EN.550.661: Nonlinear Optimization I

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Homework Assignment #1

Starred exercises require the use of MATLAB.

Exercise 1.1: Compute $f'(x)$, $\nabla f(x)$, and $\nabla^2 f(x)$ for the following functions $f : \mathbb{R}^n \to \mathbb{R}$.

(a) $f(x) = \frac{1}{2}x^T H x$, where $H \in \mathbb{R}^{n\times n}$ is a constant matrix. What if H is symmetric?

(b) $f(x) = b^T A x - \frac{1}{2}x^T A^T A x$, where $A \in \mathbb{R}^{m\times n}$ is a constant matrix and $b \in \mathbb{R}^m$ is a constant vector.

(c) $f(x) = \|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$

(d) $f(x) = \|Ax - b\|_2$, where $A \in \mathbb{R}^{m\times n}$ is a constant matrix and $b \in \mathbb{R}^m$ is a constant vector.

Exercise 1.2: Let $\mathcal{S} \subseteq \mathbb{R}^n$, $f : \mathcal{S} \to \mathbb{R}$, and suppose that $x + \alpha s \in \mathcal{S}$ for all $\alpha \in [0, 1]$.

(a) By defining $\theta(\alpha) = f(x + \alpha s)$ and using the integration-by-parts (Newton) formula

$$\theta(1) - \theta(0) = \int_0^1 \theta'(\alpha)\, d\alpha,$$

show that

$$|f(x+s) - f(x) - g(x)^T s| \le \tfrac{1}{2}\gamma^L(x)\|s\|_2^2$$

whenever $f \in \mathcal{C}^1$ has a Lipschitz continuous gradient $g(x)$ with constant $\gamma^L(x)$ on $\mathcal{S}$.

(b) Justify the integration-by-parts formula

$$\theta(1) - \theta(0) - \theta'(0) = \int_0^1 (1-\alpha)\theta''(\alpha)\, d\alpha.$$

Hence, show that

$$|f(x+s) - f(x) - g(x)^T s - \tfrac{1}{2}s^T H(x) s| \le \frac{1}{6}\gamma^Q(x)\|s\|_2^3,$$

whenever $f \in \mathcal{C}^2$ has a Lipschitz continuous Hessian $H(x)$ with constant $\gamma^Q(x)$ on $\mathcal{S}$.

Exercise 1.3: Let $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a *convex* function and $0 \le \alpha_i \in \mathbb{R}$ for $i = 1, \ldots, k$.

(a) Prove that

$$f(x) = \sum_{i=1}^k \alpha_i f_i(x)$$

is a convex function.

(b) Prove that

$$f(x) = \max\big(f_1(x), f_2(x), \ldots, f_k(x)\big)$$

is a convex function.

Exercise 1.4: Let x^* be a *local* minimizer of a *convex* function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Prove that x^* is, in fact, a *global* minimizer of f. Is x^* a *unique* global minimizer?