Computer Vision and Image Processing

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Shape from Shading

Shape from Shading

- recovery of surface shape from a single image
- linear reflectance map and reflectance map rotationally symmetric
- image irradiance equation - nonlinear first order partial differential equation
- traditional method using growing characteristic strips, sequential process
- for parallel algorithm - formulate a minimization problem that leads to a relaxation algorithm on a grid
  - minimize the integral of the difference between the observed brightness and that predicted for the estimated shape

Rotationally Symmetric Reflectance Maps

- light source is distributed in a rotationally symmetric fashion about the viewer
- e.g. hemispherical sky; viewer is looking straight down from above, point source at essentially the same place as the viewer

$$R(p, q) = f(p^2 + q^2)$$

- f is monotonic and differentiable, with inverse f^{-1}

$$p^2 + q^2 = f^{-1}(E(x, y))$$

- the direction of steepest ascent makes an angle θ_s with the x-axis where $\tan\theta_s = q/p$

$$\cos\theta_s = p/\sqrt{p^2+q^2} \quad \text{and} \quad \sin\theta_s = q/\sqrt{p^2+q^2}$$

- slope of the surface in the direction of steepest ascent

$$m(\theta_s) = \sqrt{p^2+q^2} = \sqrt{f^{-1}(E(x, y))}$$

Reflectance Map

- given brightness we can find the slope of the surface but not the direction of steepest ascent
- if the direction of steepest ascent, given by (p, q) is known
- a small step of length $\delta\xi$ in the direction of steepest ascent; the changes in x and y

$$\delta x = p/\sqrt{p^2+q^2}\delta\xi \text{ and } \delta y = q/\sqrt{p^2+q^2}\delta\xi$$

- the changes in z

$$\delta z = m\delta\xi = \sqrt{p^2+q^2}\delta\xi = \sqrt{f^{-1}(E(x,y))}\delta\xi$$

- simplify by taking a step of length $\sqrt{p^2+q^2}\delta\xi$ rather than $\delta\xi$

$$\delta x = p\delta\xi \quad \delta y = q\delta\xi \quad \delta z = (p^2+q^2)\delta\xi = f^{-1}(E(x,y))\delta\xi$$

- problem: need to determine the values of p and q at the new point in order to continue the solution
- need to develop equations for the changes δp and δq in p and q

Brightness Gradient

- planar surface patch gives rise to a region of uniform brightness in the image
- nonzero brightness gradient $(E_x, E_y)^T$ occurs only where the surface is curved
- for brightness gradient, differentiate the image irradiance equation with respect to x and y

$$E(x,y) = f(p^2+q^2)$$

$$r = \frac{\partial^2 z}{\partial x^2} \quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad t = \frac{\partial^2 z}{\partial y^2}$$

- using the chain rule for differentiation

$$E_x = 2(pr+qs)f' \text{ and } E_y = 2(ps+qt)f'$$

- to determine the changes δp and δq occasioned by the step $(\delta x, \delta y)$ in the image plane

$$\delta p = r\delta x + s\delta y \text{ and } \delta q = s\delta x + t\delta y$$

Characteristic Curves

- substituting δx and δy

$$\delta p = (pr+qs)\delta\xi = \frac{E_x}{2f'}\delta\xi \text{ and } \delta q = (ps+qt)\delta\xi = \frac{E_y}{2f'}\delta\xi$$

- as $\delta\xi \to 0$, obtain differential equations

$$\dot{x} = p \quad \dot{y} = q \quad \dot{z} = (p^2+q^2) \quad \dot{p} = \frac{E_x}{2f'} \quad \dot{q} = \frac{E_y}{2f'}$$

  dot denotes differentiation with respect to ξ
- Given starting values, these five differential equations can be solved numerically to produce a curve on the surface of the object
- these curves are called characteristic curves (in this case they happen to be the curves of steepest ascent)

$$\ddot{x} = \frac{E_x}{2f'} \quad \ddot{y} = \frac{E_y}{2f'} \quad \dot{z} = f^{-1}(E(x,y))$$

  these equations can be solved numerically

Recovering Depth

- In general, knowing the coordinates of a particular point on the surface the solution can be extended; by taking a small step $(\delta x, \delta y)$, the change in depth

$$\delta z = p\delta x + q\delta y$$

- the problem is
  - p and q are not known
  - image irradiance equation provides only one constraint
  - if we know p and q at given point (x, y) but we need new values of p and q at point $(x+\delta x, y+\delta y)$

$$\delta p = r\delta x + s\delta y \text{ and } \delta q = s\delta x + t\delta y$$

$$\begin{pmatrix} \delta p \\ \delta q \end{pmatrix} = \mathbf{H} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

