Probability for Learning

- Probability for classification and modeling concepts.
- Bayesian probability
  - Notion of probability interpreted as partial belief
- Bayesian Estimation
  - Calculate the validity of a proposition
    - Based on prior estimate of its probability
    - and New relevant evidence

Bayes Theorem

- **<u>Goal:</u>** To determine the most probable hypothesis, given the data *D* plus any initial knowledge about the prior probabilities of the various hypotheses in *H*.

Bayes Theorem

Bayes Rule: $$P(h \mid D) = \frac{P(D \mid h)P(h)}{P(D)}$$

- P(h) = prior probability of hypothesis h
- P(D) = prior probability of training data D
- P(h|D) = probability of h given D (posterior density)
- P(D|h) = probability of D given h (likelihood of D given h)

Dilemma at the movies

This person dropped their ticket in the hallway.

Do you call out

"Excuse me, ma'am!"

or

"Excuse me, sir!"

You have to make a guess.

Dilemma at the movies

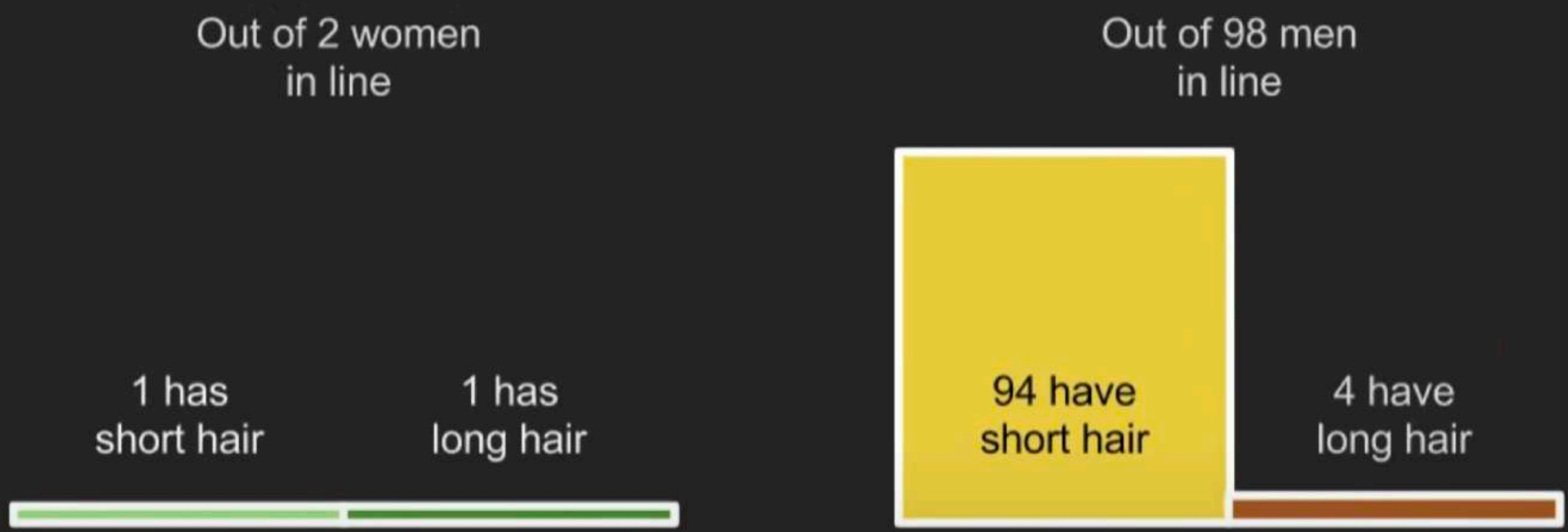
What if they're standing in line for the men's restroom?

Bayesian inference is a way to capture common sense.

It helps you use what you know to make better guesses.


Put numbers to our dilemma
Out of 100 women at the movies
50 have short hair
50 have long hair
Out of 100 men at the movies
96 have short hair
4 have long hair

Put numbers to our dilemma



But there are 98 men and 2 women in line for the men's restroom.


Out of 100 people
at the movies
50 are women
50 are men
25 women have short hair
25 women have long hair
48 men have short hair
2 men have long hair


Out of 100 people
In line for the men's
restroom
2 are
women
98 are men
One
woman
has
short
hair
One
woman
has
long
hair
94 men
have
short hair
4 men have long hair

Translate to math

P(something) = # something / # everything

P(woman) = Probability that a person is a woman

= # women / # people

= 50 / 100 = **.5**

P(man) = Probability that a person is a man

= # men / # people

= 50 / 100 = **.5**

Translate to math

P(something) = # something / # everything

P(woman) = Probability that a person is a woman

= # women / # people

= 2 / 100 = **.02**

P(man) = Probability that a person is a man

= # men / # people

= 98 / 100 = **.98**

Out of 100 people
In line for the men's
restroom

2 are
women

98 are men

Conditional probabilities

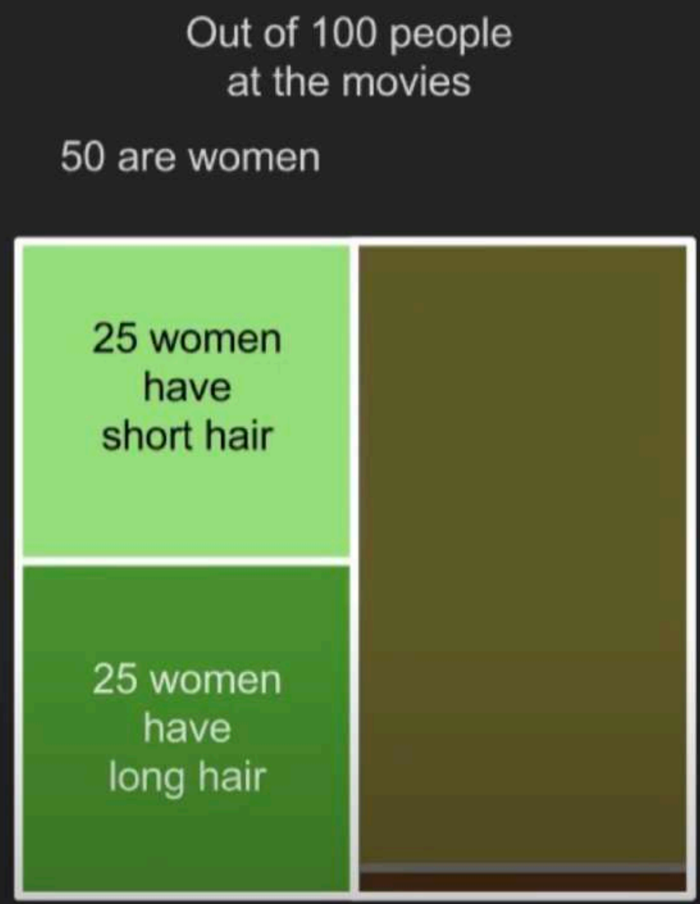
P(long hair | woman)

If I know that a person is a woman, what is the probability that person has long hair?

P(long hair | woman)

= # women with long hair / # women

= 25 / 50 = **.5**



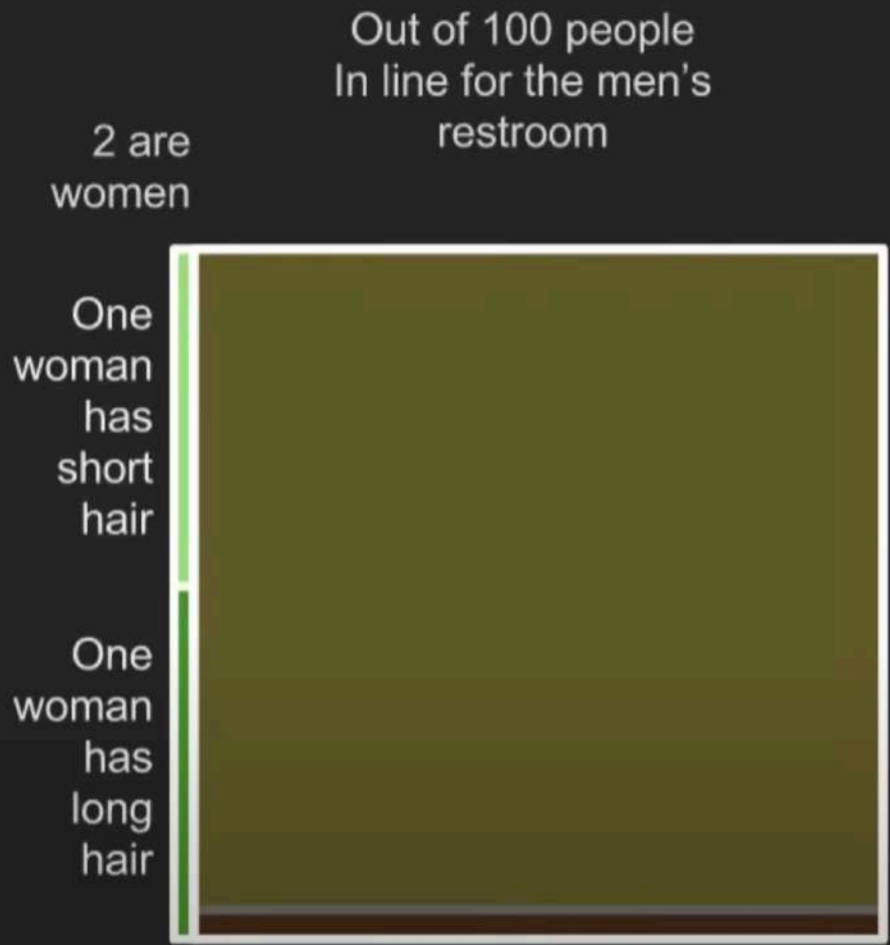
Conditional probabilities

This doesn't change when we consider people in line.

P(long hair | woman)

= # women with long hair / # women

= 1 / 2 = .5



Conditional probabilities

If I know that a person is a man, what is the probability that person has long hair?

P(long hair | man)

= # men with long hair / # men

= 2 / 50 = **.04**

Whether in line or not.

