Gradient Descent

Problem Statement

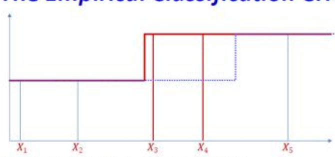
- Given a training set of input-output pairs $(\boldsymbol{X}_1, \boldsymbol{d}_1), (\boldsymbol{X}_2, \boldsymbol{d}_2), \ldots, (\boldsymbol{X}_N, \boldsymbol{d}_N)$
- Minimize the following function

$$Loss(W) = \frac{1}{N}\sum_i div(f(X_i; W), d_i)$$

w.r.t W

- This is problem of function minimization
  - An instance of optimization

The *Empirical Classification* error

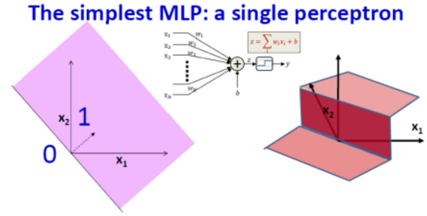


- The obvious error metric in a classifier is binary
  - The classifier is either right (error=0) or wrong (error=1)
    - Either $f(X; \boldsymbol{W}) = d$, or $f(X; \boldsymbol{W}) \neq d$

$$EmpiricalError(\boldsymbol{W}) = \frac{1}{N}\sum_{i=1}^{N} 1(f(X_i; \boldsymbol{W}) \neq d_i)$$

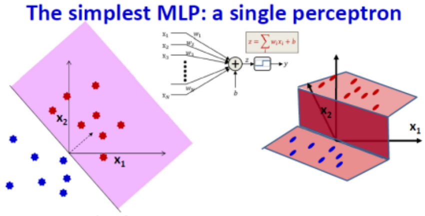
- **Learning the classifier:** Minimizing the count of misclassifications

The simplest MLP: a single perceptron



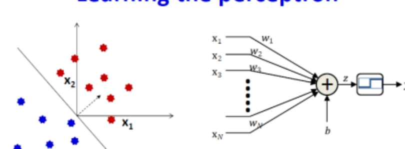
- Learn this function
  - **A step function across a hyperplane**

The simplest MLP: a single perceptron



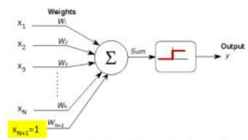
- Learn this function
  - **A step function across a hyperplane**
  - **Given only samples from it**

Learning the perceptron



- Given a number of input output pairs, learn the weights and bias
  - $y = \begin{cases} 1 & if \ \sum_{i=1}^{N} w_i X_i + b \geq 0 \\ 0 & otherwise \end{cases}$ Boundary: $\sum_{i=1}^{N} w_i X_i + b = 0$
  - Learn $W = [w_1 .. w_N]^T$ and b, given several (X, y) pairs

Restating the perceptron



- Restating the perceptron equation by adding another dimension to X

$$y = \begin{cases} 1 \ if \ \sum_{i=1}^{N+1} w_i X_i \geq 0 \\ 0 \ otherwise \end{cases}$$

where $X_{N+1} = 1$

- Note that the boundary $\sum_{i=1}^{N+1} w_i X_i = 0$ is now a hyperplane through origin

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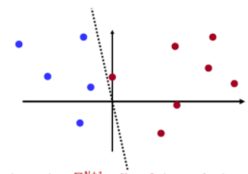
The Perceptron Problem



- Find the hyperplane $\sum_{i=1}^{N+1} w_i X_i = 0$ that perfectly separates the two groups of points

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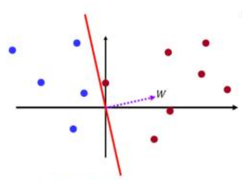
The Perceptron Problem



When the inner product of two vectors is zero, what would be the angle between two vectors?