  $\mathbf{H}$ is Hessian matrix of second partial derivatives

$$\mathbf{H} = \begin{pmatrix} r & s \\ s & t \end{pmatrix}$$

Recovering Depth

- Hessian provides information on the curvature of the surface
- For small surface inclinations, its determinant is the Gaussian curvature
- trace of the Hessian is the Laplacian of depth; for small surface inclinations is twice the so-called mean curvature
- to use Hessian matrix for computing δp and δq; need to know the second partial derivatives of z
- to keep track of them we need still higher derivatives
- Differentiating image irradiance equation with respect to x and y using the chain rule

$$E_x = rR_p + sR_q \text{ and } E_y = sR_q + tR_q$$

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \mathbf{H} \begin{pmatrix} R_p \\ R_q \end{pmatrix}$$

- to solve $\mathbf{H}$ (three unknowns r, s, and t) we have two equations

Recovering Depth

- while we can not continue the solution in an arbitrary direction, we can do so in a specially chosen direction

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} R_p \\ R_q \end{pmatrix} \delta\xi$$

$$\begin{pmatrix} \delta p \\ \delta q \end{pmatrix} = \mathbf{H} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \mathbf{H} \begin{pmatrix} R_p \\ R_q \end{pmatrix} \delta\xi = \begin{pmatrix} E_x \\ E_y \end{pmatrix} \delta\xi$$

- if the direction of the change in the image plane is parallel to the gradient of the reflectance map, then the change in (p, q) can be computed
- the direction of the change in gradient space is parallel to the gradient in the image

$$\dot{x} = R_p \quad \dot{y} = R_q \quad \dot{z} = pR_p + qR_q \quad \dot{p} = E_x \quad \dot{q} = E_y$$

dots denote differentiation with respect to ξ

- A solution of these differential equations is a curve on the surface; parameter ξ will vary along this curve

Recovering Depth

- curve traced out by the solutions of the five differential equations are called characteristic curves
- their projections in the image are called base characteristics
- solutions for x, y, z, p and q form a characteristic strip and define surface orientation along this curve
- to obtain the whole surface patch together characteristic strips
- to start solution, it needs a point where initial values are given
- given an initial curve on the surface, a solution for the surface can be obtained as long as this curve is nowhere parallel to any of the characteristics
- on this curve starting values of p and q can be obtained using image irradiance equation $E(x, y) = R(p, q)$ and known derivatives of z along the curve
- i.e. initial curve given in terms of a parameter η, $x(\eta)$, $y(\eta)$ and $z(\eta)$ then along the curve

$$\frac{\partial z}{\partial \eta} = p\frac{\partial x}{\partial \eta} + q\frac{\partial y}{\partial \eta}$$

Singular Points

- Normally an initial curve along with the image of an object not given
- any points where surface orientation can be determined directly?
- Say $R(p, q)$ has a unique isolated maximum at (p_0, q_0)

$$R(p, q) < R(p_0, q_0) \quad \forall (p, q) \neq (p_0, q_0)$$

- at some point (x_0, y_0) in the image $E(x_0, y_0) = R(p_0, q_0)$
- at this point the gradient (p, q) is uniquely determined to be (p_0, q_0)
- we could start the solution at such a singular point, but at a maximum of $R(p, q)$ the R_p and R_q are zero
- solution will not move from such a point because $\dot{x}$ and $\dot{y}$ are zero
- bypass this by constructing a small "cap" at this point and start the solution at the edge of this cap