Out of 100 people at the movies

50 are men

2 men have long hair

Conditional probabilities

P(A | B) is the probability of A, given B.

"If I know B is the case, what is the probability that A is also the case?"

P(A | B) is not the same as P(B | A).

P(cute | puppy) is not the same as P(puppy | cute)

If I know the thing I'm holding is a puppy, what is the probability that it is cute?

If I know the the thing I'm holding is cute, what is the probability that it is a puppy?

Joint probabilities

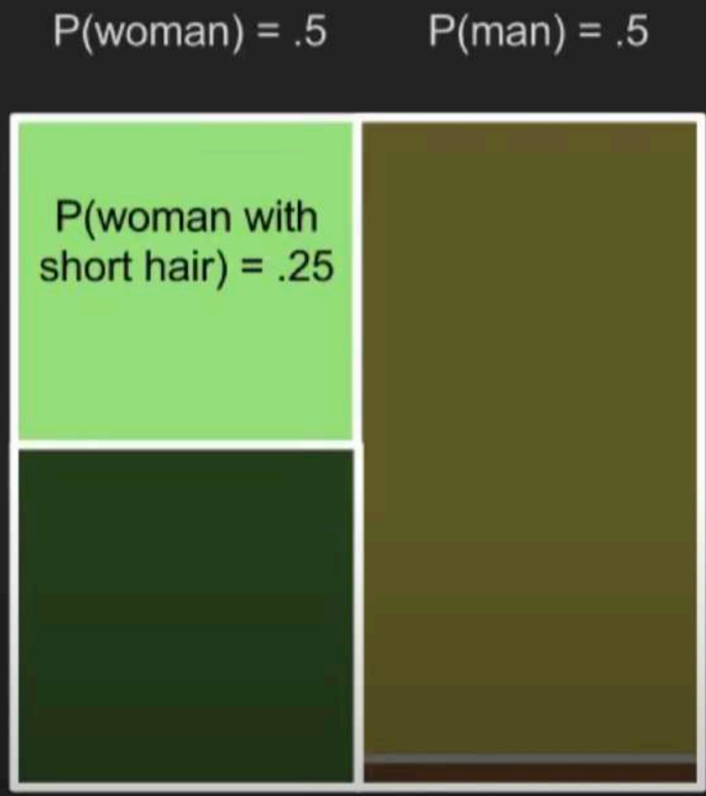
What is the probability that a person is both a woman and has short hair?

P(woman with short hair)

= P(woman) * P(short hair | woman)

= .5 * .5 = **.25**

Out of probability of 1

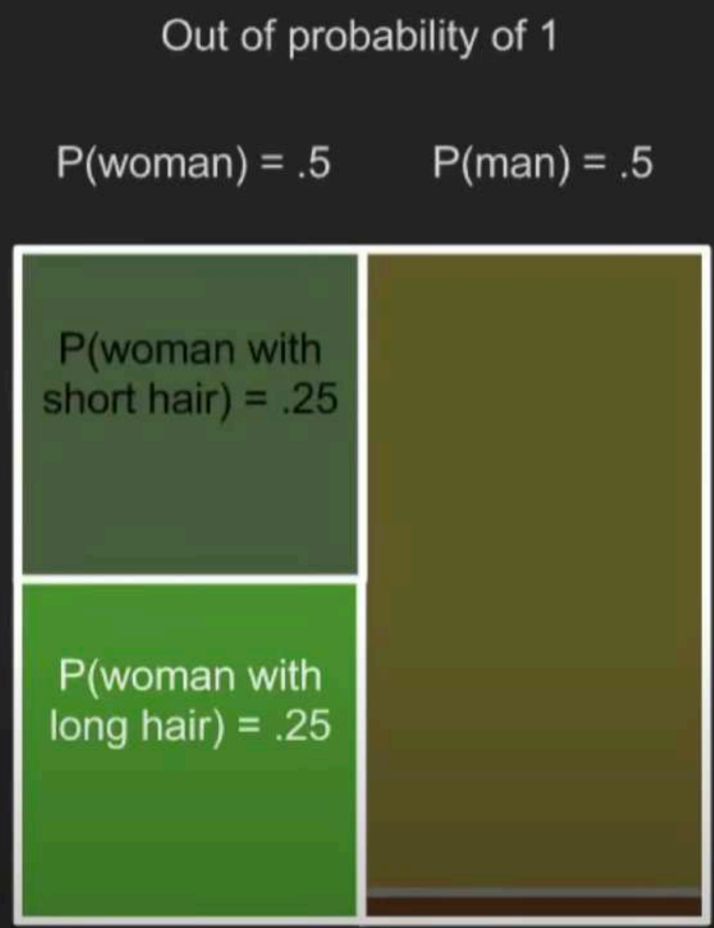


Joint probabilities

P(woman with long hair)

= P(woman) * P(long hair | woman)

= .5 * .5 = **.25**



Joint probabilities

P(man with short hair)

= P(man) * P(short hair | man)

= .5 * .96 = **.48**

Out of probability of 1

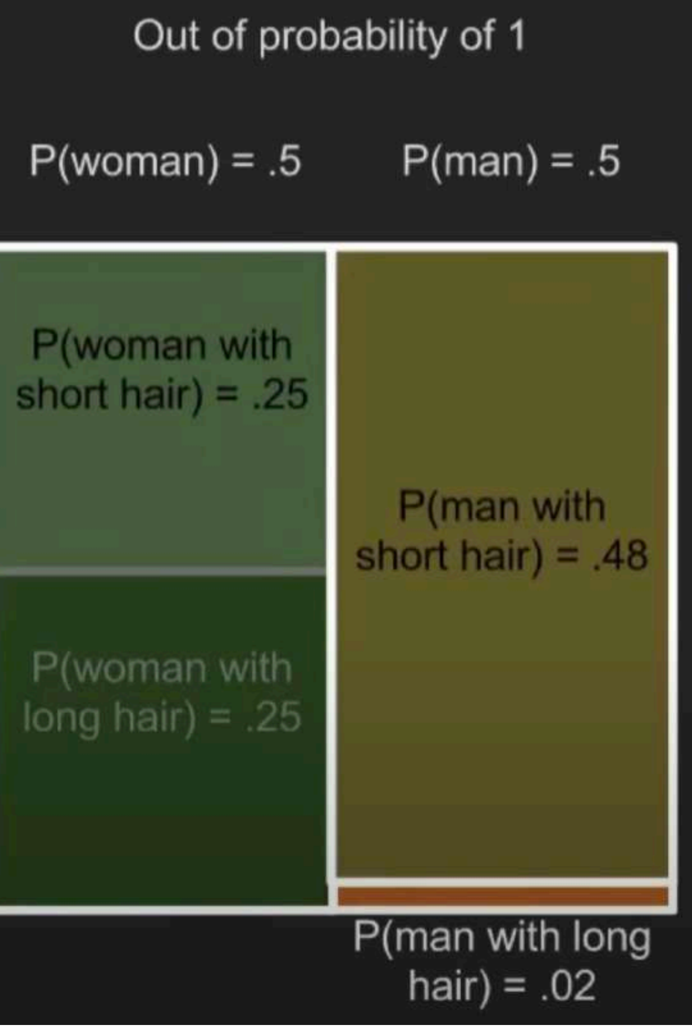
P(woman) = .5 P(man) = .5

Joint probabilities

P(man with long hair)

= P(man) * P(long hair | man)

= .5 * .04 = **.02**



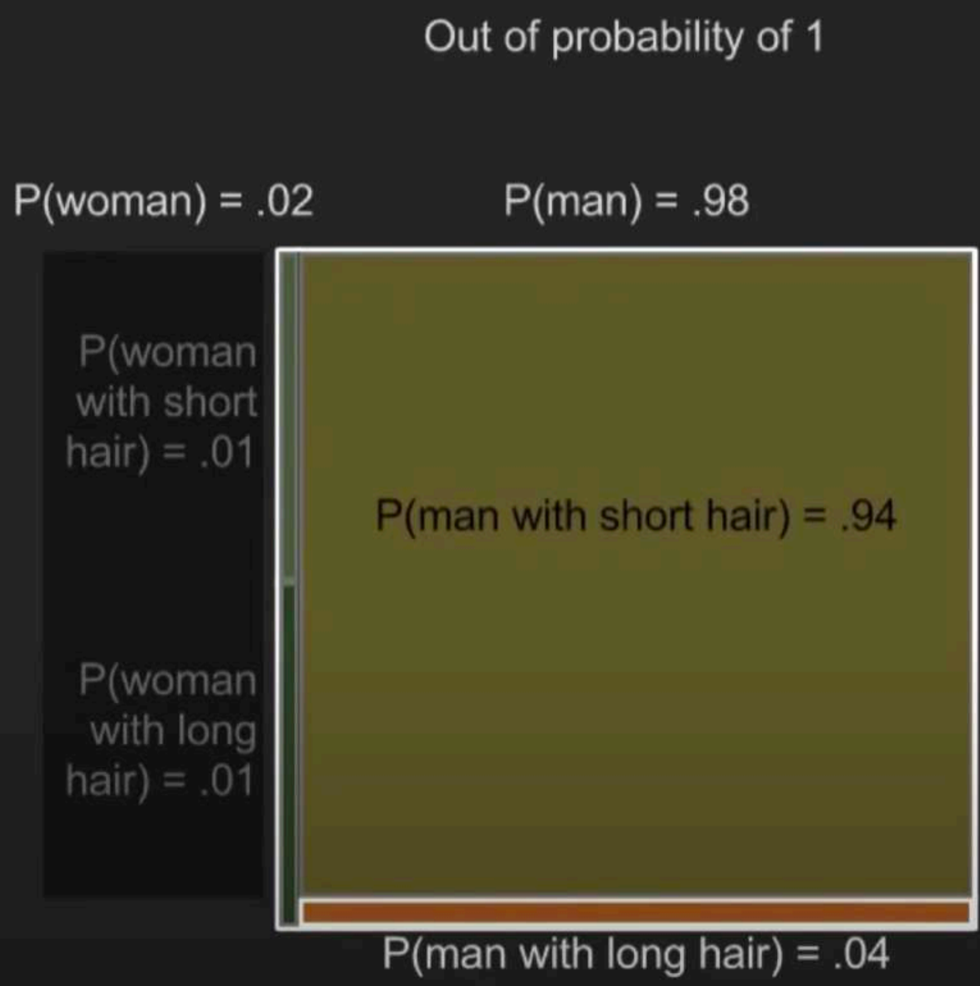
Joint probabilities

If P(man) = .98 and P(woman) = .02, then the answers change.

