Design and Analysis of Algorithms, MTech-I (1^{st} semester)

Chapter 2: Divide and Conquer: Quicksort

September 21, 2022

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1. The Quicksort Background
2. The Quicksort Operation and Algorithm
3. The Quicksort Performance Analysis
4. The Randomized Quicksort

About Quicksort

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- Almost inplace, uses only a small stack. What is inplace sorting algorithm?
- complexity $O(n\ lg\ n)$ on an average - expected complexity.

About Quicksort....

Name	Average	Worst	Memory	Stable	Method
Bubble sort	$O(n^2)$	$O(n^2)$	$O(1)$	Yes	Exchanging
Selection sort	$O(n^2)$	$O(n^2)$	$O(1)$	No	Selection
Insertion sort	$O(n^2)$	$O(n^2)$	$O(1)$	Yes	Insertion
Merge sort	$O(n \log n)$	$O(n \log n)$	$O(n)$	Yes	Merging
Quicksort	$O(n \log n)$	$O(n^2)$	$O(1)$	No	Partitioning
Heapsort	$O(n \log n)$	$O(n \log n)$	$O(1)$	No	Selection

Figure: Sorting algorithms: Complexities

About Quicksort ...

- carefully tuned version is likely to run significantly faster...
- e.g. qsort routine in C++'s STL.
- its average case complexity is considered to be better than that of the Mergesort........ why ?

Quicksort Operation

Operates in two phases

- Partition phase
  - Divides the work into half
- Sort phase
  - Conquers the halves

Partition

- It has hard division and easy combination. . . .
- how does that relate to merge-sort

Quicksort Partitioning

Partitioning

- works on the partitioning of the input array around a pivot

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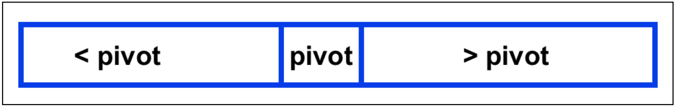

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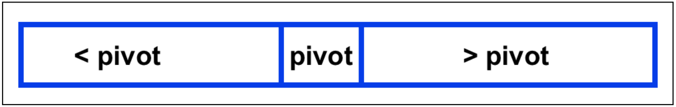

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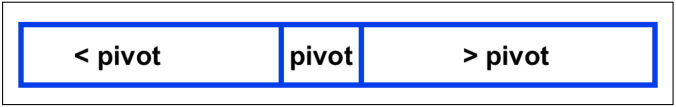

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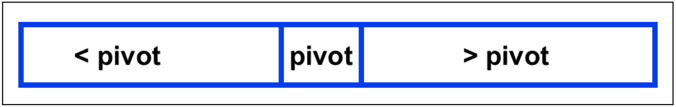
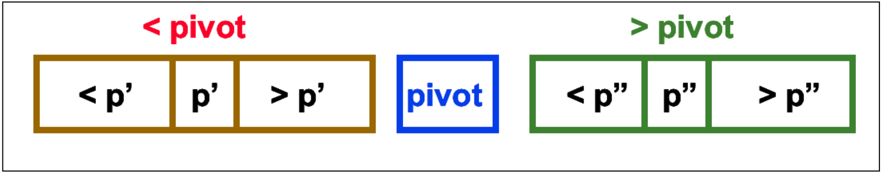


- So, where does the pivot fit in in the ultimate output?
- while partitioning, where do we place/position the pivot element ?
- most research done on partitioning. . . ..

Conquering



- Apply the same algorithm to each half

The Partition Routine...

Basic Recursive Quicksort

If the size of the list n is 0 or 1, return the list. Otherwise:

1. Choose one of the items in the list as a **pivot**
2. Next, **partition** the remaining items into two disjoint sublists, such that alll the items greater than the pivot are greater than the pivot and all elements less than the pivot are less than the pivot.
3. Finally, return the result of quicksort as the *head* sublist, followed by the pivot, followed by the result of the quicksort of the *tail* sublist.

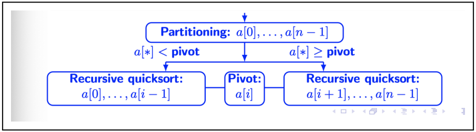


Figure: Quicksort partitioning

The Quicksort Pseudocode

```
Algorithm QUICKSORT(A, low, high)
/* Termination condition! */
1. If (low < high)
2.              pivot = PARTITION(A, low, high);
3.              QUICKSORT(A, low, pivot -1);
4.              QUICKSORT(A, pivot+1, high);
```

- Divide routine ?
- Conquer routine ?

