Fourier Series:

- Any function that periodically repeats itself can be expressed as the sum of sines and cosines of different frequencies each multiplied by a different coefficient

- This sum is known as Fourier Series

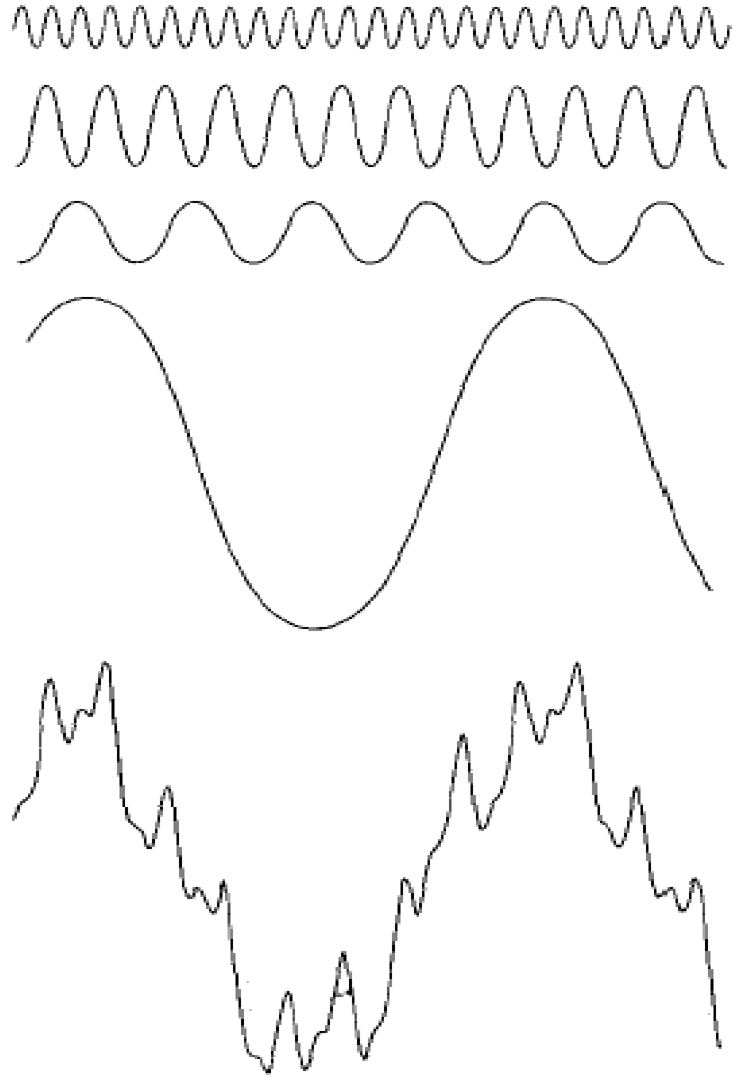


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Fourier Transform:

- Even functions that are not periodic can be expressed as the integrals of sines and cosines multiplied by a weighing function

- This is known as Fourier Transform

Inverse transform:

- A function expressed in either a Fourier Series or transform can be reconstructed completely via an inverse process with no loss of information

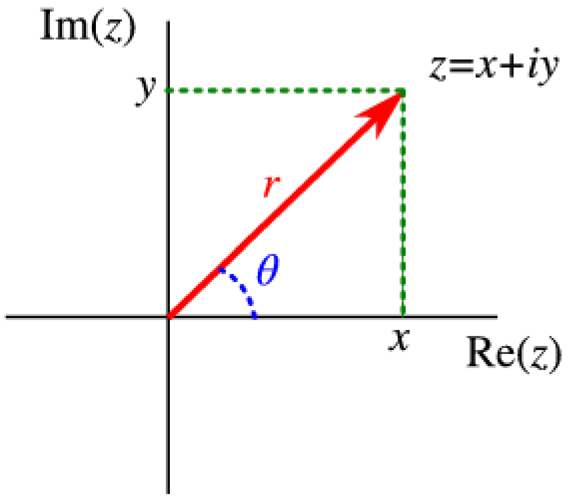
- This is one of the important characteristics of these representations because they allow us to work in the Fourier Domain and then return to the original domain of the function

Complex numbers:

A complex number, C, is defined as

$$C = R + jI$$

$$C^* = R - jI$$



$$C = |C|(\cos\theta + j\sin\theta)$$

$$|C| = \sqrt{R^2 + I^2}$$

$$\theta = \arctan(I/R)$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$C = |C|e^{j\theta}$$

Fourier Series:

- If *f(t)* is a periodic function of a continuous variable *t*, with period *T*

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

Where:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Impulses

Unit impulse of a continuous variable t located at t=0 :

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Sifting Property of impulses:

Sifting property :

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$ (if *f(t)* continuous at *t=0*)

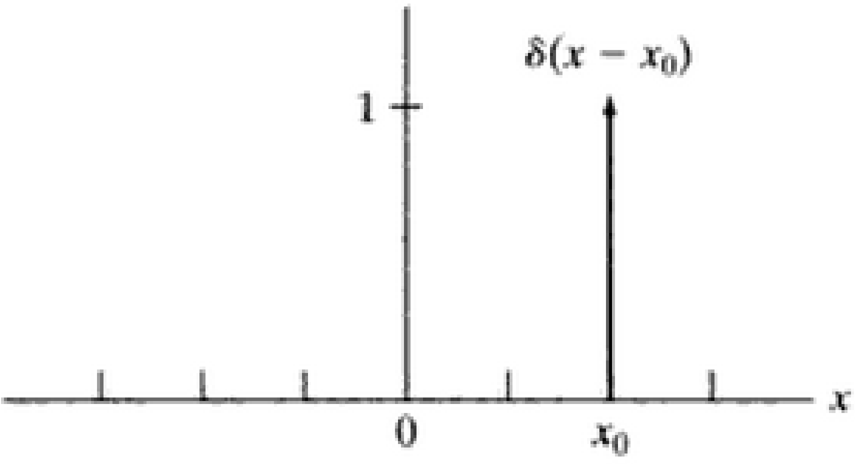
$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

Unit discrete impulse:

Unit discrete impulse located at x=0 :

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$



Sifting property of discrete Impulses:

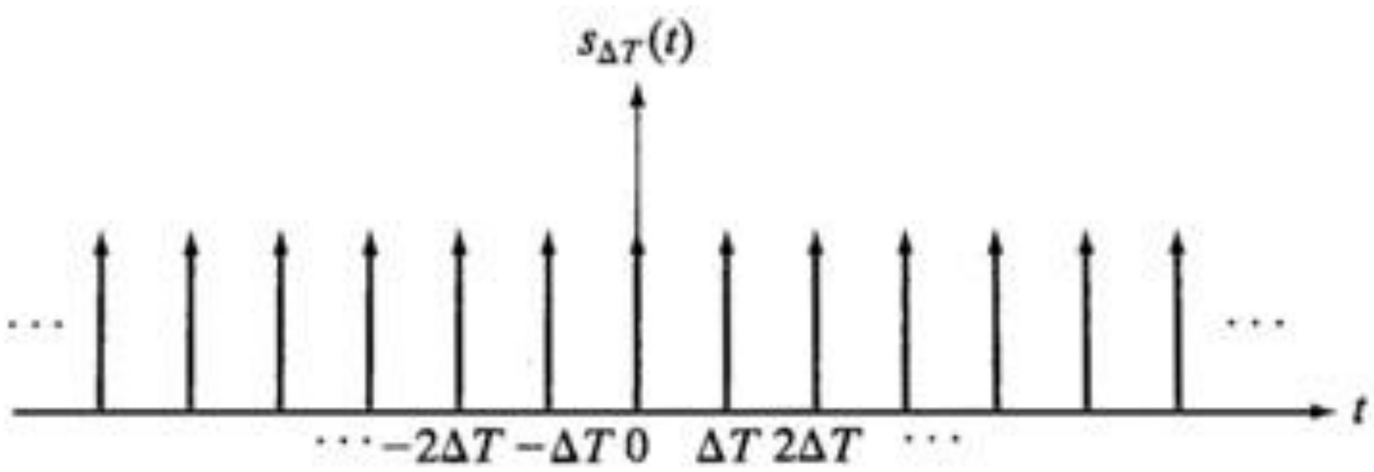
$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

Impulse train :

- sum of infinitely many periodic impulses ΔT units apart:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



The Fourier Transform:

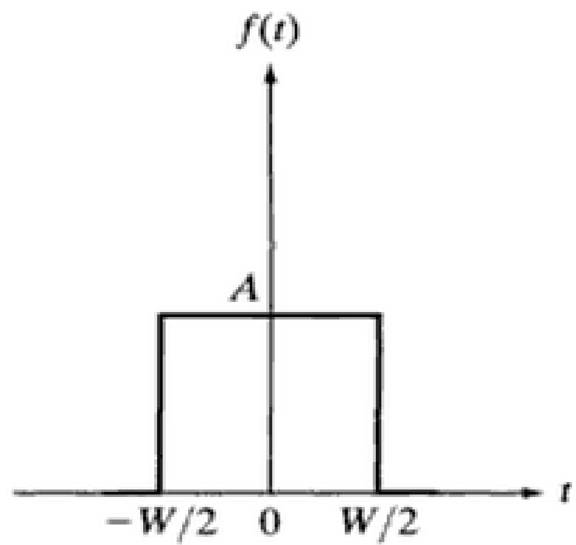
Fourier Transform of a continuous function *f(t)* of a continuous variable *t* :

$$FT\left[f(t)\right] = F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

Inverse Fourier Transform:

$$FT^{-1}\left[F(\mu)\right] = \quad f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

Example:1 Find Fourier transform of function shown in figure below.

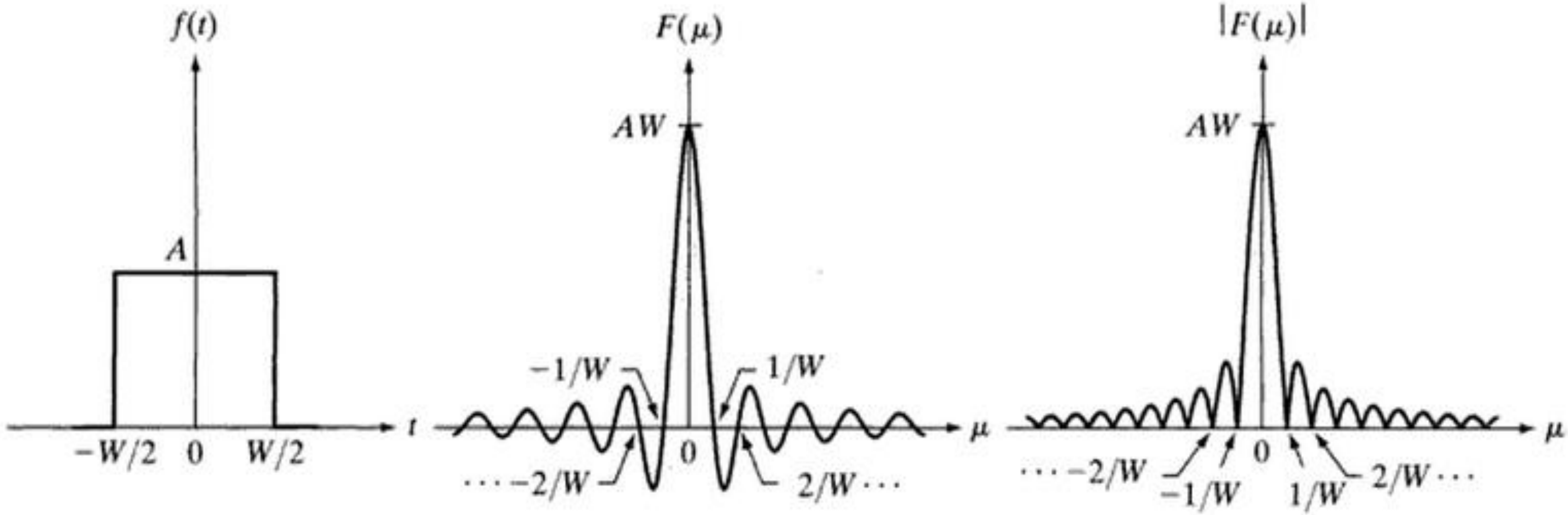


$$F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}\, dt$$

$$= \frac{-A}{j2\pi\mu}\Big[e^{-j2\pi\mu t}\Big]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\Big[e^{-j\pi\mu W} - e^{j\pi\mu W}\Big]$$

$$= \frac{A}{j2\pi\mu}\Big[e^{j\pi\mu W} - e^{-j\pi\mu W}\Big]$$

$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$$



a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

$$|F(\mu)| = AT\left|\frac{\sin(\pi\mu W)}{(\pi\mu W)}\right|$$

Example 2 : Fourier Transform of a unit impulse located at the origin:

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt$$

$$= e^{-j2\pi\mu 0} = e^{0} = 1$$

$$\boxed{\delta(t) \leftrightarrow 1}$$

Example 3 : Fourier Transform of a unit impulse located at $t = t_0$:

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu t_0}$$

$$= \cos(2\pi\mu t_0) - jsin(2\pi\mu t_0)$$

$$\boxed{\delta(t - t_0) \leftrightarrow e^{-j2\pi\mu t_0}}$$

Example : 4 Find Inverse Fourier transform of $\delta(\mu - \mu_0)$

$$F^{-1}\left(\delta(\mu - \mu_0)\right) = \int_{-\infty}^{\infty} \delta\,(\mu - \mu_0) e^{j2\pi\mu t} d\mu$$

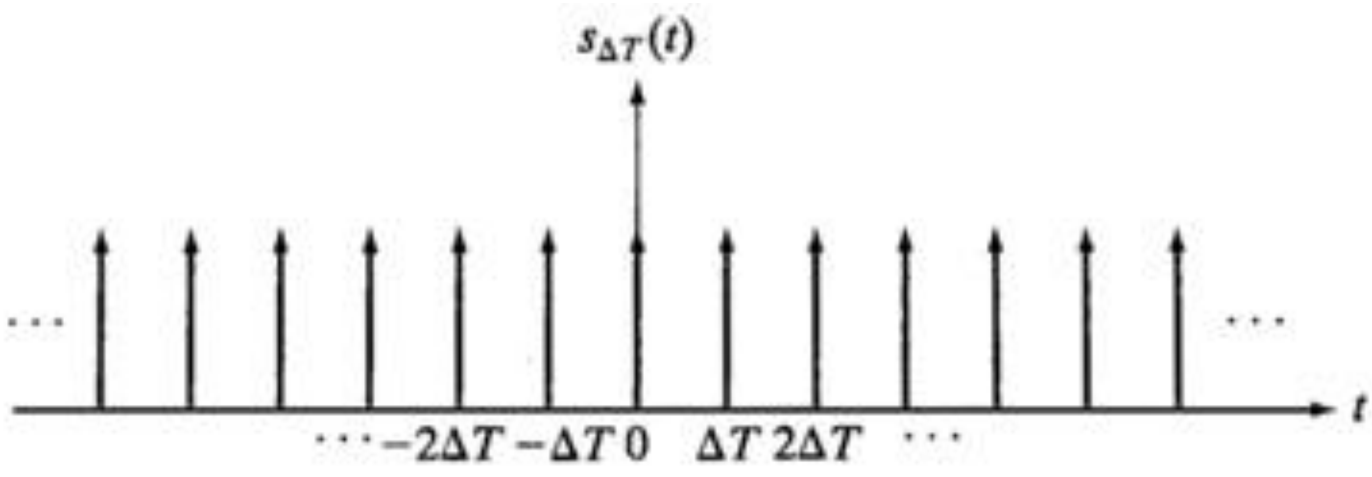
$$F^{-1}\left(\delta(\mu - \mu_0)\right) = e^{j2\pi\mu_0 t}$$

$$\boxed{e^{j2\pi\mu_0 t} \leftrightarrow \delta(\mu - \mu_0)}$$

Example 2 : Fourier Transform $S(\mu)$ of an impulse train with period ΔT

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt$$



$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt$$

$$= \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{2\pi n}{\Delta T}t} dt$$

$$= \frac{1}{\Delta T}$$

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

$$s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{T}t}$$

$$\boxed{e^{j2\pi\mu_0 t} \leftrightarrow \delta(\mu - \mu_0)}$$

$$\Im\left\{e^{j\frac{2\pi n}{\Delta T}t}\right\} = \delta\left(\mu - \frac{n}{\Delta T}\right)$$

So, $S(\mu)$, the Fourier transform of the periodic impulse train $s_{\Delta T}(t)$, is

$$\begin{aligned}
S(\mu) &= \Im\left\{s_{\Delta T}(t)\right\} \\
&= \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} \\
&= \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} \\
&= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)
\end{aligned}$$

Convolution:

Convolution of functions *f(t)* and *h(t)*, of one continuous variable *t* :

$$f(t)\bigstar h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

$$FT\left\{f(t)\bigstar h(t)\right\} = H(\mu)F(\mu)$$

Fourier Transform pairs:

$$f(t)\bigstar h(t) \Longleftrightarrow H(\mu)F(\mu)$$

$$f(t)h(t) \Longleftrightarrow H(\mu)\bigstar F(\mu)$$

Proof:

$$\Im\{f(t)\star h(t)\} = \int_{-\infty}^{\infty}\boxed{\left[\int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\,d\tau\right]}e^{-j2\pi\mu t}\,dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\,e^{-j2\pi\mu t}\,dt\right]d\tau$$

Time shifting property of Fourier transform:

$$\Im\{h(t-\tau)\} = H(\mu)e^{-j2\pi\mu\tau}$$

$$
\begin{aligned}
\Im\left\{f(t) \star h(t)\right\} &= \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)\, e^{-j2\pi\mu\tau}\right] d\tau \\
&= H(\mu) \int_{-\infty}^{\infty} f(\tau)\, e^{-j2\pi\mu\tau}\, d\tau \\
&= H(\mu)\, F(\mu)
\end{aligned}
$$

2-D Impulse:

For Continuous variables t and z :

$$\delta(t, z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

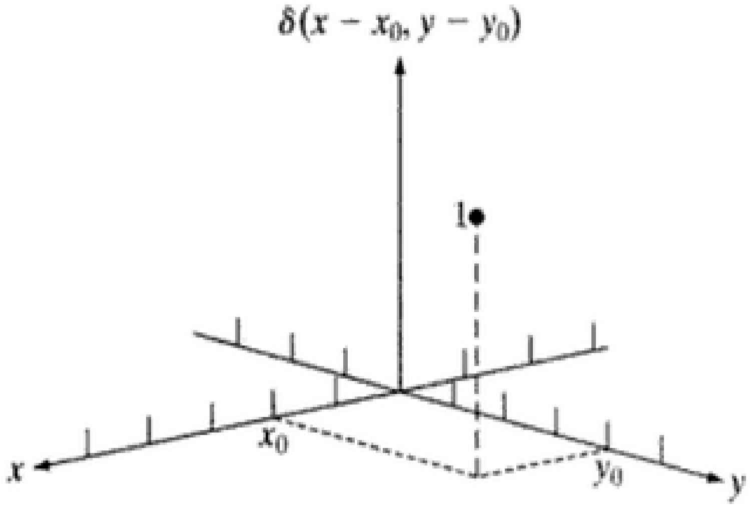
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t, z)\, dt\, dz = 1$$

Sifting property:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\,\delta(t, z)\, dt\, dz = f(0, 0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\,\delta(t - t_0, z - z_0)\, dt\, dz = f(t_0, z_0)$$

Discrete 2-D impulse:



$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

Sifting property:

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\delta(x, y) = f(0, 0)$$

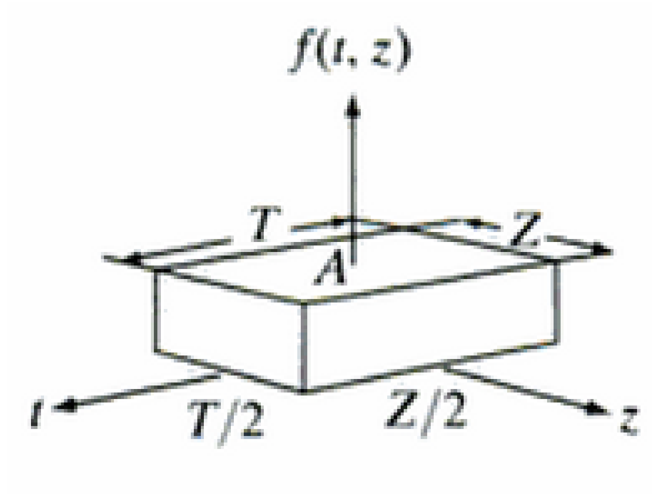
$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\delta(x - x_0, y - y_0) = f(x_0, y_0)$$

2-D continuous Fourier transform pair:

$$F(\mu, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)\, e^{-j2\pi(\mu t + vz)}\, dt\, dz$$

$$f(t, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu, v)\, e^{j2\pi(\mu t + vz)}\, d\mu\, dv$$

Example: find Fourier transform of a 2-D function:



$$F(\mu, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t, z)e^{-j2\pi(\mu t+\nu z)}\,dt\,dz$$

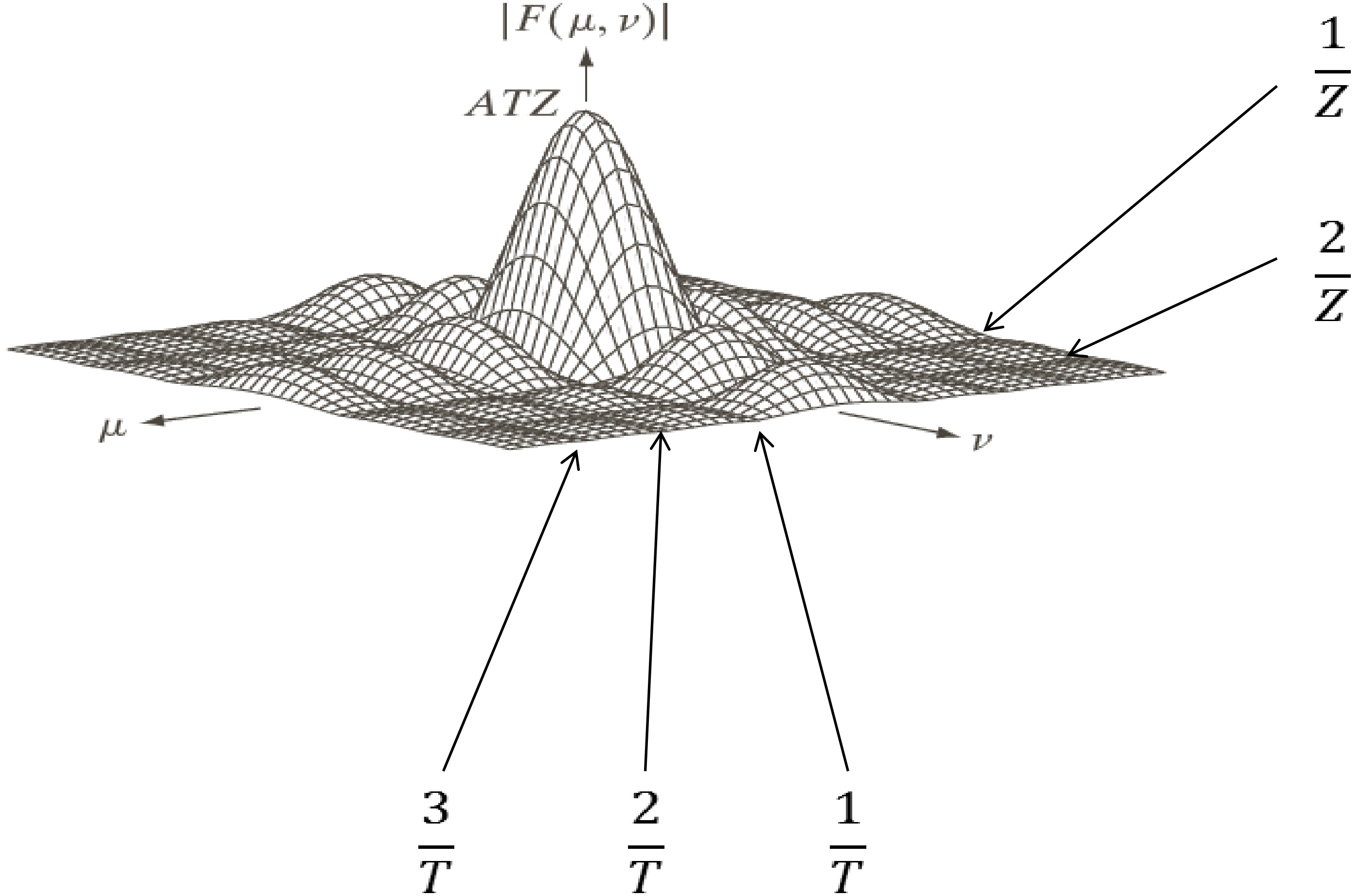
$$= \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} Ae^{-j2\pi(\mu t+\nu z)}\,dt\,dz$$

$$= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]$$

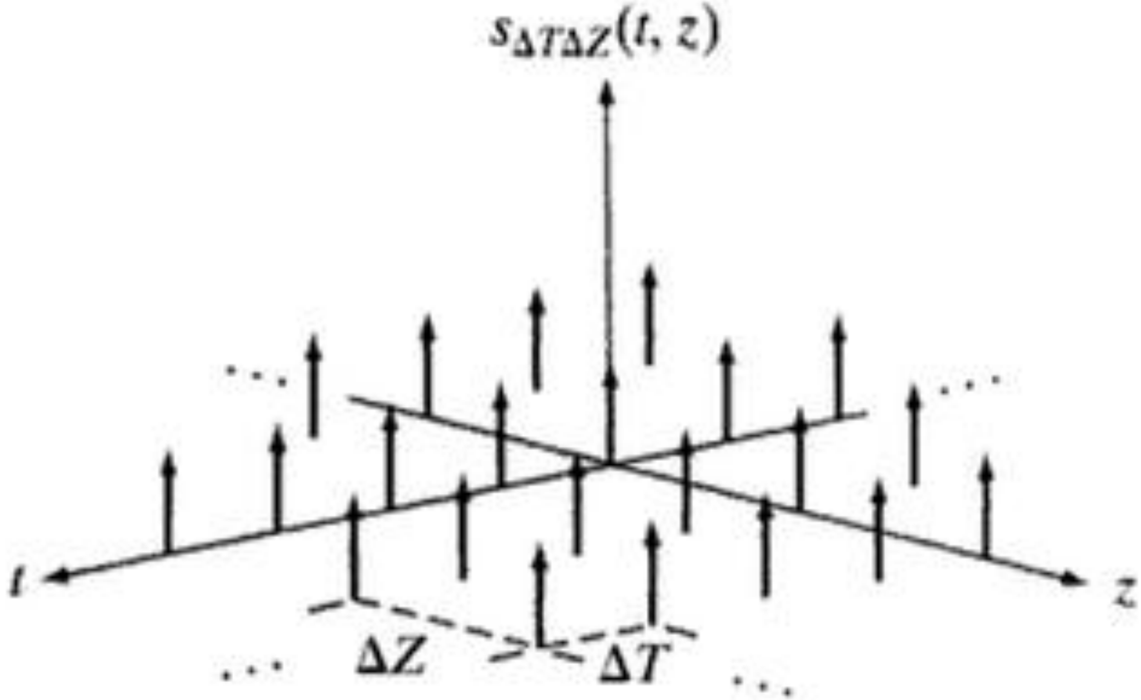
The magnitude (spectrum) is given by the expression

$$|F(\mu, \nu)| = ATZ\left|\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right|\left|\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right|$$

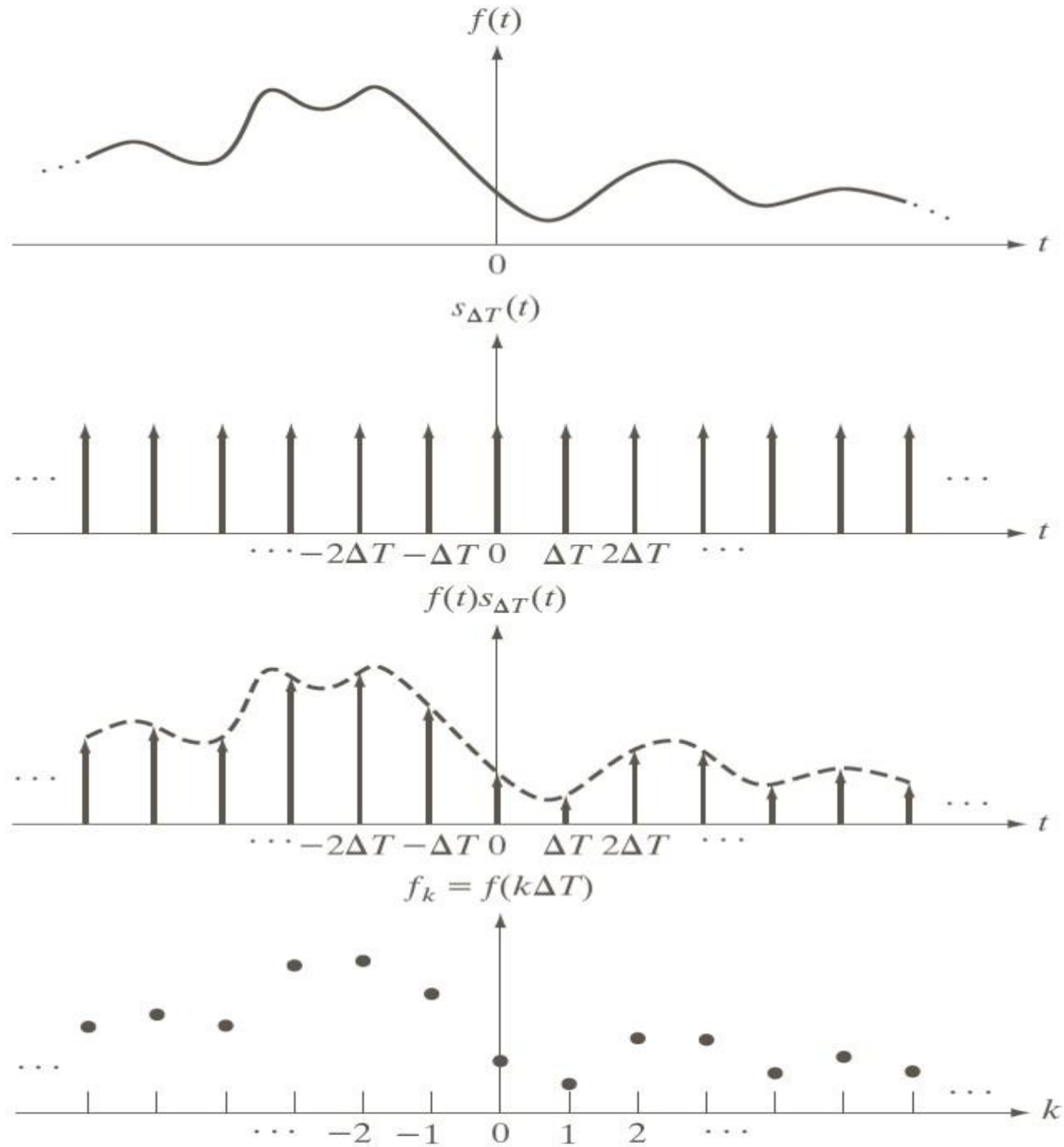
Decaying lobs: inversely proportional to T and Z at shown locations.



2-D impulse train:



Sampling and the Fourier Transform of Sampled Functions:



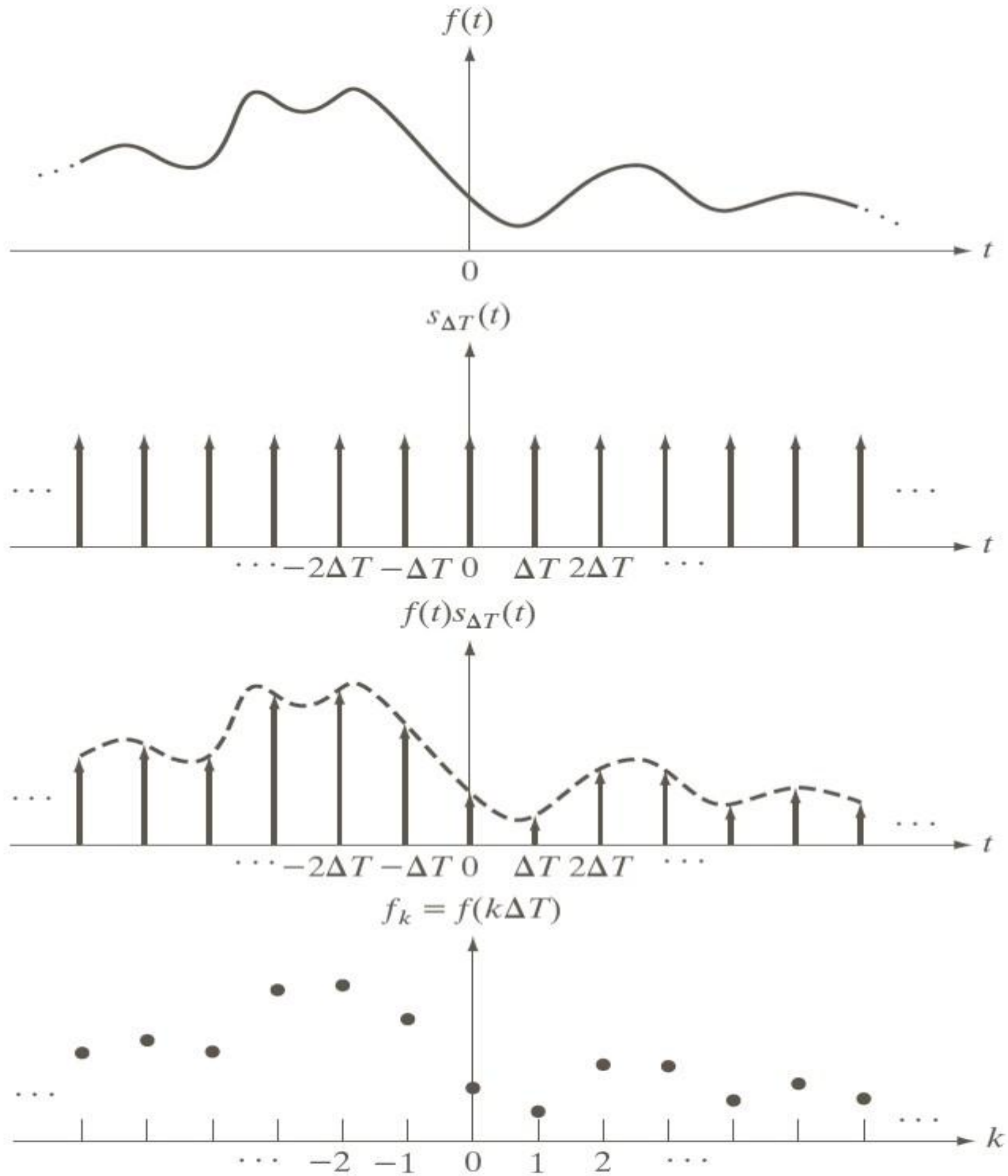
value from available sampled function:

$$
\begin{aligned}
f_k &= \int_{-\infty}^{\infty} f(t)\,\delta(t - k\Delta T)\,dt \\
&= f(k\Delta T)
\end{aligned}
$$

Sampling and the Fourier Transform of Sampled Functions:

$$s_{\Delta T} = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

$$f(t)s_{\Delta T} = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

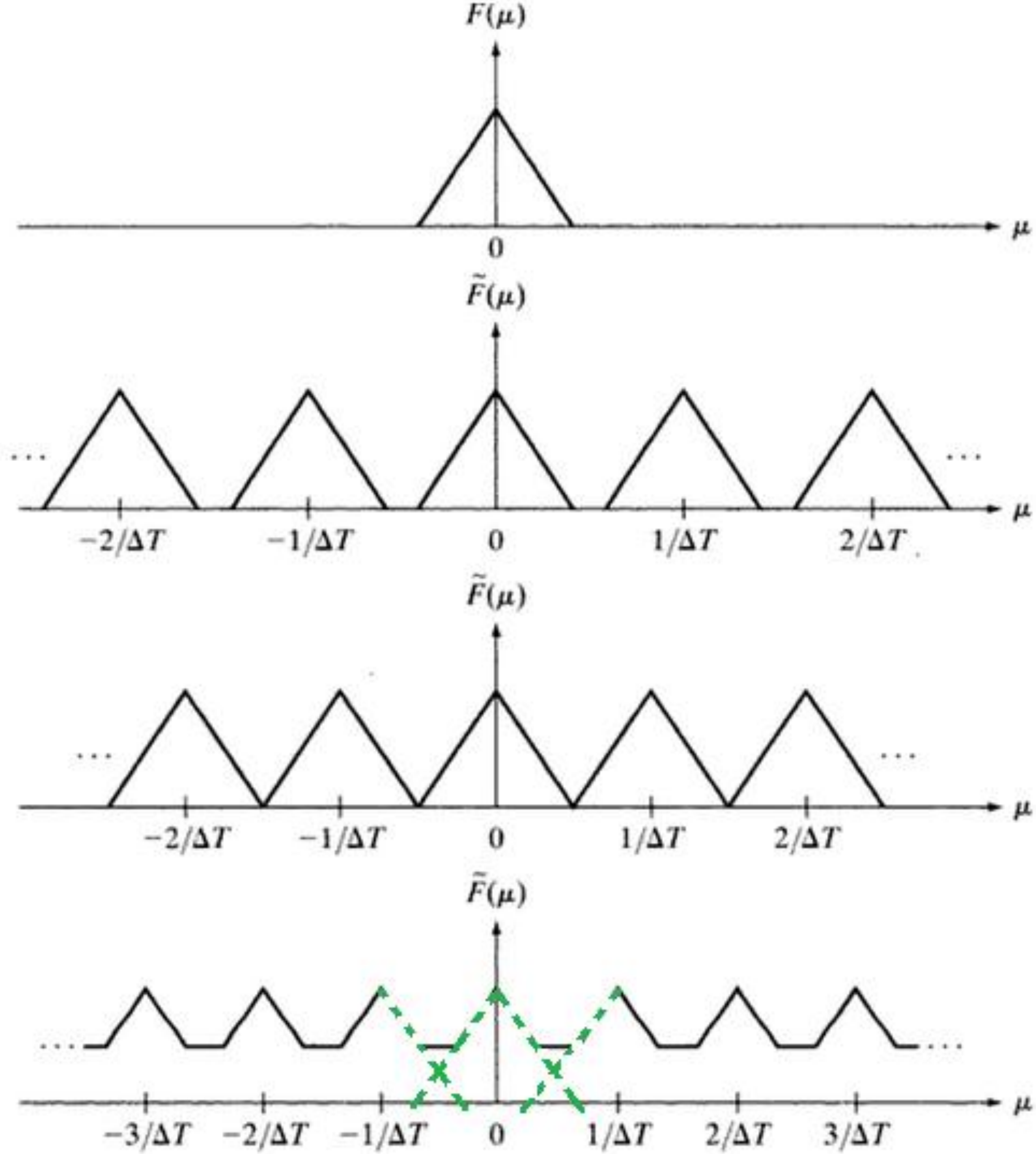


$$\tilde{f}(t) = f(t)s_{\Delta T} = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$\begin{aligned}\tilde{F}(\mu) &= \Im\left\{\tilde{f}(t)\right\} \\ &= \Im\left\{f(t)s_{\Delta T}(t)\right\} \\ &= F(\mu) \star S(\mu)\end{aligned}$$

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right) \qquad f(t)\star h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

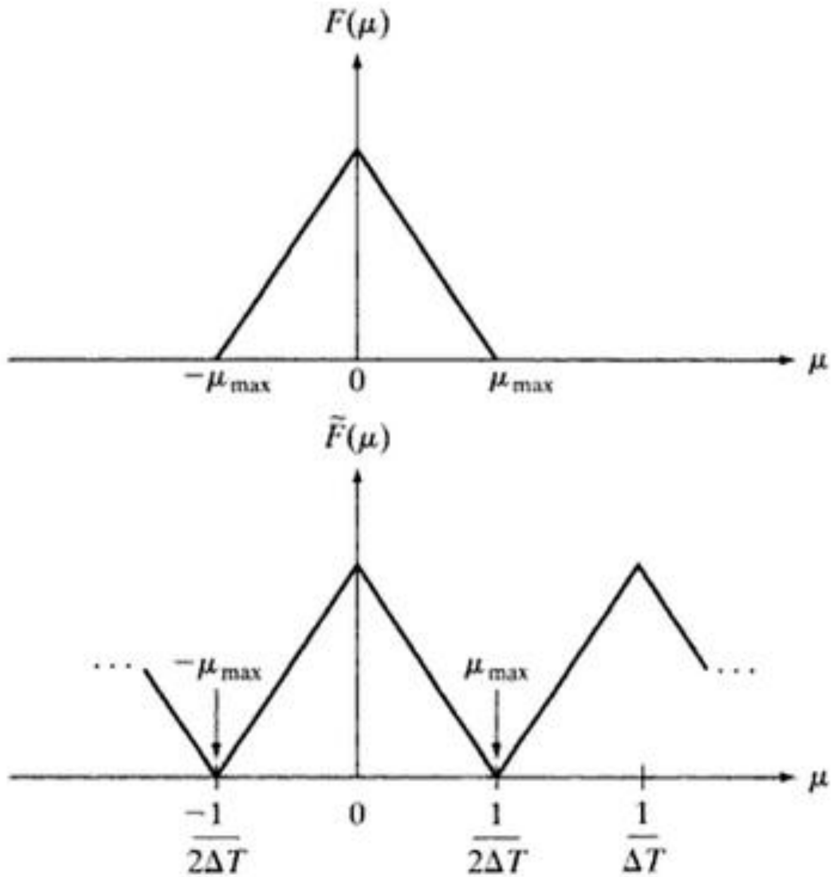
$$
\begin{aligned}
\tilde{F}(\mu) &= F(\mu) \star S(\mu) \\
&= \int_{-\infty}^{\infty} F(\tau)\, S(\mu - \tau)\, d\tau \\
&= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)\, \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)
\end{aligned}
$$



a
b
c
d

FIGURE 4.6
(a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

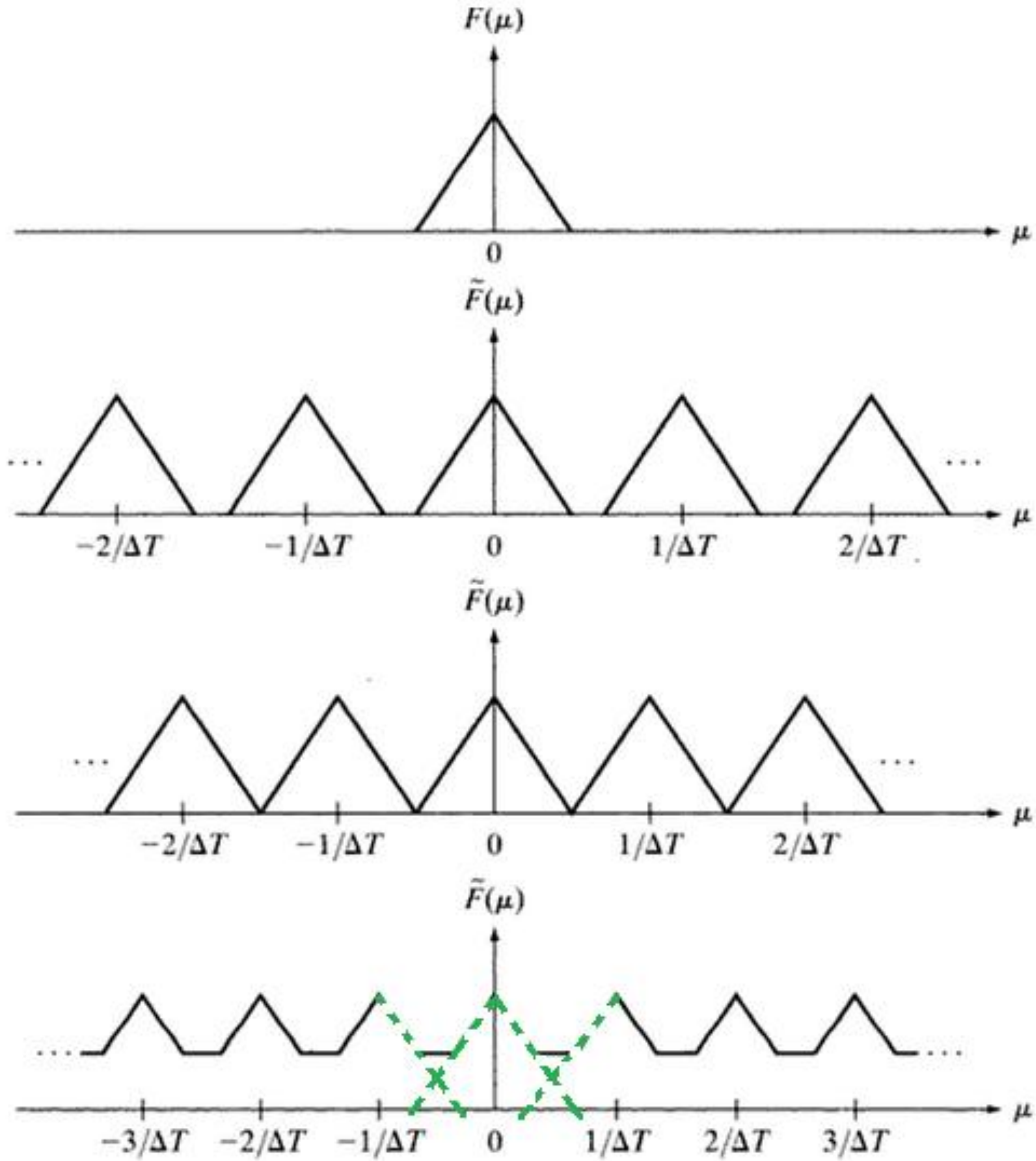
Assuming f(t) is band limited to μ_{max}:



a
b

FIGURE 4.7
(a) Transform of a band-limited function.
(b) Transform resulting from critically sampling the same function.