P(man with long hair)

= P(man) * P(long hair | man)

= .96 * .04 = **.04**

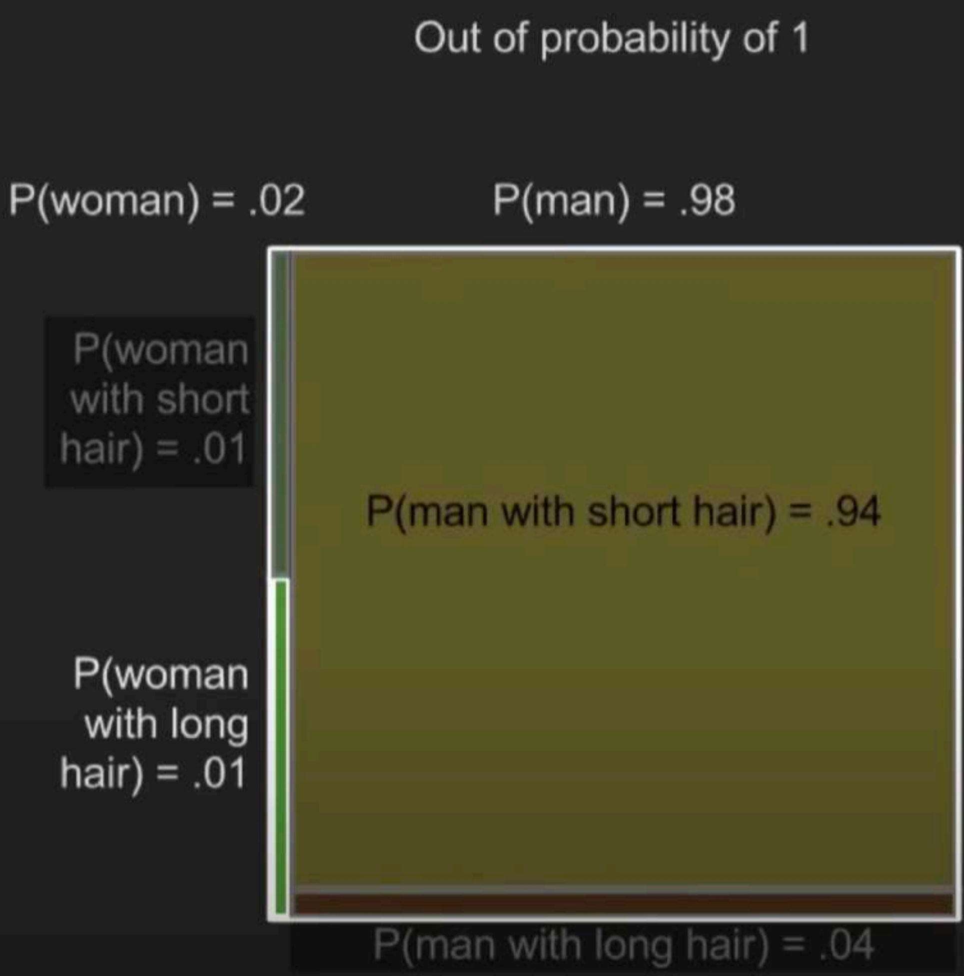


Joint probabilities

P(woman with long hair)

= P(woman) * P(long hair | woman)

= .02 * .5 = **.01**



Joint probabilities

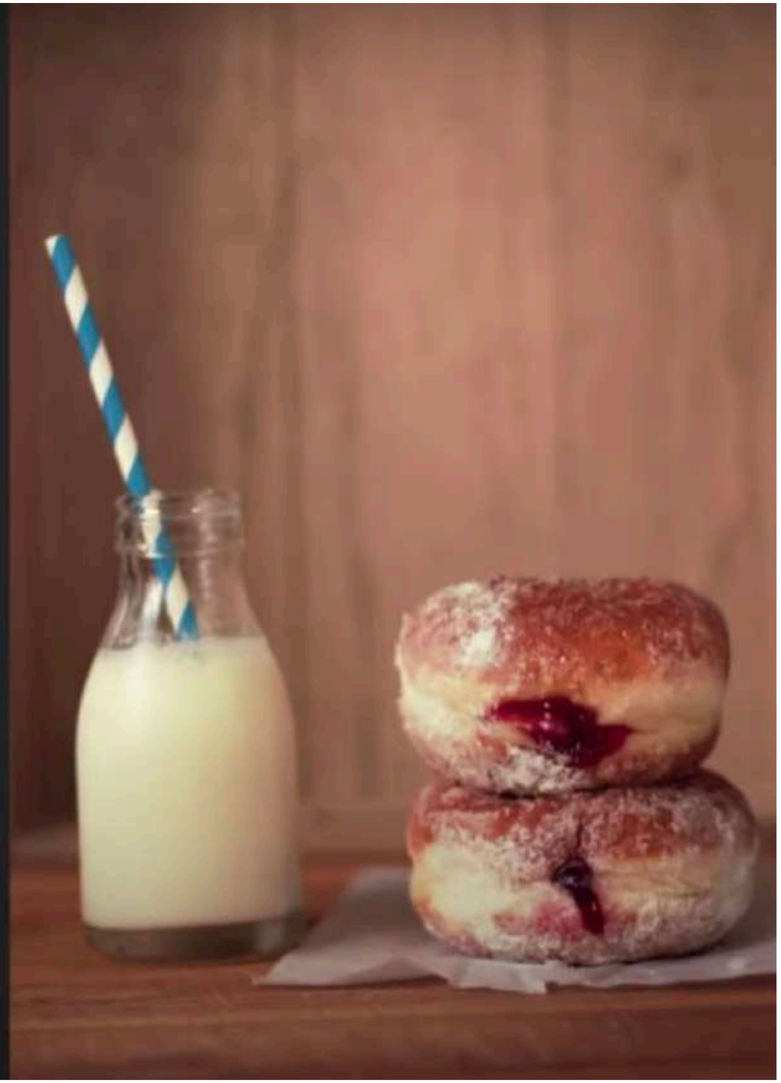
P(A and B) is the probability that both A and B are the case.

Also written P(A, B) or P(A ∩ B)

P(A and B) is the same as P(B and A)

The probability that I am having a jelly donut with my milk is the same as the probability that I am having milk with my jelly donut.

P(donut and milk) = P(milk and donut)

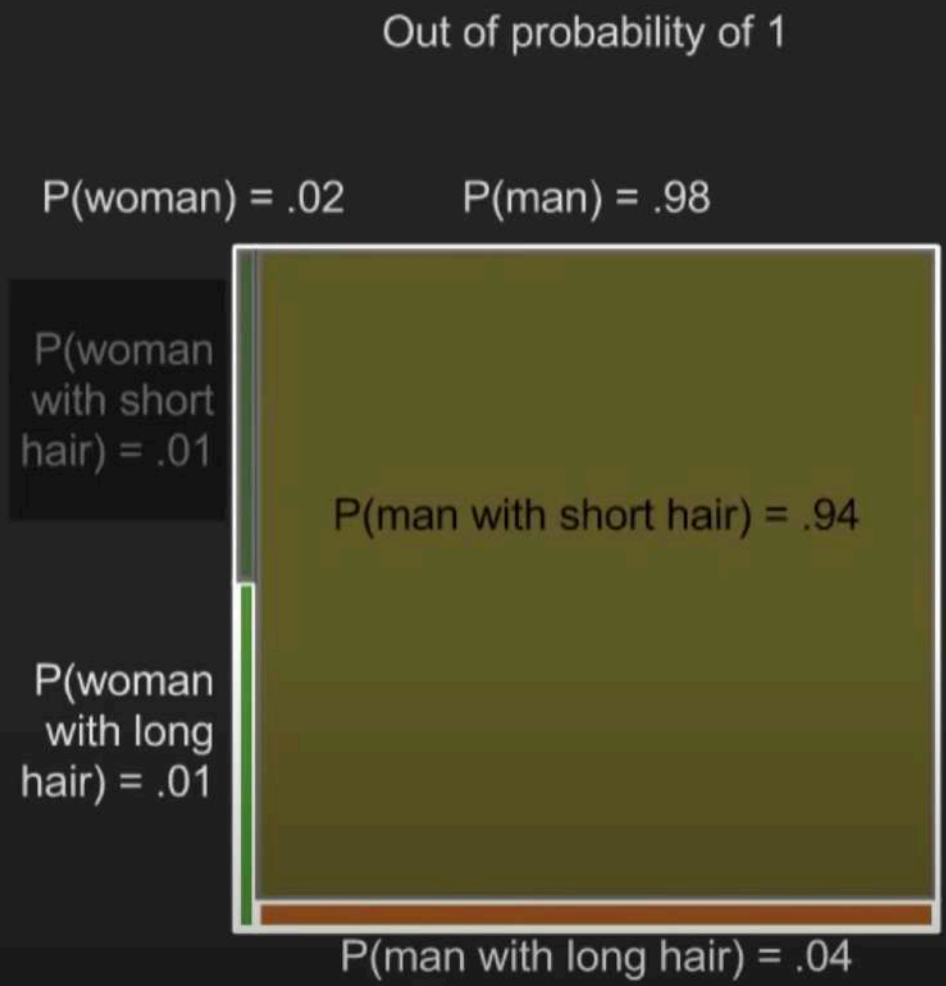


Marginal probabilities

P(long hair) = P(woman with long hair) +

P(man with long hair)

