

## Multi User Detection

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M. A. Zaveri  
Computer Engineering Department  
Sardar Vallabhbhai National Institute of Technology, Surat  
mazaveri@coed.svnit.ac.in



## Single User System

- transmitting a sequence  $\{b[0], b[1], \dots, b[M-1]\}$
- $(b[i] = \pm 1)$  or a finite alphabet of complex numbers
- linear modulation using a signaling waveform

$$x(t) = \sum_{i=0}^{M-1} b[i]w_i(t)$$

$w_i(\cdot)$  modulation waveform associated with the  $i^{th}$  symbol, for example



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$$w_i(t) = Ap(t - iT)e^{j(w_c + \phi)}$$

$A > 0$   $\phi \in (-\pi, \pi)$  and  $p(\cdot)$  baseband pulse shape

$$p(t) = p_T(t) \triangleq \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$



## Single User System

- or spreading waveform for DSSP system

$$p(t) = \sum_{j=0}^{N-1} c_j \psi(t - jT_c)$$

$N$  is the spreading gain and

- $c_0, c_1, \dots, c_{N-1}$  is a pseudorandom spreading code ( $c_j \in \{+1, -1\}$ ),



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- $\psi(\cdot)$  chip waveform and  $T_c \triangleq T/N$  chip interval
- chip waveform may be a unit-energy rectangular pulse of duration  $T_c$ :  
 $\psi(t) = p_{T_c}(t)$
- repeat the same spreading code in every symbol interval
- system with long spreading codes, the periodicity is much longer than a single symbol interval and varies spreading code from symbol to symbol

$$p_i(t) = \sum_{j=0}^{N-1} c_j^i \psi(t - jT_c)$$



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- OFDM: baseband pulse shape is a unit pulse  $p_T$ ,
- intercarrier spacing is  $1/T$  cycles per second, and the phases are orthogonal at this spacing



## Multiple-Access Techniques

- **radio resources shared among multiple users**
- FDMA - frequency band available is divided into subbands and
  - allocated to individual user, users do not transmit signals within other subbands
- TDMA - time is divided into equal-length intervals,
  - each user is allowed to transmit throughout the entire allocated frequency band during a given slot
- **FDMA allows each user to use part of the spectrum all of the time**
- **TDMA allows each user to use all of the spectrum part of the time**



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- TDMA allows each user to use all of the spectrum part of the time
- FDMA and TDMA systems are intended to assign orthogonal channels to all active users by giving each, for exclusive use, a slice of the available frequency band or transmission time
- channels are said to be orthogonal because interference between user does not



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- for a system of  $K$  users

$$x_k(t) = \sum_{i=0}^{M-1} b_k[i] w_{i,k}(t) \quad k = 1, 2, \dots, K$$

$w_{i,k}(\cdot)$  represents  $i$ th modulation waveform of user  $k$



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- if they are with time-slotted amplitude pulses  $\{p_k(\cdot)\}$  - TDMA
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- another aspect of wireless network
  - ▶ ambient noise, propagation losses, multipath, interference
  - ▶ properties arising from the use of multiple antennas
- ambient noise - thermal motion of electrons on the antenna and the receiver electronics and from background radiation sources
- modeled as a very wide bandwidth and no particular deterministic structure (e.g. AWGN)



## Multi User System

- propagation losses: diffusive losses and shadow fading
- diffusive losses due to open nature of wireless channel, energy decreases with the square of the distance between antenna and source
- shadow fading results from the presence of objects, modeled by an attenuation in signal amplitude that follows a log-normal distribution
- multipath - multiple copies of a transmitted signal are received at the receiver



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- multipath - multiple copies of a transmitted signal are received at the receiver
- multipath is manifested in several ways
  - ▶ degree of path difference relative to the wavelength of propagation
  - ▶ degree of path difference relative to the signaling rate
  - ▶ relative motion between the transmitter and receiver
  - ▶ results into Rayleigh fading or frequency-selective fading or time-selective fading



## Multi User System

- multipath from scatterers that are spaced very close together will cause a random change in the amplitude of the received signal
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- when there is relative motion between the transmitter and receiver, this fading depends on time - time-selective fading



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- **multipath from scatterers** that are spaced very close together will cause a **random change in the amplitude of the received signal**
- resulting received amplitude is often modeled as being a **complex Gaussian random variable**
- **random amplitude whose envelope has a Rayleigh distribution** - termed as **Rayleigh fading**
- when the scatterers are spaced so that the **difference in their corresponding path lengths** are significant **relative to a wavelength of the carrier** and **add constructively or destructively**
- this is a **fading depends on the wavelength of radiation** - **frequency-selective fading**
- when there is **relative motion** between the transmitter and receiver, this **fading depends on time** - **time-selective fading**
- when the **difference in path lengths** is such that **time delay of arrival along different paths** is significant **relative to a symbol interval**, results in dispersion of the transmitted signal and causes **ISI**



