Chapter 1: Part II: Asymptotic Notations

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Design and Analysis of Algorithms
IIT Jammu, Jammu

The Big-Oh Notation

- def: for a given function g(n), we say that O(g(n)) = f(n) — if there exists positive constants c and n_0 such that, $0 \leq f(n) \leq cg(n)$, for all $n \geq n_0$

$f(n) = O(g(n)) \Rightarrow$

f(n)is dominated in the growth by g(n) i.e. f(n) is of the order at the most g(n) i.e. g(n) grows at least as fast as f(n)

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- The Big-oh defines an upper bound for a function within a constant factor i.e. except for a constant factor and a finite number of exceptions, f is bounded above by g.

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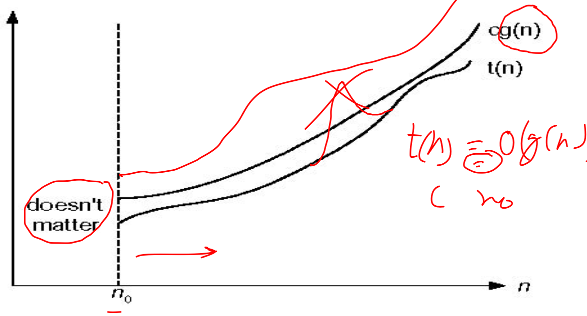
$p(n)$



$t(n) = O(g(n))$

$c \quad n_0$

Figure 2.1 Big-oh notation: $t(n) \in O(g(n))$

The Big-Oh Notation

O(1)

$\lim_{n \to \infty}$

- Can $f(n)$ grow faster than $g(n)$?
- Can $g(n)$ grow faster than $f(n)$?

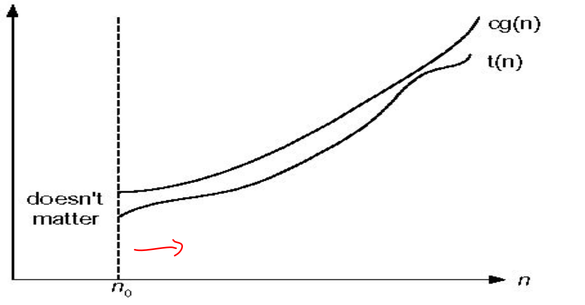


Figure 2.1 Big-oh notation: $t(n) \in O(g(n))$

- What does the growth rate imply ?

The Big-Oh Notation Illustrations

Function	notation in O
$f(n) = 5n + 8$ $O(1)$	$f(n) = O(?)$ n $O(n^2)$ $O(n^3)$
$f(n) = n^2 + 3n - 8$	$f(n) = O(?)$ 2^n
$F(n) = 12n^2 - 11$	$f(n) = O(?)$
$F(n) = 5*2^n + n^2$	$f(n) = O(?)$
$f(n) = 3n + 8$	$F(n) = O(n^2)?$ y
$f(n) = 5n + 8$	$f(n) = O(1)?$

The Big-Oh Notation

$f(n) = 5n + 8$ p.t. $f(n) = O(n)$

- allows us to keep track of the leading term while ignoring smaller terms

$c, \; n \geq n_0$

$f(n) = 5n + 8$

$\leq 5n + 8n$

$= 13n$

$\therefore c = 13$ and $n_0 = 1$ we have

$f(n) \leq c \cdot n \quad \therefore f(n) = O(n)$

The Big-Oh Notation

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- Consider that f(n)= O(n) & g(n)= O(n^2). Is f(n)= O(g(n)) saying the same as reverse i.e. g(n) = O(f(n))?
- The symbol = is not proper truly it is ϵ which should be used i.e. f(n) ϵ O(g(n)

The Big-Oh notation...

- When O notation bounds the worst case running time of an algorithm, by implication we also bound the running time of an algorithm on EVERY input.

$O(n^2)$! → n

Abuse

Technically, it is abuse to say that the running time of insertion sort is $O(n^2)$. Why?

Wrong

The Big-Oh notation...