= .01 + .04 = **.05**



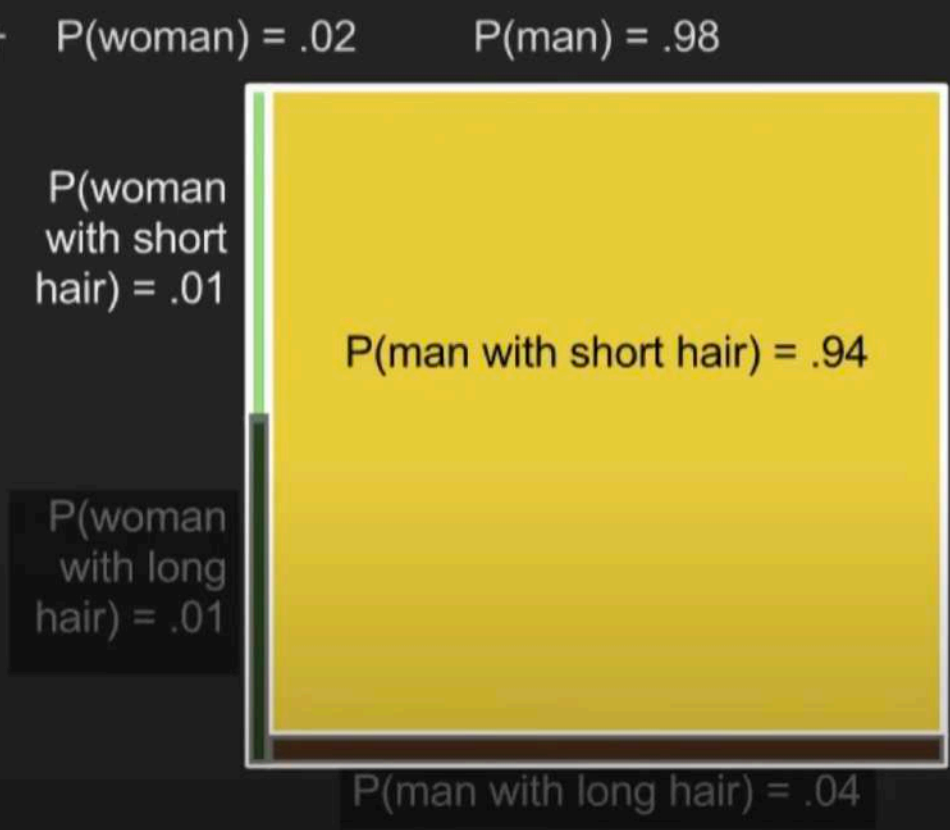
Marginal probabilities

Out of probability of 1

P(short hair) = P(woman with short hair) +

P(man with short hair)

= .01 + .94 = **.95**



What we really care about

We know the person has long hair.
Are they a man or a woman?

P(man | long hair)

We don't know this answer yet.

Thomas Bayes noticed something cool

$$\boxed{P(\text{man with long hair})} = P(\text{long hair}) * P(\text{man} \mid \text{long hair})$$

$$\boxed{P(\text{long hair and man})} = P(\text{man}) * P(\text{long hair} \mid \text{man})$$

Because $\boxed{P(\text{man and long hair})} = \boxed{P(\text{long hair and man})}$

Thomas Bayes noticed something cool

P(man with long hair) = P(long hair) * P(man | long hair)

P(long hair and man) = P(man) * P(long hair | man)

Because P(man and long hair) = P(long hair and man)

P(long hair) * P(man | long hair) = P(man) * P(long hair | man)

P(man | long hair) = P(man) * P(long hair | man) / P(long hair)

$P(A \mid B) = P(B \mid A) * P(A) / P(B)$

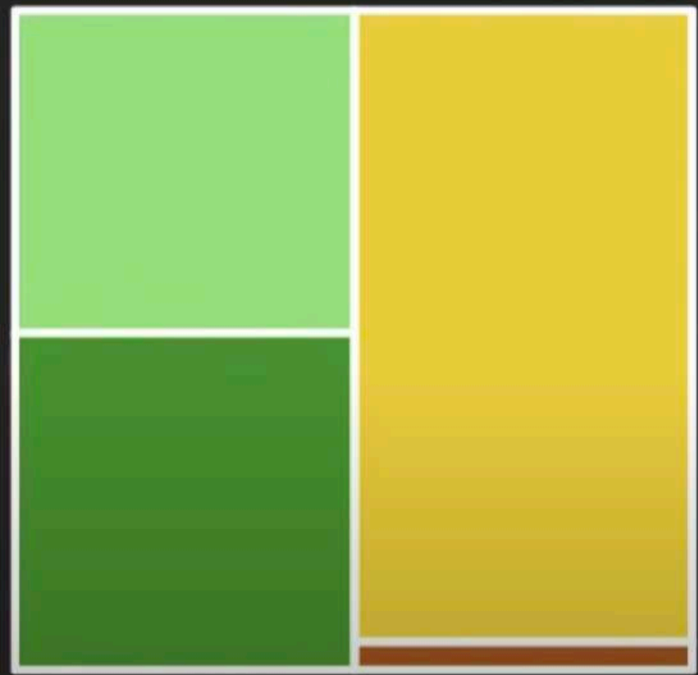
Back to the movie theater, this time with Bayes

$$P(\text{man} \mid \text{long hair}) = \frac{P(\text{man}) * P(\text{long hair} \mid \text{man})}{P(\text{long hair})}$$

$$= \frac{P(\text{man}) * P(\text{long hair} \mid \text{man})}{P(\text{woman with long hair}) + P(\text{man with long hair})}$$

$$P(\text{man} \mid \text{long hair}) = \frac{.5 * .04}{.25 + .02} = .02 / .27 = \boxed{.07}$$

P(woman) = .5 P(man) = .5



P(long hair | man) = .04

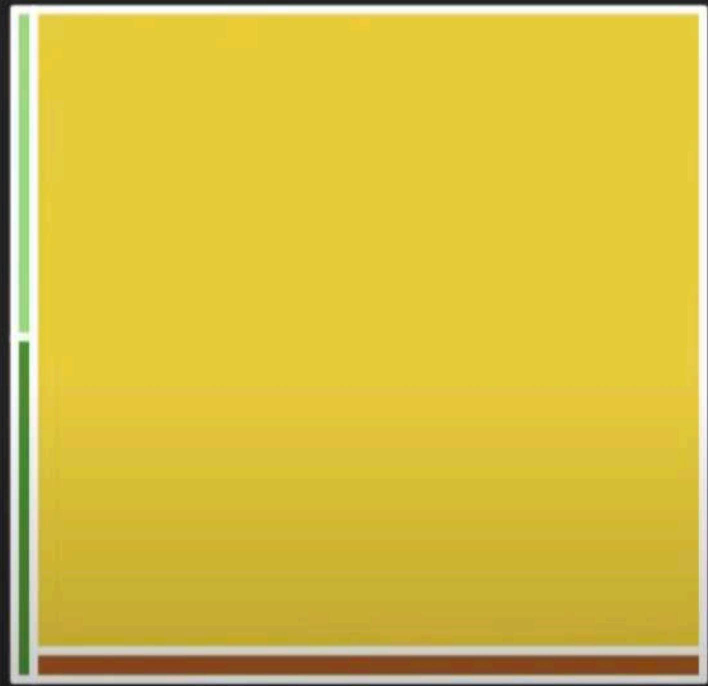
P(long hair | woman) = .5

Back to the movie theater, this time with Bayes

$$P(\text{man} \mid \text{long hair}) = \frac{P(\text{man}) * P(\text{long hair} \mid \text{man})}{P(\text{long hair})}$$

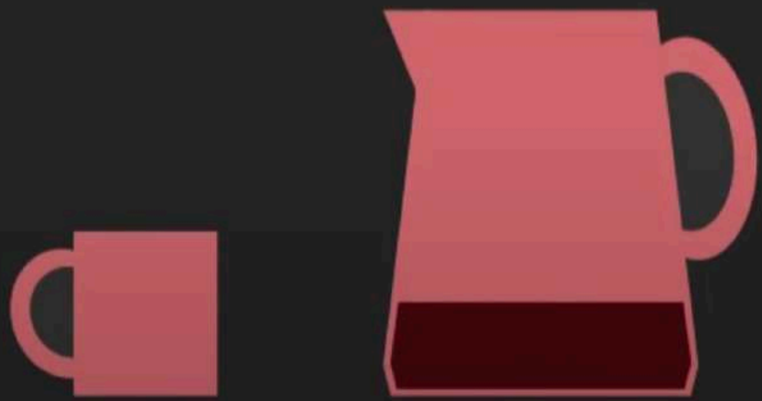
$$= \frac{P(\text{man}) * P(\text{long hair} \mid \text{man})}{P(\text{woman with long hair}) + P(\text{man with long hair})}$$

$$P(\text{man} \mid \text{long hair}) = \frac{.98 * .04}{.01 + .04} = .04 / .05 = \boxed{.80}$$



Probability distributions

Probability is like a pot with just one cup of coffee left in it.



Probability distributions

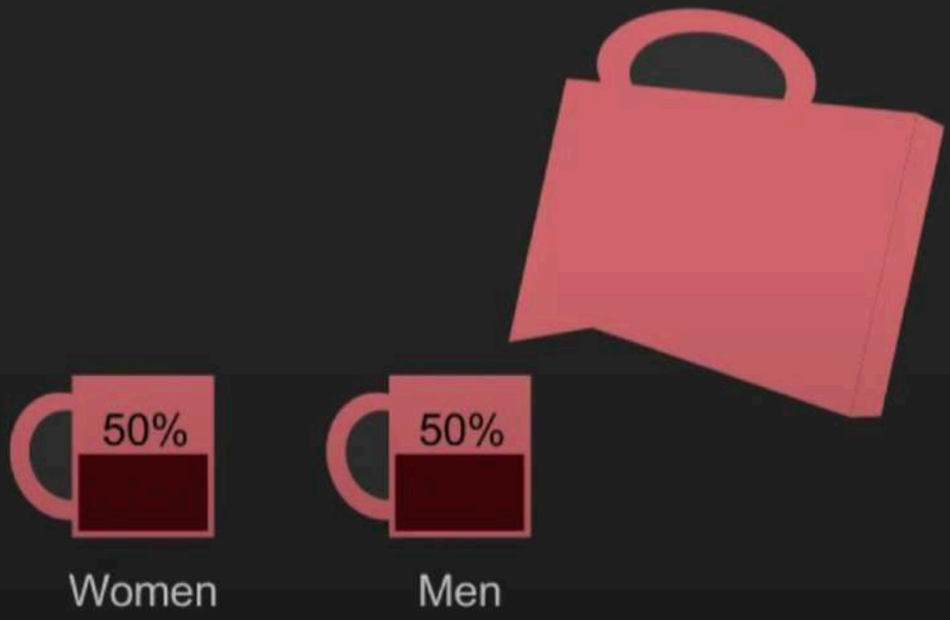
If you only have one cup, you can fill it completely.

