Markov Random Field

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- Visual modeling - Computational view point
- Define objective function for the optimal solution to a vision problem features

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- optimization process using spatial local interactions makes parallel and local computations possible

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  - Data fusion, Active contours

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- MRF provides foundation for multi-resolution computation
- Equivalence between MRF and Gibbs distributions by Hammersley and Clifford (1971) and Besag (1974)
- Model mathematically sound and tractable in Bayesian framework Geman and Geman (1984)

Applications of MRF Image Modeling

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- medical, textured, document images - different optimization criteria

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- regions with nearly constant gray levels assumed to be corrupted by iid Gaussian noise $Y_s = X_s + W_s \; \forall s \in \Omega$
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- document image segmentation: segment into text, picture, graph and background regions

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- sonar image segmentation: speckle noise in sonar images
- low depth of field image segmentation: distinguish focused object from defocused background

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- Sites not spatial regular considered as irregular e.g. features extracted from images

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- Rectangular lattice for a 2D image of size $n \times n$

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- Discrete case $\mathcal{L} = \{1, \ldots, M\} = \{l_1, \ldots, l_M\}$
- Continuous label correspond to a real line $\mathcal{R}$ or an interval label may take a vector or matrix value

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- $f_i = f(i)$ a function with domain S and image $\mathcal{L}$ mapping from S to $\mathcal{L}$

$$f : S \to \mathcal{L}$$

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- Perceptual grouping of detected features - grouped into connected or disconnected - LP3
- Feature based object matching and recognition - LP3
- Pose estimation - LP4

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- or globally as the joint probability $P(f)$
- If labels are independent one another (no context)
  - the joint probability is the product of the local ones

$$P(f) = \prod_{i \in S} P(f_i)$$

  - conditional independence

$$P(f_i|\{f_{i'}\}) = P(f_i) \quad i' \neq i$$

  - A global labeling f can be computed by considering each label f_i locally.

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$\mathcal{N}_i$ the set of sites neighboring i
- the neighboring relationship has the following properties
  - a site is not neighboring to itself $i \notin \mathcal{N}_i$
  - the neighboring relationship is mutual $i \in \mathcal{N}_{i'} \Rightarrow i' \in \mathcal{N}_i$

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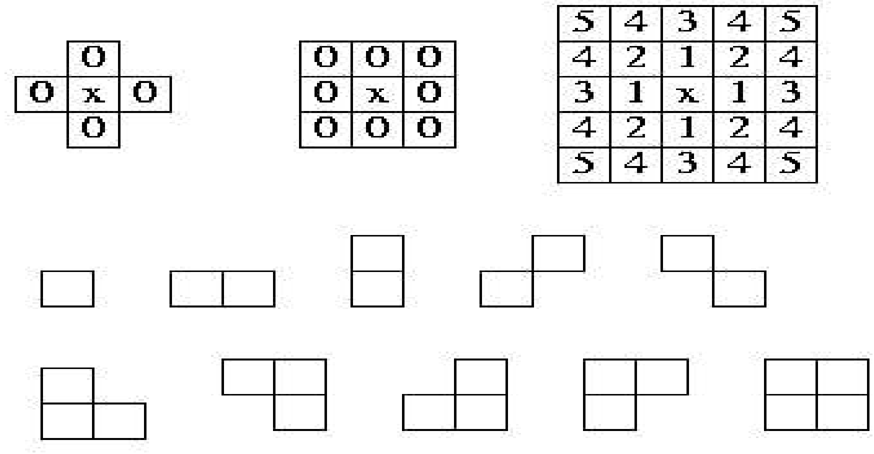
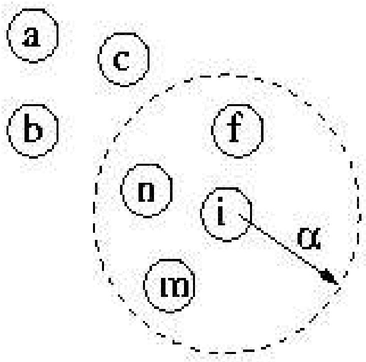
- as the order of the neighborhood system increases, the number of cliques grow rapidly and computationally expensive
- Cliques for irregular sites do not have fixed shapes as those for a regular lattice and their types are depicted by the number of involved sites

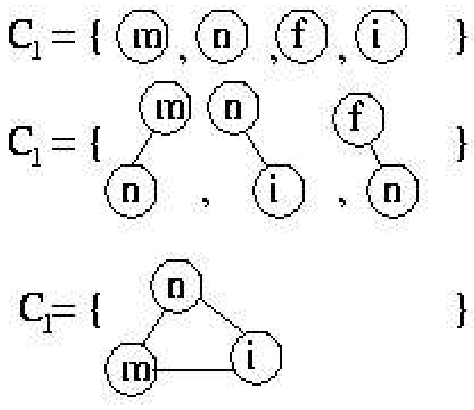
MRF: Neighborhood and Cliques on Regular Site



