GROUP 30

Numerical Modelling of Oil Spillage over Water Surface using Convection-Diffusion Equation

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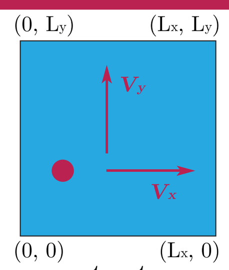
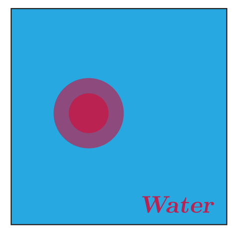
Problem Statement

An oil spill is the release of a liquid petroleum hydrocarbon into the environment, especially the marine ecosystem, due to human activity, and is a form of pollution. Spilled oil and seawater are considered in-compressible fluids, and it is assumed that there is no phase transition and slip at the oil–water interface



$t = t_o$

Goal

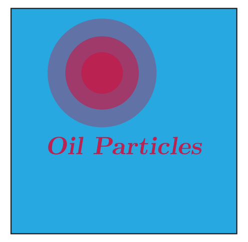


$t > t_o$

The goal of this project is to stimulate diffusion of oil in water and the convection of oil movement of oil with respect to the velocity of water) during oil spillage. A theoretical and mathematical model of convection- diffusion of oil in water(2-D) will be developed and the resulting system of equations will be solved numerically on a computer.

Physical Model

In the case of convection, the water is in motion with a certain velocity, and we use a reference frame attached to the moving water, which has velocities Vx and Vy. The diffusion process is governed by the Diffusion Coefficient D, while the convection part is dependent on the velocity field of the medium **V**. The whole problem is viewed in the 2D plane with entire medium being enclosed within a boundary Lx and Ly.



$t >> t_o$

Assumptions

A. The problem is considered in 2D and the diffusion and convection of oil particles is considered planar.
B. The domain is infinite in all 3 directions, and as the problem is 2D, no variation in z-direction is considered.
C. Each spatial point is assumed to have constant velocity field and in steady state.
D. In a common situation, the diffusion coefficient is constant, there are no sources or sinks, and the velocity field describes an in-compressible flow(i.e., it has zero divergence). Hence the R term is 0.

Governing Equations

Conservation of Mass Equation

$$\frac{d}{dt}\int_V C(x,t)\,dV + \int_A (u\cdot C)\cdot dA = 0$$

Convection-Diffusion Equation

$$\text{Diffusion} = -D\cdot\nabla^2 C$$

$$\frac{\partial C}{\partial t} + u\cdot\nabla C = D\cdot\nabla^2 C$$

$$\boxed{\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) - V_x\frac{\partial C}{\partial x} - V_y\frac{\partial C}{\partial y}}$$

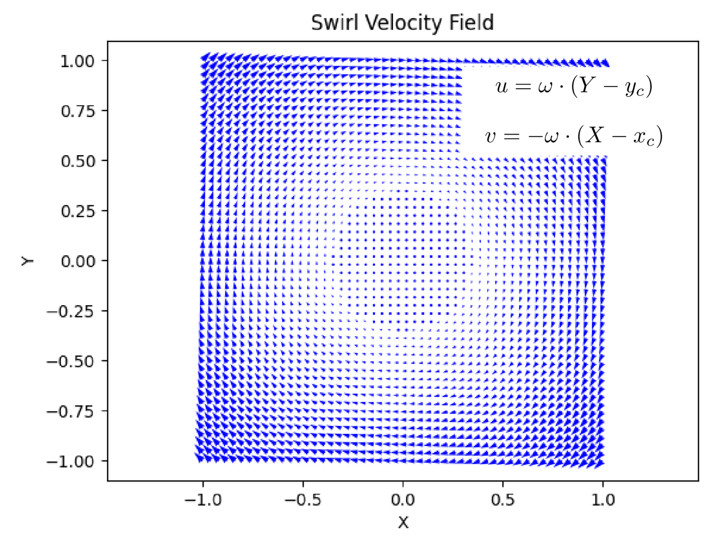
Velocity Field

2D Velocity Field:

$$\boxed{\vec{\mathbf{V}}(x,y,t) = (u(x,y,t), v(x,y,t))}$$

For our analysis, we have considered the diffusion-advection process in three velocity profiles:

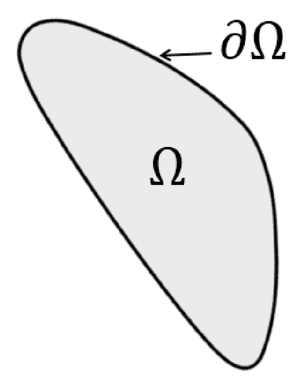
1. Zero Velocity Field (u = v = 0)
2. Constant Velocity Field (u = v = constant)
3. Tilted Velocity Field (u = Vcos(Θ), v = Vsin(Θ))



Boundary Condition

Dirichlet Boundary Condition

$$C(x,y,t) = 0 \quad \text{for } (x,y) \in \partial\Omega$$



where:

- $C(x,y,t)$ is the concentration of oil at position (x,y) and time t.
- $\partial\Omega$ represents the boundary of the computational domain Ω.

