no need of additions.

$$E(\text{sum } 7) = 7 \cdot P(7)$$

6 cases $\left\{ \begin{array}{l} 1, 6 \\ 6, 1 \\ 3, 4 \\ 4, 3 \end{array} \right.$

Q, A player tosses 3 coins. He wins 500 rupees if 3 heads occur, 300 rs if 2 Heads occurs, 100 rs if only 1 head occur. On the other hand he looses 1500 rs If 3 tails occurs Find value of the game.

X : No :of head possibility

~~$E(X) = 500 \times \frac{1}{8} +$~~

X	3	2	1	0
P(x)	1/8	3/8	3/8	1/8

VALUE OF GAME = GAIN - LOSE = Gain x its prob - Lose x its prob.

$$E(X) = 500 \times \frac{1}{8} + 300 \times \frac{3}{8} + 100 \times \frac{3}{8} - 1500 \times \frac{1}{8}$$

$$= \frac{1400 + 300 - 1500}{8}$$

$$= \frac{1700 - 1500}{8}$$

$$= \frac{200}{8} = \underline{25}$$

NOTE :

If Game is said to be fair, the expected value of game is said to be 0. (no Loss and no Gain).

Q. A man has given n keys of which one fits the lock. He tries them successively without replacement to open the lock. What is the probability that the lock will be open at the r^{th} trial. Also determine mean and variance.

Note: with Replacement implies that it is Independent events.

without Replacement implies that it is Dependent Events.

prob of opening the lock in 1^{st} trial $= 1/n$

" " 2^{nd} trial $= 1/(n-1)$

" " 3^{rd} trial $= 1/(n-2)$

⋮

prob of opening lock 1^{st} success in 2^{nd} trial $= \left(1 - \frac{1}{n}\right)\frac{1}{n-1}$

$$= \frac{n-1}{n} \times \frac{1}{\cancel{n-1}} = \frac{1}{n}$$

prob of opening lock in 1^{st} success in 3^{rd} trial $= \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n-1}\right) \times \frac{1}{n-2}$

$$= \frac{\cancel{n-1}}{n} \times \frac{\cancel{n-2}}{\cancel{n-1}} \times \frac{1}{\cancel{n-2}}$$

$$= \frac{1}{n}$$

Hence

prob of opening lock 1^{st} success r^{th} trial $= 1/n$

Variance of n natural numbers $V(x) = \frac{n^2-1}{12}$

consider a value eg:

If keys numbered from 100 — 999

$$\text{prob}\left(\underbrace{1^{st}\text{ success in}}\ 450^{th}\text{ trial without replacement}\right) = \frac{1}{900}$$ (100 - 999 means 0 → 900)

$$\text{Prob}\left(1^{st}\text{ success in } 450^{th}\text{ trial with replacement}\right) = \left(1-\frac{1}{900}\right)^{449}\frac{1}{900}$$

ie in without replacements, they are dependent, $\frac{899}{900}\times\frac{898}{899}\times\frac{897}{898}$ $= 1/900$

~~they cancel~~ they loss probabilities of $(r-1)$ trials cancel each other and get only $1/900$.

But in with replacement, each trial is independent event, Hence each ~~loss prob~~ of $(r-1)$ loss probabilities we need to multiply

$\therefore q^{r-1} p$

Note: The probability for the 1^{th} success in the r^{th} trial with replacement is

$$\boxed{P\left(1^{st}\text{ success in } r^{th}\text{ trial with replacement}\right) = q^{r-1}p}$$

q → Failure prob

p → Success prob

Q. if x is said to be a Continuous variable and its probability function ϕ

$$f(x) = Kx^2 \quad 0<x<1$$

(i) Find the value of K.

(ii) Find Mean & variance.

(i) ~~$\Sigma f()$~~ $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 Kx^2\,dx = 1$$

$$K\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$\frac{K}{3} = 1$$

$$\underline{\underline{K = 3}}$$

(ii) Mean $= E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$= 3\int_0^1 x^3\,dx$$

$$= 3\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{4}$$

(iii) Variance $V(x) = E(X^2) - (E(x))^2$

$$E(X^2) = \int_0^1 x^2 f(x)\, dx = 3\int_0^1 x^4\, dx = \underline{\underline{\frac{3}{5}}}$$

$$\text{Variance} = \frac{3}{5} - \left(\frac{3}{4}\right)^2$$

$$= \frac{3}{5} - \frac{9}{16}$$

$$= \frac{48-45}{80} = \underline{\underline{\frac{3}{80}}}$$

Q. If x is a continous r.v, and $f(x) = kx^2 e^{-x}$, ~~where~~ $0 < x < \infty$

(i) Find the value k

(ii) Mean & Variance.

(i) $$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{\infty} kx^2 e^{-x}\, dx = 1$$

Use Gamma Function,

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx \quad , n > 0$$

$$\Gamma n = (n-1)\Gamma n-1$$

$$= (n-1)! \text{ (only when } n \text{ is a integer)}$$

$$\Gamma 1 = 1$$

$\Gamma 0$ = doesnot exist.

So integral becomes.

$$K\,\Gamma(3) = 1$$

$$K\,2! = 1$$

$$\underline{\underline{K = 1/2}}$$

(ii) Mean

$$E(X) = \int_0^\infty x\, f(x)\, dx$$

$$= \frac{1}{2}\int_0^\infty x\, x^2 e^{-x}\, dx$$

$$= \frac{1}{2}\int_0^\infty x^3 e^{-x}\, dx$$

$$= \frac{1}{2}\,\Gamma(4)$$

$$= \frac{1}{2} \times 3! = \underline{\underline{3}}$$

Variance,

$$E(X^2) = \int_0^\infty x^2\, f(x)\, dx$$

$$= \frac{1}{2}\int_0^\infty x^2 \cdot x^2 e^{-x}\, dx$$

$$= \frac{1}{2}\int_0^\infty x^4 e^{-x}\, dx$$

$$= \frac{1}{2}\,\Gamma(5)$$

$$= \frac{1}{2}\,4! = 12$$

$$\text{Variance} = 12 - 3^2$$

$$= 12 - 9 = \underline{\underline{3}}$$

Q. $f(x) = |x| \quad -1 < x < +1$

$V(x)$ Find variance.

$$V(x) = E(x^2) - (E(x))^2$$

$$E(x) = \int_{-1}^{1} x f(x)\, dx = \int_{-1}^{1} x|x|\, dx.$$

$$= -\int_{-1}^{0} x^2\, dx + \int_{0}^{1} x^2\, dx$$

$$= \underline{\underline{0}}.$$

odd fn x even fn
$\therefore$ = $\underline{\underline{\text{odd fn}}}$

$\underline{\underline{\int_{-a}^{a} \text{odd fn}\, dx = 0}}$

$$E(x^2) = \int_{-1}^{+1} x^2 f(x)\, dx$$

$$= \int_{-1}^{+1} x^2 |x|\, dx$$

$$= 2 \times \int_{0}^{1} x^3\, dx$$

$$= 2 \times \left[\frac{x^4}{4}\right]_0^1$$

$$= \cancel{2}\ \underline{\underline{2/4}} = 1/2$$

$x^2|x| \rightarrow$ even x even
$\Downarrow$
even

$\int_{-a}^{a} \text{even fn} = 2\int_{0}^{a} \text{even fn}.$

$$\text{Variance} = 2/4 - 0^2 = \cancel{8}\ \underline{\underline{1/2}}$$

Q, If X & Y are the r.v's mean of X is 10, Variance of X = 25. Find positive values of a, b, such that Y = aX − b has Expectation is zero and Variance is 1.

$$E(Y) = E[aX - b] = 0$$

$$a E[X] - E[b] = 0$$

$$a \times 10 - b = 0$$

$$10a - b = 0$$

$$\underline{\underline{b = 10a}}$$

$$V[Y] = V[aX - b] = a^2 V[X] = 1$$

$$= a^2 \, 25 = 1$$

$$\underline{\underline{a = \frac{1}{5}}} \quad \text{Given +ve}$$

$$b = 10a$$

$$= 10 \times \frac{1}{5}$$

$$= \underline{\underline{2}}$$

$$\underline{\underline{a = 1/5, \quad b = 2}}$$

BIVARIATE DATA

Let x, y be two discrete r.v.s,

Their probability together given by Joint probability Mass Function (JPMF)

Let x, y be two continous r.v.s

Their probability together given by Joint Probability Density Function (JPDF)

Case (i) Continous R.V's.

→ If x and y are two continous r.v's, and its probability function is known as Joint probability density function is denoted by $f(x,y)$.

→ The marginal density functions are

$$\boxed{\begin{aligned} f(x) &= \int_y f(x,y)\, dy \\ f(y) &= \int_x f(x,y)\, dx \end{aligned}}$$

→ If x and y are 2-D continous Independent r.v's ~~and its probability function is known as Joint~~ Iff,

$$\boxed{f(x,y) = f(x)\, f(y)}$$

ie,

$$\boxed{JPDF = MDf(x) \cdot MDf(y)}$$

Joint Distribution Function (F) or Cumulative Distribution Function

$F(x,y)$ given by JDF

$$\frac{d^2}{dx\,dy} F(x,y) = f(x,y)$$

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{x} f(x,y)\,dx\,dy.$$

⇒ Conditional probability

$$f(x/y) = \frac{f(x,y)}{f(y)}, \quad (f(y) \neq 0)$$

$$E(x/y) = \frac{E(x,y)}{E(y)}, \quad (E(y) \neq 0)$$

Case (ii) Discrete R.V's.

If x and y are two dimensional r.v's and its probability function is known as joint probability function is denoted by $P(x,y)$.

The marginal mass functions are

$$P(x) = \sum_{y} P(x,y)$$

$$P(y) = \sum_{x} P(x,y)$$

Q. If x and y are 2-D continuous r.v's and its probability function is given by

$$f(x,y) = x \cdot y$$

$$0 < x < 1$$

$$0 < y < 1$$

(i) $E(X)$; ~~$E(x)$~~ $V(Y)$

(ii) $E(xy)$; $\mathrm{cov}(X,Y)$

(iii) $f(x/y)$; $E(X/y)$

(iv) check X & Y are independent or not

(i) $f(x) = \int_y f(x,y)\,dy = \int_0^1 x \cdot y\,dy = x \cdot y^2/2 \Big|_0^1 = \frac{x}{2}$

$$f(y) = \frac{y}{2}$$

$$E(x) = \int_0^1 x \cdot f(x)\,dx = \int_0^1 x^2/2\,dx = \left[\frac{x^3}{6}\right]_0^1 = 1/6$$

$$E(y) = \int_0^1 y\,f(y)\,dy = \int_0^1 \frac{y^2}{2}\,dy = \left[\frac{y^3}{6}\right]_0^1 = 1/6$$

$$E(y^2) = \int_0^1 y^2 f(y)\,dy = \int_0^1 y^2 \cdot y/2\,dy = 1/8$$

$$V(Y) = \frac{1}{8} - \left(\frac{1}{6}\right)^2 = \frac{1}{8} - \frac{1}{36} = \frac{36-8}{8 \times 36} = \frac{28}{8 \times 36} = 7/72$$

$$E(x\cdot y) = \int_{x=0}^{1}\int_{x=0}^{1} x\cdot y \; f(x,y)\, dx\,dy.$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} x\cdot y \;(x\cdot y)\, dx\, dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} x^2 y^2 \, dx\, dy$$

$$= \int_0^1 \frac{x^2}{3}\, dx = \frac{1}{9}$$

Covariance.

$$\text{CoV}(X,Y) = \cancel{E(X\cdot)}\; E(X\cdot Y) - E(X)\cdot E(Y)$$

$$= 1/9 - 1/6 \times 1/6 = 1/9 - 1/36 = \frac{4-1}{36} = \frac{3}{12}$$

~~(iv)~~ (iii)

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{x\cdot y}{y/2} = 2x$$

$$E(X/Y) = \frac{E(x\cdot y)}{E(y)} = \frac{1/9}{1/6} = 2/3$$

$$f(x,y) = f(x)\, f(y)$$

$x\cdot y \neq \frac{x}{2}\cdot y/2$ $\quad\therefore$ x & y are Dependent r.v's.

Q. If x and y are 2-D Discrete r.v's and its joint probability mass function is

X \ Y	-1	0	+1
-1	1/4	0	0
0	0	1/2	0
+1	0	0	1/4

(i) Find $P\left(X+Y=2 / X-Y=0\right)$

$$P\left(X+Y=2 / X-Y=0\right) = \frac{P(X+Y=2 \cap X-Y=0)}{P(X-Y=0)}$$

$$= \frac{P(X=1, Y=1)}{P(X=-1, Y=-1) + P(X=0, Y=0) + P(X=1, Y=1)}$$

$$= \frac{1/4}{1/4 + 1/2 + 1/4}$$

$$P\left(X+Y=2 / X-Y=0\right) = \underline{\underline{1/4}}$$

BINOMIAL DISTRIBUTION

Definition : If x is said to be a binomial random variable. It allows the values from 0 to n with the parameters (n, p) and its probability mass function is

$$B(x, n, p) = p(x) = \begin{cases} {}^{n}C_{x}\, p^{x} q^{n-x}, & 0 \leqslant x \leqslant n. \\ & p+q = 1 \\ & q = 1-p \\ 0, & \text{otherwise} \end{cases}$$

Conditions

- Observations are independent, (n is small).
- The probability of success is constant (p is large)
- Mean is greater than the variance.

PROPERTIES

$$E(x) = \text{Mean} = np$$

$$V(x) = \mu_2 = npq$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3}$$

$$\beta_1 = \frac{(1-2p)^2}{npq}$$

$$\gamma_1 = \frac{1-2p}{\sqrt{npq}}.$$

∴ In het, $p = \frac{1}{2} \longrightarrow$ symmetry.

$p < \frac{1}{2} \longrightarrow$ positively skewed.

$p > \frac{1}{2} \longrightarrow$ Negatively skewed.

Moment Generating Function (MGF)

$$\boxed{M_X(t) = E\left[e^{tx}\right] = (q+pe^t)^n}$$

⇒ ~~Momentu~~ Moment Generating Function is used for the taking the sum and difference of 2 r.v's ~~w~~ along with the Binomial Distribution.

Characteristic Function.

$$\boxed{\phi_X(t) = E\left[e^{itx}\right] = (q+pe^{it})^n}$$

Characteristic ~~equation~~ is used for finding ~~cumulative~~ and

· ratio of 2 r.v's with Binomial distribution.

Note:

→ $P = 1/2 \Rightarrow \mu_3 = 0 \Rightarrow \beta_1 = 0$

Then the Curve is Symmetry.

→ If $P < 1/2$, then the curve is Positively skewed.

, If $P > 1/2$, then the curve is Negatively skewed.

The moment Generating function is used to find addition and differences b/w the r.v's with their corresponding probability function.

The characteristic function is used to finding the convolution and ratio b/w the r.v's with their probability function.

Sum of Independent Binomial r.v's is also a Binomial random variables.

Q. Find the probability of getting a 9 exactly 2 in 3 times with a pair of dice.

$n = 3$

$x = 2$

$p(\text{sum } 9) = (5,4)(4,5)(6,3)(3,6)$

$= \frac{4}{36} = 1/9$

$q = 8/9$

required prob $= {}^nC_x \, p^x \, q^{n-x}$

$= {}^3C_2 \left(\frac{1}{9}\right)^2 \left(8/9\right)^1$

$= \frac{8}{243}$

Q. The probability of man hitting the target is 1/3.

(i) If he fires five times, what is the probability of his hitting the target at least twice.

(ii) How many time must he fire so that the probability of his hitting the target atleast once is more than 90%

(i) $n = 5, \quad p = 1/3, \quad q = 2/3$

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X=1) + P(X=0)]$$

In binomial distribution

$$\boxed{\begin{aligned} P(X=0) &= q^n \\ P(X=n) &= p^n \end{aligned}}$$ Always.

$$P(X \geq 2) = 1 - \left[\left(\tfrac{2}{3}\right)^5 + {}^5C_1 \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^4\right]$$

$$= 1 - \left[\left(\tfrac{2}{3}\right)^4 \left[\tfrac{2}{3} + \tfrac{5}{3}\right]\right]$$

$$= \frac{131}{243}$$

ii) $P(X \geq 1) > 90\%$

$$1 - P(X=0) > 0.9$$

$$P(X=0) < 0.1$$

$$q^n < 0.1$$

$$\left(\frac{2}{3}\right)^n < 0.1$$

Taking log on both sides

Here asked for number of trials

Q. If X and Y are the binomial r.v.

$$X \sim B(2, p)$$

$$Y \sim B(4, p)$$

If $P(X \geq 1) = 5/9$

Find $P(Y \geq 1) = ?$

$$P(X \geq 1) = 5/9$$

$$1 - P(X = 0) = 5/9$$

$$1 - (1-p)^n = 5/9$$

$$(1-p)^n = 4/9$$

$$q^n = 4/9$$

Give $n_x = 2$

$$q^2 = (1-p)^2 = 4/9$$

$$1 - p = 2/3$$

$$p = 1/3$$

For $Y \sim B(4, 1/3)$

$$P(Y \geq 1) = 1 - P(Y = 0)$$

$$= 1 - (q)^4$$

$$= 1 - \left(\frac{2}{3}\right)^4 = 1 - \frac{16}{81} = \frac{65}{81}$$

Q, 2 dies are rolled 120 times. Find the average no: of times in which the number on the first die exceeds the no on the second die.

$n = 120$

$p = ?$

For finding p,

no: on first die exceeds the second die,

~~(2,1) (3,1) (~~

equal case $\rightarrow 6/36$, remaining $\frac{30}{36}$, Half will be first die > second die.

$\therefore \quad p = 15/36$.

$\therefore$ Average = Mean = $E[X] = np = 120 \times \frac{15}{36} = \underline{\underline{50}}$

Q, If x is a binomial r.v and $E(X) = 4$

$V(X) = 4/3$

Find (i) $P(x \leq 2)$

(ii) comment on β_1

i) $P(X \leq 2)$

$E(x) = 4$

$np = 4$

$npq = \frac{4}{3}$

$q = 1/3$ $\quad p = 2/3$ $\quad \frac{2n}{3} = 4$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \left(\frac{1}{3}\right)^6 + {}^6C_1 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + {}^6C_2 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{3^6}\left[1 + 12 + 60\right] = \underline{\underline{73/729}}$$

(ii) Given Data is $-$vely skewed since p value is more than $\frac{1}{2}$. $\quad p = 2/3 > 1/2$.

