Design and Analysis of Algorithms, MTech-I (1^{st} semester)

Chapter 3: Greedy Algorithm Design Technique - I

September 7, 2022

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- The basic paradigm

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- The basic paradigm
- The greedy control abstraction

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- Elements of greedy strategy

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- The greedy control abstraction
- Elements of greedy strategy
- Characteristics
- Some optimization problems

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Introduction

Michael Douglas in the Wall Street

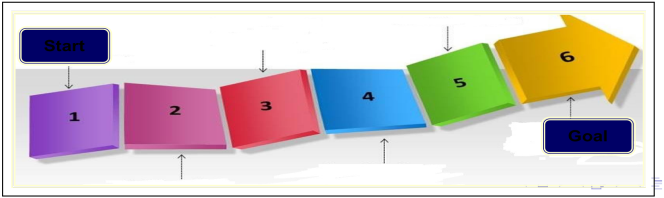
Greed is good, greed is right, greed works!!!

- Let us try to verify whether is it really so. . . . . How ?
- Take a number of computational problems investigate the pros & cons of short-sighted greed
- How to define a greedy algorithm ?

The Basic Greedy Criterion

In a greedy algorithm design

- build up the small solution in small steps, while working in stages to optimize some underlying criterion
- Selection of the next step is myopic and irreversible ....????
- we build up the small solution in small steps.......................... while working in stages
- The goal is to optimize some underlying criterion
- Selection of the next step is myopic and irreversible



The Basic Greedy criterion...

Requirements

- The next step selected must ensure feasibility. How ?
- Such selection must lead to an optimal solution. How ?
- What are optimization problems ?

Terminologies

Greedy Design terms

- greedy criterion
- optimal solution
- feasible solution
- suboptimal solution
- constraints
- optimization problems
- heuristics
- bounded performance
- approximation algorithms

The Thirsty Baby problem

The Problem

An intelligent baby wants to quench her thirst. . . .!!! She has a defined number of liquids available in a defined amount, to her disposal, each with a defined satisfaction quotient.
Her capacity to drink all the liquids in all, is bounded.
The objective is to maximize her thirst, while drinking a combination of liquids. . . . . .

- How to formalize the problem description ?
- Using the mathematical notations how do we state : the inputs, outputs, constraint function, optimizing function

Greedy algorithm characteristic

Irrevocability

What is irrevocability in this case ?

The Container Loading Problem

The Problem description

A Cargo Train Bogey is to be loaded with containers each having a specific weight, so as to maximize the no of containers, such that the maximum weight carrying capacity of the bogey is not exceeded. . . . . .

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The Formalism

- Let w_i - the weight of the container i
- Let C - the maximum cargo carrying capacity of a ship
- Let x_i be a boolean variable (1=container loaded , 0= container not loaded)
- The problem is to assign values x_i such that $\Sigma_{i=1}^{n}(x_i)$ is maximized subject to the constraint $\Sigma_{i=1}^{n}(w_i) \leq C$

The Container Loading Problem: An illustration

- Given that n $=8$, i $=[1,2,3,4,5,6,7,8]$, $\Sigma w_i = [100,200, 50, 90, 150, 50, 20, 80]$, C $=400$
- What is the order of loading ?
- What is the Σx_i?

The Container Loading Problem: Algorithm

```
for i=1 to n
        x[i] = 0
t ← ALLOCATE_MEMORY(n)
IndirectSort(w, t, n)
while (i <=n) and w[t[i]] ≤ C
        x[t[i]] = 1
        C = C − w[t[i]]
        i=i+1
delete t
```

Combinatorial Optimization problems

- Broadly the problem of finding a solution that either minimizes or maximizes the value of a particular parameter, is always subject to certain constraints.
- Combinatorial Optimization problems
    - if the parameter to be optimized is discrete such as an integer, a permutation or graph from a finite (or possibly countable infinite) set.

