P, NP Theory - II

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How to we deal with the hard problems ?

If we are not able to solve these problems,

at least

we would be happy

if we were able to verify a given solution

to be optimal or not…….

in polynomial time ???

Nondeterminism

- Deterministic Algorithms
  - determinism - the property that the result of every operation is uniquely defined.
  - that can be implemented on the actual machine
- Deterministic Polynomial Algorithms
  - Deterministic algorithms that can be solved reasonably fast
    - whose complexity is bounded by $O(n^c)$, c a positive constant, n input size as seen before.

Nondeterminism

- Theoreticians treat even $n^{million}$ as solvable reasonably fast
  - even while practically their execution time is very long
- That is the reason why
  - we have to conceptually conceive non-determinism.

Understanding Nondeterminism

- A nondeterministic algorithm can be understood using three new functions
  - Choice(S)
    - Takes a set S as input & arbitrarily chooses one of the elements of the set S
    - e.g. x=choice(1,n)
      - x is assigned any one of the integers in [1…n] ….can be qualified in any manner we desire…
    - takes time O(1)
  - Failure()
    - signals an unsuccessful completion of the algorithm
    - takes time O(1)
  - Success()
    - signals a successful completion of the algorithm
    - takes time O(1)

Which function(s) amongst these is/are deterministic ? Nondeterministic ?

Nondeterministic Search Algorithm

- Problem
  - Searching for an element x in a given set of elements A[n], n >=1.
  - Output An index j, such that A[j]=x OR j=0 if x is not in A.
  - Design the algorithm **NonDeterministicSearch(x, A)**

```
Algorithm NonDeterministicSearch(x, A)
1. j = Choice(1,n);
2. if A[j]=x, then write(j); Success();
3. write(0); Failure();`
```

- Algorithm has nondeterministic complexity O(1). Why ?

Nondeterministic Sorting Algorithm

- Problem
  - Sorting A[i], 1<=i<=n, an unsorted array of positive integer
  - Output A[i], sorted in nondecreasing order

```
Algorithm NonDeterministicSorting(A, n)
1.  for i= 1 to n, B[i]=0;
2.  for i= 1 to n {
3.     j=Choice(1,n);
4.     if B[j] ≠ 0 then Failure();
5.     B[j] = A[i];
6.  }
7.  for i=1 to n-1 //Verify the order
8.     if B[i]>B[i+1] then Failure();
9.  write B[1..n];
10.Success();
```

What is the overall Nondeterministic Complexity ??

Why Nondeterminism ?

- Nondeterministic algorithms in computational complexity theory.
  - are typically studied only on a theoretical framework.
  - outcome of an operation is not uniquely defined…..
    - but is limited to specified sets of possibilities. i.e. these can allow for multiple choices to be available for the next step of computation…….
      - at every step, creating many possible computational paths.
  - these algorithms do *not* arrive at a solution for every possible computational path,
    - but are guaranteed to arrive at a correct solution for some path i.e., if the right choices are made underway.

…….contd

Why Nondeterminism ?……

- Nondeterministic algorithms in computational complexity theory.
  - the choices can be interpreted as guesses in a search process.
  - a large number of real-life problems can be naturally stated/solved in this way
    - hence nondeterminism can help define commonality amongst the problems.

Problem Boolean Satisfiability

- Input
  - A boolean formula F in CNF
- Goal
  - Check if F is satisfiable or not.
    - e.g. if `F=(x1+x2)` can we assign at least one set of values to the literals of the formula so that F=1.
- How to solve this problem deterministically ?
  - What could be the Brute-force approach ?
  - Time complexity ??
    - $O(2^n|F|)$
  - Cannot do any better than that.
- Hence, the nondeterministic solution……………..