Q, If X is a binomial random variable, then find the value of

$$\sum_{x=0}^{n} \left(\frac{x}{n}\right) {}^nC_x \, p^x q^{n-x}$$

$$= \frac{1}{n} \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$$= \frac{1}{n} \boxed{\sum_{x=0}^{n} x \; p(x)}$$

$$= \frac{1}{n} \times \text{Mean}.$$

$$= \frac{1}{\cancel{n}} \times \cancel{n}p$$

$$= \underline{\underline{p}}$$

POISSON DISTRIBUTION

Probability function is given by

$$\boxed{P(X ; \lambda>0) = p(x) = \begin{cases} \dfrac{e^{-\lambda} \lambda^{x}}{x!} , & \lambda>0 \\ & 0 \leq x < \infty \\ 0 , & \text{otherwise.} \end{cases}}$$

→ It is used when observations are HIGH and success Probability is Low.

→ It is Used to Find RARE OCCURANCES.

→ poisson's equation is <u>time dependent</u> distribution.

→ ie, it is a Evolutionary Process.

Used to find Defect Probability.

Used to find Arrival Rate.

<u>finition</u>

f X is said to be poisson r.v defined in the interval $0 < x < \infty$ with a parameter λ ($\lambda>0$) and ts probability mass function is

$$P(X ; \lambda>0) = p(x) = \begin{cases} \dfrac{e^{-\lambda} \lambda^{x}}{x!} & \lambda>0 \\ & 0 \leq x < \infty \\ 0 & \text{otherwice.} \end{cases}$$

Conditions

- observations are infinitely large, $(n \to \infty)$
- probability of success is very small $(p \to 0)$
- $np = \lambda \Rightarrow p = \frac{\lambda}{n}$

Then $$P(x : n, p) = \frac{e^{-np} (np)^x}{x!}$$

It is approximation of binomial.

POISSON PROCESS

$$\boxed{P(x ; \lambda, t > 0) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}}$$

PROPERTIES

1. $E(X) = \text{Mean} = \lambda$

$V(X) = \mu_2 = \lambda$

$\mu_3 = \lambda$

$\lambda > 0$

$\therefore \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{1}{\lambda}$ ie, Poisson's distribution is always positively skewed.

It can never by symmetry

= positively skewed

2. MGF

$$M_X(t) = E\left[e^{tX}\right] = e^{\lambda(e^t - 1)}$$

3. Characteristic function.

$$\phi_x(t) = E\left[e^{itx}\right] = e^{\lambda(e^{t} - 1)}$$

te:

In ~~possi~~ poisson's ~~equ~~ distribution.

Mean = Variance = parameter λ.

It is always +vely skewed.

Sum of the independent poisson r.v's is also a oisson's r.v.

e Difference b/w the independent poisson's r.v's is not a poisson random variable.

Q. A telephone switch board recieves 20 calls on an avg during an hour. Find the probability for a period of 5 min,

(i) No call is recieved.

(ii) Exactly 3 calls are recieved.

(iii) Atleast 2 calls are recieved.

reach
arrive
error
defect } go for poisson's distributn.

Soln

Q avg = λ (for 60 min)

For 60 min $\rightarrow \lambda = 20$ calls

For 1 min $\rightarrow \frac{20}{60} = \frac{1}{3} = \lambda$

For 5min = $\frac{1}{3} \times 5 = 1.65 = \lambda$

(i) $P(x=0) = \frac{e^{-1.65}(1.65)}{0!}$

$= e^{-1.65}$

(ii) $P(x=3) = \frac{e^{-1.65}(1.65)^3}{3!}$

(iii) $P(x \geq 2) = 1 - P(X<2)$

$= 1 - [P(x=0) + P(x=1)] = 1 - \left[e^{-1.65} + \frac{1.65\, e^{-1.65}}{1!} \right]$

2, If X is a poisson r.v, then Find the Value of

$$\sum_{x=0}^{\infty}\left(\frac{x}{\lambda}\right)\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\cancel{\frac{1}{\lambda} \times \lambda =}$$

$$\frac{1}{\lambda}\sum_{x=0}^{\infty} x\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)$$

$$= \frac{1}{\lambda} \times E(X)$$

$$= \frac{1}{\lambda} \times \lambda = \underline{\underline{1}}$$

NORMAL DISTRIBUTION (GAUSSIAN)

$$\boxed{N(x;\mu,\sigma^2) = f(x) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & -\infty < x < \infty \\ & -\infty < \mu < \infty \\ & 0 < \sigma < \infty \\ 0, \text{ otherwise.} \end{cases}}$$

Definition: If X is said to be a normal r.v defined in the interval $-\infty < x < \infty$ with mean equal to μ and variance is equal to σ^2, Then the r.v is known as normal r.v. And its density function is

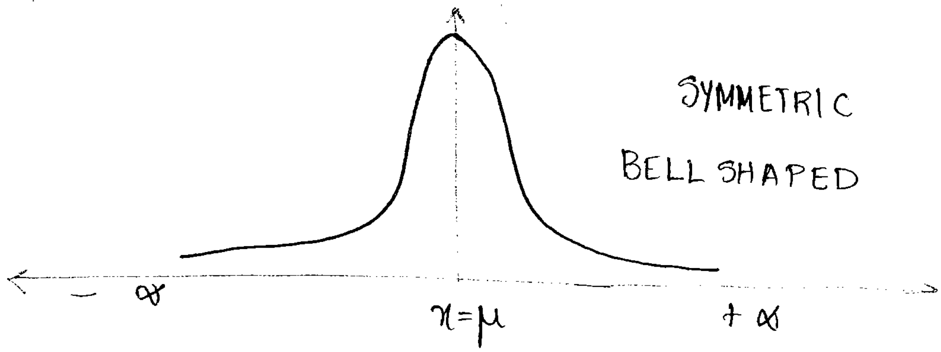
$$N(x;\mu,\sigma^2) = f(x) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & -\infty < x < \infty \\ & -\infty < \mu < \infty \\ & 0 < \sigma < \infty \\ 0, \text{ otherwise.} \end{cases}$$

STANDARD NORMAL RANDOM VARIABLE

If X is a normal random variable with mean = 0 and variance = 1, then the random variable is known as standard normal random variable. It density function is

$$\boxed{N(0,1) = f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}}$$

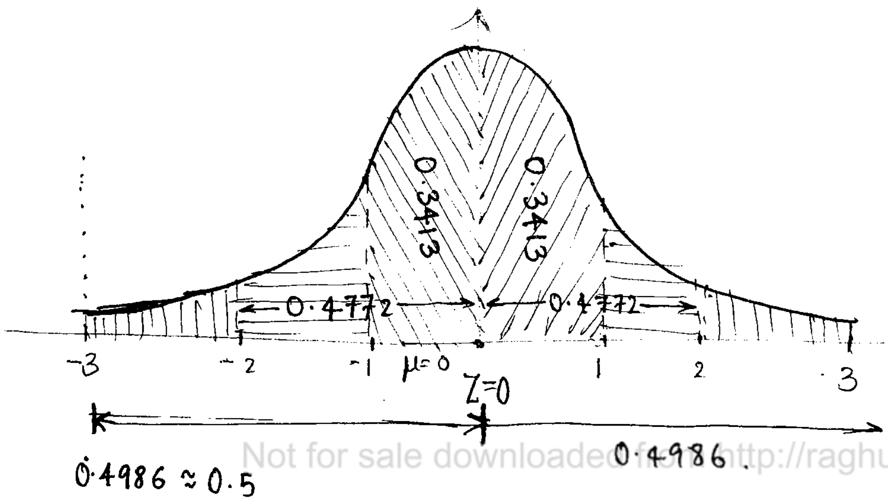
Mathematically a standard normal r.v is denoted by Z and is equal to $Z = \dfrac{X - E(X)}{\sqrt{V(X)}}$ & $\boxed{-3 \leq Z \leq +3}$



SYMMETRIC
BELL SHAPED

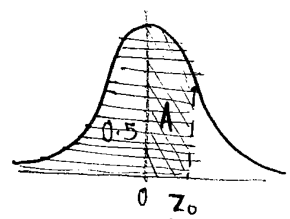
Normal distribution.

Standard Normal Distribution

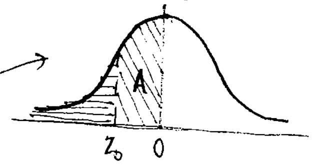


Areas under Normal curve

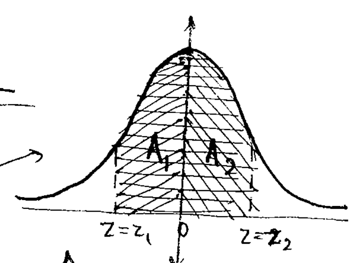
$P(Z \leq Z_0) = 0.5 + A \quad (Z_0 \text{ +ve})$.



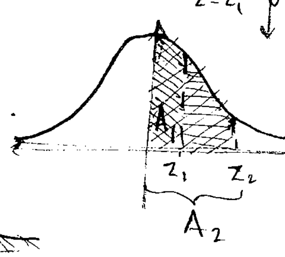
$P(Z \leq Z_0) = 0.5 - A \quad (Z_0 \text{ -ve})$.



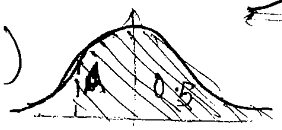
$(Z_1 \leq Z \leq Z_2) = A_1 + A_2 \quad (Z_1 \text{ -ve and } Z_2 \text{ +ve})$



$(Z_1 \leq Z \leq Z_2) = A_2 - A_2 \quad (Z_1 \text{ and } Z_2 \text{ +ve/-ve})$.



$(Z \geq Z_0) = 0.5 + A \quad (Z_0 \text{ -ve})$



$(\cancel{Z \geq Z_0}) = \cancel{0.5 - A} \quad (Z_0 \text{ +ve})$

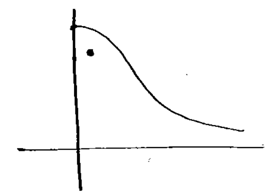
Q, If X is normally distributed with mean = 20 and std deviation 3.33. Find the probability b/w

$P(21.11 \leq x \leq 26.66)$. The area under the curve,

$Z_0 = 0$ to $Z = 0.33$ is 0.1293.

$$Z_1 = \frac{X_1 - \mu}{\sigma} \qquad Z_2 = \frac{X_2 - \mu}{\sigma}$$

$$Z_1 = \frac{1}{3} = 0.33 \qquad Z_2 = 2$$



$$P(21.11 \leq x \leq 26.66) = P(0.33 < Z < 2)$$

$$= 0.4772 - 0.1293$$

$$= \underline{\underline{0.3479}}$$

Q, If X is normally distributed with mean = 30 and std deviation is 5. Find $P(|x-30| > 5)$

$P(|X-30| > 5) = 1 - P(|x-30| < 5)$

$$\Rightarrow -5 < x - 30 < 5$$

$$P(25 < x < 35)$$

$$Z_1 = \frac{25 - 30}{5} = \frac{-5}{5} = \underline{\underline{-1}}$$

$$Z_2 = \frac{35 - 30}{5} = \frac{5}{5} = \underline{\underline{1}}$$

$$P(-1 < Z < 1) = A_1 + A_2 = \underline{\underline{0.6826}}$$

2. A die is rolled 180 times. Using the normal distribution, find the probability that the face 4 will turn up atleast 35 times.

$$P(X \geq 35) = ?$$

We can use Binomial Distribution with $n = 180$, $p = 1/6$, $q = 5/6$.

$$E(X) = np = \frac{180}{6} = 30$$

$$V(X) = npq = \frac{180}{6} \times \frac{5}{6} = 25$$

$$Z = \frac{X - \mu}{\sigma} = \frac{35 - 30}{5}$$

$$= 1$$

$$P(Z \geq 1) = 0.5 - 0.3413$$

$$= 0.1587$$

Properties

$E(X) = \text{Mean} = \mu$

$V(X) = \text{Variance} = \mu_2 = \sigma^2$

$\mu_3 = 0 \Rightarrow \beta_1 = 0 \Rightarrow$ Symmetry.

MGF

$$M_X(t) = e^{\left(t\mu + \frac{t^2\sigma^2}{2}\right)}$$

(iv) characteristic

$$\phi_X(t) = e^{\left(it\mu - \frac{t^2\sigma^2}{2}\right)}$$

Properties of std Normal distribution.

→ $X \sim N(0,1)$

→ $E(X) = 0$

→ $V(X) = 1$

→ $M_X(t) = e^{t^2/2}$

→ $\phi_X(t) = e^{-t^2/2}$

→ Sum of the independent normal r.variable is also a normal random variable.

→ The difference the independent normal r.v s is also a normal random variable (Linear combination).

UNIFORM DISTRIBUTION [RECTANGULAR]

Definition: If X is a uniform r.v in the interval $a < x < b$ $(a<b)$ and it probability density function is

$$\boxed{f(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ & a<b \\ 0 & \text{otherwise} \end{cases}}$$

MEAN $= E(X) = \dfrac{a+b}{2}$

VARIANCE $V(X) = \dfrac{(b-a)^2}{12}$

If X is uniform random variable in the interval $-a < x < a$ and its density function is

$$\boxed{f(x) = \frac{1}{2a}}$$

$$\boxed{\text{Mean} = 0}$$

$$\boxed{\text{Variance} = \frac{a^2}{3}}$$

The shape of uniform curve is rectangular.

CORRELATION / REGRESSION

A $= 0.8461$
B $= 1.1154$

CORRELATION

Karl pearson's Correlation.

The relation b/w the two dimensional r.v in bivariate data is known as regression (The degree of relation b/w the two variables is known as ~~regression~~ correlation.)

Types of correlation (i) Positive Correlation.

(ii) Negative Correlation.

Positive Correlation: If the changes in the both the variables are in the same direction (increasing or decreasing) Then those variables are known as positively correlated variables.

Negative Correlation: If the changes in the one variable is affecting the changes of other variable in reverse direction, then those variables are known as negatively correlated.

Karl pearson's Correlation equ.

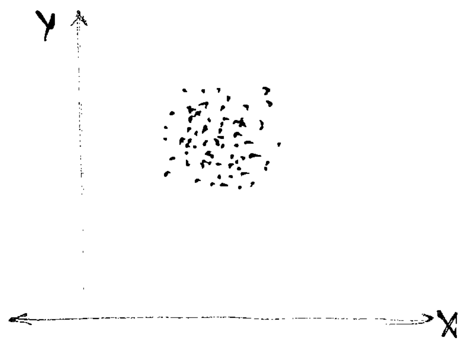
$$r(x,y) = \frac{Cov(x,y)}{\sqrt{x}\,\sqrt{y}} \quad \text{where } Cov(x,y) = \frac{1}{n}\sum xy - \bar{x}\cdot\bar{y}$$

$-1 \leq r \leq 1$

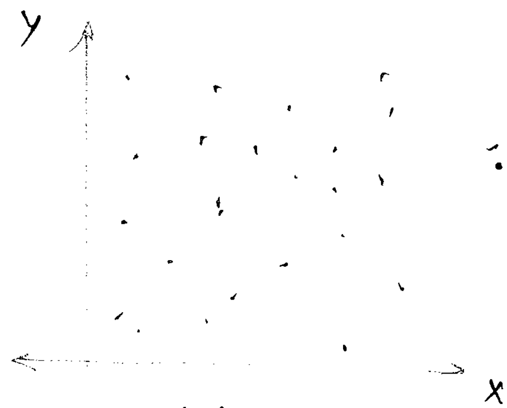
SCATTER DIAGRAM

It is a graphical representation of correlation. If the points are very close or thick on the XY plane then those points are correlated points.

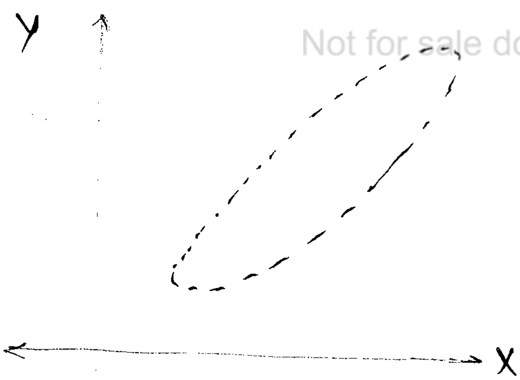
If the points are widely spreaded, then they are said to be uncorrelated.



CORRELATED

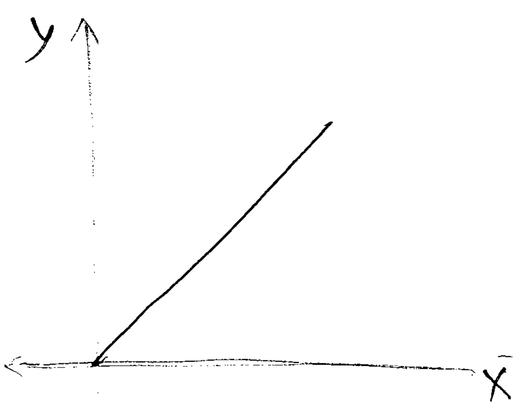


UNCORRELATED

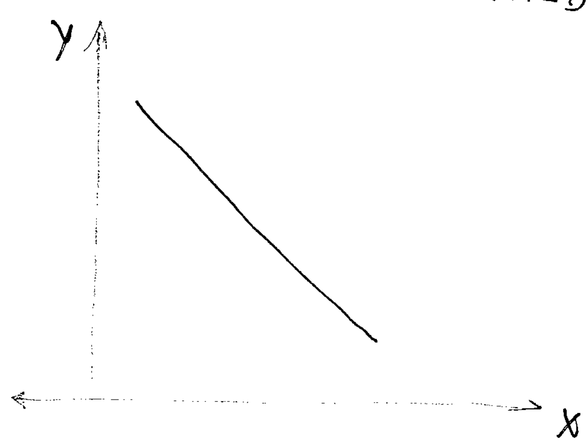


POSITIVELY CORRELATED

NEGATIVELY CORRELATED



PERFECT POSITIVELY



PERFECT NEGATIVELY

Note: $\Rightarrow$ If X & Y are independent r.v's then covariance ~~of~~ is zero.

ie, $\boxed{Cov(X,Y) = 0 \implies r(X,Y) = 0}$

But converse of the statement is not true.

$\Rightarrow$ Correlation coefficient is independent of origin as well independent of change of scale.

REGRESSION $\rightarrow$ Y ON X, X ON Y (Simple reggression)

Definition: The linear relationship b/w the 2-D random variable is known as regression.

LINES OF REGRESSION

$$Y - \bar{Y} = r \cdot \frac{\sigma_y}{\sigma_x} (X - \bar{X}) \qquad \text{Y ON X}$$

$$X - \bar{X} = r \cdot \frac{\sigma_x}{\sigma_y} (Y - \bar{Y}) \qquad \text{X ON Y}$$

$$\left.\begin{aligned} \text{where } r \cdot \frac{\sigma_y}{\sigma_x} &= b_{YX} \\ \& \ r \cdot \frac{\sigma_x}{\sigma_y} &= b_{XY} \end{aligned}\right\} \text{Regression Coefficient.}$$

→ Correlation coefficient is the Geometric mean b/w regression coefficients.

$$\boxed{r = \pm \sqrt{b_{yx}\, b_{xy}}}$$

Note: Both the regression coefficients must have a same sign.

ie, If both +ve $\Longrightarrow$ r is +ve.
If both −ve $\Longrightarrow$ r is −ve

If $b_{yx} > 1 \Rightarrow b_{xy} < 1$ (vice versa).