Combinatorial Optimization Problems

- Formally, a combinatorial optimization problem is a quadruple viz. $< I, f, m, g >$, where

  - I - is a set of problem instances
  - $f(x)$ - is the set of feasible solutions, given a specific problem instance $x \epsilon I$
  - m(x,y) - given an instance x and a feasible solution of x, $m(x, y)$ denotes the measure of y, which is usually a positive real.
  - g is the goal function, and is either min or max

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The Knapsack Problem

- is a specialization of the Container loading problem

definition

Given n objects each with weight w_i and a value v_i a knapsack with maximum weight carrying capacity W, the goal is to optimize the value

$$\sum_{i=1}^{n} x_i * v_i, \quad 1 \leq i \leq n$$

such that

$$\sum_{i=1}^{n} w_i * x_i \quad \leq W$$

- What could be the domain of the values x_i ?
- How is the output optimization function and the W related ?

The Fractional Knapsack Problem : Illustrations

Instance 1

- Let $n = 5$, $W = 100$ and w[i] $= <$ 10 20 30 40 50 $>$
  v[i] $= <$ 20 30 66 40 60 $>$
- What are the values of

$$\sum_{i=1}^{5} w[i]$$

  and

$$\sum_{i=1}^{5} v[i]$$

  ?
- Compute the values for the *Maximum-Value-first, Minimum-Weight-first, and Maximum-value-density-per-weight-first* approaches ?
- What is the optimal answer ?
- What should be the correct approach to solve the problem ?

The 0/1 Knapsack Problem : Illustrations

Instance 1

- i=[1 2 3] $v =< 20\ 15\ 15 >$ $w =< 100\ 10\ 10 >$ $and\ W = 105$
- i= [1 2] $v =< 10\ 20 >$ $w =< 5\ 100 >$ $and\ W = 25$
- i=[1 2 3] $v =< 20\ 15\ 15 >$ $w =< 40\ 25\ 25 >$ $and\ W = 30$

- Apply minimum weight/max value density criterion
- Apply minimum weight criterion
- Apply maximum value density criterion
- Compare with the optimal solutions
- What is the inference to be drawn ?

The Greedy Control Abstraction

```
Algorithm Greedy(Type a[], int n)
1.solution = EMPTY;
2.i=1;
3.for i=1 to n
4.        Type x = select(a);
5.        if feasible(solution, x)
6.        solution=solution U x;
7.return solution
```

Applications

- Optimal solutions
  - simple scheduling problems
  - change making
  - Minimum Spanning Tree (MST)
  - Single-source shortest paths
  - Huffman codes
- Approximations
  - Traveling Salesman Problem (TSP)
  - Knapsack problem
  - other combinatorial optimization problems

Shortest Interv

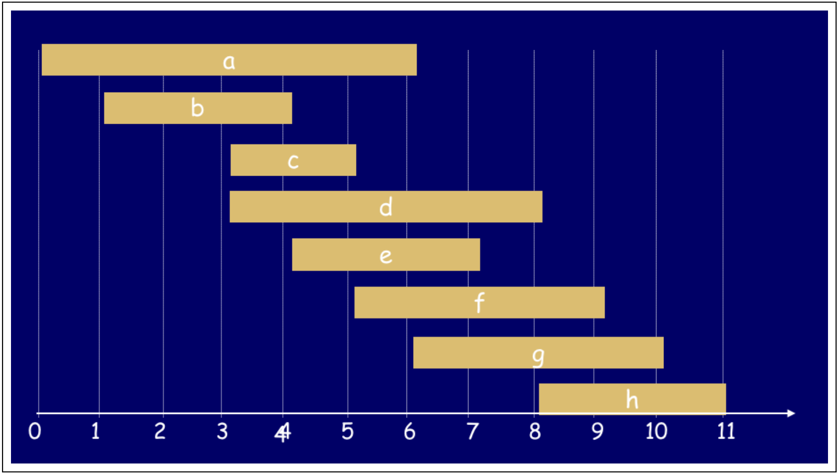
Scheduling#1: Simple Activity Selection

Also is a type of Interval Scheduling

The Problem

- Given n activities each with a defined start time s_i and finish time f_i, the problem is to select a maximal set of mutually compatible activities
- Mutually compatible activities: if each activity i occurs during the half open intveral $[s_i, f_i)$, then they are compatible if, $[s_i,f_i)$ and $[s_j,f_j)$ do not overlap ?
- When do $[s_i,f_i)$ and $[s_j,f_j)$ not overlap ?

