Pingala Series

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let α and β ($\alpha > \beta$) be the roots of the equation $z^2 - z - 1 = 0$. Define,

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4
$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Download the Python code using

```
$ wget https://github.com/NiharikaKute/PINGALA
    /blob/main/codes/1.py
```

and run it using

```
$ python3 1.py
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution: Download the Python code using

```
$ wget https://github.com/NiharikaKute/
    PINGALA/blob/main/codes/2_1.py
```

and run it using

```
$ python3 2_1.py
```

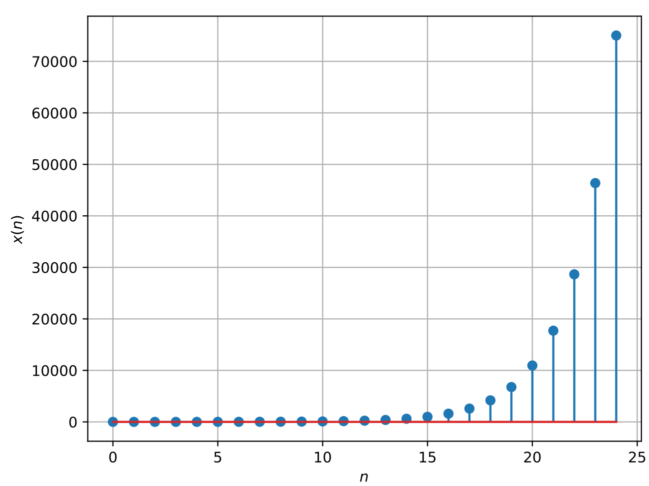

Fig. 2.1: Plot of $x(n)$

2.3 Find $X^+(z)$.
Solution: Taking the one-sided Z-transform on both sides of (2.2),

$$\mathcal{Z}^+\left[x(n+2)\right] = \mathcal{Z}^+\left[x(n+1)\right] + \mathcal{Z}^+\left[x(n)\right] \tag{2.3}$$

$$z^2X^+(z) - z^2x(0) - zx(1) = zX^+(z) - zx(0) + zX^+(z) \tag{2.4}$$

$$\left(z^2 - z - 1\right)X^+(z) = z^2 \tag{2.5}$$

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.6}$$

$$= \frac{1}{\left(1 - \alpha z^{-1}\right)\left(1 - \beta z^{-1}\right)}, \quad |z| > \alpha \tag{2.7}$$

2.4 Find $x(n)$.
Solution: Expanding $X^+(z)$ in (2.7) using partial fractions, we get

$$X^+(z) = \frac{1}{(\alpha - \beta)\, z^{-1}}\left[\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}}\right] \tag{2.8}$$

$$= \frac{1}{(\alpha - \beta)}\sum_{n=0}^{\infty}\left(\alpha^n - \beta^n\right) z^{-n+1} \tag{2.9}$$

$$= \sum_{n=1}^{\infty}\frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1} \tag{2.10}$$

$$= \sum_{k=0}^{\infty}\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} z^{-k} \tag{2.11}$$

where $k := n + 1$. Thus,

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) = a_{n+1}u(n) \tag{2.12}$$

2.5 Sketch

$$y(n) = x\left(n - 1\right) + x\left(n + 1\right), \quad n \geq 0 \tag{2.13}$$

Solution: Download the Python code using

```
$ wget https://github.com/NiharikaKute/
    PINGALA/blob/main/codes/2_2.py
```

and run it using