Sufficient separation guaranteed if : $\frac{1}{\Delta T} > 2\mu_{max}$



Oversampled

$$\frac{1}{\Delta T} > 2\mu_{max}$$

Critically sampled

$$\frac{1}{\Delta T} = 2\mu_{max}$$

Undersampled

$$\frac{1}{\Delta T} < 2\mu_{max}$$

Sampling theorem:

Sampling Theorem:

A continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function

Sampling at:

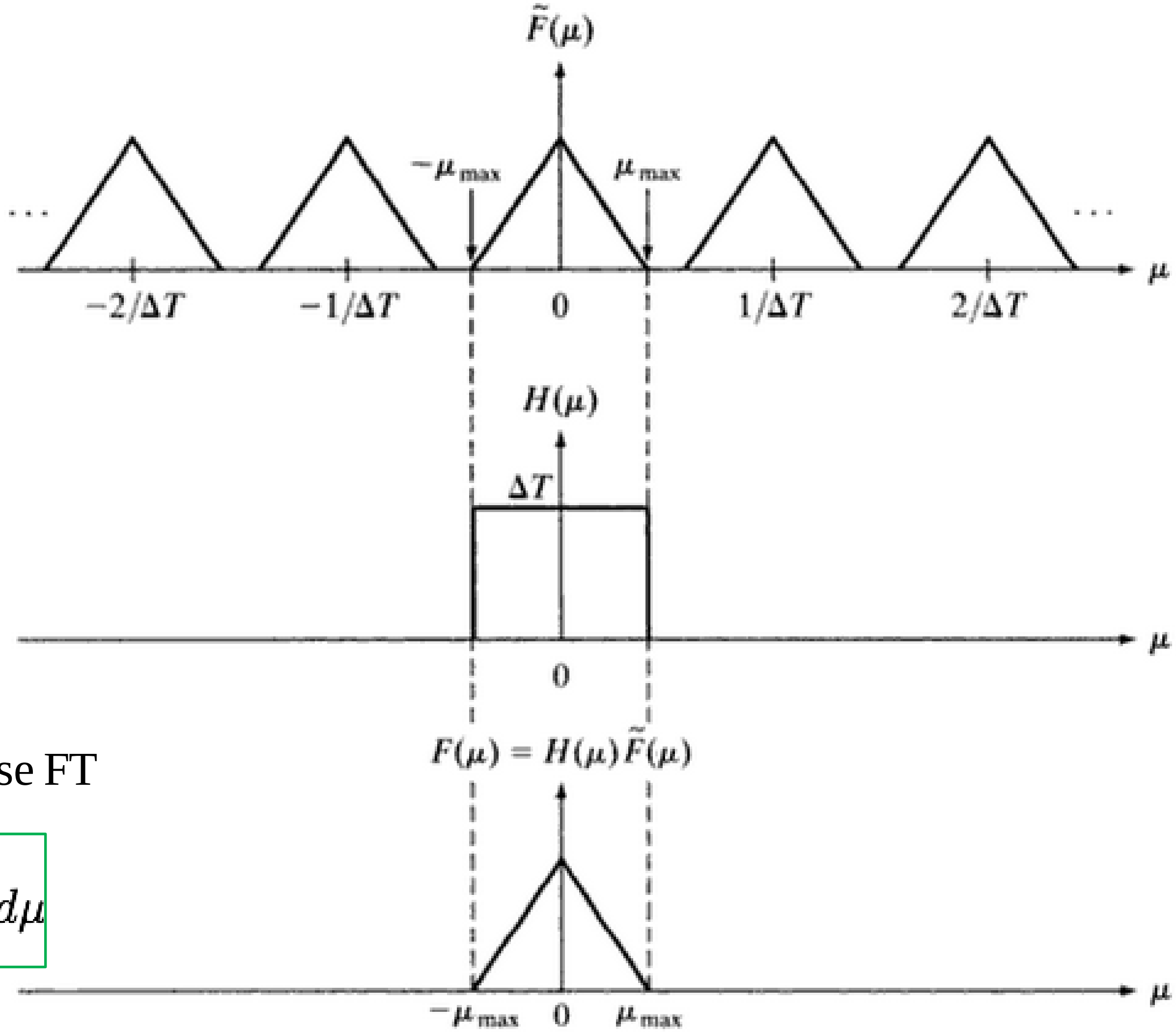
$$\frac{1}{\Delta T} = 2\mu_{max}$$

Nyquist rate

Signal recovery:

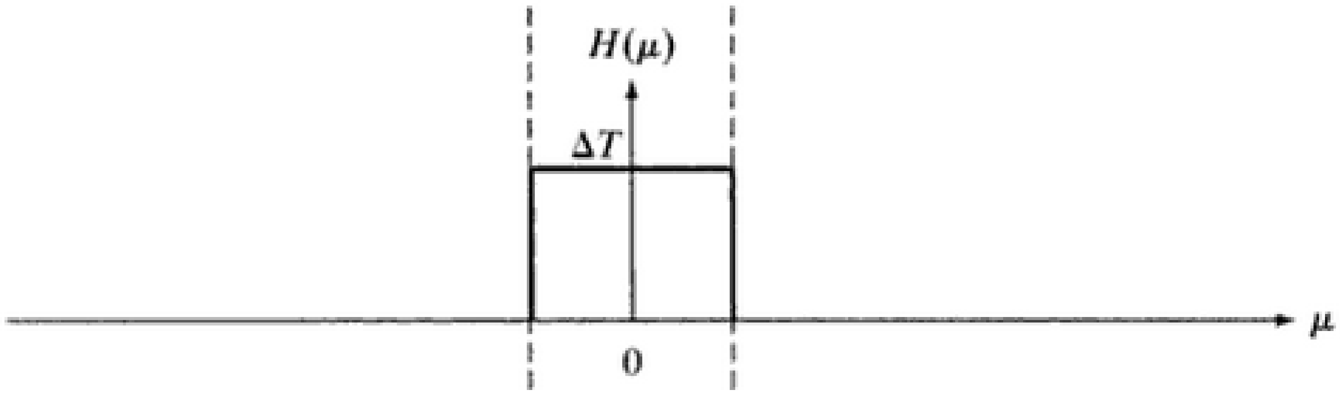
a
b
c

FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.



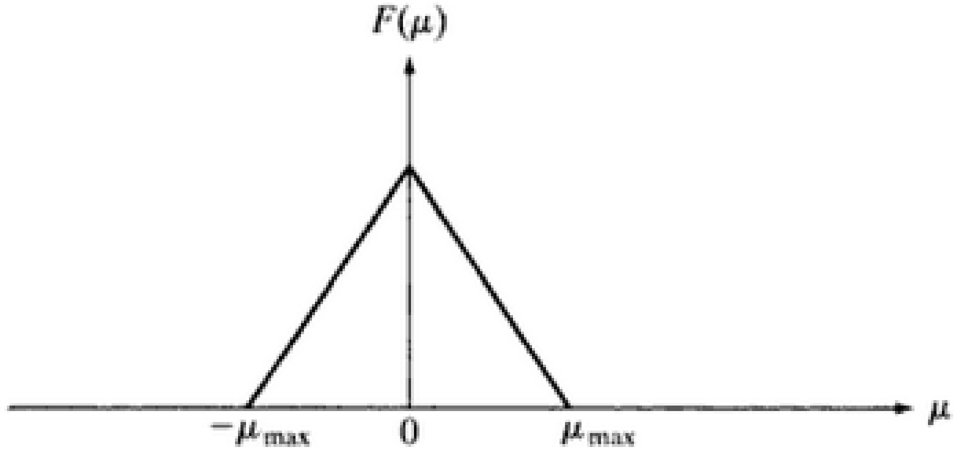
Recover *f(t)* using the inverse FT

$$\boxed{f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu}$$

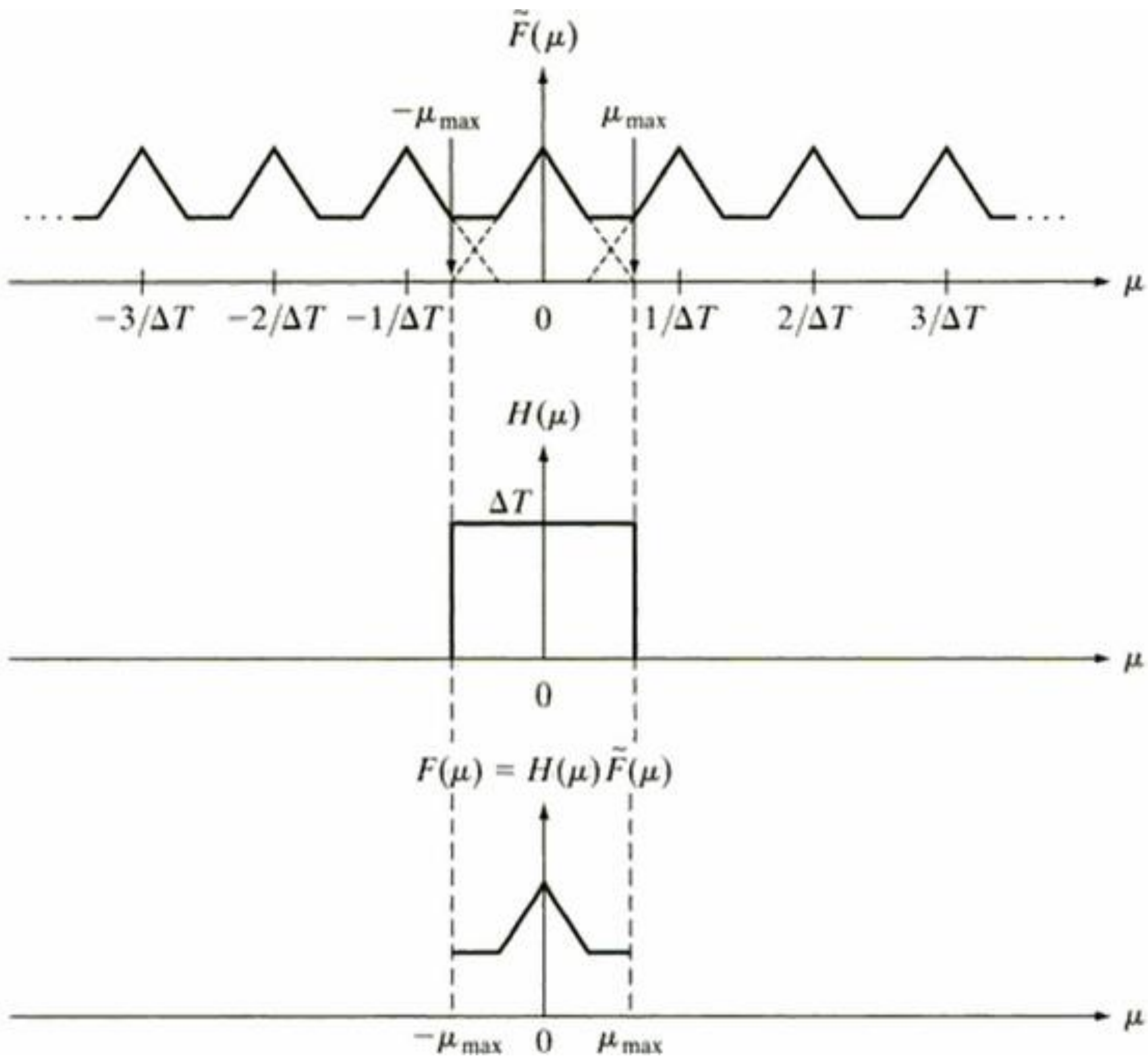


$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \leq \mu \leq \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$

- H(μ) is called a low pass filter.
- It is *ideal- rapid transition from ΔT to 0 and reverse*
- This characteristic is not achievable by physical electronic components.
- Also known as **Reconstruction filters**.



- Assumed: f(t) is band limited to μ_{max}
- Band limited signal can not be time limited. f(t) must extend from $-\infty$ to ∞
- This prevents Perfect recovery!!



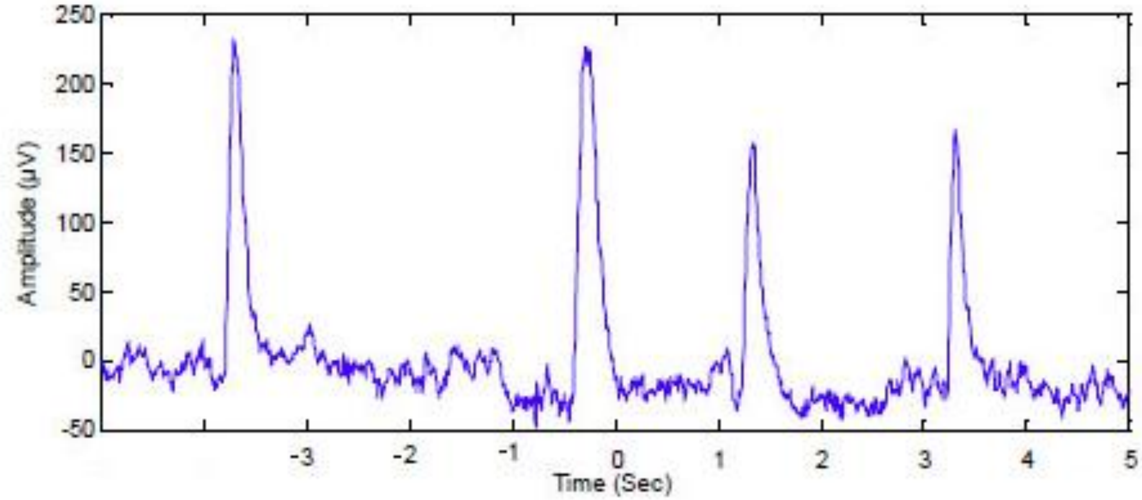
This effect, caused by under-sampling a function, is known as *frequency aliasing* or simply as *aliasing*. the term *alias*, which means "a false identity."

- Unfortunately, aliasing is present in almost all cases(except for some special cases)
- Band limited signal is time unlimited signal.
- e.g. sinc↔rect pair.
- The moment we limit our signal in time, *which we have to do in practice*, infinite frequency components are introduced.
- How?

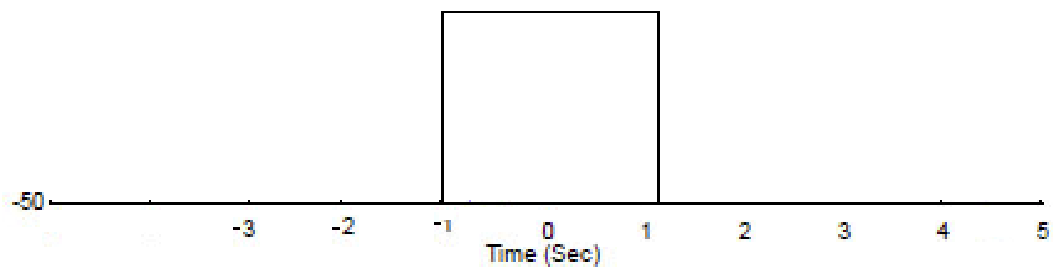
- Suppose we have to limit our f(t) in interval [0,T]
- It is equivalent to multiplication of f(t) with h(t) where

$$h(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

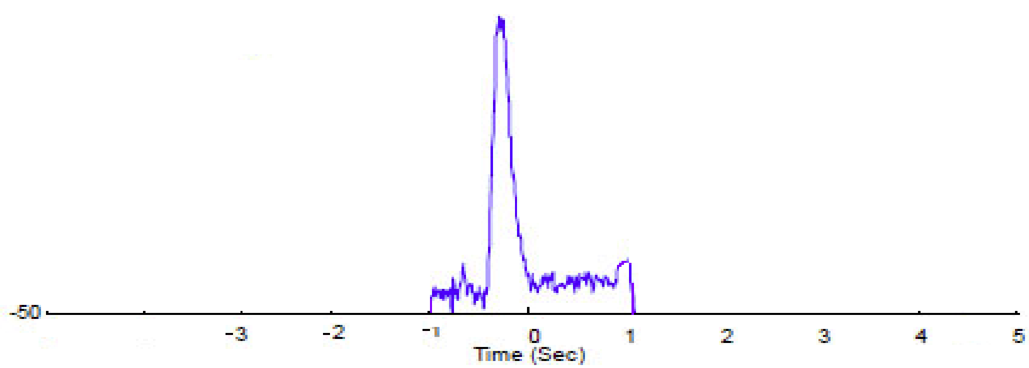
f(t)


Amplitude (µV)
250
200
150
100
50
0
-50
-3
-2
-1
0
1
2
3
4
5
Time (Sec)

h(t)


-50
-3
-2
-1
0
1
2
3
4
5
Time (Sec)

f(t)h(t)

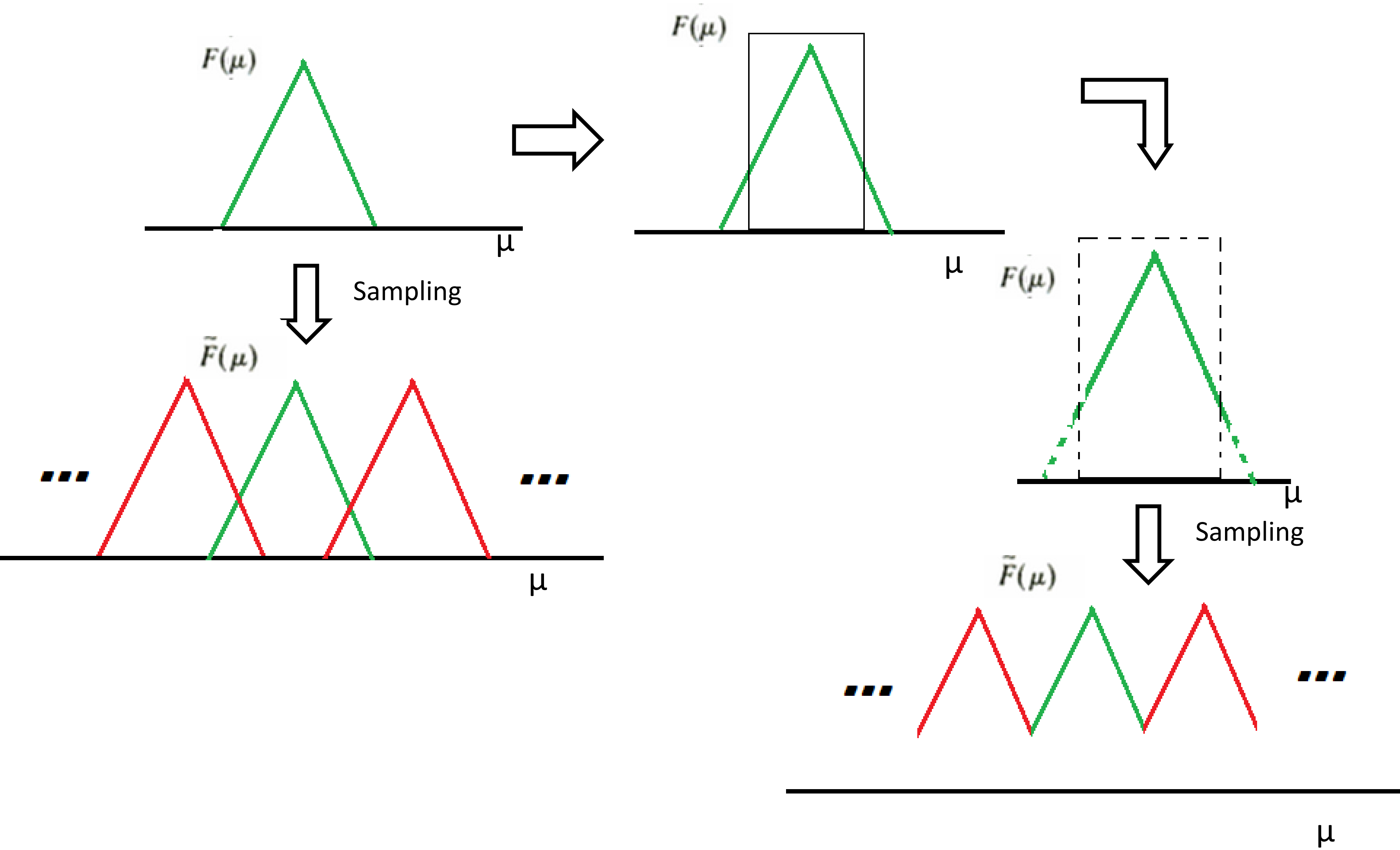

-50
-3
-2
-1
0
1
2
3
4
5
Time (Sec)

$$f(t)h(t) \Longleftrightarrow F(\mu) \bigstar H(\mu)$$

- In our case H(μ) is sinc function.
- Sliding sinc across any band limited F(μ) will yield components extending to infinity.
- So no function can be band limited.
- If function is band limited it must extend from -∞ to ∞ in time

Anti aliasing:

- Effect of aliasing can be *reduced* by smoothing input function and attenuating its higher frequencies *before* sampling- How?



$F(\mu)$
μ
Sampling
$\tilde{F}(\mu)$
μ
$F(\mu)$
μ
$F(\mu)$
μ
Sampling
$\tilde{F}(\mu)$
μ

Illustration of aliasing:

$$\frac{1}{\Delta T} \geq 2 * \max frequency$$

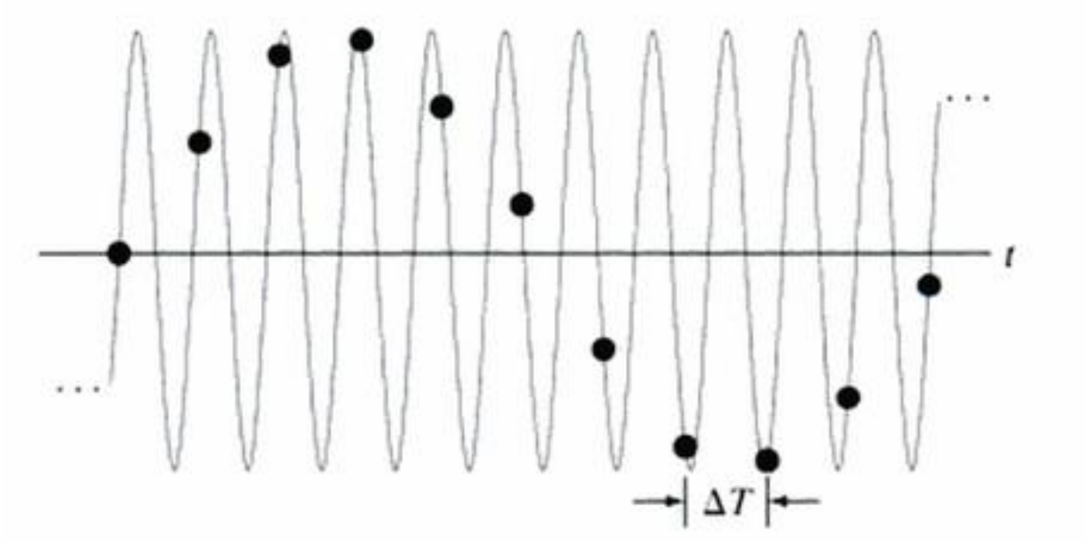
$$\frac{1}{\Delta T} \geq 2 * \frac{1}{Period}$$

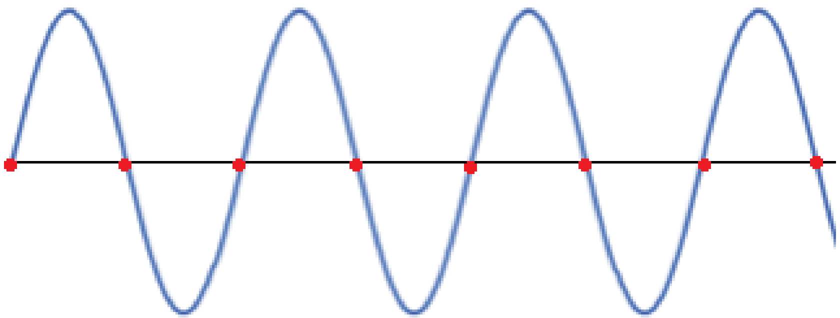
$$\Delta T \leq \frac{Period}{2}$$

$sometimes\ equal\ sign\ is\ not\ sufficient.$

$sampling\ rate\ should\ exceed\ twice\ the\ highest\ frequency$

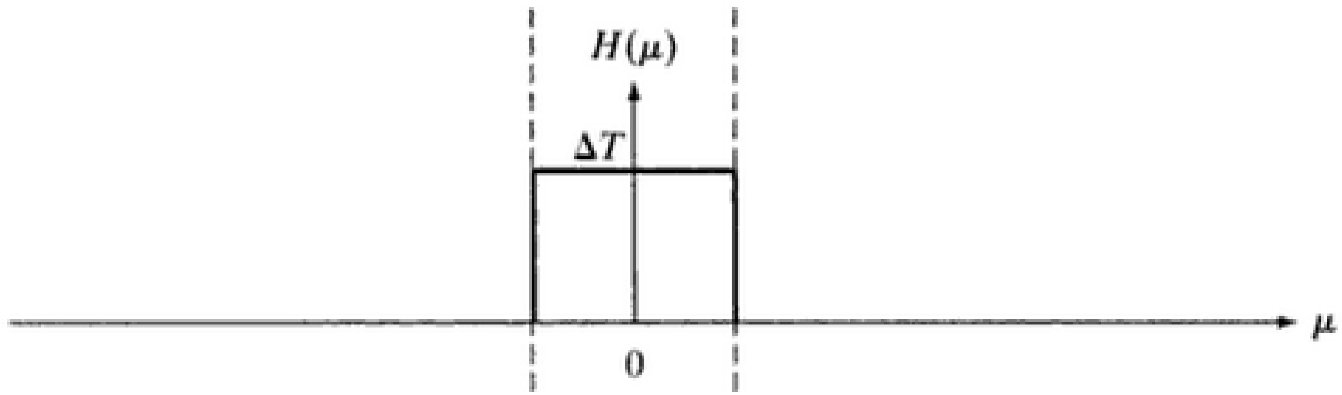
Illustration of aliasing:





Function Recovery from sampled Data

$$
\begin{aligned}
f(t) &= \Im^{-1}\{F(\mu)\} \\
&= \Im^{-1}\{H(\mu)\tilde{F}(\mu)\} \\
&= h(t) \star \tilde{f}(t)
\end{aligned}
$$

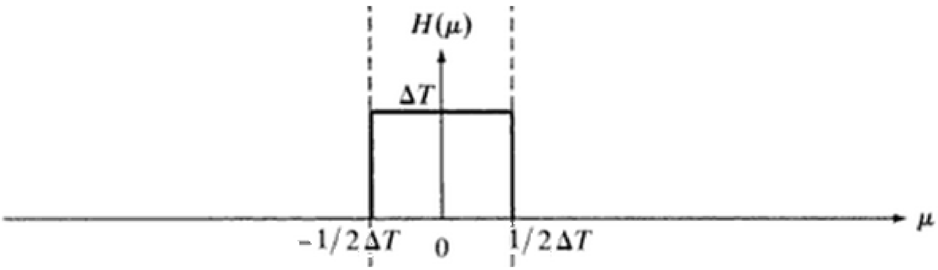


$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \leq \mu \leq \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$

Sampling at: $$\frac{1}{\Delta T} = 2\mu_{max}$$ *Nyquist rate*

$$f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{j2\pi\mu t} d\mu$$

$$h(t) = \int_{-\frac{1}{2\Delta T}}^{\frac{1}{2\Delta T}} \Delta T\, e^{j2\pi\mu t} d\mu$$



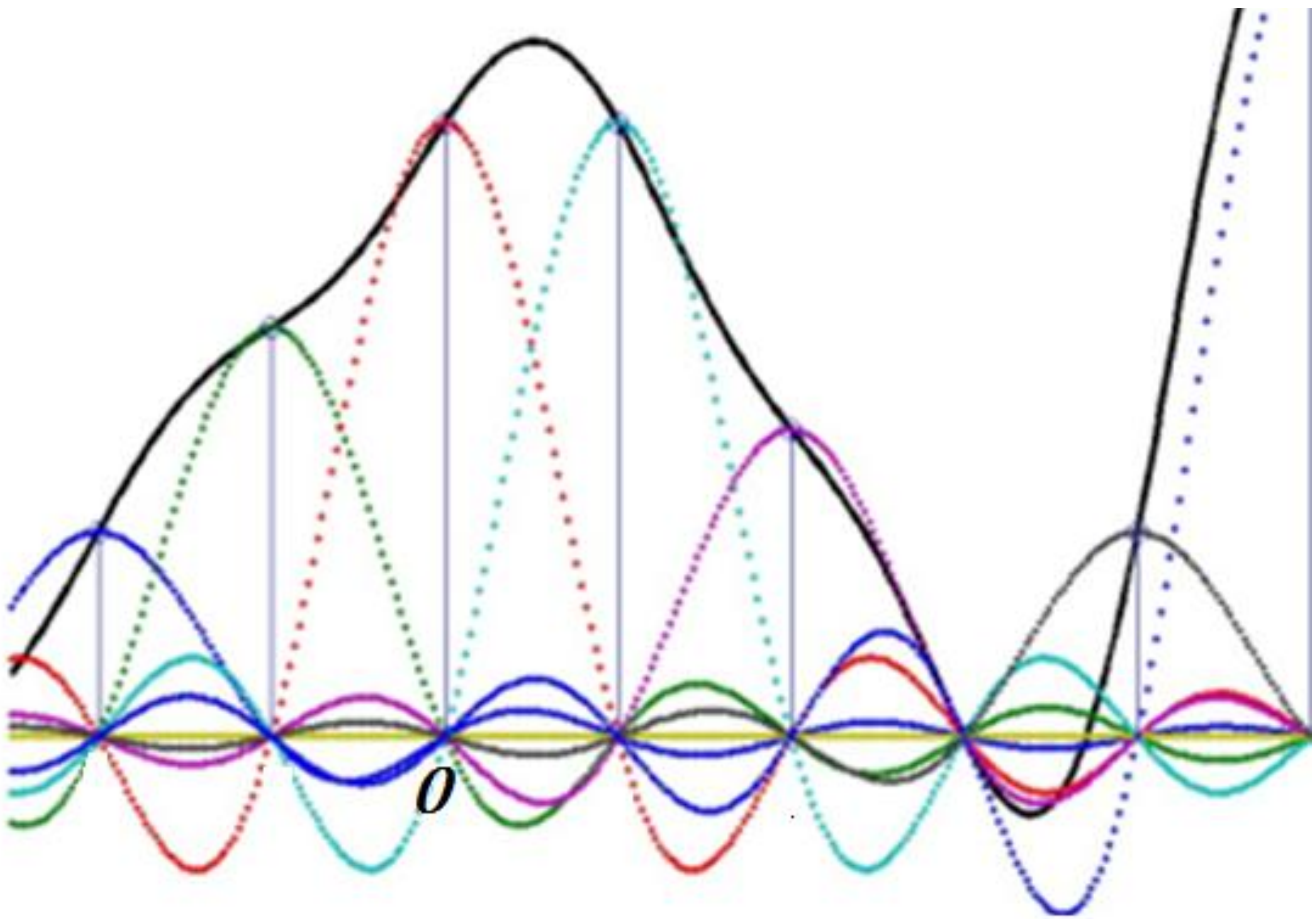
$$h(t) = \Delta T \left(\frac{e^{j2\pi\mu t}}{j2\pi t}\right)_{-\frac{1}{2\Delta T}}^{\frac{1}{2\Delta T}}$$

$$h(t) = \Delta T \left(\frac{e^{j2\pi\frac{1}{2\Delta T}t} - e^{-j2\pi\frac{1}{2\Delta T}t}}{j2\pi t}\right)$$

$$h(t) = \Delta T \left(\frac{1}{\pi t}\right) \sin\left(\frac{\pi}{\Delta T} t\right)$$

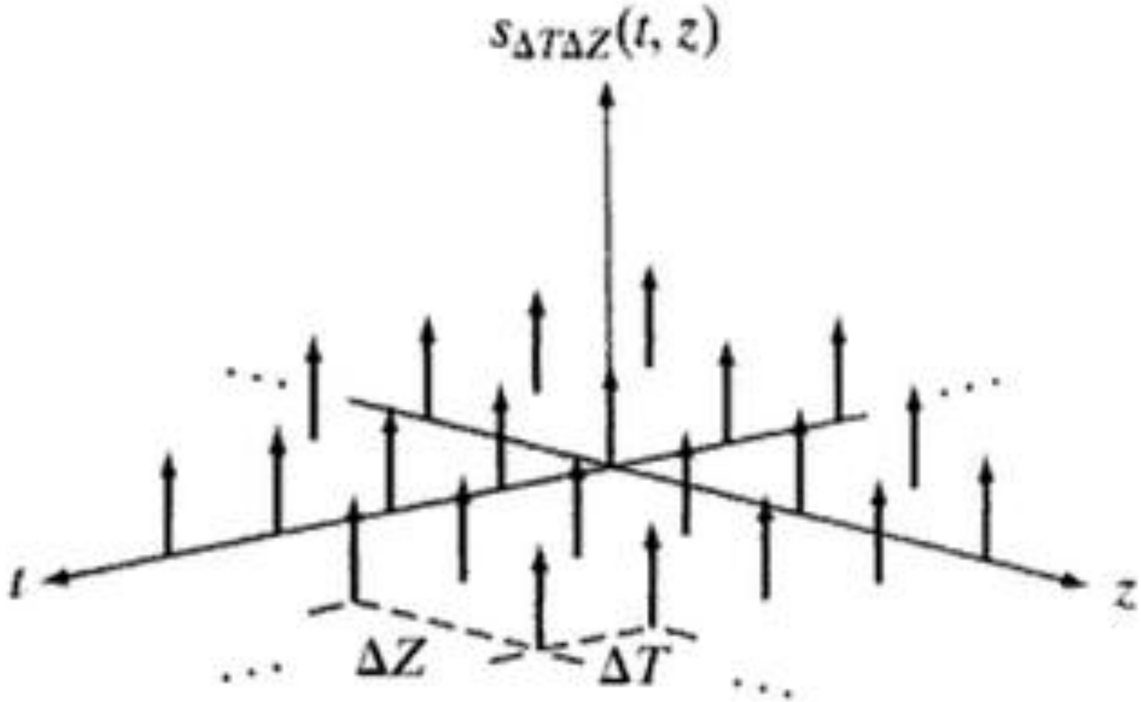
$$h(t) = \mathrm{sinc}\,(\frac{\pi}{\Delta T} t)$$

$$
\begin{aligned}
f(t) &= h(t) \star \tilde{f}(t) \\
&= \int_{-\infty}^{\infty} h(\tau)\tilde{f}(t-\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} \frac{\sin(\pi\tau/\Delta T)}{(\pi\tau/\Delta T)} \sum_{n=-\infty}^{\infty} f(t-\tau)\,\delta(t-n\Delta T-\tau)\,d\tau \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin(\pi\tau/\Delta T)}{(\pi\tau/\Delta T)} f(t-\tau)\,\delta(t-n\Delta T-\tau)\,d\tau \\
&= \sum_{n=-\infty}^{\infty} f(n\Delta T) \frac{\sin[\pi(t-n\Delta T)/\Delta T]}{[\pi(t-n\Delta T)/\Delta T]} \\
&= \sum_{n=-\infty}^{\infty} f(n\Delta T)\,\mathrm{sinc}\,[(t-n\Delta T)/\Delta T].
\end{aligned}
$$


0

2-D impulse train:

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$



2-D sampling and sampling theorem

Function $f(t, z)$ is *band-limited*

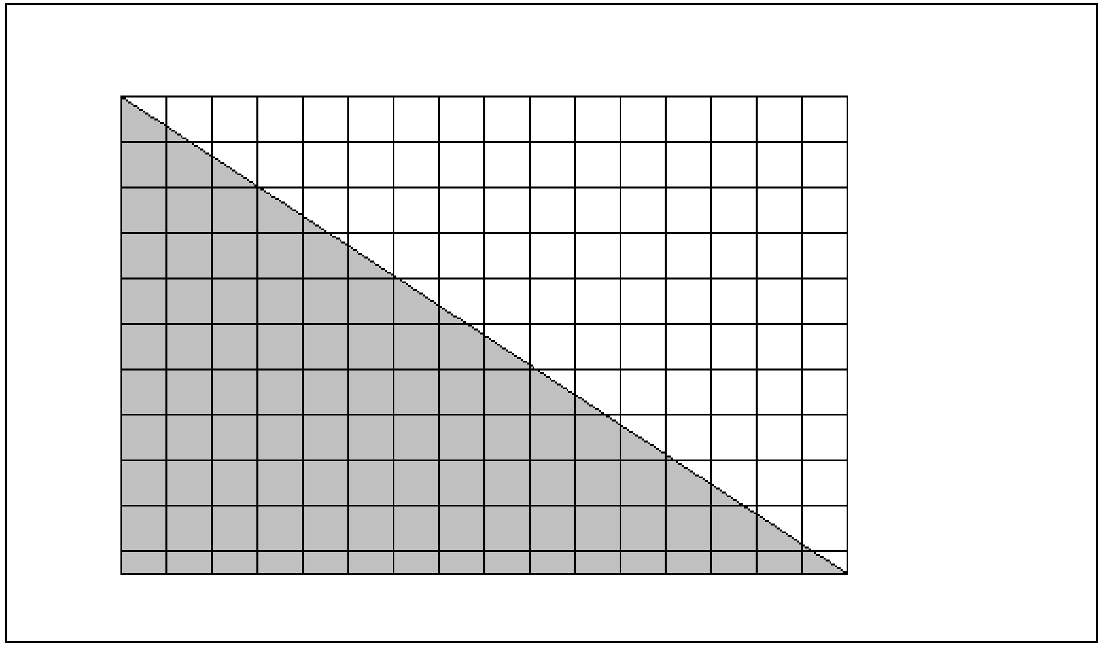
$$F(\mu, \nu) = 0 \quad \text{for } |\mu| \geq \mu_{\max} \text{ and } |\nu| \geq \nu_{\max}$$

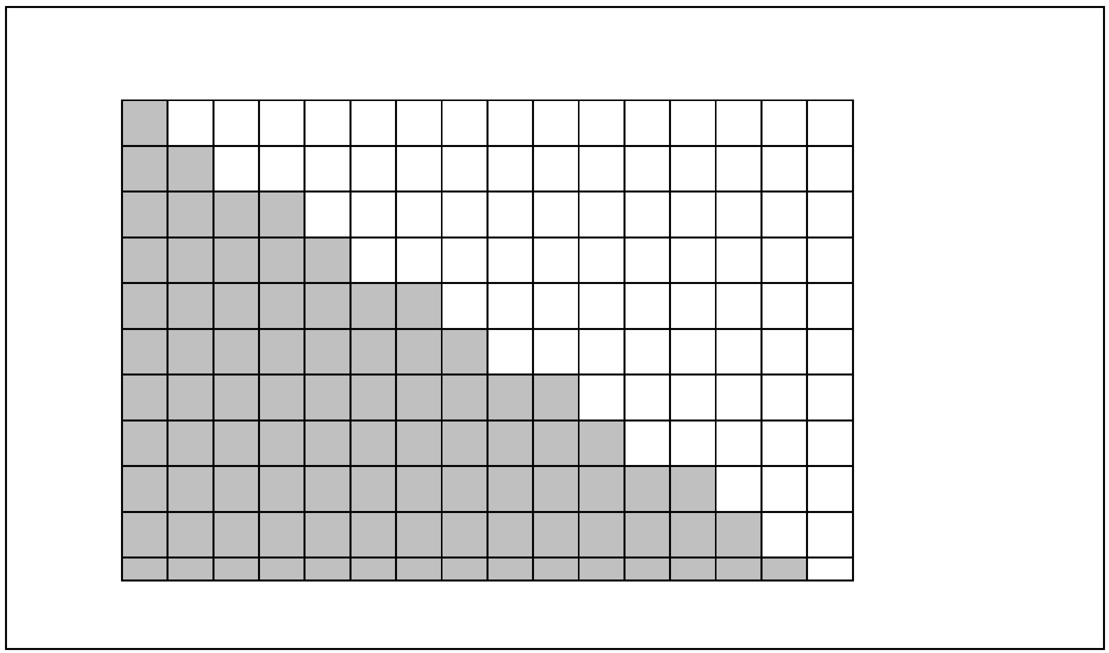
The *two-dimensional sampling theorem* states that a continuous, band-limited function $f(t, z)$ can be recovered with no error from a set of its samples if the sampling intervals are

$$\Delta T < \frac{1}{2\mu_{\max}} \qquad \frac{1}{\Delta T} > 2\mu_{\max}$$

$$\Delta Z < \frac{1}{2\nu_{\max}} \qquad \frac{1}{\Delta Z} > 2\nu_{\max}$$

Sampling:





Over sampling & under sampling in frequency domain:

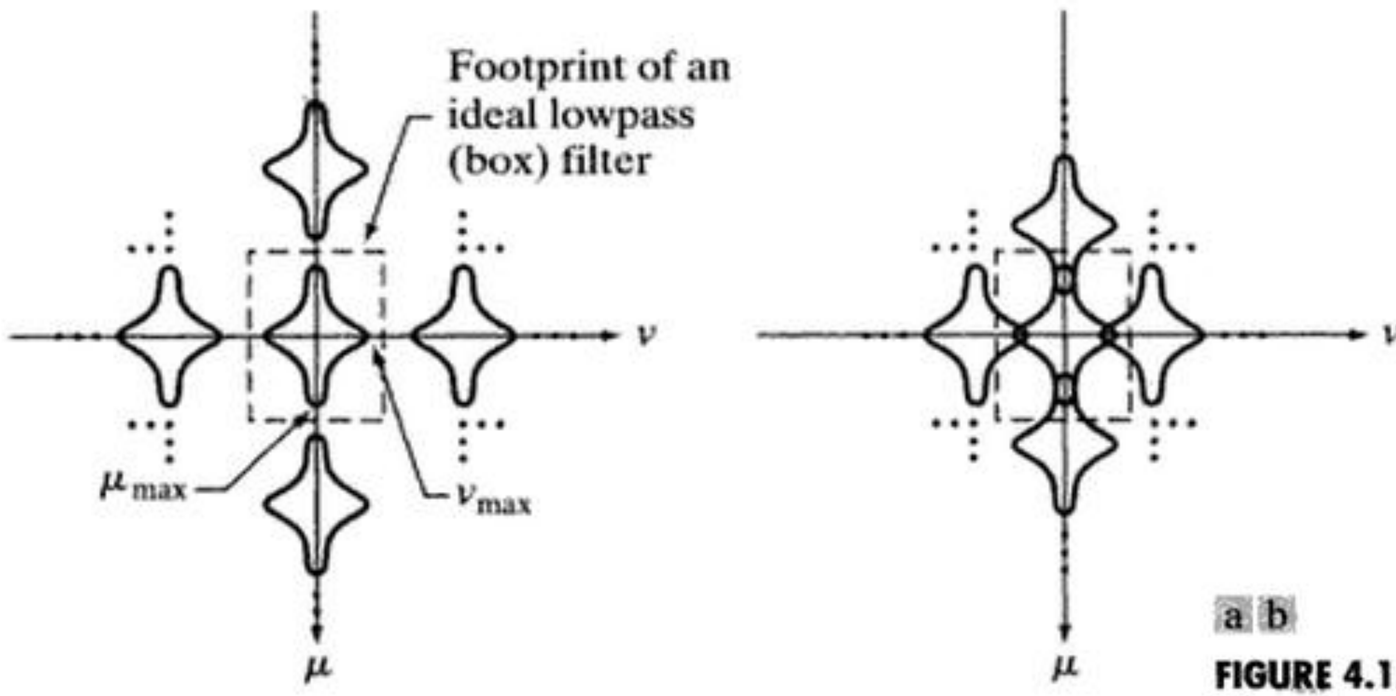


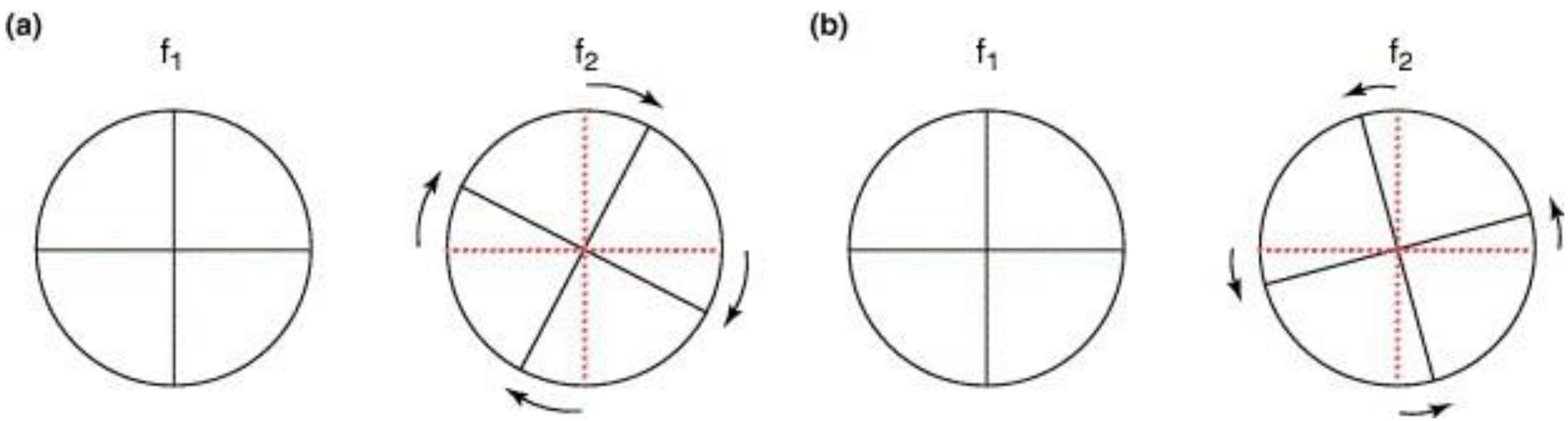
FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

Aliasing:

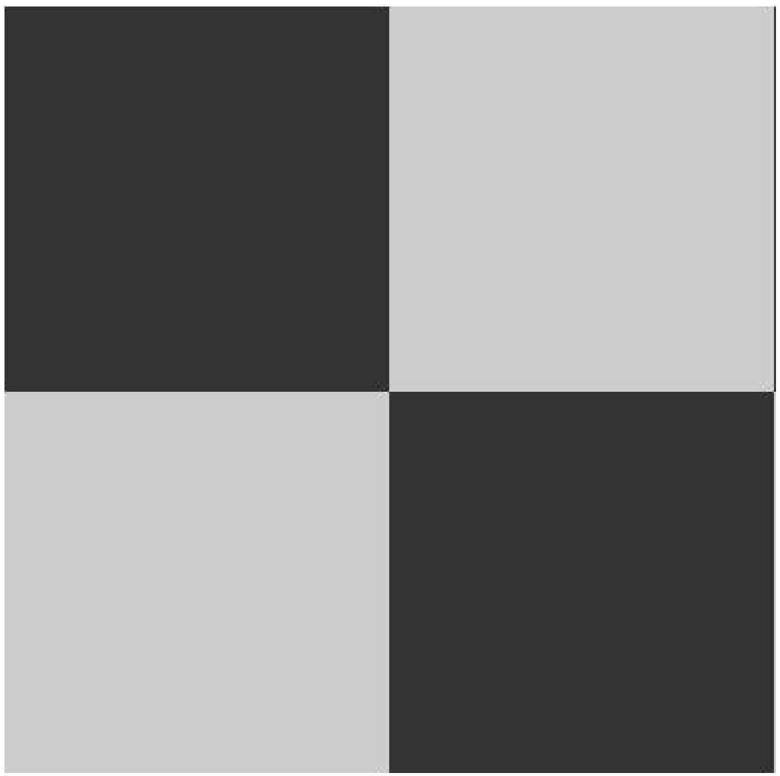
- $f(t,z)$ is band limited only if it extends infinitely in both coordinate directions.
- Limiting in time makes it band unlimited.
- So aliasing is always present in image.