- When O notation bounds the worst case running time of an algorithm, by implication we also bound the running time of an algorithm on EVERY input.
- this is not so when using other notations i.e. the worst case $\theta(n^2)$ or θ(n) does not apply to every input.

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- definition: the function f(n)= Ω (g(n)) is true iff there exists positive constants c and n_0 such that $f(n) >= c(g(n))$ for all n - n $\geq$ n_0 i.e. $0 <= c(g(n)) <= f(n)$.

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$\Omega(n) \Rightarrow$

f(n) always dominates the growth of g(n) i.e. f(n) is of the order at least g(n) i.e. g(n) grows at the most as fast as f(n).

The Big-Ω notation...

The Big Omega

- Can f(n) grow faster than g(n)?
- Can g(n) grow faster than f(n)?

} if f(n) = o(g(n))

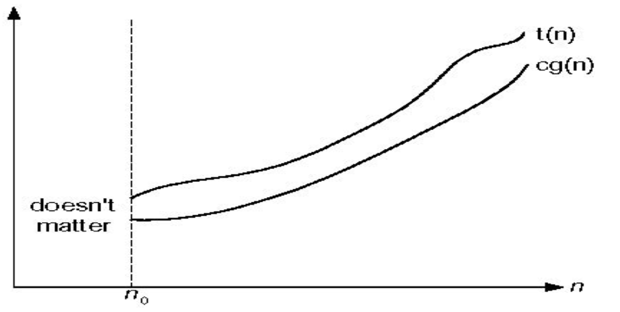

Fig. 2.2 Big-omega notation: $t(n) \in \Omega(g(n))$

The Big-Oh Notation Illustrations

Function	notation in Ω
$f(n) = 3n + 8$	$f(n) = \Omega(?)$ n
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$F(n) = 12n^2 - 11$	$f(n) = \Omega(?)$ n^2
$F(n) = 6*2^n + n^2$	$f(n) = \Omega(?)$ 2^n
$f(n) = 3n + 8$	$f(n) = \Omega(n^2)$?
$f(n) = 5n + 8$	$f(n) = \Omega(1)$? Yes

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- if f(n)= $O(\Omega(n))$, what is the lower bound ?
- Do we specify how tight this lower bound is?

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- When we say $f(n)=\Omega(g(n))$, does it mean that $g(n)=O(f(n))$?

$f(n) \geq c \cdot g(n))$

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When we use Ω-notation to describe a lower bound,

f(n) = Ω g(n))

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- Can we say that the running time of insertion sort is $\Omega(n^2)$?
- Can we say that the running time of insertion sort is O(n)?

The Big-Θ notation...

- Neither the big-O notation nor the big-Ω notation describe the asymptotically tight bounds.
- Θ-notation to express tighter bounds - used to specify the exact order of growth of functions.
- def: we say that f(n)=Θ((g)n) iff there exists positive constants c_1 and c_2 and a number n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$
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The Big-Θ Notation...

Given f(n)=Θg(n) → $f(n) = O(g(n))$

- Can $f(n)$ grow faster than $g(n)$?

$f(n) = \Omega(g(n))$

NO

The Big-Θ Notation...

Given f(n)=Θg(n)

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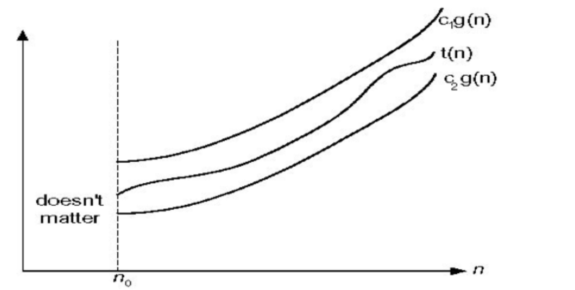

Figure 2.3 Big-theta notation: $t(n) \in \Theta(g(n))$

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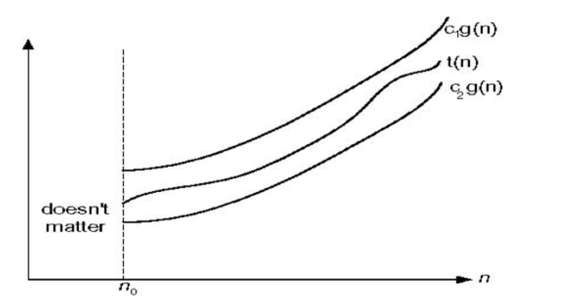