- Find the hyperplane $\sum_{i=1}^{N+1} w_i X_i = 0$ that perfectly separates the two groups of points
  - Let vector $W = [w_1, w_2, \ldots, w_{N+1}]^T$ and vector $X = [x_1, x_2, \ldots, x_N, 1]^T$
  - $\sum_{i=1}^{N+1} w_i X_i = W^T X$ is an inner product
  - $W^T X = 0$ is the hyperplane comprising all Xs orthogonal to vector W
    - Learning the perceptron = finding the weight vector W for the separating hyperplane
    - W points in the direction of the positive class

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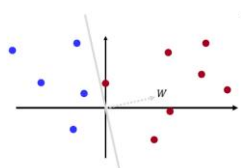
The Perceptron Problem



- Learning the perceptron: Find the weights vector W such that the plane described by $W^T X = 0$ perfectly separates the classes
  - $W^T X$ is positive for all red dots and negative for all blue ones

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Perceptron Algorithm: Summary

- Cycle through the training instances
- Only update W on misclassified instances
- If instance misclassified:
  - If instance is positive class (positive misclassified as negative)

    $$W = W + X_i$$

  - If instance is negative class (negative misclassified as positive)

    $$W = W - X_i$$

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Perceptron Learning Algorithm

- Given N training instances $(X_1, y_1), (X_2, y_2), \ldots, (X_N, y_N)$
  - $y_i = +1$ or -1

Using a +1/-1 representation for classes to simplify notation

- Initialize W
- Cycle through the training instances:
- do
  - For $i = 1..N_{train}$

    $$O(X_i) = sign(W^T X_i)$$

    - If $O(X_i) \neq y_i$

      $$W = W + y_i X_i$$

- until no more classification errors

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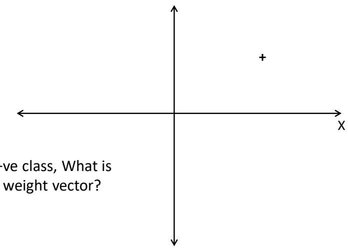
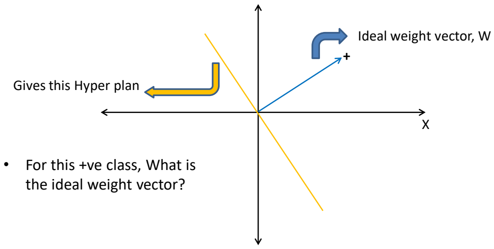
Ideal Weight vector



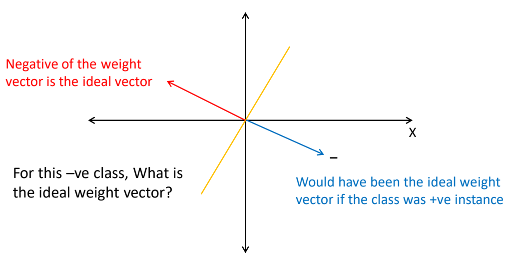
- For this +ve class, What is the ideal weight vector?

Ideal Weight vector



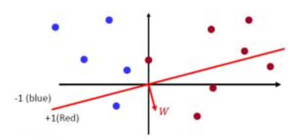
- For this +ve class, What is the ideal weight vector?

Ideal Weight vector



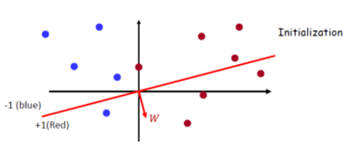
- For this –ve class, What is the ideal weight vector?

A Simple Method: The Perceptron Algorithm

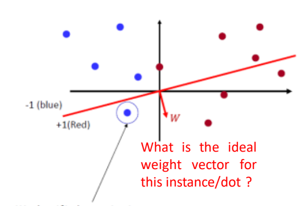


- **Initialize:** Randomly initialize the hyperplane
  - I.e. randomly initialize the normal vector W
- **Classification rule** $sign(W^T X)$
  - Vectors on the same side of the hyperplane as W will be assigned +1 class, and those on the other side will be assigned -1
- The random initial plane will make mistakes

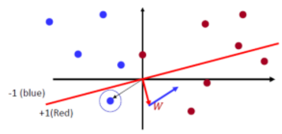
Perceptron Algorithm



Perceptron Algorithm

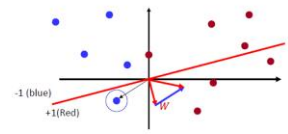


Perceptron Algorithm



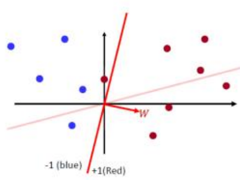
Misclassified *negative* instance, *subtract* it from W