Aliasing in image:

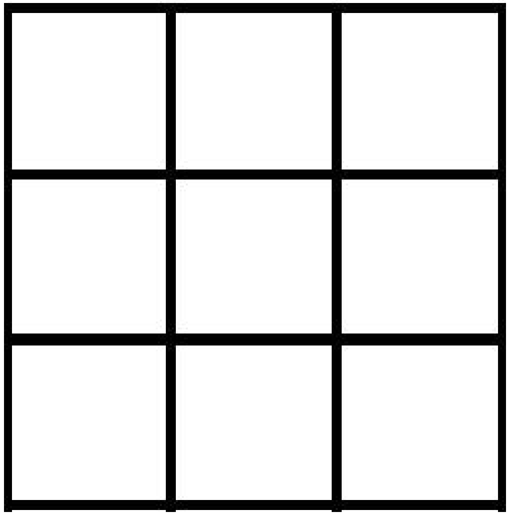
- Two important manifestations of aliasing in images:
  - Spatial aliasing: occurs due to under sampling
  - Temporal aliasing: “ wagon wheel”



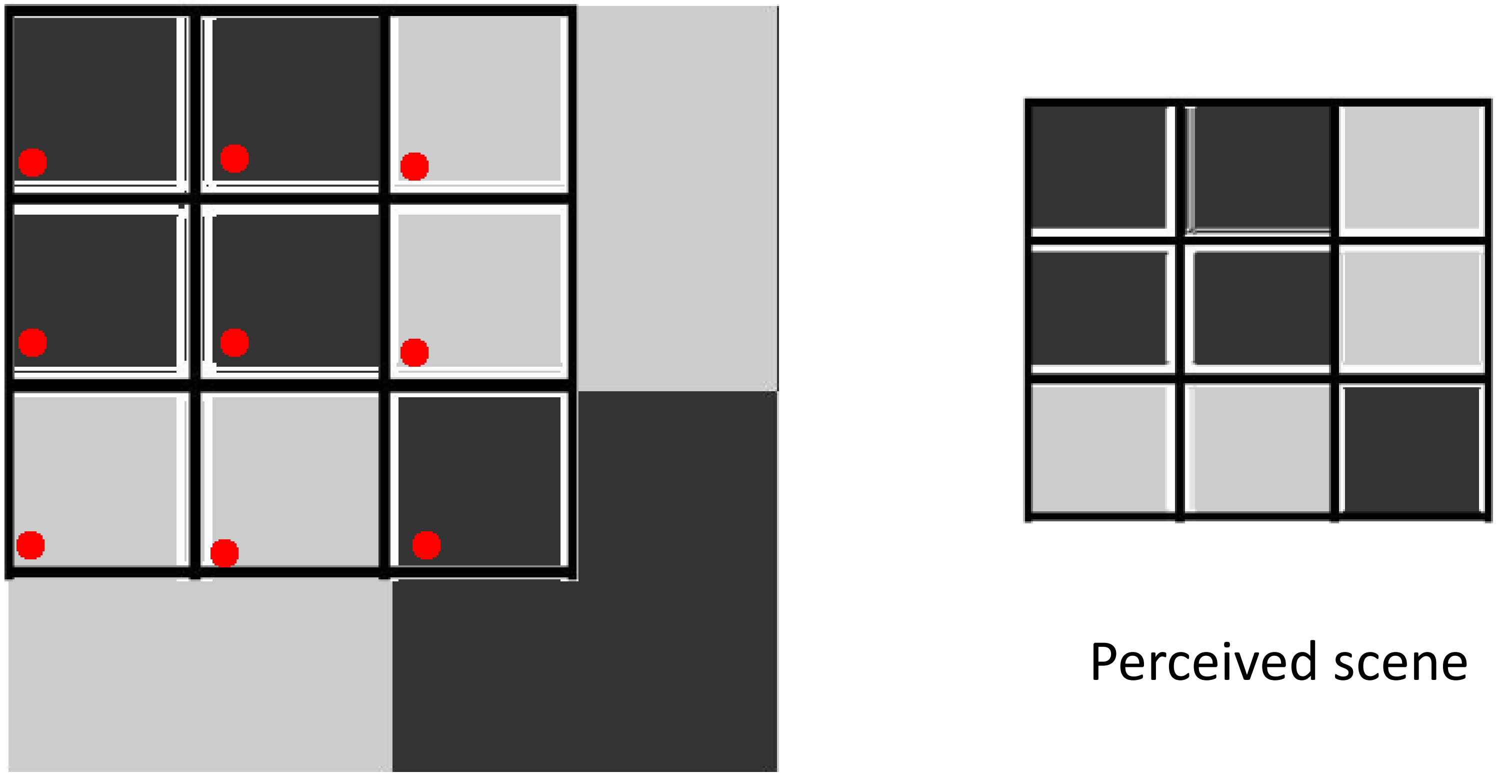
Case : 1



Original scene

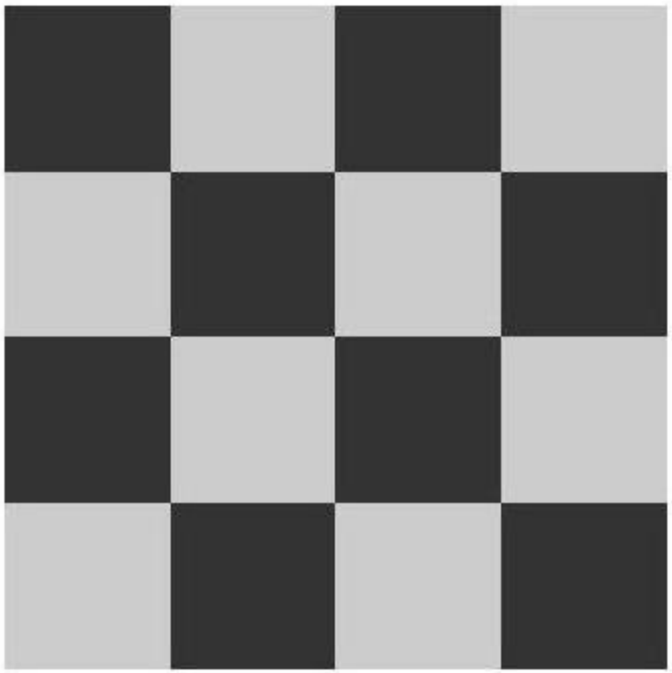


Pixel grid

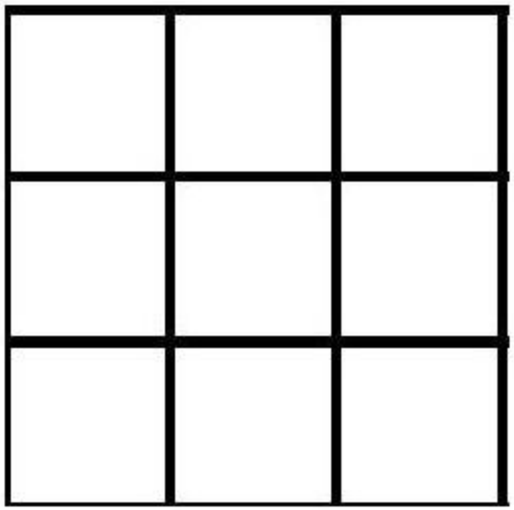


Perceived scene

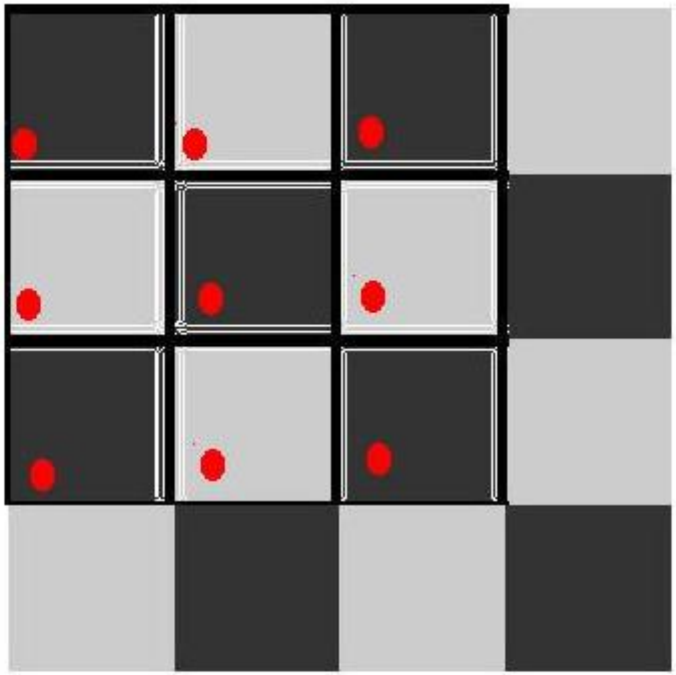
Case: 2

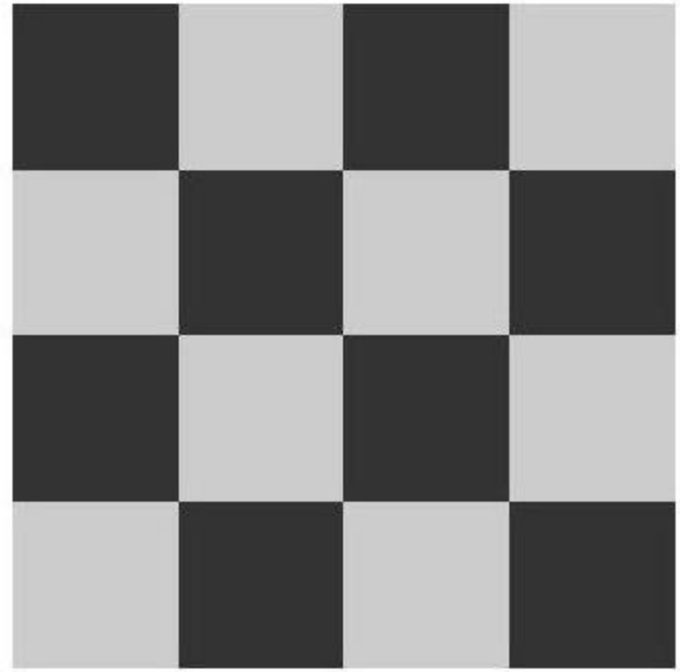


Original scene



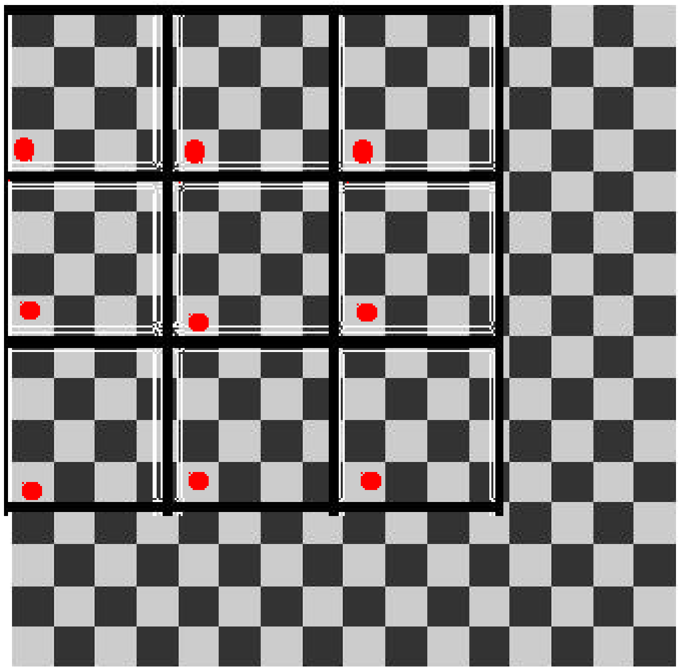
Pixel grid



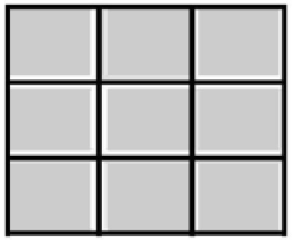


Perceived scene

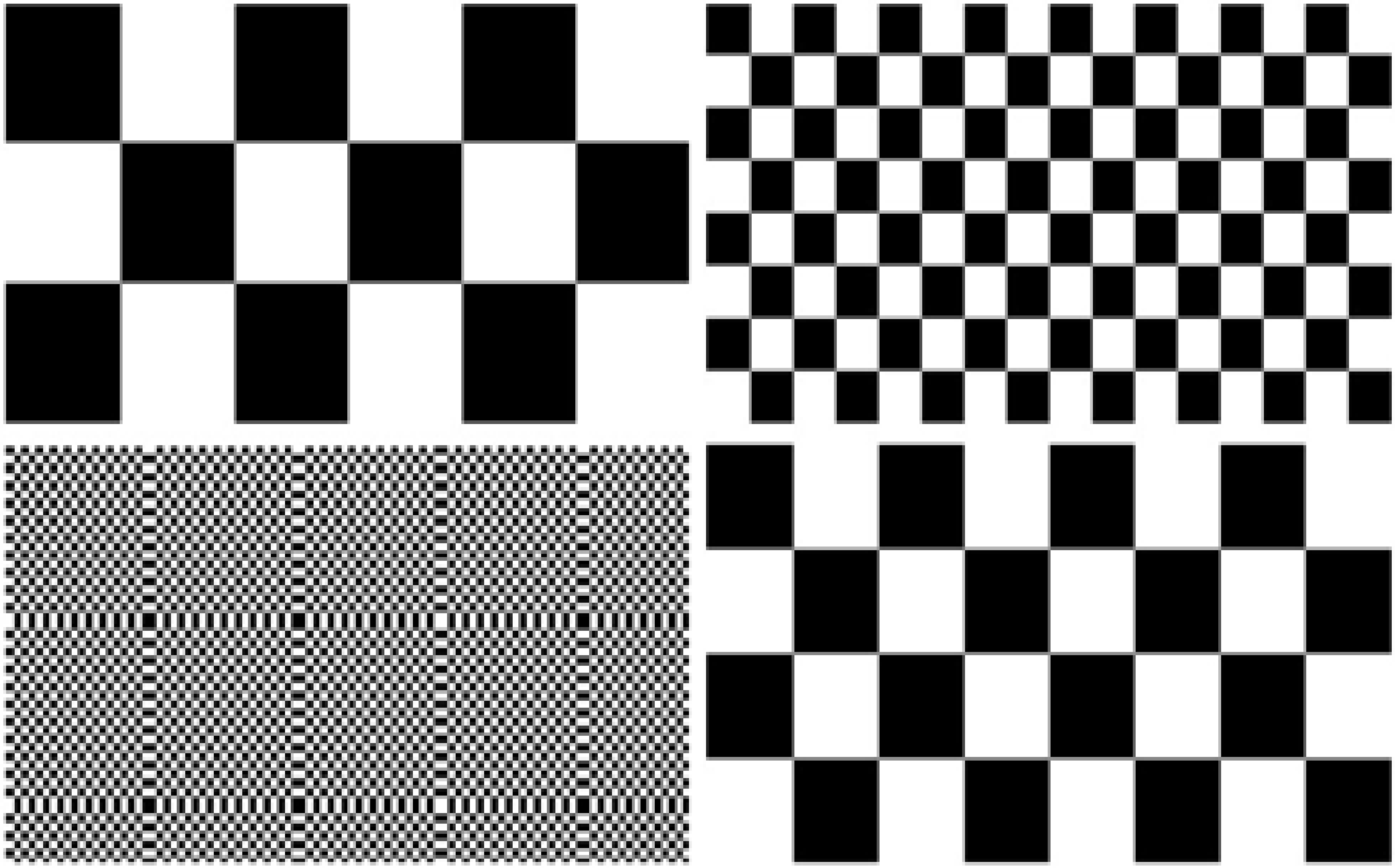
Case : 3



Original scene



Perceived scene



Anti-aliasing in images:

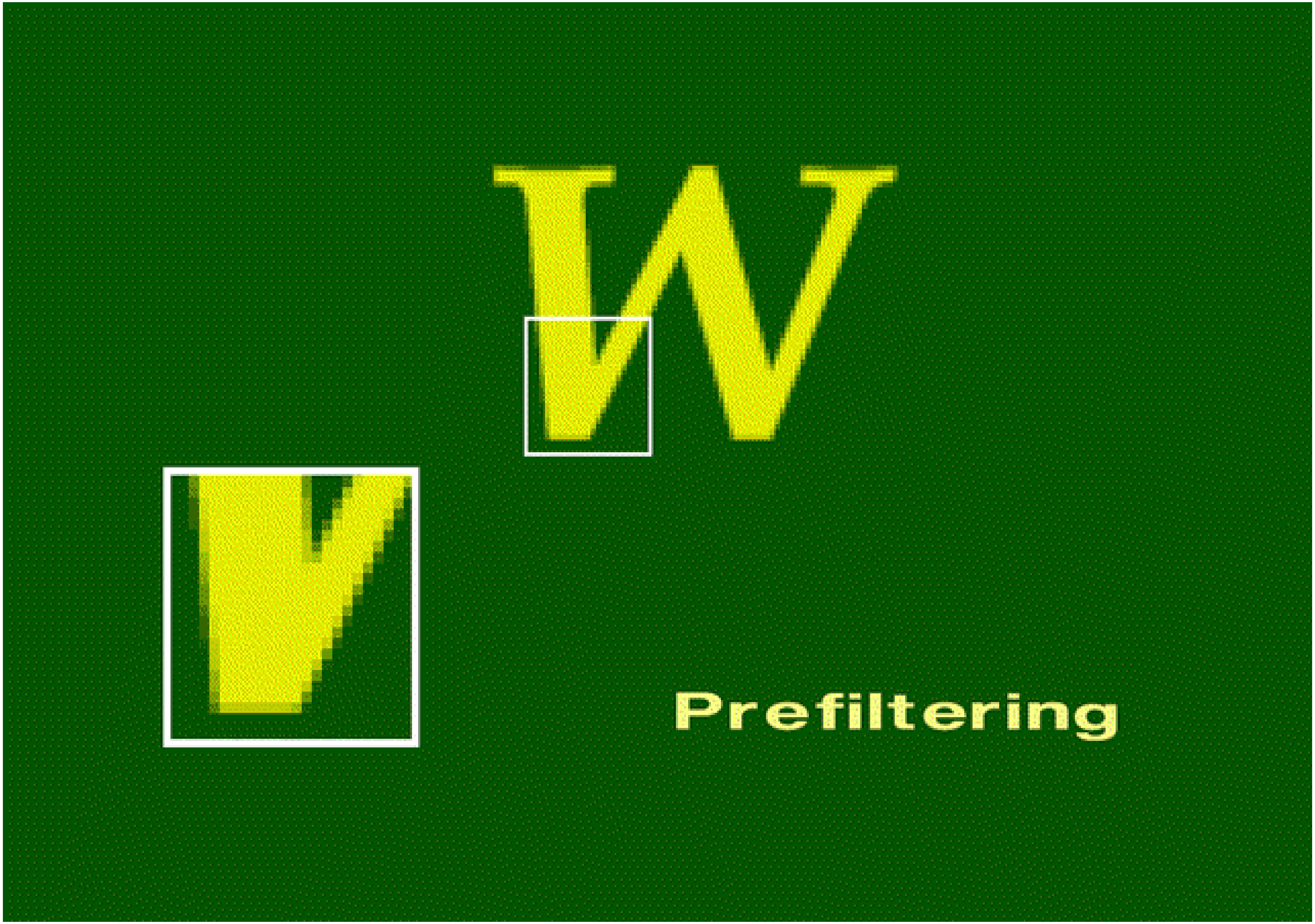
- Slightly defocusing the scene.
- It is to be done at the front end- before the image is sampled.
- Digital cameras have anti-aliasing filtering built in, either in the lens or on the surface of the sensor it self.
- Difficult to illustrate aliasing using images obtained with such cameras.
- For re-sampling- reduce artifacts (aliasing)- blurring

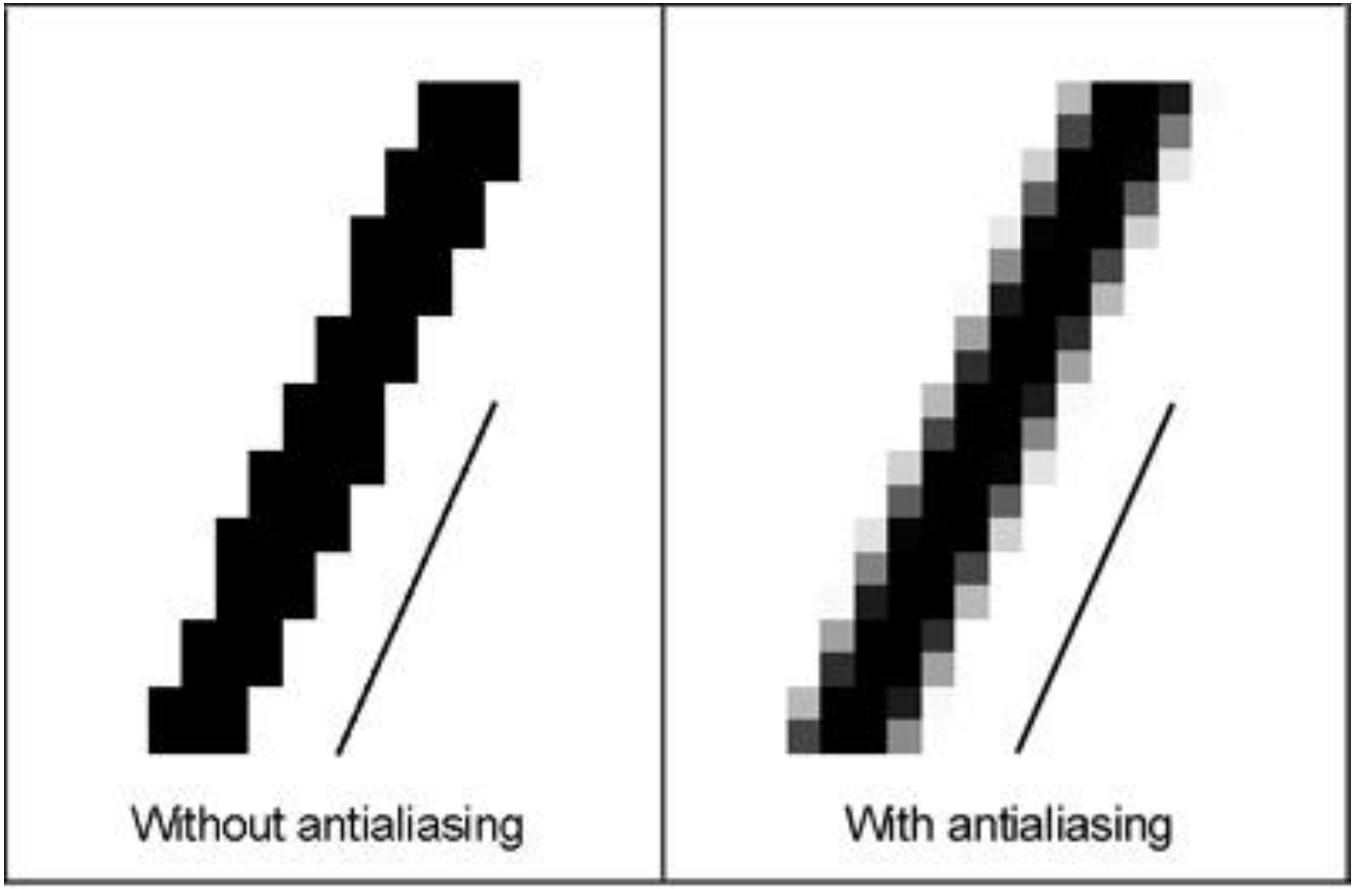
Interpolation and re-sampling:

- Zooming-over sampling and shrinking-under sampling
- Nearest neighbor, bilinear, bicubic
- Zoom by factor -2 using pixel replication-nearest neighbor
- Image shrinking- row and column deletion
- Grid analogy for non integer factor
- Blurring before resampling


original image
under sampled
LPF
undersampling after filtering

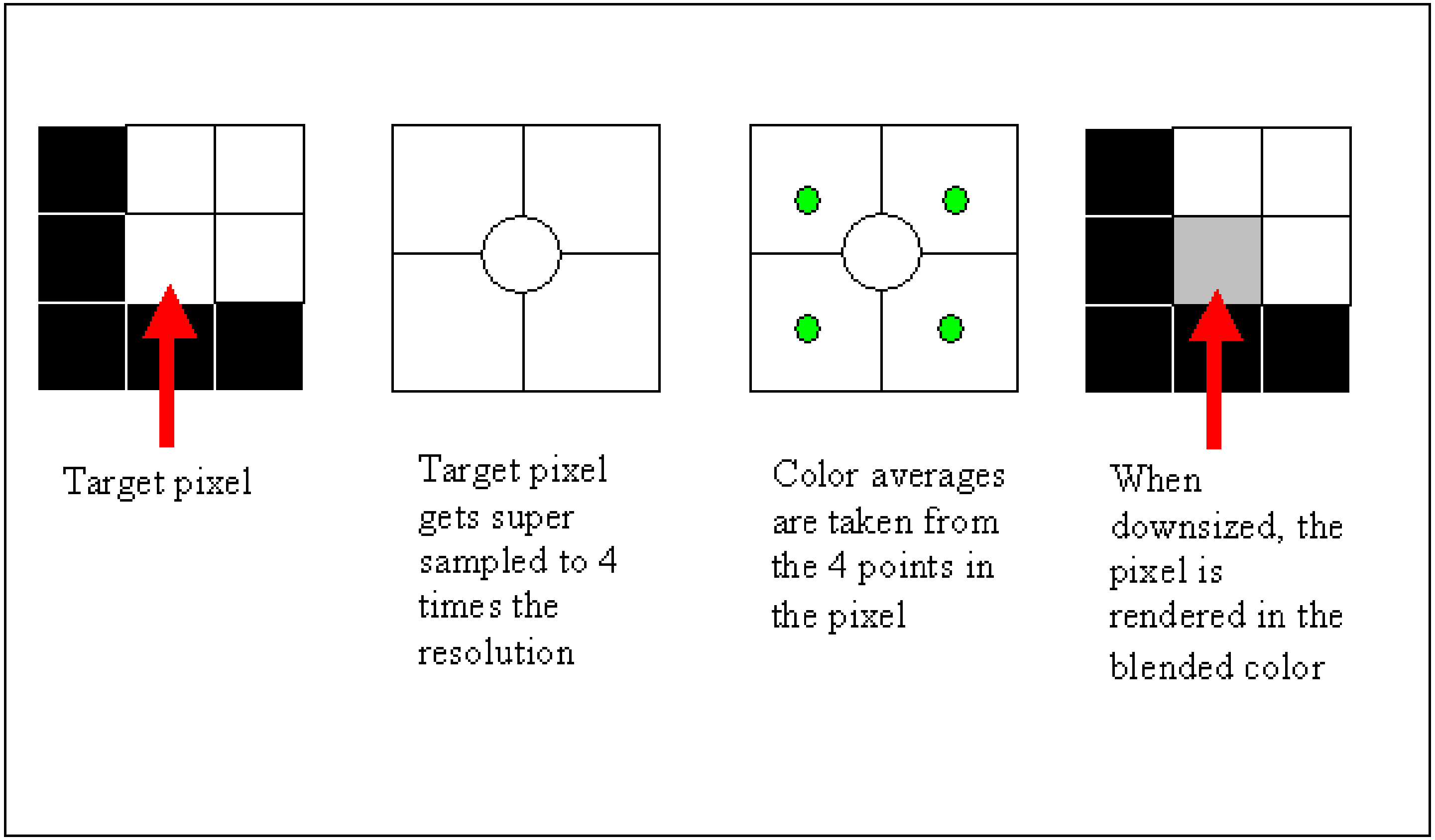
No antialiasing

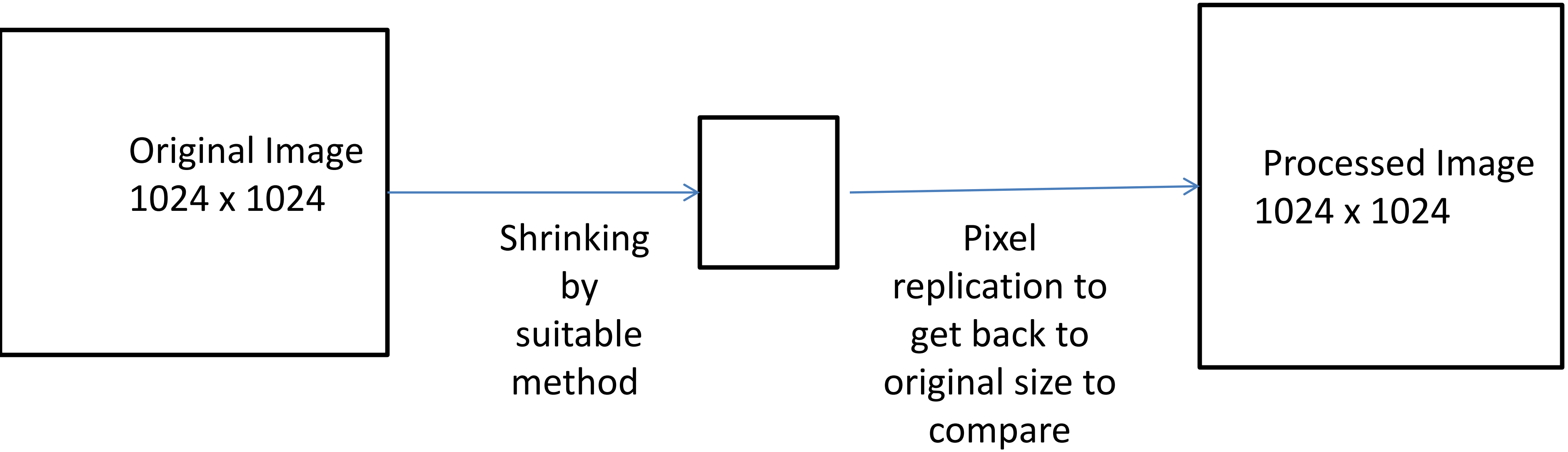

Prefiltering


Without antialiasing
With antialiasing

Super sample:

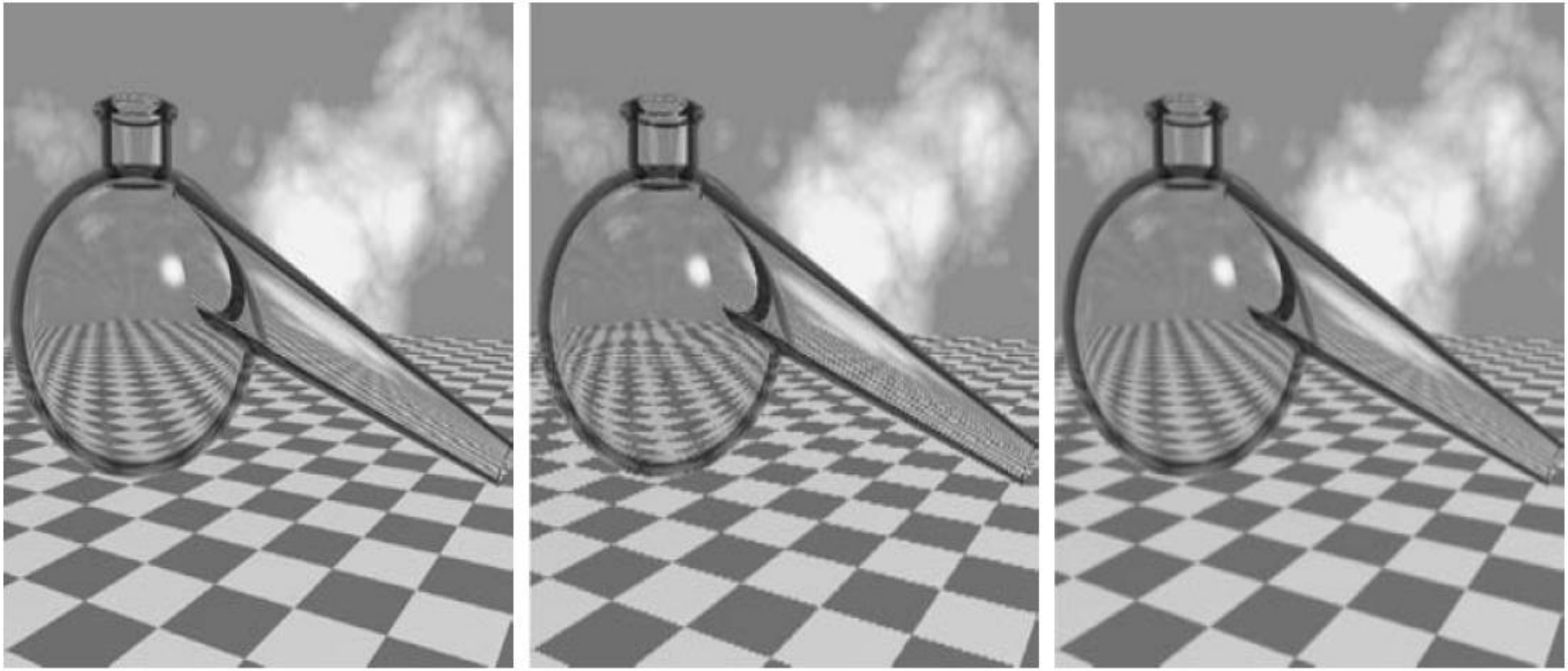
- Alternate method to reduce aliasing




Original Image
1024 x 1024
Shrinking by suitable method
Pixel replication to get back to original size to compare
Processed Image
1024 x 1024

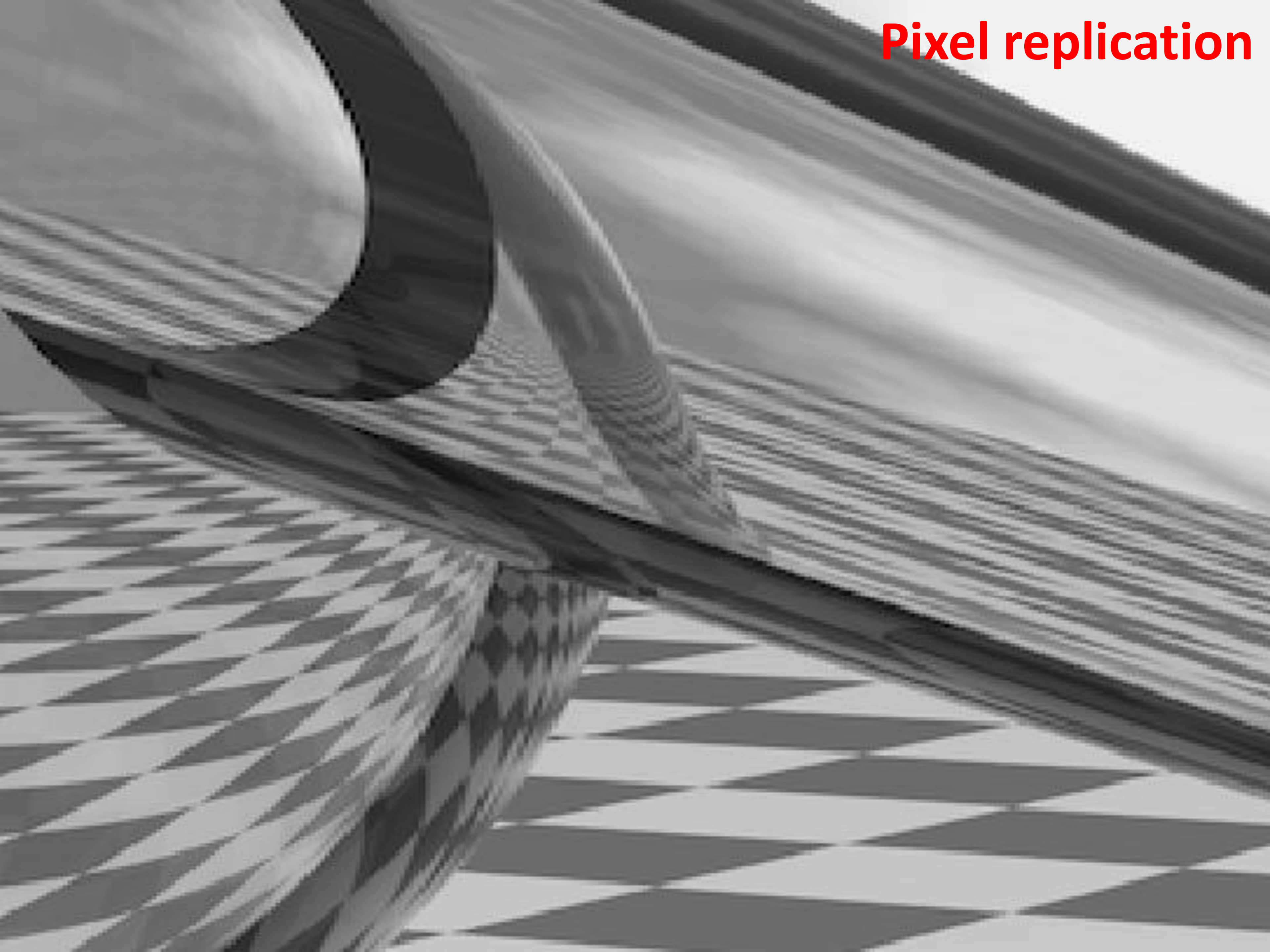
a b c

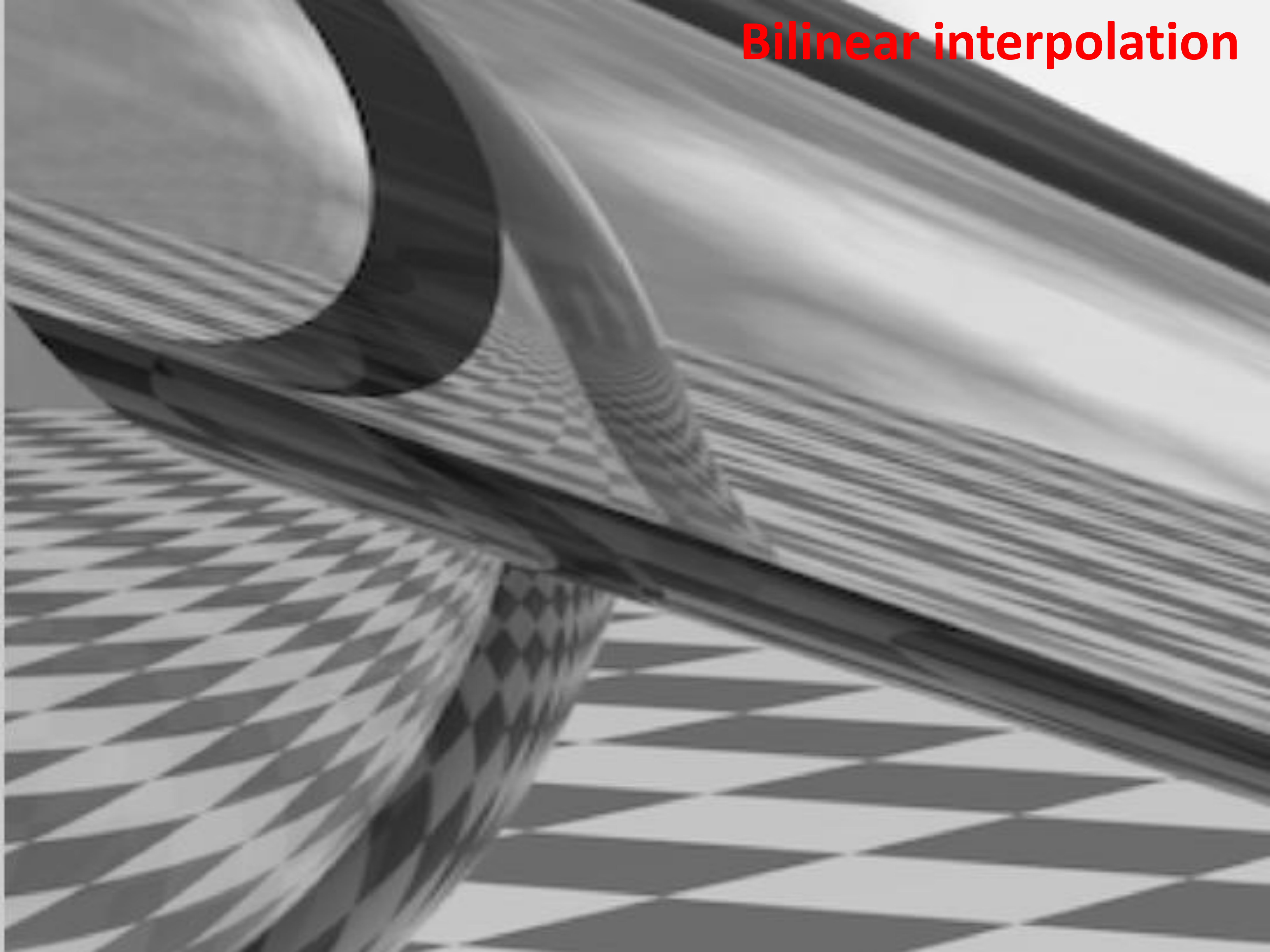
FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)



a b c

FIGURE 4.18 Illustration of jaggies. (a) A 1024×1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5×5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)


Pixel replication

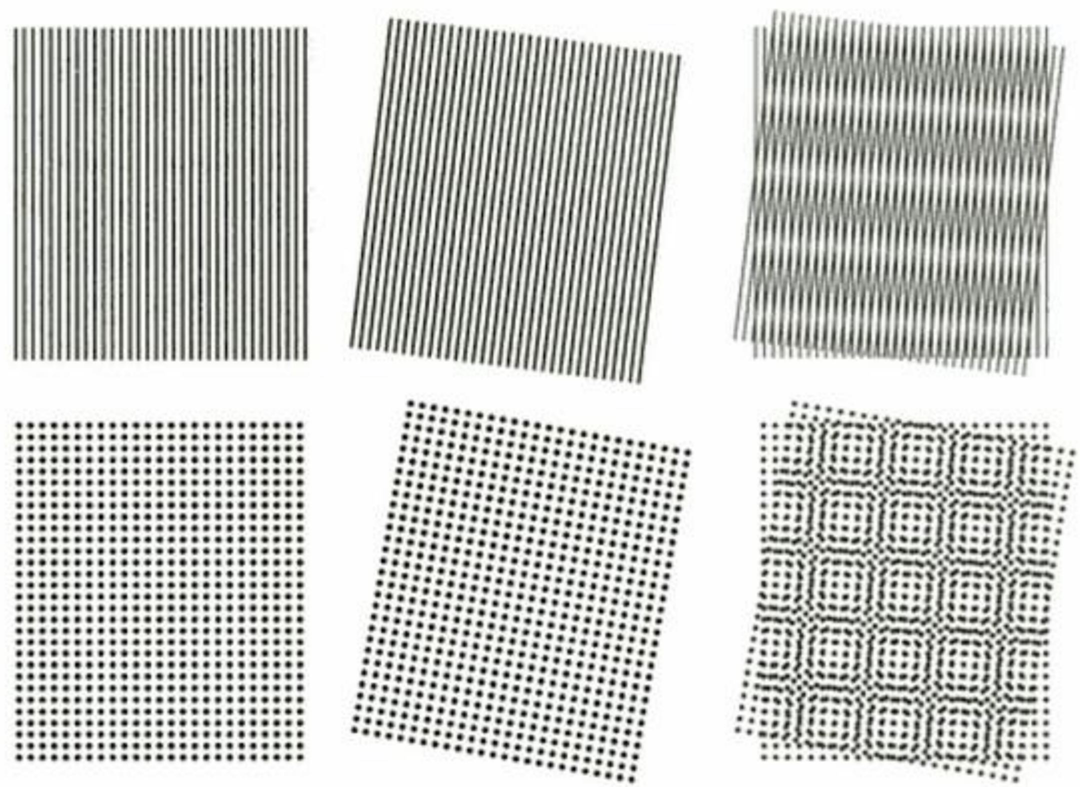

Bilinear interpolation

Moirè Patterns:

- Beat patterns produce between gratings of approximately equal spacing.

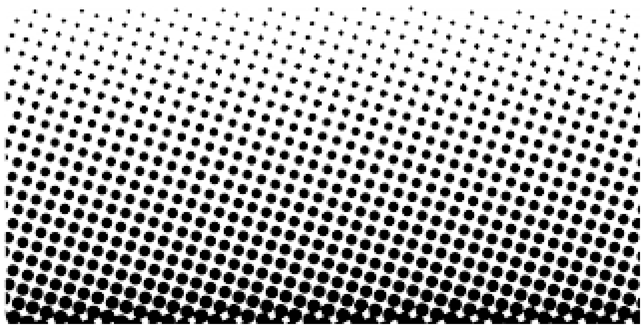
a b c
d e f

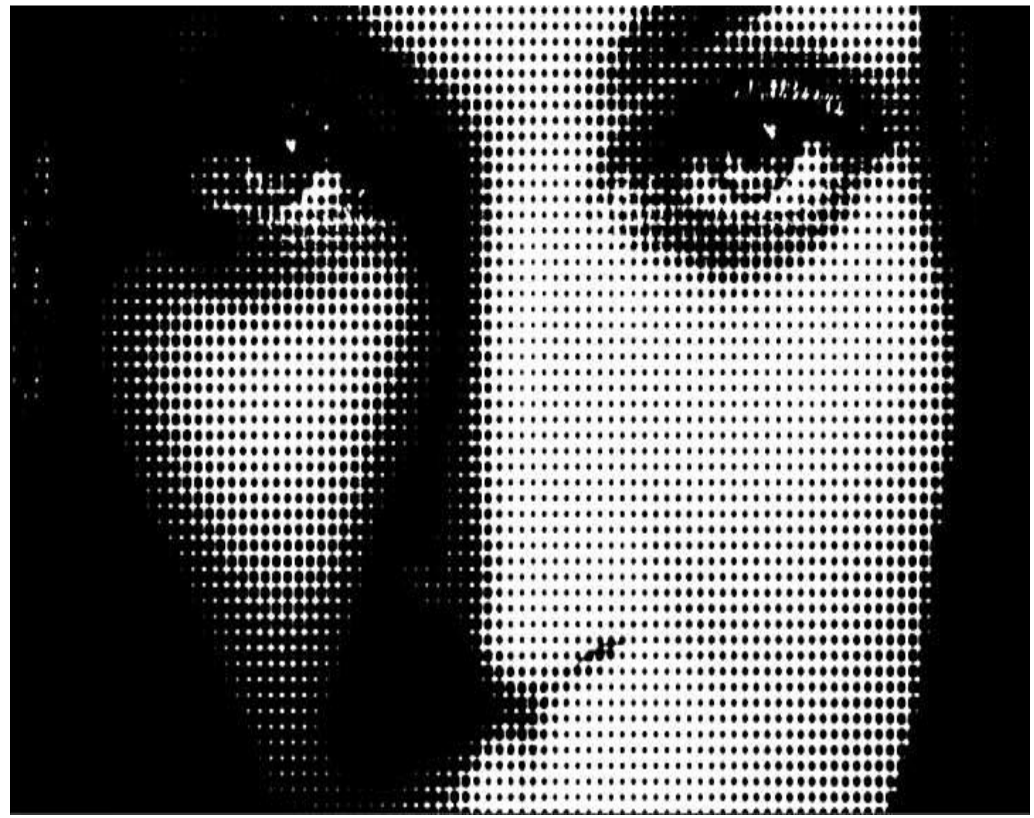
FIGURE 4.20 Examples of the moiré effect. These are ink drawings, not digitized patterns. Superimposing one pattern on the other is equivalent mathematically to multiplying the patterns.



- Moirè Patterns are generated from sampling scenes with periodic or nearly periodic components whose spacing is comparable to spacing between samples.

Half tone images:





- Print media like newspapers and magazines uses such images.

FIGURE 4.21
A newspaper image of size 246×168 pixels sampled at 75 dpi showing a moiré pattern. The moiré pattern in this image is the interference pattern created between the $\pm 45°$ orientation of the halftone dots and the north–south orientation of the sampling grid used to digitize the image.

Fig shows a newspaper image sampled at 400 dpi to avoid moiré effects.