Activity Selection Illustration



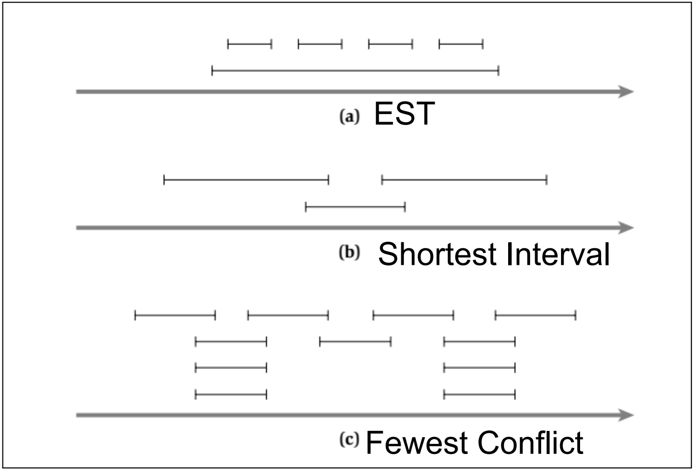
Greedy Choices/Variations

Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

- Earliest start time - Consider jobs in ascending order of start time s_j.
- Earliest finish time - Consider jobs in ascending order of finish time f_j.
- Shortest interval - Consider jobs in ascending order of interval length $f_j - s_j$
- Fewest conflicts - For each job, count the number of conflicting jobs c_j. Schedule in ascending order of conflicts c_j.

Which one of these strategies work ?

Failure Cases



Dry-run of the algoithm

Dry-run

Animation in the PPT demointervalscheduling.ppt

Solution approach

- Greedy algorithm choices
- Consider the jobs in increasing order of finish time.
- Take each job provided it is compatible with the ones already taken.

Algorithm Simple_ActivitySelection(Job, s_i, f_i)
/*A=Set of selected mutually compatible jobs*/
1. Sort jobs by finish times so that $f_1 \leq f_2 \leq \ldots . \leq f_n$.
2. A $\leftarrow \phi$
3. for j = 1 to n
4. if $f_j \leq s_{(j+1)}$ /*(job (j+1) is compatible with A)*/
5. A = A $\cup$ {j}
6. return Selected_Jobs

Complexity

Time taken by the algorithm to execute?

Analysis and Proof: Why the algorithm works?: Proof by Contradiction

- Proof: (by contradiction) ....i.e. $m > k$ in the following:

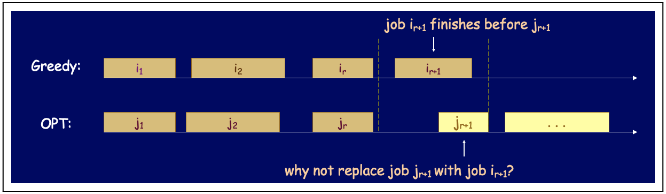


Figure: There is a request j_{k+1} in the possible set of job requests after the job i_k ends ?