Perceptron Algorithm



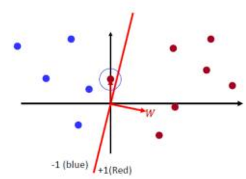
The new weight

Perceptron Algorithm



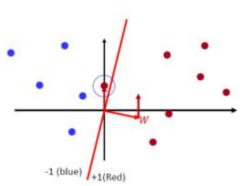
The new weight (and boundary)

Perceptron Algorithm



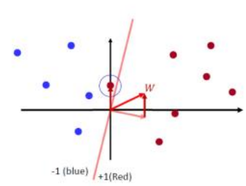
Misclassified *positive* instance

Perceptron Algorithm



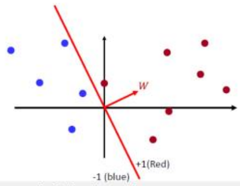
Misclassified *positive* instance, *add* it to W

Perceptron Algorithm



The new weight vector

Perceptron Algorithm



The new decision boundary
Perfect classification, no more updates, we are done

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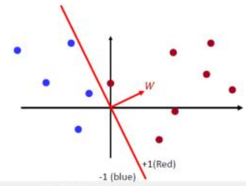
Perceptron Algorithm



The new decision boundary
Perfect classification, no more updates, we are done

If the classes are linearly separable, guaranteed to converge in a finite number of steps

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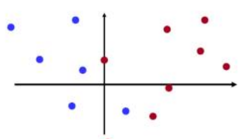
The Perceptron Solution: when classes are not linearly separable

Key: Red 1, Blue = -1



- When classes are not linearly separable, not possible to find a separating hyperplane
  - No "support" plane for reflected data
  - Some points will always lie on the other side
- Model does not support perfect classification of this data
- **Perceptron algorithm will never converge**

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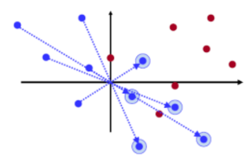
A simpler solution

Key: Red 1, Blue = -1



- *Reflect* all the negative instances across the origin
  - Negate every component of vector X
- If we use class $y \in \{+1, -1\}$ notation for the labels (instead of $y \in \{0,1\}$), we can simply write the "reflected" values as $X' = yX$
  - Will retain the features X for the positive class, but reflect/negate them for the negative class

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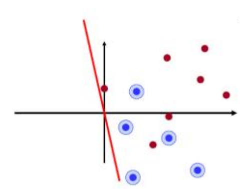
The Perceptron Solution

Key: Red 1, Blue = -1



- Learning the perceptron: Find a plane such that all the modified (X') features lie on one side of the plane
  - Such a plane can always be found if the classes are linearly separable

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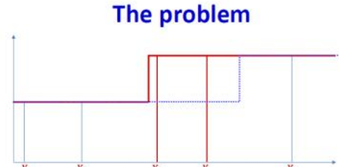
The problem



- Our binary error metric is not useful
  - To improve the classifier we must move the blue dotted line left
  - But if we move it only slightly, moving it either right or left results in no change in error

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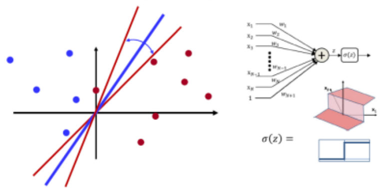
Why this problem?



- The perceptron is a flat function with zero derivative everywhere, except at 0 where it is non-differentiable
  - You can vary the weights a *lot* without changing the error
  - There is no indication of which direction to change the weights to reduce error

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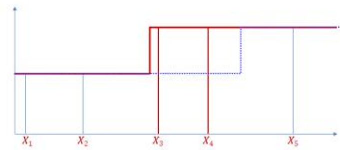
The solution



- Change our way of computing the mismatch such that modifying the classifier slightly lets us know if we are going the right way or not
  - This requires changing both, our activation functions, and the manner in which we evaluate the mismatch between the classifier output and the target output
  - Our mismatch function will now not actually count errors, but a *proxy* for it

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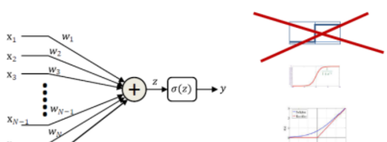
Solution: Differentiable activation



Activation functions $\sigma(z)$

- Let's make the neuron differentiable, *with non-zero derivatives over much of the input space*
  - Small changes in weight can result in non-negligible changes in output
  - This enables us to estimate the parameters using gradient descent techniques..