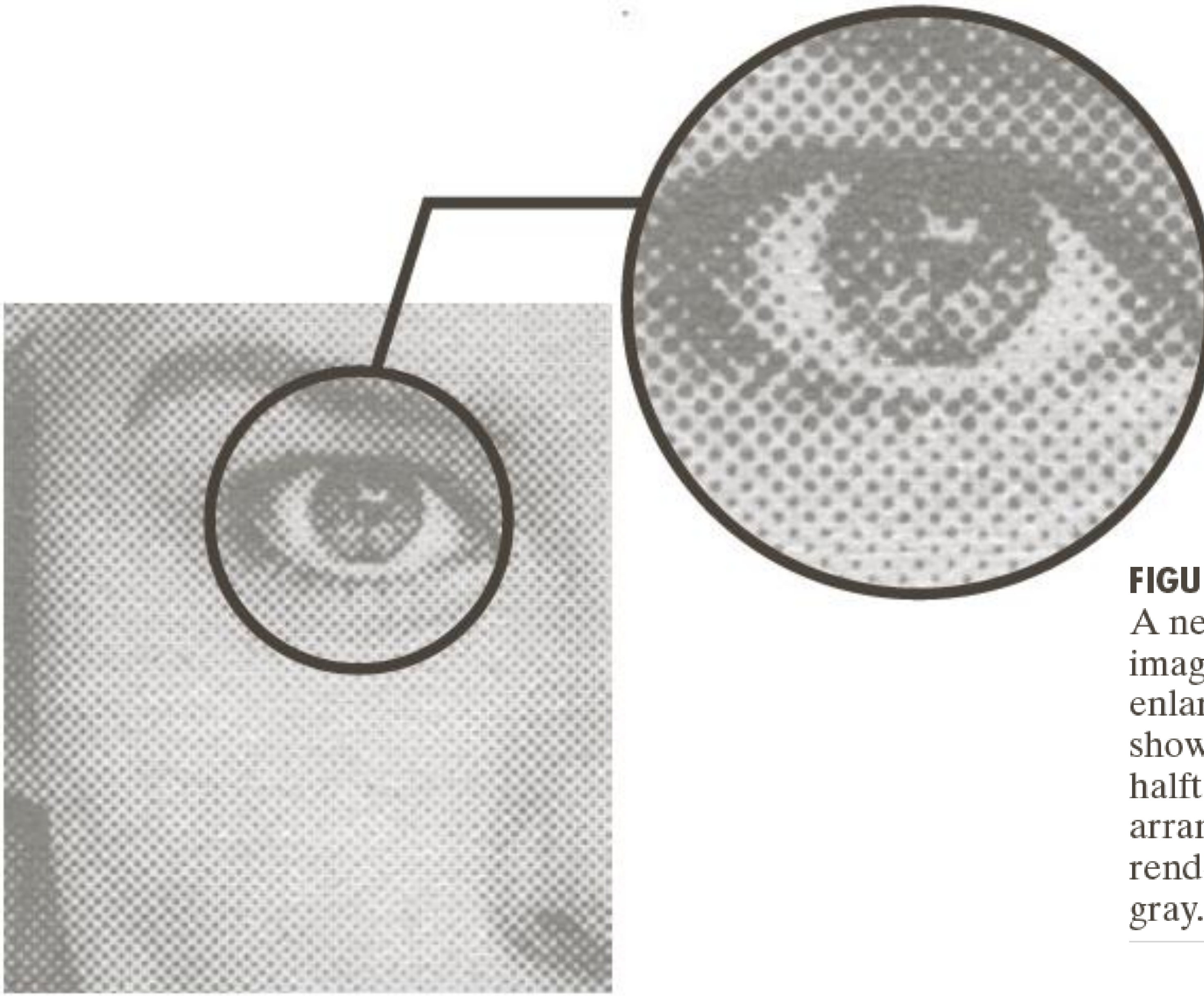


FIGURE 4.22
A newspaper image and an enlargement showing how halftone dots are arranged to render shades of gray.

DFT:

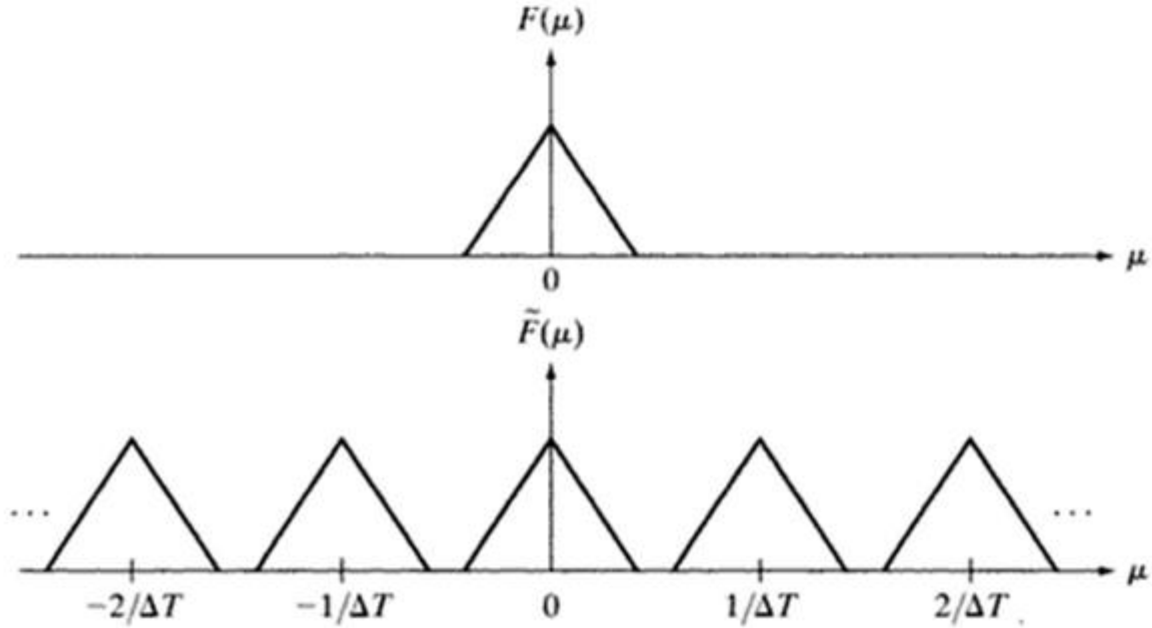
$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-j2\pi\mu t}\, dt$$

$$\tilde{f}(t) = f(t)s_{\Delta T} = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$\begin{aligned}
\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-j2\pi\mu t}\, dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\,\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt \\
&= \sum_{n=-\infty}^{\infty} f_n\, e^{-j2\pi\mu n\Delta T}
\end{aligned}$$

$$f_n = f(n\Delta T)$$

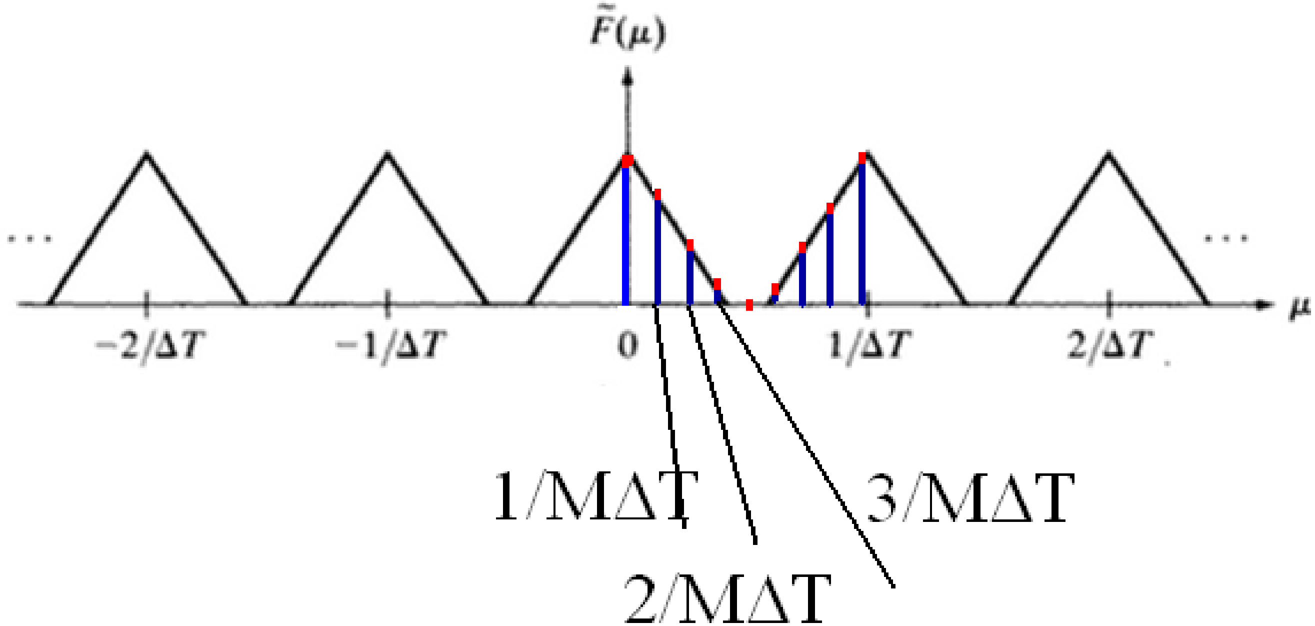
- Fourier transform is periodic function with period $1/\Delta T$



- If Fourier transform is sampled to M equally spaced samples in period $1/\Delta T$.
- What is the separation between samples?

Samples	**Space**
M	**$1/\Delta T$**
1	**(?)**

space between two samples $=1/M\Delta T$



- Samples should be taken at frequencies

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \dots, M-1$$

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \dots, M-1$$

- DFT:

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

- IDFT:

$$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{\,j2\pi mn/M} \qquad n = 0, 1, 2, \ldots, M-1$$

- With different notations:

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1$$

- F(u+M)? And f(x+M)?

$$F(u) = \sum_{x=0}^{M-1} f(x)\, e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

$$F(u+M) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi(u+M)x/M}$$

$$= \sum_{x=0}^{M-1} f(x) e^{-j2\pi Mx/M} e^{-j2\pi ux/M}$$

$$= \sum_{x=0}^{M-1} f(x)\, e^{-j2\pi ux/M}$$

$$= F(u)$$

$$F(u) = F(u + kM)$$

$$f(x) = f(x + kM)$$

The discrete equivalent of the convolution

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m)$$

circular convolution,

for $x = 0, 1, 2, \ldots, M-1$. Because in the preceding formulations the functions are periodic, their convolution also is periodic.

Relationship between frequency and space intervals:

If $f(x)$ consists of M samples of a function $f(t)$ taken ΔT units apart,

$$T = M\Delta T$$

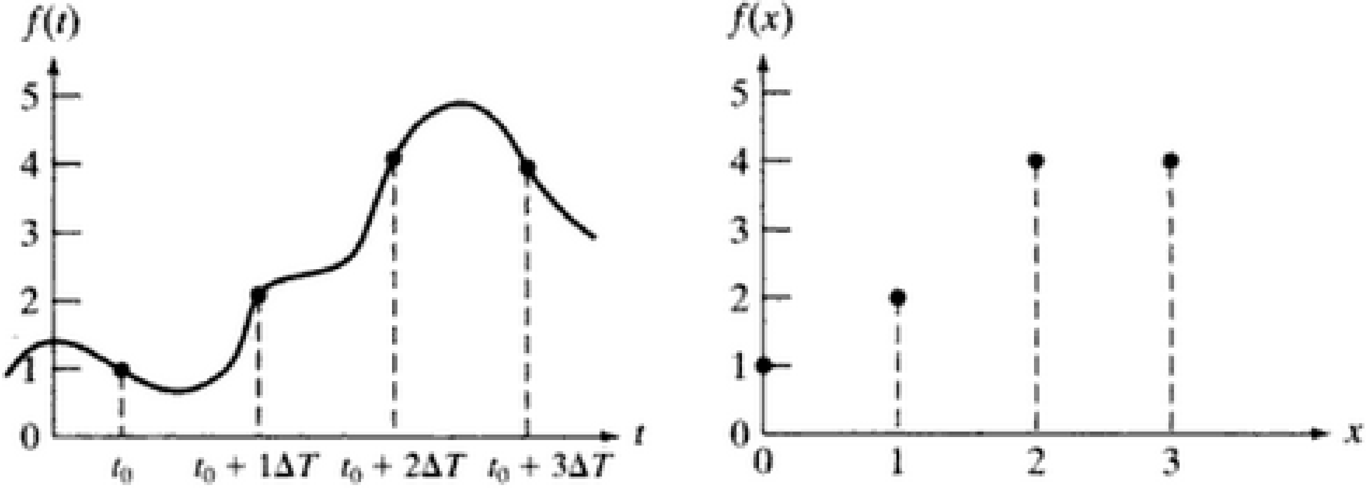
The corresponding spacing, Δu, in the discrete frequency domain

$$\Delta u = \frac{1}{M\Delta T} = \frac{1}{T}$$

The entire frequency range spanned by the M components of the DFT is

$$\Omega = M\Delta u = \frac{1}{\Delta T}$$

Example:



$$F(0) = \sum_{x=0}^{3} f(x) = \left[f(0) + f(1) + f(2) + f(3)\right]$$

$$= 1 + 2 + 4 + 4 = 11$$

The next value of $F(u)$ is

$$F(1) = \sum_{x=0}^{3} f(x) e^{-j2\pi(1)x/4}$$

$$= 1e^{0} + 2e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2} = -3 + 2j$$

Similarly, $F(2) = -(1 + 0j)$ and $F(3) = -(3 + 2j)$. Observe that *all* values of $f(x)$ are used in computing *each* term of $F(u)$.

If instead we were given $F(u)$ and were asked to compute its inverse, we would proceed in the same manner, but using the inverse transform. For instance,

$$
\begin{aligned}
f(0) &= \frac{1}{4}\sum_{u=0}^{3} F(u)e^{j2\pi u(0)} \\
&= \frac{1}{4}\sum_{u=0}^{3} F(u) \\
&= \frac{1}{4}[11 - 3 + 2j - 1 - 3 - 2j] \\
&= \frac{1}{4}[4] = 1
\end{aligned}
$$

2-D Discrete transform pair:

DFT:

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

where $f(x, y)$ is a digital image of size $M \times N$.
values of the discrete variables u and v in the ranges $u = 0, 1, 2, \ldots, M-1$ and $v = 0, 1, 2, \ldots, N-1$.

IDFT:

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

Relationship between frequency and space intervals:

$$\Delta u = \frac{1}{M\Delta T}$$

$$\Delta v = \frac{1}{N\Delta Z}$$

Translation property of Fourier transform:

$$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$$

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(x_0u/M + y_0v/N)}$$

Proof:

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$FT = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{j2\pi\left(\frac{xu_0}{M}+\frac{yv_0}{N}\right)} e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$FT = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{x(u-u_0)}{M}+\frac{y(v-v_0)}{N}\right)}$$

$$= F(u-u_0, v-v_0)$$

DFT: A complex no:

Polar Coordinate Representation of FT

- The Fourier transform of a real function is generally complex and we use polar coordinates:

$$F(u,v) = R(u,v) + j \cdot I(u,v)$$

Polar coordinate

$$F(u,v) = |F(u,v)| \exp(j\phi(u,v))$$

Magnitude: $$|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{1/2}$$

Phase: $$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

- Power spectrum is defined as:

$$\begin{aligned} P(u,v) &= |F(u,v)|^2 \\ &= R^2(u,v) + I^2(u,v) \end{aligned}$$

Translation property:

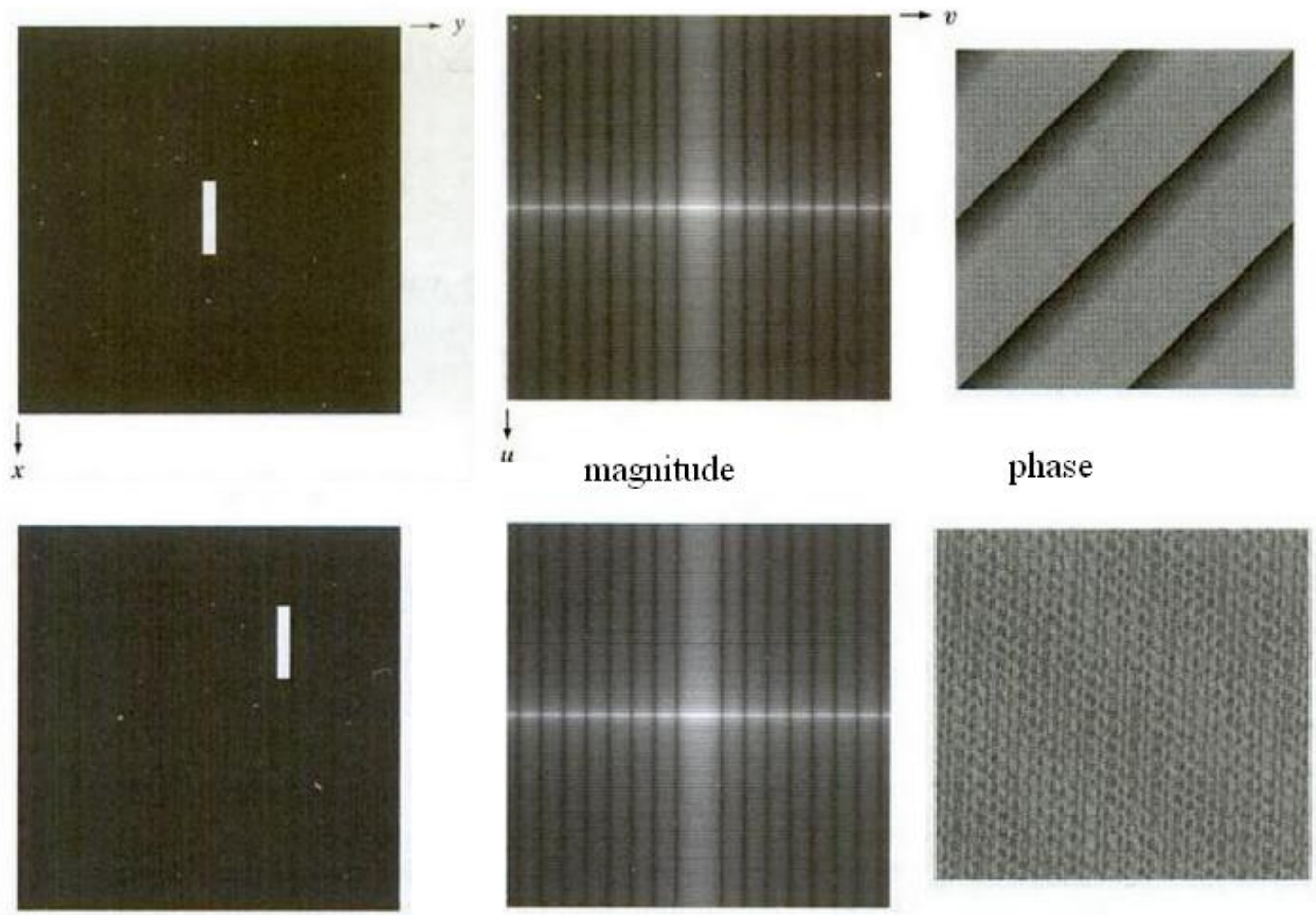
$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)\, e^{-j2\pi(x_0 u/M + y_0 v/N)}$$

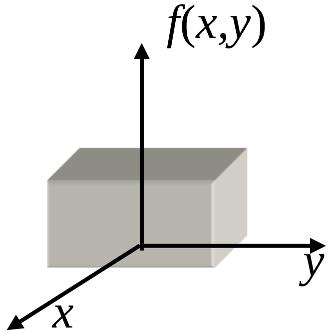
$$f(x - x_0, y - y_0) \Leftrightarrow |F(u,v)| \exp(j\phi(u,v))\, e^{-j2\pi(x_0 u/M + y_0 v/N)}$$

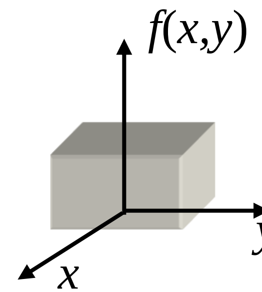
$$F(u,v) = |F(u,v)| \exp(j\phi(u,v))$$

What is new magnitude and phase?

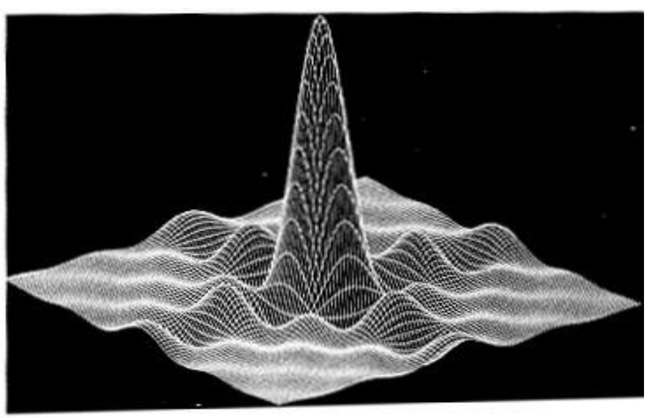
Translation has no effect on magnitude of DFT but on phase of DFT:



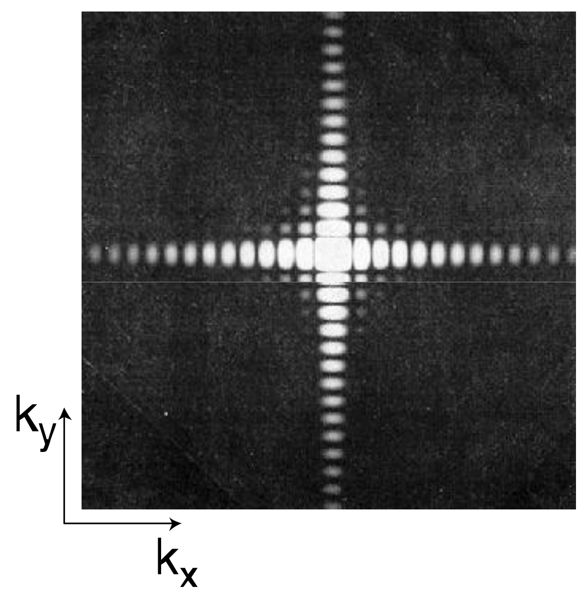




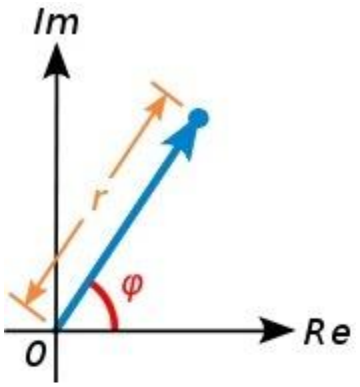
FT$\{f(x,y)\}$



FT$\{f(x,y)\}$



Rotational property



Using the polar coordinates

$$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$$

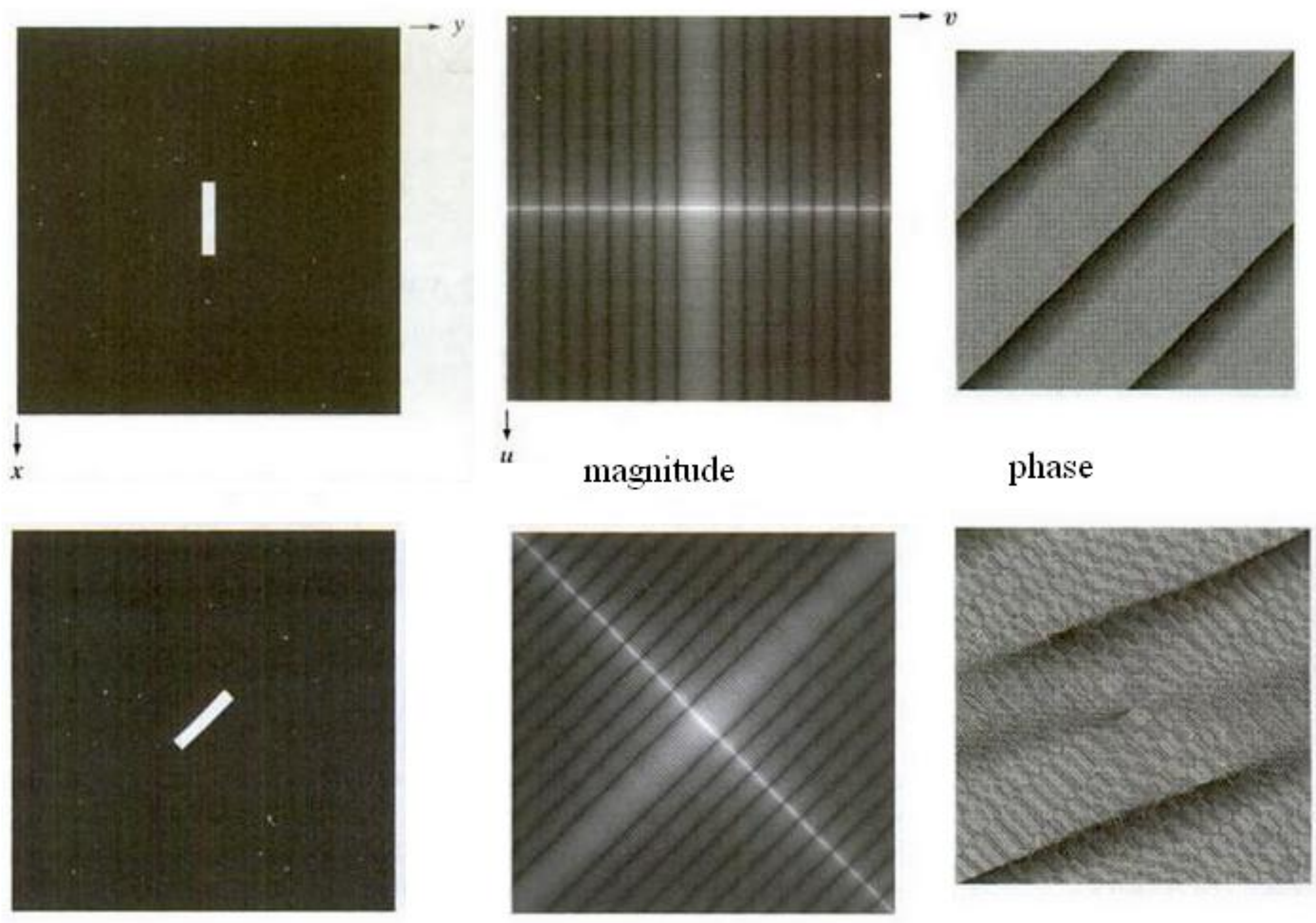
$$f(r,\theta) \Leftrightarrow F(\omega,\varphi)$$

Property says that-

$$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$$

which indicates that rotating $f(x, y)$ by an angle θ_0 rotates $F(u, v)$ by the same angle. Conversely, rotating $F(u, v)$ rotates $f(x, y)$ by the same angle.

Rotational property:



Significance of spectrum and phase:

The components of the spectrum of the DFT determine the amplitudes of the sinusoids that combine to form the resulting image.

The phase is a measure of displacement of the various sinusoids with respect to their origin.

Recall:

Consider the compact Fourier series

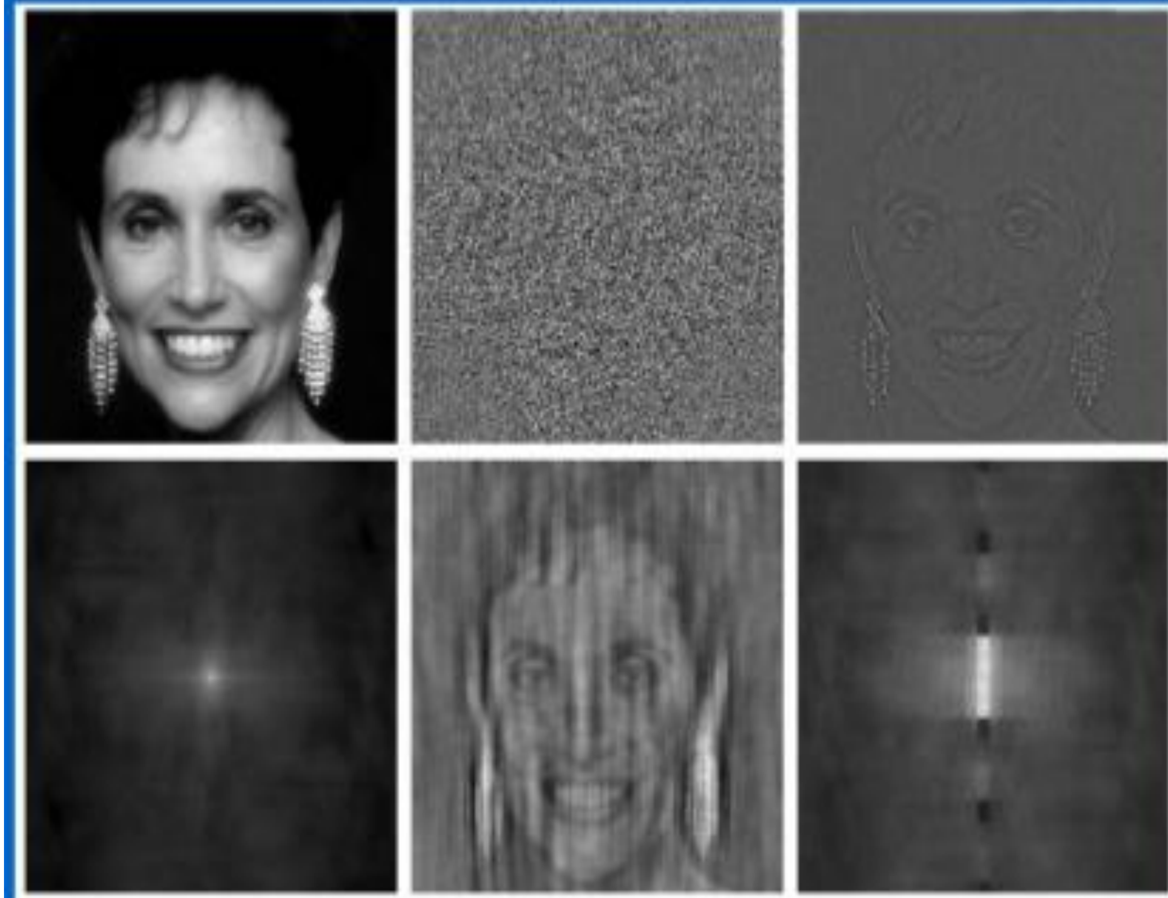
$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n w_o t + \theta_n)$$

This equation can represents a periodic signal g(t) of

frequencies: $0(dc), w_o, 2w_o, 3w_o, \ldots\ldots, nw_o$

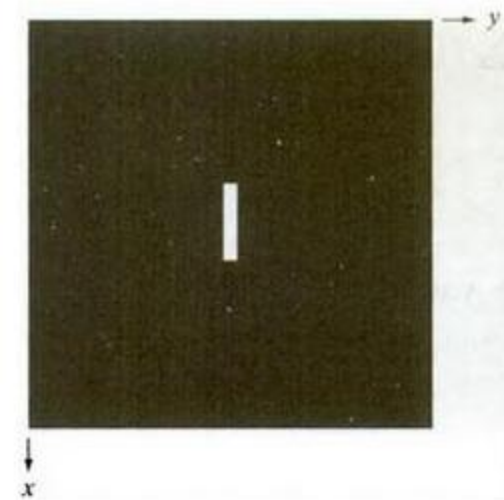
Amplitudes: $C_0, C_1, C_2, C_3, \ldots\ldots, C_n$

Phases: $0, \theta_1, \theta_2, \theta_3, \ldots\ldots \theta_n$



a b c
d e f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.



Conclusion:

Thus, while the magnitude of the 2-D DFT is an array whose components determine the intensities in the image, the corresponding phase is an array of angles that carry much of the information about where discernable objects are located in the image.

Periodicity:

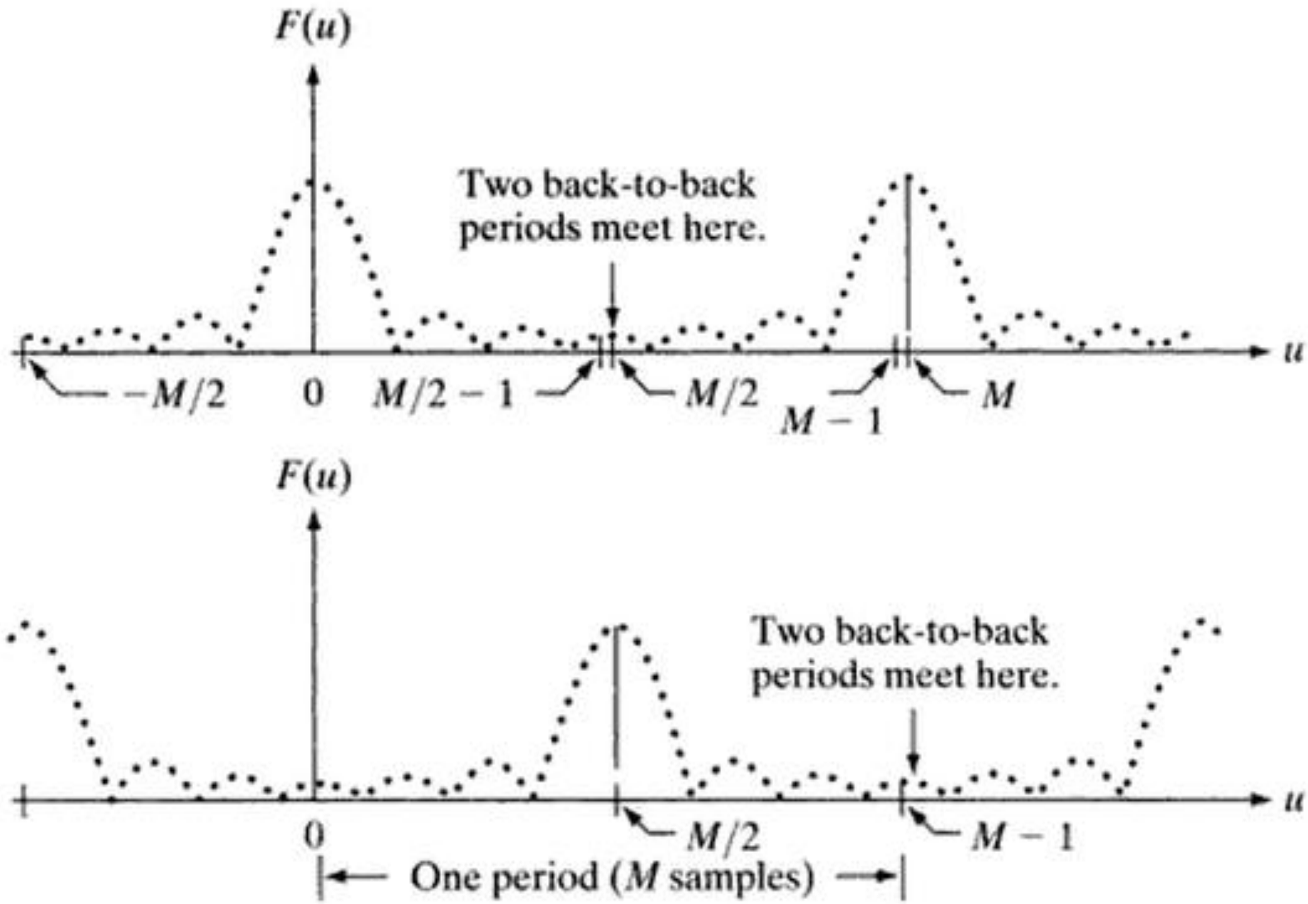
$$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N) = F(u + k_1M, v + k_2N)$$

and

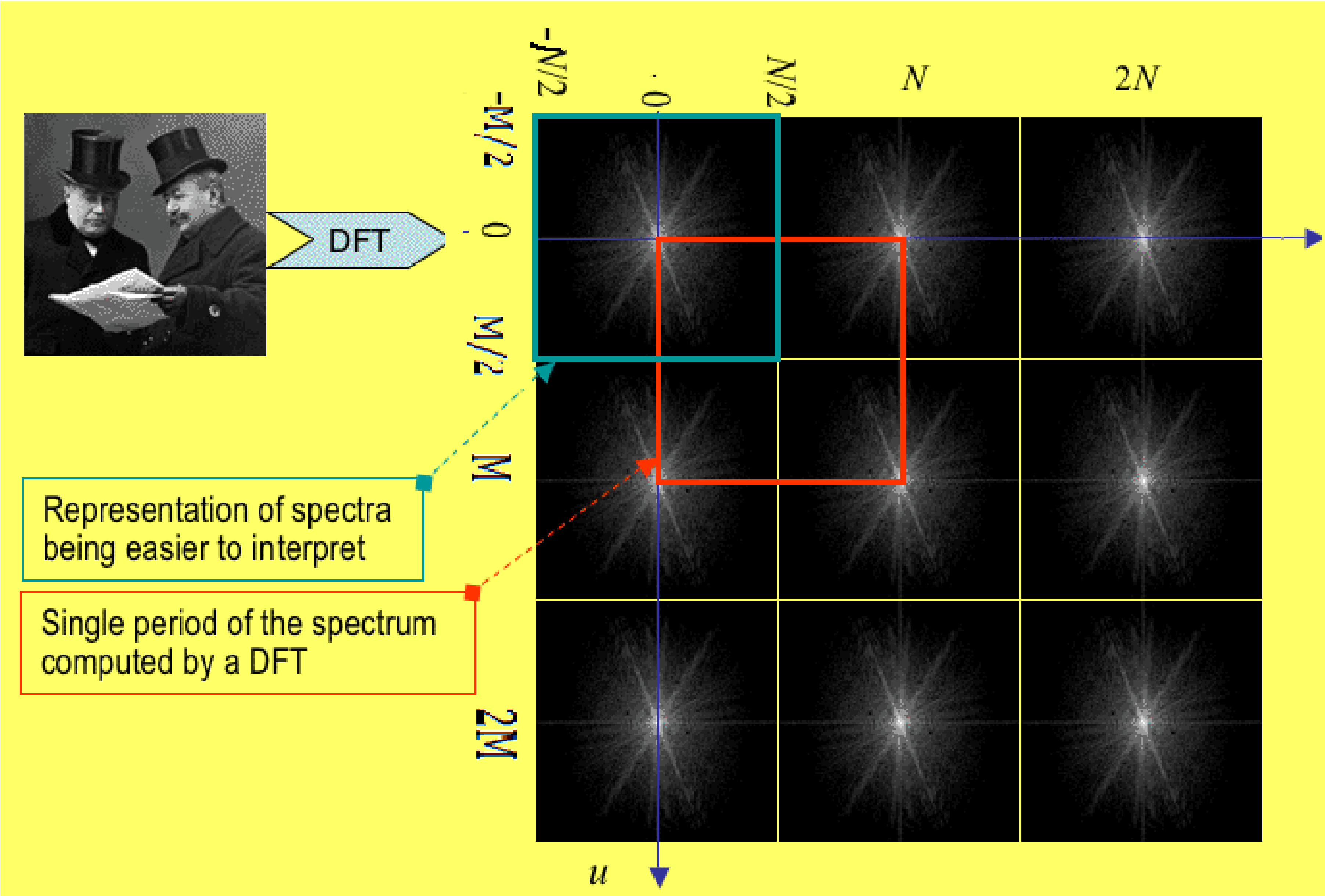
$$f(x, y) = f(x + k_1M, y) = f(x, y + k_2N) = f(x + k_1M, y + k_2N)$$

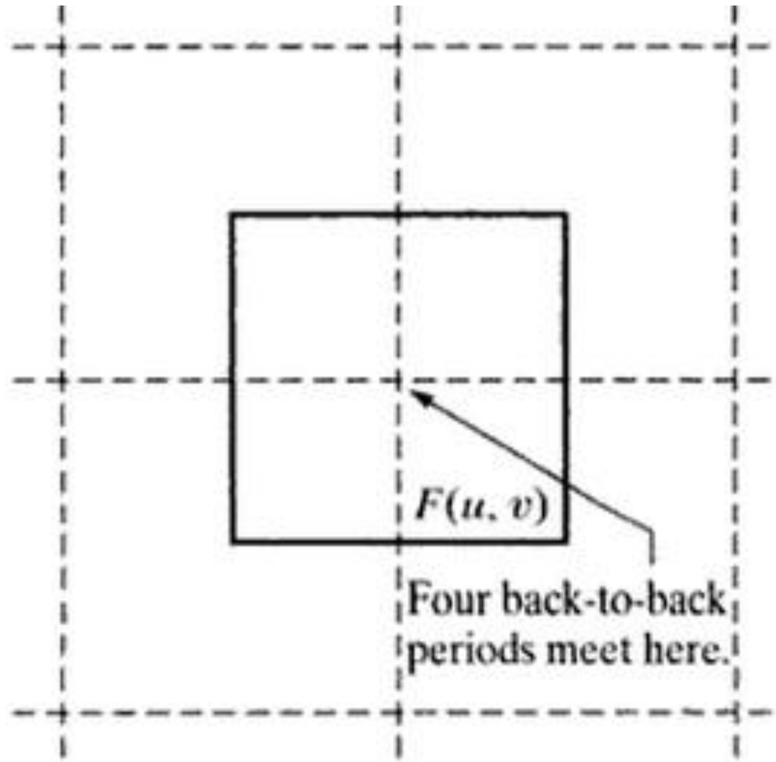
where k_1 and k_2 are integers.

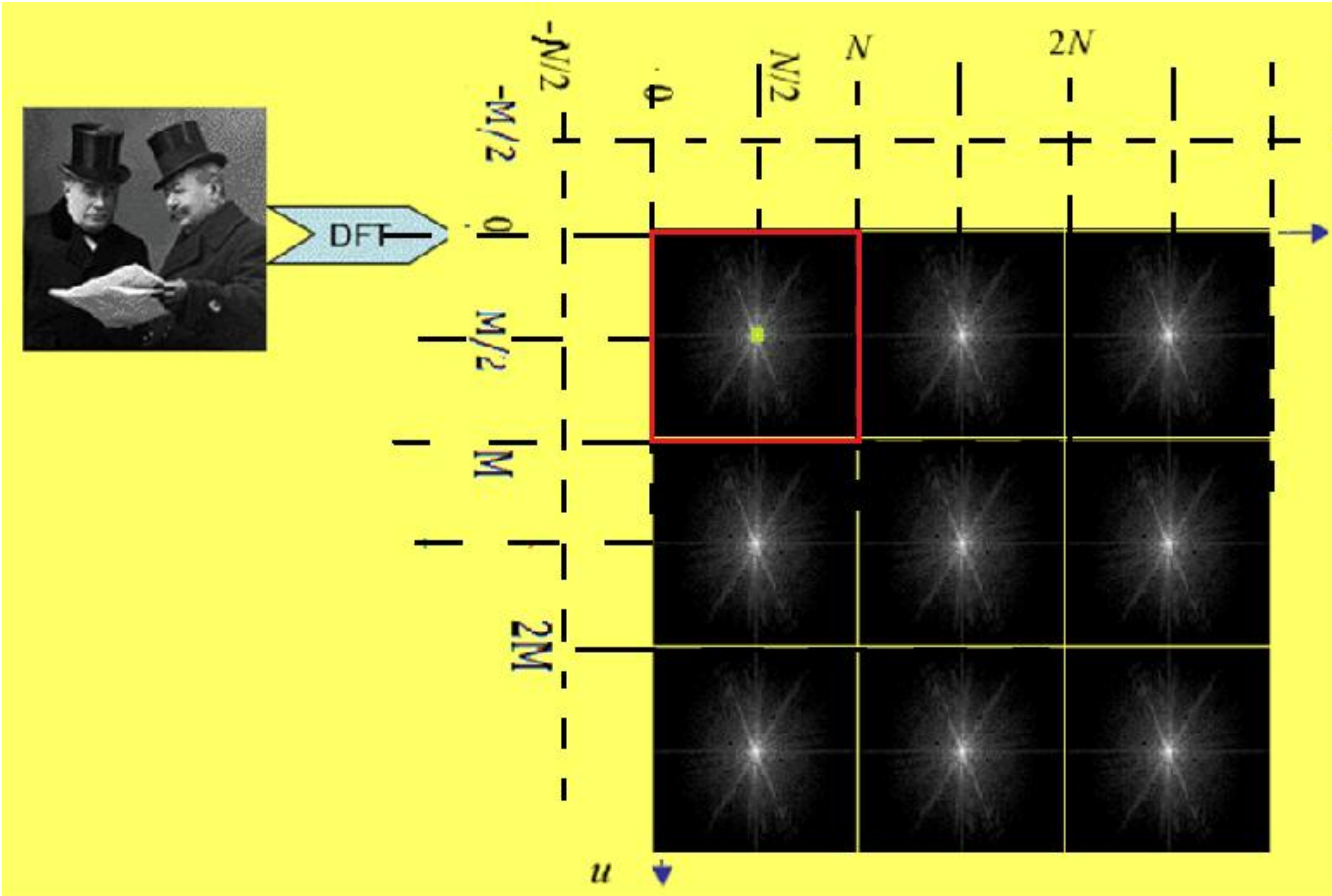
- In 1-D DFT consider 0 to (M-1) samples

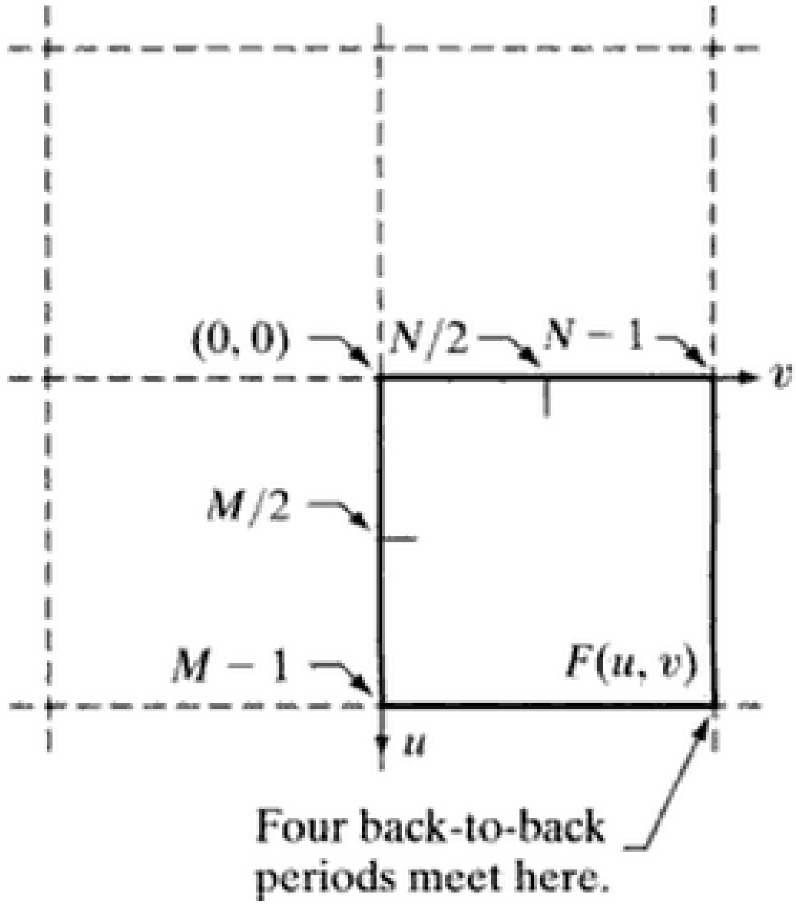


- Conclusion: We need to shift in frequency


DFT
-N/2
0
N/2
N
2N
-M/2
0
M/2
M
2M
u
Representation of spectra being easier to interpret
Single period of the spectrum computed by a DFT


F(u, v)
Four back-to-back periods meet here.


DFT
-N/2
0
N/2
N
2N
-M/2
0
M/2
M
2M
u



(0, 0)
N/2
N − 1
v
M/2
M − 1
F(u, v)
u
Four back-to-back
periods meet here.

Shift in frequency:

- How to get shift in frequency domain?
- By multiplying exponential in spatial domain

$$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$$

- What is shift amount here? u0 and v0?

