Computer Vision and Image Processing

Mukesh A. Zaveri
Computer Engineering Department
Sardar Vallabhbhai National Institute of Technology, Surat
mazaveri@coed.svnit.ac.in

Freqency Domain Presentation and Processing

- French mathematician Jean Baptiste Joseph Fourier
- any function that periodically repeats itself, can be expressed as
- sum of sines / cosines of different frequencies
- known as Fourier series
- functions that are not periodic, can be expressed as
- integral of sines / cosines multiplied by a weighing function
- known as Fourier transform
- inverse process; reconstruction with no loss of information

Frequency Domain: Frequency Transform

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx \quad f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

- discrete Fourier transform

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \quad x,u = 0,1,2,\ldots,M-1$$

$$f(x) = \sum_{x=0}^{M-1} F(u) e^{j2\pi ux/M} \quad x,u = 0,1,2,\ldots,M-1$$

both equations are multiplied by $1/\sqrt{M}$

Fourier transform

- $e^{j\theta} = \cos\theta + j\sin\theta$

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\left[\cos 2\pi ux/M - j\sin 2\pi ux/M\right]$$

- domain over which the values of $F(u)$ range - called the frequency domain
- M terms of $F(u)$ - called freqency components
- $F(u) = |F(u)|e^{-j\phi(u)}$ where $|F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2}$ called spectrum of Fourier transform
- phase spectrum of transform $\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$
- power spectrum $P(u) = |F(u)|^2$ - called spectral density

Discrete Fourier transform

- spectrum centered at $u = 0$; multiplying $f(x)$ by $(-1)^x$ before taking the transform
- $K = 8$ points, $M = 1024$ vs. $K = 16$ points and $M = 1024$
- height of spectrum doubled as area under the curve is doubled
- the number of zeros in the spectrum in the same interval doubled as the lenght of function doubled
- $f(x)$ having M samples, not necessarily taken at integer value of x in the interval $[0, M-1]$, but equally spaced, first point x_0;$f(x_0)$
- next sample $f(x_0 + \triangle x)$, kth sample $f(x_0 + k \triangle x)$ and final sample $f(x_0 + [M-1] \triangle x)$

Discrete Fourier transform

- $f(k)$ notation for $f(x_0 + k \triangle x)$, i.e., $f(x) \triangleq f(x_0 + x \triangle x)$
- u always starts at 0 frequency, values of u: $0, \triangle u, 2 \triangle u, \ldots, [M-1] \triangle u$ $F(u) \triangleq F(u \triangle u)$
- $\triangle x$ and $\triangle u$ are inversely related $\triangle u = \frac{1}{M \triangle x}$
- two dimensional DFT $f(x, y)$ of size $M \times N$

$$F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

$x = 0, 1, 2, \ldots, M-1$ and $y = 0, 1, 2, \ldots, N-1$

DFT

- u and v are transform or frequency variables
- x and y are spatial or image variables
- multiply input image function $(-1)^{x+y}$ prior to computing Fourier transform
- shifting origin of $F(u, v)$ to freqeuncy coordinates $(M/2, N/2)$
- $F(0, 0)$ is located at $u = M/2$ and $v = N/2$

$$\mathfrak{F}\left[f(x, y)(-1)^{x+y}\right] = F(u - M/2, v - N/2)$$

$$F(0, 0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

dc component of the spectrum, Fourier transform at the origin is equal to the average gray level of the image

DFT

- if $f(x, y)$ is real, Fourier transform is conjugate symmetric $F(u, v) = F^*(-u, -v)$
- $|F(u, v)| = |F(-u, -v)|$ spectrum of Fourier transform is symmetric
- centering property and conjugate symmetry: circularly symmetric filters in the freqency domain
- $\triangle u = \frac{1}{M \triangle x}$ and $\triangle v = \frac{1}{N \triangle y}$
- log transform, $c = 0.5$, plotting Fourier transform

Filtering in Frequency Domain

- each term $F(u,v)$ contains all values of $f(x,y)$, modified by values of exponential terms
- frequency related to rate of change
- frequencies in Fourier transform associated with patterns of intensity variations in an image
- low freqencies correspond to slowly varying components of an image
- e.g. correspond to smooth variations
- high freqencies correspond to faster gray level changes in the image
- e.g. edges, abrupt changes in gray level such as noise
- $H(u,v)$ filter surpresses certain frequencies and leaving other unchanged

$$G(u,v) = H(u,v)F(u,v)$$

Filtering in Frequency Domain

- F complex quantity, H generally real, are called zero phase shift filters
- filters do not change phase of the transform
- filtered image $\mathfrak{F}^{-1}[G(u,v)]$, is also complex
- image and filter function are real, imaginary components of inverse transform should all be zero
- due to round off erros inverse DFT has imaginary components
- $F(0,0)$ set to zero, average value of an image becomes zero, notch filter

$$H(u,v) = \begin{cases} 0 & \text{if } (u,v) = (M/2, N/2) \\ 1 & \text{otherwise} \end{cases}$$