Probability distributions

If you have two cups, you have to decide how to share (distribute) it.

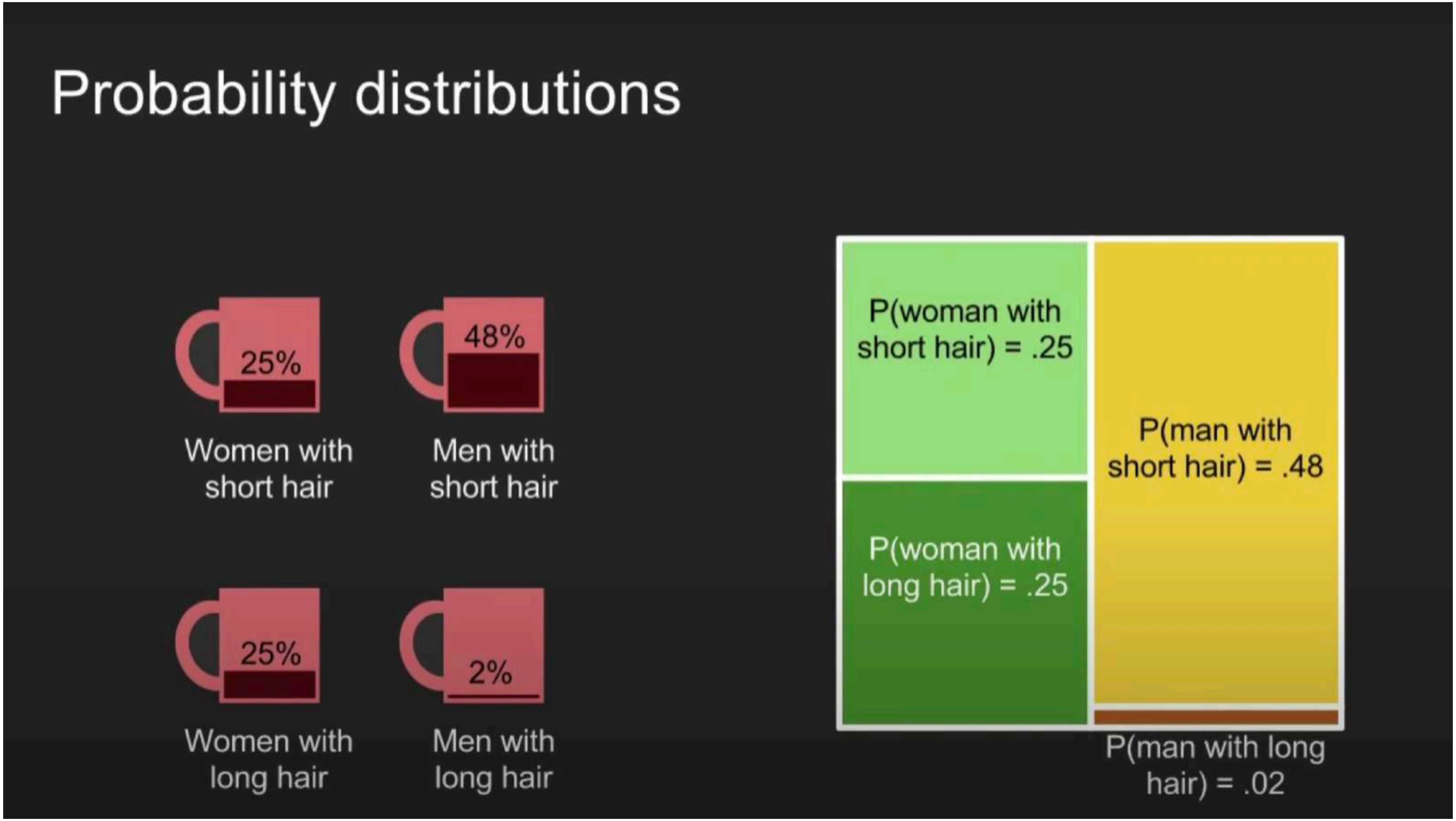
Probability distributions

Our people are distributed between two groups, women and men.



Probability distributions

We can distribute them more.



Probability distributions
25%
Women with short hair
48%
Men with short hair
25%
Women with long hair
2%
Men with long hair
P(woman with short hair) = .25
P(woman with long hair) = .25
P(man with short hair) = .48
P(man with long hair) = .02

Probability distributions

It's helpful to think of probabilities as beliefs

Probability distributions

Flipping a fair coin

Probability distributions

Rolling a fair die

Bayes Theorem

Bayes Rule: $$P(h \mid D) = \frac{P(D \mid h)P(h)}{P(D)}$$

- P(h) = prior probability of hypothesis h
- P(D) = prior probability of training data D
- P(h|D) = probability of h given D (posterior density)
- P(D|h) = probability of D given h (likelihood of D given h)

Maximum A Posteriori (MAP) Hypothesis

$$P(h \mid D) = \frac{P(D \mid h)P(h)}{P(D)}$$

The Goal of Bayesian Learning: the most probable hypothesis given the training data (Maximum A Posteriori hypothesis)

$$\begin{aligned} h_{MAP} &= \arg\max_{h \in H} P(h \mid D) \\ &= \arg\max_{h \in H} \frac{P(D \mid h)P(h)}{P(D)} \\ &= \arg\max_{h \in H} P(D \mid h)P(h) \end{aligned}$$

Compute ML Hypo

$$h_{ML} = \arg\max_{h \in H} p(D \mid h)$$

$$= \arg\max_{h \in H} \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{d_i - h(x_i)}{\sigma}\right)^2}$$

$$= \arg\max_{h \in H} \sum_{i=1}^{m} -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\left(\frac{d_i - h(x_i)}{\sigma}\right)^2$$

$$= \arg\min_{h \in H} \sum_{i=1}^{m} (d_i - h(x_i))^2$$

Bayes Optimal Classifier

Question: Given new instance x, what is its most probable classification?

- $h_{MAP}(x)$ is not the most probable classification!

Example: Let P(h1|D) = .4, P(h2|D) = .3, P(h3 |D) =.3

Given new data x, we have h1(x)=+, h2(x) = -, h3(x) = -

What is the most probable classification of x ?

Bayes optimal classification:

$$\arg\max_{v_j \in V} \sum_{h_i \in H} P(v_j \mid h_i) P(h_i \mid D)$$

where V is the set of all the values a classification can take and v_j is one possible such classification.

Example:

P(h1| D) =.4,	P(-|h1)=0,	P(+|h1)=1	$\sum_{hi \in H} P(+ \mid h_i) P(h_i \mid D) = .4$
P(h2|D) =.3,	P(-|h2)=1,	P(+|h2)=0	
P(h3|D)=.3,	P(-|h3)=1,	P(+|h3)=0	$\sum_{hi \in H} P(- \mid h_i) P(h_i \mid D) = .6$

