Chapter 3: Greedy Algorithm Design Technique - II

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- Let us attempt to understand the rationale theoretically.....

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    - Then, how many bits are required to encode each character in our restricted alphabet?

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Obviously the variable length code desired for longer strings.

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- Now delete the corresponding set of bits from the front of the message and iterate.

Decoding using a prefix codes: Examples

- Consider the alphabet S = a, b, c, d, e and the encoding $\gamma(a) = 11$ $\gamma(b) = 01$ $\gamma(c) = 001$ $\gamma(d) = 10$ and $\gamma(e) = 000$. Is this code a prefix code ? IF so, show to what the encoding represented by 0010000011101 decodes to ?

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- Consider the alphabet S = a, b, c, d and the encoding $\gamma(a) = 0$ $\gamma(b) = 10$ $\gamma(c) = 110$ $\gamma(d) = 111$. How would the string 1101001101000 be decoded ?

Prefix codes and its TBL

- How to compute the total bits required to encode a string using prefix code ?
- The total bits of a prefix code encoded string c is
  - the sum over all symbols of frequency of each character c, times the number of bits of encoding of c.
- Given that C is the alphabet, c is the prefix code function i.e. c(x) is the encoding of any character x ϵC, f(x) is the frequency of occurrence of a character c ϵC, we have

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Total Bit Length

$\sum_{x \epsilon C} f(x)|c(x)|$ for every $x \epsilon C$

Prefix codes : Data structure to implement?

- For our goal i.e. to find a prefix code representation that has the lowest possible average bits per letter.
- We anyway need suitable data structure to represent variable length code.

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- A binary tree representation can give optimal code. But let us try to delve deeper and formalize....

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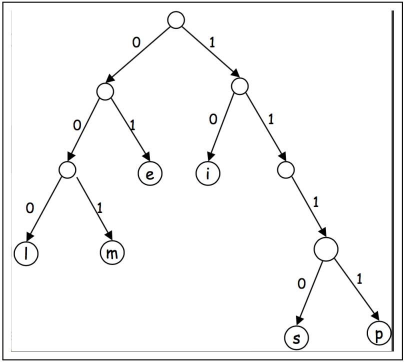
Prefix codes and Full binary trees

- Consider now another tree as shown for the code viz.



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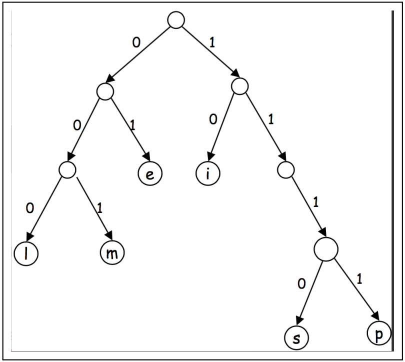

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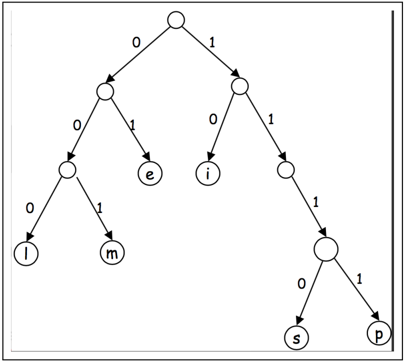

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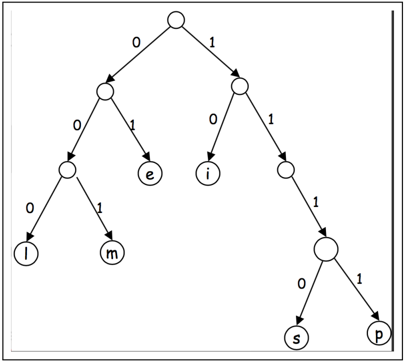

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- *Can this prefix code be made more efficient*? Why is it not efficient ? How ?

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  - Proof to be worked out by contradiction...
- Where in the tree of an optimal prefix code should letters be placed with a high frequency?

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  - a = 0, x = 10, u = 110, z = 111 with the same frequency as above

Constructing the tree: Huffman Encoding

- Defining Huffman tree
- Construct the Huffman tree for the following

a	b	c	d	e	f
6	2	3	3	4	9

The Huffman Tree Construction pseudocode

```
The Algorithm Huffman-Tree(C)
1. n = |C|
2. for i = 1 to n-1
3.        do z = ALLOCATE_NODE()
4.        x = left(z) = EXTRACT_MIN(Q)
5.        y = right(z) = EXTRACT_MIN(Q)
6.        f[z] = f[x] + f[y]
7.        INSERT(Q, z)
8. return EXTRACT_MIN(Q)
```

Why (n-1) in line 2?
Why the last line ?

The Huffman Tree Construction pseudocode

```
The Algorithm HEAP–EXTRACT–MIN(A)
1. if heap_size[A] < 1
2.    then return error heap underflow
3. min = [A]
4. A[1] = A[heap−size[A]]
5. heap−size[A]=heap−size[A]−1
5. heap−size[A] = heap−size[A] − 1
6. HEAPIFY(A, 1)
7. return min
```

Time Complexity

- HEAP-EXTRACT-MIN & INSERT take O(lg n) time on Q with n objects
- Therefore, $T(n) = \sum_{i=1}^{n} lg\ i = O(\lg(n!)) = O(n \lg n)$

Optimal Merge Patterns

Problem definition

To merge n sorted files in such a way that the fewest number of comparisons are required while merging them.

Say Given files x_1, x_2, x_3, x_4, x_5 with $|x1|$=20, $|x2|$=30, $|x3|$=10, $|x4|$=5, $|x5|$=30. Show haw can they be optimally merged.

