Design and Analysis of Algorithms, MTech-I (1st semester)

Chapter 6: NP Theory - II

October 7, 2022

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Broad Contents of the talks

- Talk1: Understanding How to Analyze an Algorithm. Efficient and Inefficient Algorithms. Complexity Classes of Problems.

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- Talk2: Understanding and Working with Problem Reductions.
- Talk3: Non-determinism, Working with NPHard, NPComplete.

Topics to be discussed

A. **Talk1: Algorithm Analysis, Problem Complexity Classes.**
    1. What does solving problems algorithmically, mean?
    2. Analyzing an algorithm
    3. How to relate time complexity to input size?
    4. Classifying problems
    5. A Motivating Example to illustrate hardness
    6. Some Hard Problems

Topics to be discussed...

- B **Talk2: Relating Problem Hardness & Polynomial Reductions**
  1. Reductions
  2. How can we relate hardness of two problems ?
  3. How can we relate solvability of two problems ?
  4. Polynomial Reduction of one problem to the other
  5. Polynomial Equivalence of one problem to the other
  6. Three methods of reductions: illustrations

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  6. Summarizing

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  - the classification of the problems viz. NP-Hard and NP-Complete
  - how one problem can be reduced to another. . . . . . rendering them to be similar in nature to one another. . . . .

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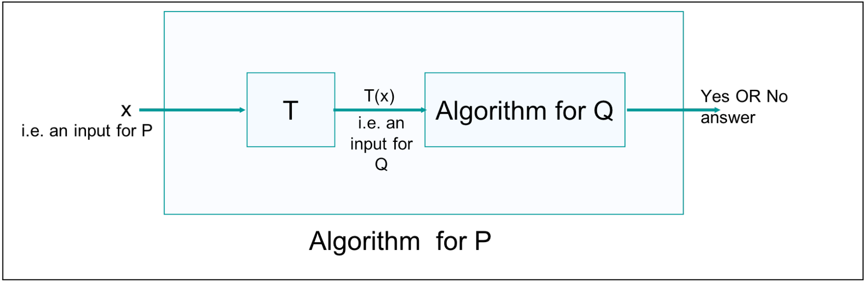

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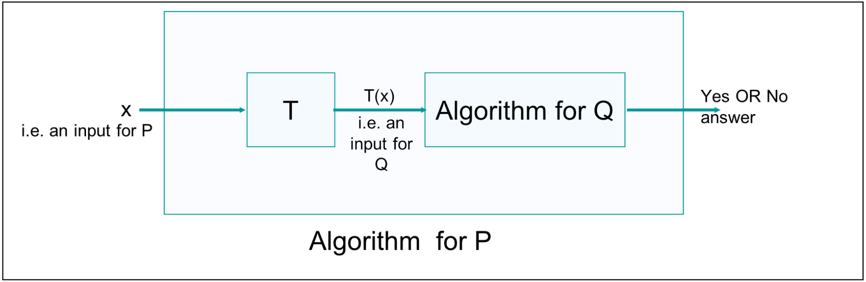

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- Note that, here T is nothing but a converter..........

Reductions, formally defining

- def: Let P_1 and P_2 be two problems. Then problem P_1 reduces to P_2 (i.e. $P_1 \ \alpha \ P_2$)if and only if there is a way to solve P_1 by a deterministic polynomial time algorithm P_2 i.e. using a deterministic algorithm that also solves P_2 in polynomial time.
- P reduces to Q is denoted as either P α_P Q OR P $\leq_P$ Q

Understanding Reductions, further

- Informally, given that a problem P_1 is reducible to P_2 (i.e. $P_1 \leq_P P_2$), it implies that if P_2 is solvable then P_1 also is solvable.

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- This means that

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Intuitively, how can we use reduction to prove a new problem to be hard ?

Figure: ???

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- Let T_2 select only the i^{th} value from these and return as the answer.