features: $X = (X_1, X_2, ..., X_n)$

label: Y

$$P(Y = y | X = (x_1, x_2, ..., x_n))$$

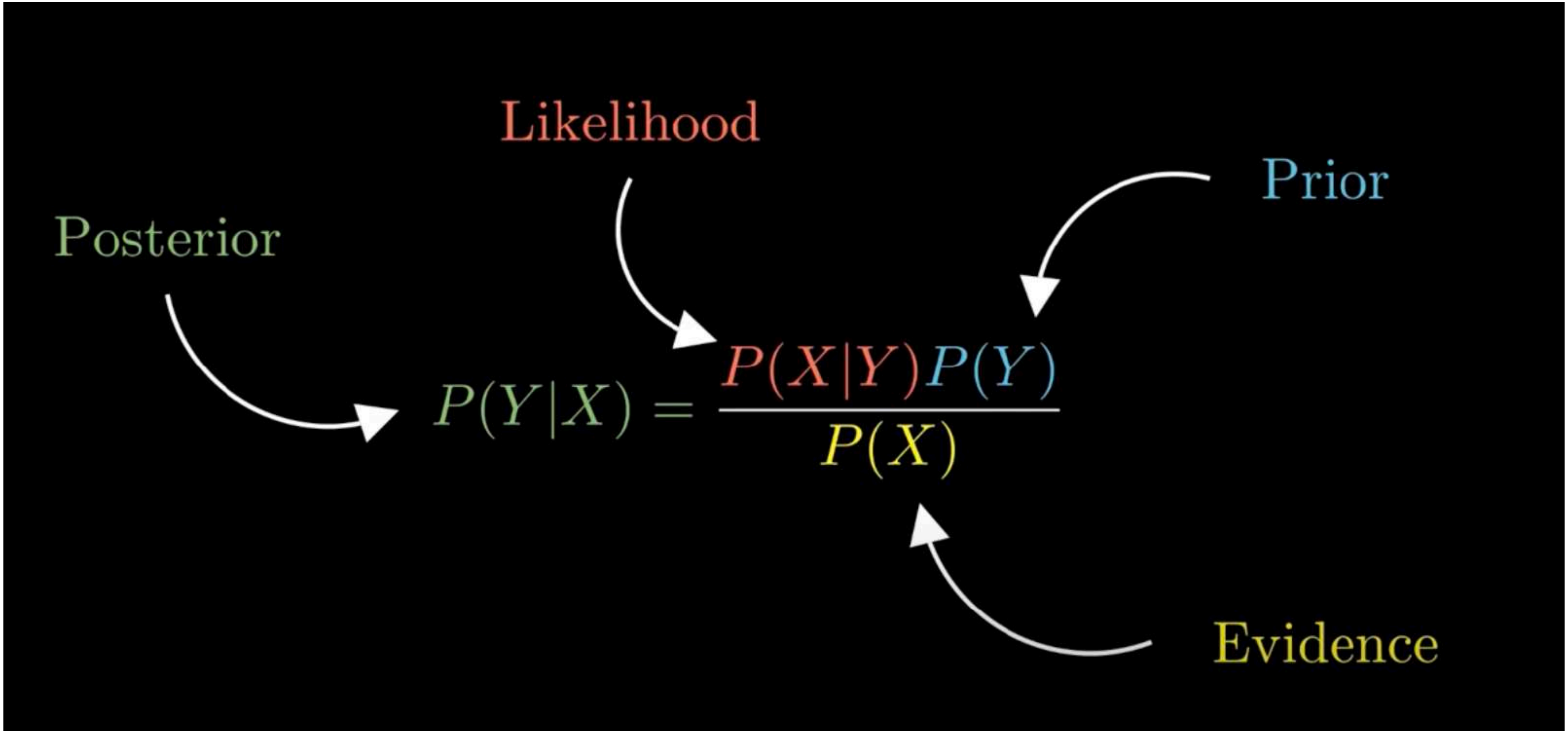
for what value of y

$$P(Y = y|X = (x_1, x_2, ..., x_n))$$

is maximum

but...$P(Y|X)$ is hard to find!

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$


Likelihood
Prior
Posterior
$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$
Evidence

X_1	X_2	Y
0	0	0
0	1	1
1	2	1
0	0	1
2	2	0
1	1	0
0	2	1
2	0	0
2	1	0
1	0	0

$$X_1, X_2 \in \{0, 1, 2\}$$
$$Y \in \{0, 1\}$$

Estimate the value of Y given that $X = (0, 2)$

X_1	X_2	Y
0	0	0
0	1	1
1	2	1
0	0	1
2	2	0
1	1	0
0	2	1
2	0	0
2	1	0
1	0	0

Let's compute $P(Y=0|X=(0,2))$ and $P(Y=1|X=(0,2))$ ...

$$P(Y=0) = \frac{\#Y=0}{\#Y=0 + \#Y=1} = \frac{6}{10}$$

$$P(Y=1) = \frac{\#Y=1}{\#Y=0 + \#Y=1} = \frac{4}{10}$$

$$P(X=(0,2)|Y=1) = \frac{1}{4}$$

$$P(X=(0,2)|Y=0) = 0$$

$$P(X=(0,2)|Y=0) * P(Y=0) = 0 * \frac{6}{10} = 0$$

$$P(X=(0,2)|Y=1) * P(Y=1) = \frac{1}{4} * \frac{4}{10} = \frac{1}{10}$$

X_1	X_2	Y
0	0	0
0	1	1
1	2	1
0	0	1
2	2	0
1	1	0
0	2	1
2	0	0
2	1	0
1	0	0

Let's compute $P(Y = 0|X = (0,2))$ and $P(Y = 1|X = (0,2))$ ...

$$P(Y = 0) = \frac{\#Y = 0}{\#Y = 0 + \#Y = 1} = \frac{6}{10}$$

$$P(Y = 1) = \frac{\#Y = 1}{\#Y = 0 + \#Y = 1} = \frac{4}{10}$$

$$P(X = (0,2)|Y = 1) = \frac{1}{4}$$

$$P(X = (0,2)|Y = 0) = 0$$

Estimated value of Y is 1 given $X = (0,2)$

X_1	X_2	Y
0	0	0
0	1	1
1	2	1
0	0	1
2	2	0
1	1	0
0	2	1
2	0	0
2	1	0
1	0	0

Let's compute $P(Y=0|X=(0,2))$ and $P(Y=1|X=(0,2))$ ...

$$P(Y=0) = \frac{\#Y=0}{\#Y=0+\#Y=1} = \frac{6}{10}$$

$$P(Y=1) = \frac{\#Y=1}{\#Y=0+\#Y=1} = \frac{4}{10}$$

$$P(X=(0,2)|Y=1) = \frac{1}{4}$$

$$P(X=(0,2)|Y=0) = 0$$

Let's assume X_1, X_2 are independant!

X_1	X_2	Y
0	0	0
0	1	1
1	2	1
0	0	1
2	2	0
1	1	0
0	2	1
2	0	0
2	1	0
1	0	0

Let's compute $P(Y=0|X=(0,2))$ and $P(Y=1|X=(0,2))$ ...

$$P(Y=0) = \frac{\#Y=0}{\#Y=0+\#Y=1} = \frac{6}{10}$$

$$P(Y=1) = \frac{\#Y=1}{\#Y=0+\#Y=1} = \frac{4}{10}$$

$$P(X=(0,2)|Y=1) = P(X_1=0|Y=1) * P(X_2=2|Y=1)$$

$$P(X=(0,2)|Y=0) = P(X_1=0|Y=0) * P(X_2=2|Y=0)$$

X_1	X_2	Y
0	0	0
0	1	1
1	2	1
0	0	1
2	2	0
1	1	0
0	2	1
2	0	0
2	1	0
1	0	0

Let's compute $P(Y=0|X=(0,2))$ and $P(Y=1|X=(0,2))$ ...

$$P(Y=0) = \frac{\#Y=0}{\#Y=0+\#Y=1} = \frac{6}{10}$$

$$P(Y=1) = \frac{\#Y=1}{\#Y=0+\#Y=1} = \frac{4}{10}$$

$$P(X=(0,2)|Y=1) = P(X_1=0|Y=1) * P(X_2=2|Y=1) = \frac{3}{4}\ \frac{2}{4}$$

$$P(X=(0,2)|Y=0) = P(X_1=0|Y=0) * P(X_2=2|Y=0) = \frac{1}{6}\ \frac{1}{6}$$

$$4/10 * \frac{3}{4} * \frac{2}{4} > \frac{1}{6} * \frac{1}{6} * 6/10$$

don't forget to multiply the priors
i.e. P(Y=1) and P(Y=0)

X_1	X_2	Y
0	0	0
0	1	1
1	2	1
0	0	1
2	2	0
1	1	0
0	2	1
2	0	0
2	1	0
1	0	0

Naïve Bayes

Bayes rule:

$$P(Y = y_k | X_1 \ldots X_n) = \frac{P(Y = y_k) P(X_1 \ldots X_n | Y = y_k)}{\sum_j P(Y = y_j) P(X_1 \ldots X_n | Y = y_j)}$$

Assuming conditional independence among X_i's:

$$P(Y = y_k | X_1 \ldots X_n) = \frac{P(Y = y_k) \prod_i P(X_i | Y = y_k)}{\sum_j P(Y = y_j) \prod_i P(X_i | Y = y_j)}$$

So, classification rule for $X^{new} = \langle X_1, \cdots, X_n \rangle$ is:

$$Y^{new} \leftarrow \arg\max_{y_k} \; P(Y = y_k) \prod_i P(X_i^{new} | Y = y_k)$$

Example

- Example: Play Tennis

PlayTennis: training examples

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Example

Learning Phase

Outlook	Play=*Yes*	Play=*No*
Sunny	2/9	3/5
Overcast	4/9	0/5
Rain	3/9	2/5

Temperature	Play=*Yes*	Play=*No*
Hot	2/9	2/5
Mild	4/9	2/5
Cool	3/9	1/5

Humidity	Play=*Yes*	Play=No
High	3/9	4/5
Normal	6/9	1/5

Wind	Play=*Yes*	Play=*No*
Strong	3/9	3/5
Weak	6/9	2/5

$P(\text{Play}=Yes) = 9/14$ $\qquad$ $P(\text{Play}=No) = 5/14$

Example

Test Phase

- Given a new instance, predict its label

  $\mathbf{x}'$=(Outlook=*Sunny,* Temperature=*Cool,* Humidity=*High,* Wind=*Strong*)

- Look up tables achieved in the learning phrase

$P(\text{Outlook}=Sunny \mid \text{Play}=Yes) = 2/9$

$P(\text{Temperature}=Cool \mid \text{Play}=Yes) = 3/9$

$P(\text{Huminity}=High \mid \text{Play}=Yes) = 3/9$

$P(\text{Wind}=Strong \mid \text{Play}=Yes) = 3/9$

$P(\text{Play}=Yes) = 9/14$

$P(\text{Outlook}=Sunny \mid \text{Play}=No) = 3/5$

$P(\text{Temperature}=Cool \mid \text{Play}==No) = 1/5$

$P(\text{Huminity}=High \mid \text{Play}=No) = 4/5$

$P(\text{Wind}=Strong \mid \text{Play}=No) = 3/5$

$P(\text{Play}=No) = 5/14$

- Decision making with the MAP rule

$$P(Yes \mid \mathbf{x}') \approx [P(Sunny \mid \text{Yes})P(Cool \mid Yes)P(High \mid \text{Yes})P(Strong \mid \text{Yes})]P(\text{Play}=Yes) = 0.0053$$

$$P(No \mid \mathbf{x}') \approx [P(Sunny \mid \text{No})\, P(Cool \mid \text{No})P(High \mid No)P(Strong \mid No)]P(\text{Play}=No) = 0.0206$$

Given the fact $P(Yes \mid \mathbf{x}') < P(No \mid \mathbf{x}')$, we label $\mathbf{x}'$ to be "*No*".