- u0=M/2
- v0=N/2

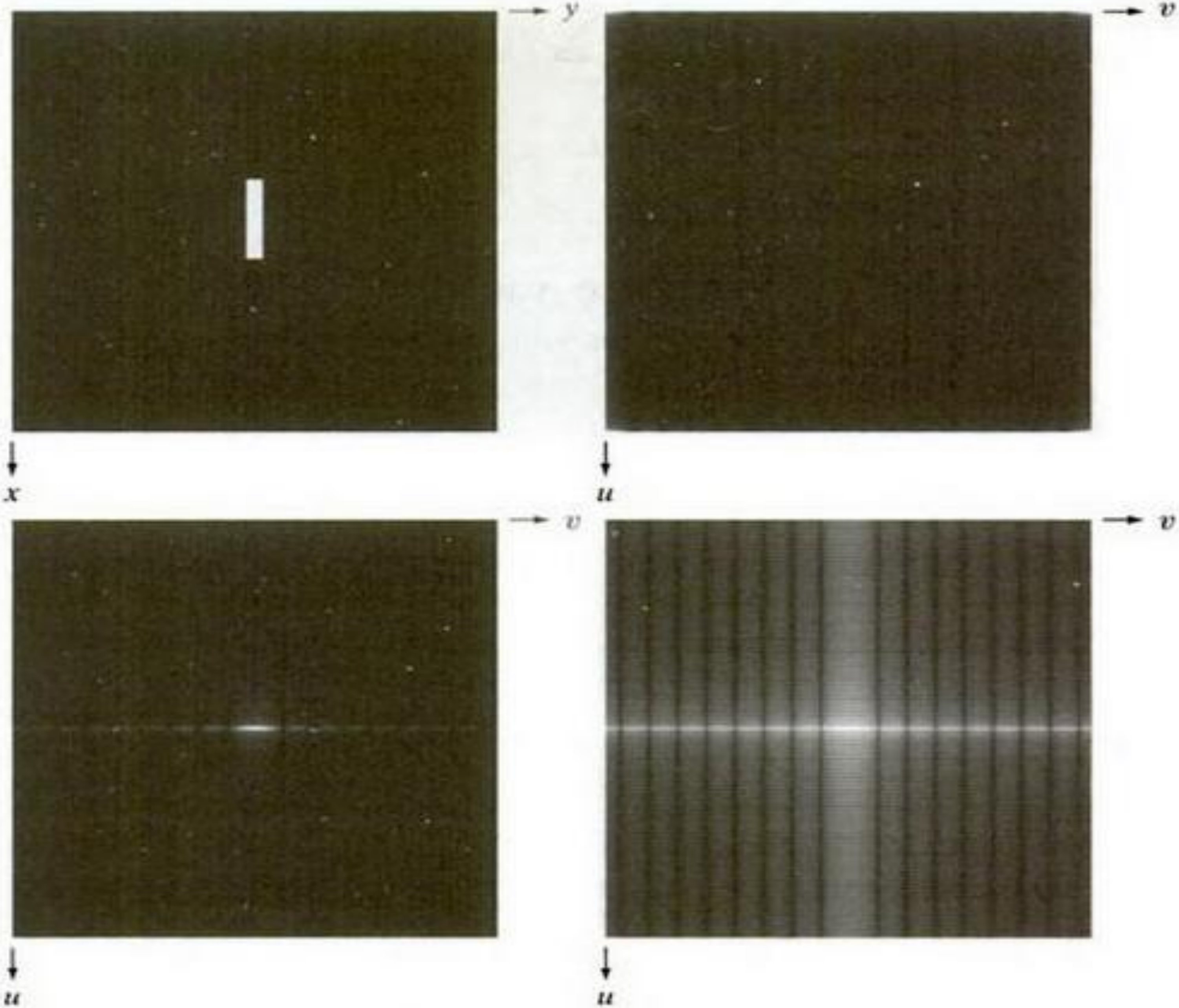
$$f(x,y)e^{j2\pi\left(\frac{x}{M}\left(\frac{M}{2}\right)+\frac{y}{N}\left(\frac{N}{2}\right)\right)} \leftrightarrow F(u-\frac{M}{2}, v-N/2)$$

$$f(x,y)e^{j2\pi\left(\frac{x}{M}\left(\frac{M}{2}\right)+\frac{y}{N}\left(\frac{N}{2}\right)\right)} = f(x,y)e^{j2\pi\left(\frac{x}{2}+\frac{y}{2}\right)}$$

$$= f(x,y)e^{j\pi(x+y)}$$

$x + y$ *is an integer so* $e^{j\pi(x+y)}$ *is* $(-1)^{x+y}$

$$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$$



$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$F(0,0) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

$$F(0,0) = MN\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$
$$= MN\bar{f}(x,y)$$

where $\bar{f}$ denotes the average value of f. Then,

$$|F(0,0)| = MN|\bar{f}(x,y)|$$

F(0,0): usually largest component, dc component

2-D circular convolution:

$$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n)$$

for $x = 0, 1, 2, \ldots, M - 1$ and $y = 0, 1, 2, \ldots, N - 1$.

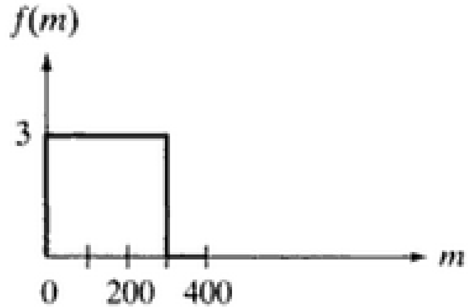
2-D convolution theorem:

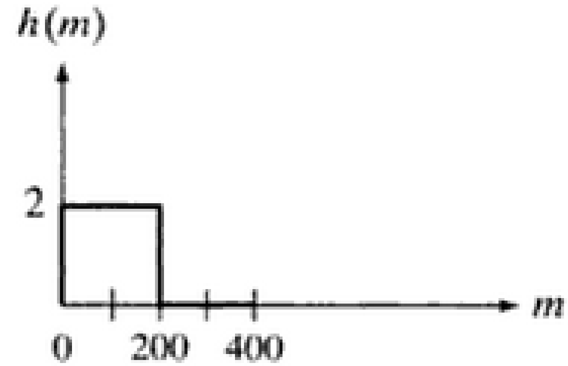
$$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$$

$$f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$$

We use DFT and IDFT… so need to deal with periodicity issue.

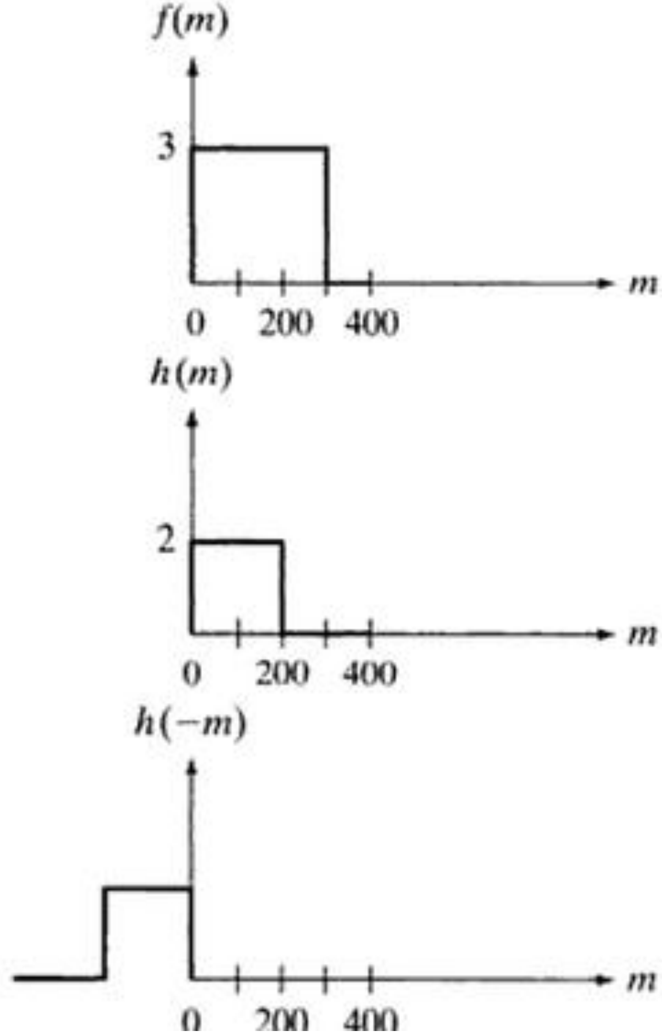
Consider 1-D case:

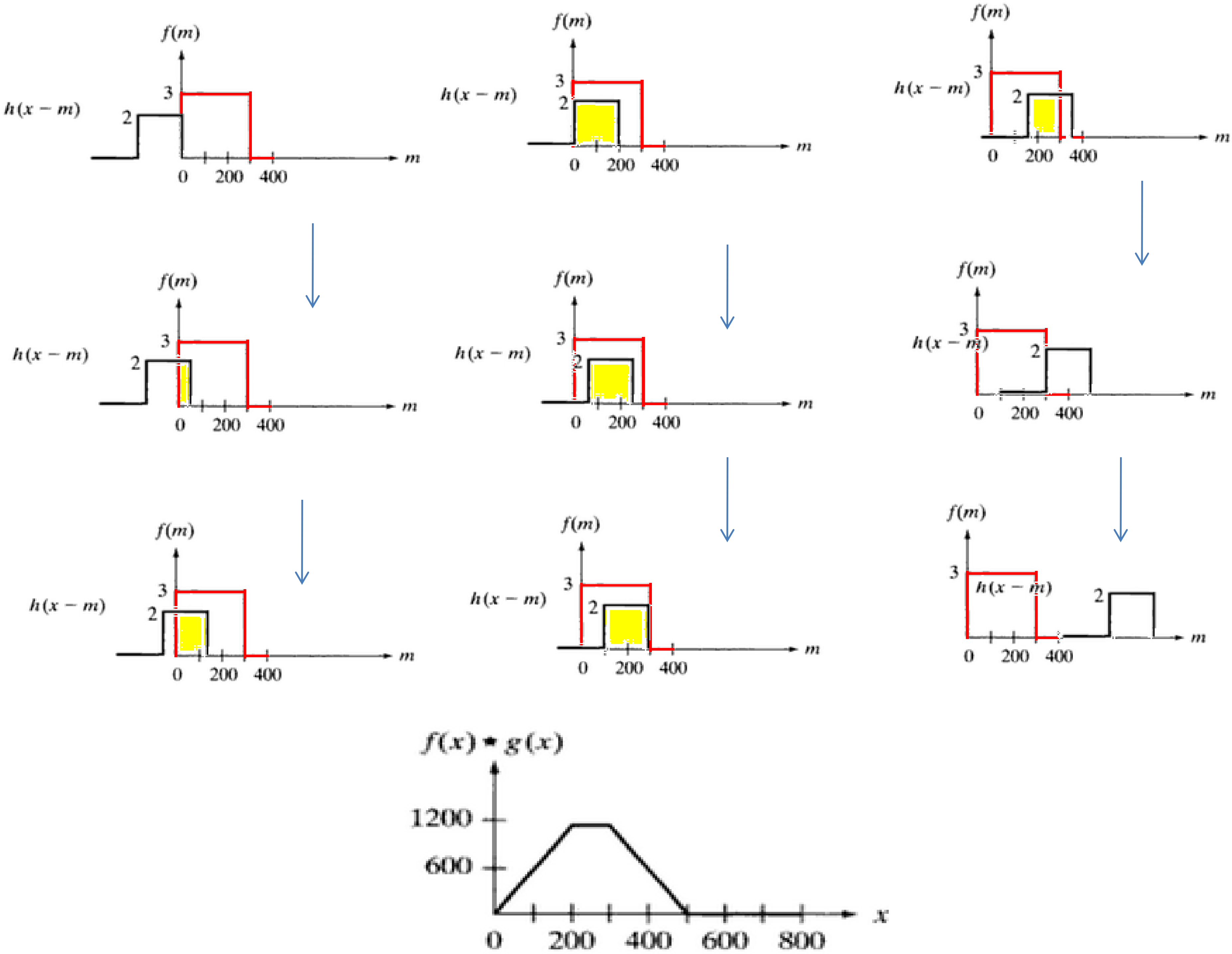




$$f(x) \star h(x) = \sum_{m=0}^{399} f(m)h(x - m)$$

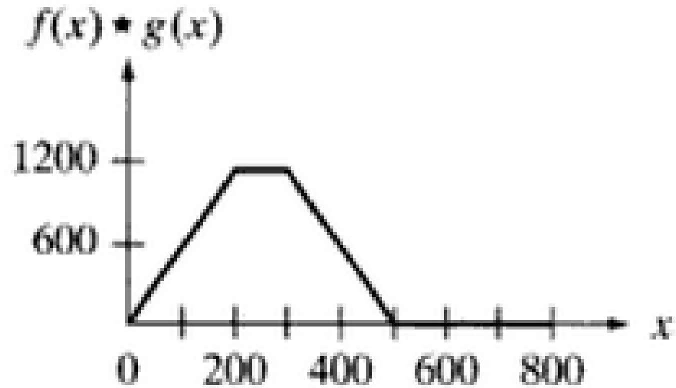
$$f(x) \star h(x) = \sum_{m=0}^{399} f(m)h(x - m)$$





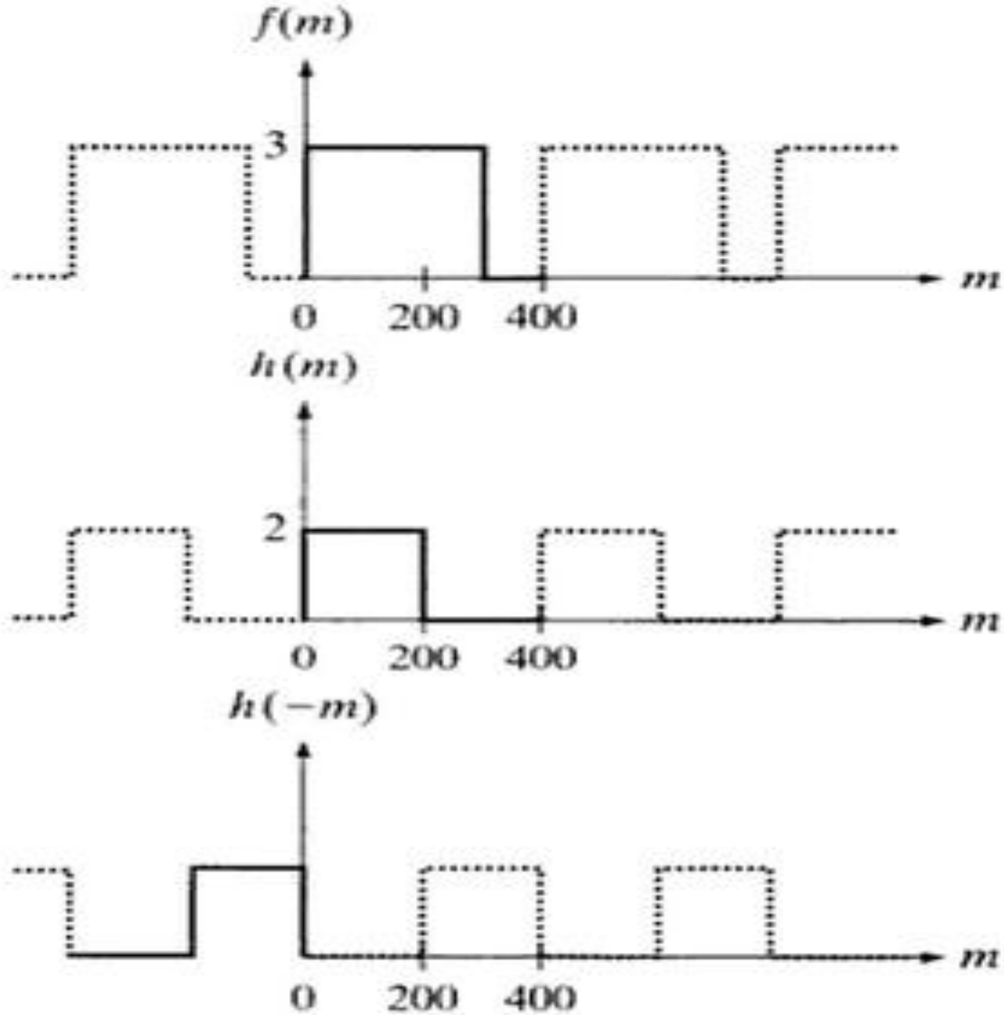
f(m)
h(x − m)
3
2
0
200
400
m
f(x) * g(x)
1200
600
0
200
400
600
800
x

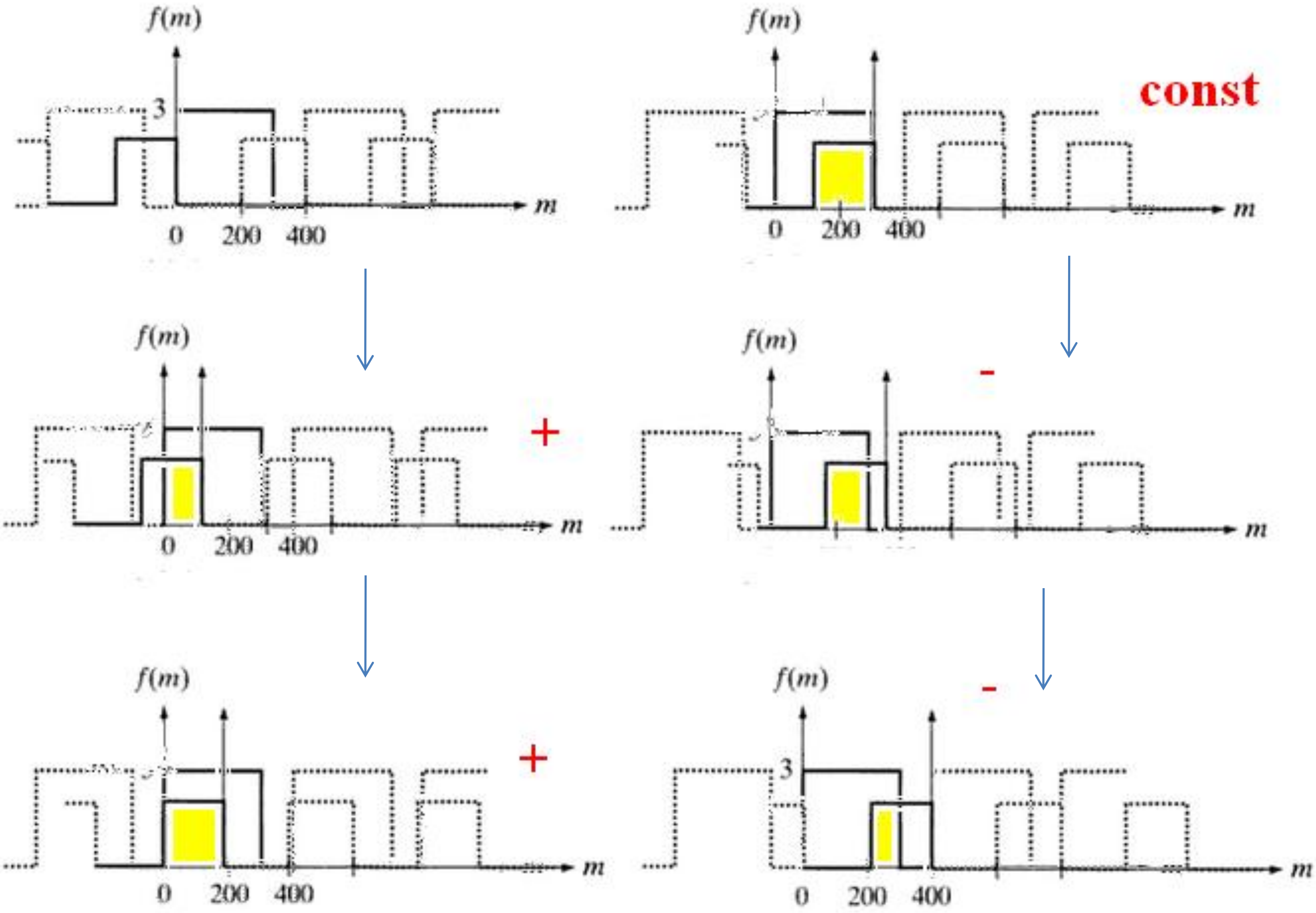
Convolution result:

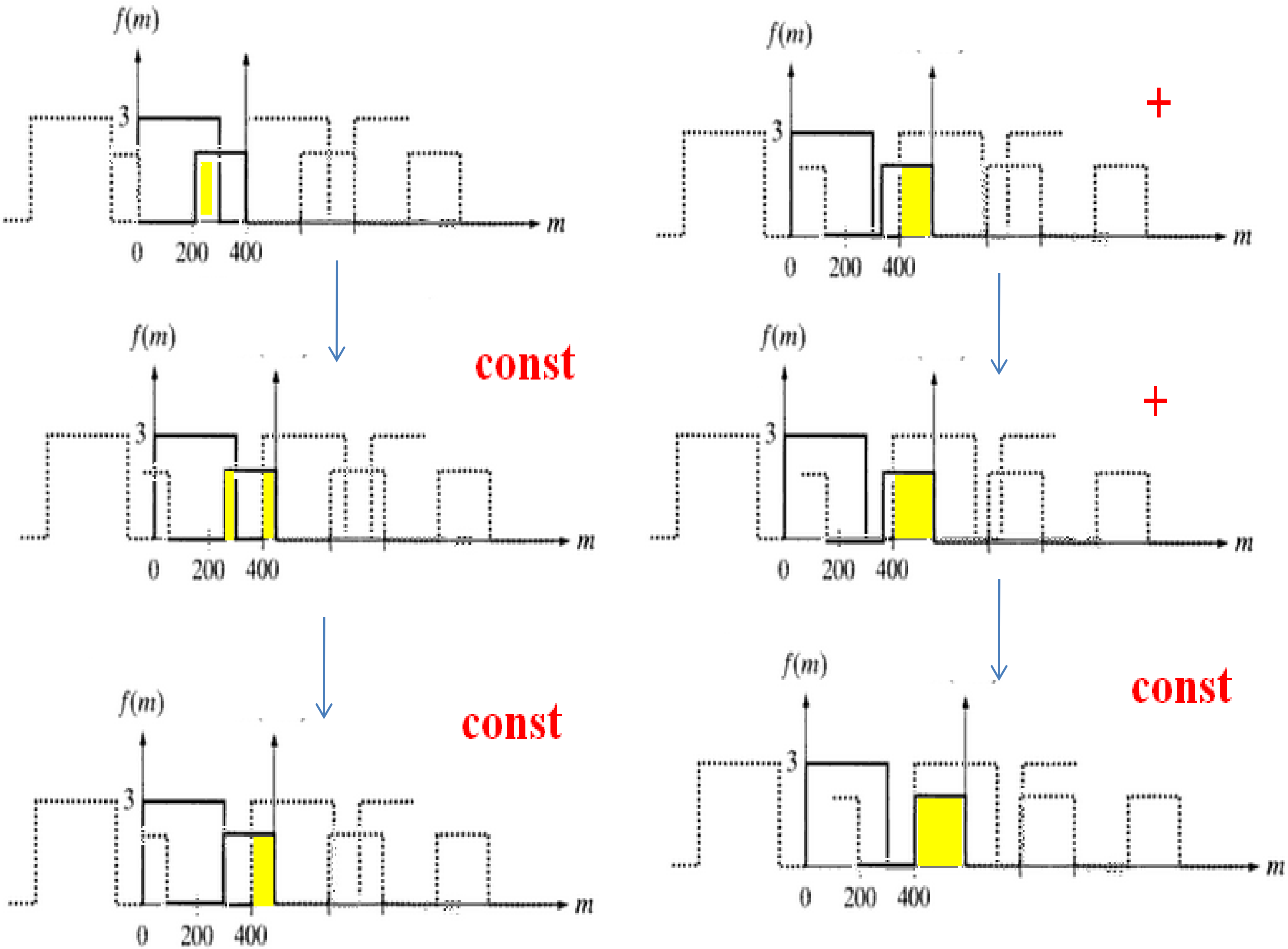


Span of output?- 800 pt: 0-799

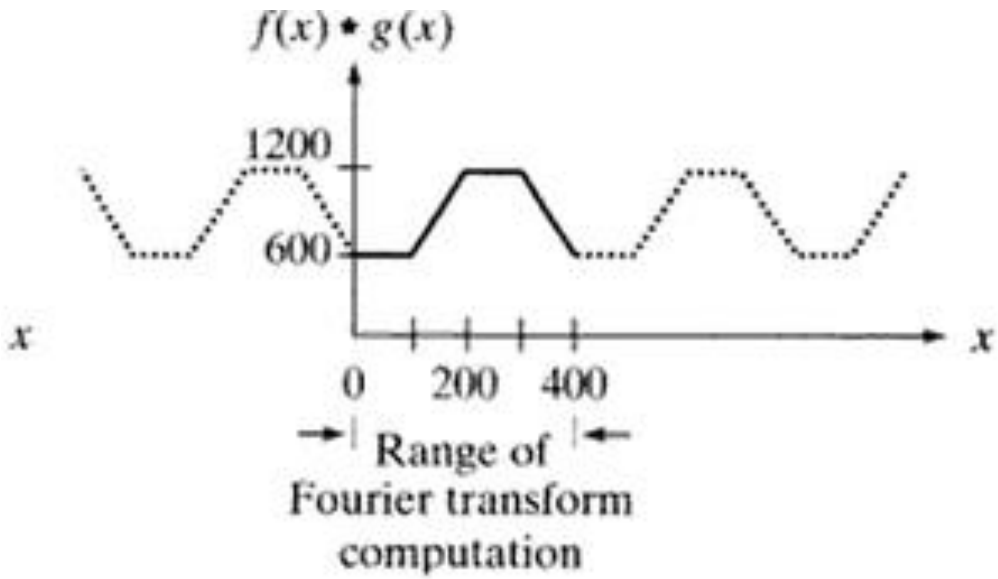
Periodicity of function:

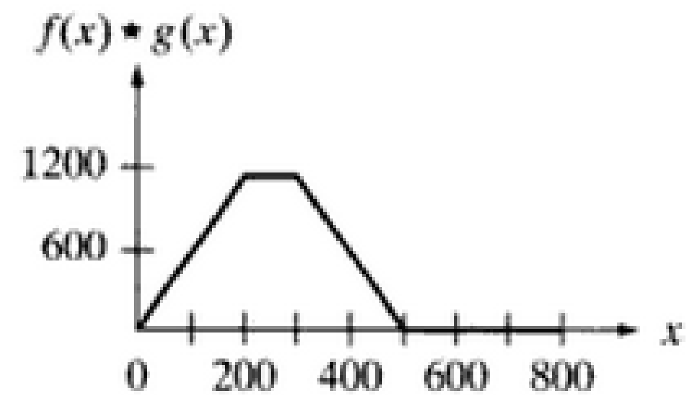



f(m)
3
0
200
400
m
const
+
-


f(m)
3
0
200
400
m
const
const
+
+
const

Result: Original:



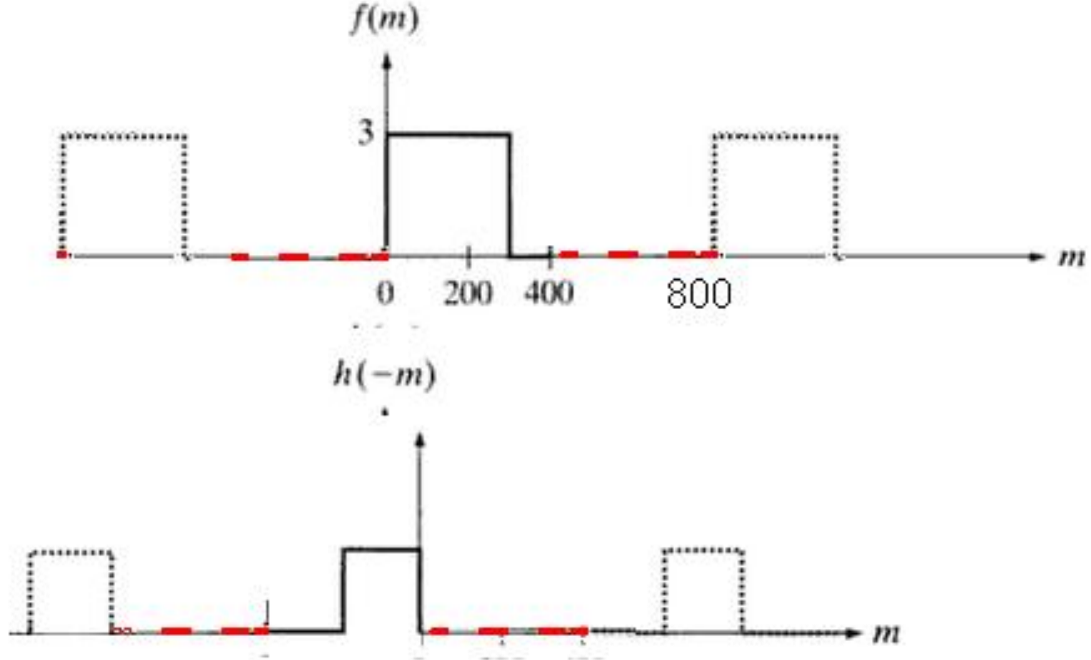


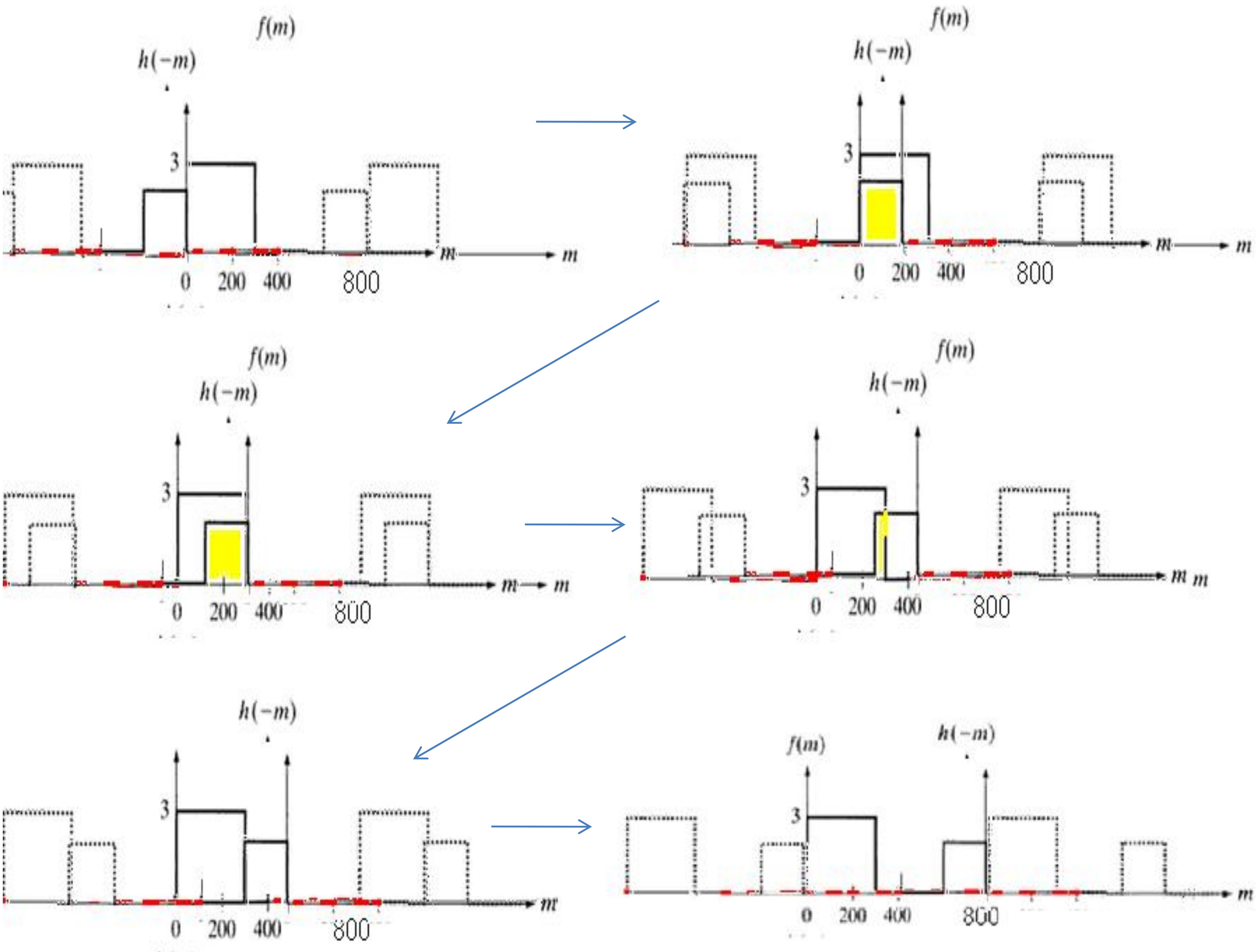
Wrap around error

Reason?- closeness in period and hence overlapping

Solution?

Padding:





f(m)
h(−m)
3
0
200
400
800
m

Amount of pading?

- f➔ size: A
- h➔size: B
- New and equal size after padding is P
- In our case value of A, B and P?
- A➔ 400
- B➔ 400
- P➔ 799 (minimum)

Zero padding :

$$P \geq A + B - 1$$

2-D?

Let $f(x, y)$ and $h(x, y)$ be two image arrays of sizes $A \times B$ and $C \times D$ pixels, respectively.

$$f_p(x, y) = \begin{cases} f(x, y) & 0 \leq x \leq A - 1 \quad \text{and} \quad 0 \leq y \leq B - 1 \\ 0 & A \leq x \leq P \quad \text{or} \quad B \leq y \leq Q \end{cases}$$

and

$$h_p(x, y) = \begin{cases} h(x, y) & 0 \leq x \leq C - 1 \quad \text{and} \quad 0 \leq y \leq D - 1 \\ 0 & C \leq x \leq P \quad \text{or} \quad D \leq y \leq Q \end{cases}$$

with

$$P \geq A + C - 1 \qquad \text{and} \qquad Q \geq B + D - 1$$

The resulting padded images are of size $P \times Q$.

If both arrays are of the same size, $M \times N$, then we require that

$$P \geq 2M - 1$$

and

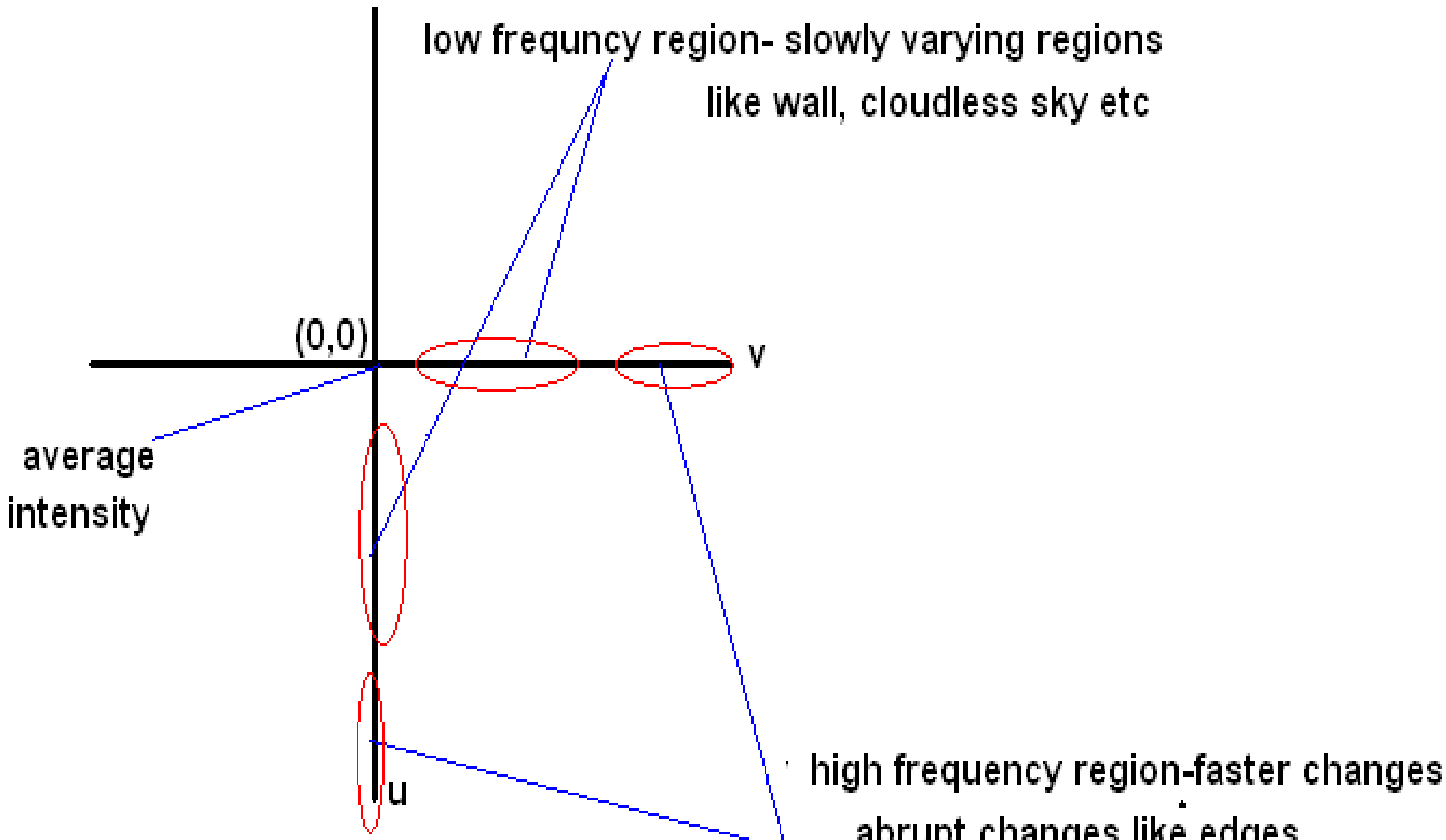
$$Q \geq 2N - 1$$

As rule, DFT algorithms tend to execute faster with arrays of even size, so it is good practice to select P and Q as the smallest even integers that satisfy the preceding equations. If the two arrays are of the same size, this means that P and Q are selected as twice the array size.

Basics of filtering in frequency domain

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

- Each term in frequency contain all terms of spatial domain.
- Direct association of each image component with its frequency domain is impossible except some trivial cases
- But some general statements for FT components and frequency features



low frequncy region- slowly varying regions
like wall, cloudless sky etc
(0,0)
v
average
intensity
u
high frequency region-faster changes,
abrupt changes like edges

Visual analysis of DFT:

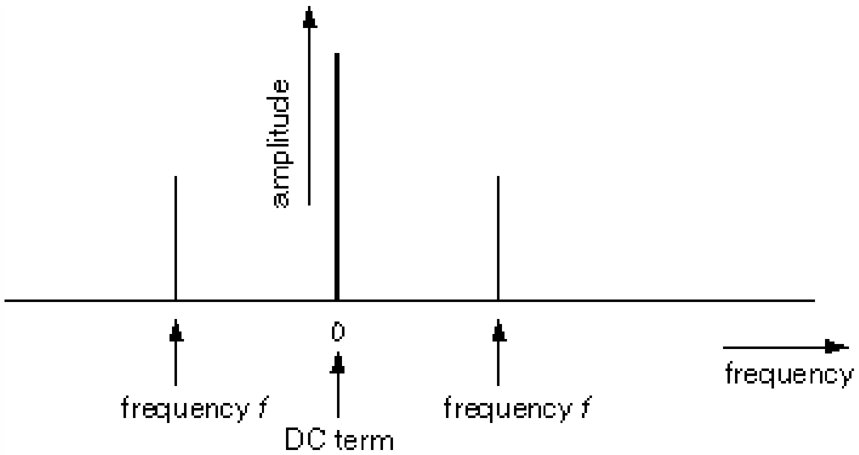
- Visual analysis of phase component is not generally not much useful.
- Intensity component gives some guidelines

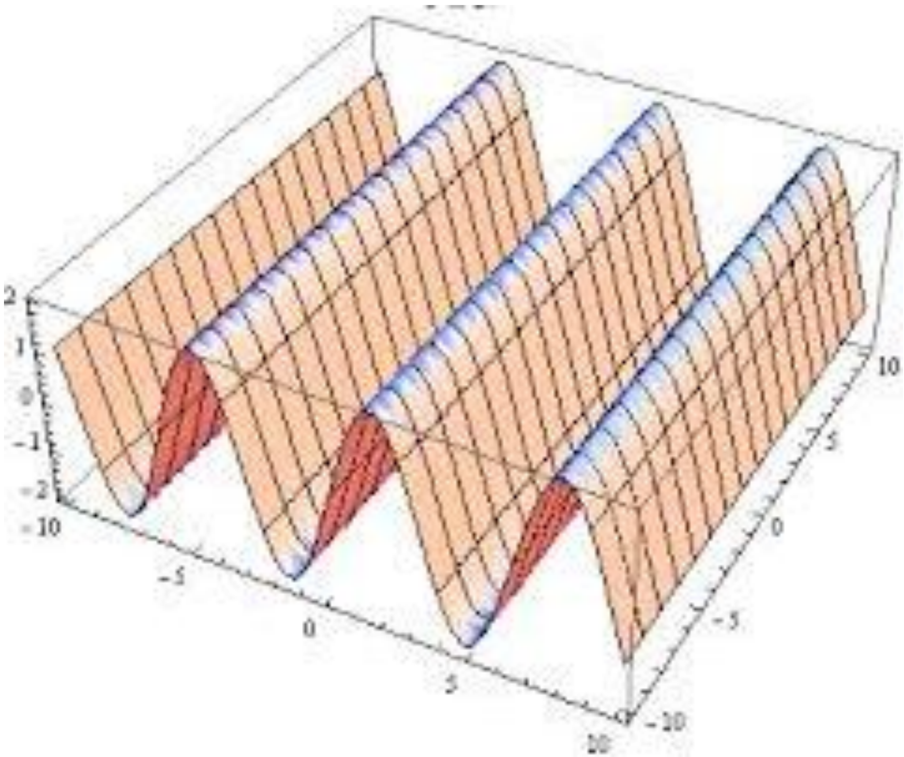
$$\boxed{e^{j2\pi\mu_0 t} \leftrightarrow \delta(\mu - \mu_0)}$$

$$G(f) = \Im\{\cos(2\pi A t)\} = \int_{-\infty}^{\infty} \frac{e^{i2\pi A t} + e^{-i2\pi A t}}{2} e^{-i2\pi f t}\, dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i2\pi A t} e^{-i2\pi f t}\, dt + \int_{-\infty}^{\infty} e^{-i2\pi A t} e^{-i2\pi f t}\, dt\right]$$

$$= \frac{1}{2}\left[\delta(f - A) + \delta(f + A)\right]$$





Sinusoidal wave in 2d

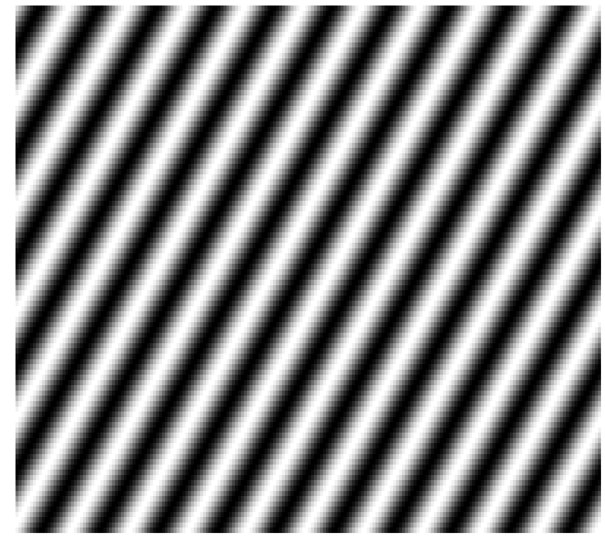
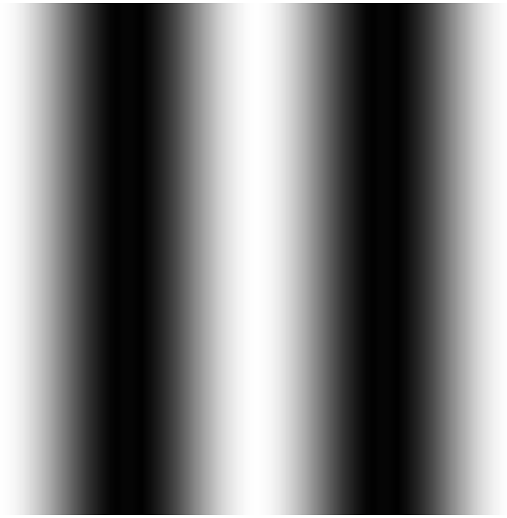
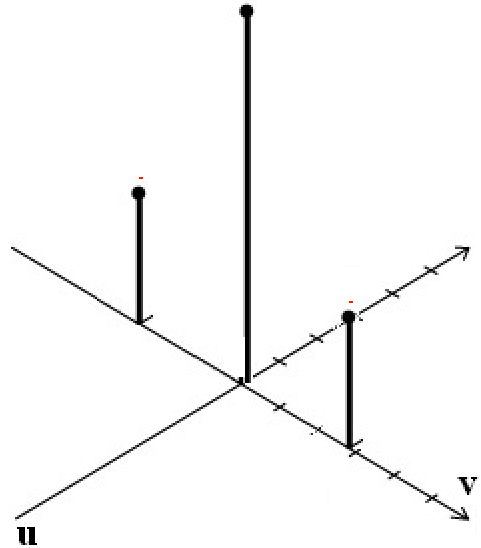


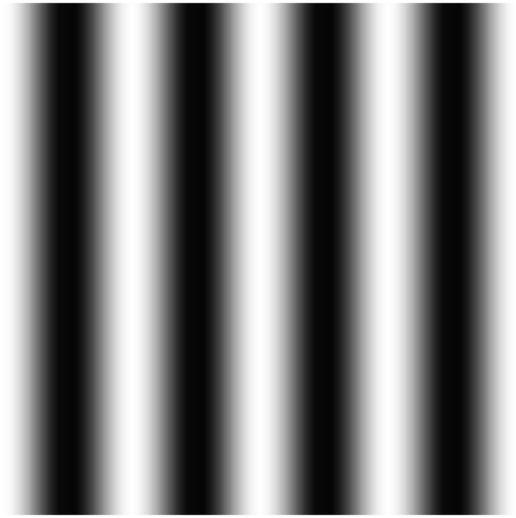
image of sine wave

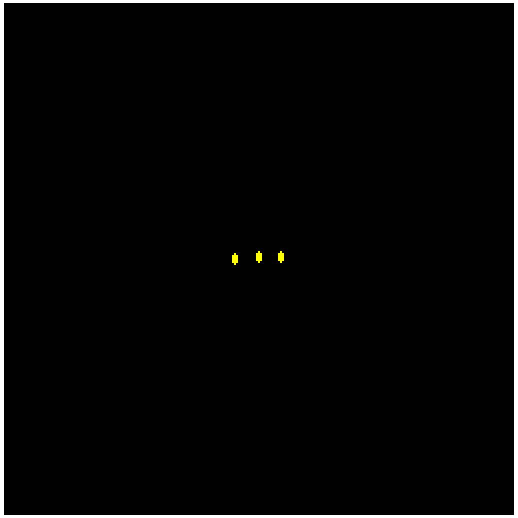




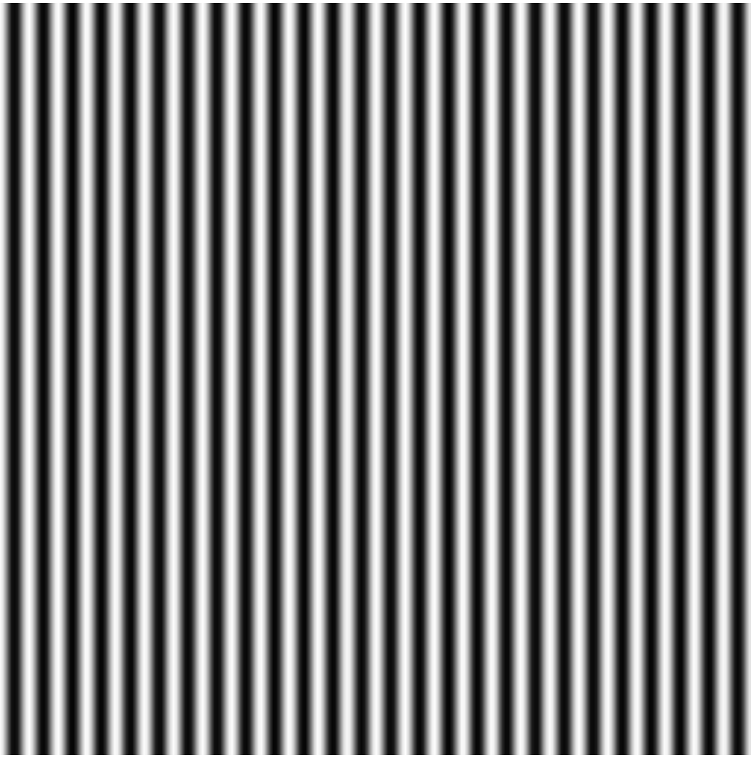
Slow variation – observe location of frequency components