- Upper: Neighborhood 4, 8, ... n^{th}
- Lower: Cliques

MRF: Neighborhood and Cliques on Irregular Site





- Left: Neighborhood
- Right: Cliques

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- for a continuous $\mathcal{L}$, the same is represented by probability density functions, $p(F_i = f_i)$ and $p(F = f)$

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- for a continuous $\mathcal{L}$, the same is represented by probability density functions, $p(F_i = f_i)$ and $p(F = f)$
- F is said to be a Markov random field on S with respect to a neighborhood system $\mathcal{N}$ if and only if the following two conditions are satisfied:
  1. $P(f) > 0, \forall f \in \mathbb{F}$ (positivity)
  2. $P(f_i | f_{S-\{i\}}) = P(f_i | f_{\mathcal{N}_i})$ (Markovianity)

  where $S - \{i\}$ is the set difference, $f_{S-\{i\}}$ denotes the set of labels at the sites in $S - \{i\}$ and

$$f_{\mathcal{N}_i} = \{f_{i'} | i' \in \mathcal{N}_i\}$$

  stands for the set of labels at the sites neighboring i

MRF: Other Property

- the positivity condition is assumed because the joint probability $P(f)$ of any random field is uniquely determined by its local conditional probabilities, if positivity condition is satisfied
- the Markovianity depicts the local characteristics of F
- the label at a site is dependent only on those at the neighboring sites, i.e, neighboring labels have direct interactions on each other

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- the Markovianity depicts the local characteristics of F
- the label at a site is dependent only on those at the neighboring sites, i.e, neighboring labels have direct interactions on each other
- MRF have other properties as homogeneity and isotropy. It is said to be homogeneous if $P(f_i|f_{N_i})$ is regardless of the relative position of site i in S
- Equivalence between Markov random fields and Gibbs distribution provides a mathematically tractable means of specifying the joint probability of an MRF

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$$P(f) = Z^{-1} \times e^{-\frac{U(f)}{T}} \qquad Z = \sum_{f \in \mathbb{F}} e^{-\frac{U(f)}{T}}$$

is a normalizing constant called the partition function, T is a constant called the temperature which shall be assumed to be 1 and $U(f)$ is the energy function

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- the energy

$$U(f) = \sum_{c \in \mathcal{C}} V_c(f) \tag{1}$$

is a sum of clique potentials $V_c(f)$ over all possible cliques $\mathcal{C}$
- the value of $V_c(f)$ depends on the local configuration on the clique c

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- to calculate a Gibbs distribution, it is necessary to evaluate the partition function Z which is the sum over all possible configuration in $\mathbb{F}$
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- several approximation methods exist for solving this problem

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- given T and $U(f)$, a class of patterns can be generated by sampling the configuration space $\mathbb{F}$ according to $P(f)$
- for discrete labeling problem, a clique potential $V_c(f)$ can be specified by a number of parameters
- for continuous labeling problem, $V_c(f)$ is continuous function of f_c

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$$U(f) = \sum_{i\in S} V_1(f_i) + \sum_{i\in S} \sum_{i'\in\mathcal{N}_i} V_2(f_i, f_{i'})$$

- $\{i, i'\}$ and $\{i', i\}$ are two distinct cliques in $\mathcal{C}_2$ because the sites in a clique are ordered

GRF: Equivalence

- the conditional probability can be written as

$$P(f_i|f_{\mathcal{N}_i}) = \frac{e^{-\left[V_1(f_i)+\sum_{i' \in \mathcal{N}_i} V_2(f_i,f_{i'})\right]}}{\sum_{f_i \in \mathcal{C}} e^{-\left[V_1(f_i)+\sum_{i' \in \mathcal{N}_i} V_2(f_i,f_{i'})\right]}}$$

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- the theorem states that F is an MRF on S with respect to $\mathcal{N}$ if and only if F is a GRF on S with respect to $\mathcal{N}$
- this theorem provides a simple way of specifying the joint probability

Markov-Gibbs Equivalence

- one can specify the joint probability $P(F = f)$ by specifying the clique potential functions $V_c(f)$ and
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- in optimization problems, an objective function is in the form of an energy function and is to be minimized
- as the quantitative cost measure, an energy function defines the minimal solution as its minimum, usually a global one
- formulate an energy function so that the correct solution is embedded as the minimum

MRF Image Modeling: Example

- image modeled as a random field $Y = \{Y_s : s \in L\}$
- $L = \{(i,j) | 0 \leq i \leq N_1 - 1, 0 \leq j \leq N_2 - 1\}$ index set, a set of site indices on 2-D discrete $N_1 \times N_2$ rectangular integer lattice

