Approximation Algorithms

1

Combinatorial Optimization problems

- The problems in which the goal is to find the "best" object from within some finite space of objects, e.g.
  - Shortest path:
    - Given a graph with edge costs and a pair of nodes, find the shortest path (least costs) between them
  - Traveling salesman:
    - Given a complete graph with nonnegative edge costs, find a minimum cost cycle visiting every vertex exactly once
  - Maximum Network Lifetime:
    - Given a wireless sensor networks and a set of targets, find a schedule of these sensors to maximize network lifetime

2

Optimization Problems: Formalization

- Given:
  - A problem instance I.
- Goal:
  - With respect to a defined parameter, we wish
    - to find the optimum solution …..that may be maximum or minimum
    - i.e. to minimize (or maximize) some cost function c(S) for a "solution" S to the problem instance I.
- e.g.
  - INDEPENDENT SET……maximum number of vertices in a graph..
  - Shortest Path…..minimum cumulative edges weights in a path…..
  - TSP…..smallest number of edges that span a specific vertices

3

Classes of Discrete Optimization Problems

- Class I
  - problems have polynomial-time algorithms for solving the problems optimally.
  - e.g. Minimum Spanning Tree problem, SSSP problem….
- Class 2 …NP-hard problems
  - No polynomial-time algorithm is known….
  - it is very likely that there is none.
  - e.g. Traveling Salesman Problem

8

Dealing with NP problems

- However, NP problems need solutions in real-life
- Up to now…..
  - the best algorithm for solving a hard optimization problem requires exponential time in the worst case.
- Thus, the question is ……How to deal with NP….*ness* ?

9

Dealing with NP problems

- However, NP problems need solutions in real-life
- Up to now…..
  - the best algorithm for solving an NP-complete problem requires exponential time in the worst case.
- Thus, the question is ……How to deal with NP….*ness* ?

10

Dealing with limitations of algorithm computations…

- Three main directions to solve NP-hard discrete optimization problems:
  - Integer programming techniques
  - Heuristics
  - Approximation algorithms

11

Approximation : Solution for NP problems

- To reduce the time required for solving a problem…….
  - we can relax the problem, and
  - obtain a feasible solution "close" to an optimal solution
- We compromise on optimality for a good feasible solution…… No heuristics!
- In Hochbaum's words…..
  - "Trading-off optimality in favour of tractability is the paradigm of approximation algorithms".

12

Approximation vs Heuristics

- Ideally,
  - we would like to have a guarantee on how close to optimal, the solution given by our algorithm is…...
    - e. g. the case of Euclidean TSP algorithm …….
      - a tour whose length at most a factor ρ times the minimum length of a tour, for a (hopefully small) value of ρ.
- Heuristics
  - may produce good solutions but do not come with a guarantee on the quality of their solution.

13

Approximation Ratios

- Basic terminology
  - Let, OPT(I) denote the value of an optimal solution to the problem under consideration for input I.
  - e.g.
    - OPT(G) in TSP denotes……
    - OPT(H) in the independent-set problem denotes……

16

Approximation ratio, informally

- The approximation ratio ρ is defined as the ratio
  - $\frac{result\ obtained\ by\ the\ algorithm}{the\ optimal\ cost\ or\ profit}$
- Typically this ratio is greater than one……
  - e.g. in TSP…...
  - e.g. in an airline reservation example……

17

Approximation ratio, informally…

- A k-approximation algorithm is an algorithm with approximation ratio k
  - e.g. both the instances above are 2-approximation algorithms.

18

Approximation Ratio: Minimization problem

- Consider a minimization problem.
  - Let OPT(I)= value of the optimal solution….a positive integer.
  - consider an algorithm A that gives an output solution A(I), on an instance I.
- Then, the approⁿ ratio of the algorithm A is $\rho = \max_I \frac{A(I)}{OPT(I)}$
- That is, $OPT(I) \leq A(I)$
  - however, closer the value of the ρ is to one, better is the approⁿ algorithm A(I)

19

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20

Approximation Ratio

$1 \leq \text{Max}\,(A(I)/OPT(I), OPT(I)/A(I)) \leq \rho(n)$

21

Formal definition of AA

- def:
  - An ρ-*approximation algorithm* is a polynomial-time algorithm that always produces a solution of value within ρ times the value of an optimal solution.
  - that is, for any instance of the problem

    $A(I)/OPT(I) \leq \rho$, ……..*for a minimization problem*

    $OPT(I)/A(I) \leq \rho$, ……..*for a maximization problem*
- ρ is called the *approximation guarantee* (or *factor*) of the algorithm.