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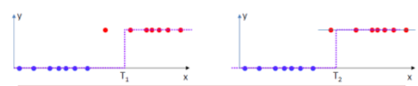
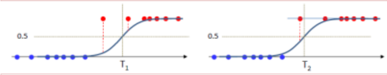
Differentiable Mismatch function



- Threshold activation: shifting the threshold from T_1 to T_2 does not change classification error
  - Does not indicate if moving the threshold left was good or not



- Smooth, continuously varying activation: Classification based on whether the output is greater than 0.5 or less
  - Quantify *how much* the output differs from the desired target value (0 or 1)
  - Moving the function left or right changes this quantity, even if the classification error itself doesn't change

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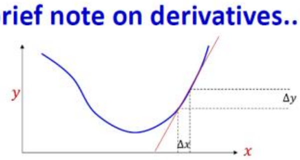
A brief note on derivatives..

derivative



- A derivative of a function at any point tells us how much a minute increment to the *argument* of the function will increment the *value* of the function
  - For any $y = f(x)$, expressed as a multiplier α to a tiny increment Δx to obtain the increments Δy to the output

$$\Delta y = \alpha \Delta x$$

  - Based on the fact that at a fine enough resolution, any smooth, continuous function is locally linear at any point

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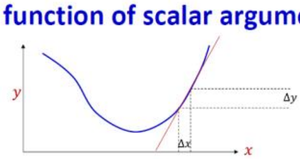
Scalar function of scalar argument



- When x and y are scalar

$$y = f(x)$$

  - Derivative:

$$\Delta y = \alpha \Delta x$$

  - Often represented (using somewhat inaccurate notation) as $\frac{dy}{dx}$
  - Or alternately (and more reasonably) as $f'(x)$

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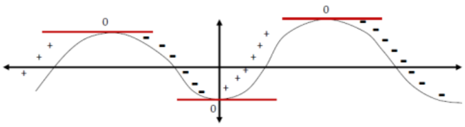
Scalar function of scalar argument



- Derivative $f'(x)$ is the *rate of change* of the function at x
  - How fast it increases with increasing x
  - The magnitude of f'(x) gives you the steepness of the curve at x
    - Larger |f'(x)| → the function is increasing or decreasing more rapidly
- It will be positive where a small increase in x results in an *increase* of f(x)
  - Regions of positive slope
- It will be negative where a small increase in x results in a *decrease* of f(x)
  - Regions of negative slope
- It will be 0 where the function is locally flat (neither increasing nor decreasing)

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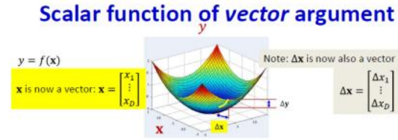
Multivariate scalar function: Scalar function of *vector* argument



$$\Delta y = \alpha \Delta \mathbf{x}$$

- Giving us that α is a row vector: $\alpha = [\alpha_1 \quad \cdots \quad \alpha_D]$

$$\Delta y = \alpha_1 \Delta x_1 + \alpha_2 \Delta x_2 + \cdots + \alpha_D \Delta x_D$$

- The *partial* derivative α_i gives us how y increments when *only* x_i is incremented
- Often represented as $\frac{\partial y}{\partial x_i}$

$$\Delta y = \frac{\partial y}{\partial x_1} \Delta x_1 + \frac{\partial y}{\partial x_2} \Delta x_2 + \cdots + \frac{\partial y}{\partial x_D} \Delta x_D$$

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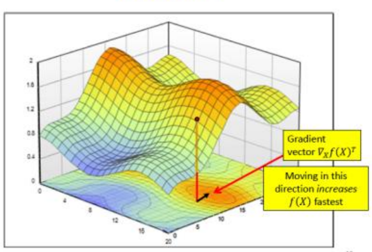
Gradient



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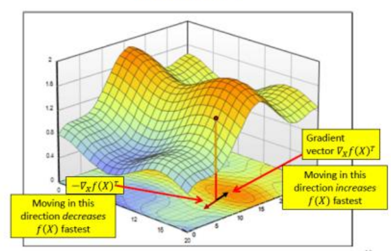
Gradient