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- y_s may be feature values depending on the application; DCT or DWT coefficients

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- site is not a neighbor of itself
- clique is a subset of sites in N_l $c \in N_l$ is a clique of every pair of distinct sites in c are neighbors
- random field X is considered to be an MRF on S if and only if $P(X = x) > 0$ and
  $P(X_l = x_l | X_r = x_r, r \neq s) = P(X_l = x_l | X_r = x_r, r \in N_l)$
- difficult to determine the above characteristics in practice

MRF Image Modeling: Example

- MRF has the form

$$P(X = x) = \frac{1}{Z} e^{-U(X)/T}$$

- $X = x$ is a realization from $X = \{X_1, \ldots, X_m\}$, i.e., $x = \{x_1, \ldots, x_m\}$ is a set of random field X and

$$U(X) = \sum_L V_c(x)$$

  is global energy function and it is given by the sum of clique potentials $V_c(x)$, over all possible cliques
- choice of energy function is arbitrary

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- MRF has the form

$$P(X = x) = \frac{1}{Z} e^{-U(X)/T}$$

- $X = x$ is a realization from $X = \{X_1, \ldots, X_m\}$, i.e., $x = \{x_1, \ldots, x_m\}$ is a set of random field X and

$$U(X) = \sum_{L} V_c(x)$$

  is global energy function and it is given by the sum of clique potentials $V_c(x)$, over all possible cliques
- choice of energy function is arbitrary
- for example, several definition of $U(X)$ in the framework of image segmentation

MRF Image Modeling: Example

- general expression for the energy function

$$U(X) = \sum_{L} V_c(ci) + \sum_{c \in N_l} V_c(ci, cj)$$

- known as Potts model $V_c(ci)$ external field that weighs the relative importance of different classes
- simplified Potts model with no external energy $V_c(ci) = 0$
- local spatial transitions are taken into account and all the classes in the label segmentation, X_{opt}, as near as possible to the real image X^*

$$V_c(ci, cj) = \begin{cases} -\beta & \text{if } ci = cj \\ 0 & \text{otherwise} \end{cases}$$

$$V_c(ci, cj) = \begin{cases} -\frac{\beta \sigma_i^2}{(\sigma_i^2 + (y_{ci} - y_{cj})^2 \times d_{cicj})} & \text{if} \quad ci \neq cj \\ -\beta & \text{if} \quad ci = cj \end{cases}$$

MRF-MAP Segmentation

- y_{ci} and y_{cj} are the pixel intensities of ci and cj, $d_{cicj} = 1 \ or \sqrt{2}$ represents the distance between the two pixels
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- $P(X|Y)$ is the posteriori, $P(Y|X)$ is the conditional probability density of the image Y, $P(X)$ is the prior density of labelling X

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- based on conditional independent assumption of Y the conditional density $P(Y|X) = \prod_L P(y_l|x_l)$

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$$P(Y|X) = \prod_L \left[\frac{1}{\sqrt{2\pi\sigma_l^2}} \exp\left\{-\frac{(y_l - \mu_{xl})^2}{2\sigma_l^2}\right\}\right] \propto \exp\left[-\sum_L \frac{(y_l - \mu_{xl})^2}{2\sigma_l^2}\right]$$

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- similarly prior density of MRF takes the form of

$$P(X) = \frac{1}{Z} \exp\left[-\sum_L \sum_{c \in N_l} \frac{V_c(ci, cj)}{T} \right]$$

$$P(X|Y) \propto \exp\left(-\sum_L \left[\frac{(y_l - \mu_{xl})^2}{2\sigma^2} + \sum_{c \in N_l} \frac{V_c(ci, cj)}{T} \right] \right)$$

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- image Y is then segmented by finding the field of labels X

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$$\begin{aligned} X_{opt} &= \min \arg_{x \in L} U(X|Y) \\ &= \min \arg \left(\sum_L \left[\frac{(y_l - \mu_l)^2}{2\sigma^2} + \frac{1}{T} \sum_{c \in N_l} V_c(ci, cj) \right] \right) \end{aligned}$$

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- optimization method: simulated annealing (SA), iterated conditional model (ICM) used to find the solution

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- optimization method: simulated annealing (SA), iterated conditional model (ICM) used to find the solution
- SA is slow but guarantee a global minimum solution
- ICM likely to reach local minima and no guarantee that a global minimum of energy function can be obtained, provides much faster convergence
- ICM iteratively decrease the energy by visiting and updating the pixel

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$$X_l^{(k+1)} = \arg\max P(X_l^{(k)} | Y, X_r^{(k)}, r = l)$$

- starting from the initial state, keep on running on the procedure above until either the predefined number of iterations is reached or the label of X does not change

Thank You