Shape from Shading: Example

- consider power series near a singular point

$$R(p,q) = \frac{1}{2}(p^2 + q^2)$$

- it has unique isolated minimum, singular point at the origin such that $E(0,0) = 0$, so $(p,q) = (0,0)$ at this point
- say surface is smooth enough, z as a power series (without first order terms) and ignoring higher-order terms near the origin

$$z = z_0 + \frac{1}{2}(ax^2 + 2bxy + cy^2) \quad p = ax + by \quad \text{and} \quad q = bx + cy$$

$$E(x,y) = \frac{1}{2}(a^2 + b^2)x^2 + (a+c)bxy + \frac{1}{2}(b^2 + c^2)y^2$$

Shape from Shading: Example

- task is to determine a, b, and c given the image brightness and its derivatives near the origin

$$z = z_0 - \frac{1}{2}(ax^2 + bxy + \frac{1}{2}cy^2) \quad p = ax + by \quad \text{and} \quad q = bx + cy$$

- gives rise to the same shading pattern; so at least two solutions possible

$$E_x = (a^2 + b^2)x + (a+c)by \quad E_y = (a+c)bx + (b^2 + c^2)y$$

- $(E_x, E_y)^T = (0,0)^T$ at $(x,y) = (0,0)$; we can not use brightness gradient to recover the shape

$$E_{xx} = a^2 + b^2 \quad E_{xy} = (a+c)b \quad E_{yy} = b^2 + c^2$$

- three second order polynomials in three unknowns can have up to eight solutions
- more than one shape might give rise to the same shading, since the nonlinear equations containing the coefficients of the power series near the singular point can have more than one solution

Relaxation Method

- method of characteristic strip expansion suffers from
  - sensitive to measurement noise
  - prevent adjacent characteristics from crossing over each other
- Minimization in continuous case
  - objective is to find two functions $f(x,y)$ and $g(x,y)$ that ensure that the image irradiance equation,

$$E(x,y) = R_s(f,g)$$

    is satisfied. $R_s(f,g)$ reflectance map expressed in stereographic coordinates
  - what we want? $f(x,y)$ and $g(x,y)$ to correspond to a smooth surface
  - to measure smoothness, choose function that penalizes rapid changes in f and g; try to minimize

$$e_s = \int\int_I ((f_x^2 + f_y^2) + (g_x^2 + g_y^2))dxdy$$

Relaxation Method: Solution

- minimize e_s subject to the constraint that f and g must satisfy the image irradiance equation
- in practice, there are errors in both the measurements of irradiance and the determination of reflectance map
- try to minimize the error

$$e_i = \int\int_I (E(x,y) - R_s(f,g))^2 dxdy$$

- overall, minimize $e_s + \lambda e_i$ λ weights the errors in the image irradiance equation relative to the departure from smoothness
- λ should be made large if brightness measurements are very accurate and small if they are very noisy

Relaxation Method: Solution

- minimization of an integral of the form

$$\int\int F(f,g,f_x,f_y,g_x,g_y)dxdy$$

- the corresponding Euler equations are

$$F_f - \frac{\partial}{\partial x}F_{f_x} - \frac{\partial}{\partial y}F_{f_y} = 0 \quad F_g - \frac{\partial}{\partial x}F_{g_x} - \frac{\partial}{\partial y}F_{g_y} = 0$$

$$F = (f_x^2 + f_y^2) + (g_x^2 + g_y^2) + \lambda(E(x,y) - R_s(f,g))^2$$

- aim is to minimize the integral of F and Euler equations for this

$$\nabla^2 f = -\lambda(E(x,y) - R_s(f,g))\frac{\partial R_s}{\partial f} \quad \nabla^2 g = -\lambda(E(x,y) - R_s(f,g))\frac{\partial R_s}{\partial g}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

- result is a couple pair of elliptic second-order partial differential equations and can be solved by iterative methods

Minimization in the discrete case

- measure the departure from smoothness at the point (i,j) by

$$s_{i,j} = \frac{1}{4}((f_{i+1,j} - f_{i,j})^2 + (f_{i,j+1} - f_{i,j})^2 + ((g_{i+1,j} - g i,j)^2 + (g_{i,j+1} - g_{i,j}^2))$$