If the regression coefficients are equal, their variances also equal.

$$\boxed{b_{xy} = b_{yx}} \Rightarrow r \cdot \frac{\sigma_x}{\sigma_y} = r \cdot \frac{\sigma_y}{\sigma_x}$$

$$\Rightarrow \boxed{\sigma_y^2 = \sigma_x^2}$$

Regression equations are passes through the point $\bar{X}, \bar{Y}$

Regression Coefficient is independent of change of origin as well as dependent of change of scale.

Angle b/w Regression lines

$$\boxed{\theta = \tan^{-1}\left(\frac{1-r^2}{|r|} \cdot \frac{\sigma_x \sigma_y}{(\sigma_x)^2+(\sigma_y)^2}\right)}$$

$r = 0 \Rightarrow \theta = \pi/2$

$r = 1 \Rightarrow \theta = 0$ or π

Q, The regression equation are $3x + 2y = 1$.
$2x + 4y = 0$

(i) to Find r?

(ii) $\bar{X}, \bar{Y}$

whichever the coefficient in the expression is higher then it is the DEPENDENT VARIABLE.

consider $\uparrow 3x + 2y = 1$

Dependant variable

$\therefore$ X on Y

X on Y	Y on X
$3x+2y=1$	$2x+4y=0$
$3x = 1-2y$	$4y = -2x$
$x = 1/3 - 2/3\, y$	$y = -2/4\, x$
$b_{xy} = -2/3$	$b_{yx} = -1/2$

— Both –ve Hence r also –ve ,

$$r = \sqrt{\cdot 2/3 \times 1/2}$$

$$r = -\sqrt{1/3}$$

$$r = -1/\sqrt{3}$$

(ii) Means $\bar{x}$ & $\bar{y}$ satisfies eqn.

$$6\bar{x} + 4\bar{y} = 2$$

$$2\bar{x} + 4\bar{y} = 0$$

$$4\bar{x} = 2$$

$$\bar{x} = 1/2$$

$$\bar{y} = -1/4$$

$\therefore$ Means $(\bar{x}, \bar{y}) = (1/2, -1/4)$

$x - 2y = 2$

$3x - y = 1$

(i) r (ii) $\bar{x}, \bar{y}$

Y on X

$x - 2y = 2$

$x - 2 = 2y$

X on Y

$3x = 1 + y$

$x = 1/3 + y/3$

$$r = \sqrt{1/2 \times 1/3}$$

$$r = \frac{1}{\sqrt{6}}$$

(ii) $\bar{x} - 2\bar{y} = 2$

$3\bar{x} - \bar{y} = 1$

$3\bar{x} - 6\bar{y} = 6$

$3\bar{x} - \bar{y} = 1$

$-5\bar{y} = 5$

$\bar{y} = -1$

$3\bar{x} + \cancel{1} = \cancel{1}$

$\bar{x} = 0$

$(\bar{x}, \bar{y}) = (0, -1)$

25/08/10

Transformation Theory:

Laplace Transform:

$f(t) \ \forall t$

$$I(f(t)) = \int_{-\infty}^{\infty} K(s,t)\, f(t)\, dt = \bar{f}(s)$$

s may be real (or) Complex

[Bilateral L.T.]

$K(s,t)$ kernel of the Integral Transform

(1) $K(s,t) = e^{-st} \longrightarrow$ L.T

(2) $K(s,t) = e^{ist}$ (or) $e^{-ist} \longrightarrow$ F.T

$K(s,t) = e^{-st}$

(1) $$L\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} \cdot f(t)\, dt$$

↳ Bilateral L.T

$e^{-st} \rightarrow$ kernel of the L.T

$\rightarrow \quad f(t) \quad \forall t \geq 0$

$$L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)\, dt$$

unilateral L.T

Condition for Existency:

If $\int_0^\infty |e^{-st} f(t)\,dt| < \infty$ i.e Convergent integral. (7)

Sufficient Conditions for Existency: (8)

(1) $f(t)$ is piecewise (Sectionally) continous.

(2) $f(t)$ is a function of Exponential order i.e

$$\lim_{t \to \infty} e^{-st} f(t) \longrightarrow \text{finite (or) zero.}$$

Ga

Class A function if it satisfies both (1) & (2) Conditions. (1)

(2)

Linearity property

$$L\{a f(t) \pm b g(t)\} = a\, L\{f(t)\} \pm b L\{g(t)\}$$ (3)

(4)

Laplace Transform of some Elementary functions: (5)

(1) $L\{1\} = 1/s \quad ; s > 0$ (6)

(2) $L\{e^{at}\} = \dfrac{1}{s-a} \quad ; s > a$

(3) $L\{e^{-at}\} = \dfrac{1}{s+a} \quad ; s > -a$

(4) $L\{\sin at\} = \dfrac{a}{s^2+a^2} \quad ; s > 0$

(5) $L\{\cos at\} = \dfrac{s}{s^2+a^2} \quad ; s > 0$

(6) $L\{\sinh at\} = \frac{a}{s^2-a^2}$; $s > |a|$

(7) $L\{\cosh at\} = \frac{s}{s^2-a^2}$; $s > |a|$

(8) $L[t^n] = \frac{n!}{s^{n+1}}$; $n \in Z^+$

$\frac{\Gamma(n+1)}{s^{n+1}}$; $n \notin Z^+$.

onditions.

Gamma function

(1) $\Gamma(n+1) = n\Gamma n$

(2) $\Gamma(n+1) = n(n-1)(n-2)(n-3)\ldots$ if n is +ve rational number

(3) $\Gamma(n+1) = n!$ if $n \in Z^+$

(4) $\Gamma 1 = 1$ & $\Gamma\frac{1}{2} = \sqrt{\pi}$

(5) $\Gamma n = \frac{\Gamma(n+1)}{n}$ if 'n' is a -ve rational number

(6) $\Gamma 0, \Gamma{-1}, \Gamma{-2} \ldots$ are undefined.

Cont.. first shifting theorem of L.T :

If $L\{f(t)\} = \bar{F}(s)$

$\Rightarrow$ (1) $L\{e^{at} f(t)\} = \bar{F}(s-a)$

(2) $L\{e^{-at} f(t)\} = \bar{F}(s+a)$

Second shifting theorem of L.T :

If $L\{f(t)\} = \bar{F}(s)$ and $G(t) = \begin{cases} f(t-a) & t \geq a \\ 0 & t < a \end{cases}$

then $L\{G(t)\} = e^{-as} \cdot L\{f(t)\}$

(or)

$e^{-as} \cdot \bar{F}(s)$

Change of Scaling :

$L\{f(t)\} = \bar{F}(s) \Rightarrow L\{f(at)\} = \frac{1}{a} \bar{F}(s/a)$

$L\{f(t/a)\} = a \bar{F}(as)$

multiplication by t^n ($n \in Z^+$) :

(1) $L\{t\ f(t)\} = -\frac{d}{ds}\{L\{f(t)\}\}$

(2) $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[L\{f(t)\}]$

Division by t^n ($n \in z^+$):

① $$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L\{f(t)\}\, ds$$

② $$L\left\{\frac{f(t)}{t^n}\right\} = \int_s^\infty L\{f(t)\}\, (ds)^n$$

L.T of derivatives:

① $$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

② $$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0)$$

L.T of integrals:

$$L\left\{\int_0^t f(t)\, dt\right\} = \frac{L\{f(t)\}}{s}$$

$$L\left\{\int_0^t f(t)(dt)^n\right\} = \frac{L\{f(t)\}}{s^n}$$

L.T of periodic fun

If a function is a periodic fun in the period of T. i.e $f(t+T) = f(t)$ $\forall t$

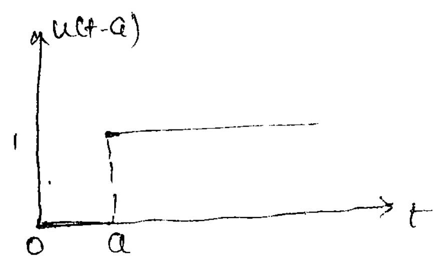
$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}$$

Unit Step fun | Heavisides fn:

⑥

$u(t-a)$ (or) $H(t-a)$ (or) $u_a(t)$

$$= \begin{cases} 1 & ; t \geq a \\ 0 & ; t < a \end{cases} \quad a \geq 0$$



uni

① $L\{u(t-a)\} = \frac{e^{-as}}{s}$

② $L\{u(t)\} = \frac{1}{s}$

③ $L\{f(t-a)\,u(t-a)\} = e^{-as} \cdot L\{f(t)\}$

* ④ $L\{f(t)\,u(t-a)\} = e^{-as} \cdot L\{f(t+a)\}$

⑤ $f(t) = \begin{cases} f_1(t) & 0 < t < a \\ f_2(t) & t \geq a \end{cases}$

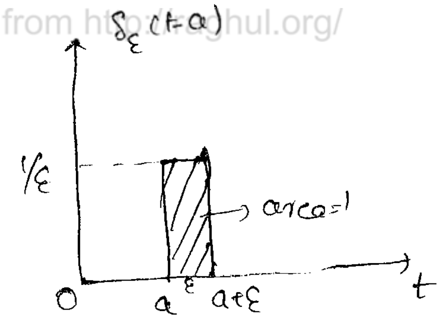
then $f(t) = f_1(t) + \{f_2(t) - f_1(t)\}\,u(t-a)$

(6) $f(t) = \begin{cases} f_1(t) & 0 < t < a_1 \\ f_2(t) & a_1 \le t < a_2 \\ f_3(t) & a_2 \le t < a_3 \\ \cdots & \cdots \\ f_n(t) & t \ge a_{n-1} \end{cases}$ then

$$f(t) = f_1(t) + \{f_2 - f_1\} u(t-a_1) + \{f_3 - f_2\} u(t-a_2) + \cdots + [f_n - f_{n-1}] u[t-a_{n-1}]$$

unit impulse fn / Dirac delta fun

force applied in a short time is called impulse



$$\delta(t-a) = \lim_{\varepsilon \to 0} \{ \delta_\varepsilon(t-a)$$

$$= \lim_{\varepsilon \to 0} \begin{cases} 1/\varepsilon & a \le t \le a+\varepsilon \\ 0 & \text{elsewhere} \end{cases}$$

$u(t-a)$

$$= \begin{cases} \infty & t = a \\ 0 & t \ne a \end{cases}$$ such that $\int_0^\infty \delta(t-a)\,dt = 1$

$L\{\delta(t-a)\} = e^{-as}$

$L\{\delta(t)\} = 1$

$L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$

$L\{u'(t-a)\} = L\{\delta(t-a)\}$

$\int_{-\infty}^{\infty} f(t)\,\delta'(t-a)\,dt = -f'(a)$ (4) L

1) $L\{t^{7/2} + t^{-5/2} + t^3\}$

$$= \frac{\Gamma(7/2+1)}{s^{9/2}} + \frac{\Gamma(-5/2+1)}{s^{-3/2}} + \frac{3!}{s^4}$$

$$= \frac{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\Gamma(1/2)}{s^{9/2}} + \frac{\Gamma(-3/2)}{s^{-3/2}} + \frac{6}{s^4}$$ (5) L

$$= \frac{105\sqrt{\pi}}{s^{9/2}} + \frac{4/3\sqrt{\pi}}{s^{-3/2}} + \frac{6}{s^4} \quad \left[\because \Gamma(-3/2) = \frac{\Gamma(-3/2+1)}{-3/2} = \frac{\Gamma(-1/2+1)}{-3/2\times -1/2} = \frac{4}{3}\sqrt{\pi}\right]$$

2) $L\{1/t^2\}$ → ~~Dean~~ Does not exist (6) L

3) $L\{(\sqrt{t} - \frac{1}{\sqrt{t}})^2\}$

$= L\{t + 1/t - 2\}$

Does not exist

(4) $L\left\{\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^3\right\}$

$$L\left\{t^{3/2}-t^{-3/2}+3t^{-1/2}-3t^{1/2}\right\}$$

$$\frac{\Gamma\left(3/2+1\right)}{s^{5/2}}-\frac{\Gamma\left(-3/2+1\right)}{s^{-1/2}}+3\frac{\Gamma\left(-1/2+1\right)}{s^{1/2}}-3\frac{\Gamma\left(1/2+1\right)}{s^{3/2}}$$

$$\frac{3/2\times\frac{1}{2}\sqrt{\pi}}{s^{5/2}}-\frac{\Gamma\left(-1/2\right)}{s^{-1/2}}+3\frac{\Gamma\left(-1/2+1\right)}{s^{1/2}}-3\frac{\Gamma\left(1/2+1\right)}{s^{3/2}}$$

$$\frac{3/4\sqrt{\pi}}{s^{5/2}}+\frac{2\sqrt{\pi}}{s^{-1/2}}+\frac{3\sqrt{\pi}}{s^{1/2}}-\frac{3/2\sqrt{\pi}}{s^{3/2}}$$

$$\left[\because \Gamma\left(-1/2\right)=\frac{\Gamma\left(-1/2+1\right)}{-1/2}=-2\sqrt{\pi}\right]$$

(5) $L\left\{a^t\right\}$

$$L\left\{e^{\log a^t}\right\}$$

$$e^{\log x}=x$$

$$\log e^x=x$$

$$=L\left\{e^{t(\log a)}\right\}$$

$$=\frac{1}{s-\log a}$$

$$\Gamma\left(-3/2\right)=\frac{\Gamma\left(-3/2+1\right)}{-3/2}$$

$$\left[\frac{\Gamma\left(-1/2+1\right)}{-3/2\times-1/2}=\frac{4}{3}\sqrt{\pi}\right]$$

(6) $L\left\{\sin 2t\cos 3t\right\}$

$$\frac{1}{2}L\left\{2\sin 2t\cos 3t\right\}$$

$$\frac{1}{2}L\left\{\sin 5t-\sin t\right\}$$

$$\frac{1}{2}\left[\frac{5}{s^2+25}-\frac{1}{s^2+1}\right]$$

(7) $L\{\cos t \cos 2t \cos 3t\}$

$\frac{1}{2} L\{2\cos t \cos 2t \cos 3t\}$

$\frac{1}{2} L\{(\cos 3t + \cos t)\cos 3t\}$

$\frac{1}{4} L\{2\cos 3t \cos 3t + 2\cos t \cos 3t\}$

$\frac{1}{4} L[\cos 6t + \cos 0 + \cos 4t + \cos 2t\}$

$\frac{1}{4} L[1 + \cos 6t + \cos 4t + \cos 2t]$

$\frac{1}{4}\left[\frac{1}{s} + \frac{6}{s^2+36} + \frac{4}{s^2+16} + \frac{2}{s^2+4}\right\}$

(10) L {

(11) L

(8) $L\{\sin^3 2t\}$

$\cos 3t = 4\cos^3 t - 3\cos t$

$\sin 3t = 3\sin t - 4\sin^3 t$

$L\left\{\frac{3\sin 2t - \sin 6t}{4}\right)$

(12) L

$\frac{1}{4}\left[\frac{3\times 2}{s^2+4} - \frac{6}{s^2+36}\right]$

$= \frac{6}{4}\left[\frac{32}{(s^2+4)(s^2+36)}\right]$

(9) $L\{\sinh^3 2t\}$

$\sinh 3t = 3\sinh t + 4\sinh^3 t$

$\cosh 3t = 4\cosh^3 t - 3\cosh t$

$L\left\{\frac{\sinh 6t - 3\sinh 2t}{4}\right\}$

$$\frac{1}{4}\left[\frac{6}{s^2-36}-\frac{3\times 2}{s^2-4}\right]$$

$$\frac{1}{4}\left(\frac{6}{s^2-36}\right)-\frac{3}{4}\left(\frac{2}{s^2-4}\right)$$

(10) $L\{(\sin t-\cos t)^2\}$

$$L[1-\sin 2t]$$

$$\frac{1}{s}-\frac{2}{s^2+4}$$

(11) $L[\sin(\omega t\pm\alpha)]$

$$=L[\sin\omega t\cos\alpha\pm\cos\omega t\sin\alpha]$$

$$=\frac{(\cos\alpha)\omega\pm(\sin\alpha)\cdot\omega}{(s^2+\omega^2)}$$

st

$\sin^3 t$

(12) $L\{\sinh at\sin at\}$

$$L\left[\frac{e^{at}-e^{-at}}{2}\sin at\right]$$

$$\frac{1}{2}L\{e^{at}\sin at-e^{-at}\sin at\}$$

(First shi. Theorem)

$$\frac{1}{2}\left[\frac{a}{(s-a)^2+a^2}-\frac{a}{(s+a)^2+a^2}\right]$$

$\sinh t+4\sinh^3 t$

$\sinh^3 t-3\cosh t$

$$=\frac{a}{2}\left[\frac{4as}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{2a^2 s}{s^4 + 4a^4}$$

$$= \frac{2a^2 s}{(s^2+2a^2)^2 - (2as)^2}$$

(13) $L\{\sin\sqrt{t}\}$

$e^{\log \sin\sqrt{t}}$

$\frac{1}{2}\log$

(15)

Sol

$$= L\left\{ t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5/2!} - \cdots \right\} \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \frac{\Gamma(1/2+1)}{s^{3/2}} - \frac{\Gamma(3/2+1)}{3!\, s^{5/2}} + \frac{\Gamma(5/2+1)}{5!\, s^{7/2}} - \cdots$$

$$= \frac{1/2\sqrt{\pi}}{s^{3/2}} - \frac{3/2 \times 1/2 \times \sqrt{\pi}}{3!\, s^{5/2}} + \frac{5/2 \times 3/2 \times 1/2\sqrt{\pi}}{5!\, s^{7/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \frac{1}{3!(4s)^3} + \cdots \right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}.$$

(16)

(14) $L\{e^{t^3}\}$

$\frac{\Gamma(3+1)}{3}$

Sol

$$= L\left\{ 1 + t^3 + \frac{t^6}{2!} + \frac{t^9}{3!} + \cdots \right\}$$

$$= \frac{1}{s} + \frac{3!}{s^4} + \frac{6!}{2!\, s^7} + \frac{9!}{3!\, s^{10}} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(3n)!}{n!\, s^{3n+1}}$$

ratio test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \begin{cases} l > 1 & \text{cgt} \\ l < 1 & \text{dgt} \\ l = 1 & \text{fails} \end{cases}$$

= Divergent

$L\{e^{t^3}\} \rightarrow$ ~~does~~ not Exist.