The Partition pseudocode

```
Algorithm PARTITION(A, low, high)
1.      pivot = A[low]
2.      left  = low
3.      right = high+1
4.      while ( TRUE )
5.          do repeat right=right − 1;
6.          until A[right] ≤ pivot
7.          do repeat left=left +1;
6.          until A[left] ≥ pivot
/* right is final position for the pivot */
8.          if left < right
9.                  SWAP(A[left], A[right]);
10.         else
11.                 SWAP(pivot, A[right])
12.                 return right
```

Illustrating Quicksort: Tutorial Problem 1

Work up the algorithm on the following input...

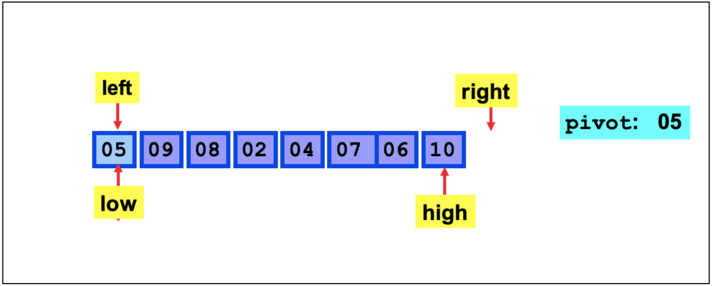


Figure: Run Quicksort - number of swaps & comparisons ?

Illustrating Quicksort:Tutorial Problem 2

Work up the algorithm on the following input.......

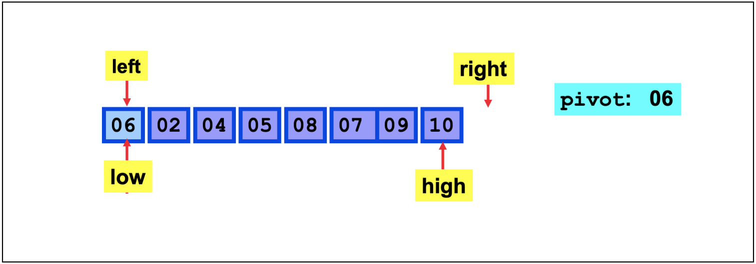


Figure: Run Quicksort - number of comparisons=8 & number of swaps=0??!!!

Illustrating Quicksort: Tutorial Problem 3

Work up the algorithm on the following input.......

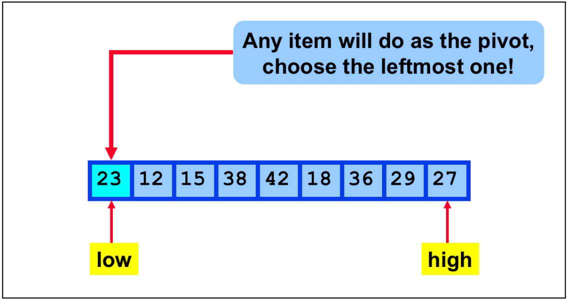


Figure: Run Quicksort - number of swaps=1 & comparisons=10 ?

Illustrating Quicksort: Tutorial Problem 4

Work up the algorithm on the following input

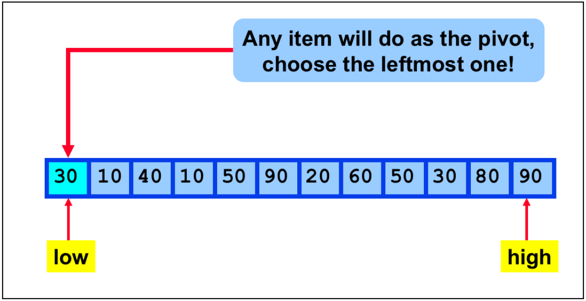


Figure: Run Quicksort - number of swaps & comparisons in each case ?

Illustrating Quicksort: Tutorial Problem 4

Work up the algorithm on the following input

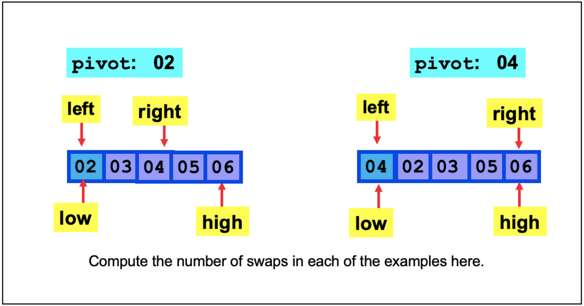


Figure: Quicksort on LHS - number of swaps = 0 (plus a futile last swap) & comparisons=6 ???. Whereas in RHS - number of swaps = 0 & comparisons=6 ????