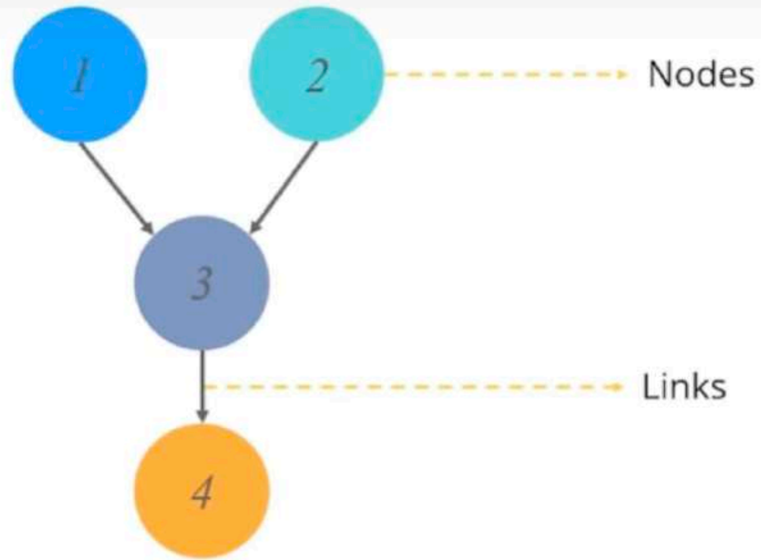
Naïve Bayes: Assumptions of Conditional Independence

Often the X_i are not really conditionally independent

- We can use Naïve Bayes in many cases anyway
  - often the right classification, even when not the right probability

What Is A Bayesian Network?

A Bayesian Network falls under the category of Probabilistic Graphical Modelling (PGM) technique that is used to compute uncertainties by using the concept of probability.



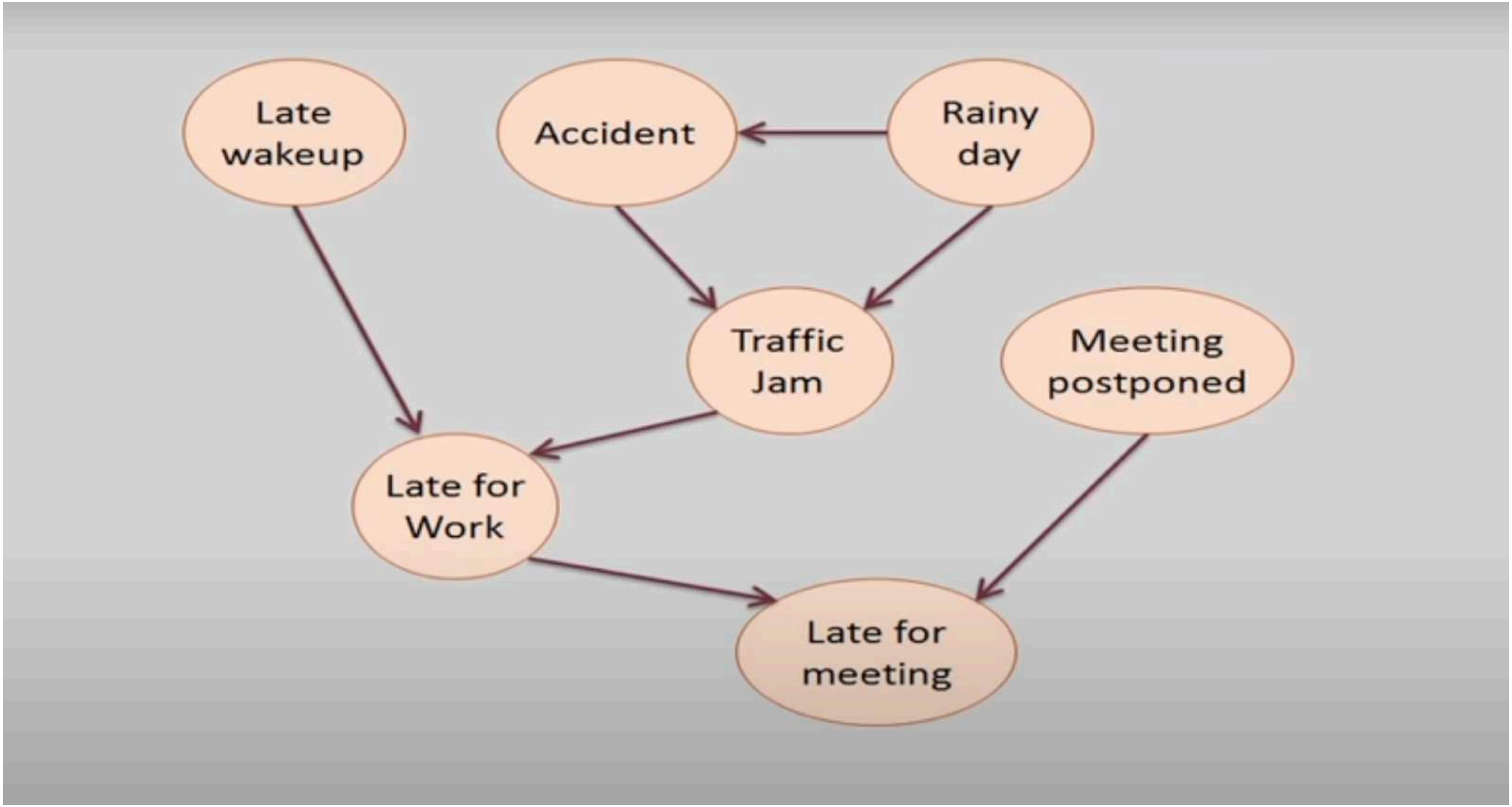
A DAG models the uncertainty of an event occurring based on the Conditional Probability Distribution (CDP) of each random variable

What Is A Directed Acyclic Graph?

A Directed Acyclic Graph is used to represent a Bayesian Network and like any other statistical graph, a DAG contains a set of nodes and links, where the links denote the relationship between the nodes.

Probabilistic Graphical Models

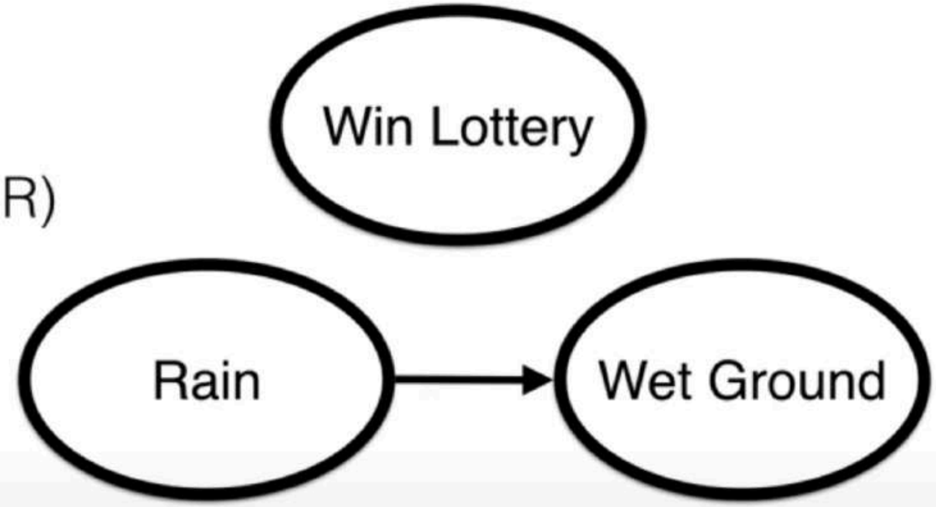
- PGMs represent probability distributions
- They encode conditional independence structure with graphs
- They enable graph algorithms for inference and learning


Late wakeup
Accident
Rainy day
Traffic Jam
Meeting postponed
Late for Work
Late for meeting

Bayesian Networks

$P(L, R, W)$

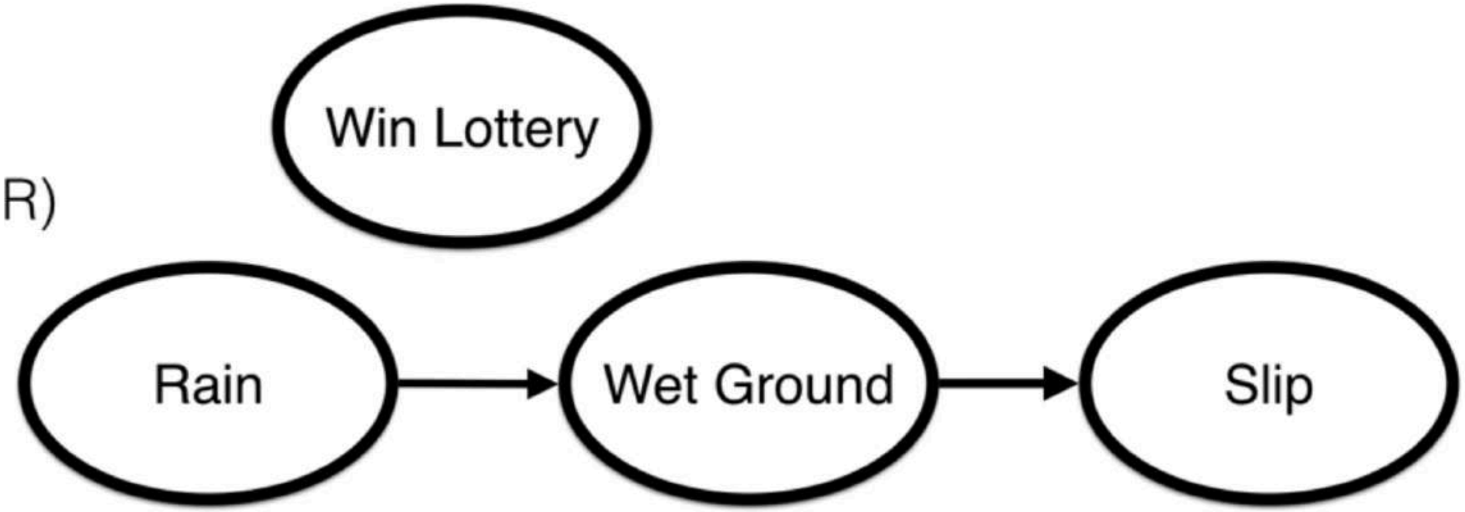
$= P(L)\, P(R)\, P(W \mid R)$



Bayesian Networks

P(L, R, W)

= P(L) P(R) P(W | R)