in $\sqrt{t}$

$\frac{1}{2}$ ~~log~~

(15) $L\{f(t)\} = \frac{e^{-1/s}}{s}$ $L\left\{\int_0^t f(3t)\, e^{-3t}\, dt\right\}$

Sol

Scaling

$$L\{f(3t)\} = \tfrac{1}{3} \cdot \bar{f}(s/3) = \tfrac{1}{3} \frac{e^{-1/s/3}}{s/3}$$

F.S. Th

$$L\left[e^{-3t} f(3t)\right] = \frac{e^{-3/s+3}}{s(s+3)}$$

Integral

$$L\left\{\int_0^t e^{-3t} f(3t)\, dt\right\} = \frac{e^{-3/s+3}}{s(s+3)}$$

(16) $f(t) = \begin{cases} \cos(t - \pi/3) & t \geq \pi/3 \\ 0 & t < \pi/3 \end{cases}$ $L\{f(t)\} = ?$

Sol

$$\frac{e^{-\pi/3\, s}\, s}{s^2+1}$$

$e^{-\pi/3\, s}\ \underline{s}$

(17) $f(t) = \begin{cases} (t-2)^2 & t \geq 2 \\ 0 & t < 2 \end{cases}$ $L\{f(t)\}$

$$e^{-2s} \frac{2!}{s^3}$$

(18) $f(t) = \begin{cases} t^2 & t \geq 2 \\ 0 & t < 2 \end{cases}$ $L\{f(t)\}$

$$L\{t^2 u(t-2)\} = e^{-2s} L\{(t+2)^2\}$$

$$= e^{-2s} \cancel{e^{2s}} \cdot \frac{2!}{\cancel{s^3}} L\{t^2+2t+4\}$$

$$= e^{-2s}\left[\frac{2!}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$$

(21) L

(19) $L\{4\sin(t-3)u(t-3)\}$

$$= e^{-3s} \cdot L\{4\sin(t+3-3)\}$$

$$= e^{-3s} \cdot \frac{4}{s^2+1}$$

$$L\{f(t)u(t-a)\} = e^{-as} L\{f(t+a)\}$$

(20) $L\{t^2 \sin at\}$

(21) L [

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} L\{f(t)\}$$

$$= (-1)^2 \frac{d^2}{ds^2} L\{\sin at\}$$

$$= \frac{d^2}{ds^2}\left\{\frac{a}{s^2+a^2}\right\} = \frac{d}{ds}\left[\frac{a(-2s)}{(s^2+a^2)^2}\right]$$

$$= -2a \frac{d}{ds}\left(\frac{s}{(s^2+a^2)^2}\right)$$

$$= -2a \frac{(s^2+a^2)^2 \cdot 1 - s\, 2(s^2+a^2)\cdot 2s}{(s^2+a^2)^4}$$

$$= -2a\left[\frac{(s^2+a^2)^2 - 4s^2(s^2+a^2)}{(s^2+a^2)^4}\right]$$

$$= -2a\,(\cancel{s^2+a^2})\left[\frac{s^2+a^2-4s^2}{(s^2+a^2)^{\cancel{4}\,3}}\right]$$

$$= -2a\left(\frac{a^2-3s^2}{(s^2+a^2)^3}\right)$$

(21) $L\left[t^6 e^{-3t}\right]$

$$(-1)^6 \frac{d^{\cancel{6}}}{ds^{\cancel{6}}}\left(e^{-3t}\right)$$

$$L(t^6) = \frac{6!}{s^7}$$

$$L\left(t^6 e^{-3t}\right) = \frac{6!}{(s+3)^7}$$

(22) $L\left(t e^{-2t}\cos 2t\right)$

$$L(t\cos 2t) = (-1)\frac{d}{ds} L[\cos 2t]$$

$$= (-1)\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= (-1)\frac{(s^2+4) - 2s^2}{(s^2+4)^2}$$

$$= -\frac{(4-s^2)}{(s^2+4)^2}$$

$= e^{-as}$

$L\{f(t+a)\}$

$$L\left[e^{-2t}\,t\cos 2t\right] = \frac{-(4-s^2)}{((s+2)^2+4)^2}$$

$$= \frac{-(4-(s+2)^2)}{((s+2)^2+4)^2}$$

$$= \frac{s^2-4s}{(s^2+4s+8)^2}$$

(23) $L\left[\frac{t^{n-1}}{1-e^{-t}}\right]$

$$L\left[t^{n-1}(1-e^{-t})^{-1}\right]$$

$$L\left[t^{n-1}(1+e^{-t}+e^{-2t}+e^{-3t}+\cdots)\right]$$

$$= L\left[t^{n-1}\sum_{k=0}^{\infty} e^{-kt}\right]$$

$$= L\left[\sum_{k=0}^{\infty} e^{-kt}\,t^{n-1}\right] \qquad \text{F.S.Th}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma n}{(s+k)^n}$$

(24) $L\left[\frac{\sin at}{t}\right]$

$$\int_s^{\infty} L\{\sin at\}\,ds$$

$$\int_s^{\infty} \frac{a}{s^2+a^2}\,ds = \left[\tan^{-1}(s/a)\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}(s/a) \text{ (or) } \cot^{-1}(s/a)$$

(or)

$$\tan^{-1}(a/s)$$

(25) L

(26) L [

(27) L

$\rightarrow L\left[\frac{\sin t}{t}\right] = \cot^{-1}(s)$ (or) $\tan^{-1}(1/s)$

(25) $L\left(\frac{\cos at}{t}\right)$

$$\int_s^{\infty} L[\cos at] = \int_s^{\infty} \frac{s}{s^2+a^2}$$

$$= \frac{1}{2}\int_s^{\infty} \frac{2s}{s^2+a^2}$$

$$= \frac{1}{2}\left(\log(s^2+a^2)\right)_s^{\infty}$$

$= \infty -$ finite $\Rightarrow$ <u>does not exist</u>

(26) $L\left[\frac{e^{\pm at}}{t}\right] \longrightarrow$ does not exist

(27) $L\left(\frac{e^{-at}-e^{-bt}}{t}\right)$

$$\int_s^{\infty}\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$\left[\log(s+a)-\log(s+b)\right]_s^{\infty}$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^{\infty} = \left[\log\left(\frac{1+a/s}{1+b/s}\right)\right]_s^{\infty}$$

$$= \log 1 - \log\frac{s+a}{s+b}$$

$$= -\log\frac{s+a}{s+b} \text{ (or) } \log\frac{s+b}{s-a}$$

v) $\cot^{-1}(s/a)$

v)

(27) $\to L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \log\left(\frac{s+b}{s+a}\right)$

(28) $L\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

(29) $L\left(\frac{\sinh at}{t}\right) = \frac{1}{2}\log\left(\frac{s+a}{s-a}\right)$

(30) $L\left(\frac{\cosh at}{t}\right) =$ Does not Exist

(31) $L\left[\left(\frac{\sin t}{t}\right)^2\right]$

$L\left[\frac{1-\cos 2t}{2t^2}\right]$

consider $L\left[\frac{1-\cos 2t}{t}\right] = \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$

$(\because a=0 \; b=2)$

Lt $s\to\infty$ $s \cdot \frac{1}{\infty}$

$L\left[\frac{1-\cos 2t}{t^2}\right] = \frac{1}{2}\int_s^\infty \log\left(1+\frac{4}{s^2}\right) ds$ $\to$

Integration by parts

$= \frac{1}{2}\int_s^\infty \log\left(1+\frac{4}{s^2}\right) \times 1 \, ds$

$= \frac{1}{2}\left[\log\left(1+\frac{4}{s^2}\right)\cdot s - \int \frac{1}{1+4/s^2} \times -\frac{8}{s^3} \times s \, ds\right]_s^\infty$

$= \left[\frac{1}{2} s \cdot \log\left(1+\frac{4}{s^2}\right)\right]_s^\infty - \frac{1}{2}\times -8 \int_s^\infty \frac{s^2}{s^2+4}\cdot\frac{1}{s^3}\cdot s \, ds$

$= (\quad) - \frac{1}{2} s \log(1+4/s^2) + 4 \times \frac{1}{2} [\tan^{-1} s/2]_s^\infty$

$= (0) - \quad " \quad + 2\cot^{-1}(s/2)$

$\lim_{s\to\infty} s \log(1+4/s^2) = 0$

$\lim_{s\to\infty} s^2 \log(1+4/s^2) = 4$

$\lim_{s\to\infty} s^3 \log(1+4/s^2) = \infty$

$\lim_{s\to\infty} s \cdot \log(1+4/s^2) = s\left[\frac{4}{s^2} - \frac{(4)^2}{2s^4} + \frac{4^3}{3s^6} \dots\right]$

$= 2)$

$\rightarrow \quad \lim_{t\to\infty} (1+1/x)^x = e$

$\lim_{t\to 0} (1+x)^{1/x} = e$

$\frac{8}{s^3} \times s\,ds \Big]_s^\infty$

$0 \times \infty$

$(1)^\infty$

$(\infty)^0$ Indetermination forms.

$(\infty)^\infty$

$\infty - \infty$

$\frac{\infty}{\infty}$

$\frac{1}{s^3} \cdot 8\,ds.$

1) $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$ if $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}\, e^{-1/4s}}{2s^{3/2}}$

sol

$$f(t) = \sin\sqrt{t} \Rightarrow f(0) = 0$$

$$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

$$L[f'(t)] = sL\{f(t)\} - f(0)$$

$$L\left(\frac{\cos\sqrt{t}}{\cancel{2}\sqrt{t}}\right) = s \times \frac{\sqrt{\pi}\, e^{-1/4s}}{\cancel{2}s^{3/2}} - 0$$

$$= \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

2) $L\left\{\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$

$$L\left[\int_0^t f(t)\,dt\right] = \frac{L[f(t)]}{s}$$

$$L[f(t)] = L\left(\frac{e^{-t}\sin t}{t}\right)$$

$$= \cot^{-1}(s+1) \text{ or } \tan^{-1}\left(\frac{1}{s+1}\right)$$

$$\Rightarrow \frac{\cot^{-1}(s+1)}{s}$$

3) $L\left(\int_0^t \frac{1-e^{-t}}{t}\,dt\right)$

$$\frac{L\left(\frac{1-e^{-t}}{t}\right)}{s}$$

(4) f

sol

(5) f

f

$L\left[\frac{1-e^{-t}}{t}\right]$

$a=0 \quad b=1$

$= \log\left(\frac{s+1}{s}\right)$

$L\left\{\frac{1-e^{-t}}{t}\right\} = \frac{1}{s}\log\left(\frac{s+1}{s}\right)$

(4) $f(t) = \begin{cases} \sin t & t \geq \pi \\ \cos t & t < \pi \end{cases}$ $\quad L[f(t)] = ?$

Sol $\quad f(t) = \cos t + [\sin t - \cos t]\, u(t-\pi)$

$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s} L[\sin(t+\pi) - \cos(t+\pi)]$

$-\sin t \qquad -\cos t$

$= \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{-1}{s^2+1} + \frac{s}{s^2+1}\right]$

(5) $f(t) = |t+1| + |t-1|$ for $t \geq 0$ $\quad L[f(t)] = ?$

$f(t) = \begin{cases} (t+1) - (t-1) & 0<t<1 \\ (t+1) + (t-1) & t \geq 1 \end{cases}$

$\left(\frac{1}{s+1}\right)$

$f(t) = \begin{cases} 2 & 0<t<1 \\ 2t & t \geq 1 \end{cases}$

$f(t) = 2 + [2t-2]\, u(t-1)$

$L[f(t)] = \frac{2}{s} + e^{-s} L[\cancel{2(t+1)-2}]$ $\quad 2t$

$= \frac{2}{s} + e^{-s}\frac{2}{s^2}$

a) $f(t) = |t-1| + |t-2| \quad t \geq 0$

$$= -(t-1) - (t-2) \quad 0 < t < 1$$

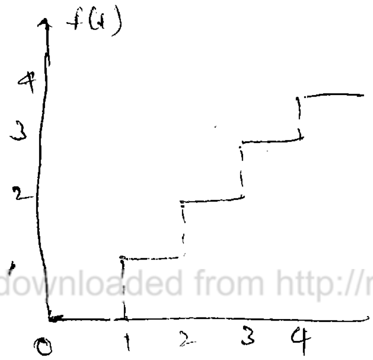
$$= (t-1) - (t-2) \quad 1 < t < 2$$

$$= (t-1) + (t-2) \quad t \geq 2$$

b) $L\{[t]\} = ?$ $\longrightarrow$ stair case fun / floor fun / Integral fun

$[t]$ = Greatest Integer $\leq t$

$$= \begin{cases} 0 & 0 \leq t < 1 \\ = 1 & 1 \leq t < 2 \\ = 2 & 2 \leq t < 3 \\ = 3 & 3 \leq t < 4 \\ = 4 & 4 \leq t \leq 5 \\ \ldots & \end{cases}$$



$$\int_1^5 [x]\, dx = \cancel{0} + 2 + 3 + 4 + \cancel{0} = 10$$

$$\int_1^{100} [x]\, dx = 1 + 2 + 3 + \cdots + 99$$

$$= \frac{n(n+1)}{2} = \frac{99 \times 100}{2}$$

integral part

$[0.5] = 0$

$[0.9] = 0$

$[-2.5] = -3$

$[-3 + 0.5]$

$[4.5] = [4 + 0.5]$ $\downarrow$ integral part.

$$f(t) = 0 + (1-0)\,u(t-1) + (2-1)\,u(t-2) + (3-2)\,u(t-3) + \dots$$

$$= u(t-1) + u(t-2) + u(t-3) \dots$$

$$L\{f(t)\} = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s} + \dots$$

$$= \frac{e^{-s}}{s}\left[1 + e^{-s} + e^{-2s} + e^{-3s} + \dots\right]$$

gral fun

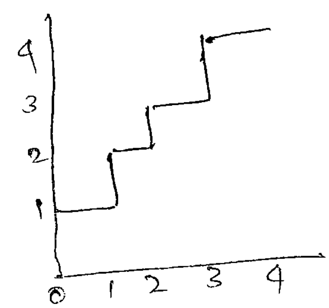
$$= \frac{e^{-s}}{s}\left(1 - e^{-s}\right)^{-1}$$

$$= \frac{e^{-s}}{s(1-e^{-s})}$$

Staircase / ceiling fun:

(t) or $\lceil t \rceil$

$$= \begin{cases} 1 & 0 \le t < 1 \\ 2 & 1 \le t < 2 \\ 3 & 2 \le t < 3 \\ \dots \end{cases}$$



9

$\frac{\times 100}{2}$

$$f(t) = u(t) + u(t-1) + u(t-2) + u(t-3) \dots$$

$$L\{f(t)\} = \frac{1}{s}\left[1 + e^{-s} + e^{-2s} + \dots\right]$$

$$= \frac{1}{s}\left(1 - e^{-s}\right)^{-1}$$

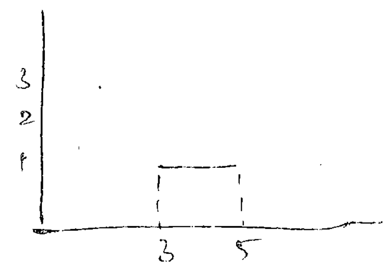
$$\frac{1}{s(1-e^{-s})}$$

(7) or $\lceil t \rceil$ = least integer $\geq t$

$$\lceil 0.5 \rceil = 1 \rightarrow \text{ceiling}$$

$$\lfloor 0.5 \rfloor = 0 \rightarrow \text{floor}$$

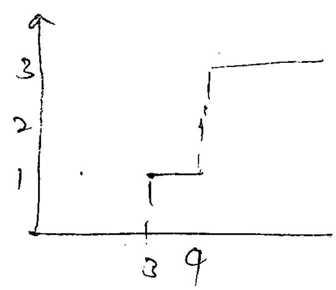
Q)



$$f(t) = u(t-3) - u(t-5)$$

$$L(f(t)) = \frac{e^{-3s}}{s} - \frac{e^{-5s}}{s}$$

Q)

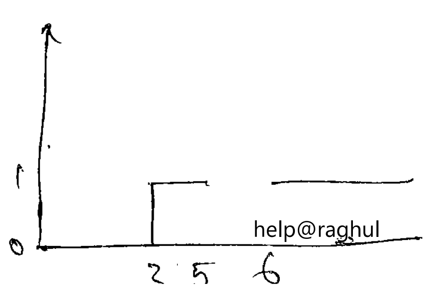


$$f(t) = 0 + u(t-3) + (3-1)\,u(t-4)$$

$$f(t) = u(t-3) + 2u(t-4)$$

$$L(f(t)) = \frac{e^{-3s}}{s} + \frac{2e^{-4s}}{4}$$

Q)



does not exist.

(4) $f(t) = \begin{cases} \sin\omega t & 0 < t < \pi/\omega \\ 0 & \pi/\omega \le t \le 2\pi/\omega \end{cases}$ $L\{f(t)\} = ?$

Sol

$$L\{f(t)\} = \frac{\int_0^{2\pi/\omega} e^{-st} \cdot f(t)\,dt}{\left(1 - e^{-s \cdot 2\pi/\omega}\right)}$$

(if periodic or $f(t + 2\pi/\omega) = f(t) \ \forall t$ if given exists it not not exist.)