The Partition pseudocode...:Another approach

```
Algorithm PARTITION(A, low, high)
1.      pivot = A[low]
2.      left  = low+1
3.      right=high
4.      while ( left < right )
5.              while( A[left] < pivot ) left++;
6.              while( A[right]  > pivot ) right--;
7.      if ( left < right ) SWAP(A[left], A[right]);
/* right is final position for the pivot */
8.   SWAP(A[right], pivot);
9.              return right;
```

Dry run of the new PARTITION routine: Homework Assignment

Run the PARTITION routine on the following input instance:
- A(5,9,8,2,4,7,6,10)
- A(23, 12, 15)
- A(10,20,30)

Compute the number of swaps & comparisons ? in each case.

The Partition pseudocode as in the text

```
Algorithm CLRS–PARTITION(A, low, high)
1. pivot = A[high]
2. left = low − 1
3. for right=low to (high −1)
5.              if A[right] <= pivot
6.                      left = left +1
7.                      swap(A[left], A[right])
8. swap(A[left+1], pivot)
9. return left+1
```

Dry run of the PARTITION routine

Run the PARTITION routine on the following input instance:
- A(2,8,7,1,3,5,6,4)
- A(5,9,8,2,4,7,6,10)

The Quicksort Performance

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- On the other side
  - a D&C algorithm is most efficient when the division is as even as possible.
  - i.e. quicksort best case occurs when two partitions are of size n/2 each
  - on an average the performance of quicksort is actually $O(n\ lg\ n)$

The Quicksort Performance Analysis

What is the Recurrence relation ?

- Partition $\Longrightarrow$ Check every item once $\theta(n)$

The above recurrence solves to $\Theta(n\ lg\ n)$. How ?
The best case running time of quick sort is $\Theta(n\ g\ n)$

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What is the Recurrence relation ?

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- In general, if the partition is such that it produces two partitions - where one is of size i, then the other would be of size $n-(i-1)$.
- The generic recurrence of quicksort is as follows:
  $T(n) \;=\; T(i) + T(n-(i-1)) + \theta(n)$

The above recurrence solves to $\Theta(n\ lg\ n)$. How ?
The best case running time of quick sort is $\Theta(n\ g\ n)$

The Quicksort Performance Analysis...

- Partition $\implies$ Check every item once $\theta(n)$
- Conquer $\implies$ Conquer data in each half of roughly size $n/2$
- If we assume partitions of size $\lfloor n/2 \rfloor$ and $\lceil (n-1)/2 \rceil$, then ?
- if we tolerate the sloppiness from ignoring the floor and ceiling and for partitions of from subtracting 1, then ?
- T(n) $= 2T(n/2) + \Theta(n)$
- The above recurrence solves to $\Theta(n\ lg\ n)$. How ?
- The best case running time of quick sort is $\Theta(n\ g\ n)$
- Let us see how.......

The Quicksort Performance Analysis: Using Recursion Tree

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  4. Therefore, the sum of sizes of the children at any level is less than or equal to n - this implies that the work done at each level is O(n).

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  4. Therefore, the sum of sizes of the children at any level is less than or equal to n - this implies that the work done at each level is O(n).
  5. Now, the key question is as we saw in Sr no 4 above, what is going to be the height of the tree for a specific input?