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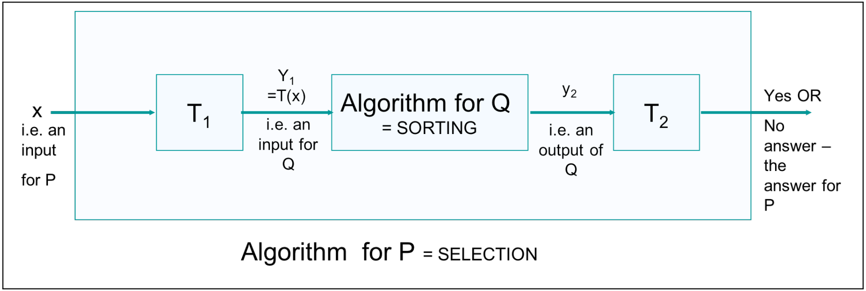

Figure: Solving SELECTION using SORTING and T1 and T2

Reductions, another example

- Consider Two problems
  - SumPair - Compute the sum of two numbers
  - SumList - Compute the sum of n numbers in a vector
- Is it possible to use SumList to find the sum of n numbers in a vector i.e. the answer to SumPair ?
- Hence, we would say $Sum_{Pair} \leq_P Sum_{List}$
- However, in this case we can also say that $Sum_{List} \leq_P Sum_{Pair}$
- Therefore, we can say that $Sum_{Pair} \equiv_P Sum_{List}$

Polynomial Time Reduction

- It is essential to ensure that the functions T_1, T_2 and the algorithm for Q are all polynomial time to ensure Polynomial Time Reducibility.
- That is, reducibility is useful only if it is polynomial time reducibility.
- def: We say that a problem P_1 polynomially reduces to another problem P_2 only if the time for CONVERTER 1 plus the time for CONVERTER 2 is DETERMINISTIC POLYNOMIAL TIME.

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Reduction By Simple Equivalence

Vertex Cover $\equiv_P$ IS

- Minimal Vertex Cover of a graph is a minimum subset of the vertices of G which contains at least one of the two endpoints of each edge in G

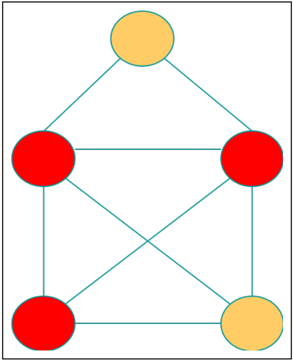

Figure: What is the Vertex Cover?

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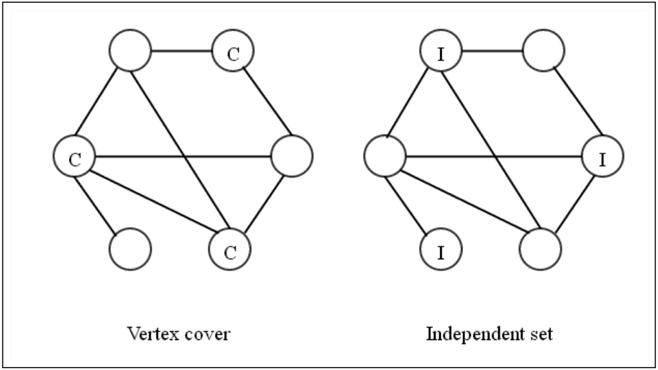

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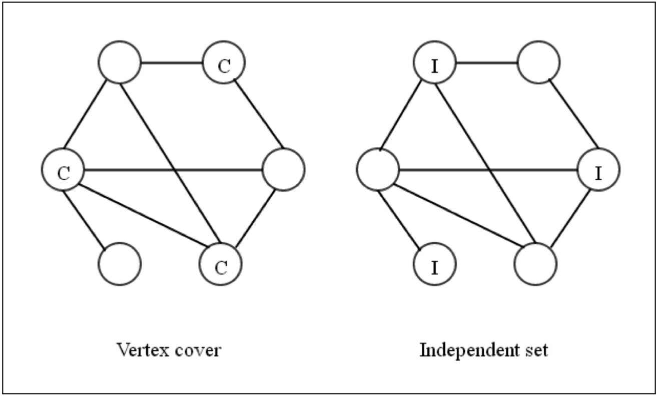

Figure: Independent Set?