22

Approximation Scheme

- ρ may be a function of the input size.
- When an algorithm A achieves an approximation ratio ρ we call it a ρ - approximation algorithm.
- An approximation scheme for an optimization problem
  - is an approximation algorithm that takes as inputs
    - an instance of the problem and
    - a value $\in > 0$,
  - such that for a fixed $\in$, the scheme is a $(1 + \in)$ - approximation algorithm.

23

Characteristics of AAs

- Approximation algorithms …..….
  - are time-efficient ……
    - however, are sometimes not as efficient as heuristics
  - don't guarantee optimal solution……,
    - however guarantee a "good" solution within some factor of the optimum
  - rigorous mathematical analysis is required to prove the approximation guarantee
  - often use algorithms for related problems as subroutines

24

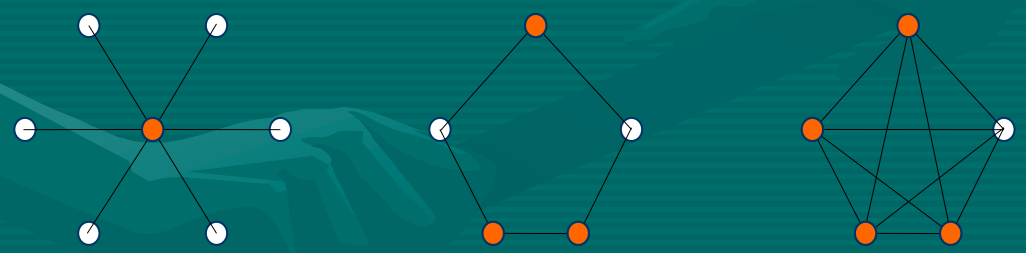
The Vertex Cover Problem

- VERTEX COVER
  - informally a subset of vertices that include all the edges…..an edge is covered if one of its endpoint is chosen.
- Given a graph G = (V, E) and an integer k,
  - is there a subset of vertices $S \subseteq V$ such that for each edge, at least one of its endpoints is in S?

25

The Vertex Cover Problem…

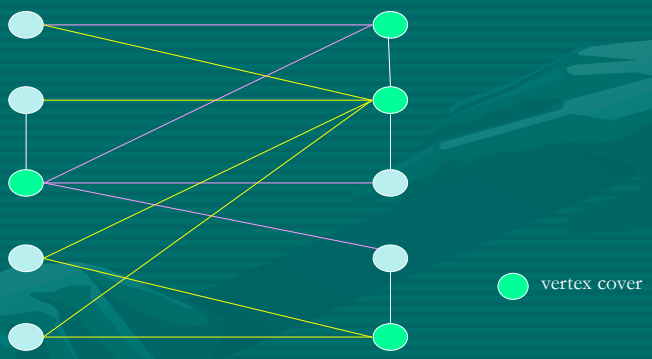
- The Minimum Vertex Cover
  - Find a vertex cover with minimum number of vertices.
  - Given a graph G = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?



26

The Vertex Cover Problem

- Ex. Is there a vertex cover of size ≤ 4? Yes. size ≤ 3? No.