The Quicksort Performance Analysis: Recursion Tree

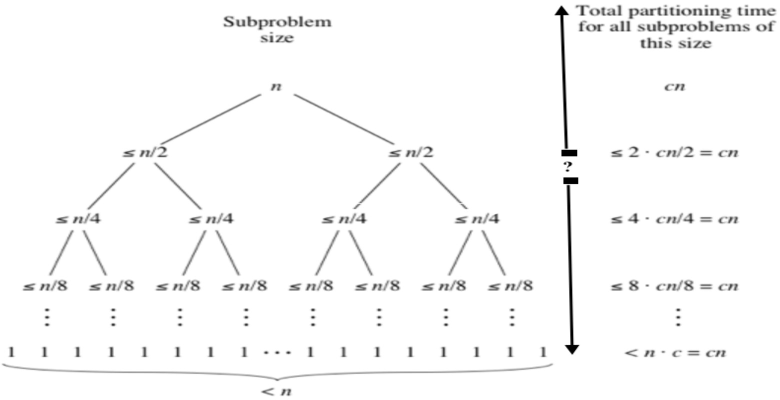


Figure: Recursion Tree: Quicksort Recursion Tree

The Quicksort Performance Analysis: Best case: Recursion Tree

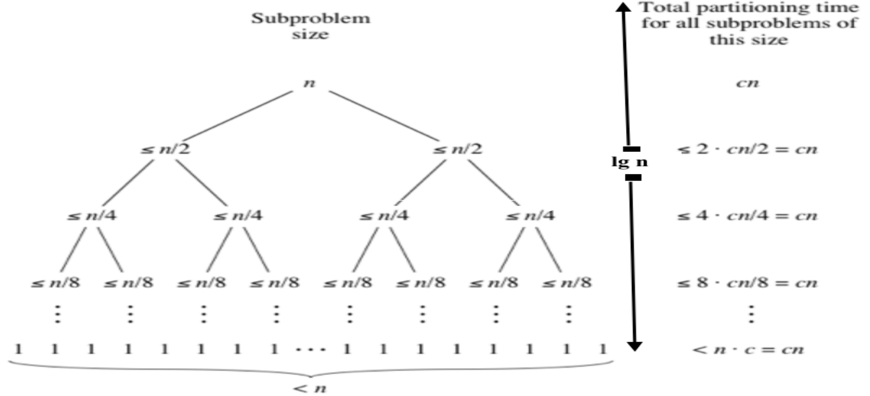


Figure: Recursion Tree: Quicksort Best case

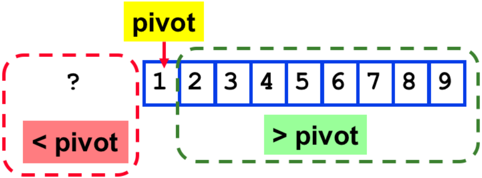
- How can we argue that the height of the tree is lg n?
- But now the question is how to prove this result ?

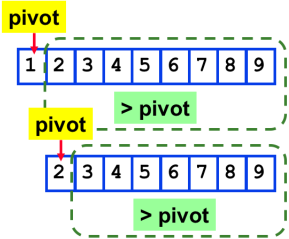
The Quicksort Performance Analysis: Best case

- Prove that the recurrence $T(n) = 2T(n/2) + \Theta(n)$ has the solution $T(n) = \Theta(n \; lg \; n)$
  - Can be done using the Master's theorem - case II applies. Left as an exercise.
  - Can be proven using the Principle of Mathematical Induction - left as an exercise.
- Proving using telescoping- substitution.

The Quicksort Performance Analysis...

- But there is a catch
- Each partition produces a problem of size 0 and one of size (n-1)
- Number of partitions?





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Recurrence Relation on uneven partitioning

$$T(n) = T(n-1) + T(0) + \theta(n)$$

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Recurrence Relation on uneven partitioning

$T(n) = T(n-1) + T(0) + \theta(n)$

- We have to ponder as to what could be T(0)?
- Also, if we assume that the original call takes cn time for some constant c, then $\theta(n) = cn$
- Then the recursion tree takes the form as shown on the next slide... ... ...

The Quicksort Performance: Uneven Partitioning...

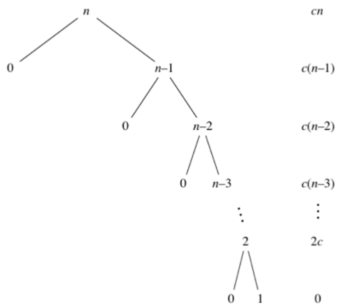

Figure: Quicksort Unbalanced partitioning at each level

Solving the recurrence for the worst case

- Given the recurrence relation as $T(n) = T(n-1) + T(0) + \theta(n)$
- Solving the recurrence....
  - What did we estimate, would be the value of T(0) ?
  - What would be the cost of PARTITION(A, low, high) routine with (n-1) elements in the first recursive call?
  - What would be the cost of PARTITION(A, low, high) routine with (n-2) elements in the second recursive call?
  - Solving the recurrence by iterative method.....

The Quicksort Worst Case

- Prove that the recurrence $T(n) = T(n-1) + \Theta(n) + T(0)$ has the solution $T(n) = \Theta(n^2)$
- Conclusion: Quicksort is as bad as bubble or insertion sort...

The Quicksort Worstcase Performance...

- Thus, if the partitioning is maximally unbalanced at every recursive level of the algorithm, the running time is $\theta(n^2)$
- Therefore the worst-case running time of quicksort is no better than that of insertion sort.
- Moreover, the $\theta(n^2)$running time occurs when the input array is already completely sorted.
    - a common situation in which insertion sort runs in O(n) time.