Nondeterministic Boolean Satisfiability

- Phase1

```
1. for i=1 to n do
2.    use Choice() to determine the value of xi's
      that will satisfy F
```

- Phase2

```
1. Check if F is satisfied under the above
   criterion
```

- Runtime
  - Phase1……O(1) * n i.e. O(n)
  - Phase2……O(|F|)
  - Total………O(n + |F|)

Problem k-Clique

- Input G(V,E), k
- Output "Yes", if G has a clique of size k and "No" otherwise
  - e.g.

k-Clique Deterministic approach

- Input G(V,E), k
- Output "Yes", if G has a clique of size k and "No" otherwise
- Naïve/Deterministic approach
  - For every subset of k nodes, check if these k nodes form a clique
  - Time complexity
    - $\binom{n}{k}\binom{k}{2}$
    - i.e. $O(n^k k^2)$
- The issue is can we do better than this ?

k-Clique Non-deterministic approach

```
Phase 1
   Using the Choice(), we investigate whether "for every
   node in the graph is this node a k-clique?"
   Guess k nodes that form a clique

Phase 2
   Verify whether the above nodes indeed form a k-clique.
```

- Time Complexity
  - Phase I takes k units of time…..
  - Phase two takes (k choose 2 i.e.) units of time
  - O(k + (k choose 2)) i.e. $O(k^2)$.

Complexity Classes

- A Complexity Class
  - Is a collection of decision problems (???) all of which can be solved in SAME resource bounds
    - i.e. within same time, space or randomness….
- The question is

Why do we bring in the discussion and then focus on decision problems and not on search problems ?

Different versions of problems

- Combinatorial Optimization problems
  - Answer is a number
- Decision problems
  - Answer is yes or no
- Construction problems
  - Answer is some object (set of vertices, function, …)

Classes of Problems

- Combinatorial Optimization problems
  - how to we define these ?
- Output

Our Focus on decision problems

Classes of Problems

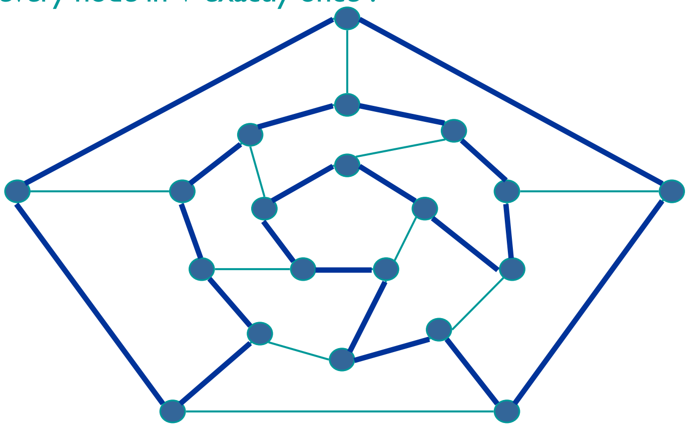
- Decision problems
  - variation of optimization problems
  - answer yes/no to a question
- Statement of a decision problem has two parts
  - instance description
  - question with variables defined in instance description with actual yes/no asked
- Many problems will have decision and optimization versions.
- For example ??

Illustrating the difference

- Clique
  - optimization
    - instance For an undirected graph G = (V, E)
    - question does G contain a clique of k vertices ?
      - say running time $O(k^2n^k)$.
      - Is it polynomial ?......depends on the nature of k
  - Decision
    - instance For an undirected graph G = (V, E) and an integer k
    - question does G contain a clique of k vertices ?
      - say running time $O(k^2n^k)$.
      - Is it polynomial ? ......depends on the nature of k

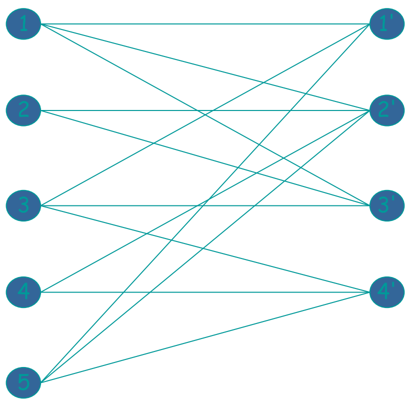
Hamiltonian Cycle

- given an undirected graph G = (V, E), does there exist a simple cycle Γ that contains every node in V exactly once ?