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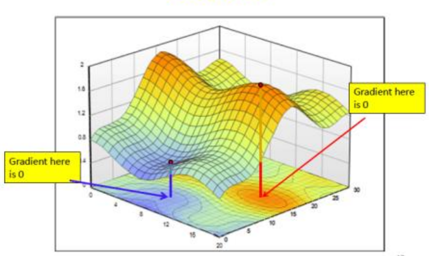
Gradient



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Gradient

- At any location X, there may be many directions in which we can step, such that f(X) increases
- The direction of the gradient is the direction in which the function increases fastest

The Hessian

- The Hessian of a function $f(x_1, x_2, \dots, x_n)$ is given by the second derivative

$$\nabla_x^2 f(x_1,\dots,x_n) := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdot & \cdot & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdot & \cdot & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdot & \cdot & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

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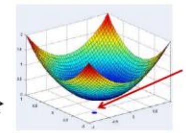
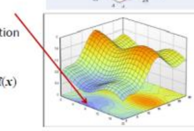
The problem of optimization



- General problem of optimization: Given a function $\mathbf{f}(x)$ of some variable x …
- Find the value of x where $\mathbf{f}(x)$ is minimum



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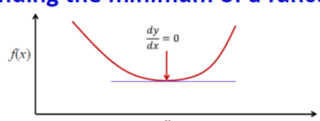
Finding the minimum of a function



- Find the value x at which $f'(x) = 0$
  - Solve

$$\frac{df(x)}{dx} = 0$$

- The solution is a "turning point"
  - Derivatives go from positive to negative or vice versa at this point
- But is it a minimum?

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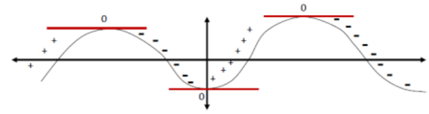
Turning Points



- Both *maxima* and *minima* have zero derivative
- Both are turning points

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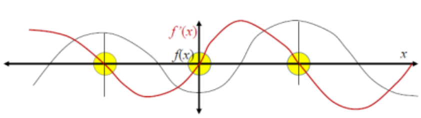
Derivatives of a curve



- Both *maxima* and *minima* are turning points
- Both *maxima* and *minima* have zero derivative

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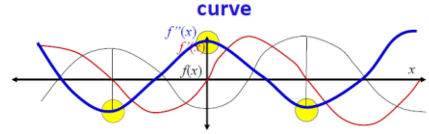
Derivative of the derivative of the curve



- Both *maxima* and *minima* are turning points
- Both *maxima* and *minima* have zero derivative
- The *second derivative* $f''(x)$ is –ve at maxima and +ve at minima!

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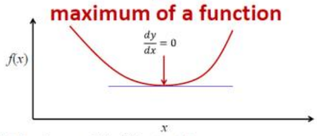
Solution: Finding the minimum or maximum of a function



- Find the value x at which $f'(x) = 0$: Solve
$$\frac{df(x)}{dx} = 0$$
- The solution x_{soln} is a ***turning point***
- Check the double derivative at x_{soln} : compute
$$f''(x_{soln}) = \frac{df'(x_{soln})}{dx}$$
- If $f''(x_{soln})$ is positive x_{soln} is a minimum, otherwise it is a maximum

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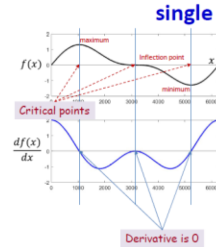
A note on derivatives of functions of single variable



- All locations with zero derivative are *critical* points
  - These can be local maxima, local minima, or inflection points
- The *second* derivative is
  - Positive (or 0) at minima
  - Negative (or 0) at maxima
  - Zero at inflection points

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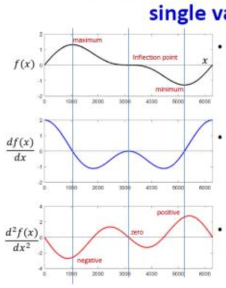
A note on derivatives of functions of single variable



- All locations with zero derivative are *critical* points
  - These can be local maxima, local minima, or inflection points
- The *second* derivative is
  - ≥ 0 at minima
  - ≤ 0 at maxima
  - Zero at inflection points
- It's a little more complicated for functions of multiple variables..