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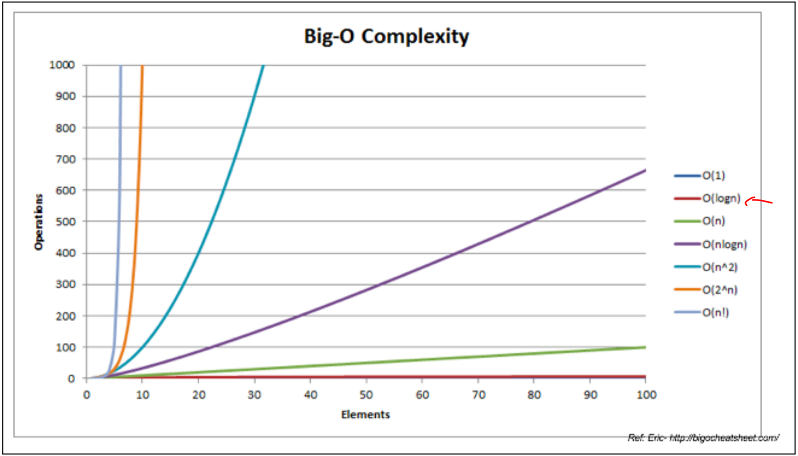
$f(n) = 3n + 8$	$f(n) = \theta(n^2)$?
$f(n) = 5n + 8$	$f(n) = \theta(1)$?

The Asymptotic Classes



Complexity of Data Structures

Ref: Eric- http://bigocheatsheet.co

Data Structure	Time Complexity								Space Complexity
	Average				Worst				Worst
	Access	Search	Insertion	Deletion	Access	Search	Insertion	Deletion	
Array	O(1)	O(n)	O(n)	O(n)	O(1)	O(n)	O(n)	O(n)	O(n)
Stack	O(n)	O(n)	O(1)	O(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Singly-Linked List	O(n)	O(n)	O(1)	O(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Doubly-Linked List	O(n)	O(n)	O(1)	O(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Skip List	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)	O(n)	O(n)	O(n)	O(n log(n))
Hash Table	-	O(1)	O(1)	O(1)	-	O(n)	O(n)	O(n)	O(n)
Binary Search Tree	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)	O(n)	O(n)	O(n)	O(n)
Cartesian Tree	-	O(log(n))	O(log(n))	O(log(n))	-	O(n)	O(n)	O(n)	O(n)
B-Tree	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)
Red-Black Tree	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)
Splay Tree	-	O(log(n))	O(log(n))	O(log(n))	-	O(log(n))	O(log(n))	O(log(n))	O(n)
AVL Tree	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)

Complexities of Sorting Algorithms

linear time sorting type of sorting algorithm.

Algorithm	Time Complexity			Space Complexity
	Best	Average	Worst	Worst
Quicksort	O(n log(n))	O(n log(n))	O(n^2)	O(log(n))
Mergesort	O(n log(n))	O(n log(n))	O(n log(n))	O(n)
Timsort	O(n)	O(n log(n))	O(n log(n))	O(n)
Heapsort	O(n log(n))	O(n log(n))	O(n log(n))	O(1)
Bubble Sort	O(n)	O(n^2)	O(n^2)	O(1)
Insertion Sort	O(n)	O(n^2)	O(n^2)	O(1)
Selection Sort	O(n^2)	O(n^2)	O(n^2)	O(1)
Shell Sort	O(n)	O((nlog(n))^2)	O((nlog(n))^2)	O(1)
Bucket Sort	O(n+k)	O(n+k)	O(n^2)	O(n)
Radix Sort	O(nk)	O(nk)	O(nk)	O(n+k)

Ref: Eric- http://bigocheatsheet.com/

Lower Bound on Sorting

- To sort n elements, one needs information about the sequence of these elements
  i.e. given two elements, the relative order could be

$n = 100$

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- Draw a decision tree showing the number of comparisons of any n-distinct elements - say for n =3.