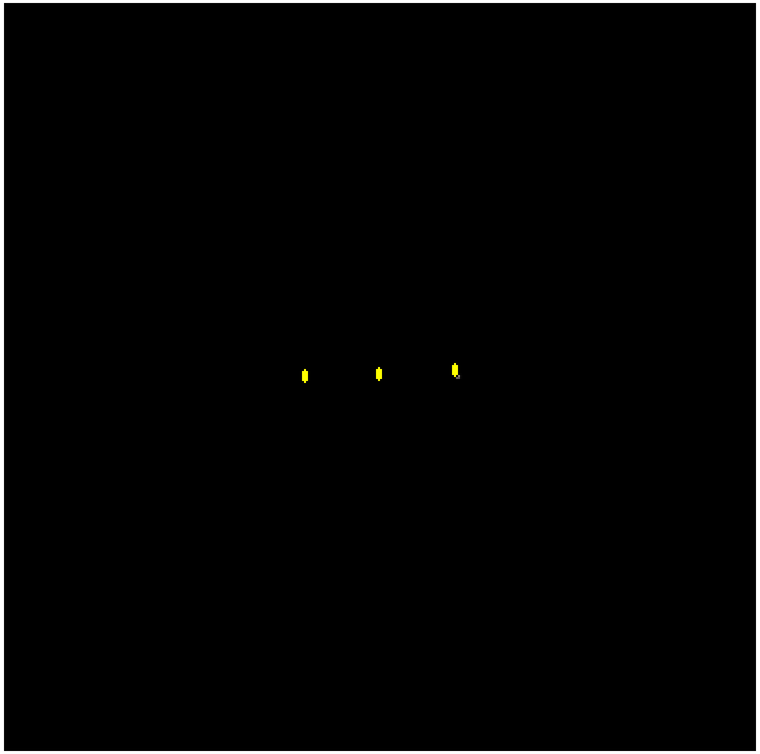
u



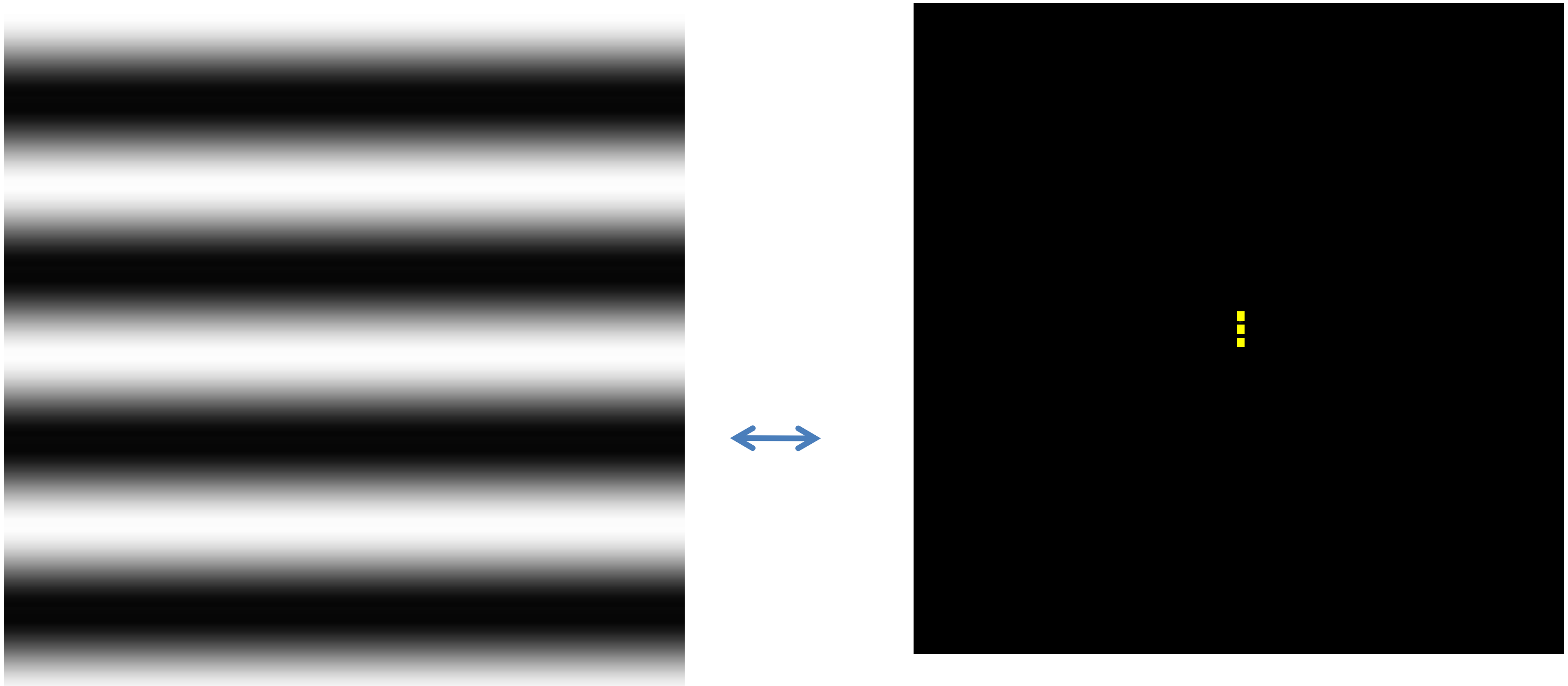


Comparatively still Slow variation – observe location of frequency components

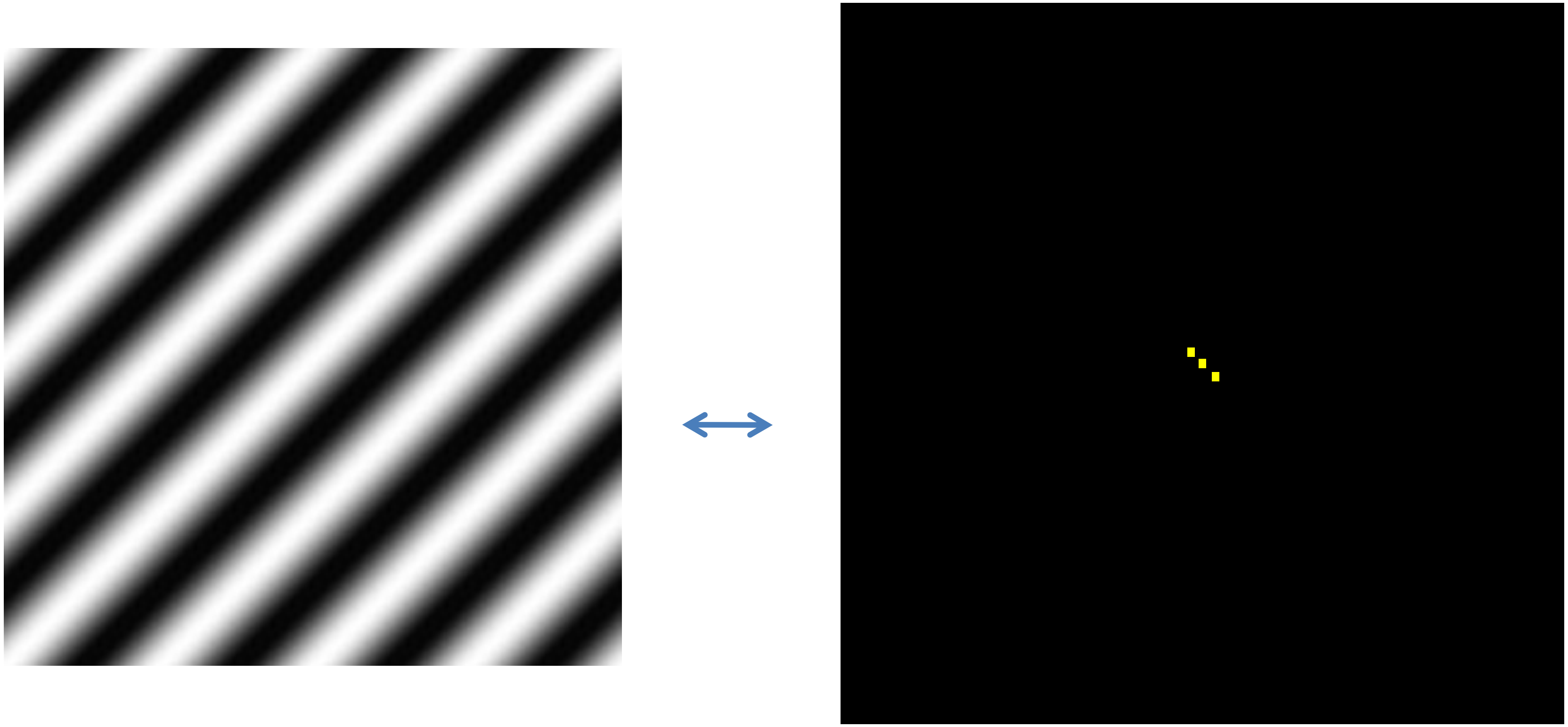


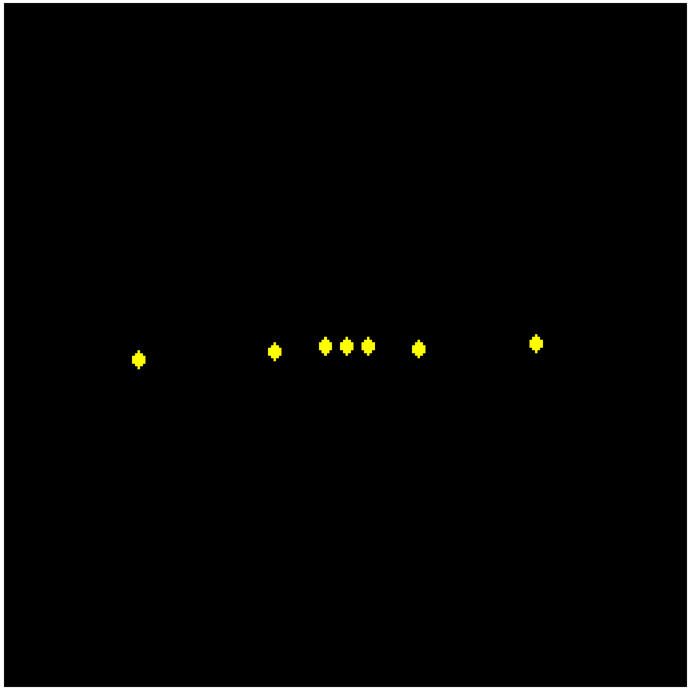


Fast variation – observe location of frequency components
Far region points high frequency and near region (w.r.t origin) points low frequency

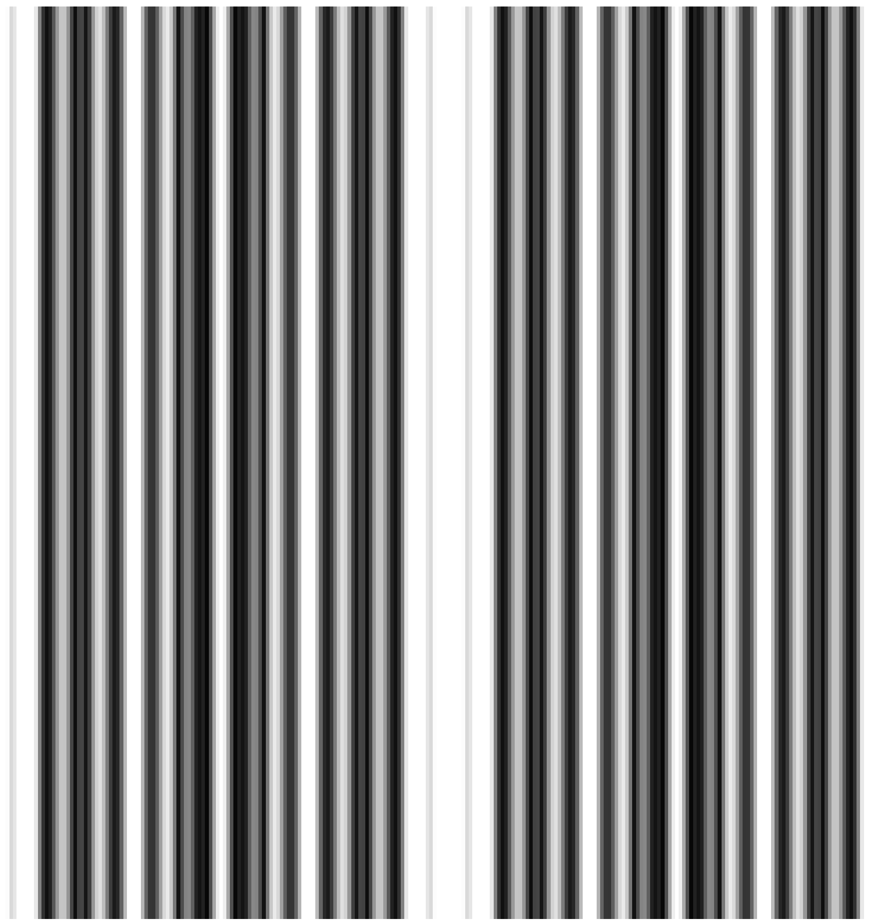


What is the direction of a sine wave? And intensity change?

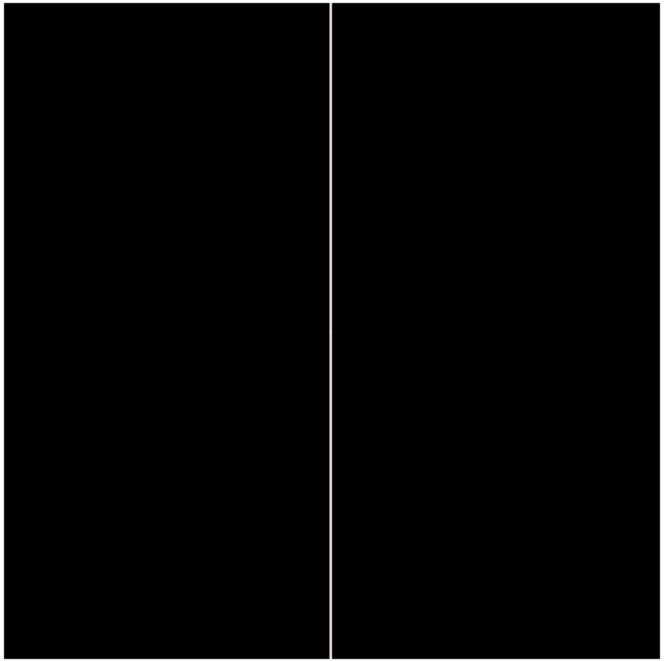




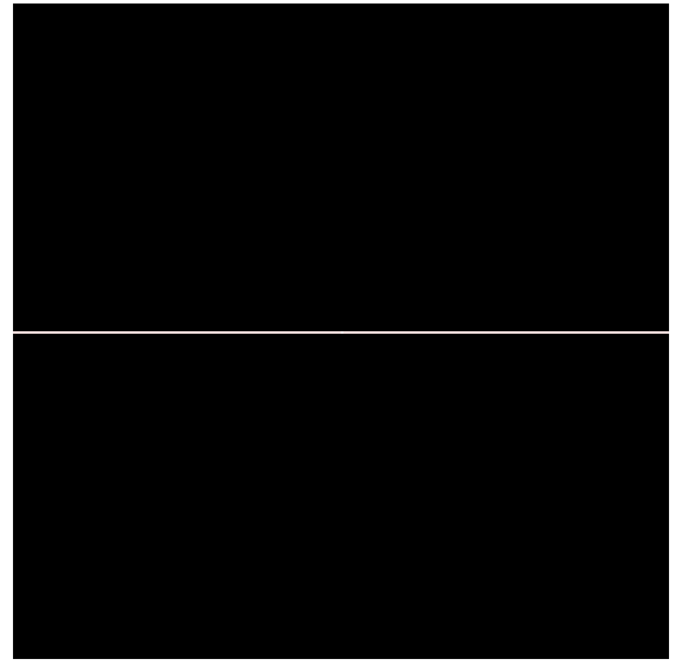
What is your interpretation?
How many sine waves? Direction of variation?



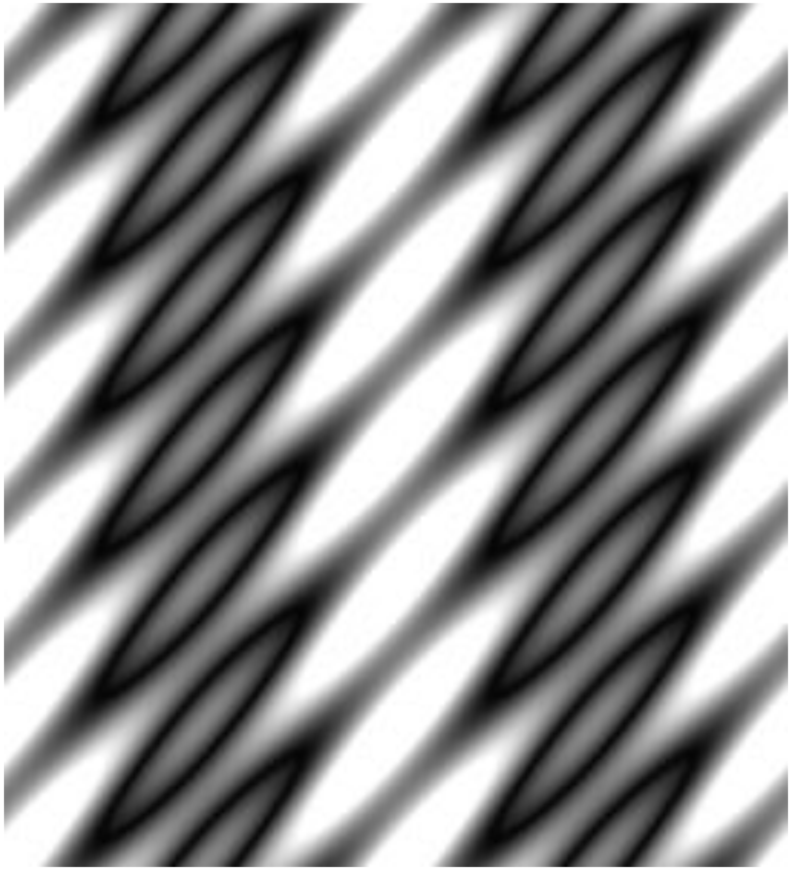
Image

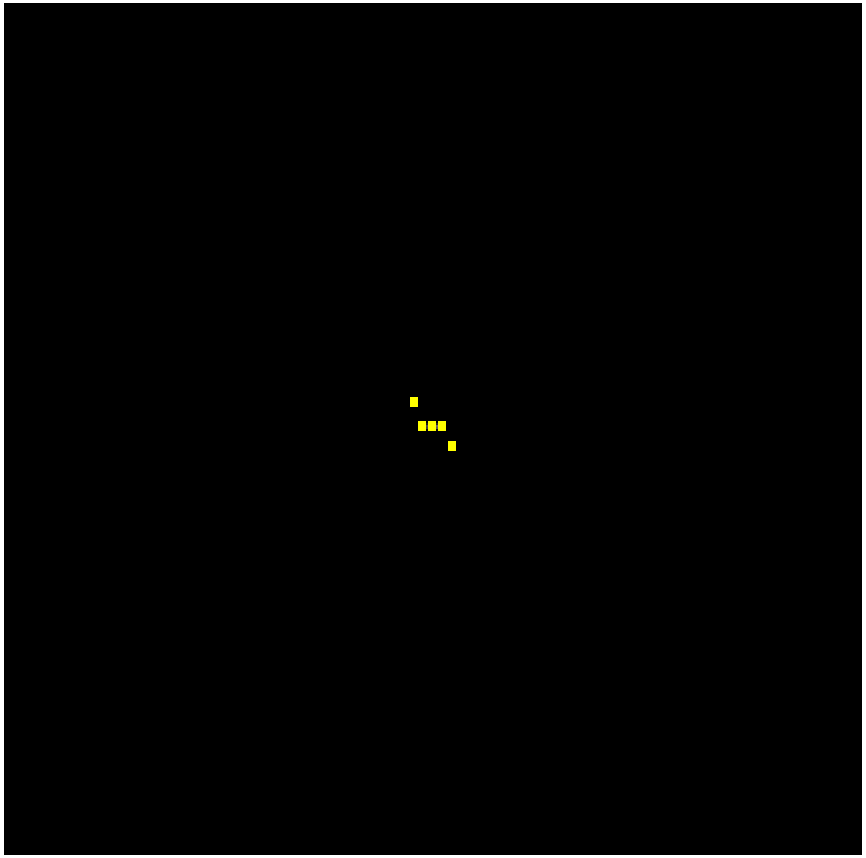


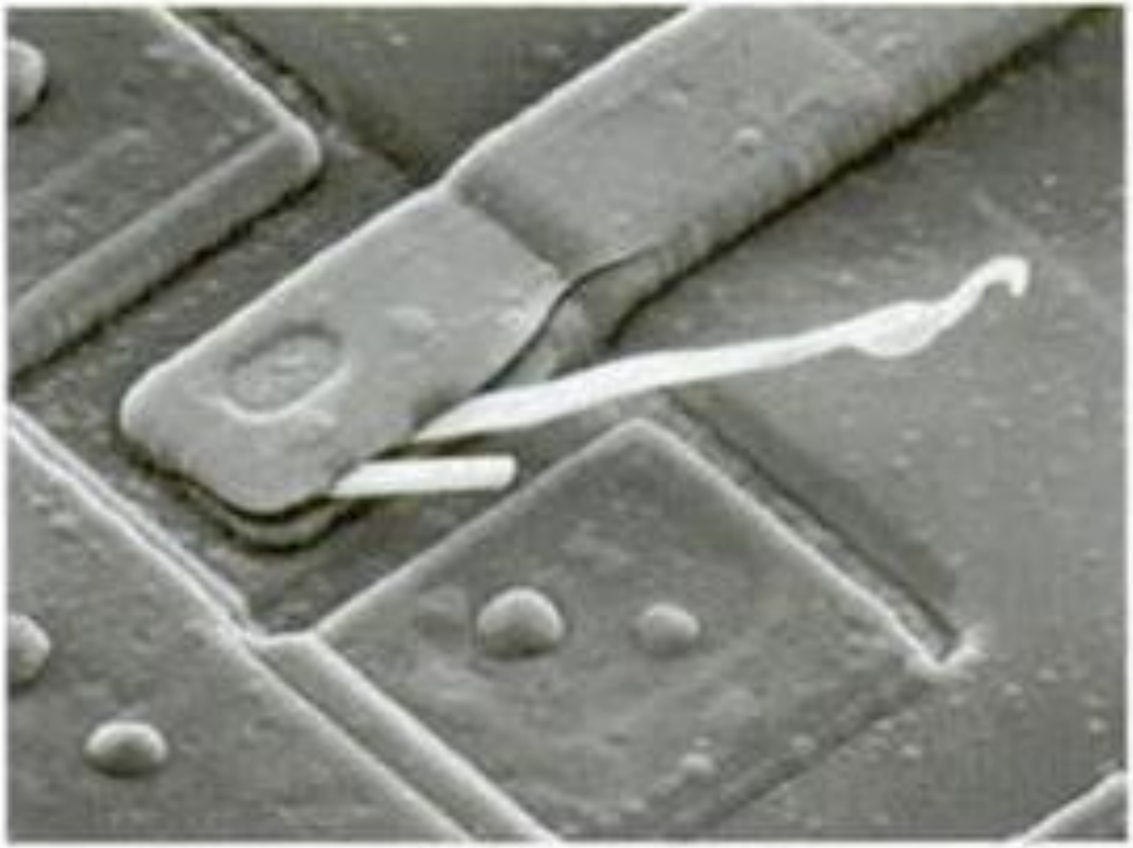
DFT



Extending previous concept of adding sine waves of higher and higher frequency
➔eventually end up with a horizontal line across middle of the FFT image and one bright vertical band through the middle of the original image as mostly all lobs are cancelled out.







What is expected DFT?

Frequency domain filtering fundamentals:

$$g(x, y) = \Im^{-1}[H(u, v)F(u, v)]$$

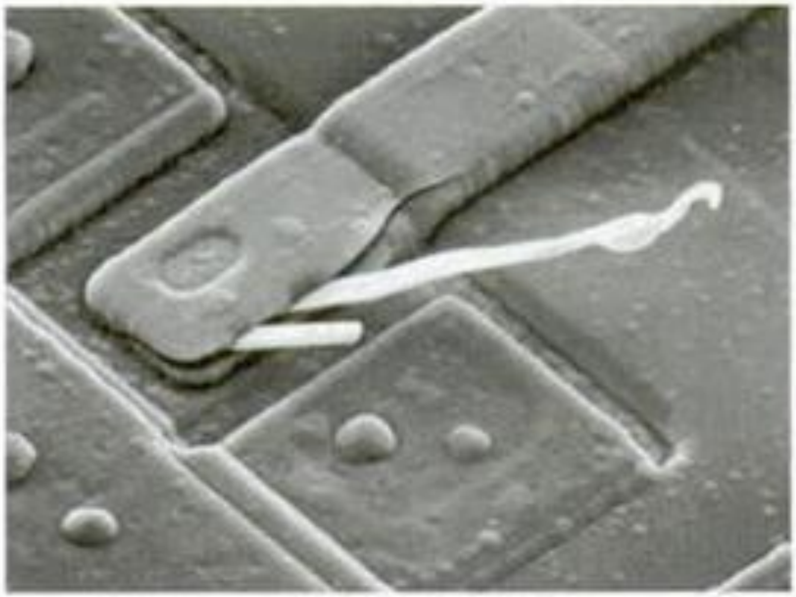
- H is DFT of filter function.
- F is DFT of input image
- f, g, H, F all are of dimension MxN

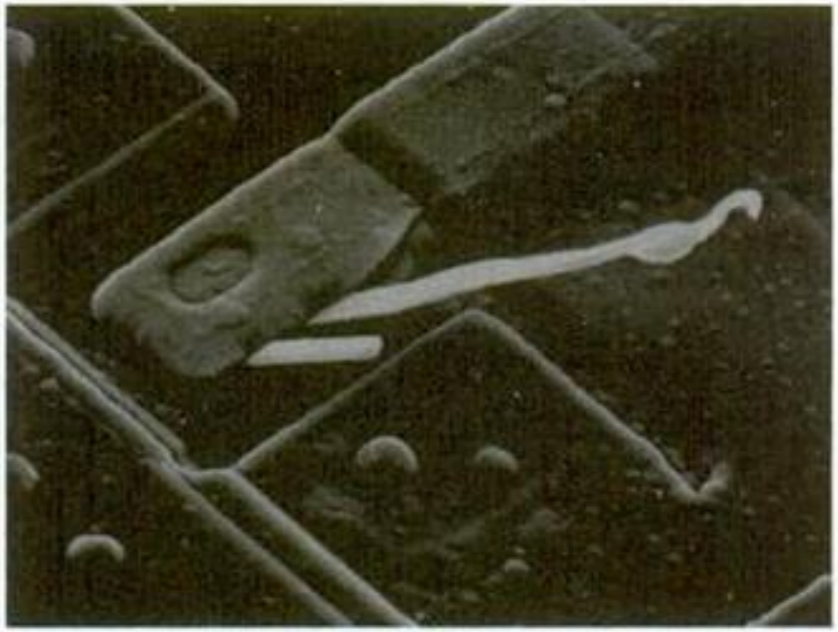
Simplest filtering

$$\boxed{\text{DFT}} \quad \text{X} \quad \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \cdot\, 0\, \cdot & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

F(u,v) H(u,v)

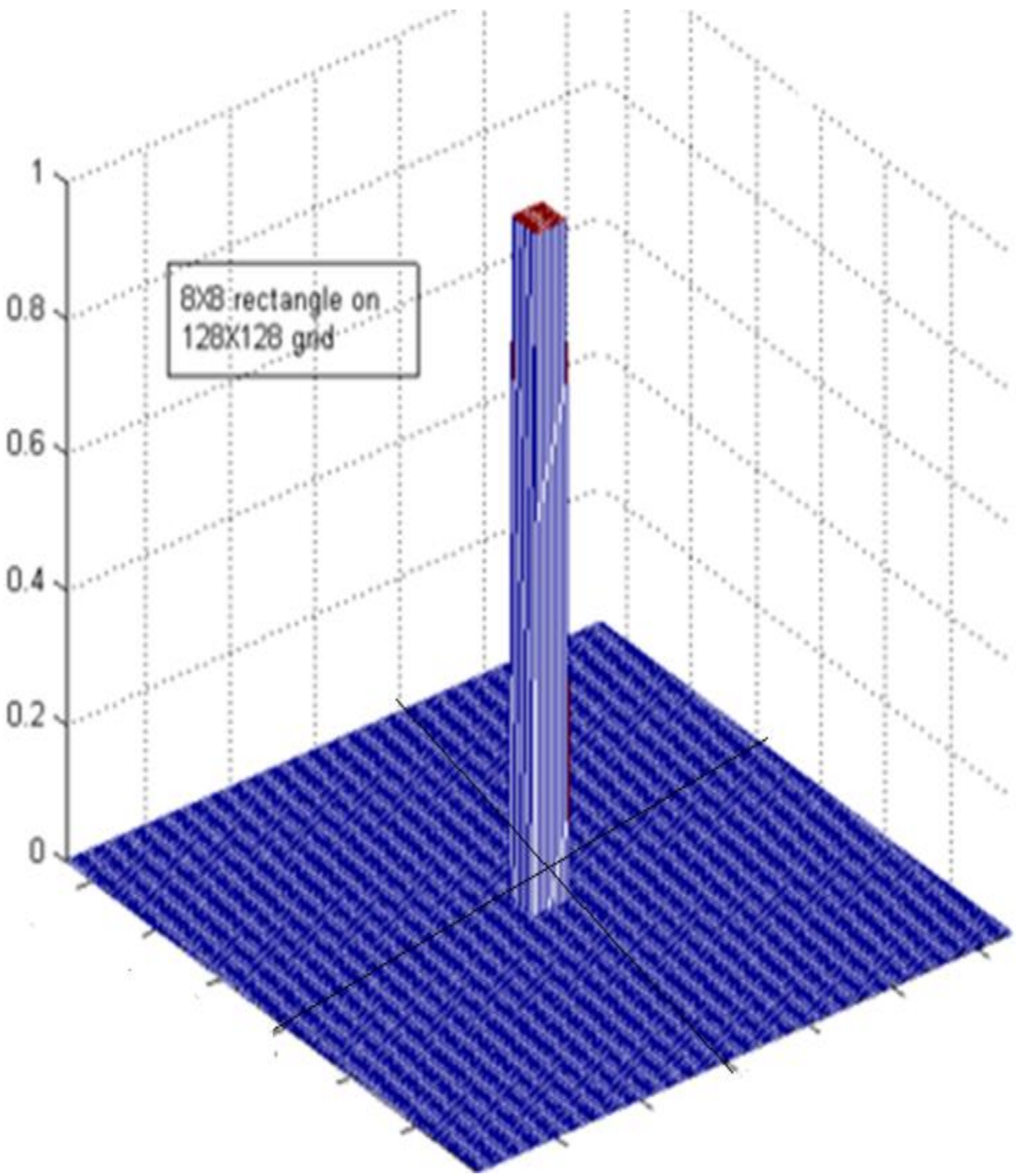
Expected result?

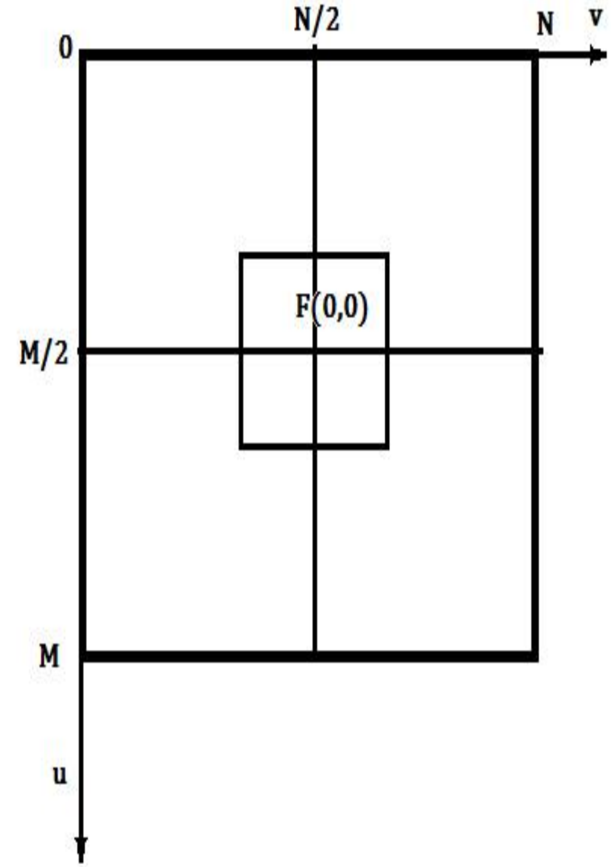


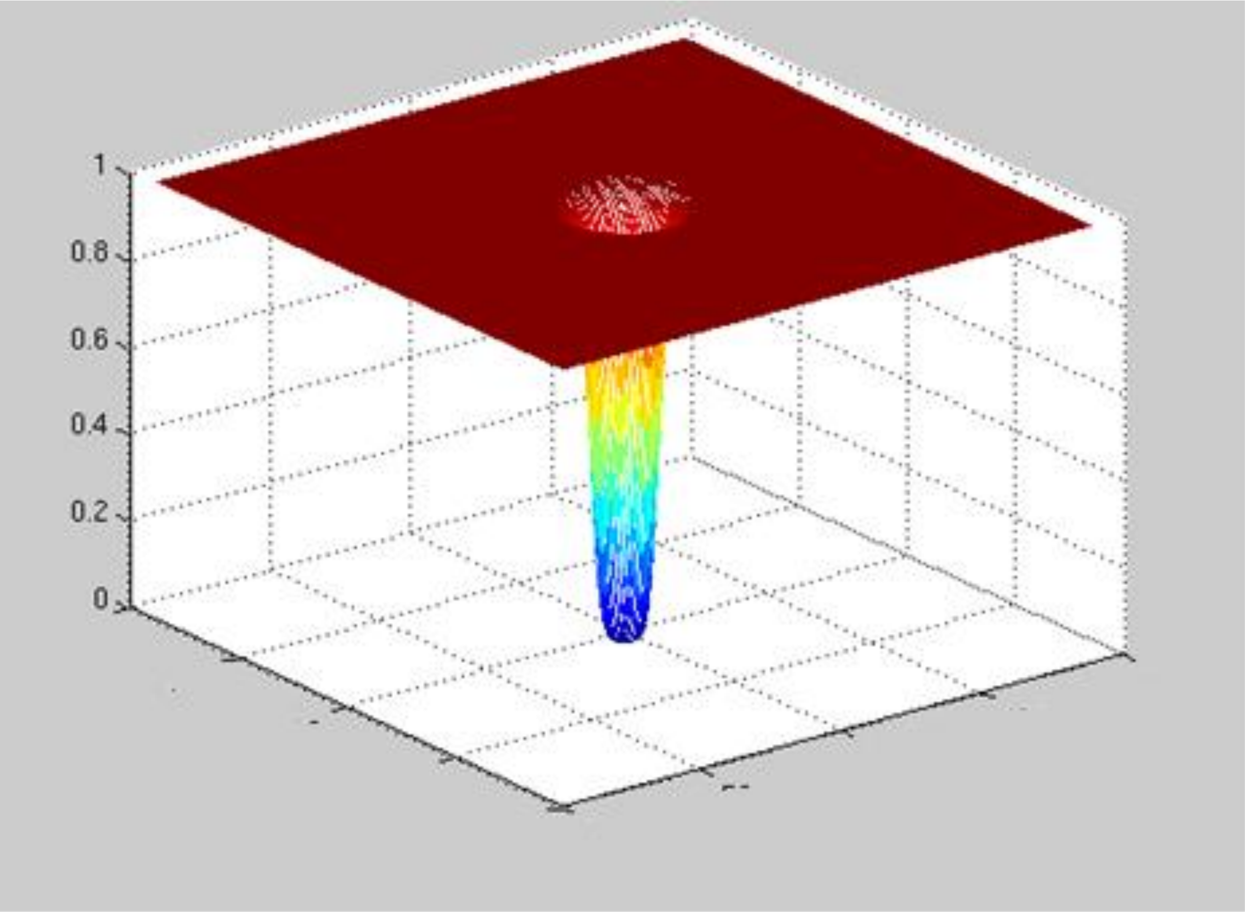


As expected : DC term is rejected i.e image average intensity is made 0
p.s: Negative intensities are cropped to 0 so not a true representation of image.

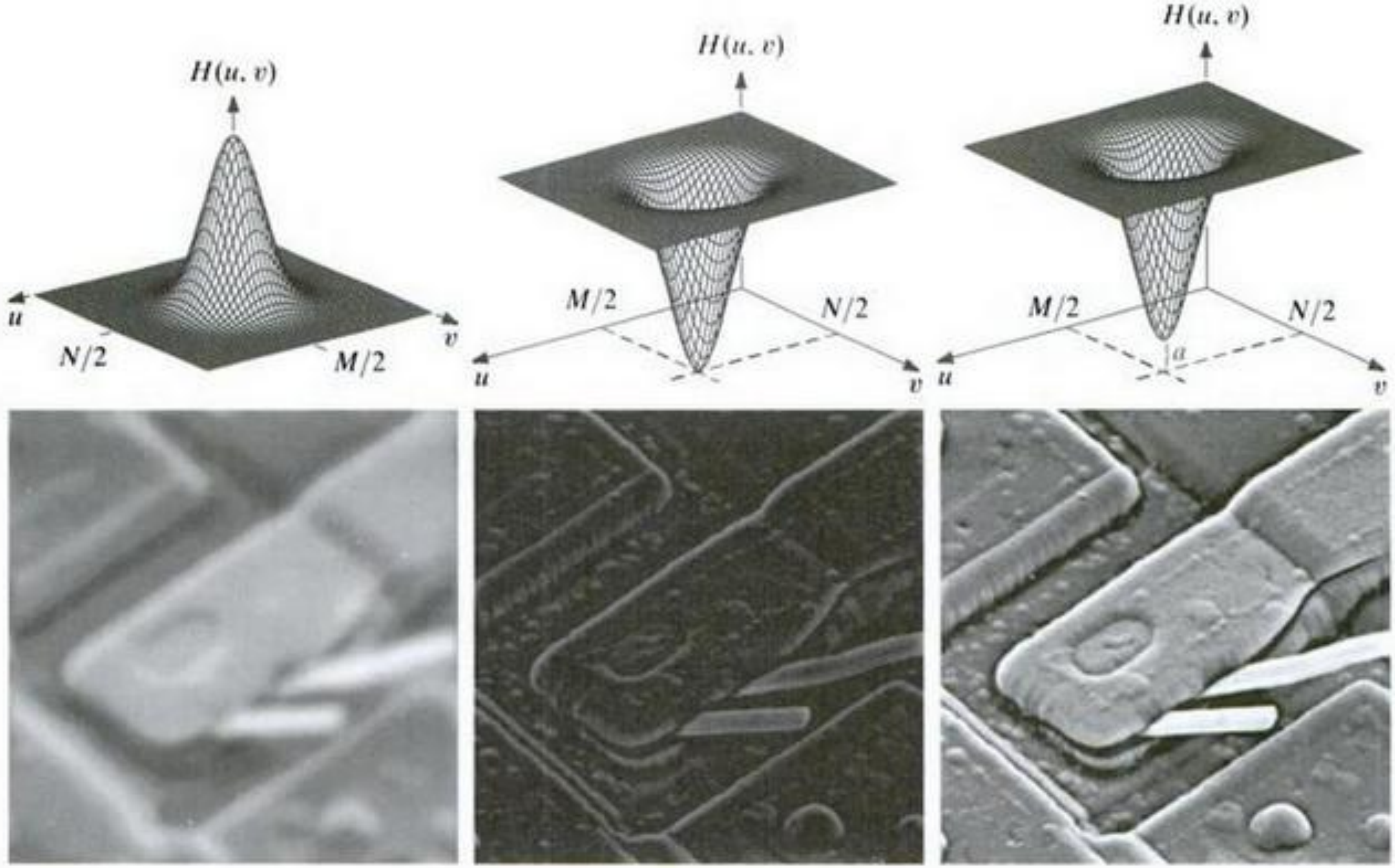
- Low frequencies – less changes : wall, sky
- High frequencies- sharp edges
- Pass low frequencies- attenuate edges- blurring
- High pass filter-sharp edges only but Dark images? Why?
- Adding a small constant will preserve tonality


1
0.8
0.6
0.4
0.2
0
8X8 rectangle on
128X128 grid


N/2
N
v
0
F(0,0)
M/2
M
u



1
0.8
0.6
0.4
0.2
0



a b c
d e f

FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

4.7.3 Summary of Steps for Filtering in the Frequency Domain

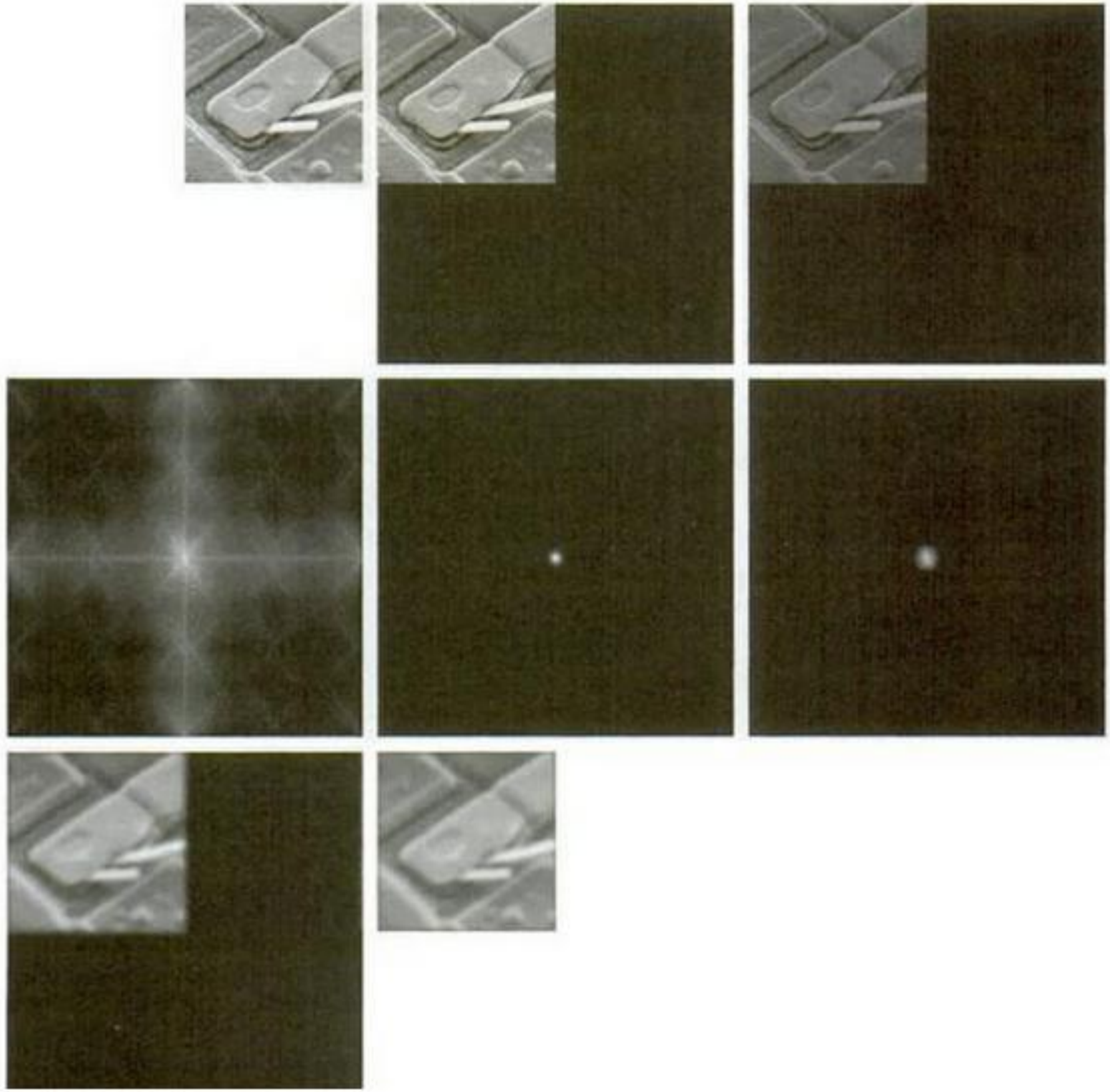
The material in the previous two sections can be summarized as follows:

1. Given an input image $f(x, y)$ of size $M \times N$, obtain the padding parameters P and Q from Eqs. (4.6-31) and (4.6-32). Typically, we select $P = 2M$ and $Q = 2N$.
2. Form a padded image, $f_p(x, y)$, of size $P \times Q$ by appending the necessary number of zeros to $f(x, y)$.
3. Multiply $f_p(x, y)$ by $(-1)^{x+y}$ to center its transform.
4. Compute the DFT, $F(u, v)$, of the image from step 3.
5. Generate a real, symmetric filter function, $H(u, v)$, of size $P \times Q$ with center at coordinates $(P/2, Q/2)$.† Form the product $G(u, v) = H(u, v)F(u, v)$ using array multiplication; that is, $G(i, k) = H(i, k)F(i, k)$.
6. Obtain the processed image:

$$g_p(x, y) = \left\{\text{real}\left[\Im^{-1}[G(u, v)]\right]\right\}(-1)^{x+y}$$

   where the real part is selected in order to ignore parasitic complex components resulting from computational inaccuracies, and the subscript p indicates that we are dealing with padded arrays.
7. Obtain the final processed result, $g(x, y)$, by extracting the $M \times N$ region from the top, left quadrant of $g_p(x, y)$.

As noted earlier, centering helps in visualizing the filtering process and in generating the filter functions themselves, but centering is not a fundamental requirement.



a b c
d e f
g h

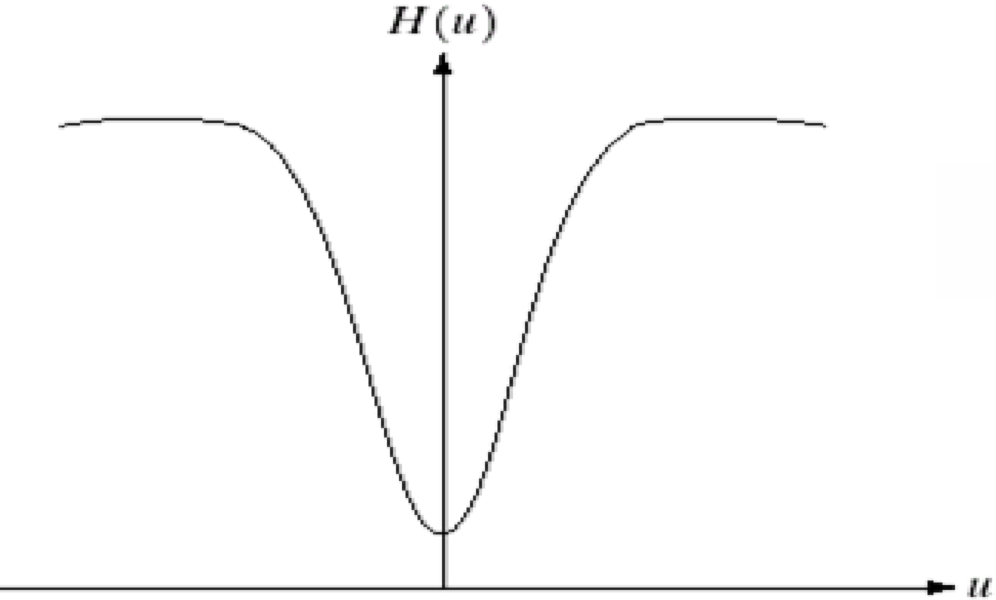
FIGURE 4.36
(a) An $M \times N$ image, f.
(b) Padded image, f_p of size $P \times Q$.
(c) Result of multiplying f_p by $(-1)^{x+y}$.
(d) Spectrum of F_p. (e) Centered Gaussian lowpass filter, H, of size $P \times Q$.
(f) Spectrum of the product HF_p.
(g) g_p, the product of $(-1)^{x+y}$ and the real part of the IDFT of HF_p.
(h) Final result, g, obtained by cropping the first M rows and N columns of g_p.