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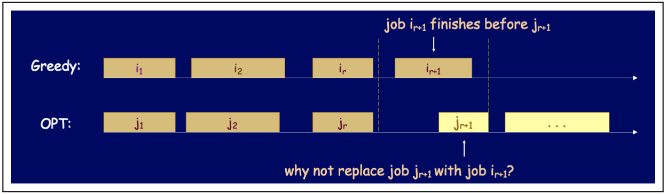

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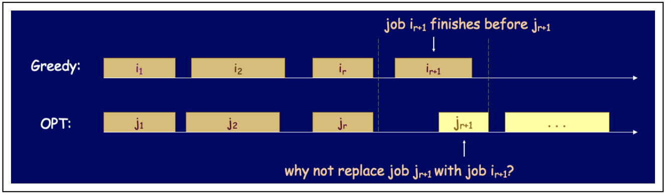

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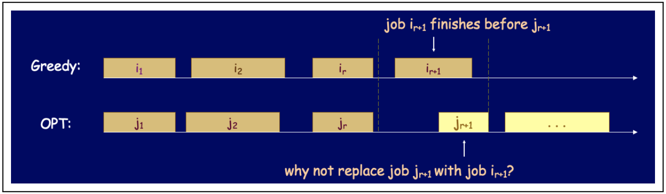

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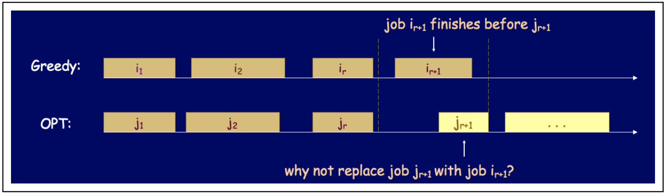
  - Note that $m > k$



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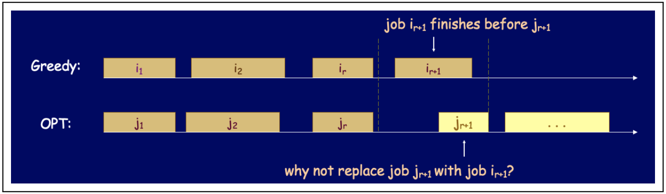
  - How are $f(i_k)$ and $f(j_k)$ related?



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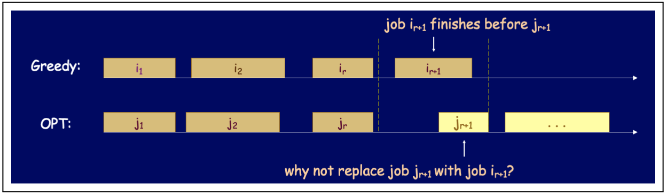
  - Note that $m > k$
  - How are $f(i_k)$ and $f(j_k)$ related?
  - The possible set of job requests still contains the request $f(j_{k+1})$ after all the requests $i_1, i_2, ...i_k$ end or are deleted. Why ?



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- For the interval scheduling problem, let the set $A = i_1, i_2, i_3, ...i_k$ with $|A| = k$ be the set of intervals returned by the algorithm as the answer and

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- The goal is to prove that k = m i.e. to prove that the r^{th} accepted request in the algorithm's finishes no later than the r th request in the optimal schedule.
- What is this equivalent to proving ?

Analysis and Proof by Mathematical Induction

- For the interval scheduling problem, let the set $A = i_1, i_2, i_3, ...i_k$ with $|A| = k$ be the set of intervals returned by the algorithm as the answer and
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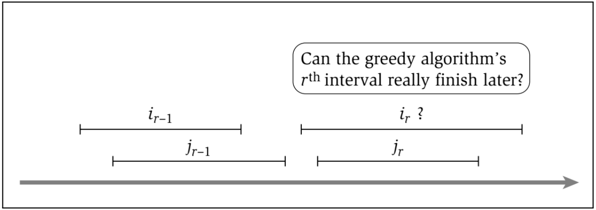

- How are $fj_{(r-1)}$ and $sj_{(r-1)}$ related ?
- How are $fi_{(r-1)}$ and sj_r related ?

Machine scheduling problem II

Minimize Average Completion time

Jobs $j_1, j_2, j_3 .... j_n$ with running times $t_1, t_2, t_3, ... t_n$ to be scheduled on a single processor such as to minimize the avg completion time

Process ID	Execution Time in units
J_1	15
J_2	8
J_3	3
J_4	10

How to implement the above algorithm ? What is the time complexity?