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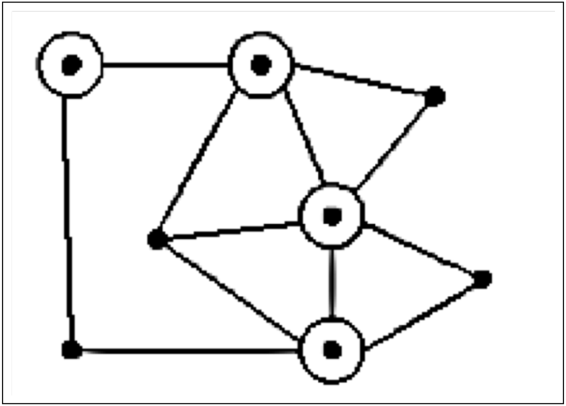

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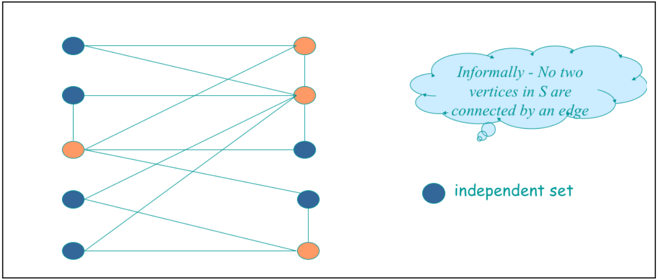

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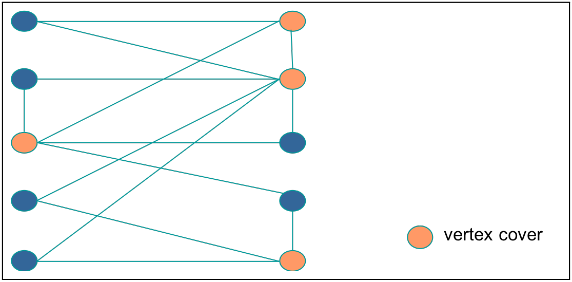
- Minimum VERTEX COVER : informally a set of vertices that include all the edges. . . ..
- Given a graph G = (V, E) and an integer k, is there a subset of vertices S $\leq$ V such that $|S| \leq$ k, and for each edge, at least one of its endpoints is in S?
- Is there a vertex cover of size $\leq$ 4? Yes. size $\leq$ 3? No.



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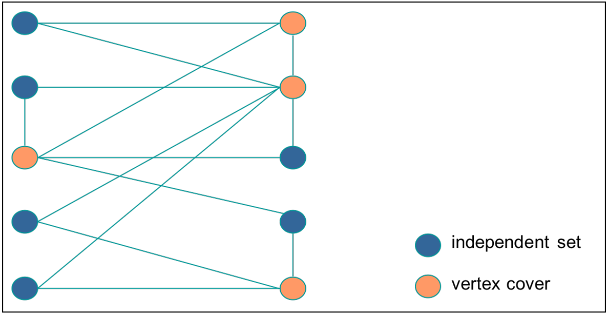
- Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.
- We show S is an independent set iff V - S is a vertex cover.



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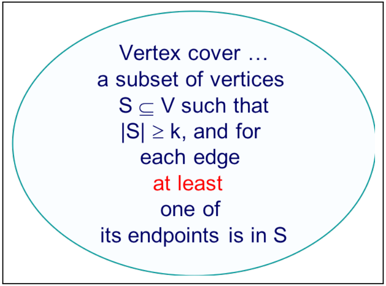

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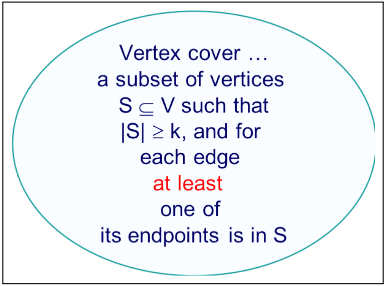

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- Well, the interval scheduling problem is equivalent to finding the maximum independent set in this intersection graph.
- Finding a maximum independent set is NP-hard in general graphs, but it can be done in polynomial time in the special case of intersection graphs (ISMP).

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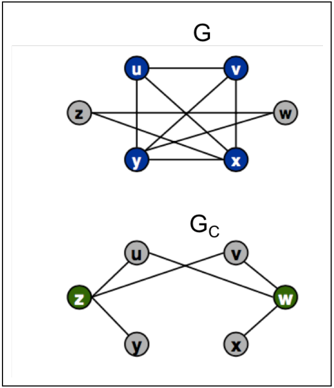

Figure: K-Clique? Vertex Cover?