## Multi User System

- **wideband signaling methods** such as **spread spectrum** - a countermeasure to **frequency-selective fading**
- **dividing a high-rate signal into many parallel lower-rate signal** - OFDM mitigates channel dispersion on high-rate signals
- **multiple access interference (MAI)** - arising from other signals in the same network as
- the signal of interest (if signals received) are not orthogonal to one another
- **co-channel interference** - due to signals different networks but operating in same frequency band
- the above phenomena can be incorporated into a general analytical model for a wireless **multiple-access channel**



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$$r(t) = \sum_{k=0}^K \sum_{i=0}^{M-1} b_k[i] \int_{-\infty}^{\infty} g_k(t, u) w_{i,k}(u) du + i(t) + n(t)$$





## Multi User System

- $g_k(t, u)$  impulse response of a linear filter representing the channel between the  $k$ th transmitter and the receiver
- $i(\cdot)$  co-channel interference and  $n(\cdot)$  ambient noise, in general, all are random processes



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$$g_k(t, u) = \sum_{l=1}^{L_k} \alpha_{l,k} \delta(t - u - \tau_{l,k})$$

$L_k$  number of paths between user  $k$  and the receiver,  $\alpha$  and  $\tau$  gain and delay ( $l$ th path of  $k$ th user)

- model includes frequency-selective fading;
- relative delays will cause constructive and destructive interference at the receiver, depending on the wavelength of propagation and
- Rayleigh fading also using path gains



## Multi User System

- composite modulation waveform associated with  $b_k[i]$ :

$$f_{i,k}(t) = \int_{-\infty}^{\infty} g_k(t, u) w_{i,k}(u) du$$

- if these waveforms are not orthogonal for different values of  $i$ , ISI will result



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- if the composite waveform for different values of  $k$  are not orthogonal, MAI will result
- this can happen in CDMA when pseudorandom code sequences used by different users are not orthogonal,
- this happens in FDMA and TDMA due to the effects of multipath or asynchronous transmission



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- model can be generalized for multiple antennas at the receiver



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$$\mathbf{r}(t) = \sum_{k=1}^K b_k[i] \int_{-\infty}^{\infty} \mathbf{g}_k(t, u) w_{i,k}(u) du + \mathbf{i}(t) + \mathbf{n}(t)$$

- $p$ th component of  $\mathbf{g}_k(t, u)$  is the impulse response of the channel between user  $k$  and the  $p$ th element of the receiving array

$$\mathbf{g}_k(t, u) = \sum_{l=1}^{L_k} \alpha_{l,k} \delta(t - u - \tau_{l,k})$$

- multiple antennas at both the transmitter and receiver called multiple-input/multiple-output (MIMO) systems



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- multiple antennas at both the transmitter and receiver called multiple-input/multiple-output (MIMO) systems
- channel transfer functions are matrices with the number of rows equal to the number of receiving antennas and the number of columns equal to the number of transmitting antennas at each source



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- basic receiver signal processing
- matched filter / RAKE receiver
- say single user  $K = 1$  channel impulse  $g_1(\cdot, \cdot)$  is known to receiver,



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$$r(t) = \sum_{i=0}^{M-1} b_1[i] f_{i,1}(t) + n(t)$$

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$$r(t) = b_1[0] f_{0,1}(t) + n(t)$$

- optimal inferences about the symbol  $b_1[0]$  using likelihood function of observations conditional on the symbol  $b_1[0]$



## Multi User Detection

$$\mathcal{L}(r(\cdot)|b_1[0]) = \exp \left\{ \frac{1}{\sigma^2} \left[ 2\Re \left\{ b_1^*[0] \int_{-\infty}^{\infty} f_{0,1}^*(t) r(t) dt \right\} - |b_1[0]|^2 \int_{-\infty}^{\infty} |f_{0,1}(t)|^2 dt \right] \right\}$$

asterik - complex conjugation  $\Re(\cdot)$  real part of argument

- optimal inferences about  $b_1[0]$  can be made using ML or MAP
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$$\hat{b}_1[0] = \arg \left\{ \max_{b \in \mathcal{A}} \mathcal{L}(r(\cdot)|b_1[0] = b) \right\}$$

$$= \arg \left\{ \max_{b \in \mathcal{A}} \left[ 2\Re \left\{ b^* \int_{-\infty}^{\infty} f_{0,1}^*(t) r(t) dt \right\} - |b|^2 \int_{-\infty}^{\infty} |f_{0,1}(t)|^2 dt \right] \right\}$$