What are the various record moves required merging these files in different patterns ?

- Consider the game of guessing a chosen object from n possibilities (say, an integer between 1 and n) by asking questions answerable by yes or no.
- Different strategies for playing this game can be modeled by decision trees

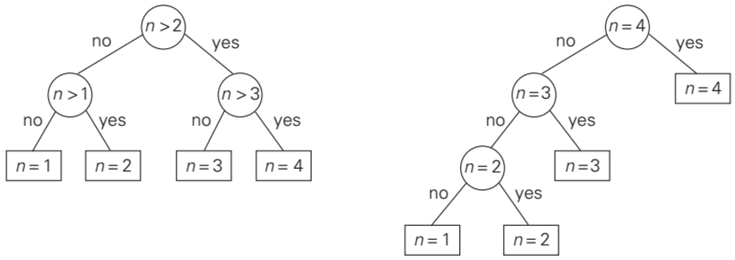

FIGURE 9.13 Two decision trees for guessing an integer between 1 and 4.

Prefix codes and Full binary trees

- Exercise 16-3.4: Lemma1: Prove that we can also express the total cost of a tree for a code as the sum, over all internal nodes, of the combined frequencies of the two children of the node.

Prefix codes and Full binary trees

- Exercise 16-3.4: Lemma1: Prove that we can also express the total cost of a tree for a code as the sum, over all internal nodes, of the combined frequencies of the two children of the node.
- Theorem2: Prove that the optimal data compression that is achievable by any character code can always be achieved with a prefix code.

Correctness Proof: Huffman coding...

The Huffman's algorithm is correct

To prove that the greedy algorithm HUFFMAN is correct, we show that the problem of determining an optimal prefix code exhibits the greedy-choice and optimal-substructure properties.

- Step 1: Show that this problem satisfies the greedy choice property, that is, if a greedy choice is made by Huffman's algorithm, an optimal solution remains possible.
- Step 2: Show that this problem has an optimal substructure property, that is, an optimal solution to Huffman's algorithm contains optimal solution to subproblems.
- Step 3: Conclude correctness of Huffman's algorithm using step 1 and step 2.

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  - Greedy Choice Property: Let c be an alphabet in which each character c has frequency f[c]. Let x and y be two characters in C having the lowest frequencies. Then there exists an optimal prefix code for C in which the codewords for x and y have the same length and differ only in the last bit.

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- Lemma 1 implies that process of building an optimal tree by mergers can begin with the greedy choice of merging those two characters with the lowest frequency.

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- We have already proved that TBL(T) i.e. the total cost of the tree constructed is the sum of the costs of its mergers (internal nodes) of all possible mergers.
- At each step Huffman chooses the merger that incurs the least cost

Correctness Proof: Huffman coding

Lemma 1

Let C be an alphabet in which each character c ϵ C has frequency $f(c)$. Let x and y be two characters in C having the lowest frequencies. Then there exists an optimal prefix code for C in which the codewords for x and y have the same length and differ only in the last bit.

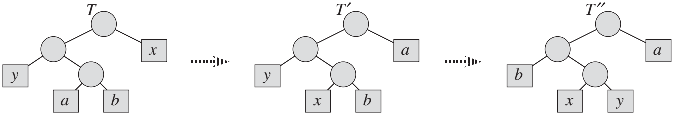

Figure 16.6 An illustration of the key step in the proof of Lemma 16.2. In the optimal tree T, leaves a and b are two siblings of maximum depth. Leaves x and y are the two characters with the lowest frequencies; they appear in arbitrary positions in T. Assuming that $x \neq b$, swapping leaves a and x produces tree T', and then swapping leaves b and y produces tree T''. Since each swap does not increase the cost, the resulting tree T'' is also an optimal tree.

Correctness Proof: Huffman coding...

The Huffman's algorithm is correct

To prove that the greedy algorithm HUFFMAN is correct, we show that the problem of determining an optimal prefix code exhibits the greedy-choice and optimal-substructure properties.

- Step 2: Show that this problem has an optimal substructure property, that is, an optimal solution to Huffman's algorithm contains optimal solution to subproblems.
  - Let T be a full binary tree representing an optimal prefix code over an alphabet C, where frequency f[c] is define for each character c belongs to set C. Consider any two characters x and y that appear as sibling leaves in the tree T and let z be their parent. Then, considering character z with frequency f[z] = f[x] + f[y], tree T‘ = T - x, y represents an optimal code for the alphabet C‘ = C - x, yUz.
- Step 3: Conclude correctness of Huffman's algorithm using step 1 and step 2.

Lemma 2

A Tutorial Problem

- A long string consists of the four characters A, C, G, T; they appear with frequencies 31%, 20%, 9% and 40% respectively. Run the Huffman coding procedure from class to derive an optimal code for these characters. What is the average number of bits per character in your code?
  - Prove that in the Huffman coding scheme, if some character occurs with frequency more than 2/5, then there is guaranteed to be a codeword of length 1.
  - Prove also that if all characters occur with frequency less than 1/3, then there is guaranteed to be no codeword of length 1.
  - Suppose the frequencies of letters in an n-letter alphabet are $f_1, f_2, f_3, ...f_n$ what relationship between these frequencies leads to the longest possible codeword? You should give an intuitive justification for your answer, but you need not provide a formal proof.

- IT is necessary to include the coding table into the encoded text to make its decoding possible.
- This drawback can be overcome by using dynamic Huffman encoding in which the coding tree is updated each time a new symbol is read from the source text.
- Modern schemes like Lempel-Ziv algorithms assign codewords not to individual symbols but to strings of symbols, allowing them to achieve better and more robust compressions in many applications.

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