Hamiltonian Cycle (contd)

- HAM-CYCLE given an undirected graph G = (V, E), does there exist a simple cycle Γ that contains every node in V.



Traveling Salesperson problem

- Minimum Tour problem
  - Optimization problem
    - Given a complete, weighted graph find a minimum-weight Hamiltonian cycle
  - Decision problem
    - Given a complete, weighted graph and an integer k, is there a Hamiltonian cycle with total weight at most k?

Contents

Class P

- P
  - the class of decision problems that are solvable deterministically in O(p(n)), where $p(n)$ is a polynomial on n
  - the class of decision problems that are polynomially bounded.

Solving the Decision Problems

- Can every decision problem be solved in polynomial time ?
  - Obviously not !!
  - Some decision problems cannot be solved at all by any algorithm
    - Undecidable problems e.g. The Halting problem
  - Proving that the Halting problem can not be solved at all…..
  - Are there any decidable but intractable problems ?
    - Yes, they are

Solving the Decision Problems…

- However, as seen before…..the important aspect is
  - that there are decidable problems for which neither the polynomial time algorithm has been found nor impossibility of doing so can be proved

Class NP

- NP – Nondeterministic Polynomial and not Nonpolynomial !!
  - the set of all the decision problems that can be solved using nondeterministic algorithms in polynomial time
  - Loosely
    - the class of decision problems for which a given proposed solution for a given input

    can be checked quickly in polynomial time.

Class NP

- Recollect that
  - A nondeterministic algorithm is a two stage procedure that takes as input an instance i of a decision problem
    - Guessing (nondeterministic) stage
      - an arbitrary string S is generated that can be thought of as a candidate solution to the given instance i
    - Verification (Deterministic) stage
      - a deterministic algorithm takes both i and S as its input and outputs yes if S represents a solution to instance i; otherwise returns no.

Class NP (contd)

- Thus, NP can also be thought of as
  - the collection of problems that have polynomially i.e. efficiently verifiable solutions
    - that can be solved in polynomial time on a machine that can pursue infinitely many paths of the computation in parallel

Certification in NP

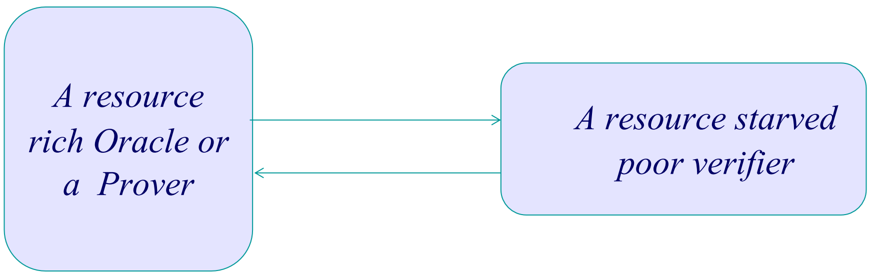
- def
  - Algorithm C(s, t) is a certifier for problem X if for every string s, s ∈ X there exists a string t such that C(s, t) = yes.

"certificate" or "witness"

C(s, t) is a poly-time algorithm and |t| ≤ p(|s|) for some polynomial p(·).

- Certification algorithm intuition.
  - Certifier views things from "managerial" viewpoint.
  - Certifier doesn't determine whether s ∈ X on its own; rather, it checks a proposed proof t that s ∈ X.
- NP
  - Decision problems for which there exists a poly-time certifier.

Prover Verifier View



Certifiers-Certificates Composite

- COMPOSITES. Given an integer s, is s composite?
- Certificate
  - A nontrivial factor t of s. Note that such a certificate exists iff s is composite. Moreover $|t| \le |s|$.
- Certifier

```
boolean C(s, t) {
   if (t ≤ 1 or t ≥ s)
      return false
   else if (s is a multiple of t)
      return true
   else
      return false
}
```

- Instance. s = 437,669.
- Certificate. t = 541 or 809.
- Conclusion. COMPOSITES is in NP.