Machine scheduling problem II...

Minimize Average Completion time

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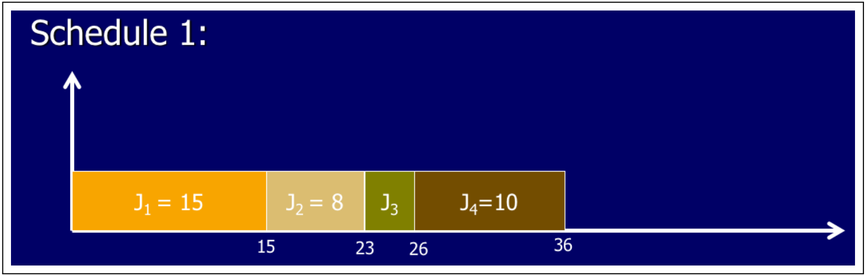


Shortest Job First Scheduling

Proof of the algorithm

Prove that the shortest job first assignment for JJobs $j_1, j_2, j_3........j_n$ with running times $t_1, t_2, t_3, ...t_n$ to be scheduled on a single processor such as to minimize the avg completion time is an optimal assignment.

Machine scheduling problem III

Minimum Average Completion time with multiprocessors

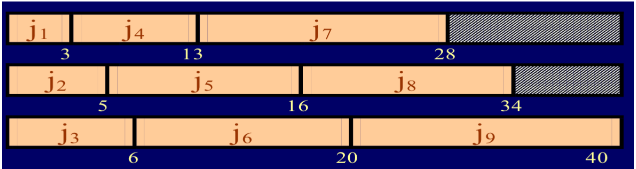
- Jobs $j_1, j_2, j_3 .... j_n$ with running times $t_1, t_2, t_3, ... t_n$ to be scheduled on multiprocessors such as to minimize the average completion time

j1	j2	j3	j4	j5	j6	j7	j8	j9
3	5	6	10	11	14	15	18	20

Machine scheduling problem III

Minimum Average Completion time with multiprocessors

- Jobs $j_1, j_2, j_3 \ldots\ldots\ldots j_n$ with running times $t_1, t_2, t_3, \ldots\ldots\ldots t_n$ to be scheduled on multiprocessors such as to minimize the average completion time



M_1	M_2	M_3	M_1	M_2	M_3	M_1	M_2	M_3
j_1	j_2	j_3	j_4	j_5	j_6	j_7	j_8	j_9
3	5	6	10	11	14	15	18	20
3	5	6	13	16	20	28	34	40

Machine Scheduling Problem IV

Minimizing the Final Completion time with multiprocessors

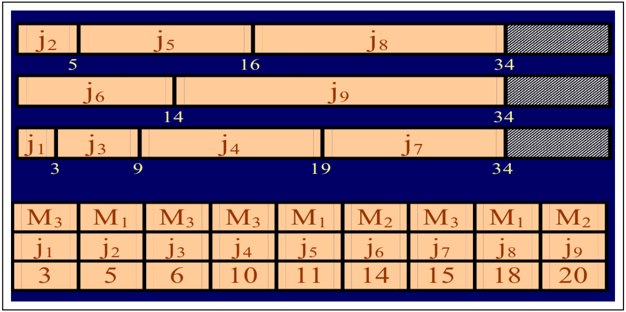
- Jobs $j_1, j_2, j_3........j_n$ with running times $t_1, t_2, t_3, ...t_n$ to be scheduled on multiprocessors such that
  - no machine processes more than one process at a time
  - no process is executed by more than one machine
  - there is non-preemptive scheduling
  - the final completion time is minimized.

- What is the final completion time ?
- How to solve this for the given jobmix ?

j_1	j_2	j_3	j_4	j_5	j_6	j_7	j_8	j_9
3	5	6	10	11	14	15	18	20

Machine Scheduling Problem IV...

Understanding what is Final Completion time....



- What is the final completion time for this job mix?
- How to solve this ?

Machine Scheduling Problem IV...