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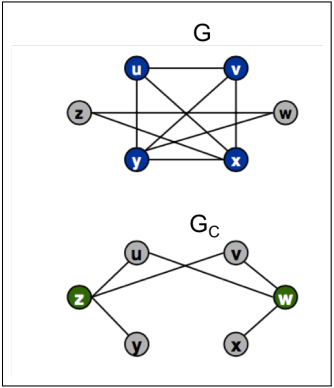

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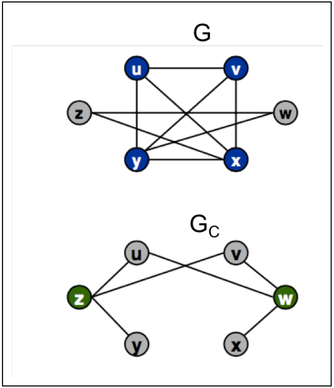

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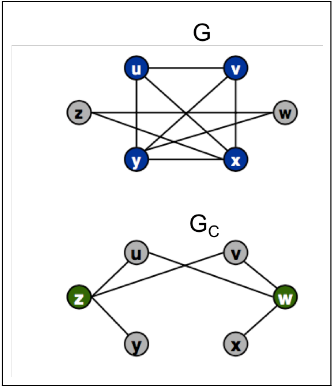

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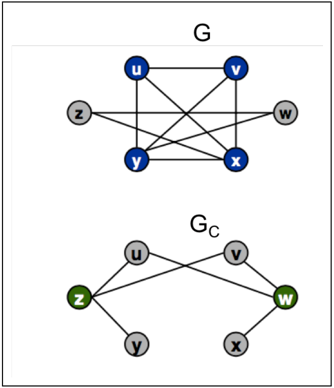

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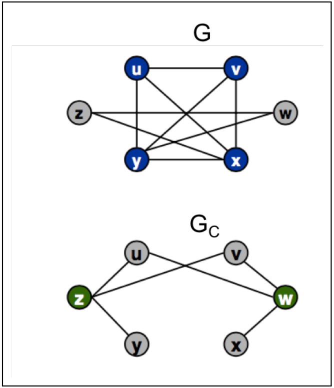

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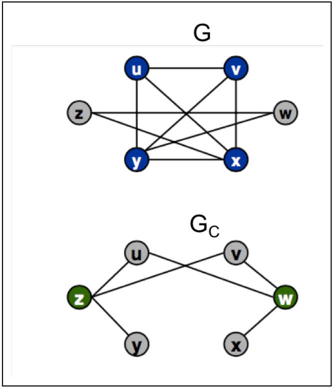

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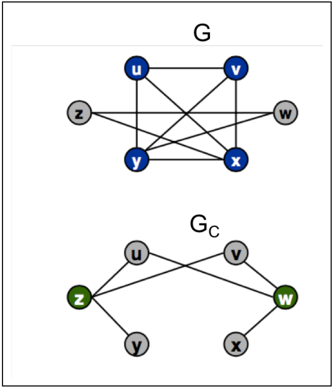
  - Lastly, what about the sizes k and $|V| - k$? Keep in mind that the vertex sets of both G and Gc are the same i.e. V.



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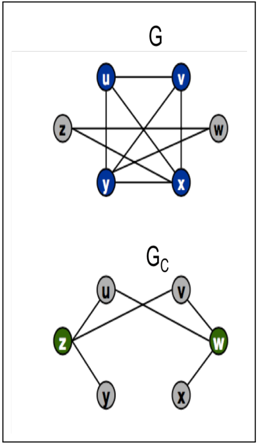

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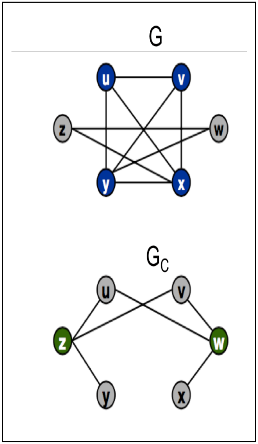

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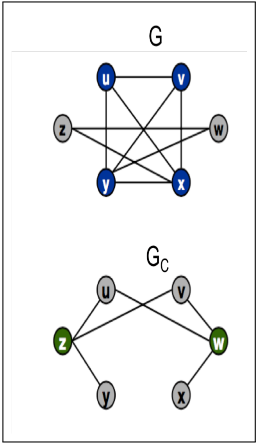