↳ periodic function defination

$$= \frac{\int_0^{\pi/\omega} e^{-st} \sin\omega t + 0}{1 - e^{-s \cdot 2\pi/\omega}}.$$

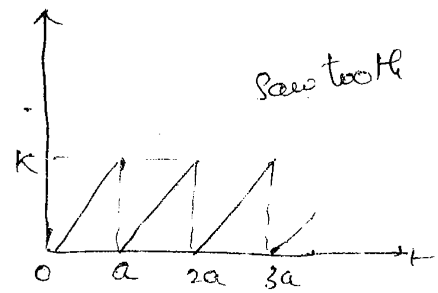
$$= \frac{1}{1 - e^{-\frac{2\pi}{\omega} \cdot s}} \left[\frac{e^{-st}}{s^2 + \omega^2} \left(-s\sin\omega t - \omega\cos\omega t \right) \right]_0^{\pi/\omega}$$

$$= \frac{1}{\left(1 - e^{-2\pi/\omega s}\right)} \left[\frac{e^{-s \cdot \pi/\omega}}{s^2 + \omega^2} \left[-\omega(-1) \right] - \frac{1}{s^2 + \omega^2}(-\omega) \right]$$

$$= \frac{\omega}{\left(s^2 + \omega^2\right)} \left[\frac{\cancel{\left(1 + e^{-\pi/\omega s}\right)}}{\left(1 - e^{-\pi/\omega s}\right)\cancel{\left(1 + e^{-\pi/\omega s}\right)}} \right]$$

$$= \frac{\omega}{s^2 + \omega^2} \left[\frac{1}{1 - e^{-\pi/\omega s}} \right]$$

Q)



$L\{f(t)\} = ?$ $\quad f(t+a) = f(t)\ \forall t$

$T = a$

$f(t) = \frac{k}{a}t \ ; \ 0<t<a$

$$L[f(t)] = \frac{\int_0^a k/a\, t e^{-st} dt}{(1-e^{-as})}$$

$$= \frac{\frac{k}{a}\left[t\frac{e^{-st}}{(-s)} - (-1)\cdot\frac{e^{-st}}{(-s)^2}\right]_0^a}{(1-e^{-as})}$$

$$= \frac{k}{a(1-e^{-as})}\left(-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}\right)$$

$$= \frac{-ke^{-as}}{s(1-e^{-as})} + \frac{k}{as^2}\frac{\cancel{(1-e^{-as})}}{\cancel{(1-e^{-as})}}$$

$$= \frac{-ke^{-as}}{s(1-e^{-as})} + \frac{k}{as^2}.$$

Q) $f(t) = \sin\omega t \quad 0 \le t \le 2\pi/\omega \qquad f(t+2\pi/\omega) = f(t)\ \forall t$

Sol

$$\frac{\omega}{s^2+\omega^2}$$

Q) $\int_0^\infty \frac{S}{}$

Q) $\int_0$

Q) $\int_0$

(a) $\int_0^\infty \frac{\sin t}{t}\,dt$

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\,dx$$

$$L\left\{\frac{\sin t}{t}\right\} = \cot^{-1}(1/s)$$

↓ Def

$$\int_0^\infty e^{-st} \cdot \frac{\sin t}{t}\,dt = \cot^{-1}(1/s)$$

put $s=0$

$$\int_0^\infty \cdot \frac{\sin t}{t}\,dt = \cot^{-1}(0) = \pi/2$$

(b) $\int_0^\infty \frac{e^{-3t} - e^{-2t}}{t}\,dt$

$$= L\left(\frac{e^{-3t} - e^{-2t}}{t}\right)_{\text{at } s=0}$$

$$= \left(\log\left(\frac{s+2}{s+3}\right)\right)_{\text{at } s=0} = \log\left(\frac{2}{3}\right)$$

(c) $\int_0^\infty t e^{-3t} \cos 2t\,dt$

$$\int_0^\infty e^{-3t}\, t\cos 2t\,dt$$

$$L\left[t\cos 2t\right]_{\text{at } s=3}$$

$$\left(-\frac{d}{ds}\left[\frac{s}{s^2+4}\right]\right)_{s=3}$$

$$-\left(\frac{(s^2+4)(1) - s\cdot 2s}{(s^2+4)^2}\right) = -\left(\frac{13-18}{(13)^2}\right) = \frac{5}{169}$$

8) $\int_{-\infty}^{\infty} \sin t\, \delta(t-\pi/6)\, dt$

$$= \int_{0}^{\infty} \sin t\, \delta(t-\pi/6)\, dt$$

$$= L\left[\sin t\, \delta(t-\pi/6)\right]_{at\ s=0}$$

$$\left(e^{-\pi/6 s} \cdot \sin \pi/6\right)_{at\ s=0} = \frac{1}{2}$$

$$\boxed{\int_{-\infty}^{\infty} f(t)\, \delta(t-a)\, dt = f(a)}$$

8) $\int_0$

5) $\int_{0}^{\infty} t^2 u(t-2)\, dt$

$$L\left[t^2 u(t-2)\right]_{at\ s=0}$$

$$\left[e^{-2s} \cdot L\left\{(t+2)^2\right\}\right]_{at\ s=0}$$

$$= e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]_{at\ s=0}$$

$\Downarrow$

undefined

9) $\int_{0}^{\infty} \sin t\, u(t-\pi/6)\, dt$

$$\left[e^{-\pi/6 s} L\left\{\sin(t+\pi/6)\right\}\right]_{at\ s=0}$$

$$\left(e^{-\pi/6 s} \cdot \left[\frac{1}{s^2+1} \times \frac{\sqrt{3}}{2} + \frac{s}{s^2+1} \cdot \frac{1}{2}\right]\right)_{at\ s=0}$$

$$= 1 \times \frac{\sqrt{3}}{2}$$

8) $\int_0^\infty \int_0^t e^{-t} \frac{\sin u}{u} \, du \, dt$

$$\int_0^\infty \int_0^t e^{-t} \frac{\sin u}{u} \, du \, dt$$

$$\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} \, du \cdot dt$$

$$L\left[\int_0^t \frac{\sin u}{u} du\right] \text{ at } s=1$$

$$= \left(\frac{1}{s} \cot^{-1}(s)\right)_{s=1}$$

$$= \cot^{-1}(1) = \pi/4$$

$\Big]\Big]_{\text{at } s=0}$

Inverse Laplace Transformations

If $L\{f(t)\} = \bar{f}(s) \Rightarrow f(t) = L^{-1}\{\bar{f}(s)\}$

Linearity: $L^{-1}\{a\bar{f}(s) \pm b\bar{g}(s)\} = aL^{-1}\{\bar{f}(s)\} + bL^{-1}\{\bar{g}(s)\}$

(1) $L^{-1}(1) = \delta(t)$

(2) $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

(3) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

(4) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

(5) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

(6) $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

(7) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

(8) $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$ if $n \in Z^+$

$\frac{t^{n-1}}{\Gamma n}$ if $n \notin Z^+$

First

(1) L

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First shifting theorem of I.L.T:

$\left\{\bar{g}(s)\right\}$

(1) $L^{-1}\{\bar{F}(s-a)\} = e^{at} L^{-1}\{\bar{F}(s)\}$

(2) $L^{-1}\{\bar{F}(s+a)\} = e^{-at} L^{-1}\{\bar{F}(s)\}$

second shifting theorem:

If $L^{-1}\{\bar{F}(s)\} = f(t)$

$$L^{-1}\{e^{-as}\bar{F}(s)\} = f(t-a)\,u(t-a)$$

(or)

$$\begin{cases} f(t-a) & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

change of scaling:

If $L^{-1}\{\bar{F}(s)\} = F(t)$

(1) $L^{-1}\{\bar{F}(s/a)\} = a\,f(at)$

(2) $L^{-1}\{\bar{F}(as)\} = \frac{1}{a} f(t/a)$

multiplication by s^n in (L^{-1})

(1) $L^{-1}\{s\bar{F}(s)\} = f'(t)$ provided $f(0) = 0$

(2) $L^{-1}\{s^n \bar{F}(s)\} = f^n(t)$ it provided $f(0) = f'(0)$ $= f^{n-1}(0) = 0$

Division by s^n ($n \in Z^+$)

① $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t L^{-1}\{\bar{f}(s)\} dt$

② $L^{-1}\left\{\frac{\bar{f}(s)}{s^n}\right\} = \int_0^t L^{-1}\{\bar{f}(s)\} (dt)^n$

* I.L.T of Derivatives:

① $L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = -t\, L^{-1}\{\bar{f}(s)\}$

② $L^{-1}\left\{\frac{d^n}{ds^n}\bar{f}(s)\right\} = (-t)^n\, L^{-1}\{\bar{f}(s)\}$

I.L.T of integrals:

$$L^{-1}\left\{\int_s^\infty \bar{f}(s) ds\right\} = \frac{1}{t} L^{-1}\{\bar{f}(s)\}$$

Convolution property:

If $L^{-1}\{\bar{f}(s)\} = f(t)$ & $L^{-1}\{\bar{g}(s)\} = g(t)$

$L^{-1}\{\bar{f}(s) \cdot \bar{g}(s)\} = f(t) * g(t)$

(Convolution is commutative)

$$= \int_0^t f(x)\, g(t-x)\, dx.$$

Inte

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Q) I

Initial value theorem:

If $L\{f(t)\} = \bar{F}(s)$ then

$$\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ s\bar{F}(s).$$

Final value theorem:

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s\bar{F}(s)$$

→ The final value theorem is applicable if the poles of $\bar{F}(s)$ lie in the left plane i.e left side of y-axis.

Q) $L^{-1}\left[\frac{1}{\sqrt{2s+3}}\right]$

$$L^{-1}\left[\frac{1}{(2s+3)^{1/2}}\right]$$

$$L^{-1}\left[\frac{1}{(2(s+3/2))^{1/2}}\right] = L^{-1}\left[\frac{1}{\sqrt{2}}\frac{1}{(s+3/2)^{1/2}}\right]$$

$$= \frac{1}{\sqrt{2}} L^{-1}\left[\frac{1}{(s+3/2)^{1/2}}\right]$$

$$= \frac{1}{\sqrt{2}} e^{-3/2 t} L^{-1}\left[\frac{1}{s^{1/2}}\right]$$

$$= \frac{1}{\sqrt{2}} e^{-3/2 t} \frac{t^{-1/2}}{\Gamma(1/2)}$$

$$= \frac{e^{-3/2 t}}{\sqrt{2\pi t}}$$

Q) $L^{-1}\left[\frac{3s+5}{16s^2+9} - \frac{4s+1}{9s^2-16} + \frac{5}{5s-1}\right]$

$$= \frac{1}{16} L^{-1}\left[\frac{3s}{s^2+9/16} + \frac{5}{s^2+9/16}\right] - \frac{1}{9} L^{-1}\left[4\frac{s}{s^2-16/9} + \frac{1}{s^2-16/9}\right] + \frac{1}{5} L^{-1}\left[\frac{1}{s-1/5}\right]$$

$$= \frac{1}{16}\left[3\cos\frac{3}{4}t + 5\frac{1}{3/4}\sin 3/4 t\right] - \frac{1}{9}\left[4\cosh 4/3 t + \frac{1}{4/3}\sinh 4/3 t\right] + e^{1/5 t}.$$

Q) $L^{-1}\left(\frac{1}{s^{3/2}} + \frac{1}{s^{-1/2}} + \frac{1}{s^3}\right)$

$$\frac{t^{3/2-1}}{\Gamma 3/2} + \frac{t^{-1/2-1}}{\Gamma -1/2} + \frac{t^{3-1}}{\Gamma 3}$$

$$= \frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}} + \frac{t^{-3/2}}{-2\sqrt{\pi}} + \frac{t^2}{2!}$$

Q) $L^{-1}\left(\frac{s}{(s+a)^n}\right)$

$$L^{-1}\left(\frac{s+a-a}{(s+a)^n}\right)$$

$$L^{-1}\left[\frac{1}{(s+a)^{n-1}} - \frac{a}{(s+a)^n}\right]$$

$$= e^{-at}\left[\frac{t^{n-2}}{\Gamma n-1} - \cdot \frac{a\,t^{n-1}}{\Gamma n}\right]$$

$$= \frac{e^{-at}\,t^{n-2}}{\Gamma n-1}\left[1 - \frac{a\,t}{\Gamma n-1}\right]$$

$\frac{s}{s^2 + 16/9} +$

$$\text{Q)} \quad L^{-1}\left[\frac{s^2}{(s-a)^2}\right]$$

$$L^{-1}\left[\frac{(s+a)-a)^2}{(s-a)^2}\right]$$

4/3 t+

$$L^{-1}\left[\frac{(s+a)^2 + 2a(s+a) + a^2}{(s-a)^2}\right]$$

$$L^{-1}\left[1 + \frac{2a}{(s-a)} + \frac{a^2}{(s-a)^2}\right]$$

$$= e^{at} L\left[1 + \frac{2a}{s} + \frac{a^2}{s^2}\right] =$$

$$= e^{at}\cdot\left[\delta(t) + 2a + a^2 t\right]$$

$$\text{Q)} \quad L^{-1}\left[\frac{s}{s^2-4s+3}\right]$$

$s^2 + 4s + 4$

$$L^{-1}\left[\frac{s-2+2}{(s-2)^2 - 1^2}\right] = e^{2t}\cdot \cancel{\cos ht}$$

$$L^{-1}\left[\frac{s-2}{(s-2)^2 - 1^2} + \frac{2}{(s-2)^2 - 1^2}\right] = e^{+2t}\left[\cos ht - 2\sin ht\right]$$

(8)
$$L^{-1}\left[\frac{s}{(s-3)(s-1)}\right]$$

$$L^{-1}\left[\frac{3/2}{s-3} - \frac{1/2}{s-1}\right]$$

$$= \tfrac{3}{2}e^{3t} - \tfrac{1}{2}e^{t}$$

9)
$$L^{-1}\left[\frac{s^2+s-2}{s(s^2+2s-8)}\right]$$

$$L^{-1}\left(\frac{s^2+s-2}{s(s+4)(s-2)}\right)$$

$16-4-2$ $\frac{10}{-4\times-6}$ $\frac{5}{12}$
-4×-6

$$= L^{-1}\left(\frac{1/4}{s} + \frac{5/12}{s+4} + \frac{1/3}{s-2}\right)$$

$$= \tfrac{1}{4} + e^{-4t}\tfrac{5}{12} + \tfrac{1}{3}e^{2t}$$

6)
$$L^{-1}\left[\frac{s}{s^2-4s+1}\right]$$

$(s-2)^2$
$s^2-4s+4-$

$$L^{-1}\left(\frac{(s-2)+2}{(s-2)^2-3}\right)$$

$$L^{-1}\left(\frac{s-2}{(s-2)^2-(\sqrt{3})^2} + \frac{2}{(s-2)^2-(\sqrt{3})^2}\right)$$

$$e^{2t}\cosh\sqrt{3}t + \frac{2}{\sqrt{3}}e^{2t}\sinh\sqrt{3}t$$

$$= e^{2t}\left[\cosh\sqrt{3}t + \frac{2}{\sqrt{3}}\sinh\sqrt{3}t\right]$$

5) L^{-1}

Sol

6) L^{-1}

Sol

9)

Q) $\bar{L}^1\left\{\frac{3s+5}{s^2-5s+3}\right\}$

Sol $\bar{L}^1\left\{\frac{3(s-5/2)+\frac{15}{2}+5}{(s-5/2)^2-\frac{25}{4}+3}\right\}$

$\bar{L}^1\left\{\frac{3(s-5/2)+25/2}{(s-5/2)^2-13/4}\right\}$ $\bar{f}(s-a)$

F.S.Th

$e^{5/2t}\ \bar{L}^1\left\{3\cosh\frac{\sqrt{13}}{2}t+\frac{25}{2}\times\frac{1}{\sqrt{13}/2}\times\sinh\frac{\sqrt{13}}{2}t\right\}$

$16-9-2$ $\frac{10\ 5}{-4x-6}$ $-4x-6$ $\overline{12}$

Q) $\bar{L}^1\left\{\frac{1}{s^3(s^2+1)}\right\}$

$L\ \frac{t^2}{2}*\sin t$

Sol $\bar{L}^1\left[\frac{1}{s}\cdot\frac{1}{s^2}\frac{1}{(s^2+1)}\right]$ $=\int_0^t\frac{x^2}{2!}\sin(t-x)\,dx$

$\bar{L}^1\left\{\frac{1}{s}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)\right\}$

$\int_0^t(t-\sin t)\,dt$

$=\left(\frac{t^2}{2}+\cos t\right)_0^t=\frac{t^2}{2}+\cos t-1$

(or) $\int_0^t\int_0^t\int_0^t\sin t\,(dt)^3$

$^2-4s+4-$

Q) $\bar{L}^1\left\{\frac{1}{(s+3)(s^2+1)}\right\}$

$e^{-3t}*\sin t=\sin t\times e^{-3t}$

$$= \int_0^t \sin x \cdot e^{-3(t-x)}\,dx$$

$$= e^{-3t}\int_0^t e^{3x}\sin x\,dx$$

$$= e^{-3t}\left[\frac{e^{3x}}{10}(3\sin x-\cos x)\right]_0^t$$

$$= e^{-3t}\left[\frac{e^{3t}}{10}(3\sin t-\cos t)-\frac{1}{10}(-1)\right]$$

$$= \frac{1}{10}(3\sin t-\cos t)+\frac{1}{10}.$$

9) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ 9) L^{-1}

$\frac{1}{a}\sin at * \cos at$

$$= \int_0^t \frac{1}{a}\sin at\,\cos a(t-x)\,dx$$

$$= \frac{1}{2a}\int_0^t\left(\sin at+\sin(2ax-at)\right)dx$$

$$= \cancel{\frac{1}{2a}}\ -\cancel{\frac{\cos at}{a}} + \cancel{\cos(2ax-at}$$

$$= \frac{1}{2a}\left[x\sin at-\frac{\cos(2ax-at)}{2a}\right]_0^t$$ Slly

(9)

$$= \frac{1}{2a}\left[t\sin at-\cancel{\frac{\cos at}{2a}}+\cancel{\frac{\cos(-at)}{2a}}\right]$$

$$= \frac{1}{2a}\left[t\sin at\right]$$

Corr

$+$

s

$\frac{}{0}(-1)]$

$$F(s) = \frac{1}{s^2+a^2}$$

$$\frac{d}{ds}F(s) = -2\frac{s}{(s^2+a^2)^2}$$

$$L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t\,L^{-1}\{F(s)\}$$

$$L^{-1}\left[\frac{-2s}{(s^2+a^2)^2}\right] = -t\,\frac{1}{a}\sin at$$

$$= \frac{1}{2a}\left[t\sin at\right]$$

Q) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

$$= L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right]$$

$$= \int \frac{t\sin at}{2a}\,dt$$

$$= \frac{1}{2a}\left[t\cdot\frac{(-\cos at)}{a} - (1)\left(\frac{-\sin at}{a^2}\right)\right]_0^t$$

$$= \frac{1}{2a^3}\left[-at\cos at + \sin at\right]$$

Illy

Q) $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$

$$L^{-1}\left[s\cdot\frac{s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{dt}\left[\frac{t\sin at}{2a}\right]$$

$$= \frac{1}{2a}\left(at\cos at + \sin at\right)$$

Q) $L^{-1}\left\{\frac{s}{(s^2+a)(s^2+b)}\right\}$

$\cos at * \frac{1}{b}\sin bt$

$\int_0^t \frac{1}{b}\sin bt \cos a(t-x)\, dx$

$\frac{1}{2b} \int_0^t \left\{\sin(at+bx-ax) + \sin(bt-at+ax)\right\} dx$

$\frac{1}{2b} \int_0^t \sin(at+(b-a)x) + \frac{1}{2b}\int_0^t \sin(t(b-a)+a) dx$

$\frac{1}{2b}\left[- \cos(at \right.$

$L^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\}$

$\frac{1}{(b^2-a^2)} \cdot L^{-1}\left\{\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right\}$

$\frac{1}{b^2-a^2}\left\{\cos at - \cos bt\right\}$

Q) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

$\frac{1}{(b^2-a^2)} L^{-1}\left\{\frac{s^2}{s^2+a^2} - \frac{s^2}{s^2+b^2}\right\}$