Minimizing the Final Completion time with multiprocessors

- What is the final completion time for this job mix?

j1	j2	j3	j4	j5	j6	j7	j8	j9
3	5	6	10	11	14	15	18	20

- What is the final completion time for this job mix?
- How to solve this ?

Minimizing the final completion time

TThe problem is NP-hard

- No polynomial time algorithm to run in $O(n^k m^l)$ for any constants k and l
- With an approximation algorithm the schedule lengths are though not optimal but at the most $(\frac{4}{3} - \frac{1}{3m})$ of the optimal schedule

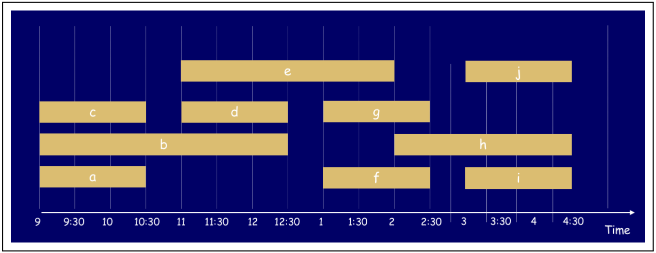
Review

Reviewing the summary of the scheduling variations discussed

Machine Scheduling V

Scheduling a class time table

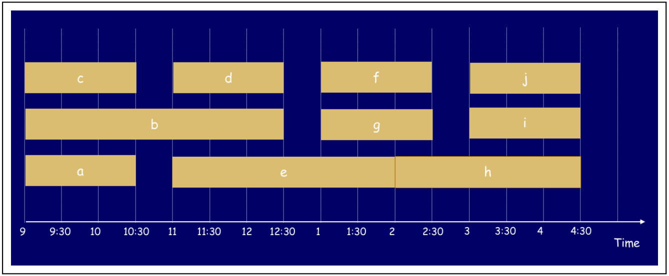
- But the goal is to minimize the number of classrooms used
- Assume that a lecture j starts at s_j and finishes at f_j, then the goal is to find the minimum number of classrooms to schedule all lectures, so that no two occur at the same time in the same room.
- An example schedule with 4 classrooms to schedule 10 lectures



Machine Scheduling V...

Scheduling a class time table

- Lecture j starts at s_j and finishes at f_j.
- This schedule uses only 3.



Machine Scheduling V...

One Solution approach

- Arrange the lectures in their increasing order of start times
- Keep track of availability times of classrooms i.e. let availability time of a classroom be M1.
- Then, assign the lecture i_1, completing at time t_1, mark its availability time as t_1 and check compatibility, when scheduling the next lecture l_2

Machine Scheduling V...

A typical schedule and approach

task	start_ time	finish_ time	time required	availability time now on respective classroom	scheduling order
A	0	2	2	2(M1)	1
B	3	7	4	7(M1)	3
C	4	7	3	7(M3)	4
D	9	11	2	11(M3)	7
E	7	10	3	10(M1)	6
F	1	5	4	5(M2)	2
G	6	8	2	8(M2)	5

Solution stragtegy

Algorithm approach

- Consider lectures in increasing order of start time
- assign lecture to any compatible classroom.
- For each classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

Solution approach

```
Algorithm Classroom_Scheduling(Interval[], s[], f[]))
1. Sort intervals by starting time so that \\
```

$$s_1 \leq s_2 \leq s3 \leq ...s_(n-1) \leq s_n$$

```
2. d ← 0
3. for j = 1 to n
4. if (lecture j is compatible with some classroom k)
5.        schedule lecture j in classroom k
6. else
7.         allocate a new classroom d + 1
8.        schedule lecture j in classroom d + 1
9.        d ← (d+1)
```

Complexity

Time taken by the algorithm to execute?