The Quicksort Performance: Balanced Partitioning

- Assume that the partitioning is such that the pivot is neither the median of the input array, nor it is skewed i.e. the least or the highest element.

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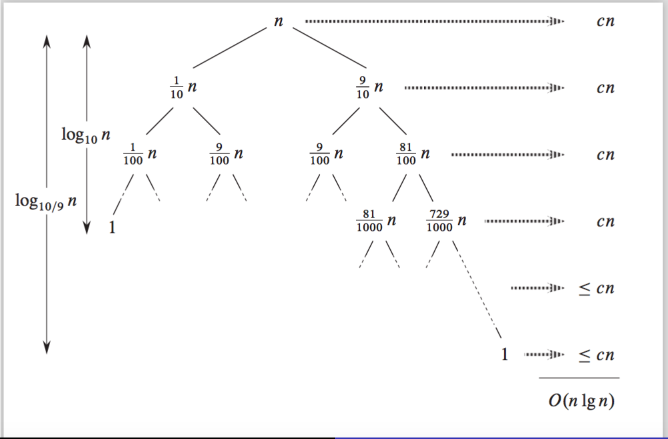

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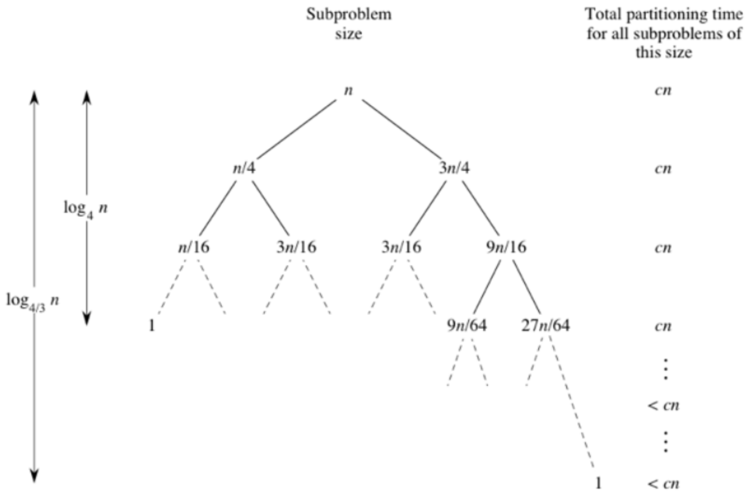

The Quicksort Performance: Balanced Partitioning...

The Balanced PArtitioning case

- In fact, any split of constant proportionality yields a recursion tree of depth $\theta(n \; lg \; n)$ where the cost at each level is $O(n)$.
- The running time is therefore $O(n \; lg \; n)$ whenever the split has constant proportionality.

The Quicksort Performance Analysis...

Thus, the Quicksort Performance

- appears to be the same as Heapsort/Mergesort
- quicksort is generally faster
- fewer comparisons or shorter inner loop
- But there is a catch... ... ...

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- when we run quicksort on a random input array, the partitioning is highly unlikely to happen in the same way at every level
  - it is expected that some of the splits will be reasonably well balanced and that some will be fairly unbalanced.
- Exercise 7.2-6 CLRS : Show that about 80 percent of the time PARTITION produces a split that is more balanced than 9 to 1, and about 20 percent of the time it produces a split that is less balanced than 9 to 1.

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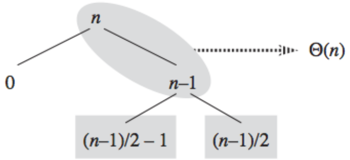

Figure: (a) Bad Split followed by a Good split

The Quicksort Average Case: Good and Bad splits distribution...

- Now consider another average-case execution of PARTITION, in which a single level of partitioning that produces two subarrays of size $(n-1)/2$ *and* $((n-1)/2)-1$ indeed the best case.

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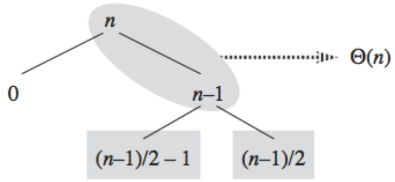

Figure: (b) Bad Split followed by a Good split

The Quicksort Average Case: Good and Bad splits distribution...

- Is the cost of PARTITION in figure (a) worse than that in the figure (b) ?
- Each partition produces a problem of size 0 and one of size (n-1).
- Cost of partitions?