$\frac{1}{b^2-a^2} L^{-1}\left\{\left(1-\frac{a^2}{s^2+a^2}\right) - \left(1-\frac{b^2}{s^2+b^2}\right)\right\}$

$$\frac{1}{b^2-a^2}\left[-a\sin at + b\sinh bt\right]$$

$+ax)\} dx$

$(b-a)+a\} dx$

9) $$L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\}$$

$$s^4+s^2+1 = (s^2+1)^2 - s^2$$
$$= (s^2+1-s)(s^2+1+s)$$

$$L^{-1}\left\{\frac{s}{(s^2+1-s)(s^2+1+s)}\right\}$$

$$\frac{1}{2} L^{-1}\left\{\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right\}$$

$$= \frac{1}{2} L^{-1}\left\{\frac{1}{(s-1/2)^2-1/4+1} - \frac{1}{(s+1/2)^2-1/4+1}\right\}$$

F.S.the

$$= \frac{1}{2}\left[e^{t/2}\frac{1}{\sqrt{3}/2}\sin\frac{\sqrt{3}}{2}t - e^{-t/2}\frac{1}{\sqrt{3}/2}\sin\frac{\sqrt{3}}{2}t\right]$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{3}/2}\sin\frac{\sqrt{3}}{2}t\left[e^{t/2}-e^{-t/2}\right]$$

$$= \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\,(2\sinh t/2)$$

9) $$L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$$

s

$$L^{-1}\left\{\frac{s}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right\}$$

$s^2)\}$

$$= \frac{1}{4a}L^{-1}\left\{\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right\}$$

$$\frac{1}{4a} \quad L^{-1}\left[\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right]$$

$$\frac{1}{4a}\left[e^{at}\frac{\sin at}{a} - e^{-at}\frac{\sin at}{a}\right]$$

$$= \frac{1}{4a^2}\sin(at)\,(2\sinh at)$$

$$= \frac{\sin(at)\sinh at}{2a^2} \;//$$

Q) $L^{-1}\left\{\frac{2+5s}{e^{4s}\,s^4}\right\}$

$$L^{-1}\left[e^{-4s}\left(\frac{2}{s^4}+\frac{5}{s^3}\right)\right]$$

$$L^{-1}\left(\frac{2}{s^4}+\frac{5}{s^3}\right) = \left(2\frac{t^3}{3!}+5\frac{t^2}{2!}\right)$$

S.S theorem $f(t-a)\,u(t-a)$

$$\therefore \left[2\frac{(t-4)^3}{3!}+5\frac{(t-4)^2}{2!}\right]u(t-4)$$

Q) $L^{-1}\left\{\frac{e^{-1/\sqrt{s}}}{s}\right\}$

$$L^{-1}\left[\frac{1}{s}\left(1-\frac{1}{\sqrt{s}}+\frac{1}{2!(\sqrt{s})^2}-\frac{1}{3!(\sqrt{s})^3}+\cdots\right)\right]$$

$$= L^{-1}\left\{\frac{1}{s}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{(\sqrt{s})^k}\right\}$$

$$= L^{-1}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{s^{\frac{k}{2}+1}}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{t^{k/2}}{\Gamma\left(\frac{k}{2}+1\right)}$$

Q) $L^{-1}\left\{ \frac{1}{s - e^{-s}} \right\}$

Sol $L^{-1}\left[\frac{1}{s\left(1 - e^{-s}/s\right)} \right]$

$$L^{-1}\left\{ \frac{1}{s} \cdot \left(1 - \frac{e^{-s}}{s}\right)^{-1} \right\}.$$

$$L^{-1}\left\{ \frac{1}{s}\left(1 + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^3} + \dots\right) \right\}$$

$$= L^{-1}\left\{ \sum_{k=0}^{\infty} \frac{e^{-ks}}{s^{k+1}} \right\} = \sum_{k=0}^{\infty} \frac{(t-k)^k}{k!} u(t-k)$$

Q) $L^{-1}\left\{ \cot^{-1}(s) \right\} = \frac{\sin t}{t} \Rightarrow L^{-1}\left\{ \frac{1}{s} \cot^{-1}(s) \right\} = \int_0^t \frac{\sin t}{t} dt$ ✓

$$L^{-1}\left[\frac{1}{s} \cot^{-1}(s) \right] = \int_0^t \frac{\sin t}{t} dt.$$

Q) $L^{-1}\left\{ \log\left(\frac{s+3}{s+2}\right) \right\} = \frac{e^{-3t} - e^{-2t}}{t} dt$

$$L^{-1}\left[\frac{1}{s} \log\left(\frac{s+3}{s+2}\right) \right] = \int_0^t \frac{e^{-3t} - e^{-2t}}{t} dt$$

let $F(s) = \cot^{-1}(s)$

$$\frac{d}{ds} F(s) = \frac{-1}{s^2+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$= -\frac{1}{t} L^{-1}\left[\frac{-1}{s^2+1}\right]$$

$$= \frac{\sin t}{t}$$

(4) $L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

$$F(s) = \tan^{-1}(2/s^2)$$

$$\frac{d}{ds}F(s) = \frac{1}{1+4/s^4} \cdot \frac{-4}{s^3}$$

$$= \frac{s^4}{s^4+4} \cdot \frac{-4}{s^3}$$

$$= \frac{-4s}{s^4+4}$$

$$\therefore L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left(\frac{d}{ds}F(s)\right)$$

$$= +\frac{1}{t} L^{-1}\left(+4\frac{s}{s^4+4}\right)$$

$$= \frac{4}{t}\left[\frac{1}{2(1)}\sin t \sinh t\right]$$

$$= \frac{2}{t}(\sin t \sinh t)$$

$$\left(\because L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{2a^2}\sin at \sinh at\right)$$

9. If $F(s) = \frac{s}{s^2-2s+2}$ then $\lim_{t\to\infty} f(t) = ?$

(a) 0 (b) 1 (c) −1 (d) None

unstable function

Applications of Laplace transform to linear differential Equation:

1) $\frac{d^2y}{dt^2} + y = 0 \quad y(0) = 1 \quad y'(0) = 0$ Find the solution

$$L\{y'' + y\} = L\{0\}$$

$$s^2 \bar{y}(s) - sy(0) - y'(0) + \bar{y}(s) = 0$$

$$(s^2+1)\bar{y}(s) = sy(0) + y'(0) = s$$

$$\bar{y}(s) = \frac{s}{s^2+1}$$

$$y = L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$$

2) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 1 \quad y(0) = 0 \quad y'(0) = 1$

$$L(y'' - 4y' + 3y) = L[1]$$

$$s^2\bar{y}(s) - sy(0) - y'(0) - 4s\bar{y}(s) - 4y(0) + 3\bar{y}(s) = 1/s$$

$$\bar{y}(s)[s^2 - 4s + 3] = \frac{1}{s} + 1 = \frac{s+1}{s}$$

$$\bar{y}(s) = \frac{s+1}{s(s^2-4s+3)}$$

$$y = L^{-1}\left[\frac{s+1}{s(s-3)(s-1)}\right]$$

$$y = \frac{1}{3} + \frac{2}{3}e^{3t} - e^{t}$$

sinht]

]

sinhat)

tion

20) $(D^2-1)x = a\cosh t$; $x(0) = x'(0) = 0$

Sol

$$L\{x''-x\} = L\{a\cosh t\}$$

$$(s^2-1)\,\bar{x}(s) = a\cdot\frac{s}{s^2-1}$$

$$x(t) = L^{-1}\left\{a\,\frac{s}{(s^2-1)^2}\right\}$$

$$= \left\{\frac{a\,t\sinh t}{2}\right\}$$

Q) $\{f(t)\} = \begin{cases} 1 & 0<t<a/2 \\ -1 & a/2<t<a \end{cases}$ $\quad L\{f(t)\} = ?$

Sol

$$L\{f(t)\} = \frac{\int_0^{a/2} e^{-st}(1)\,dt + \int_{a/2}^{a}(e^{-st})(-1)\,dt}{(1-e^{-as})}$$

$$= \frac{1}{(1-e^{-as})}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right]$$

$$= \frac{1}{(1-e^{-as})}\left[-\frac{e^{-as/2}}{s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{-as/2}}{s}\right]$$

$$= \frac{1}{s(1-e^{-as})}\left[1+e^{-as} - 2e^{-as/2}\right]$$

$$= \frac{1}{s(1-e^{-as})}\,(1-e^{-as/2})^2$$

$$= \frac{1\;(1-e^{-as/2})^{2}}{s\,\cancel{(1-e^{-as/2})}\,(1+e^{-as/2})}$$

$$= \frac{1}{s} \frac{\left(e^{as/4} - e^{-as/4}\right)}{\left(e^{as/4} + e^{-as/4}\right)} = \frac{1}{s} \tanh\left(\frac{as}{4}\right)$$

Z-TRANSFORMATIONS

Z-transformation also has similar prosperties of LT But it operates only on discrete functions & not on Continous function

If define $f(n)$ for $n = 0, 1, 2, 3 \ldots$

$f(n) = 0$ for $n < 0$

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = \bar{F}(z)$$

Here z is a complex parameter

z^{-n} is kernal

Right sided Z-transform.

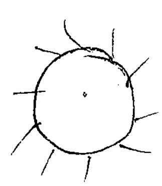
$$Z\{f(n)\} = \sum_{n=-\infty}^{-1} f(n) z^{-n}$$

left sided Z-transform.

$$Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

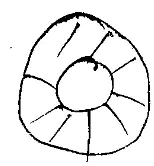
Two sided Z-transform.

→ ROC of right-sided Z-Transformation is in the form of $|z| > |a|$

i.e 

→ ROC of left sided Z-transform is in the form of $|z| < |a|$

→ ROC of two sided Z-transform is in the form of $|a| < |z| < |b|$



Z transform of some standard functions:

$$Z(1) = \frac{z}{z-1} \; ; \quad |z| > 1$$

$$Z\{a^n\} = \frac{z}{z-a} \; ; \quad |z| > |a|$$

$$Z\{(-a)^n\} = \frac{z}{z+a} \quad |z| > |a|$$

$$Z\{1\} = -\frac{z}{z-1} \quad |z| > 1$$

$$Z\{k\} = k \cdot \frac{z}{z-1} \quad |z| > 1$$

$$Z\{(-1)^n\} = \frac{z}{z+1} \; ; \quad |z| > 1$$

$$Z\{n\} = \sum_{n=0}^{\infty} n \cdot z^{-n}$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right)$$

$$= \frac{1}{z}\left((1 - 1/z)^{-2}\right) \quad \text{for } \left|\frac{1}{z}\right| < 1$$

$$= \frac{1}{z}\left(\frac{(z-1)}{z}\right)^{-2} \quad \text{for } |z| > 1$$

$$= \frac{z}{(z-1)^2} \quad |z| > 1$$

Recurrance Relation:

$$z\{n^p\} = -z \cdot \frac{d}{dz}\left(z(n^{p-1})\right); \quad p \in z^+$$

Eg

$$z\{n^2\} = -z \cdot \frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= \frac{z^2 + z}{(z-1)^3}; \quad |z| > 1$$

Q) $z\left\{\frac{1}{n}\right\}$

$$= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \left(\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots\right)$$

$$= -\log\left(1 - \frac{1}{z}\right) \quad \text{for } |z| < 1$$

(or)

$$\log\left(\frac{z}{z-1}\right)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]$$

Q) $z($

Q) $z($

Q) z

Q) z

Q) $Z\left[\frac{1}{n+1}\right]$

$$\sum_{n=0}^{\infty} z^{-n} \cdot \frac{1}{n+1}$$

$$= z \log\left(\frac{z}{z-1}\right)$$

Q) $Z\left[\frac{1}{n!}\right]$

$$\sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$$

$$= e^{1/z} \quad ; \text{ for } |z| > 0 \text{ (or) for } z \neq 0$$

Q) $Z\left\{\frac{1}{(n+1)!}\right\}$

$$= z\left(e^{1/z} - 1\right)$$

* $Z(n!)$ is not convergent.

Q) $Z\left\{\cos n\theta + i \sin n\theta\right\}$

$$= Z\left\{e^{in\theta}\right\}$$

$$= Z\left\{\left(e^{i\theta}\right)^n\right\}$$

$$= \frac{z}{z - e^{i\theta}} \quad ; \ |z| > 1$$

$$= \cancel{\frac{z(z + e^{i\theta})}{z^2 \ \ e^{2i\theta}}}$$

$$= \frac{z(z-e^{-i\theta})}{(z-e^{i\theta})(z-e^{-i\theta})}$$

$$= \frac{z^2 - z(e^{-i\theta})}{z^2 - z(e^{i\theta}+e^{-i\theta}) + e^{i\theta}\cdot e^{-i\theta}}$$

$$Z[\cos n\theta + i\sin n\theta] = \frac{z^2 - z(\cos\theta - i\sin\theta)}{z^2 - 2z\cos\theta + 1}$$

By comparing.

$$\rightarrow Z(\cos n\theta) = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

$$\rightarrow Z(\sin n\theta) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\rightarrow z\{\cosh n\theta) = \frac{z^2 - z\cosh\theta}{z^2 - 2z\cosh\theta + 1}$$

$$\rightarrow z\{\sinh n\theta\} = \frac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$$

Eg:

$$z\{\sin n\pi/2\} = \frac{z}{z^2+1}$$

$$z\{\cos n\pi/2\} = \frac{z^2}{z^2+1}$$

$$z[\sin n\theta \cos n\theta] = \frac{1}{2}z\{\sin(2n\theta)\}$$

$$= \frac{1}{2} z\{\sin n(2\theta)\}$$

Eg:

$\neq$ z

$$\frac{1}{2}\left[\frac{z\sin 2\theta}{z^2-2z\cos 2\theta+1}\right]$$

Eg: $Z\{\cos \pi\}$

$\Rightarrow Z(-1) = \frac{-z}{z-1}$; $|z|>1$

* $Z\left\{\frac{1}{n(n+1)}\right\}$

(a) $-(1-z)\log(1-1/z)$

(b) $-(1+z)\log(1-1/z)$

(c) $-(1-z)\log(1-1/z)+1$

(d) $-(1+z)\log(1-1/z)+1$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}z^{-n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)z^{-n}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}z^{-n} - \sum_{n=1}^{\infty}\frac{1}{n+1}z^{-n}$$

$$= -\log(1-1/z) - (-z\log(1-1/z)-1)$$

$$= -(1-z)\log(1-1/z)+1$$

Properties:

Damping Rule (or) change of scaling:

If $z\{u_n\} = \bar{u}(z)$ then

① $z\{a^n u_n\} = \bar{u}(z/a)$

② $z\{a^{-n} u_n\} = \bar{u}(az)$

1st shifting: (Right shifting property):

If $z\{u_n\} = \bar{u}(z)$ then

$$z\{u_{n-k}\} = z^{-k} \cdot \bar{u}(z)$$

2nd shifting: (left shifting property):

$$z\{u_{n+k}\} = z^k\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \cdots \frac{u_{k-1}}{z^{k-1}}\right]$$

multiplication by n^p $(p \in z^+)$:

If $z\{u_n\} = \bar{u}(z)$

$\Rightarrow$ ① $z\{n \cdot u_n\} = -z \cdot \frac{d}{dz}(\bar{u}(z))$

② $z\{n^p \cdot u_n\} = \left(-z\frac{d}{dz}\right)^p(\bar{u}(z))$

Division by n:

$$z\left\{\frac{u_n}{n}\right\} = -\int_0^z z^{-1}\,\bar{u}(z)\,dz$$

Convolution property:

If u_n & v_n are discrete function then

$$u_n * v_n = \sum_{m=0}^{n} u_m v_{n-m}$$

Initial value theorem:

If $Z\{u_n\} = \bar{u}(z)$ then

① $u_0 = \lim_{z \to \infty} \bar{u}(z)$

② $u_k = \lim_{z \to \infty} z\{u_{n+k}\}$

Eg: $u_1 = \lim_{z \to \infty} z\{u_{n+1}\}$

$= \lim_{z \to \infty} z\left[\bar{u}(z) - u_0\right]$

$\left.\frac{u_{k-1}}{z^{k-1}}\right)$

$u_2 = \lim_{z \to \infty} z\{u_{n+2}\}$

$= \lim_{z \to \infty} z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$

Final value theorem:

$$\lim_{n \to \infty} u_n = \lim_{z \to 1} (z-1)\,\bar{u}(z)$$

(1) $Z\left[3n^2+5\sin n\frac{\pi}{4}+3a^4\right]$

$$3\,\frac{z^2+z}{(z-1)^3}+5\,\frac{z\sin\pi/4}{z^2-2z\cos\pi/4+1}+3a^4\,\frac{z}{z-1} \quad |z|>1$$

luy

(2) $z\left\{2^{2n+3}\right\}$

$$z\left(2^{2n}\cdot 2^3\right) = z\left[8\cdot 4^n\right]$$

$$= \frac{8z}{z-4} \quad ; \quad |z|>4$$

(3) $z\left[a^{|n|}\right]$ and condition for existence

so

$$z\left\{a^{|n|}\right\} = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= az+(az)^2+(az)^3+\dots\infty+\frac{z}{z-a}$$

$$= \frac{az}{1-az}+\frac{z}{z-a}$$

$$|az|<1 \quad |z|>a$$

$$|z|<1/a \quad |z|>a$$

$$\Rightarrow |a|<|z|<\frac{1}{|a|}$$

$$\Rightarrow |a|^2<1$$

$$\Rightarrow |a|<1 \quad a^2<1$$

(1) $Z\{c$

(2) z

(4) w

(5)

(6)

$|z|>1$

Illy $Z\{2^{|n|}\} \to$ Does not Exist

$Z\{(-\frac{1}{2})^n\} \to$ Exist

	Domain	Transformation	ROC
(1) $Z\{a^n\}$	$n \geq 0$	$\frac{z}{z-a}$	$\|z\|>\|a\|$
(2) $Z\{a^n\}$	$n < 0$	$\frac{z}{a-z}$	$\|z\|<\|a\|$

(4) write $Z\{2^n+3^n\}$ & ROC

$\frac{z}{z-2}$ $|z|>2$ + $\frac{z}{z-3}$ $|z|>3$

ROC $|z|>3$

(5) write $Z\{2^n+3^n\}$ for $n<0$ & ROC

$\frac{z}{2-z}$ $|z|<2$ + $\frac{z}{z-3}$ $|z|<3$

ROC $|z|<2$

(6) $u_n = \begin{cases} 3^n & n\geq 0 \\ 2^n & n<0 \end{cases}$ then $Z\{u_n\} = ?$