Filtering in Frequency Domain

- relationship between spatial and frequency domain
- convolution theorem

$$f(x,y) * h(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m, y-n)$$

- flipping function about origin
- shift function w.r.t. other by changing (x,y)
- computing sum of products over all values of m and n

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

$$f(x,y)h(x,y) \Leftrightarrow F(u,v) * H(u,v)$$

- convolution in frequency domain - multiplication in spatial domain

Filtering in Frequency Domain

- impulse function, located at (x_0, y_0): $A\delta(x - x_0, y - y_0)$

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} s(x,y) A\delta(x - x_0, y - y_0) = As(x_0, y_0)$$

- sifting property of the impulse function
- Fourier transform of a unit impulse at the origin

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} s(x,y)\delta(x,y) = s(0,0)$$

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \delta(x,y) e^{-j2\pi(ux/M + vy/N)} = \frac{1}{MN}$$

Filtering in Frequency Domain

- impulse at origin, Fourier transform is a real constant
- impulse located elsewhere, the transform has complex value
- $f(x,y) = \delta(x,y)$

$$f(x,y) * h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\delta(m,n)h(x-m,y-n) = \frac{1}{MN}h(x,y)$$

$$\delta(x,y) * h(x,y) \Leftrightarrow \mathfrak{F}[\delta(x,y)]\,H(u,v) \quad h(x,y) \Leftrightarrow H(u,v)$$

- given a filter in frequency domain; inverse Fourier transform corresponds to filter in spatial domain
- frequency domain filter size $M \times N$, much small filter in spatial domain

Gaussian Filter

- forward and inverse Fourier transforms of Guassian function are real Guassian functions

$$H(u) = Ae^{-u^2/2\sigma^2} \qquad h(x) = \sqrt{2\pi}\sigma Ae^{-2\pi^2\sigma^2x^2}$$

- these functions reciprocal w.r.t. one another
- $H(u)$ broad profile (larger value of σ), $h(x)$ has narrow profile
- σ approaches infinity, $H(u)$ becomes a constant function and $h(x)$ an impulse
- highpass filter as difference of Gaussians

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2} \quad (A \geq B, \sigma_1 > \sigma_2)$$

$$h(x) = \sqrt{2\pi}\sigma_1 Ae^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma_2 Be^{-2\pi^2\sigma_2^2x^2}$$

Frequency Domain Filters

- $G(u,v) = H(u,v)F(u,v)$
- lowpass filters: ideal, Butterworth and Gaussian
- Butterworth filter, filter order as parameter, high value of it makes filter ideal

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

- D_0 is specified nonnegative quantity, cutoff frequency
- $D(u,v)$ is distance from point (u,v) to origin of frequency rectangle
- center is at $(u,v) = (M/2, N/2)$

$$D(u,v) = \left[(u-M/2)^2 + (v-N/2)^2\right]^{1/2}$$

Frequency Domain Filters

- cutoff frequency, compute radius that enclose specified amount of total image power
- as filter radius increases, less power is removed, reduces blurring
- "ringing" becomes finer as amount of high frequency content removed decreases
- "ringing" behavior is a charateristic of ideal filters

$$\alpha = 100\left[\sum_u\sum_v P(u,v)/P_T\right] \qquad P_T = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1}P(u,v)$$

- $h(x,y)$ obtained by $H(u,v)$ multiplied by $(-1)^{u+v}$ for centering
- inverse DFT and real part of it multiplied by $(-1)^{x+y}$
- center component responsible for blurring
- concentric components responsible for ringing characteristic

Frequency Domain Filters

- Butterworth lowpass filter of order n, cutoff frequency at distance D_0

$$H(u,v) = \frac{1}{1+[D(u,v)/D_0]^{2n}}$$

- does not have sharp discontinuity
- Gaussian lowpass filter

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

- σ measure of spread of Gaussian curve ($\sigma = D_0$)
- highpass filter $H_{hp}(u,v) = 1 - H_{lp}(u,v)$

Frequency Domain Filters

- Ideal highpass filter

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

- Butterworth highpass filter

$$H(u,v) = \frac{1}{1+[D_0/D(u,v)]^{2n}}$$

- Gaussian highpass filter

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$