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Unconstrained Minimization of function (Multivariate)

1. Solve for the X where the derivative (or gradient) equals to zero
$$\nabla_X f(X) = 0$$
2. Compute the Hessian Matrix $\nabla_X^2 f(X)$ at the candidate solution and verify that
   - Hessian is positive definite (eigenvalues positive) -> to identify local minima
   - Hessian is negative definite (eigenvalues negative) -> to identify local maxima

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Unconstrained Minimization of function (Example)

- Minimize
$$f(x_1, x_2, x_3) = (x_1)^2 + x_1(1 - x_2) + (x_2)^2 - x_2x_3 + (x_3)^2 + x_3$$
- Gradient
$$\nabla_X f^T = \begin{bmatrix} 2x_1 + 1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 + 1 \end{bmatrix}$$

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Unconstrained Minimization of function (Example)

- Set the gradient to null
$$\nabla_X f = 0 \Rightarrow \begin{bmatrix} 2x_1 + 1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- Solving the 3 equations system with 3 unknowns
$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

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Unconstrained Minimization of function (Example)

- Compute the Hessian matrix $\nabla_X^2 f = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$
- Evaluate the eigenvalues of the Hessian matrix

  $\lambda_1 = 3.414,\ \lambda_2 = 0.586,\ \lambda_3 = 2$
- All the eigenvalues are positives => the Hessian matrix is positive definite
- The point $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ is a minimum

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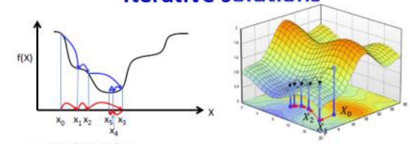
Iterative solutions



- Iterative solutions
  - Start from an initial guess X_0 for the optimal X
  - Update the guess towards a (hopefully) "better" value of f(X)
  - Stop when f(X) no longer decreases
- Problems:
  - Which direction to step in
  - How big must the steps be

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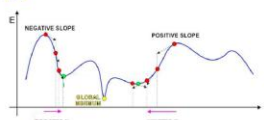
The Approach of Gradient Descent



- Iterative solution:
  - Start at some point
  - Find direction in which to shift this point to decrease error
    - This can be found from the derivative of the function
      - A negative derivative → moving right decreases error
      - A positive derivative → moving left decreases error
  - Shift point in this direction

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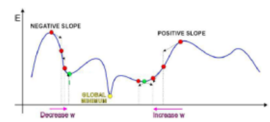
The Approach of Gradient Descent



- Iterative solution: Trivial algorithm
  - Initialize x^0
  - While $f'(x^k) \neq 0$

    $x^{k+1} = x^k - \eta^k f'(x^k)$
- η^k is the "step size"

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So far…

- Minimum of a function f(x) is when : f'(x) = 0 and the second derivative f''(x) is positive
- At any location X, there may be many directions in which we can step, such that f(X) increases
- The direction of the gradient is the direction in which the function increases fastest

Gradient descent/ascent (multivariate)

- The gradient descent/ascent method to find the minimum or maximum of a function f iteratively
  - To find a *maximum* move *in the direction of the gradient*

    $$x^{k+1} = x^k + \eta^k \nabla_x f(x^k)^T$$
  - To find a *minimum* move *exactly opposite the direction of the gradient*

    $$x^{k+1} = x^k - \eta^k \nabla_x f(x^k)^T$$
- Many solutions to choosing step size η^k

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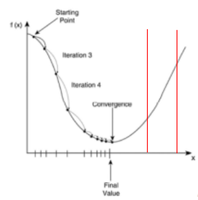
Gradient descent convergence criteria

- The gradient descent algorithm converges when one of the following criteria is satisfied

$$\boxed{\left|f(x^{k+1}) - f(x^k)\right| < \varepsilon_1}$$

- Or

$$\left\|\nabla_x f(x^k)\right\| < \varepsilon_2$$



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Overall Gradient Descent Algorithm