27

The Vertex Cover Problem is NPC

- The problem
  - of finding a vertex cover i.e. optimal vertex cover is a classical optimization problem
  - is a typical example of an NP-hard optimization problem
- However, has an approximation algorithm to sub optimally solve it i.e. one that returns a near-optimal vertex cover

28

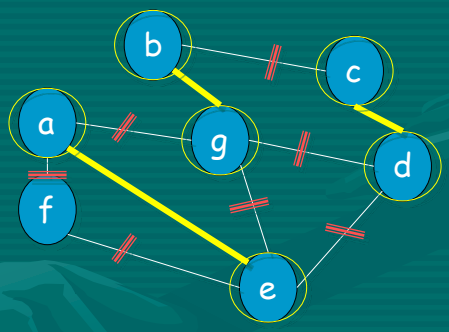
Approximate Vertex Cover

- Algorithm Outline
  - Let S be the cover….initialized to Φ
  - Pick an edge (u,v) in the graph
  - Add it's end-points u and v to S
  - Remove any edge that neighbors u or v i.e. one that is incident on either u or v.
  - Repeat until there are no edges left

29

Approximate Vertex Cover: An Example…

- First, what is the vertex cover of the graph ?
- Now, let us apply the approximation algorithm and find out…..



30

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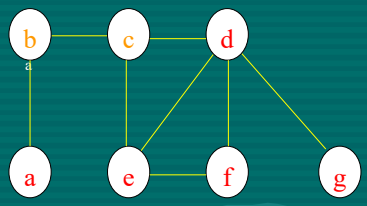
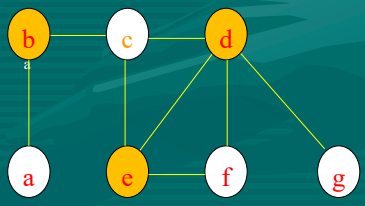


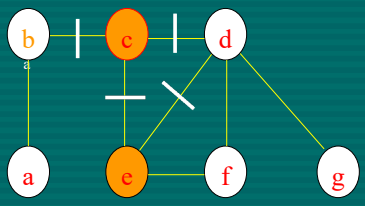
Input Graph



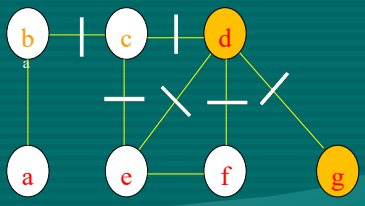
Optimal result, Size 3

31

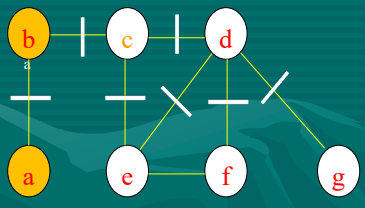
Approximate Vertex Cover: An Example…



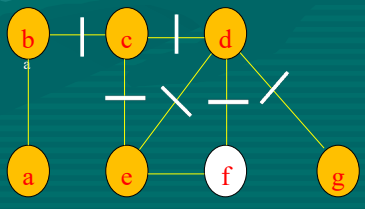
Step 1: choose edge (c,e)



Step 2: choose edge (d,g)



Step 3: choose edge (a,b)



Result : size = 6

32

Approximate Vertex Cover Algorithm

APPROX-VERTEX COVER (G)

1. C = ϕ
2. E' = E[G]
3. while E' != ϕ
4. let (u,v) be an arbitrary edge of E'
5. C = C ∪ {u,v}
6. remove from E' every edge incident on either u or v
7. return C

33

AVCA: Time Complexity

APPROX-VERTEX COVER (G)

1. C = ϕ
2. E' = E[G] } $O(n^2)$
3. while E' != ϕ

$O(n^2)$ {

4. let (u,v) be an arbitrary edge of E' } $O(1)$
5. C = C ∪ {u,v}
6. remove from E' every edge incident on either u or v } $O(n)$