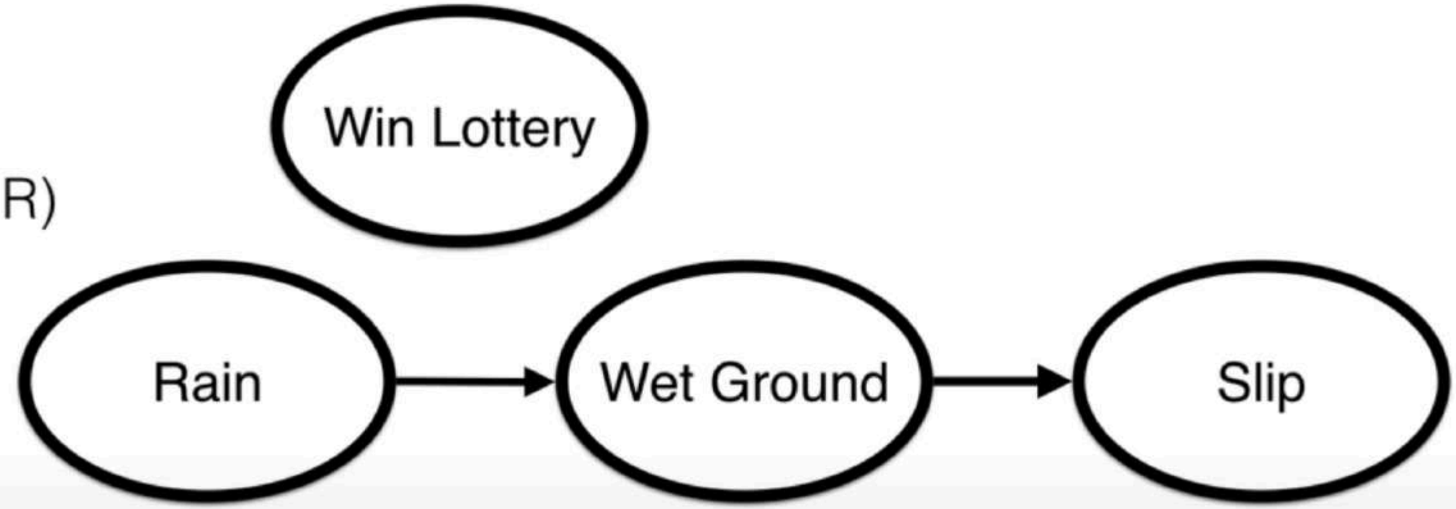


P(L, R, W, S) = P(L) P(R) P(W | R) P(S | W)

Bayesian Networks

$P(L, R, W)$

$= P(L)\, P(R)\, P(W \mid R)$



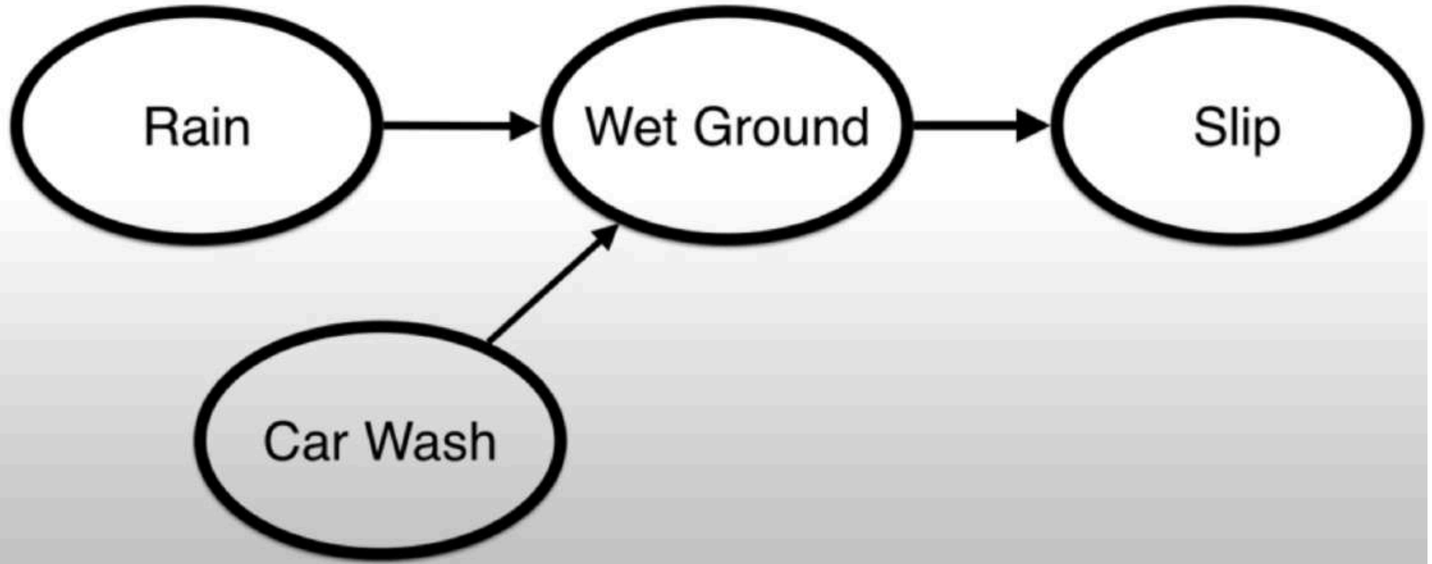
$P(L, R, W, S) = P(L)\, P(R)\, P(W \mid R)\, P(S \mid W)$

$\cancel{P(S \mid W, R)}$

Bayesian Networks

$P(R, W, S, C) = P(R)\, P(C)\, \boxed{P(W \mid C, R)}\, P(S \mid W)$

$P(X \mid \text{Parents}(X))$



Bayesian Networks

- Structure of the graph ⇔ Conditional independence relations

In general,

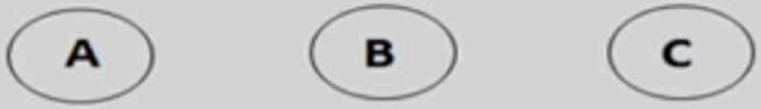
$$p(X_1, X_2, \ldots X_N) = \Pi\, p(X_i \mid \text{parents}(X_i))$$

The full joint distribution

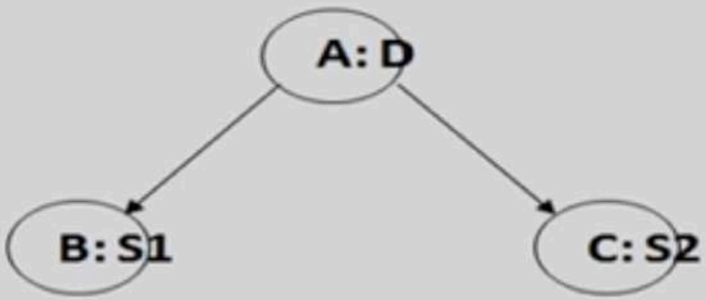
The graph-structured approximation

- Requires that graph is acyclic (no directed cycles)
- 2 components to a Bayesian network
  - The graph structure (conditional independence assumptions)
  - The numerical probabilities (for each variable given its parents)

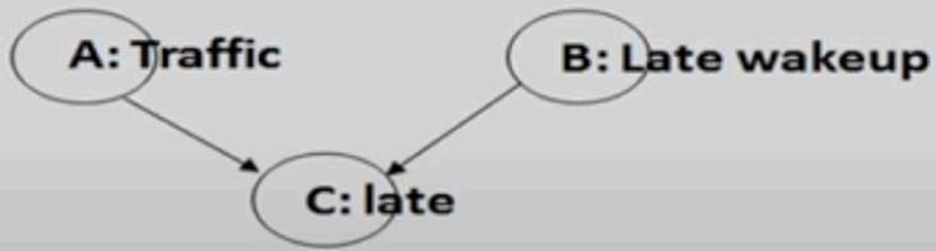
Examples



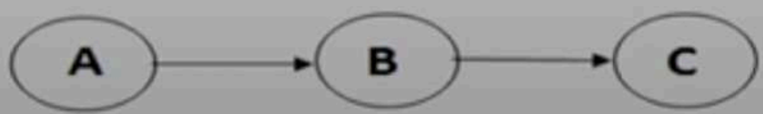
Marginal Independence:
p(A,B,C) = p(A) p(B) p(C)



Conditionally independent effects:
p(A,B,C) = p(B|A)p(C|A)p(A)
B and C are conditionally independent
Given A



Independent Causes:
p(A,B,C) = p(C|A,B)p(A)p(B)
"Explaining away"



Markov dependence:
p(A,B,C) = p(C|B) p(B|A)p(A)

Learning Bayesian Belief Networks

1. The network structure is given in advance and all the variables are fully observable in the training examples.

– estimate the conditional probabilities.

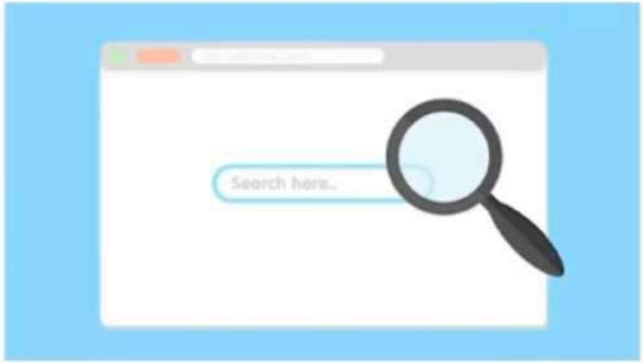
2. The network structure is given in advance but only some of the variables are observable in the training data.

– Similar to learning the weights for the hidden units of a Neural Net: Gradient Ascent Procedure

3. The network structure is not known in advance.

– Use a heuristic search or constraint-based technique to search through potential structures.

Disease Diagnosis



Optimized Web Search

Bayesian Networks Applications

Applications of Bayesian Networks

- Diagnosis: P(cause|symptom)=?
- Prediction: P(symptom|cause)=?
- Classification: P(class|data)
- Decision-making
  (given a cost function)