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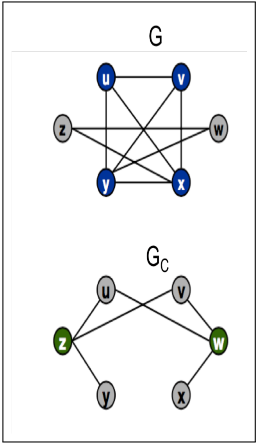

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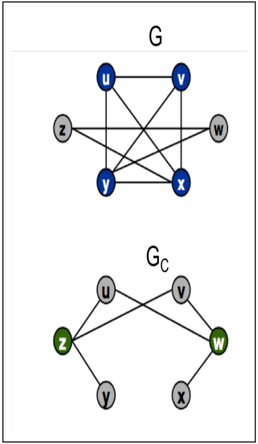

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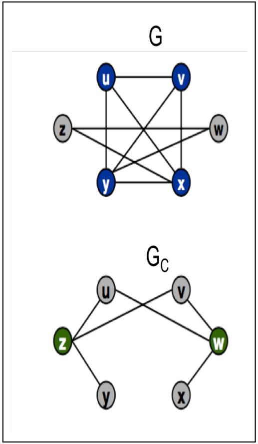

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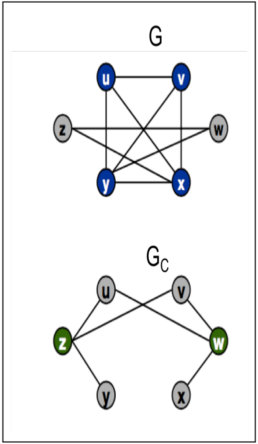

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Reduction By Using Gadgets

The Set Cover problem

- An instance of Set Cover is given by a ground set $U = x_1, x_2, x_3, \ldots\ldots.x_n$, subsets S $\subseteq$ U of that ground set, and an integer k.
- The question is, is it possible to select a collection C of at most k of these subsets such that taken together, they *cover* all of U?
- That is, is there a n set $C \subseteq 1, 2, \ldots\ldots, m$ such that $|C| = k$ and $\cup_{i \epsilon C}\ S_i =$ U ?

U = { 1, 2, 3, 4, 5, 6, 7 }
k = 2

S_a = {3, 7}　　S_b = {2, 4}

S_c = {3, 4, 5, 6}　　S_d = {5}

S_e = {1}　　S_f = {1, 2, 6, 7}

Figure: What are S_c and S_f shown in red?

The Set Cover problem: Applications

- Sample application.
  - m available pieces of software.
  - Set U of n capabilities that we would like our system to have.
  - The i_{th} piece of software provides the set $S_i \subseteq$ U of capabilities.
  - Goal achieve all n capabilities using fewest pieces of software.

U = { 1, 2, 3, 4, 5, 6, 7 }
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S_a = {3, 7} S_b = {2, 4}

S_c = {3, 4, 5, 6} S_d = {5}

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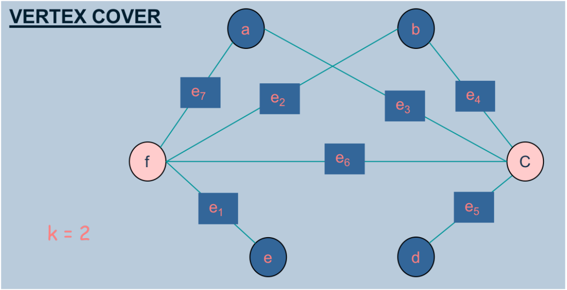
Vertex Cover $\equiv_P$ SetCover



SET COVER

U = { 1, 2, 3, 4, 5, 6, 7 }, k = 2

S_a = {3, 7}, **S_b = {2, 4},** **S_c = {3, 4, 5, 6},** **S_d = {5},** **S_e = {1},**

S_f= {1, 2, 6, 7}

Figure: Vertex Cover $\equiv_P$ SetCover.

K-Clique $\equiv_P$ SAT

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    - otherwise connect all the other vertices across different classes.

Vertex Cover $\equiv_P$ SetCover.

Gadget design for $\mu = (x_1 + \bar{x_2})(x_3 + x_4 + \bar{x_2})(\bar{x_1} + \bar{x_4})$

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