$437{,}669 = 541 \times 809$

Certifiers-Certificates 3-Satisfiability

- SAT
  - Given a CNF formula Φ, is there a satisfying assignment?
- Certificate
  - An assignment of truth values to the n boolean variables.
- Certifier
  - Check that each clause in Φ has at least one true literal.
- Ex.

instance s $\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_1 \vee x_2 \vee x_4\right) \wedge \left(\overline{x_1} \vee \overline{x_3} \vee \overline{x_4}\right)$

$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$ certificate t

- Conclusion. SAT is in NP.

Certifiers-Certificates Hamiltonian Cycle

- HAM-CYCLE
  - Given an undirected graph G = (V, E), does there exist a simple cycle C that visits every node?
- Certificate
  - A permutation of the n nodes.
- Certifier
  - Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.

- Conclusion. HAM-CYCLE is in NP.

Certifiers-Certificates k-CLIQUE

- A graph G=(V, E) is a Clique if every two nodes in V are connected by an edge
- K-CLIQUE
  - Given a graph G=(V,E), prove that k–clique is in NP.
- Proof
  - define CLIQUE = {<G,K>|G is an undirected graph with a k-clique
  - Certificate
    - the collection of k vertices of the graph
  - Certifier
    - Take as input some string c as (<G,k>, c)
    - Test whether c is a set of k nodes in G
    - Test whether G contains all edges connecting nodes in c
    - If so, proved.

Certifiers-Certificates SUBSET SUm

- Tutorial Assignment……………………….

Revisiting P, NP, EXP

- P
  - Decision problems for which there is a poly-time algorithm.
- EXP
  - Decision problems for which there is an exponential-time algorithm.
- NP
  - Decision problems for which there is a poly-time certifier.

Revisiting P, NP, EXP (contd)

- Claim. P $\subseteq$ NP.
- Proof
  - Consider any problem X in P.
    - By definition, there exists a poly-time algorithm A(s) that solves X.
    - Certificate t = ε, certifier C(s, t) = A(s).
    - Therefore, the fact.

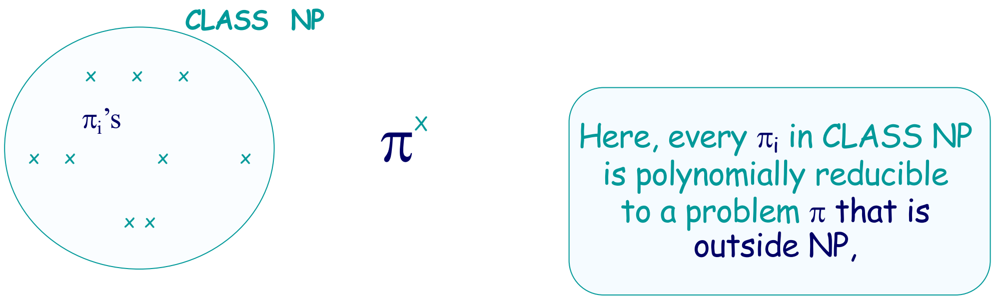
Remember, we said an Algorithm C(s, t) is a certifier for problem X;
iff for every string s, s $\in$ X there exists a string t such that C(s, t) = yes.

Revisiting P, NP, EXP (contd)

- Claim. NP $\subseteq$ EXP.
- Proof
  - Consider any problem X in NP.
    - By definition, there exists a poly-time certifier C(s, t) for X.
    - To solve input s, run C(s, t) on all strings t with $|t| \leq p(|s|)$.
    - Return `yes`, if C(s, t) returns `yes` for any of these