- Initialize:
  - x^0
  - $k = 0$
- do
  - $x^{k+1} = x^k - \eta^k \nabla_x f(x^k)^T$
  - $k = k + 1$
- while $\left|f(x^{k+1}) - f(x^k)\right| > \varepsilon$

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Preliminaries

Problem Statement

- Given a training set of input-output pairs $(\boldsymbol{X}_1, \boldsymbol{d}_1), (\boldsymbol{X}_2, \boldsymbol{d}_2), \ldots, (\boldsymbol{X}_T, \boldsymbol{d}_T)$
- Minimize the following function

$$Loss(W) = \frac{1}{T}\sum_i div(f(X_i; W), d_i)$$

w.r.t W

- This is problem of function minimization
  - An instance of optimization

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Problem Setup: Things to define

- Given a training set of input-output pairs $(\boldsymbol{X}_1, \boldsymbol{d}_1), (\boldsymbol{X}_2, \boldsymbol{d}_2), \ldots, (\boldsymbol{X}_T, \boldsymbol{d}_T)$
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Problem Setup: Things to define

- Given a training set of input-output pairs $(\boldsymbol{X}_1, \boldsymbol{d}_1), (\boldsymbol{X}_2, \boldsymbol{d}_2), \ldots, (\boldsymbol{X}_T, \boldsymbol{d}_T)$
- What are these input-output pairs?

$$Loss(W) = \frac{1}{T}\sum_i div(f(X_i; W), d_i)$$

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Problem Setup: Things to define

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What is f() and what are its parameters W?

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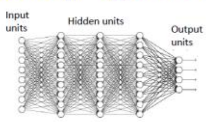
What is the divergence div()?

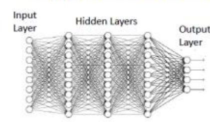
What is f() and what are its parameters W?

What is f()? Typical network



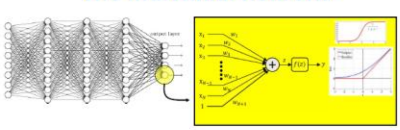
- Multi-layer perceptron
- A *directed* network with a set of inputs and outputs
  – No loops

Typical network



- We assume a "layered" network for simplicity
  – Each "layer" of neurons only gets inputs from the earlier layer(s) and outputs signals only to later layer(s)
  – We will refer to the inputs as the ***input layer***
    - No neurons here – the "layer" simply refers to inputs
  – We refer to the outputs as the ***output layer***
  – Intermediate layers are ***"hidden" layers***

The individual neurons



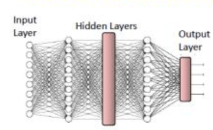
- Individual neurons operate on a set of inputs and produce a single output
  – **Standard setup:** A continuous activation function applied to an affine function of the inputs

$$y = f\left(\sum_i w_i x_i + b\right)$$

  – More generally: *any* differentiable function

$$y = f(x_1, x_2, \ldots, x_N; W)$$

Vector Activations

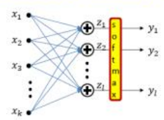


- We can also have neurons that have *multiple coupled* outputs

$$[y_1, y_2, \ldots, y_l] = f(x_1, x_2, \ldots, x_k; W)$$

  – Function $f()$ operates on set of inputs to produce set of outputs
  – Modifying a single parameter in W will affect *all* outputs

Vector activation example: Softmax



- Example: Softmax *vector* activation

$$z_i = \sum_j w_{ji} x_j + b_i$$

Parameters are weights w_{ji} and bias b_i

$$y = \frac{exp(z_i)}{\sum_j exp(z_j)}$$

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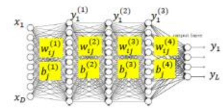
Notation



- The input layer is the 0^{th} layer
- We will represent the output of the i-th perceptron of the k^{th} layer as $y_i^{(k)}$
  - **Input to network:** $y_i^{(0)} = x_i$
  - **Output of network:** $y_i = y_i^{(N)}$
- We will represent the weight of the connection between the i-th unit of the k-1th layer and the jth unit of the k-th layer as $w_{ij}^{(k)}$
  - The bias to the jth unit of the k-th layer is $b_j^{(k)}$

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Problem Setup: Things to define

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$$Loss(W) = \frac{1}{T} \sum_i div(f(X_i; W), d_i)$$