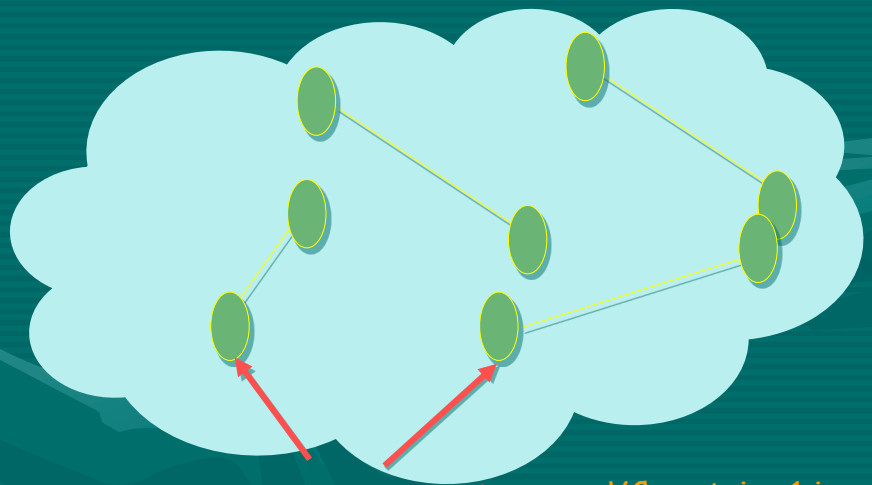
7. return C

- The algorithm runs in O(V + E) time.

34

How good the approximation is ?

Observe the set of edges our algorithm chooses



no common vertices! ⇒ any VC contains 1 in each
our VC contains both, hence at most twice as large

35

Approximate Vertex Cover…

- OPT must cover every edge so that……
  - either u or v must be in the cover
  - this implies that => OPT > ½|S|……..where S is the vertex cover
  - this implies that 2*OPT ≥ |S|
- Therefore, we have a 2-approximation.

36

Proof of Approximate Vertex Cover

- Theorem: APPROX-VERTEX-COVER is a 2-approximation algorithm.
- Proof:
  - The algorithm clearly returns a vertex cover, since it loops until all edges have been removed.
  - Each edge removed in line 6 was covered by some vertex of an edge removed in line 4.
  - Let A denote the set of edges removed in line 4:
    - The optimal cover OPT(I) must contain at least one endpoint of each of the edges in A.
  - No two edges in A share an endpoint because all edges that share an endpoint with an edge in A are removed in line 6.

37

Proof of Approximate Vertex Cover …

- Proof: …..continued
  - Since no two edges in A are covered by the same vertex in OPT(I) , we have that

    $|OPT(I)| \geq |A|$ …… (1)

    is a lower bound on the size of OPT(I).
  - Each execution of line 4 picks an edge, both of whose endpoints are added to the approximate cover C.
  - This gives us an upper bound on the size of C:
    - $|C| = 2|A|$ ……(2)
  - Combining equations (1) and (2), we get $|C| \leq 2|OPT(I)|$

38

Set Cover Problem (SCP): Illus^{tn}#0

- Consider a scenario…..
  - Say you'd like to send some message to a large list of people (e.g. the entire campus)
  - The Central Computer Centre provides you the existing available mailing-lists
    - However, the moderator of each list charges Rs 100 for each message sent
  - Hence, one would like to find the smallest set of lists that covers all recipients

Set Cover Problem (SCP): Illus^{tn}#1

- Consider a scenario…..
  - on a geometric plane, a collection of towns representing a province are marked as points.
  - the planners want to build schools for the entire province.
- Constraints
  - each school should be located in a town and
  - no one in any town should travel more than 5 kms to get to a school.
- The problem
  - What is the minimum number of schools required?

SCP : Illus^{tn}#1 - More formally ...

- For each town t……
  - let S_t = set of towns within 5 kms of it.
  - a school at t covers (itself, of course) these other towns.
- Question: How many sets S_t must be picked in order to cover all the towns in the province ?

42

SCP: Illus^{tn}#1 - More formally ...

- Formally:
  - Input: A set of elements X; and also given are sets $S_1, \cdots, S_m \subseteq X$
  - Output: A selection of the sets S_k whose union is X.
  - Cost: Number of sets S_k picked.
- The objective is to minimize the cost.

43

SCP : Illustration#2

- Consider another scenario…..
  - Execution of a project requires a certain number of skills (set X).
  - Each team member possesses a subset of the skills.
  - Select a team that together covers all the skills.
  - Objective: Minimize the number of members selected.

44

A greedy strategy for SCP

- DO UNTIL all elements of the skills set X are covered…….
  - Pick the set S_i with the largest number of uncovered elements.
- Easy observation: The greedy strategy is not optimal.
- How bad can be the greedy solution?
  - Luckily, not far from the optimal value.

45