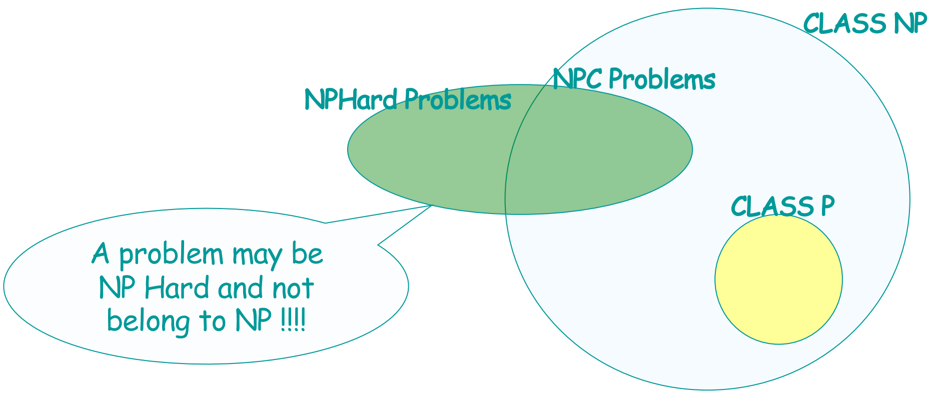
NP-Hard

- A problem π is NP-Hard if for every problem $\pi_l \in$ NP, π_l reduces to π.
  - i.e. π is at least as difficult as π_l OR
  - i.e. π is at least as difficult as every problem in class NP.
- Graphically,



NP Complete Problems

- A problem π is NPC if π is NP-Hard and $\pi \in$ NP.
- Thus, based on the above we can conceive of the Venn-diagram as below



NP Completeness

- Cook and Levin 's theory
  - There are certain problems in NP
    - whose individual complexity is related to that of the entire class.
    - i.e. if a poly time algorithm were to exist for any problem in this class, all problems in NP would be polynomial time solvable.
- These problems are called NP-complete problems.

NP Completeness

- A problem P is NP-complete if one can prove that
  - P is in NP, and
  - show that there is some other problem Q - known to be an NPC – is poly-time reducible to the problem P.
    - the hard part of that was proving the *first* example of an NP-complete problem
    - that was done by Steve Cook in Cook's Theorem

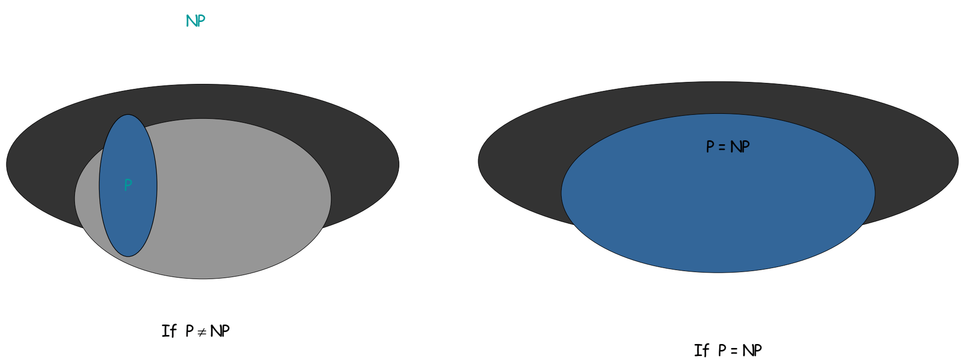
Theorem P = NP

- Fact Let π be any NPC problem. If $\pi \in P$, then P = NP.
- Proof
  - Let π_I be any problem in NP.
  - Now, if π_I is an NPC problem, then by definition of NPC, π_I is also NP-Hard and so $\pi_I \propto \pi$.
  - Now, if $\pi \in P$, then since $\pi_I \propto \pi$, π_I is also in P
  - Therefore, P = NP.

- However, the belief is that if π is any NPC problem, then π !∈ P.
- So, P != NP.

The Main Question P Versus NP

- Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]
  - Is the decision problem as easy as the certification problem?



The Main Question P Versus NP…

- Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]
  - Is the decision problem as easy as the certification problem?
- If yes
  - Efficient algorithms for 3-COLOR, TSP, FACTOR, SAT, …
- If no
  - No efficient algorithms possible for 3-COLOR, TSP, SAT, …
- Consensus opinion on P = NP? Probably no.