$Z(3^n) = \frac{z}{z-3}$ $|z|>3$ no common ROC

$Z(2^n) = \frac{z}{2-z}$ $|z|<2$

(7) $u_n = \begin{cases} 2^n & n \geq 0 \\ 3^n & n < 0 \end{cases}$ then $Z(u_n) = ?$

$$U(z) = \frac{z}{z-2} + \frac{z}{3-z}$$

$$|z| < 3 \quad \& \quad |z| > 2$$

ROC: $2 < |z| < 3$

(8) If $u_6 = \{1, 3, 5, 7, 0, 9\}$ $z\{u_6\} = ?$

$$z(u_n) = \sum_{n=0}^{5} u_n z^{-n}$$

$$= 1 + \frac{3}{z} + \frac{5}{z^2} + \frac{7}{z^3} + \frac{9}{z^5}$$

(9) If $u_6 = \{1, 3, 5, 7, 0, 9\}$
↑ ↑

$$z(u_n) = \sum_{n=-2}^{3} u_n z^{-n}$$

$$= z^2 + 3z + 5 + \frac{7}{z} + \frac{9}{z^3}$$

(10) $z\{u_k\}$. If $u_k = \frac{1}{4^k}$; $-2 \leq k \leq 2$

$$z\{u_k\} = \sum_{n=-2}^{2} u_k \cdot z^{-k}$$

$$= \sum_{k=-2}^{2} \frac{1}{4^k} \cdot z^{-k}$$

$$= 16z^2 + 4z + 1 + \frac{1}{4z} + \frac{1}{16z^2}$$

$0 < |z| < \infty$

(11) Z

(12) Z

(13) T

(11) $Z\left\{{}^{n}C_k\right\}$; $0 \le k \le n$

$$Z\left[{}^{n}C_k\right] = \sum_{k=0}^{n} {}^{n}C_k\, z^{-k}$$

$$= {}^{n}C_0 + {}^{n}C_1 \frac{1}{z} + {}^{n}C_2 \frac{1}{z^2} + \cdots + {}^{n}C_n \cdot \frac{1}{z^n}$$

$$= \left(1 + \frac{1}{z}\right)^n$$

(12) $Z\left\{{}^{(n+k)}C_n\right\}$ $\quad k \ge 0$

$$= \sum_{k=0}^{\infty} {}^{(n+k)}C_n\, z^{-k}$$

$$= \sum_{k=0}^{\infty} {}^{n+k}C_k\, z^{-k} \qquad \because {}^{n}C_r = {}^{n}C_{n-r}$$

$$= {}^{n}C_0 + {}^{n+1}C_1 \cdot \frac{1}{z} + {}^{n+2}C_2 \frac{1}{z^2} + \cdots$$

$$= 1 + (n+1) \cdot \frac{1}{z} + \frac{(n+2)(n+1)!}{2!} \frac{1}{z^2} + \cdots$$

$$= 1 - (-n-1)\frac{1}{z} + \frac{(-n-1)(-n-2)}{2!} \frac{1}{z^2} - \frac{(-n-1)(-n-2)(-n-3)}{3!} \cdot \frac{1}{z^3} + \cdots$$

$$= \left(1 + \frac{1}{z}\right)^{-(n+1)}$$

(13) $\frac{1}{n!} * \frac{1}{n!}$

$$\sum_{m=0}^{n} \frac{1}{m!} \frac{1}{(n-m)!}$$

z^{2}

$$= \frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{2!(n-2)!} + \frac{1}{3!(n-3)!} + \cdots + \frac{1}{n!}$$ (15)

$$= \frac{1}{n!}\left[1 + n + \frac{n(n-1)}{2!} + \cdots + 1\right]$$

$$= \frac{1}{n!}\left({}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n\right)$$

$$= \frac{1}{n!}(2^n)$$ (16)

IIIly:

$$\frac{1}{n!} * \frac{1}{n!} * \frac{1}{n!} = \frac{3^n}{n!} \quad ; \quad n \geq 1$$

✓ $n=1 \quad 1*1*1 = 3$

$n=0 \quad 1*1*1 = 1 \quad \times$

(17) Z

(14) $Z\{a^n n\, u(n)\}$

$$Z[a^n n\, u(n)] =$$

$$Z[a^n n]$$

$$Z\{n\} = \frac{z}{(z-1)^2}$$

Damping rule

$$Z[a^n n] = \frac{z/a}{(z/a-1)^2}$$

$$= \frac{az}{(z-a)^2}$$

unit step

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & n < 0 \end{cases}$$ (18) I

$$Z\{u(n)\} = Z\{1\}$$

$$= \frac{z}{z-1} \; ; \; |z| > 1$$

unit impulse sequence

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

$$Z\{\delta(n)\} = 1 + 0 + 0 + \cdots$$

$$= 1$$

(15) $z\left\{ a^n/n! \right\}$

$$z[1/n!] = e^{1/z}$$

$$z\{a^n/n!\} = e^{1/z/a} = e^{a/z} \quad (DR)$$

(16) $z\{ e^{-an} \cos n\theta \}$

$$= z[(e^{a})^{-n} \cos n\theta] \quad (D.R)$$

$$= \frac{(ze^{a})^2 - (ze^{a})\cos\theta}{(ze^{a})^2 - 2(ze^{a})\cos\theta + 1}$$

(17) $z\{ n\cos n\theta \}$

$$= -z \frac{d}{dz}\left(\frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1} \right)$$

$n \geq 0$

$n < 0$

(18) If $u_n = n^2$ then $z\{u_{n-1}\} = ?$

$$z\{u_{n-1}\} = z^{-1} z\{u_n\}$$

F.S. Theorem

; $|z| > 0$

$$= z^{-1} z\{n^2\}$$

quence

$$= z^{-1}\left(\frac{z^2 + z}{(z-1)^3} \right)$$

or $n = 0$

or $n \neq 0$

$$= \frac{z+1}{(z-1)^3}$$

$0 + 0 + \cdots$

1

(19) $Z[u_n] = \bar{u}(z) = \frac{z}{z-1} + \frac{z^2}{z+1}$ then $Z[u_{n-1}] = ?$

Sol

$$Z[u_{n-1}] = z^{-1}(\bar{u}(z))$$

$$= \frac{1}{z-1} + \frac{z}{z^2+1}$$

(21) z

(20) If $Z[u_n] = \frac{z}{z-1} + \frac{z}{z^2-1}$ then $Z[u_{n+2}] = ?$

$$Z[u_{n+2}] = z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$$= z^2\left(\frac{z}{z-1} + \frac{z}{z^2-1}\right)$$

$$= \frac{z^3}{z-1} + \frac{z^3}{z^2-1}$$

Initial value: $u_0 = \lim_{z\to\infty} \bar{u}(z) = \lim_{z\to\infty} \frac{z}{z-1} + \frac{z}{z^2-1}$

$$= \lim_{z\to\infty} \frac{1}{1-1/z} + \frac{1}{z-1/z}$$

$$= 1+0 = 1$$

$= 1$

$$u_1 = \lim_{z\to\infty} z(u(z) - u_0) =$$

$$= \lim_{z\to\infty} z\left[\frac{z}{z-1} + \frac{z}{z^2-1} - 1\right]$$

$\frac{2z}{1} + \frac{2z}{2z}$

$$= \lim_{z\to\infty} z\left(\frac{\cancel{z}-(\cancel{z}-1)}{z-1} + \frac{z}{z^2-1}\right)$$

$$= \lim_{z\to\infty} \frac{z}{z-1} + \frac{z^2}{z^2-1} = 2$$

$$Z[u_{n+2}] = z^2\left(\bar{u}(z) - 1 - \frac{2}{z}\right)$$

(21) $Z[u_n] = \bar{u}(z) = \dfrac{2z^2+3z+4}{(z-3)^3}$ then (u_2, u_3, u_4).

$|z| > 3$

$$|z| > 3$$
$$|z/3| > 1$$
$$\Rightarrow |3/z| < 1$$

$$= \frac{2z^2+3z+4}{(z-3)^3} = \frac{2z^2+3z+4}{z^3(1-3/z)^3}$$

$$= \frac{2z^2+3z+4}{z^3}\left(1-3/z\right)^{-3}$$

$$= \frac{2z^2+3z+4}{z^3}\left[1+3\cdot\frac{3}{z}+6\cdot\left(\frac{3}{z}\right)^2+10\left(\frac{3}{z}\right)^3+15\left(\frac{3}{z}\right)^4+\cdots\right]$$

$$= (2z^2+3z+4)\left[\frac{1}{z^3}+\frac{9}{z^4}+\frac{54}{z^5}+\frac{270}{z^6}+\cdots\right]$$

$u_2 = \text{Co-eff of } \frac{1}{z^2} = 18+3 = 21$

$u_3 = \text{Co-eff of } \frac{1}{z^3} = 54\times2+9\times3+4 =$

$u_4 = \text{Co-eff of } \frac{1}{z^4} = 270\times2+54\times3+9\times4 =$

$\frac{z}{z-1}$

$\frac{1}{z-1/z}$

$\frac{2z}{1}+\frac{2z}{2z}$

$= 2$

INVERSE TRANSFORMATION

If $Z\{f(n)\} = \bar{F}(z) \Rightarrow f(n) = Z^{-1}\{\bar{F}(z)\}$

Linearity $Z^{-1}\{a\bar{F}(z) \pm b\bar{G}(z)\} = aZ^{-1}\{\bar{F}(z)\} \pm bZ^{-1}\{\bar{G}(z)\}$

1. Standard formulae (formulae based)
2. Partial fractions
3. Convolution property
4. power series / Division method.
5. Contour integration

2. $\frac{u(z)}{z} = \frac{z}{(z+1)(z-2)} \Rightarrow \frac{u(z)}{z} = \frac{1}{(z-1)(z-2)}$

In this method split the partial fraction for $\frac{u(z)}{z}$ not for $u(z)$ directely.

3. $Z^{-1}\{\bar{U}(z)\} = u_n$ & $Z^{-1}\{\bar{V}(z)\} = v_n$

$\Rightarrow Z^{-1}\{\bar{U}(z) \cdot \bar{V}(z)\} = u_n * v_n$

$= \sum_{m=0}^{n} u_m v_{n-m}$

4. Expand $f(z)$ as power series (using Maclaren's, Taylors, Laurentz etc...) or Division method

$$\sum_n u_n z^{-n}$$

where u_n is reqd inverse ~~to~~ transform.

$\bar{g}(z)$

5. $$z^{-1}\{\bar{u}(z)\} = \frac{1}{2\pi i}\oint_C \bar{u}(z)\cdot z^{n-1}\,dz$$

By residue theorem

$= \frac{1}{2\pi i} \times 2\pi i$ [Sum of Residues at Each pole of $\bar{u}(z)\, z^{n-1}$ which are inside of C]

$$= \sum_i \gamma_i$$

$)(z-2)$

(1) $z=a$ is a simple pole i.e of order one then

$$\operatorname{Res}[f(z) : z=a] = \lim_{z \to a} (z-a) f(z)$$

con ~~for~~

(2) If $z=a$ is a pole of order m then

$$\operatorname{Res}[f(z) : z=a] = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$$

→ If C is not given then consider C consisting of all poles.

is, Taylors,

1. $\bar{z}^1\left(\frac{z}{z-1}\right) = u(n)$

2. $\bar{z}^1(1) = \delta(n)$.

3. $\bar{z}^1\left(\frac{z}{z-a}\right) = a^n u(n)$

4. $\bar{z}^1\left(\frac{z}{z+a}\right) = (-a)^n u(n)$

5. $\bar{z}^1\left[\log\left(\frac{z}{z+1}\right)\right]$

$$\bar{z}^1\left[-\log\left(\frac{z+1}{z}\right)\right] = \bar{z}^1\left[-\log(1+1/z)\right]$$

$$= -\bar{z}^1\left[\frac{1}{z} - \frac{1}{2z^2} + \frac{1}{3z^3} - \frac{1}{4z^4} + \cdots\right]$$

$$= -\bar{z}^1\left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^{-n}\right]$$

$$= -\frac{(-1)^{n-1}}{n} \quad \text{for } n \geq 1$$

$$= \frac{(-1)^{n-1}}{n} u(n)$$

$\bar{z}^1(e^{2/z})$

$$= \bar{z}^1\left(1 + \frac{2}{z} + \frac{2^2}{2!\,z^2} + \frac{2^3}{3!\,z^3} + \cdots\right)$$

$$= \bar{z}^1\left(\sum_{n=0}^{\infty} \frac{2^n}{n!} z^{-n}\right)$$

$$= \frac{2^n}{n!} u(n)$$

Convolution

$$z^{-1}\left(e^{2/z}\right) = z^{-1}\left(e^{1/z} \cdot e^{1/z}\right)$$

$$= 1/n! * \frac{1}{n!} = \frac{2^n}{n!}$$

3. $z^{-1}\left(\frac{z}{z^2-7z+10}\right)$

$$U(z) = \frac{z}{(z-5)(z-2)}$$

$$\frac{U(z)}{z} = \frac{1}{(z-5)(z-2)} = \frac{1/3}{z-5} - \frac{1/3}{z-2}$$

$$U(z) = \frac{1}{3}\frac{z}{z-5} - \frac{1}{3}\cdot\frac{z}{z-2}$$

$$z^{-1}\left[U(z)\right] = \left(\frac{1}{3}5^n - \frac{1}{3}2^n\right)u(n)$$

Contour integration method

$$z^{-1}\left(\frac{z \cdot z^{n-1} = z^n}{(z-5)(z-2)}\right)$$

$$r_1|_{z=5} = 5^n/3$$

$$r_2|_{z=2} = 2^n/-3$$

$$r_1 + r_2 = \left(\frac{5^n - 2^n}{3}\right)u(n)$$

4. $z^{-1}\left(\frac{8z^2}{(4z-1)(2z+1)}\right)$

$$z^{-1}\left(\frac{8z^2}{4(z-1/4)2(z+1/2)}\right)$$

$$z^{-1}\left[\frac{z^2 \cdot z^{n-1}}{(z-1/4)(z+1/2)} z^{n+1}\right]$$

$$r_1\Big|_{z=1/4} = \frac{(1/4)^{n+1}}{\frac{1}{4}+\frac{1}{2}} = \frac{(1/4)^{n+1}}{3/4} = \frac{1}{3}\left(\frac{1}{4}\right)^n$$

$\dot{z}$

Case ii

$$r_2\Big|_{z=-1/2} = \frac{(-1/2)^{n+1}}{-1/2-1/4} = \frac{(-1/2)^{n+1}}{-3/4} = -\frac{1}{3}\cdot(-1/2)^{n-1}$$

$$r_1+r_2 = \frac{1}{3}\left[\left(\frac{1}{4}\right)^n - \left(\frac{-1}{2}\right)^{n-1}\right] u(n) \;//$$

5. $z^{-1}\left(\frac{1}{z-2}\right)$

$$z^{-1}\left(\frac{z^{n-1}}{z-2}\right)$$

$$r_1\Big|_{z=2} = 2^{n-1}\, u(n-1)$$

6. $z^{-1}\left(\frac{1}{(z-2)^5}\right)$

2^{n-1}

$$\frac{1}{4!}(n-1)(n-2)(n-3)(n-4)\, 2^{n-5} \;//$$

z^{-1}

Methode for $z^{-1}\left(\frac{1}{z-2}\right)$

Case i: If $|z| > 2$

$$\left|\frac{2}{z}\right| < 1$$

$$\frac{1}{z-2} = \frac{1}{z(1-2/z)} = \frac{1}{z}\left(1-2/z\right)^{-1}$$

r

$= \frac{1}{2}\left[1 + (2/z)^2 + (3/2)^2 + \cdots\right]$

$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n}$

$\bar{z}\left(\frac{1}{z-2}\right) = 2^{n-1}$ for $n \geq 1$ $= 2^{n-1} u(n-1)$

$-\frac{1}{2})^{n-1}$

Case ii

If $|z| < 2$

$|\frac{z}{2}| < 1$

$\frac{1}{z-2} = \frac{1}{-2(1-z/2)} = -\frac{1}{2}(1-z/2)^{-1}$

$= -\frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$

$= -\frac{1}{2}\sum_{n=-\infty}^{0} 2^{n} z^{-n} = -\sum_{n=-\infty}^{0} 2^{n-1} z^{-n}$

$z^{-1}\left(\frac{1}{z-2}\right) = -2^{n-1}$ for $n \leq 0$

$= -2^{n-1} u(-n)$ //

$z^{-1}\left(\frac{z}{z^2-2z+2}\right)$

$z^2-2z+2 = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$z^{-1}\left(\frac{z\, z^{n-1}}{(z-1-i)(z-1+i)}\right)$

$r_1 \Big|_{z=1+i} = \frac{(1+i)^n}{\cdots}$

$$r_2 \Big|_{z=1-i} = \frac{(1-i)^n}{-2i}$$

$$r_1 + r_2 = \frac{1}{2i}\left[(1+i)^n - (1-i)^n\right] u(n)$$

a)

Q) $z^{-1}\left(\frac{z}{z^2+1}\right)$

$z^2+1 \Rightarrow z^2 = -1 \Rightarrow z = \pm i$

$$z^{-1}\left(\frac{z\, z^{n-1}}{(z-i)(z+i)}\right)$$

$$r_1 \Big|_{z=i} = \frac{(i)^n}{2i}$$

$$r_2 \Big|_{z=-i} = \frac{(-i)^n}{-2i}$$

$$r_1 + r_2 = \frac{1}{2i}\left[(i)^n - (-i)^n\right] u(n)$$

$$= \frac{\left(\cos \pi/2 + i \sin \pi/2\right)^n - \left(\cos \pi/2 - i \sin \pi/2\right)^n}{2i}$$

$$= \frac{2i \sin n\pi/2}{2i} = \sin n\pi/2$$

a)

