

• Vertex Cover = minimum subset of vertices of G which contain atleast one of two endpoints of each edges in G .

Q 28

Vertex Cover \equiv_p Independent Set. Prove.

Ans

Part 1 :-

- Claim :- Vertex Cover \equiv_p Independent Set.
- Proof :- We show S is an independent set if and only if $V - S$ is vertex cover.

Part 1 :-

- to prove that if S is independent set then $V - S$ is vertex cover. (VC of \equiv_p IS)
- Proof :-

- Consider an edge (u, v) in E [E is edge set]
- { Because S is an IS, either $u \notin S$ or $v \notin S$.
third condition was $u \neq v \notin S$, but this is not possible as (u, v) is edge present as per our consideration. }
- This implies $u \in (V - S)$ or $v \in (V - S)$
- " ", at least u or $v \in (V - S)$ for an edge (u, v) in E .
- Arguing same for all the edges in the graph, $(V - S)$ is vertex cover of graph. Hence proved.

Part 2 :-

- to prove that if $(V - S)$ is vertex cover then S is independent set (IS of \equiv_p VC)

• Proof :-

- Consider two vertices u and v in V .
- What can we argue about edge (u, v) ?

(a) either $u \in V - S'$ or $v \in V - S'$

implies $u \notin S'$ or $v \notin S'$

(b) both either $u \in V - S'$ and $v \in V - S'$

implies both $u \notin S'$ and $v \notin S'$

(d) \rightarrow It is possible for valid if

it doesn't harm IS property.

(c) both $u \notin V - S'$ and $v \notin V - S'$

\rightarrow It is not possible to form valid edge

bcz it will violate vertex cover

property, ex for valid edge $(u, v) \in E$,

if $u \notin VS$ & $v \notin VS$ then vertex who
will cover that edge (u, v) ?

- So that for all three possibilities mentioned
above, property of IS is maintained.

Hence proved.

\therefore Hence proved, $NC \equiv IS$

P

Vertex Cover \equiv_p k-cliqueClaim :- Vertex Cover \equiv_p k-clique

Proof :- We show graph $G(V, E)$ has a clique of size k iff complement graph $G_c(V, E')$ has vertex cover of size $|V| - k$.

Part-1

to prove that, if graph G has a clique of size k , then complement graph G_c has vertex cover $(V - S)$ of size $(|V| - k)$. [$\forall C \in \Delta_p$ k-clique]

Proof :-

for an edge (u, v) in G_c , there are following

cases in graph G . (Case 1 & 2)

case 1 :- $u \notin S$ & $v \notin S$.

Is it possible? No

because (u, v) is edge in G_c . So there would not be an edge (u, v) in G . and as there is not an edge betw. u and v , both of them can not be part of clique in G together.

case 2 :- either $u \in S$ or $v \in S$

Is it possible? Yes

implies either $u \notin (V - S)$ or $v \notin (V - S)$ || || $u \in (V - S)$ or $v \in (V - S)$ therefore $(V - S)$ is vertex cover in G_c w.r.t an edge (u, v) in G_c

case 3 :- $u \notin S$, possible? Yes $\rightarrow u \notin V \in (V - S)$ therefore $(V - S)$ is VC in G_c w.r.t an edge (u, v) in G_c .

We do the same for all the edges in G_c and hence the proof that G_c has vertex cover $(V-S)$ of size $(|V|-k)$.

~~Part 2~~ :-

~~to prove that~~

And for edge (u,v) not in G_c , we don't think about that because it doesn't obey definition of vertex cover.

Part 2 :-

- to prove that if graph G_c has vertex cover $(V-S)$ of size $(|V|-k)$, then complement graph G' has clique S of size k . (k -clique of V)

Proof :-

Consider an edge (u,v) in G_c .

case 1 :- if $(u,v) \in G_c$ then

either $u \in (V-S)_c$ or $v \in (V-S)_c$

i.e. if both then by (transitive)

if $u \notin (V-S)$ and $v \notin (V-S)$ then $(u,v) \notin G_c$

\Rightarrow if $u \in S$ and $v \in S$ then $(u,v) \in G$.

therefore S is clique

case 2 :- if $(u,v) \in G$ then $u \in (V-S) \cap (V-S)$

\therefore if $u \neq v \in (V-S)$ then $(u,v) \notin G_c$

\therefore if $u \neq v \in S$ then $(u,v) \in G$.

therefore S is forming clique.