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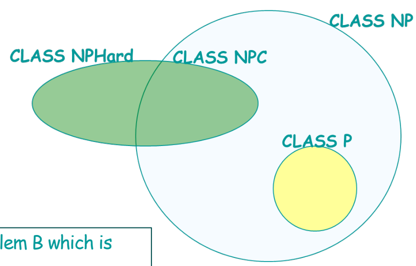
Theorem

- Prove that if a problem B is NP-complete and B $\leq_P$ C for C in NP, then C is NP-Complete.

A problem Π is NPC if it belongs to the class NP and for every problem $Π_1$ ∈ NP, $Π_1$ reduces to Π.

- Proof
  - We are given C is in NP and
    - to prove that C is NPC we must now show that every problem in class NP is polynomial time reducible to C.
  - We are given that B is NP-complete.
    - That is, every problem in NP is $\leq_P$ to B
    - But, we are given that B $\leq_P$ C
    - Therefore, every problem A in NP is $\leq_P$ C
    - Therefore, C is NP-complete



A problem A is NP-complete if one can prove that A is in NP, and there is some problem B which is already proven to be an NPC is poly-time reducible to a problem A.

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0/1 Knapsack problem: A special mention

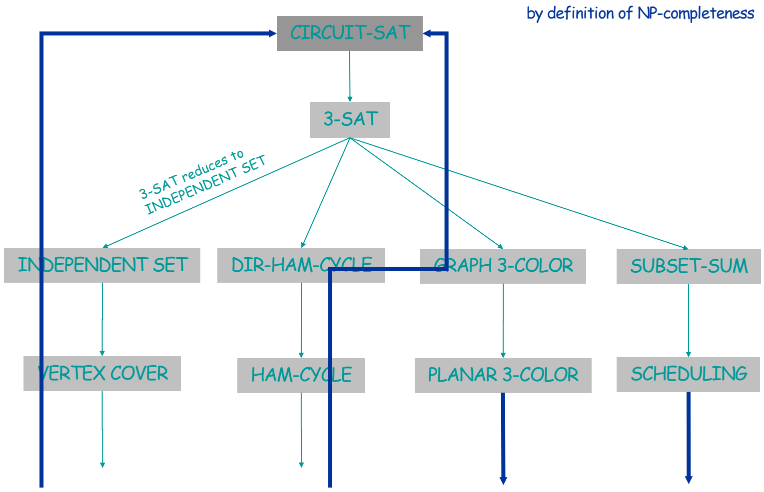
- Is 0/1 knapsack problem solvable using dynamic programming?
- Why is then its solution referred to as pseudo-polynomial time solution?
- /*write here from the notes page */

Any problems that are NPH but not NPC?

- Can there be any problem that is NPH but not NPC ?
- When can a problem be NPH but not NPC ?
- The following problems are NPH but not NPC

NP-Completeness

- Observation. All problems below are NP-complete and polynomial reduce to one another!



Some NP-Complete Problems

- Six basic genres of NP-complete problems and paradigmatic examples.
  - Packing problems SET-PACKING, INDEPENDENT SET.
  - Covering problems SET-COVER, VERTEX-COVER.
  - Constraint satisfaction problems SAT, 3-SAT.
  - Sequencing problems HAMILTONIAN-CYCLE.
  - Partitioning problems 3D-MATCHING 3-COLOR.
  - Numerical problems SUBSET-SUM, KNAPSACK.
  - Practice. Most NP problems are either known to be in P or NP-complete.
- Notable exceptions. Factoring, graph isomorphism, Nash equilibrium.

Concluding Remarks

References

- Text by CLRS
- Text by Garey and Johnson
- Text by Klienberg Tardos
- NPTEL Videos
- Special gratitude to Prof Rajsekaran, IUCEE Course Instructor at Infosys, July 2009.

To Teach is to Learn twice !!!!

Hence, Thank You !!!