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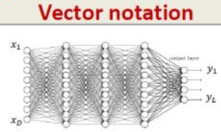
Input, target output, and actual output: Vector notation



- Given a training set of input-output pairs $(X_1, d_1), (X_2, d_2), \ldots, (X_T, d_T)$
- $X_n = [x_{n1}, x_{n2}, \ldots, x_{nD}]^\top$ is the nth input vector
- $d_n = [d_{n1}, d_{n2}, \ldots, d_{nL}]^\top$ is the nth desired output
- $Y_n = [y_{n1}, y_{n2}, \ldots, y_{nL}]^\top$ is the nth vector of *actual* outputs of the network
  - Function of input X_n and network parameters
- We will sometimes drop the first subscript when referring to a *specific* instance

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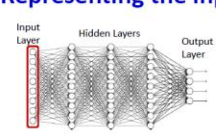
Representing the input



- Vectors of numbers
  - (or may even be just a scalar, if input layer is of size 1)
  - E.g. vector of pixel values
  - E.g. vector of speech features
  - E.g. real-valued vector representing text
  - Other real valued vectors

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Multi-class output: One-hot representations

- Consider a network that must distinguish if an input is a cat, a dog, a camel, a hat, or a flower
- We can represent this set as the following vector, with the classes arranged in a chosen order:

  [cat dog camel hat flower]^T
- For inputs of each of the five classes the desired output is:

  cat: $[1\,0\,0\,0\,0]^T$

  dog: $[0\,1\,0\,0\,0]^T$

  camel: $[0\,0\,1\,0\,0]^T$

  hat: $[0\,0\,0\,1\,0]^T$

  flower: $[0\,0\,0\,0\,1]^T$
- For an input of any class, we will have a five-dimensional vector output with four zeros and a single 1 at the position of that class
- This is a *one hot vector*

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Problem Setup: Things to define

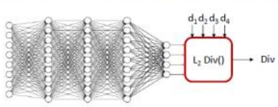
- Given a training set of input-output pairs $(\boldsymbol{X}_1, \boldsymbol{d}_1), (\boldsymbol{X}_2, \boldsymbol{d}_2), \ldots, (\boldsymbol{X}_T, \boldsymbol{d}_T)$
- Minimize the following function

$$Loss(W) = \frac{1}{T}\sum_i div(f(X_i; W), d_i)$$

What is the divergence div()?

Note: For Loss(W) to be differentiable w.r.t W, div() must be differentiable

Examples of divergence functions



- For real-valued output vectors, the (scaled) L_2 divergence is popular

$$Div(Y, d) = \frac{1}{2}\|Y - d\|^2 = \frac{1}{2}\sum_i (y_i - d_i)^2$$

  - Squared Euclidean distance between true and desired output
  - Note: this is differentiable

$$\frac{dDiv(Y, d)}{dy_i} = (y_i - d_i)$$

$$\nabla_Y Div(Y, d) = [y_1 - d_1, y_2 - d_2, \ldots]$$

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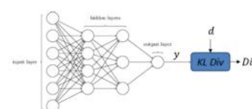
For binary classifier



- For binary classifier with scalar output, $Y \in (0,1)$, d is 0/1, the Kullback Leibler (KL) divergence between the probability distribution $[Y, 1-Y]$ and the ideal output probability $[d, 1-d]$ is popular

$$Div(Y, d) = -d\log Y - (1-d)\log(1-Y)$$

  - Minimum when $d = Y$
- Derivative

$$\frac{dDiv(Y, d)}{dY} = \begin{cases} -\frac{1}{Y} & if\ d = 1 \\ \frac{1}{1-Y} & if\ d = 0 \end{cases}$$

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Summary

- Neural nets are universal approximators
- Neural networks are trained to approximate functions by adjusting their parameters to minimize the average divergence between their actual output and the desired output at a set of "training instances"
  - Input-output samples from the function to be learned
  - The average divergence is the "Loss" to be minimized
- To train them, several terms must be defined
  - The network itself
  - The manner in which inputs are represented as numbers
  - The manner in which outputs are represented as numbers
    - As numeric vectors for real predictions
    - As one-hot vectors for classification functions
  - The divergence function that computes the error between actual and desired outputs
    - L2 divergence for real-valued predictions
    - KL divergence for classifiers