Correspondence between filtering in spatial domain and in frequency domain :

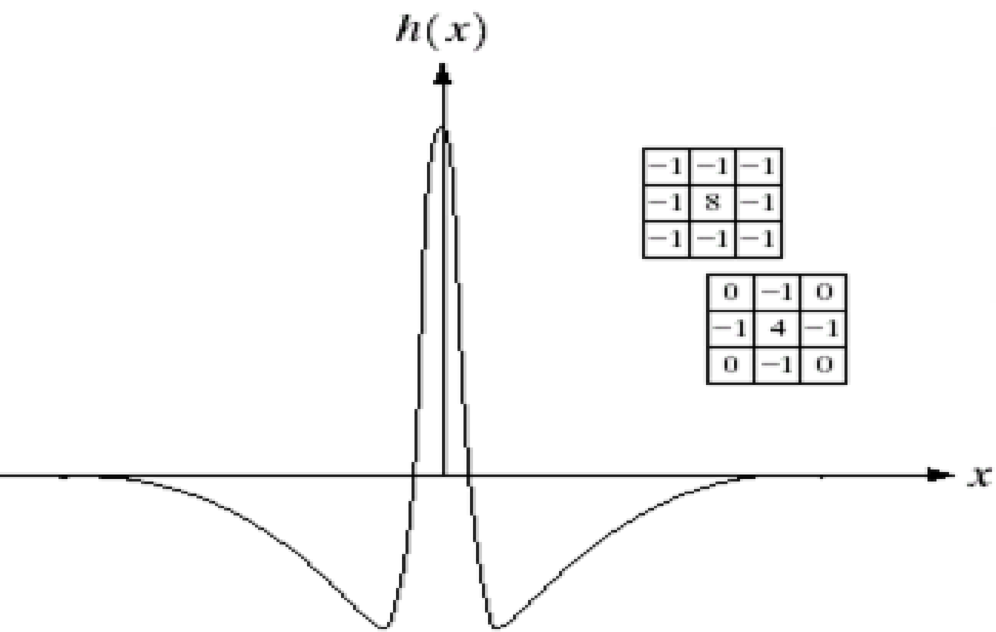
- Implementation of filtering is usually done in spatial domain using small mask.
- Concepts are more intuitive in Frequency domain.
- How to take advantage of both?

Correspondence between filtering in spatial domain and in frequency domain :

- Specify filter in frequency domain
- Take IDFT
- Use that IDFT as guide to specify small masks
- Implement filtering in spatial domain



$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$



−1	−1	−1
−1	8	−1
−1	−1	−1

0	−1	0
−1	4	−1
0	−1	0

$$h(x) = \sqrt{2\pi}\sigma_1 Ae^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma_2 Be^{-2\pi^2\sigma_2^2x^2}$$

Image smoothing using Frequency domain filter:

- Low pass filtering:

  Passes low frequency, suppresses high frequency

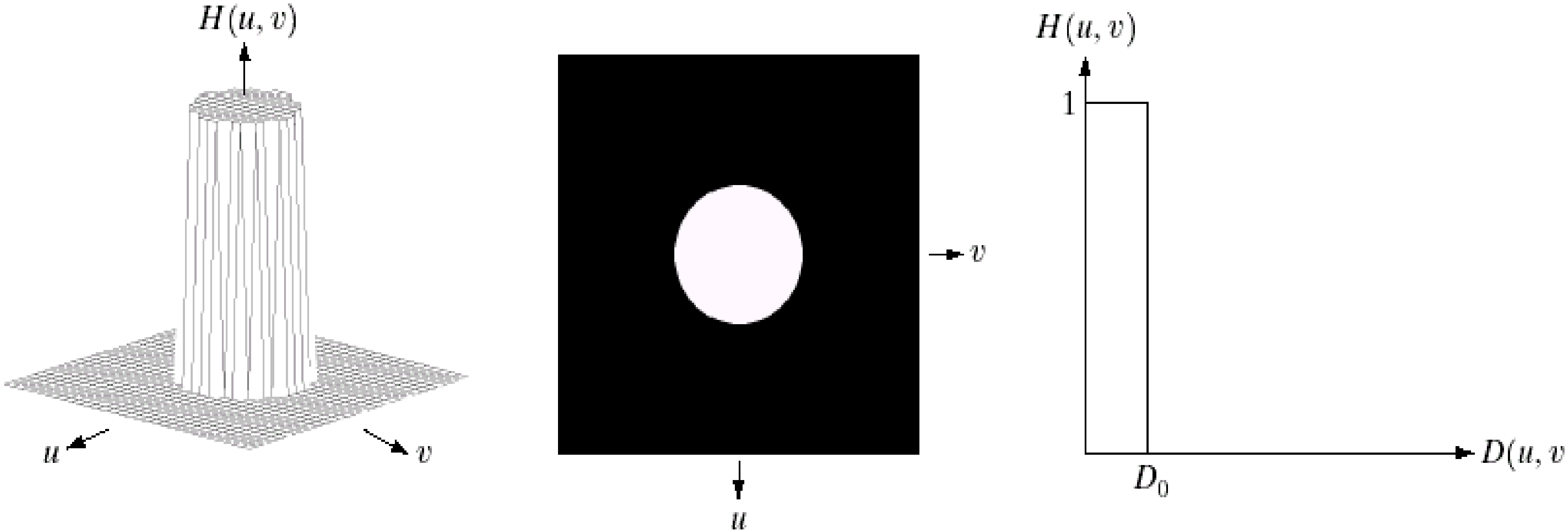
  - Ideal(very sharp cut off)
  - Gaussian(Very smooth cut off)
  - Butterworth(Transition between two extremes)

Low pass filtering:

A 2-D lowpass filter that passes without attenuation all frequencies within a circle of radius D_0 from the origin and "cuts off" all frequencies outside this circle is called an *ideal lowpass filter* (ILPF)

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \left[(u - M/2)^2 + (v - N/2)^2\right]^{1/2}$$



a b c

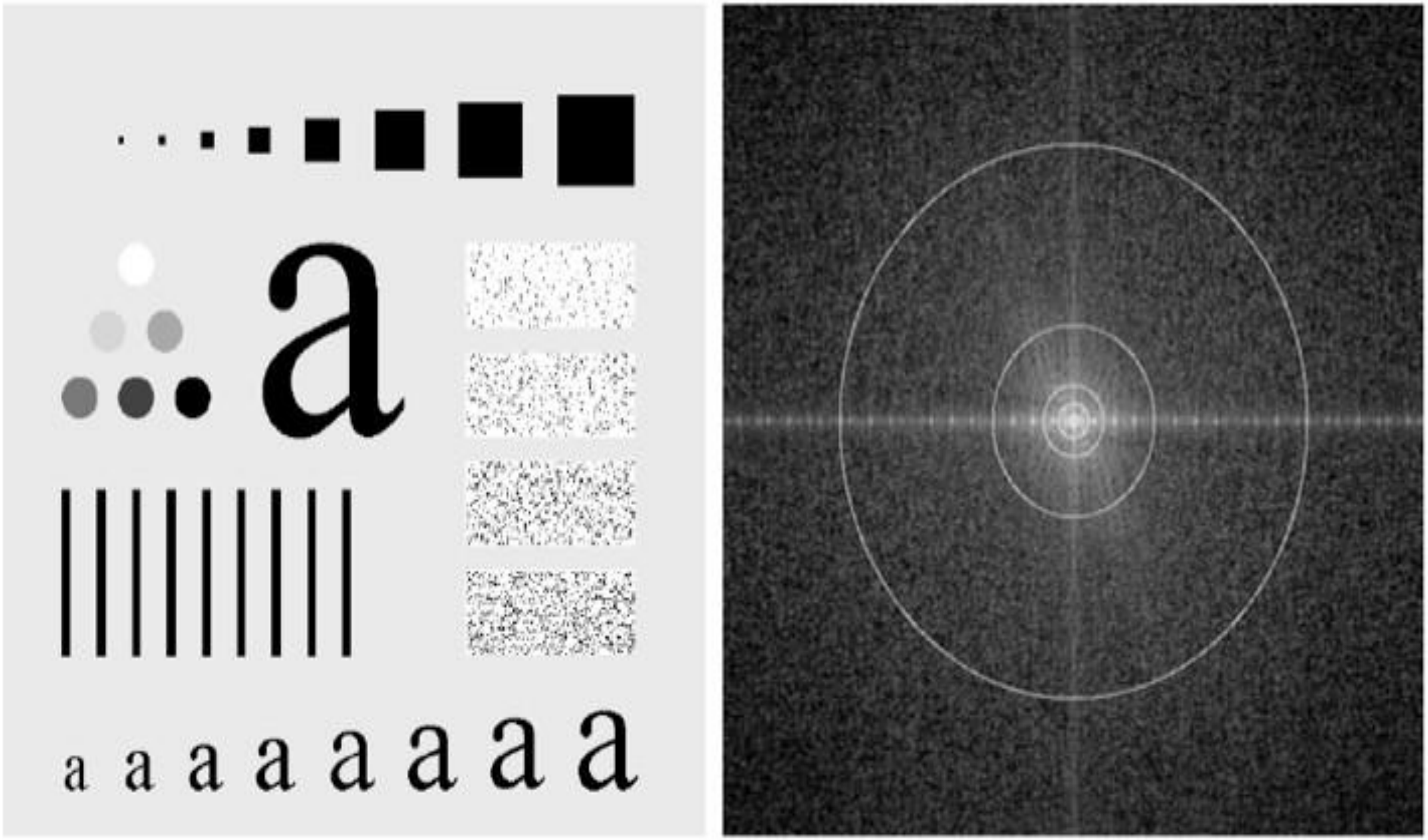
FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Total Power $\longrightarrow$ $$P_T = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\left|F(u,v)\right|^2$$

The remained percentage power after filtration

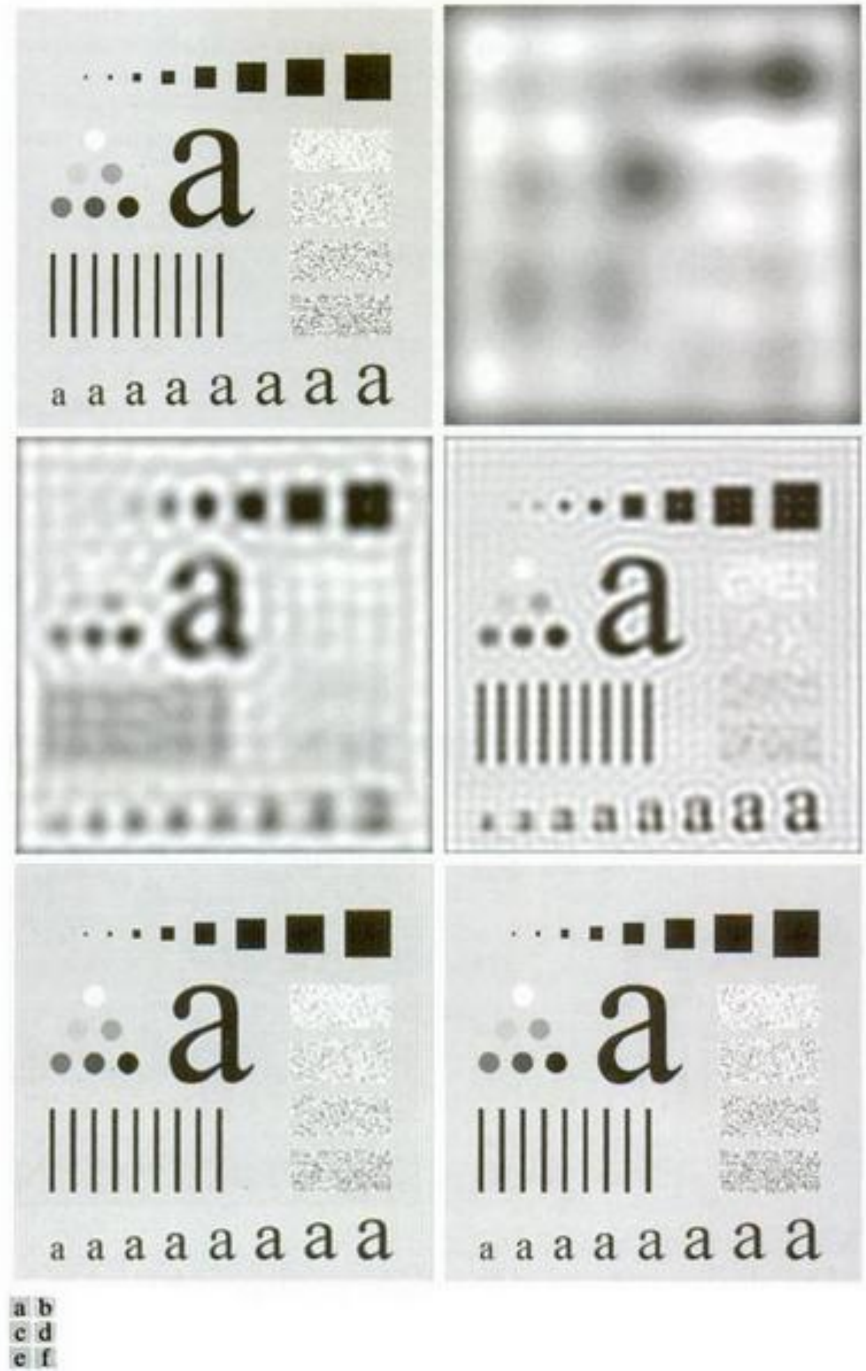
$$\alpha = 100\left[\sum_u \sum_v P(u,v)/P_T\right]$$

and the summation is taken over values of (u, v) that lie inside the circle or on its boundary.



a b

FIGURE 4.41 (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



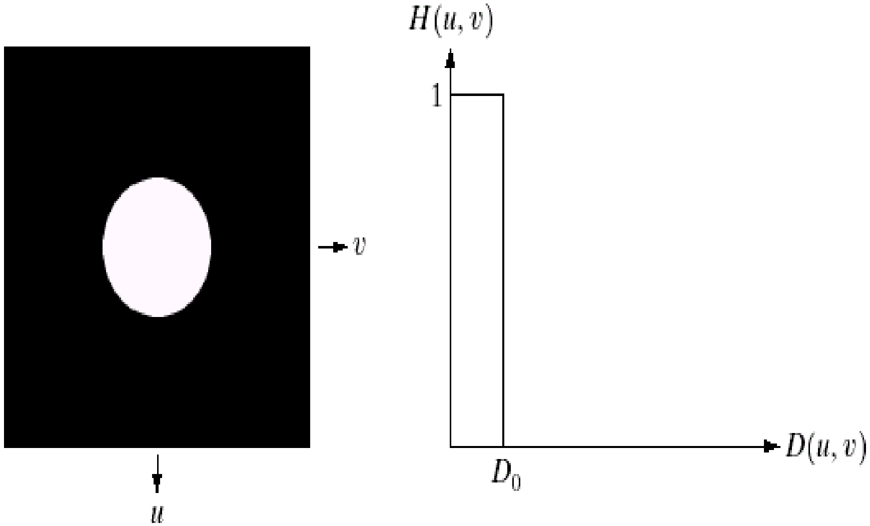
$D_0 = 10$
$\alpha = 87\%$

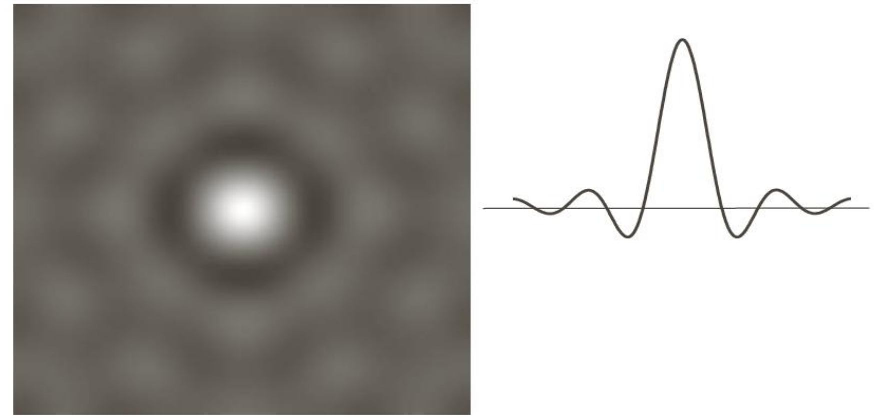
$D_0 = 30$
$\alpha = 93.1\%$

$D_0 = 60$
$\alpha = 95.7\%$

$D_0 = 160$
$\alpha = 97.8\%$

$D_0 = 460$
$\alpha = 99.2\%$





the "spread" of the sinc function is inversely proportional to the radius of $H(u, v)$, the larger D_0 becomes, the more the spatial sinc approaches an impulse which, in the limit, causes no blurring at all when convolved with the image.'

Butterworth filter:

Transfer function of a Butterworth lowpass filter (BLPF) of order *n* and with cutoff frequency at a distance D_0 from the origin:

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$

$$D(u,v) = \left[(u - M/2)^2 + (v - N/2)^2\right]^{1/2}$$

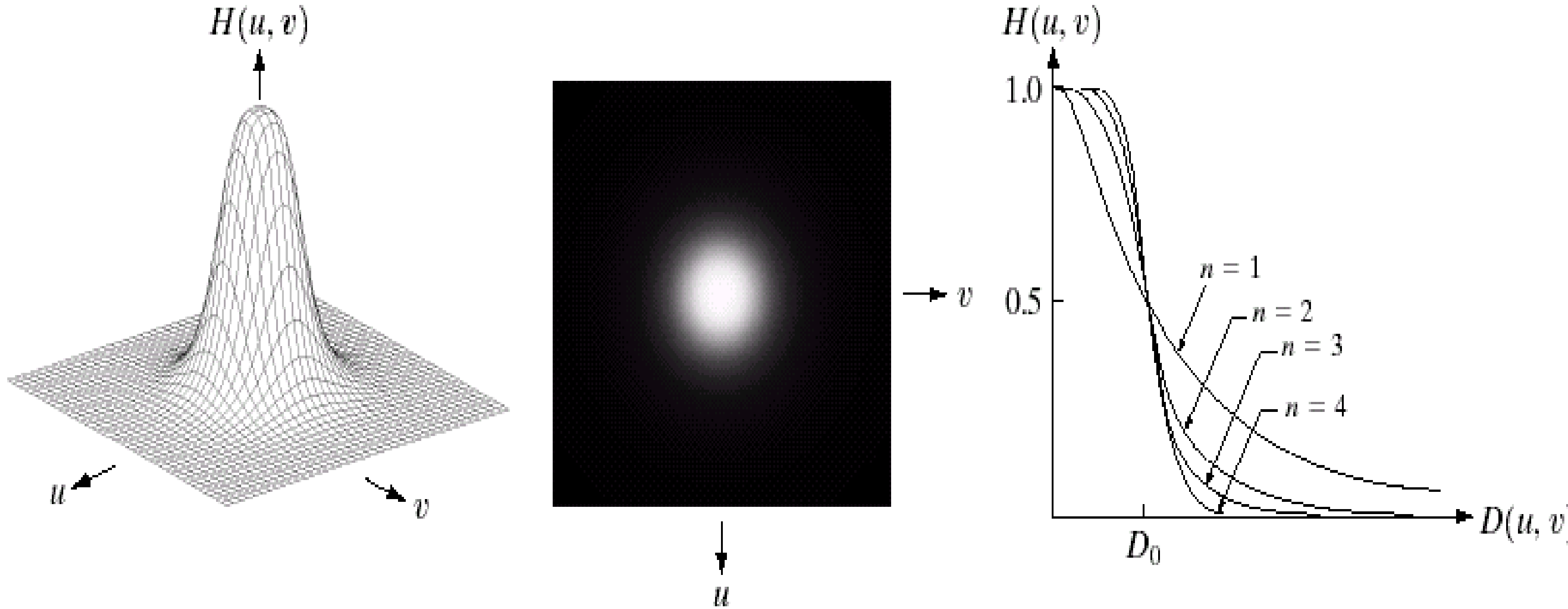
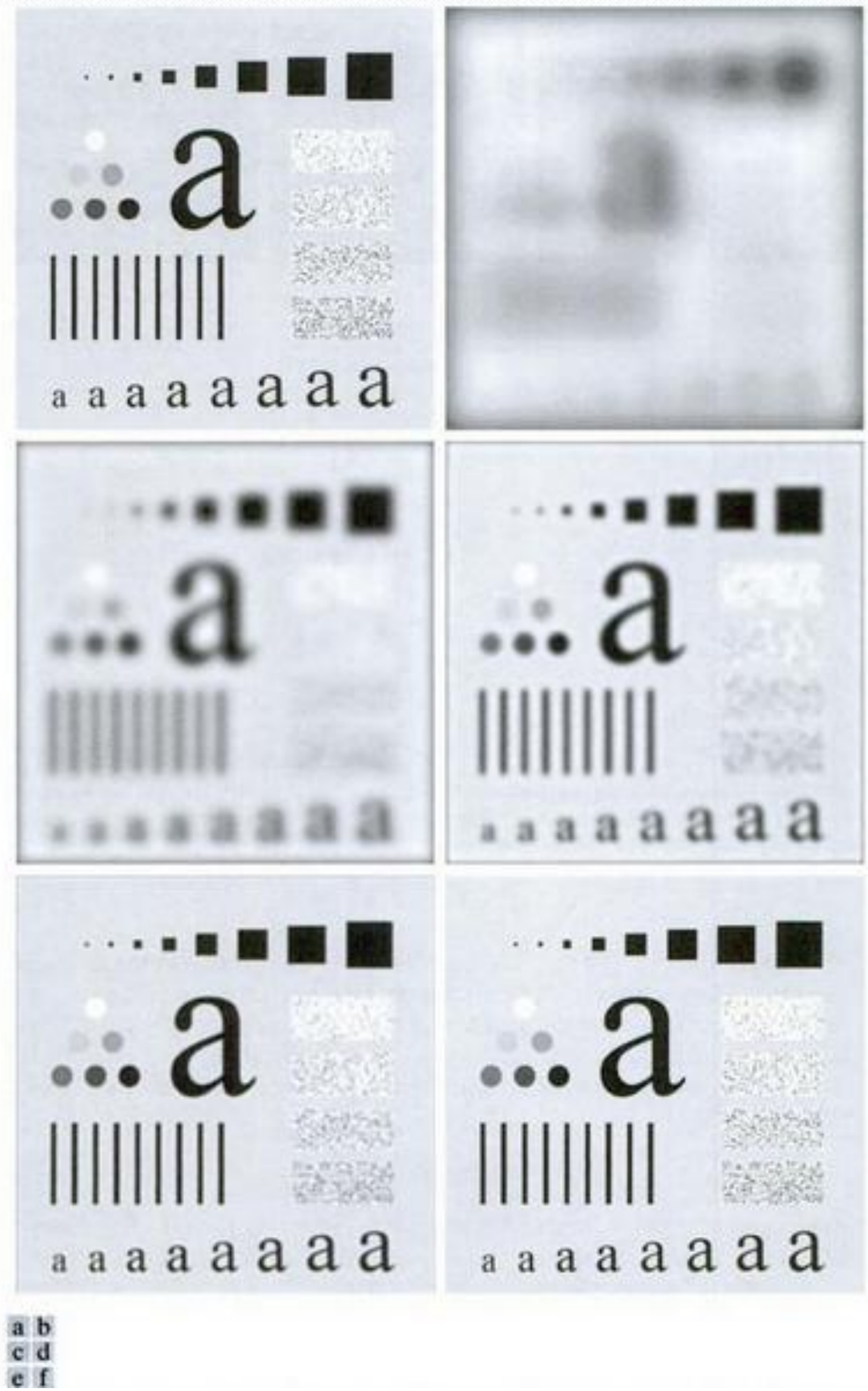


FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

n=2

D_0 =10



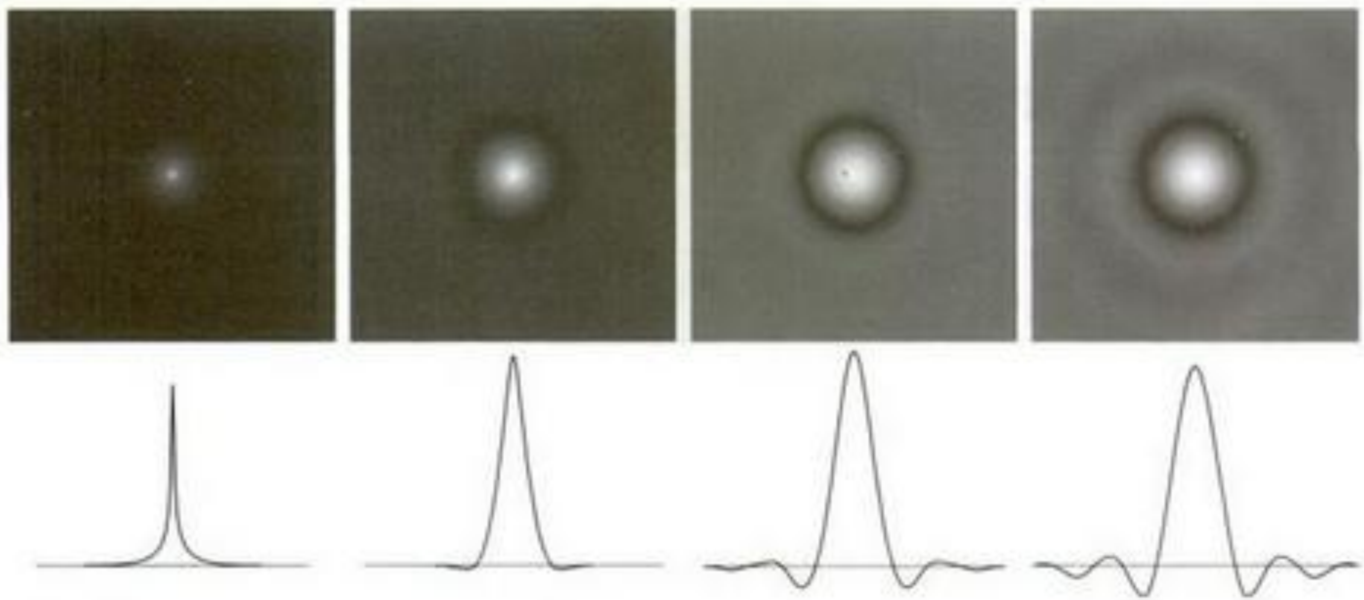
D_0 =30

D_0 =60

D_0 =160

D_0 =460

Ringing effect in butter worth filter



a b c d

FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000 × 1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

Gaussian Low Pass Filters

$$H(u,v) = e^{-\frac{D^2(u,v)}{2\sigma^2}} \xrightarrow{\sigma = D_0} H(u,v) = e^{-\frac{D^2(u,v)}{2D_0^2}}$$

$$D^2(u,v) = \left(u - \tfrac{M}{2}\right)^2 + \left(v - \tfrac{N}{2}\right)^2$$

$$H(D(u,v) = 0) = 1$$

$$H(D(u,v) = D_0) = e^{-0.5} = 0.607$$

Inverse Fourier transform of GLPF is also Gaussian

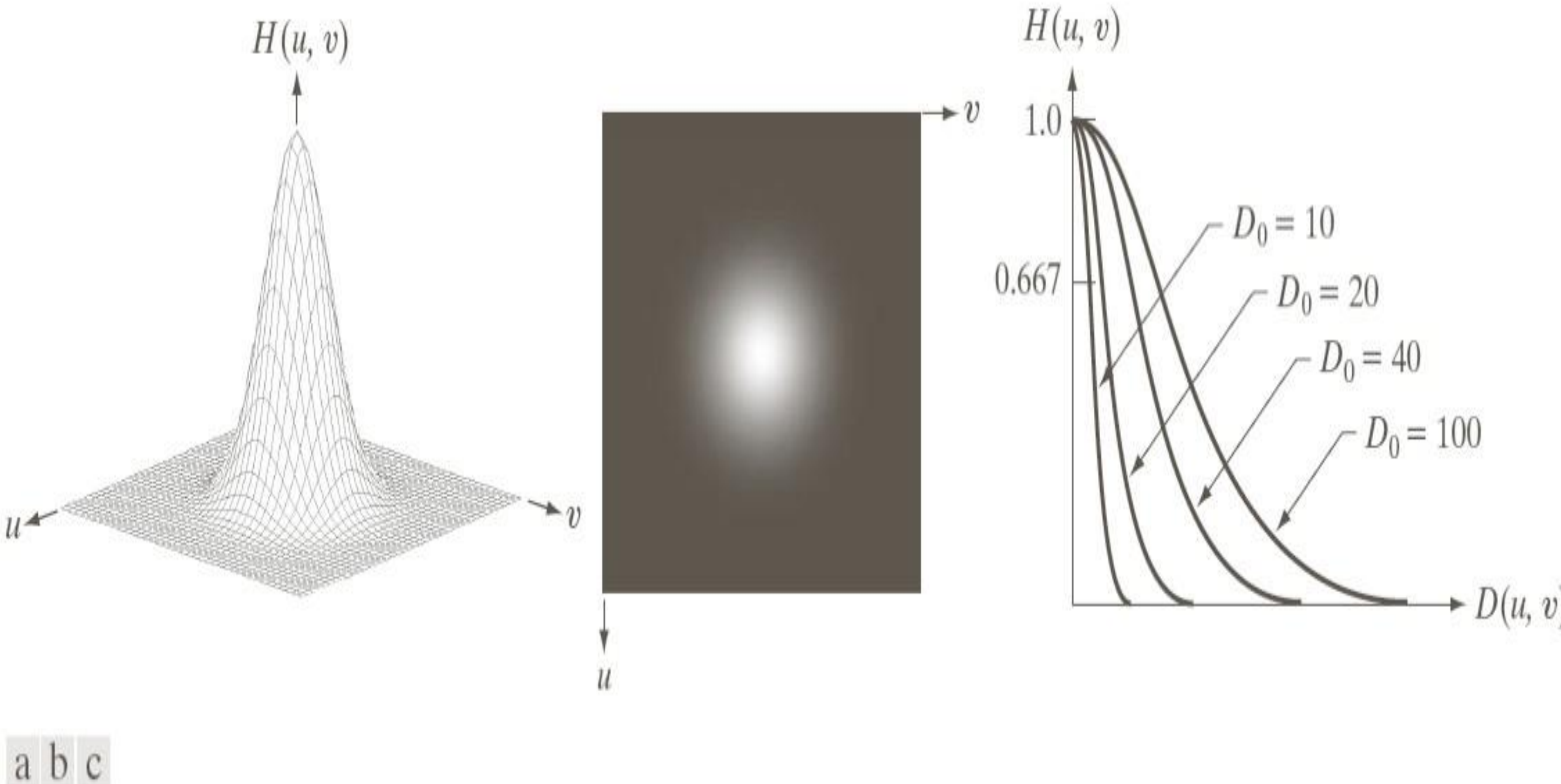
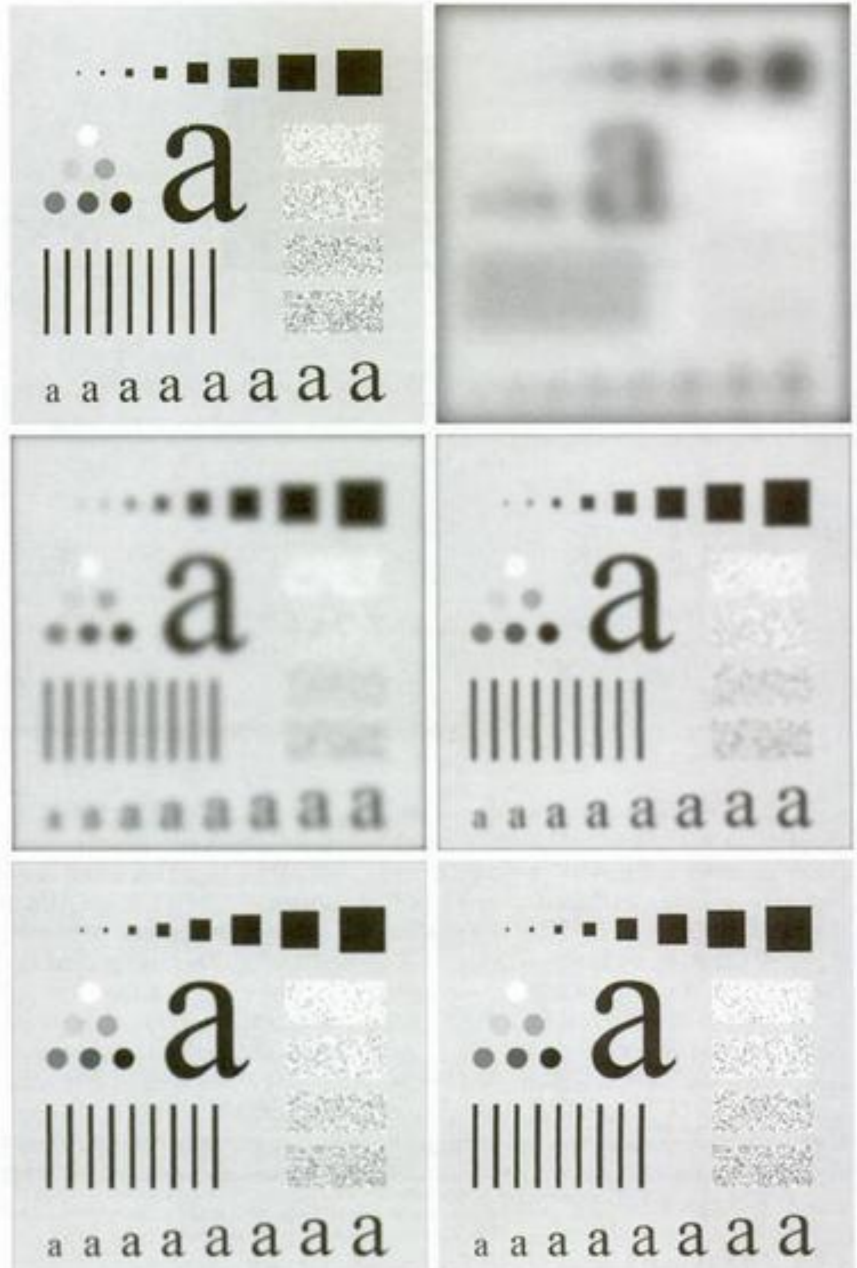


FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.



$D_0 = 10$

$D_0 = 30$

$D_0 = 60$

$D_0 = 160$

$D_0 = 460$

Comparing result for D0=30

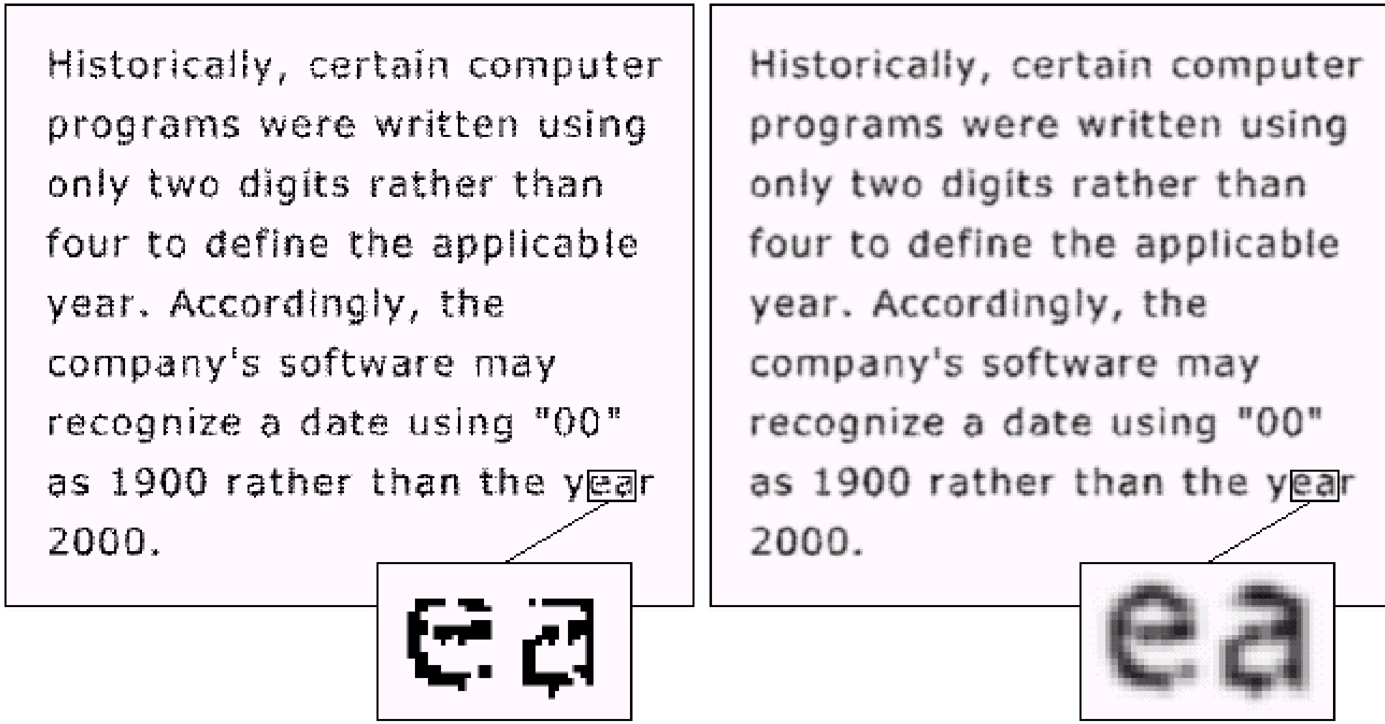
ILPF *BLPF* *GLPF*

Decreasing order of tightness, Decreasing blurring

Additional examples of LPF

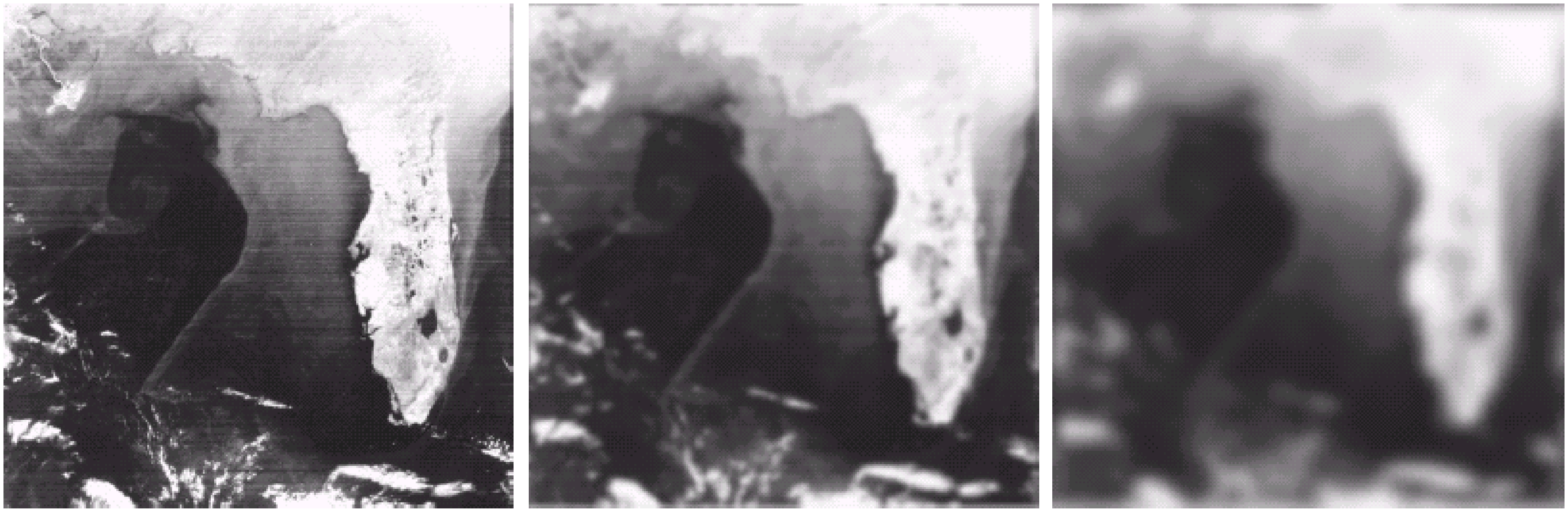
a b

FIGURE 4.19
(a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).



a b c

FIGURE 4.50 (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).



a b c

FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)

High pass filtering:

$$H_{\mathrm{HP}}(u, v) = 1 - H_{\mathrm{LP}}(u, v)$$

Ideal Highpass Filters

A 2-D *Ideal Highpass Filter* (IHPF) is defined as:

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \leq D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

Ideal Highpass Filter:

Butterworth Highpass Filter:

Gaussian Highpass Filter:

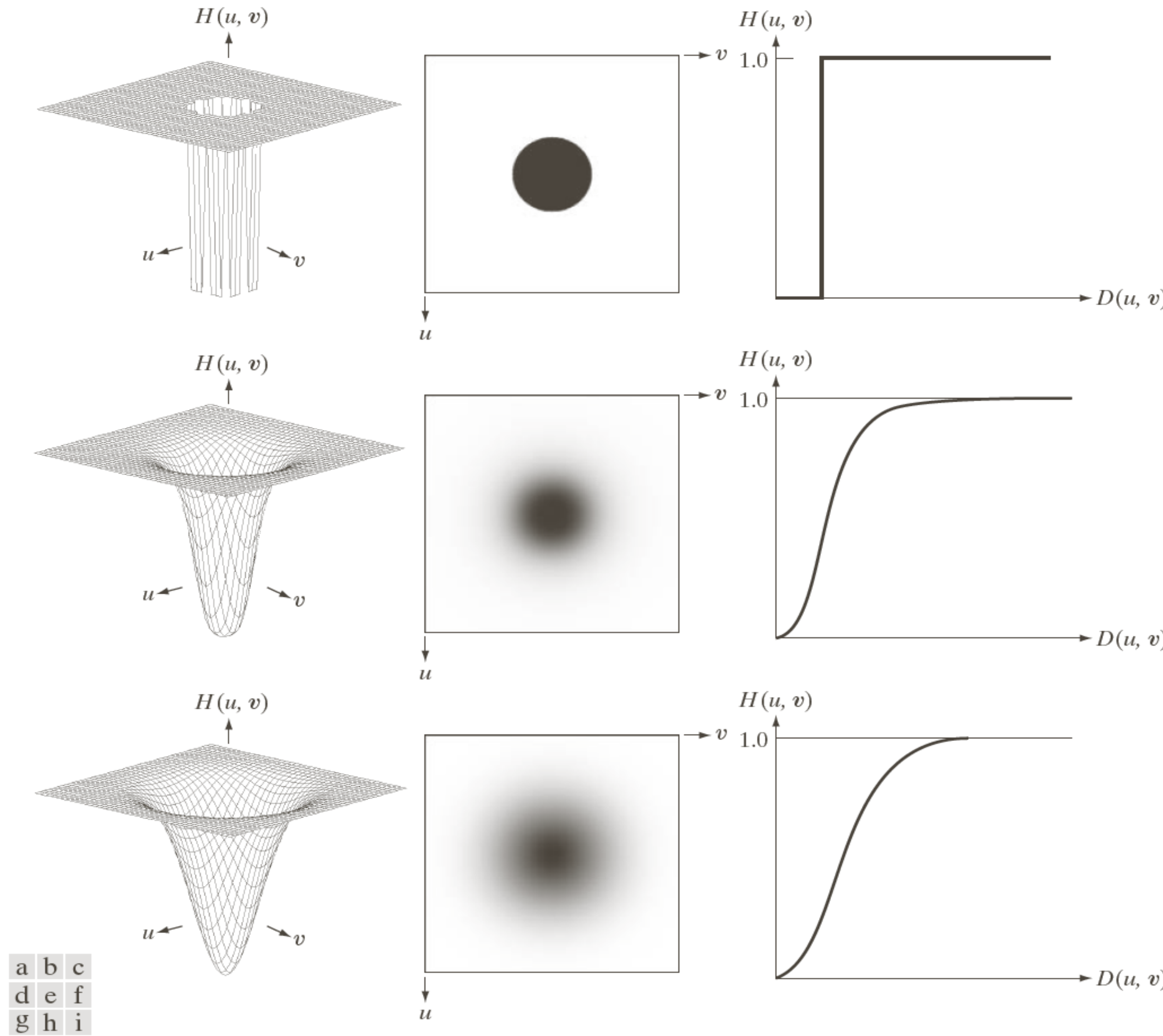
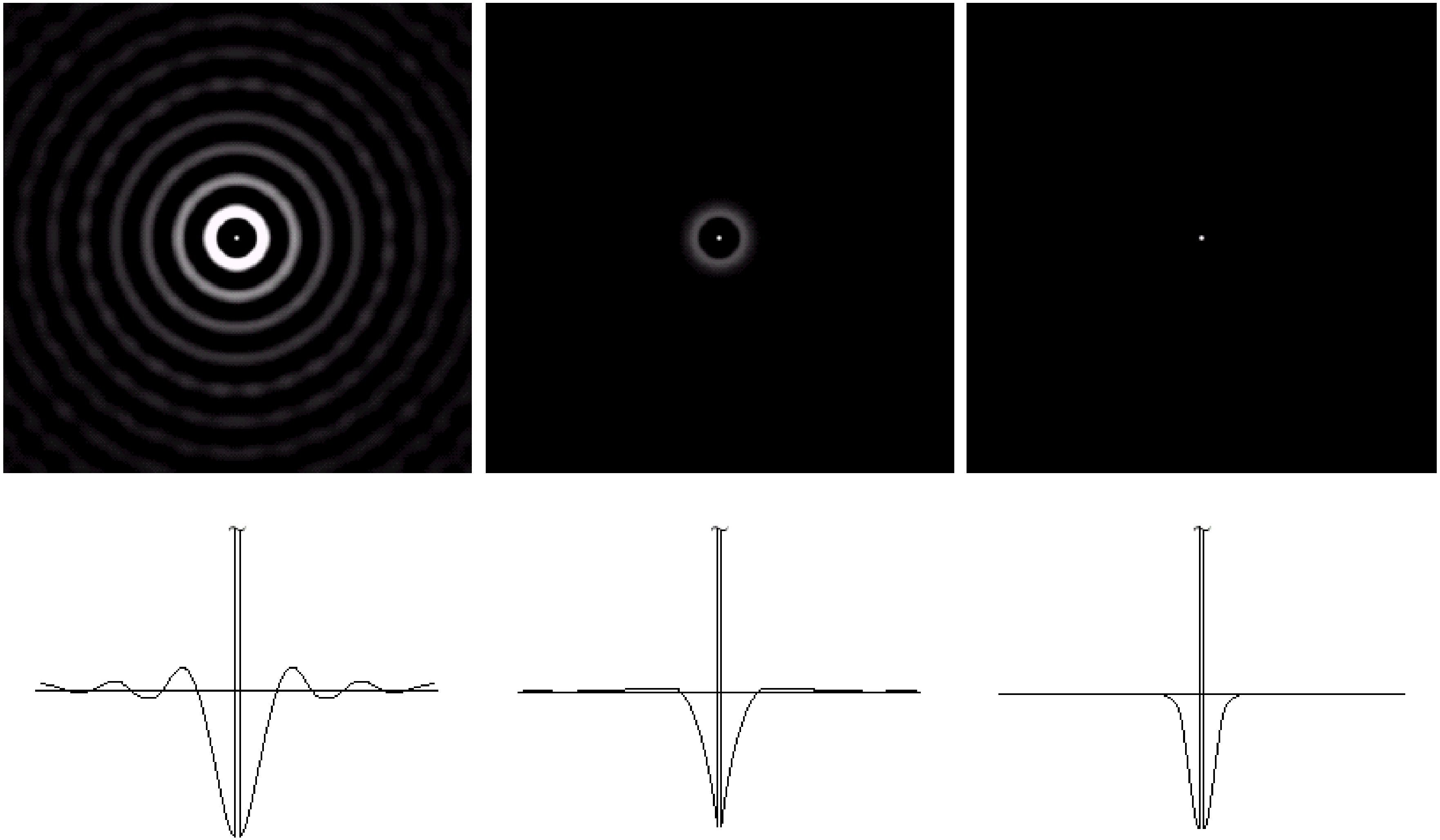
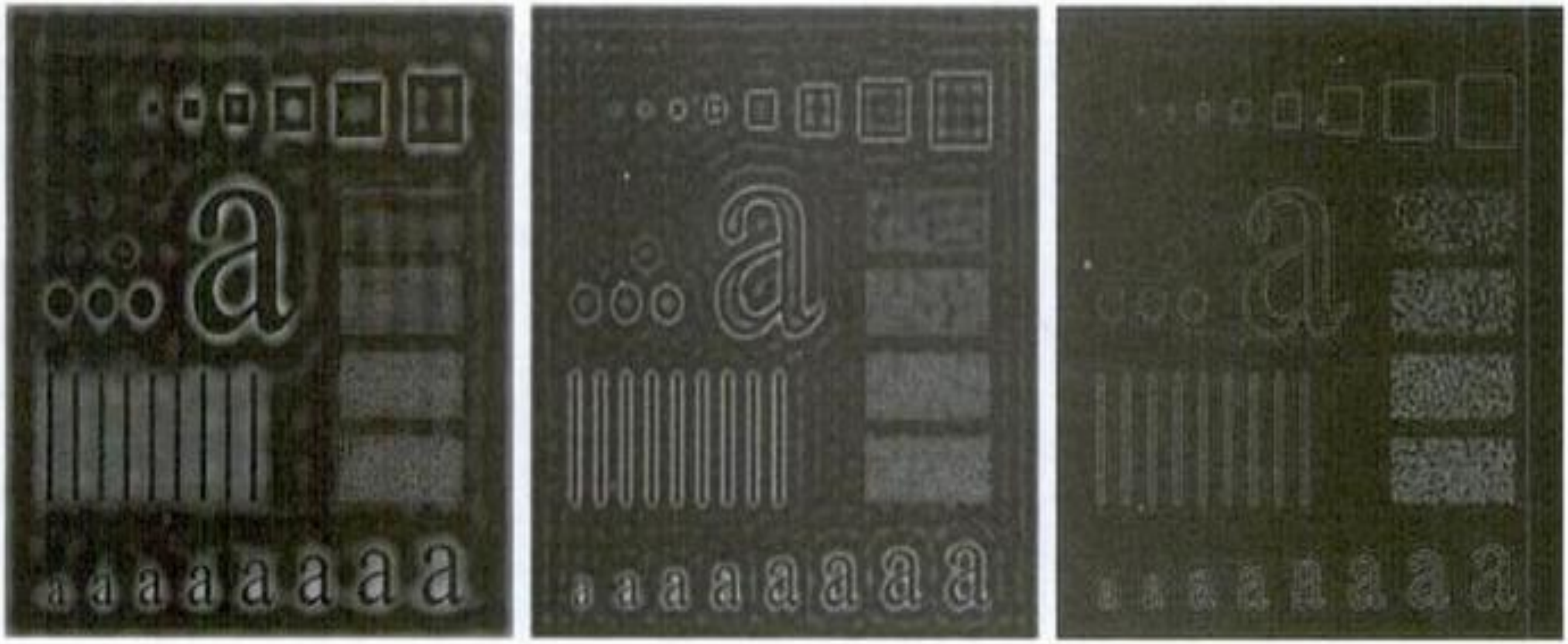


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.



a b c

FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.