We assume this for all the edges of
Hence the point that G has clique of
size k .

Hence the proof that

$$\text{VC} \equiv_{\text{P}} k\text{-clique}$$



Prove that the problem k -clique is NP-Complete.

and

Proof :- In order to prove that k -clique is an NPC, we need to prove the following :

(1) the problem k -clique belongs to class NP.
This can be done by giving certificate approach. To prove this, we will use NP-completeness of SAT.

(2) We need to select another problem (SAT in this case) to be belonging to class NP to prove that this "another problem" is polynomially reduces to k -clique problem.

→ Note that, Since underlying data structures of k -clique ($\text{graph } G(V, E)$) and that of SAT problem are different, we have to design gadget on which our subsequent proof can be based. Hence we have to design

Hence we first show the design of

(*) Proof that problem k -clique belongs to class NP using certificate-certificate approach.

↳ Certificate-Certificate Approach :- If a problem belongs to class NP, then it should have polynomial-time verifiability, that is given a certificate, we should be able to verify in polynomial time if it is soln for the problem.

↳ For k -clique problem,

Certificate :-

Using choice(), we ^{Selects} ~~choose~~ nodes in set S , (S is subgraph of G)

Verification :-

We have to check if there exists a clique of size k in graph. Hence verifying it no. of nodes in S equals k , takes $O(1)$ time. Verifying whether each vertex has an outdegree $(k-1)$ takes $O(1^2)$ time. Therefore, to check if the graph formed by k nodes in S is complete or not takes $O(k^2) \in O(n^2)$ ($\because k \leq n$).

i.e. therefore, k -clique decision problem has polynomial time verifiability hence belongs to class NP.



Now, let's us design gadget as follows :-

→ Assume that we are given SAT expression F , consisting of k clauses, ($k=3$ in our example of below)

$$F = ((x_1 + \bar{x}_2) (x_3 + x_4 + \bar{x}_2) (\bar{x}_1 + \bar{x}_4))$$

→ Our gadget graph must show the gadgets modelled by expression F .

→ Gadget Design Rules :-

- ① Consider all the literals of SAT expⁿ or group them in classes depending on which class they are in, e.g. we have 3-classes in given SAT expⁿ.
- ② The no. of vertices in graph is equal to no. of literals.
- ③ Label vertices by $(\sigma, i), (\bar{\sigma}, j)$ where $\sigma, \bar{\sigma}$ represent literals of i, j represent class number.
- ④ (a) Do not connect two vertices if they are complements of each other.
 (b) Do not connect two vertices from ~~the~~ same class.
 (c) Connect all other vertices.

i.e., Do not connect (σ, i) & $(\bar{\sigma}, j)$
if $\sigma = \bar{\sigma}$ or $i = j$.

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Proof that SAT instance \Rightarrow k-clique.

QP

* CLAIM #1 If given SAT expⁿ F with k-clauses is satisfiable then corresponding graph is k-clique.

[Proof #1] Let us assume that we are given a SAT expⁿ F with k-clauses which is satisfiable.

- Let us now pick up one literal each from a clause say $\alpha_1, \alpha_2, \dots, \alpha_k$ i.e. k literals that are all assigned value 1.
- Let us identify corresponding vertices in graph, noting that vertices must come from different clauses (since each literal is from diff clause) ... as $(\alpha_1, 1), (\alpha_2, 1), \dots, (\alpha_k, 1)$.
- The question is are all these vertices connected? We say they are, since on our design side, only vertices that are complement of each other or are vertices that are in same class are not connected; remaining all are connected.
- And since we have k-vertices, they are forming k-clique.

CLAIM #2 :- If a given graph $G(V, E)$ has a clique of size k , then corresponding SAT expression is satisfiable.

Proof #2 :- We note that, we are given a graph $G(V, E)$ with k -clique.

- Let us identify vertices of given graph G as $(d_1, 1), (d_2, 2), \dots, (d_k, k)$. Note that the vertices selected are in different clauses because only then they can be connected forming a k -clique.
- The question is that, can the corresponding literals (for vertices) be assigned with value '1' or '0'?
- We say yes because if our design is correct, if vertices are connected, they can't be complements of each other.
- And therefore we have to express an F with k clauses which will be satisfiable.

Hence k -clique problem is NP.