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  - represent each internal node by a_i:a_j in the range $1 \leq i \leq n$

Lower Bound on Sorting

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- Decision tree at a higher level of abstraction

  permutation

- For an input sequence of three elements, what would be the total number of 2-element comparisons ?

  $\frac{n!}{(n-1)!}$ $\frac{3!}{(n-2)!} = 6$

- Draw a decision tree showing the number of comparisons of any n-distinct elements - say for n =3.
  - represent each internal node by a_i:a_j in the range $1 \leq i \leq n$
  - denote each leaf by permutation ($\pi(1)$, $\pi(2)$, $\pi(3)$, $\pi(4)$.. $\pi(n)$)

Blank $a_1 \; a_2 \; a_3$: Decision - tree

$a_1 < a_2$

$a_1 < a_3 < a_2$

$a_1 < a_2$

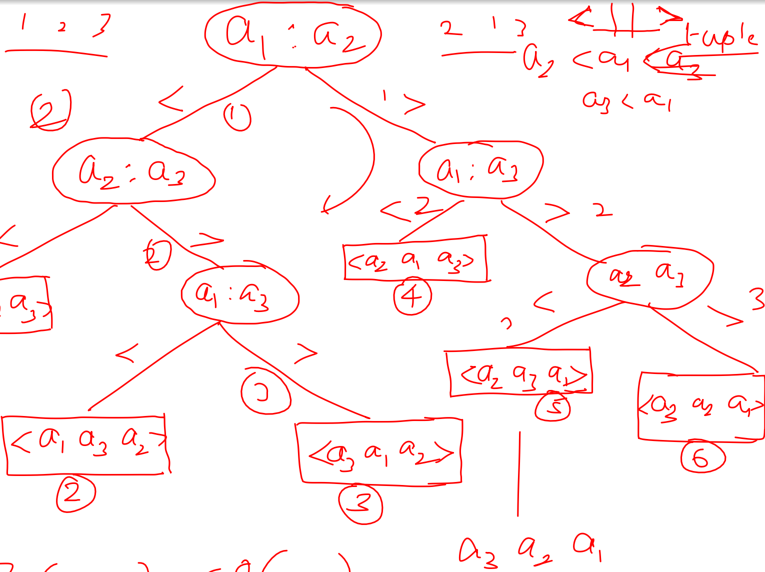
longest path

n

2 element

6

$\Omega(\quad)$ $\Theta(\quad)$



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Theorem: Any decision tree that sorts n elements has height $\boxed{\Omega(n \lg n)}$

Proof:

→ Consider a decision tree of height h that sorts n elements.

→ How many leaves does this tree have?

(a) → If the height of a tree is h, then $\boxed{\text{maximally}}$ it can have 2^h leaves.

$n! \leq \lceil 2^h \rceil$

i.e. $\lceil 2^h \rceil$ - - - - ①

(b) → if a decision tree represent sorting of n elements, then it will have n! leaves - - ②

Blank

$n! \leq [2^n]$

i.e. $\log n! \leq \log[2^n]$

$\log n! \leq n$ - - - - (i)

$\Rightarrow$ Stirling's approximation applied to our case:

$\log n \leftarrow$ $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta(1/n)\right)$

$\boxed{\log e} \leftarrow$

$n! \geq \left(\frac{n}{e}\right)^n$ - - - (ii)

$n > \log\left(\frac{n^n}{e^n}\right)$

$n = n\log n - n\log e$

$n \geq n \log n$

$= \Omega(n \log n)$

Blank

Corollary: Heapsort and Mergesort are optimal comparison based sorts.

Proof: Optimal? → the best we can do.

① Any comparison $\Omega(n \log n)$

↓ at least $n \log n$

② complexities of HS and MS

$O(n \log n)$ ‖ ③ —— $O(n^2)$

Blank

Counting sort

not comparison based

$O(n)$

Self-reading