- Top 3 circles → no strong edges so less frequency and not passed.
- Impact of convolution in spatial domain on smaller objects.

BHPF:

- A 2-D *Butterworth Highpass Filter* (BHPF) of order *n* and cutoff frequency D_0 is defined as:

$$H(u,v) = \frac{1}{1 + [D_0 / D(u,v)]^{2n}}$$

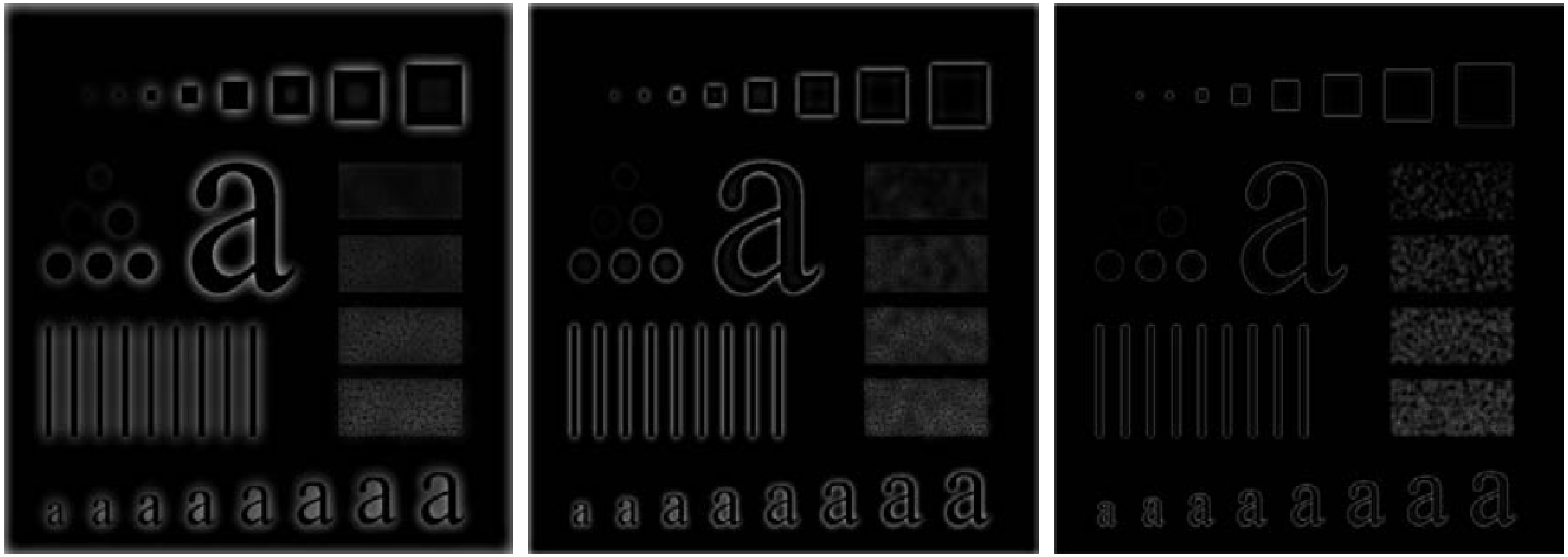
$$H_{HP} = 1 - \frac{1}{\left(1 + \left(\frac{D(u,v)}{D_0}\right)^{2n}\right)}$$

$$H_{HP} = \frac{1 + \left(\frac{D(u,v)}{D_0}\right)^{2n} - 1}{\left(1 + \left(\frac{D(u,v)}{D_0}\right)^{2n}\right)}$$

$$H_{HP} = \frac{\left(\frac{D(u,v)}{D_0}\right)^{2n}}{\left(1 + \left(\frac{D(u,v)}{D_0}\right)^{2n}\right)}$$

$$H_{HP} = \frac{\left(\frac{D(u,v)}{D_0}\right)^{2n}}{\left(\frac{D(u,v)}{D_0}\right)^{2n}\left(\frac{1}{\left(\frac{D(u,v)}{D_0}\right)^{2n}} + 1\right)}$$

$$H_{HP} = \frac{1}{\left(\left(\frac{D_0}{D(u,v)}\right)^{2n} + 1\right)}$$



a b c

FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

- Less/no ringing
- Impact of convolution in spatial domain on smaller objects.c

GHPF:

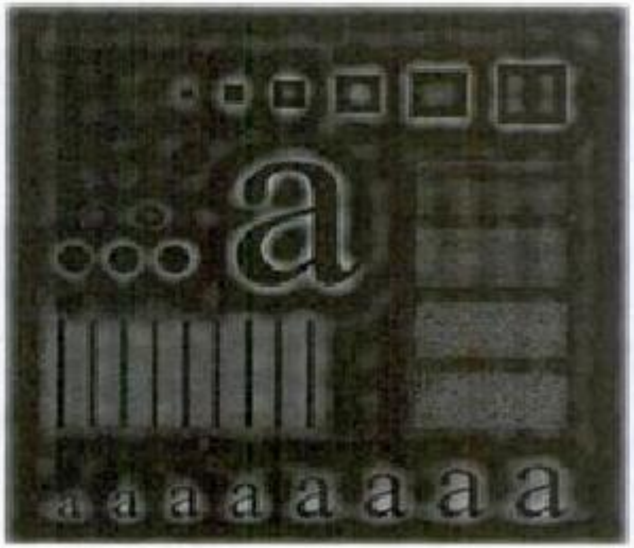
- The transfer function of the *Gaussian Highpass Filter* (GHPF) with cutoff frequency locus at a distance D_0 from the centre of the frequency rectangle is defined as:

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

a b c

FIGURE 4.56 Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30$, 60, and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

Comparison of high pass filters D0=30



IHPF

BHPF

GHPF

- Thin and smaller objects are clearer➔

Additional examples of HPF

a b c

FIGURE 4.57 (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)

Laplacian in frequency domain:

$$H(u,v) = -4\pi^2(u^2 + v^2)$$

Or, with respect to the centre of the frequency rectangle:

$$H(u,v) = -4\pi^2\left[(u - P/2)^2 + (v - Q/2)^2\right]$$

$$= -4\pi^2 D^2(u,v)$$

Laplacian in frequency domain:

- The Laplacian image is obtained by:

$$\nabla^2 f(x,y) = IDFT\left[H(u,v)F(u,v)\right]$$

- Enhancement of the image is achieved using (H(u,v) negative):

$$g(x,y) = f(x,y) - \nabla^2 f(x,y)$$

Scaling in laplacian

Introduction of large scaling factors.

Practical solution:

- Normalize $f(x,y)$ to the range [0,1] before computing the DFT
- Divide $\nabla^2 f(x,y)$ by its maximum value to bring it in range[-1,1]

Laplacian in frequency domain:

$$
\begin{aligned}
g(x, y) &= \Im^{-1}\left\{F(u, v) - H(u, v)F(u, v)\right\} \\
&= \Im^{-1}\left\{\left[1 - H(u, v)\right]F(u, v)\right\} \\
&= \Im^{-1}\left\{\left[1 + 4\pi^2 D^2(u, v)\right]F(u, v)\right\}
\end{aligned}
$$

Unsharp Masking, Highboost Filtering and High-Frequency-Emphasis Filtering:

$$g_{\text{mask}}(x, y) = f(x, y) - f_{\text{LP}}(x, y)$$

$$f_{\text{LP}}(x, y) = \Im^{-1}\left[H_{\text{LP}}(u, v)F(u, v)\right]$$

$$g(x, y) = f(x, y) + k * g_{\text{mask}}(x, y)$$

This expression defines unsharp masking when $k = 1$ and highboost filtering when $k > 1$.

$$F\big(g(x, y)\big) = F\big(f(x, y)\big) + F\big(k * g_{\text{mask}}(x, y)\big)$$

$$F(g_{mask}(x, y)) = F(f(x, y)) - F(f_{LP}(x, y))$$

$$= F(u, v) - H_{LP}(u, v)F(u, v)$$

$$F(g(x, y)) = F(u, v) + k * (F(u, v) - H_{LP}(u, v)F(u, v))$$

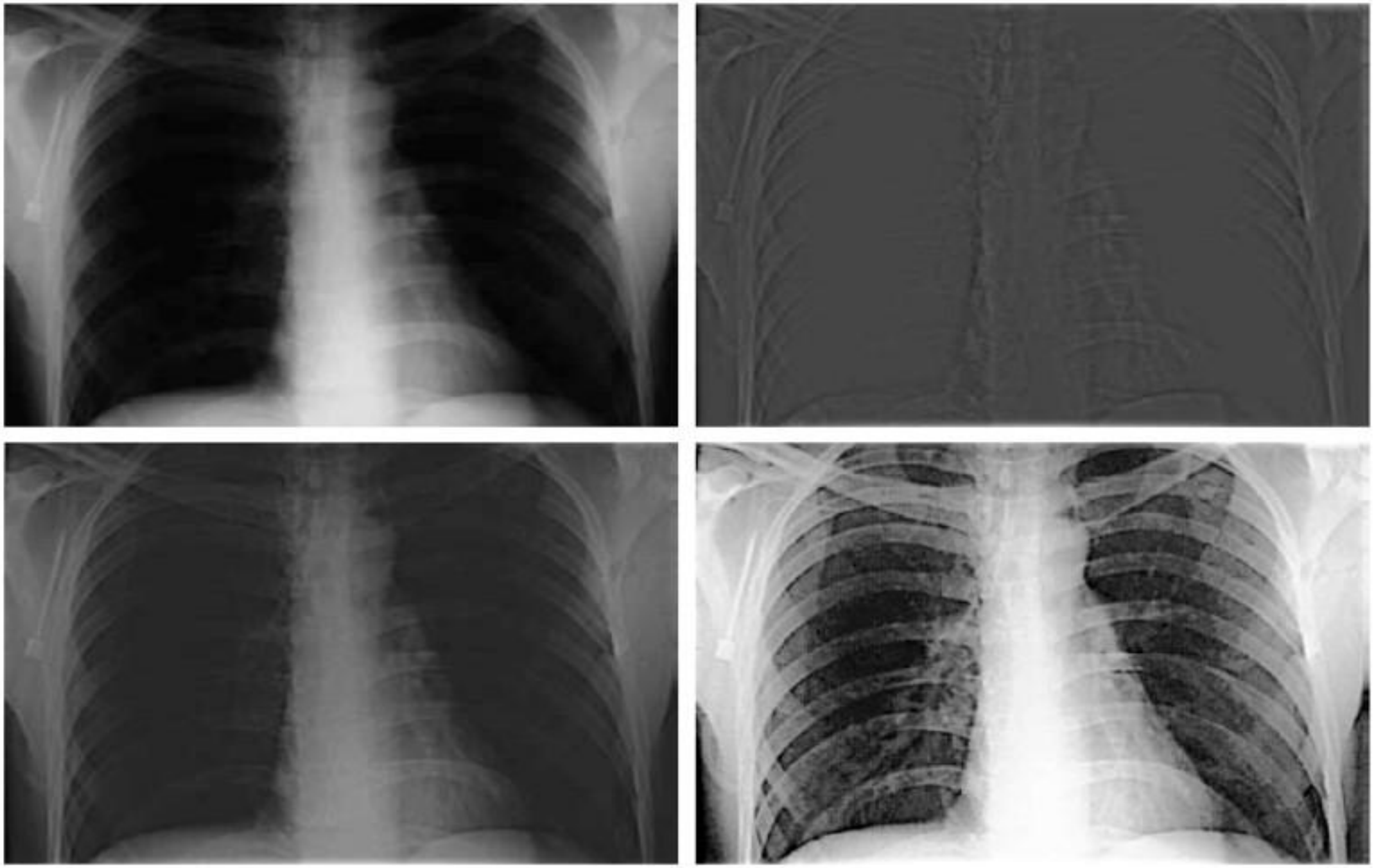
$$g(x, y) = \Im^{-1}\{[1 + k * [1 - H_{LP}(u, v)]]F(u, v)\}$$

$$g(x, y) = \Im^{-1}\{[1 + k * H_{HP}(u, v)]F(u, v)\}$$

The expression contained within the square brackets is called a *high-frequency-emphasis filter*.

$$g(x, y) = \Im^{-1}\{[k_1 + k_2 * H_{HP}(u, v)]F(u, v)\}$$

where $k_1 \geq 0$ gives controls of the offset from the origin [see Fig. 4.31(c)] and $k_2 \geq 0$ controls the contribution of high frequencies.



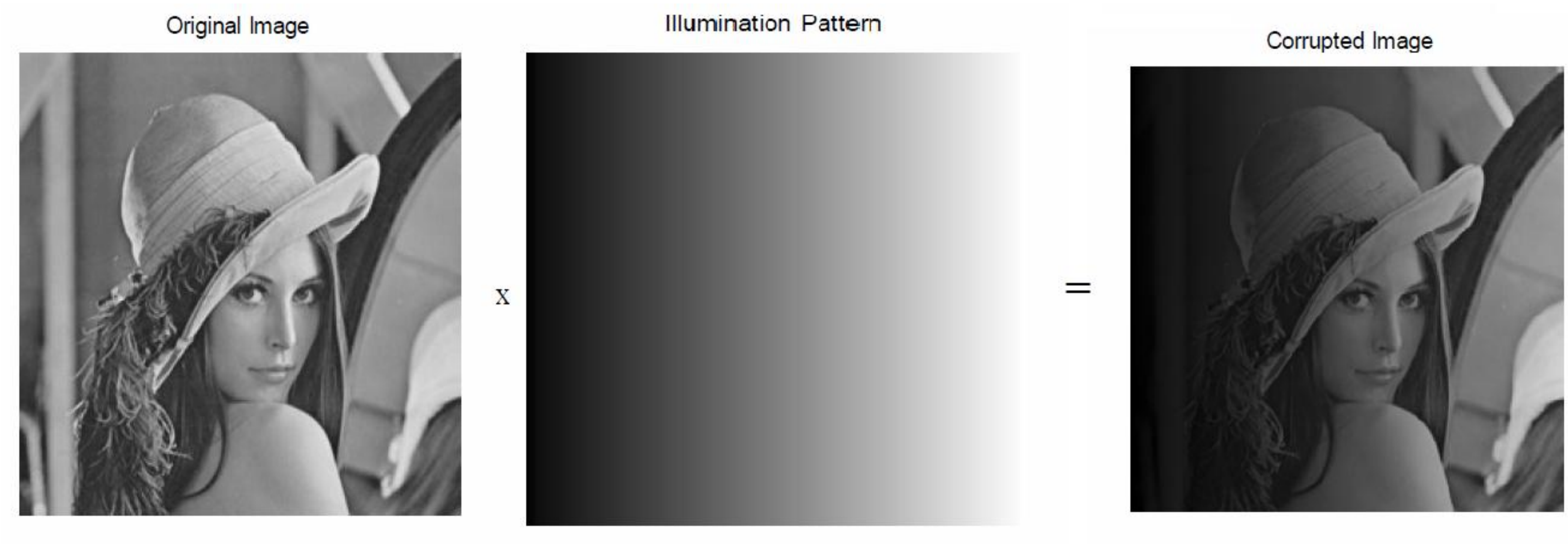
a	b
c	d

FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

Homomorphic filtering:

- Result of non uniform illumination.

Multiplicative noise:



In real world when images are poorly illuminated or corrupted by multiplicative noise – how to enhance?

Histogram equalization:

Original Image

Image restored after histogram equalization

Still non uniform illumination is present. solution?

Homomorphic filtering:

$$f(x, y) = i(x, y)r(x, y)$$

$$\Im[f(x, y)] \neq \Im[i(x, y)]\Im[r(x, y)]$$

However, suppose that we define

$$\begin{aligned} z(x, y) &= \ln f(x, y) \\ &= \ln i(x, y) + \ln r(x, y) \end{aligned}$$

Then,

$$\begin{aligned} \Im\{z(x, y)\} &= \Im\{\ln f(x, y)\} \\ &= \Im\{\ln i(x, y)\} + \Im\{\ln r(x, y)\} \end{aligned}$$

or

$$Z(u, v) = F_i(u, v) + F_r(u, v)$$

where $F_i(u, v)$ and $F_r(u, v)$ are the Fourier transforms of $\ln i(x, y)$ and $\ln r(x, y)$, respectively.

We can filter $Z(u, v)$ using a filter $H(u, v)$ so that

$$\begin{aligned} S(u, v) &= H(u, v)Z(u, v) \\ &= H(u, v)F_i(u, v) + H(u, v)F_r(u, v) \end{aligned}$$

The filtered image in the spatial domain is

$$\begin{aligned} s(x, y) &= \Im^{-1}\{S(u, v)\} \\ &= \Im^{-1}\{H(u, v)F_i(u, v)\} + \Im^{-1}\{H(u, v)F_r(u, v)\} \end{aligned}$$

By defining

$$i'(x, y) = \Im^{-1}\{H(u, v)F_i(u, v)\}$$

$$r'(x, y) = \Im^{-1}\{H(u, v)F_r(u, v)\}$$

we can express Eq. in the form

$$s(x, y) = i'(x, y) + r'(x, y)$$

Finally, because $z(x, y)$ was formed by taking the natural logarithm of the input image, we reverse the process by taking the exponential of the filtered result to form the output image:

$$\begin{aligned} g(x, y) &= e^{s(x,y)} \\ &= e^{i'(x,y)}e^{r'(x,y)} \\ &= i_0(x, y)r_0(x, y) \end{aligned}$$

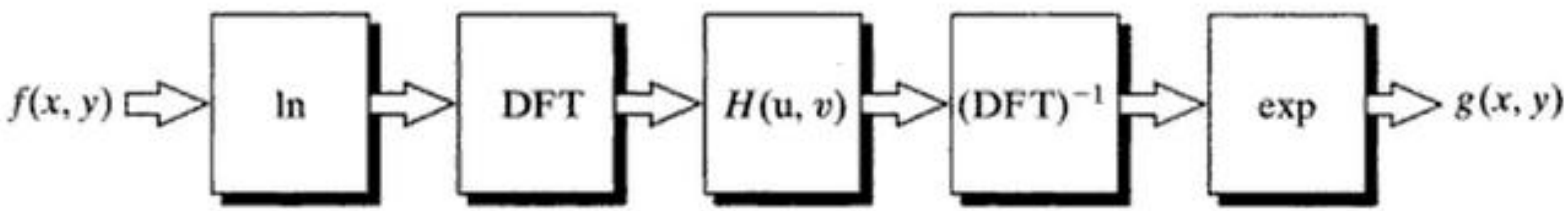
where

$$i_0(x, y) = e^{i'(x,y)}$$

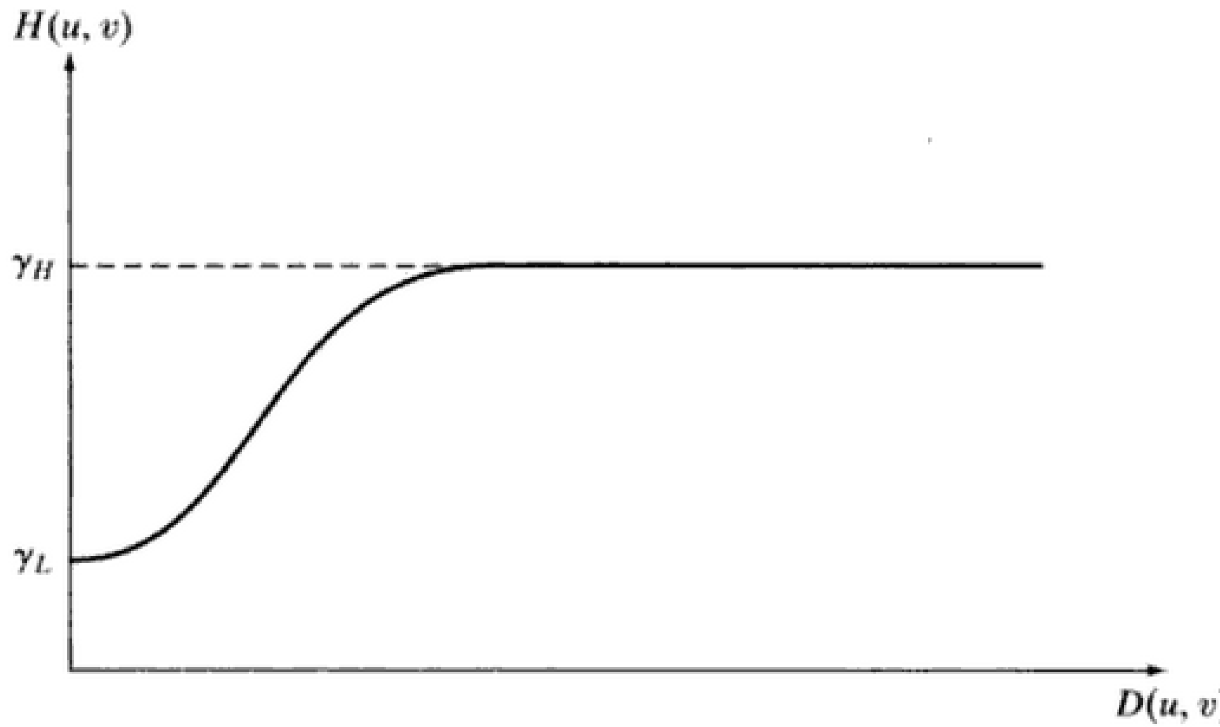
and

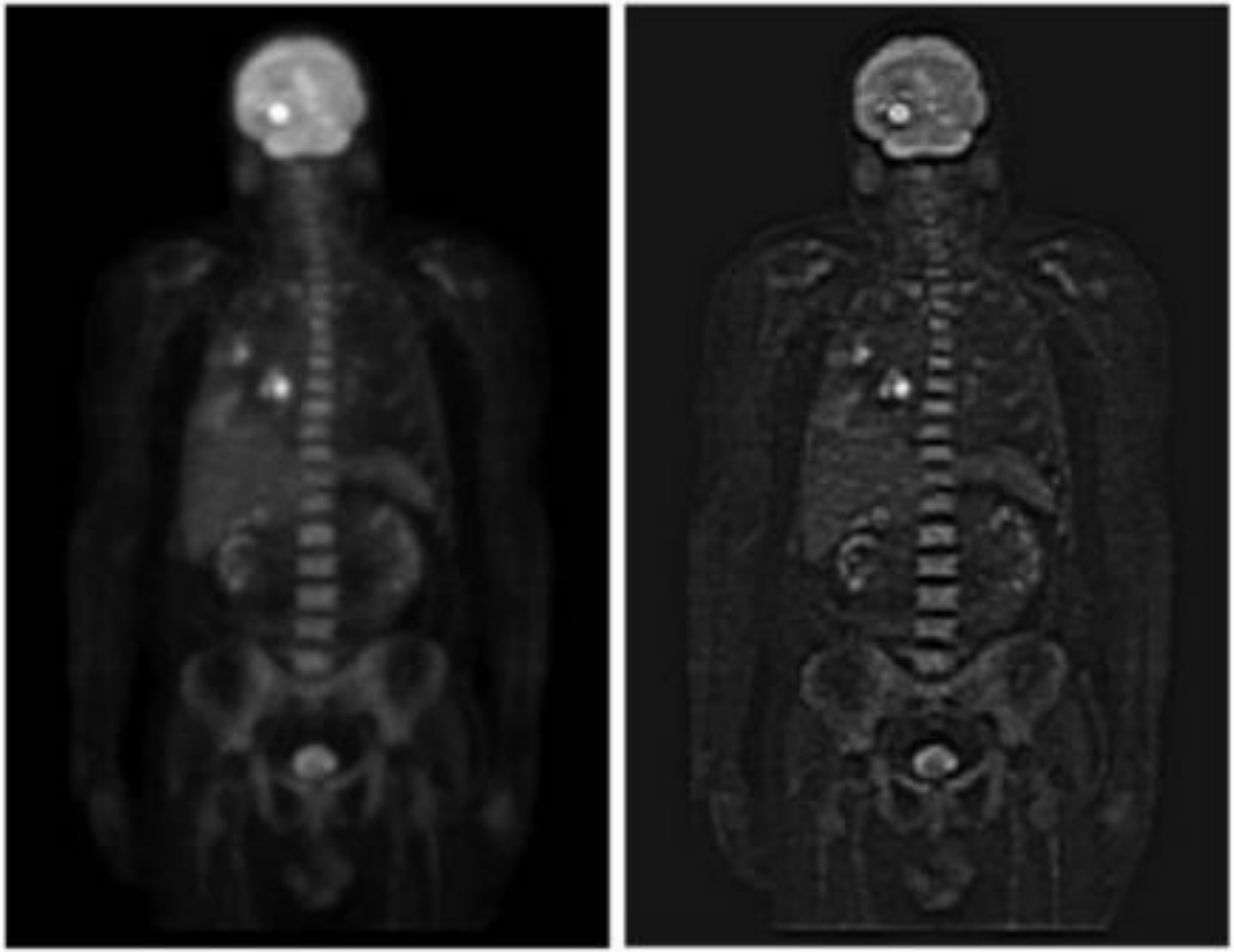
$$r_0(x, y) = e^{r'(x,y)}$$

Homomorphic filtering:



$$H(u, v) = (\gamma_H - \gamma_L)\left[1 - e^{-c[D^2(u,v)/D_0^2]}\right] + \gamma_L$$





a b

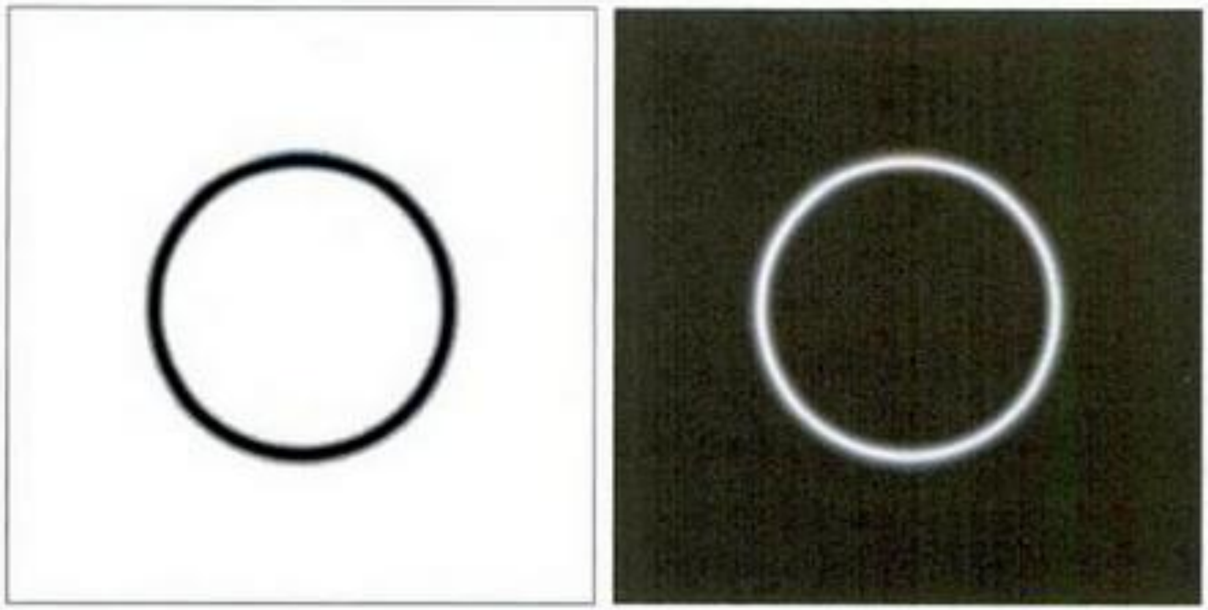
FIGURE 4.62 (a) Full body PET scan. (b) Image enhanced using homomorphic filtering. (Original image courtesy of Dr. Michael E. Casey, CTI PET Systems.)

Band pass and band reject filters:

Bandreject filters. W is the width of the band, D is the distance $D(u, v)$ from the center of the filter, D_0 is the cutoff frequency, and n is the order of the Butterworth filter. We show D instead of $D(u, v)$ to simplify the notation in the table.

Ideal	Butterworth	Gaussian
$H(u, v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \leq D \leq D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u, v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$	$H(u, v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$

Band pass and band reject filters:



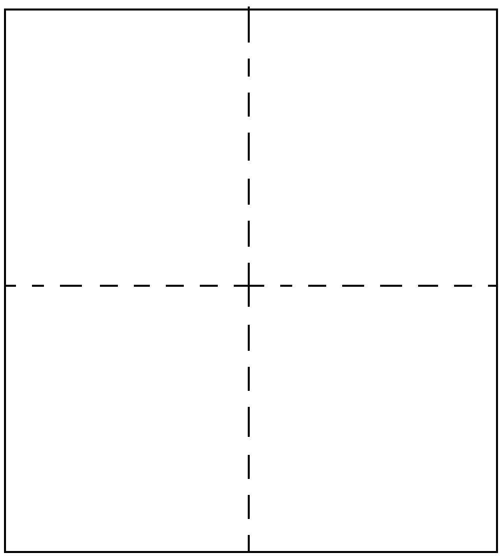
a b

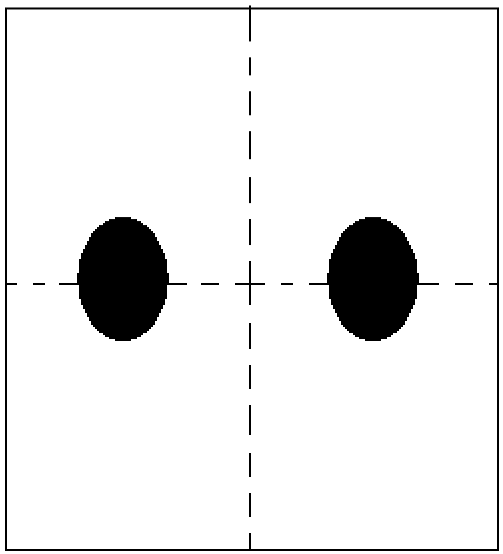
FIGURE 4.63
(a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.

$$H_{\mathrm{BP}}(u, v) = 1 - H_{\mathrm{BR}}(u, v)$$

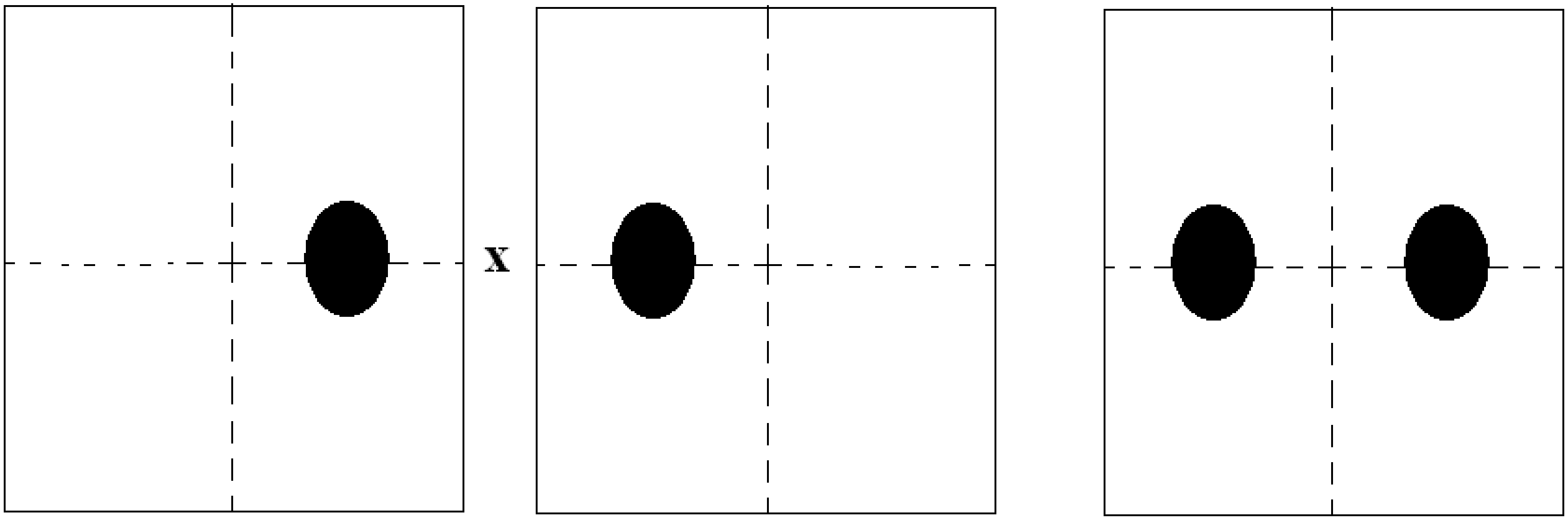
Notch filter

Notch filters are the most useful of the selective filters. A notch filter rejects (or passes) frequencies in a predefined neighborhood about the center of the frequency rectangle.





Notch Reject filter



$$H_{\mathrm{NR}}(u, v) = \prod_{k=1}^{Q} H_k(u, v) H_{-k}(u, v)$$

where $H_k(u, v)$ and $H_{-k}(u, v)$ are highpass filters whose centers are at (u_k, v_k) and $(-u_k, -v_k)$, respectively. These centers are specified with respect to the

Notch Reject filter:

For example, the following is a Butterworth notch reject filter of order n, containing three notch pairs:

$$H_{NR}(u, v) = \prod_{k=1}^{3} \left[\frac{1}{1 + [D_{0k}/D_k(u, v)]^{2n}} \right] \left[\frac{1}{1 + [D_{0k}/D_{-k}(u, v)]^{2n}} \right]$$

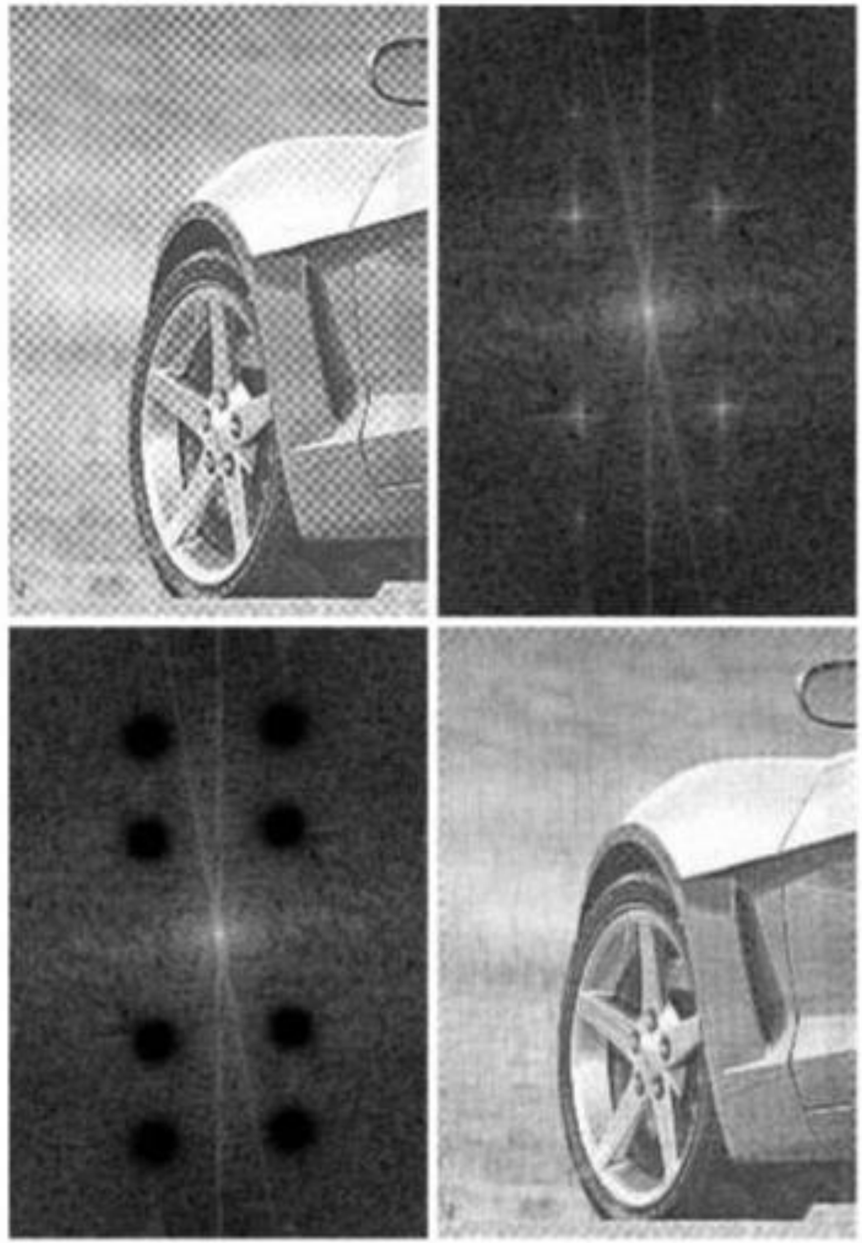
$$D_k(u, v) = \left[(u - M/2 - u_k)^2 + (v - N/2 - v_k)^2 \right]^{1/2}$$

and

$$D_{-k}(u, v) = \left[(u - M/2 + u_k)^2 + (v - N/2 + v_k)^2 \right]^{1/2}$$

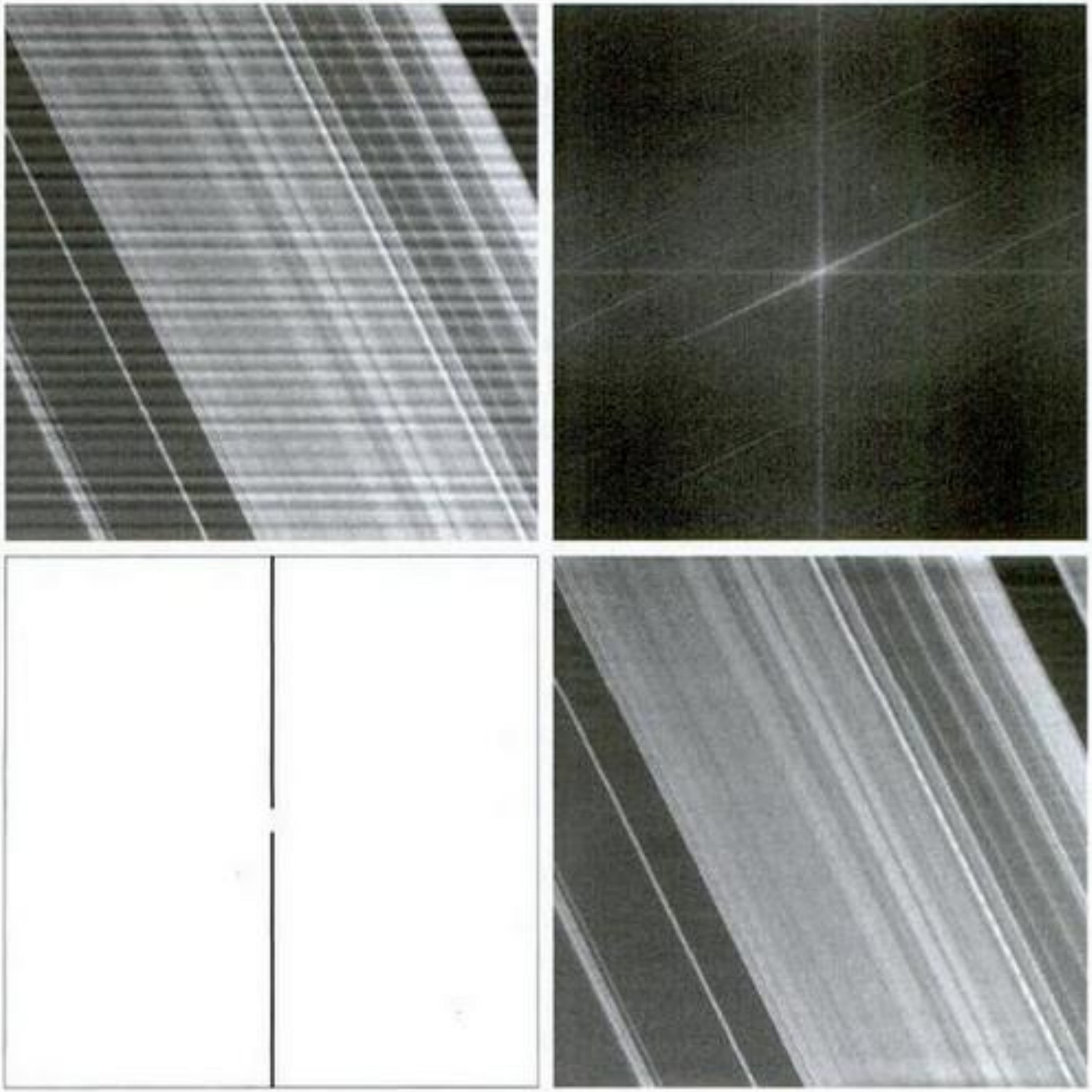
Notch pass filter:

$$H_{NP}(u, v) = 1 - H_{NR}(u, v)$$



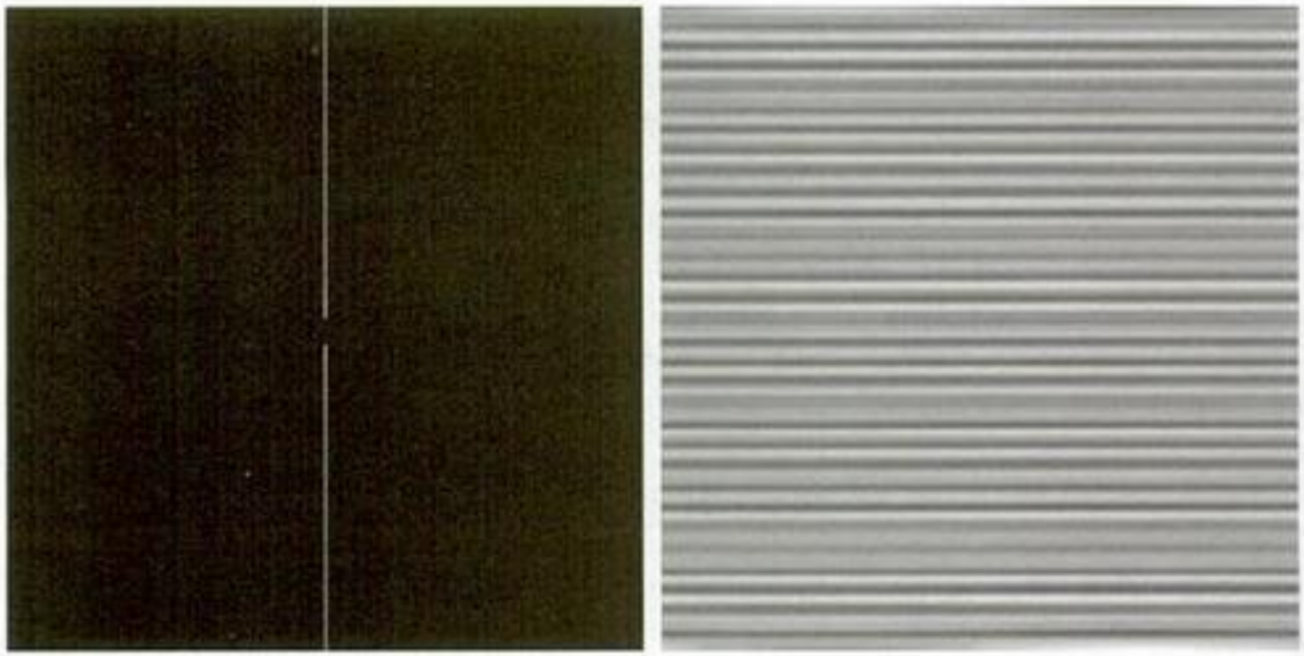
a b
c d

FIGURE 4.64 (a) Sampled newspaper image showing a moiré pattern. (b) Spectrum. (c) Butterworth notch reject filter multiplied by the Fourier transform. (d) Filtered image.



a b
c d

FIGURE 4.65
(a) 674×674 image of the Saturn rings showing nearly periodic interference. (b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern. (c) A vertical notch reject filter. (d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data. (Original image courtesy of Dr. Robert A. West, NASA/JPL.)



a b

FIGURE 4.66 (a) Result (spectrum) of applying a notch pass filter to the DFT of Fig. 4.65(a). (b) Spatial pattern obtained by computing the IDFT of (a).