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- the symbol estimate - the solution to the problem is  $\min_{b \in \mathcal{A}} |b - z|^2$

$$z = \frac{\int_{-\infty}^{\infty} f_{0,1}^*(t) r(t) dt}{\int_{-\infty}^{\infty} |f_{0,1}(t)|^2 dt}$$



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$$\hat{b}_i[0] = \text{sign}\{\Re\{z\}\} = \text{sign}\left\{\Re\left\{\int_{-\infty}^{\infty} f_{0,1}^*(t) r(t) dt\right\}\right\}$$

$\text{sign}\{\cdot\}$  denotes signum function:

$$\text{sign}\{x\} = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ +1 & x > 0 \end{cases}$$

- choices of symbol alphabet are M-ary phase shift keying MPSK and quadrature amplitude modulation QAM



## Multi User Detection

- MPSK symbol alphabet is

$$\mathcal{A} = \left\{ e^{j2\pi m/M} \mid m \in \{0, 1, \dots, M-1\} \right\}$$

or some rotation of this set around the unit circle



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- QAM symbol alphabet containing  $M \times N$  values is

$$\mathcal{A} = \{b_R + jb_I \mid b_R \in \mathcal{A}_R \text{ and } b_I \in \mathcal{A}_I\}$$

$\mathcal{A}_R$  and  $\mathcal{A}_I$  are discrete sets of amplitudes containing  $M$  and  $N$  points respectively, with  $M = N$

$$\mathcal{A}_R = \mathcal{A}_I = \left\{ \pm \frac{1}{2}, \pm \frac{3}{2}, \dots, \pm \frac{M}{4} \right\}$$

or a scaled version of this choice



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- BPSK  $\mathcal{A} = \{-1, +1\}$
- for MPSK ML symbol choice that whose angle is closest to the angle of complex number  $z$
- for QAM ML symbol estimate are decoupled with  $\Re\{b\}$
- being chosen to be the closest element of  $\mathcal{A}_R$  to  $\Re\{z\}$



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$$\begin{aligned} \hat{b}_1[0] &= \arg \left\{ \max_{b \in \mathcal{A}} P(b_1[0] = b | r(\cdot)) \right\} \\ &= \arg \left\{ \max_{b \in \mathcal{A}} [\mathcal{L}(r(\cdot) | b_1[0] = b) P(b_1[0] = b)] \right\} \end{aligned}$$

- if symbol are equiprobable ML and MAP estimates are same



## Multi User Detection

- single user, single symbol, known channel case
- the receiver signal processing task is to compute the term

$$y_1[0] = \int_{-\infty}^{\infty} f_{0,1}^* r(t) dt$$

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$$\int_{-\infty}^{\infty} f_{0,1}^* r(t) dt = (h * r)(0)$$

- convolution between  $h$  and  $r$
- $h$  is impulse response of the time invariant linear filter

$$h(t) = f_{0,1}^*(-t)$$

- this structure is called a **matched filter**,



## Multi User Detection

- the impulse response is matched to the composite waveform on which the symbol is received
- the composite signaling waveform has a finite duration so that
- $h(t) = 0$  for  $t < -D \leq 0$
- the matched filter receiver can be implemented by sampling at time  $D$
- the output of the causal filter with the impulse response

$$h_D(t) = \begin{cases} f_{0,1}^*(D - t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- if signaling waveform  $s_{0,1}(t)$  has duration  $[0, T]$  and the channel has delay spread  $\tau_d$
- the composite signaling waveform will have this property with  $D = T + \tau_d$



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- the correlator output

$$y_1[0] = \sum_{l=1}^{L_1} \alpha_{l,1}^* \int_{-\infty}^{\infty} s_{0,1}^*(t - \tau_{l,1}) r(t) dt$$

- a configuration known as RAKE receiver



## Multi User Detection

- Equalization
- there is more than one symbol in the frame  $M > 1$
- likelihood function of observations  $r(\cdot)$  conditioned on the entire frame of symbols  $b_1[0], b_1[1], \dots, b_1[M-1]$



## Multi User Detection

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- $\mathcal{L}(r(\cdot) | b_1[0], b_1[1], \dots, b_1[M-1])$

$$= \exp \left\{ \frac{1}{\sigma^2} \left[ 2\Re \left\{ \mathbf{b}_1^H \mathbf{y}_1 \right\} - \mathbf{b}_1^H \mathbf{H}_1 \mathbf{b}_1 \right] \right\}$$

- $H$  conjugate transpose - Hermitian transpose,  $\mathbf{b}_1$  column vector whose  $i$ th component is  $b_1[i]$ ,  $i = 0, 1, \dots, M-1$
- $\mathbf{y}_1$  its  $i$ th component

$$y_1[i] = \int_{-\infty}^{\infty} f_{i,1}^*(t) r(t) dt$$

- $\mathbf{H}_1$   $M \times M$  whose  $(i, j)$ th element is cross correlation between  $f_{i,1}$  and  $f_{j,1}(t)$