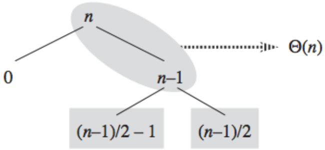

Figure: (a)

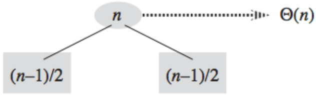

Figure: (b)

The Quicksort Average Case...

Bad split and Good split

- the $\theta(n-1)$ cost of the bad split can be absorbed into the $\theta(n)$ cost of the good split, and the resulting split is good.
- Thus, the running time of quicksort, when levels alternate between good and bad splits, is like the running time for good splits alone i.e. still $O(n\ lg\ n)$, but with a slightly larger constant hidden by the O-notation.

The Average Case Analysis of Quicksort

- Formal analysis of the average case of the quicksort. To be written on the board.

The Quicksort : Median of three pivot

- Take 3 positions say First, middle, last and choose the median.

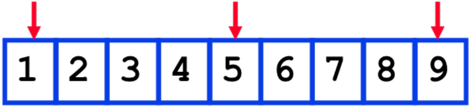

Figure: Pivot is median of three

- What is the median?
- Thus, one gets perfect partitioning yielding $O(n\ lg\ n)$ time complexity
- Since, sorted (or nearly sorted) data is common, median of 3 is a good strategy

MEdian of three does not work always

- However, even any pivot selection strategy could lead to $O(n^2)$ time
- What is the median-of-3 chosen as, in Fig (a) ?
- What is the median-of-3 chosen as, in Fig (b) ?

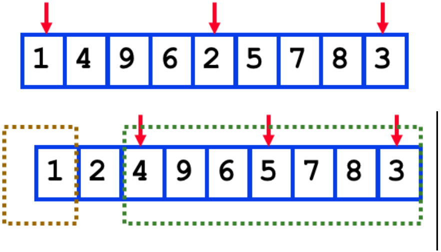

Figure: (a) and (b) Pivot is median of three

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- Different position for every partition
- On average, assume the sorted data is divided evenly yielding $O(n\ lg\ n)$ time
- However, the key requirement is that the pivot choice must take $O(1)$ time

Sorting: Reviewing

Bubble, Insertion

- are $O(n^2)$ sorts
- simple code
- May run faster for small n, n $\tilde{1}0$ (system dependent)

Quick

- is on an average $O(n\ lg\ n)$ sort, but ....
- can deteriorate to $O(n^2)$
- depends on pivot selection
- Better but not guaranteed performance

The Quicksort in comparison

- Quicksort is generally faster - an empirical comparison

n	Quick		Heap		Insert	
	Comp	Exch	Comp	Exch	Comp	Exch
100	712	148	2842	581	2595	899
200	1682	328	9736	9736	10307	3503
500	5102	919	53113	4042	62746	21083

Figure: A typical Empirical Comparison

Quicksort and Heapsort

- Quicksort
  - Generally faster
  - Sometimes $O(n^2)$
  - Better pivot selection reduces probability
  - Used when average good performance is desired
  - Commercial applications, Information systems
- Heap Sort
  - Generally slower
  - Guaranteed $O(n \; log \; n)$
  - Used for real-time systems, where time is a constraint

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    - use the median of the three approach
    - use a randomized version of the algorithm

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- we may be able to eliminate the difference between the good and the bad behaviour.

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  - e.g. Contraction algorithm for global min cut.

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  - Can always convert a Las Vegas algorithm into Monte Carlo, but no known method to convert the other way

The RANDOMIZED QUICK-SORT(S)

```
Algorithm RANDOMIZED—QUICKSORT(S)
if |S| = 0 return
choose a pivot—element  a_i ϵ S uniformly at random
for each (a ϵ S)
        if (a < a_i)
                put a in S−
        else if (a > a_i)
                put a in S+
RANDOMIZED—QUICKSORTS—)
        output a_i
RANDOMIZED—QUICKSORT(S+)
```

Randomized Algorithms

- Running time.
- Best case. Select the median element as the splitter: quicksort makes θ(n log n) comparisons.
- Worst case. Select the smallest element as the splitter: quicksort makes $\theta(n^2)$ comparisons.
- Randomize. Protect against worst case by choosing splitter at random.
- Intuition. If we always select an element that is bigger than 25% of the elements and smaller than 25% of the elements, then quicksort makes θ(n log n) comparisons.

b l a n k

b l a n k

b l a n k

b l a n k