- error in the image irradiance equation

$$r_{ij} = (E_{ij} - R_s(f_{ij}, g_{ij}))^2$$

E_{ij} is the observed image irradiance

- seek a set of values $\{f_{ij}\}$ and $\{g_{ij}\}$ that minimize

$$e = \sum_i \sum_j (s_{ij} + \lambda r_{ij})$$

Minimization in the discrete case

- differentiating e with respect to f_{kl} and g_{kl}

$$\frac{\partial e}{\partial f_{kl}} = 2(f_{kl} - \bar{f}_{kl}) - 2\lambda(E_{kl} - R_s(f_{kl}, g_{kl}))\frac{\partial R_s}{\partial f}$$

$$\frac{\partial e}{\partial g_{kl}} = 2(g_{kl} - \bar{g}_{kl}) - 2\lambda(E_{kl} - R_s(f_{kl}, g_{kl}))\frac{\partial R_s}{\partial g}$$

$$\bar{f}_{i,j} = \frac{1}{4}(f_{i+1,j} + f_{i,j+1} + f_{i-1,j} + f_{i,j-1}) \quad \bar{g}_{i,j} = \frac{1}{4}(g_{i+1,j} + g_{i,j+1} + g_{i-1,j} + g_{i,j-1}$$

- the extremum is to be found where the above derivatives of e are equal to zero
- rearranging the resulting equations by solving for f_{kl} and g_{kl}, an iterative solution method

$$f_{kl}^{n+1} = \bar{f}_{kl}^n + \lambda(E_{kl} - R_s(f_{kl}^n, g_{kl}^n))\frac{\partial R_s}{\partial f} \quad g_{kl}^{n+1} = \bar{g}_{kl}^n + \lambda(E_{kl} - R_s(f_{kl}^n, g_{kl}^n))\frac{\partial R_s}{\partial g}$$

Minimization in the discrete case

if there are n images

$$e = \int\int_I ((f_x^2 + f_y^2) + (g_x^2 + g_y^2))dxdy + \sum_{i=1}^{n} \lambda_i \int\int_I (E_i(x,y) - R_i(f,g))^2 dxdy$$

$$\nabla^2 f = -\sum_{i=1}^{n} \lambda_i(E_i(x,y) - R_i(f,g))\frac{\partial R_i}{\partial f}$$

$$\nabla^2 g = -\sum_{i=1}^{n} \lambda_i(E_i(x,y) - R_i(f,g))\frac{\partial R_i}{\partial g}$$

$$f_{kl}^{n+1} = \bar{f}_{kl}^n + \sum_{i=1}^{n} \lambda_i(E_{i;kl} - R_i(f_{kl}, g_{kl}))\frac{\partial R_i}{\partial f}$$

$$g_{kl}^{n+1} = \bar{g}_{kl}^n + \sum_{i=1}^{n} \lambda_i(E_{i;kl} - R_i(f_{kl}, g_{kl}))\frac{\partial R_i}{\partial g}$$

Recovering depth

- given p and q, the partial derivatives of $z(x, y)$ with respect to x and y; recover $z(x, y)$ by integrating along arbitrary curves in the plane

$$z(x, y) = z(x_0, y_0) + \int_{(x_0,y_0)}^{(x,y)} (pdx + qdy)$$

- p and q are recovered from noisy image data by some method
- choose $z(x, y)$ so as to minimize the error

$$\int\int_I ((z_x - p)^2 + (z_y - q)^2) dxdy$$

  p and q are the given estimates of the components of the gradient and z_x and z_y are the partial derivatives of the best-fit surface
- this is a problem in the calculus of variations

Recovering depth

- minimize integral of the form

$$\int\int F(z, z_x, z_y) dxdy$$

- Euler equation

$$F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} = 0$$

  from

$$F = (z_x - p)^2 + (z_y - q)^2$$

  we obtain

$$\frac{\partial}{\partial x}(z_x - p) + \frac{\partial}{\partial y}(z_y - q) = 0 \quad \text{or} \quad \nabla^2 z = p_x + q_y$$