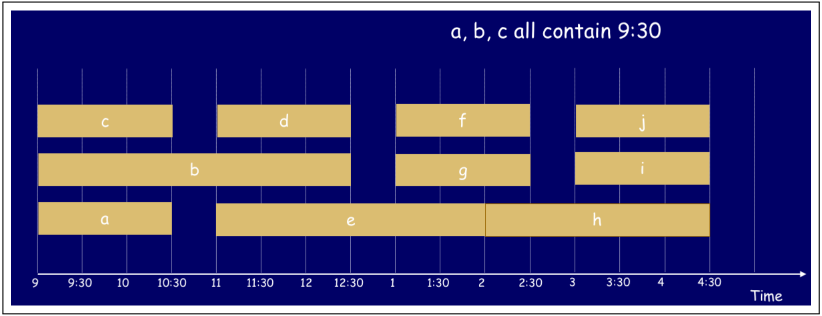
IP: Lower Bound on Optimal Solution

- def: The depth of a set of open intervals is the maximum number of the intervals contained, at a unique time
- Key observation: Prove that the number of resources (classrooms) needed is at least the depth.

Algorithm Correctness

Theorem

IF we use the greedy algorithm above, every lecture will be assigned a classroom and no two overlapping lectures will receive the same classroom

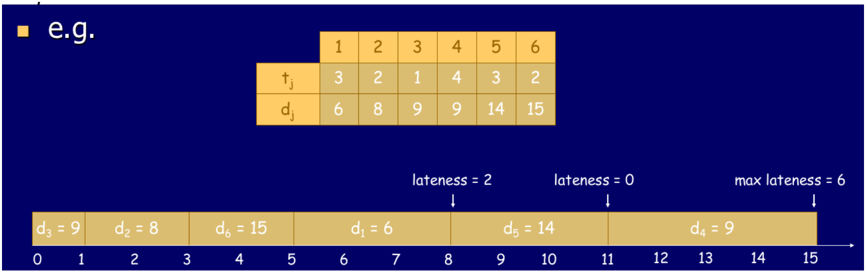


The Optimal Tape Storage Problem

- Given n files of length $m_1, m_2, m_3, m_4......m_n$ find the best order in which the files can be stored on a sequential storage device.
- e.g. if n=3 and m1=5, m2=10 and m3=3
- There can be 3! possible orderings.
- Which one is the best ?

Scheduling to minimize the lateness

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j. If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $l_j = \max 0, f_j - d_j$.
- Goal: schedule all jobs to minimize the maximum lateness L = maximum lateness



	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15

Figure: Lateness of different jobs in a schedule

How to minimize maximum lateness ?

Greedy template. Consider jobs in some order.

- Shortest processing time first: Consider jobs in ascending order of processing time t_j.
- Earliest deadline first: Consider jobs in ascending order of deadline d_j.
- Smallest slack: Consider jobs in ascending order of slack d_j - t_j.

Minimizing lateness: Counterexamples

Shortest processing time first does not work

	1	2
tj	1	10
dj	100	10

Figure: Can we and if so how can one achieve lateness 0 above?

Shortest slack time first does not work

	1	2
t_j	1	10
d_j	2	10

Figure: Can we and if so how can one achieve lateness 0 above?

Solution approach

```
Algorithm EarliestDeadlineFirst(Job[], s[], f[]))
1.Sort n jobs by deadline so that d_1 ≤ d_2 ≤ ......d_n
2.t ← 0
3.for j = 1 to n
4.      Assign job j to interval [t, t + t_j]
5.      s_j ← t, f_j ← t + t_j, t ← t + t_j
6.output intervals [s_j, f_j]
```

Proving Correctness of a greedy algorithm

- Greedy algorithm stays ahead.
  - Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.
- Exchange argument
  - Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.
- Structural
  - Discover a simple *structural* bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Two vital properties of a Greedy Algorithm

- Greedy-choice property
  - a globally optimal solution can be arrived from a locally optimal choice.
  - algorithm proceeds in a top down fashion – reducing the given problem instance into smaller ones
- Optimal Sub-structure property
  - an optimal solution to a problem contains within it other optimal solutions to smaller subproblems

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