Q) $z^{-1}\left(\frac{z^2}{(z-3)^5}\right)$

$$z^{-1}\left(\frac{z^2 \cdot z^{n-1}}{(z-3)^5} : z^{n+1}\right)$$

$$r\Big|_{z=3} = \frac{1}{4!}\left[(n+1)\, n\,(n-1)(n-2)\, 3^{n-3}\right] u(n)$$

$\underline{Q)}$ $\quad z^{-1}\left(\left(\frac{z}{z-a}\right)^2\right)$

$$z^{-1}\left(\frac{z}{z-a}\cdot\frac{z}{z-a}\right) = \frac{z}{z-a} * \frac{z}{z-a}$$

$$= a^n * a^n$$

$$= \sum_{m=0}^{n} a^{\cancel{m}}\, a^{n-\cancel{m}}$$

$$= a^n \sum_{m=0}^{n} 1$$

$$= a^n(n+1)$$

$\underline{\text{Contour:}}$

$$z^{-1}\left(\frac{z^2}{(z-a)^2}\, z^{n-1}\right)$$

$$z^{-1}\left[\frac{z^{n+1}}{(z-a)^2}\right]$$

$\underline{n(1/2)^n}$

$$r\Big|_{z=a} = \frac{1}{1!}(n+1)\cdot a^n = (n+1)a^n$$

$\underline{1/2}$

$\underline{Q)}$ $\quad z^{-1}\left(\frac{z^2}{(z-a)(z-b)}\right)$

$$z^{-1}\left(\frac{z^2\cdot z^{n-1}}{(z-a)(z-b)}\right) \quad \text{(Contour method)}$$

$$\frac{z^{n+1}}{(z-a)(z-b)}$$

$$r_1\Big|_{z=a} = \frac{a^{n+1}}{a-b}$$

$$r_2\Big|_{z=b} = \frac{b^{n+1}}{b-a}$$

$$= \frac{1}{a-b}\left[a^{n+1} - b^{n+1}\right] = \frac{\left(a^n \cdot a - b^n \cdot b\right)}{(a-b)} u(n)$$

$$= \left(\frac{a^{n+1} - b^{n+1}}{a-b}\right) u(n)$$

By convolution method

$$z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-b}\right]$$

$$a^n * b^n$$

$$\sum_{m=0}^{n} a^m b^{n-m}$$

$$b^n \left[\sum_{m=0}^{n} (a/b)^m\right]$$

$$= b^n \left[1 + a/b + (a/b)^2 + \cdots + (a/b)^n\right]$$

Sum to n terms in G.P

$$S_{n+1} = \frac{a(r^{n+1} - 1)}{r-1}$$

$$= b^n \left[1 \cdot \frac{(a/b)^{n+1} - 1}{(a/b) - 1}\right]$$

Q). If

Sol

Appli

Sole

Q) If $Z[f(n)] = \dfrac{z}{(z-1/2)(z+3)(z-2)}$ in $|z|<1$ then $F(1) = ?$

Sol

$$Z[f(n)] = \frac{z \cdot z^{n-1}}{(z-1/2)(z+3)(z-2)} = \frac{z^n}{(z-1/2)(z+3)(z-2)}$$

$$r_1\Big|_{z=1/2} = \frac{(1/2)^n}{(1/2+3)(1/2-2)}$$

Consider $|z|<1$ there $1/2$ lies within the unit circle.

$$\text{Therefore } f(n) = \frac{(1/2)^n}{(7/2)(-3/2)} = -\frac{4}{21}(1/2)^n$$

$$f(1) = -2/21$$

Applications of Z-transforms to Difference Eqns:

Solution of $u_{n+1} - u_n = 0$ & $u_0 = 1$

$$Z[u_{n+1} - u_n] = Z[0]$$

$$z[\bar{u}(z) - u_0] - \bar{u}(z) = 0$$

$$\bar{u}(z)[z-1] = z u_0 = z$$

$$\bar{u}(z) = \frac{z}{z-1}$$

$$u_n = Z^{-1}\left(\frac{z}{z-1}\right) = 1$$

(3) Solution of $u_{n+2} - 3u_{n+1} + 2u_n = 1$; $u_0 = u_1 = 0$

$$Z[u_{n+2} - 3u_{n+1} + 2u_n] = Z(1)$$

$$\cancel{Z[\bar{u}(z)]}\,(z^2 - 3z + 2)\bar{u}(z) = \frac{z}{z-1}$$

$$\bar{u}(z) = \frac{z}{(z-1)(z^2-3z+2)}$$ (32)

$$\bar{u}(z) = \frac{z}{(z-1)(z-2)(z-1)}$$

$$u_n \quad u\cancel{(z)} = Z^{-1}\left(\frac{z}{(z-1)^2(z-2)}\right)$$

$$Z^{-1}\left(\frac{z\,z^{n-1}}{(z-1)^2(z-2)} = z^n\right)$$

$$r_1 \Big|_{z=1} : \cancel{n}$$

$$r_2 \Big| \cancel{z=1} ; \cancel{n}\; n(n-1)z^{n-2}$$

$$r_1 + r_2 = \cancel{(z-1)^n} + n(n-1)z^{n-2}$$

$r \Big|_{z=1}$

$$r_1 \Big|_{z=2} = \frac{2^n}{(1)}$$

$$r_2 \Big|_{z=1} = \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz}\left(\frac{z^n}{z-2}\right)$$

$r \Big|_z$

$$= \lim_{z\to 1} \frac{(z-2)nz^{n-1} - z^n}{(z-2)^2}$$

$$= \frac{(-1)n-1}{(1-2)^2} = -(n+1)$$

$$u_n = 2^n - (n+1)$$

(32) $(u_{n+2} + 2u_{n+1} + u_n) = n$; $u_0 = u_1 = 0$

$$Z(u_{n+2} + 2u_{n+1} + u_n) = Z(n)$$

$$\bar{u}(z)(z^2 + 2z + 1) = \frac{z}{(z-1)^2}$$

$$\bar{u}_1(z) = \frac{z}{(z-1)^2 (z+1)^2}$$

$z = 1$ & -1 are second order poles.

$$u_n = Z^{-1}\left(\frac{z \cdot z^{n-1} = z^n}{(z-1)^2 (z+1)^2}\right)$$

$$r|_{z=1} = \lim_{z \to 1} \frac{d}{dz}\left[\frac{z^n}{(z+1)^2}\right]$$

$$= \lim_{z \to 1} \frac{(z+1)^2 \cdot n z^{n-1} - z^n \cdot 2(z+1)}{(z+1)^4}$$

$$= \frac{4n-4}{16} = \frac{4(n-1)}{16} = \frac{n-1}{4}$$

$$r|_{z=-1} = \lim_{z \to -1} \frac{d}{dz}\left[\frac{z^n}{(z-1)^2}\right]$$

$$= \lim_{z \to -1} \frac{(z+1)^2 \, n(z)^{n-1} - z^n \, 2(z+1)}{(z+1)^4}$$

$$= 4(-1)^{n-1} \frac{(n-1)}{1/64}$$

$$= (-1)^{n-1} \frac{(n-1)}{1/64}$$

$$r_1 + r_2 = \frac{(n-1)}{4} \left(1 + (-1)^{n-1}\right)$$

$$u_n = \frac{(n-1)}{4} \left(1 + (-1)^{n-1}\right) //$$

satisfyi

or a for

periodic

form .

$(-c, c)$

Case i :

Case ii

Fourier Transformation:

If $f(x)$ is a periodic function ~~with~~ which satisfying dirchlet condition then it can be Expanded as a fourier series. By Expanding this concept to non periodic ~~signals~~ function they can Expand as an integral form.

If a function $f(x)$ is defined in the interval $(-c, c)$ and satisfying the dirchlet condition as $c \to \infty$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x)\, dt\, d\lambda$$

↳ fourier integral

Case i: $f(x)$ is an Even function

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \cos \lambda t \cos \lambda x\, dt\, d\lambda$$

↳ fourier cosine integral

Case ii If $f(x)$ is an odd fun

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \sin \lambda t \sin \lambda x\, dt\, d\lambda$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i\lambda(t-x)}\, dt\, dx$$

↳ fourier integral in Complex form.

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ist} dt}_{\bar{F}(s)} \, ds$$ ② 1

$$\bar{F}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \, e^{-ist} dt \longrightarrow F.T$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \bar{F}(s) \, ds \longrightarrow I.F.T$$ Finite

Different forms of F.T:

① $$\bar{F}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \, e^{ist} dt \text{ — } F.T$$

② $$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{F}(s) \, e^{-ist} ds \longrightarrow I.F.T$$

① what

$$\bar{F}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \, e^{ist} dt \longrightarrow F.T$$ Sol

$$f(t) = 1 \cdot \int_{-\infty}^{\infty} \bar{F}(s) \, e^{-ist} ds \longrightarrow I.F.T.$$

Fourier Sine/Cosine Transform:

① $$F_s\{f(t)\} = \bar{F}_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin st \, dt$$
↳ F.S.T

$$f(t) = F_s^{-1}\{\bar{F}_s(s)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{F}_s(s) \sin st \, ds$$
↳ I.F.S.T

② $F_c\{f(t)\} = \bar{f}_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \, dt$

↳ F.C.T.

$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(s) \cdot \cos st \, ds$

↳ I.F.C.T.

Finite Fourier Sine/Cosine Transforms

If $f(x)$ is defined in $(0, l)$

$$\left.\begin{aligned} \bar{f}_s(s) &= \int_0^l f(t) \sin \frac{n\pi t}{l} dt \rightarrow \text{F.F.S.T} \\ \bar{f}(s) &= \int_0^l f(t) \cos \frac{n\pi t}{l} dt \rightarrow \text{F.F.C.T} \end{aligned}\right\} n \in Z^+$$

① what is the fourier integral of $f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

Sol

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos \lambda (t - x) \, dt \, d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \left[\int_{-1}^1 (1) \cos (\lambda t - \lambda x) \, dt \, d\lambda \right]$$

$$= \frac{1}{\pi} \int_0^\infty \left[\frac{\sin(\lambda t - \lambda x)}{\lambda} \right]_{t=-1}^1 d\lambda$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\frac{\sin(\lambda - \lambda x) - \sin(-\lambda - \lambda x)}{\lambda} \right] d\lambda$$

$$f(x) = \frac{1}{\pi} \int^\infty \frac{2 \sin \lambda \cos \lambda x}{\lambda} d\lambda$$

2) Find the fourier transform of $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$ (3)

Sol

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{ist} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1)\, e^{ist} dt$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{ist}}{is} \right)_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isa} - e^{-isa}}{is} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2\sin as}{s}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin as}{s} ds \quad (I.F.T)$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-ist} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{2\sin as}{s} e^{-ist} ds$$

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} e^{-ist} ds$$

$t = 0$

$$f(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} ds$$

$$\int_{-\infty}^{\infty} \frac{\sin as}{s} ds = \pi$$

$|x|<a$
$|x|>a$

(3) Fourier Sine Transform of e^{-ax}

Sol

$$\{F_s(f(x))\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{s}{a^2+s^2}$$

or

$$\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$

$$= \left[0 - \frac{1}{a^2+s^2}(-s)\right]$$

$$= \frac{s}{s^2+a^2}$$

(4) Fourier sine transform of xe^{-ax}

$$F_s\{x.e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\sin sx\,dx$$

Integration w.r.t s

$$\int F_s(s)\,ds = \sqrt{\frac{2}{\pi}}\int_0^\infty \cancel{x}e^{-ax}\left(\frac{-\cos sx}{\cancel{x}}\right)dx$$

$$= -\sqrt{\frac{2}{\pi}}\ \frac{a}{s^2+a^2}$$

Diff ... w.r.t s

$$F_s(s) = -\sqrt{\frac{2}{\pi}}\ \frac{-a\times 2s}{(s^2+a^2)^2} = \left(\sqrt{\frac{2}{\pi}}\cdot\frac{2as}{(s^2+a^2)^2}\right)$$

$$\boxed{\begin{aligned} F_S\{f(x)\} &= \sqrt{\frac{2}{\pi}}\ \frac{s}{s^2+a^2} \\ F_C\{f(x)\} &= \sqrt{\frac{2}{\pi}}\ \frac{a}{s^2+a^2} \end{aligned}}$$

: Fouri

Q) F.S.T of $\frac{e^{-ax}}{x}$

$$F_S\left\{\frac{1}{x}e^{-ax}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}e^{-ax}\sin sx\,dx$$

Diff w.r.t s

$$\frac{d}{ds}\bar{f}_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{\cancel{x}}e^{-ax}\cdot\cancel{x}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{a}{s^2+a^2}$$

Integration w.r.t s

$$\bar{f}(s) = \sqrt{\frac{2}{\pi}}\ \tan^{-1}(s/a)$$

Q) F

Q) Fourier sine transform of $\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{s}}$

$$F_S\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$$

$$F_C\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$$

$$F\left[e^{-x^2/2}\right] = e^{-s^2/2}$$

→ self reciprocal function

Fourier Sine transform of $\frac{1}{x}$?

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \cdot \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{\pi}{2} \left[\because \int_0^\infty \frac{1}{t} \sin at \, dt = \pi/2 \right]$$

Q) F.S.T of $\frac{1}{\sqrt{x}}$?

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{\sqrt{x}} \sin sx \, dx$$

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$$

$$F_s(s) = -\operatorname{Img} \int_0^\infty x^{-1/2} \left[e^{-isx}\right] dx$$

$$\Rightarrow -isx = t$$

$$x = \frac{t}{is} \Rightarrow dx = \frac{dt}{is}$$

$$F_s(s) = -\operatorname{Img} \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{t}{is}\right)^{-1/2} e^{-t} \frac{dt}{is}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{(is)^{1/2}} \cdot \Gamma \tfrac{1}{2}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi} \sqrt{s} (i)^{1/2}} \Gamma \tfrac{1}{2}$$

$$= -\operatorname{img} \text{ of } \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{s}} \frac{(-i)^{1/2}}{(i)^{1/2} (-i)^{1/2}}$$

$$= -\operatorname{Imag} \text{ of } \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{s}} \left(\cos \pi/2 - i \sin \pi/2\right)^{1/2}$$

self reciprocal function

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{\sqrt{s}} \times \sin \pi/4$$

$$= \frac{1}{\sqrt{s}}.$$

Note:

If the transformation of the function is itself, then it is called self reciprocal under sine and cosine transformation.

→ $F\{e^{-x^2/2}\} = e^{-s^2/2}$ is self-reciprocal under fourier transform.

* Solve for $f(x)$ from $\int_0^\infty f(x) \sin tx \, dx = \begin{cases} 1 & 0<t<1 \\ 2 & 1 \le t<2 \\ 0 & t>2 \end{cases}$

$$F_s\{f(t)\} = \frac{2}{\pi}\int_0^\infty f(x)\sin tx \, dx = \bar{f}(t)$$

$$= \frac{2}{\pi}\left[\int_0^1 1 \sin tx \, dt + \int_1^2 2 \sin tx \, dt\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-\cos tx}{x}\right)_0^1 + \left(\frac{-2\cos tx}{x}\right)_1^2\right]$$

$$= \frac{2}{\pi}\left[-\frac{\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right]$$

* What is $f(x)$ whose sine transform is e^{-as}

$$f(x) = F_s^{-1}\{\bar{f}(s)\}$$

$$= \frac{2}{\pi}\int_0^\infty e^{-as}\sin sx \, ds$$

$$= \frac{2}{\pi}\left[\frac{x}{x^2+a^2}\right]$$

* F.S.

Propert

1. Line

2. Ch

3. Sh

If, then

transformation.

fourier

x dx to)

$\frac{s x}{x}$]

* F.S.T of 1 $(0,\pi)$

$$f(x) = \int_0^{\pi} \frac{\sin n\pi x}{\pi}\,dx$$

$$= \int_0^{\pi} \sin nx\,dx$$

$$= -\frac{\cos nx}{n}\Big]_0^{\pi}$$

$$= \frac{1}{n}\left[-(-1)^n + 1\right]$$

$$= \frac{1}{s}\left[-(-1)^s + 1\right]$$

Properties:

1. Linearity

2. Change of Scaling

$$F\{f(at)\} = \frac{1}{a}\bar{F}\left(\frac{s}{a}\right)$$

$$F\left\{f\left(\frac{t}{a}\right)\right\} = a\bar{F}(as)$$

3. Shifting property:

if $F\{f(t)\} = \bar{F}(s)$

then $F\{f(t)e^{iat}\} = \bar{F}(s+a)$

$$F\{f(t)e^{-iat}\} = \bar{F}(s-a)$$

$$F\{f(t-a)\} = \bar{F}(s)e^{isa}$$

(4) Modulation property

$$F\{f(t)\} = \bar{F}(s)$$

$$\Rightarrow F\{f(t)\cos at\} = \frac{1}{2}\{\bar{F}(s+a) + \bar{F}(s-a)\}$$

(1) $F_s\{f(t)\sin at\} = \frac{1}{2}\{\bar{F_c}(s-a) - \bar{F_c}(s+a)\}$

(2) $F_s\{f(t)\cos at\} = \frac{1}{2}\{\bar{F_s}(s+a) + \bar{F_s}(s-a)\}$

(3) $F_c\{f(t)\sin at\} = \frac{1}{2}\{\bar{F_s}(s+a) - \bar{F_s}(s-a)\}$

(4) $F_c\{f(t)\cos at\} = \frac{1}{2}\{\bar{F_c}(s+a) - \bar{F_c}(s-a)\}$

Relation b/w Laplace and Fourier transform:

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}} L\{g(t)\}$$

$$\text{if } f(t) = \begin{cases} e^{-xt} g(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Convolution theorem:

$$\bar{F}^{1}\{\bar{F}(s)\} = f(t) \quad \& \quad \bar{F}^{1}\{\bar{g}(s)\} = g(t)$$

$$\Rightarrow \bar{F}^{1}\{\bar{F}(s)\cdot\bar{g}(s)\} = f(t) * g(t)$$

$$= \int_{-\infty}^{\infty} f(x)\cdot g(t-x)\,dx$$

Parse

(1) $\frac{1}{2}$

(2) $\frac{1}{2\pi}$

(3) $\frac{2}{\pi}$

(4) $\frac{2}{\pi}$

Parseval's Identities:

(1) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\cdot \bar{G}(s)\,ds = \int_{-\infty}^{\infty} f(x)\cdot \bar{g}(x)\,dx$$

$$F(s) \xleftrightarrow{FT} f(x)$$

$$G(s) \longleftrightarrow g(x)$$

$$\bar{G}(s) \xleftrightarrow{\text{conjugate}} G(s)$$

$$\bar{g}(x) \xrightarrow{\text{conjugate}} g(x)$$

(2) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{\infty}^{\infty} |f(x)|^2\,dx$$

(3) $$\frac{2}{\pi}\int_{-\infty}^{\infty} F_s\{f(x)\}\cdot F_s\{g(x)\}\,ds = \int_0^{\infty} f(x)\,g(x)\,ds$$

(4) $$\frac{2}{\pi}\int_0^{\infty} F_c\{f(x)\}\cdot F_c\{g(x)\}\,ds = \int_0^{\infty} f(x)\,g(x)\,dx.$$

(5) $$\int_0^{\infty}\frac{x^2}{(x^2+a^2)^2}\,dx$$

Sol

$$= \frac{\pi}{2}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\frac{x}{x^2+a^2}\sqrt{\frac{2}{\pi}}\frac{x}{x^2+a^2}\cdot dx$$

$$= \frac{\pi}{2}\int_0^{\infty} e^{-as}\cdot e^{-as}\,ds$$

$$= \frac{\pi}{2}\int_0^{\infty} e^{-2as}\,ds$$

$$= \frac{\pi}{2}\left[\frac{e^{-2as}}{-2a}\right]_0^{\infty}$$

$\pi/2 \left[\frac{1}{2a}\right] = \frac{\pi}{4a}$ 44

* $F_s\{x \cdot f(x)\} = -\frac{d}{ds}\{\bar{F}_c(s)\}$

* $F_c\{x \cdot f(x)\} = \frac{d}{ds}\{\bar{F}_s(s)\}$