## Multi User Detection

$$\mathbf{H}_1[i, j] = \int_{-\infty}^{\infty} f_{i,1}^*(t) f_{j,1}(t) dt$$

- likelihood function depends on  $r(\cdot)$  through vector  $\mathbf{y}_1$  of correlator outputs
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- maximum likelihood detection



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$$\hat{\mathbf{b}}_1 = \arg \left\{ \max_{\mathbf{b} \in \mathcal{A}^M} \left[ 2\Re \left\{ \mathbf{b}^H \mathbf{y}_1 \right\} - \mathbf{b}^H \mathbf{H}_1 \mathbf{b} \right] \right\}$$

- if  $\mathbf{H}_1$  is a diagonal matrix (all of its off-diagonal elements are zero) decouples into a set of  $M$  independent problems of single symbol type
- the solution in this case

$$\hat{b}_1[i] = \arg \max_{b \in \mathcal{A}} |b - z_1[i]|^2$$

$$z_1[i] = \frac{y_1[i]}{\int_{-\infty}^{\infty} |f_{i,1}(t)|^2 dt}$$



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$$P(b_1[0] = b | r(\cdot)) = \frac{\sum_{\{a \in \mathcal{A}^M | a_i = b\}} \mathcal{L}(r(\cdot) | \mathbf{b}_1 = \mathbf{a}) P(\mathbf{b}_1 = \mathbf{a})}{\sum_{\{a \in \mathcal{A}^M\}} \mathcal{L}(r(\cdot) | \mathbf{b}_1 = \mathbf{a}) P(\mathbf{b}_1 = \mathbf{a})}$$

- the dynamic programming solution known as maximum likelihood sequence detector
- a number of lower-complexity algorithms have been devised
- examining sufficient statistic vector  $\mathbf{y}_1$ , can be written as

$$\mathbf{y}_1 = \mathbf{H}_1 \mathbf{b}_1 + \mathbf{n}_1$$



## Multi User Detection

- $\mathbf{n}_1$  is complex Gaussian random vector with independent real and imaginary parts having
- identical  $\mathcal{N}(\mathbf{0}, \frac{\sigma^2}{2} \mathbf{H}_1)$  distributions
- above equation describes a linear model and the goal of equalization is to fit this model with data vector  $\mathbf{b}_1$
- ML and MAP are two ways but exponential complexity with exponent equal to bandwidth of  $\mathbf{H}_1$
- the vector  $\mathbf{b}_1$  takes on values from a discrete set
- one way is to fit linear model without constraining  $\mathbf{b}_1$  to be discrete and then to
- quantize the resulting (continuous) estimate of  $\mathbf{b}_1$  into symbol estimates



## Multi User Detection

- linear fit  $\mathbf{M}\mathbf{y}_1$  as continuous estimate of  $\mathbf{b}_1$
- $\mathbf{M}$  is  $M \times M$  matrix
- $i$  the symbol decision is  $\hat{b}_1[i] = q([\mathbf{M}\mathbf{y}_1]_i)$
- $[\mathbf{M}\mathbf{y}_1]_i$  denotes  $i$  the component of  $\mathbf{M}\mathbf{y}_1$
- $q(\cdot)$  denotes quantizer mapping the complex numbers to the symbol alphabet  $\mathcal{A}$
- various choice of  $\mathbf{M}$  lead to different linear equalizers
- $\mathbf{M} = \mathbf{I}_M$   $M \times M$  identity matrix
- the resulting linear detector is the common matched filter, which is optimal in the absence of ISI
- matched filter ignores ISI
- if  $\mathbf{H}_1$  is invertible choice  $\mathbf{M} = \mathbf{h}_1^{-1}$
- forces the ISI to zero

$$\mathbf{H}_1^{-1} \mathbf{y}_1 = \mathbf{b}_1 + \mathbf{H}_1^{-1} \mathbf{n}_1$$



## Multi User Detection

- is known as zero forcing equalizer ZFE
- it would be optimal, perfect decision in absence of AWGN
- a tradeoff between the extremes is effected by minimum mean square error MMSE linear equalizer
- which chooses  $\mathbf{M}$  to give an MMSE fit of the model assuming the symbols are independent of the noise
- this results in choice

$$\mathbf{M} = (\mathbf{H}_1 + \sigma^2 \sum_b^{-1})^{-1}$$

- $\sum_b$  denotes covariance matrix of the symbol  $\mathbf{b}_1$  (this will be in the form of a constant time  $\mathbf{I}_M$ )
- for multi user detection symbols sorted by symbol number and then by user number



# Thank You