```
$ python3 2_2.py
```

2.6 Find $Y^+(z)$.
Solution: Taking the one-sided Z-transform on

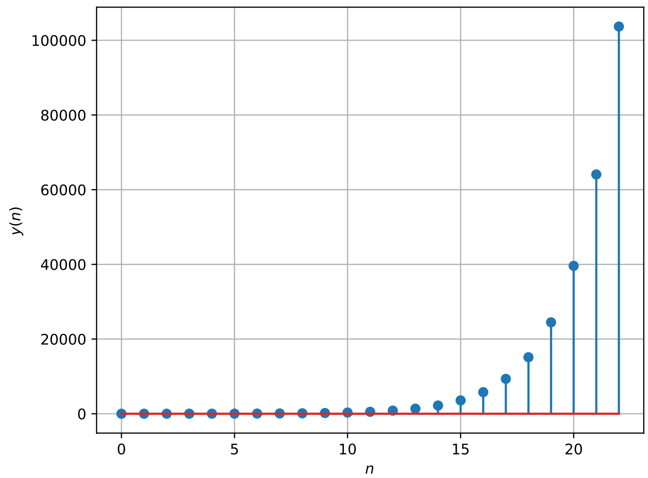

Fig. 2.2: Plot of $y(n)$

both sides of (2.13),

$$\mathcal{Z}^+\left[y(n)\right] = \mathcal{Z}^+\left[x(n+1)\right] + \mathcal{Z}^+\left[x(n-1)\right] \tag{2.14}$$

$$Y^+(z) = zX^+(z) - zx(0) + z^{-1}X^+(z) + zx(-1) \tag{2.15}$$

$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \tag{2.16}$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha \tag{2.17}$$

since $x(n) = 0 \ \forall \ n < 0$.

2.7 Find $y(n)$.
Solution: Using (2.7),

$$Y^+(z) = \left(1 + 2z^{-1}\right)\sum_{n=0}^{\infty} x(n)z^{-n} \tag{2.18}$$

$$= \sum_{n=0}^{\infty} x(n)z^{-n} + \sum_{n=1}^{\infty} 2x(n-1)z^{-n} \tag{2.19}$$

$$= x(0) + \sum_{n=1}^{\infty}\left(x(n) + 2x(n-1)\right) z^{-n} \tag{2.20}$$

Thus, $y(0) = x(0) = 1$ and for $n \geq 1$, using the fact that α and β are the roots of the equation

$z^2 - z - 1 = 0,$

$$y(n) = \frac{\left(\alpha^{n+1} - \beta^{n+1}\right) + (2\alpha^n + 2\beta^n)}{\alpha - \beta} \tag{2.21}$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) + (\alpha^n + \beta^n)}{\alpha - \beta} \tag{2.22}$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) - \alpha\beta\left(\alpha^n + \beta^n\right)}{\alpha - \beta} \tag{2.23}$$

$$= \frac{(\alpha - \beta)\left(\alpha^{n+1} + \beta^{n+1}\right)}{\alpha - \beta} \tag{2.24}$$

$$= \alpha^{n+1} + \beta^{n+1} \tag{2.25}$$

Thus, $y(n) = \alpha^{n+1} + \beta^{n+1}$ for $n \geq 0$ as $\alpha + \beta = 1$. Comparing (2.22) with the definition of b_n, we see that $y(n) = b_{n+1}$. Hence, $b_n = \alpha^n + \beta^n$.

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{3.1}$$

Solution: From (2.12), and noting that $x(n) = 0 \ \forall \ n < 0$,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.2}$$

$$= \sum_{k=-\infty}^{n-1} x(k) \tag{3.3}$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \tag{3.4}$$

$$= x(n) * u(n-1) \tag{3.5}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.6}$$

can be expressed as

$$\left[x(n+1) - 1\right] u(n) \tag{3.7}$$

Solution: From (2.12),

$$a_{n+2} - 1 = \left[x(n+1) - 1\right], \quad n \geq 0 \tag{3.8}$$

and so, using the definition of $u(n)$,

$$a_{n+2} - 1 = \left[x(n+1) - 1\right] u(n) \tag{3.9}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.10}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \tag{3.11}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.12}$$

$$= \frac{1}{10} X^+(z) \tag{3.13}$$

$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \tag{3.14}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.15}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.16}$$

and find $W(z)$.

Solution: Putting $n = k + 1$ in (3.15) and using the definition of $u(n)$,

$$\alpha^n + \beta^n = \left(\alpha^{k+1} + \beta^{k+1}\right) u(k) \tag{3.17}$$

Hence, (3.15) can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) = y(n) \tag{3.18}$$

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.19}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.20}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \tag{3.21}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.22}$$

$$= \frac{1}{10} Y^+(z) \tag{3.23}$$

$$= \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \tag{3.24}$$

3.6 Solve the JEE 2019 problem.

Solution: We know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.25}$$

But

$$x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z)z^{-1}U(z) \tag{3.26}$$

$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \tag{3.27}$$

$$= z\left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}}\right] \tag{3.28}$$

$$\overset{\mathcal{Z}}{\rightleftharpoons} z\sum_{n=0}^{\infty}(x(n) - 1)\,z^{-n} \tag{3.29}$$

$$= \sum_{n=0}^{\infty}(x(n) - 1)\,z^{-n+1} \tag{3.30}$$

$$= \sum_{n=0}^{\infty}(x(n+1) - 1)\,z^{-n} \tag{3.31}$$

$$\tag{3.32}$$

From (2.12), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{3.33}$$

We have already established the remaining options in order in the problems (3.3), (2.7), (3.5). Therefore, options 1, 2, and 3 are correct and option 4 is incorrect.