This condition implies that the concentration of oil is fixed at 0 at the domain boundaries, indicating impermeable boundaries, no inflow or outflow of oil, and containment of the spill within the modeled domain. The oil spillage is, therefore, contained within the modeled area, and no inflow or outflow of oil occurs through the boundaries. It is a suitable choice for certain modeling scenarios but may not capture the full dynamics of real-world oil spill situations.

Parameters Involved

Parameter	Value
L_x (horizontal limit on water boundary)	$1.00m$
L_y (vertical limit on water boundary)	$1.00m$
D (Diffusion Coefficient)	$10^{-5}m^2/s$
N_x (Spatial x grid points)	50
N_Y (Spatial y grid points)	50
τ_s (Total Simulation Time)	$100.00s$
Δx (X-Grid Size)	$0.02m$
Δy (Y-Grid Size)	$0.02m$
Δt (Time Step)	$0.1s$

Error Analysis

Courant–Friedrichs–Lewy (CFL) condition

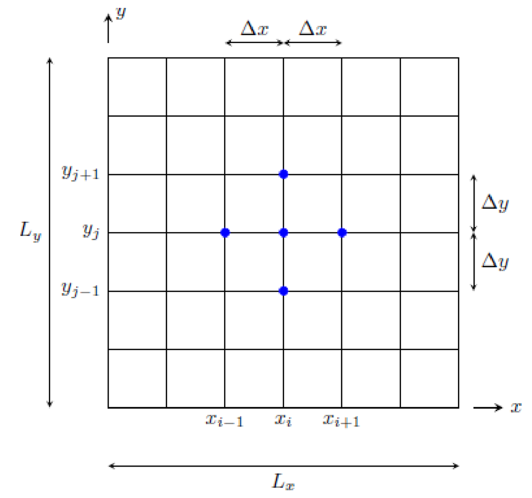
$$C_{\text{convection}} = \frac{V_x\Delta t}{\Delta x} + \frac{V_y\Delta t}{\Delta y} \leq CFL_{\max}$$

$$C_{\text{convection}} + C_{\text{diff}} \leq C_{\max}$$

$$C_{\text{diff}} = \frac{D\Delta t}{(\Delta x^2 + \Delta y^2)} \leq CFL_{max}$$

$$\boxed{\Delta t \leq \min\left(\frac{\Delta x^2}{2(|V_x|+|V_y|)+2D}, \frac{\Delta y^2}{2(|V_x|+|V_y|)+2D}\right)}$$

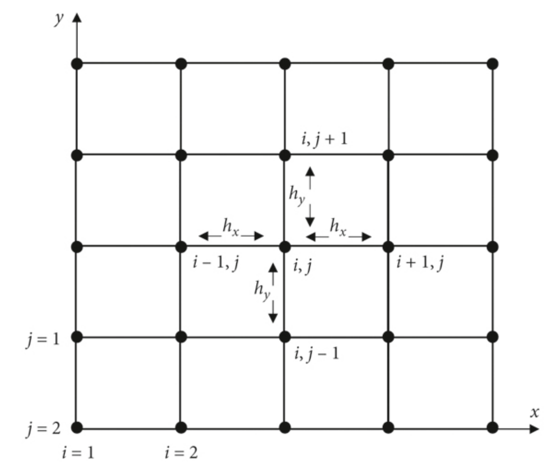
Numerical Solution



$$\frac{\partial^2 C}{\partial x^2} \approx \frac{C_{i+1} - 2C_i + C_{i-1}}{(\Delta x)^2}$$

$$\frac{\partial^2 C}{\partial y^2} \approx \frac{C_{j+1} - 2C_j + C_{j-1}}{(\Delta y)^2}$$

$$\frac{\partial C}{\partial t} \approx \frac{C^{N+1} - C^N}{\Delta t}$$



For 2-Dimensional Case:

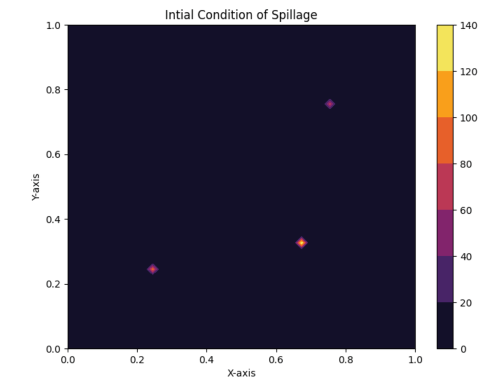
$$\frac{C^{N+1}_{i,j} - C^N_{i,j}}{\Delta t} = D\left(\frac{C^N_{i+1,j} - 2C^N_{i,j} + C^N_{i-1,j}}{(\Delta x)^2} + \frac{C^N_{i,j+1} - 2C^N_{i,j} + C^N_{i,j-1}}{(\Delta y)^2}\right)$$

$$- V_x\left(\frac{C^N_{i+1,j} - C^N_{i,j}}{\Delta x}\right) - V_y\left(\frac{C^N_{i,j+1} - C^N_{i,j}}{\Delta y}\right)$$

where $C^N_{i,j}$ represents the $x = i^{th}, y = j^{th}$ grid point in the N^{th} time step. Notice that for each time-step, we are evaluating an entire 2D array of spatial grid points! And each $C^{N+1}_{i,j}$ cell in the grid is dependent on five values from the previous time-step which enclose around the grid point $C^N_{i,j}$. Hence we can determine $C^{N+1}_{i,j}$ for all points except the BOUNDARY POINTS!

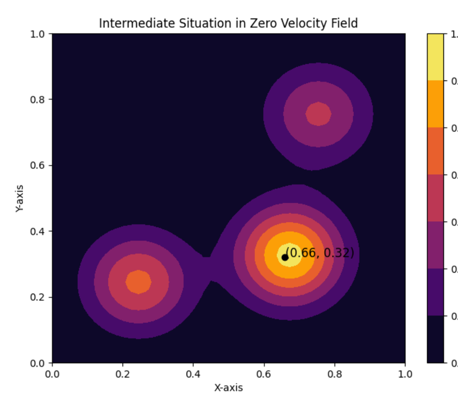
Initial Condition

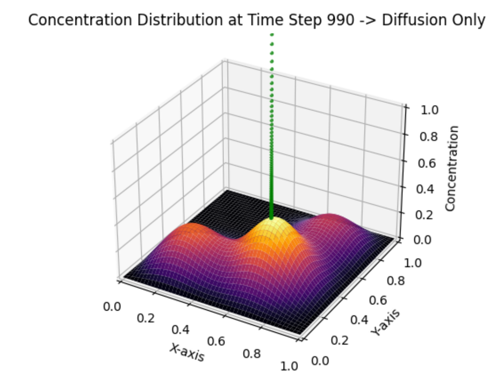
1. $C(\frac{L_x}{4\Delta x}, \frac{L_y}{4\Delta y}, t=0) = 100.00\ particles/mm^3$
2. $C(\frac{3L_x}{4\Delta x}, \frac{3L_y}{4\Delta y}, t=0) = 75.00\ particles/mm^3$
3. $C(\frac{2L_x}{3\Delta x}, \frac{L_y}{3\Delta y}, t=0) = 100.00\ particles/mm^3$

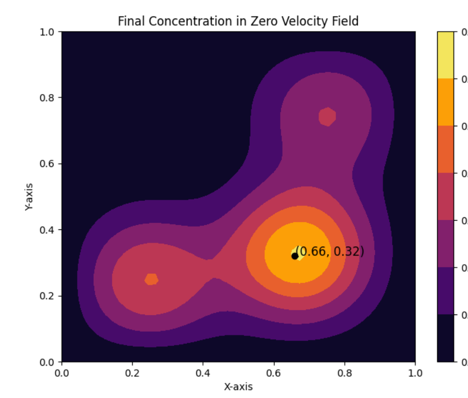


Results

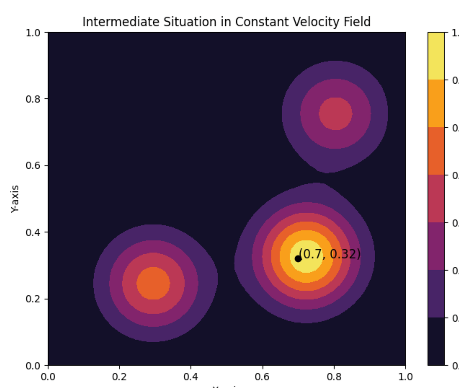
Zero Velocity Field

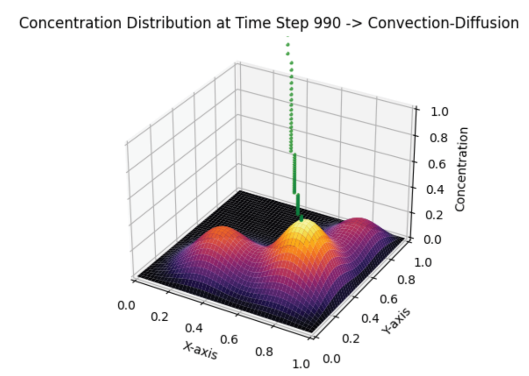


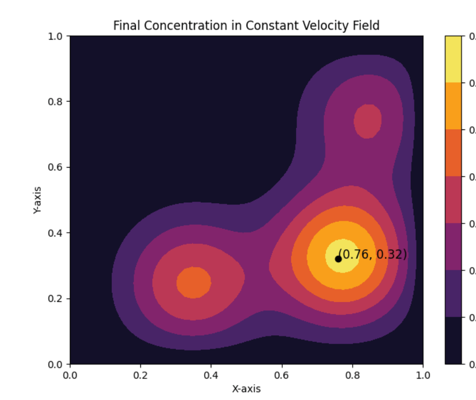




Constant Velocity Field







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