Cramér–Rao Lower Bounds for QAM Phase & Frequency Estimation

Authors : Feng Rice, Bill Cowley, Bill Moran, and Mark Rice

Niharika
2018122008

Motivation

- The aim of this paper is to estimate frequency & phase offset for Quadrature Amplitude Modulation (QAM) , PSK & PAM signals in an AWGN noise setting .

- Comparison of these bounds with existing CRLBs for unmodulated carrier wave (CW) , BPSK and QPSK .

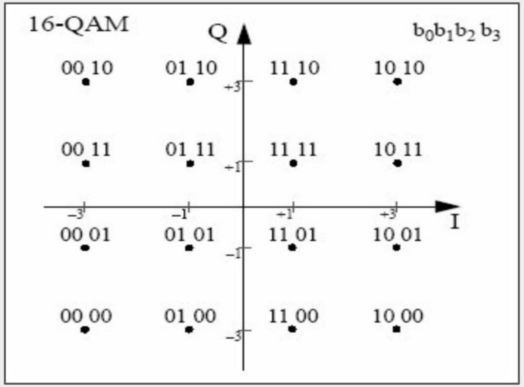
- Comparison of existing phase estimation techniques for QAM with the true CLRB .

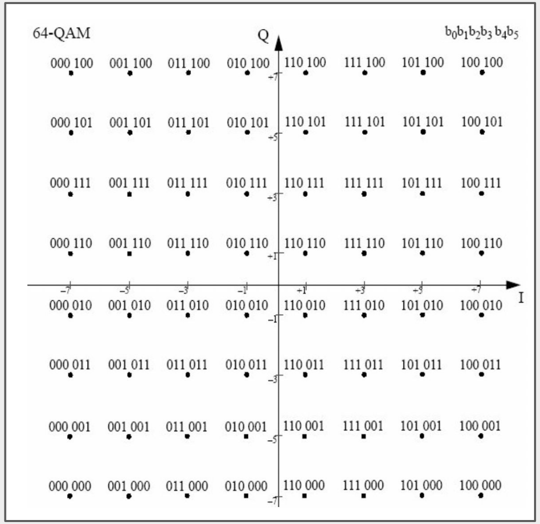
Introduction

QAM : It’s a modulation technique , that uses both multiple amplitude & phase levels for transmission . It provides high levels of spectrum usage efficiency. Two drawbacks include : Noise susceptibility & Linear amplification.

- QAM is a signal in which two carriers shifted in phase by 90 degrees (i.e. sine and cosine) are modulated and combined. As a result of their 90° phase difference they are in quadrature and this gives rise to the name. Often one signal is called the In-phase or “I” signal, and the other is the quadrature or “Q” signal.

Types of QAM	Bits Required	Error Margin	SNR for BER = 1 in 10^6
16 QAM	4	0.23	~ 10.5
64 QAM	6	0.1	~ 18.5





Derivations for CRLB

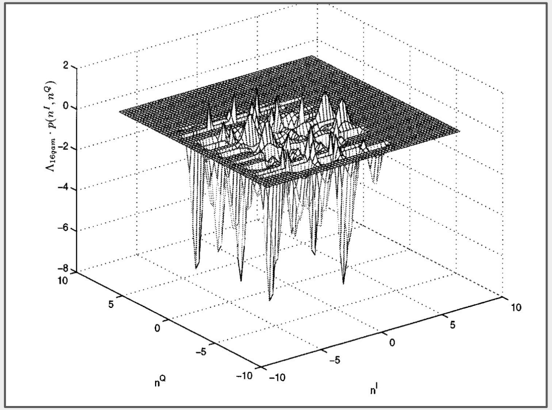
Ⅰ. CRLB for Phase Estimation

II. Frequency CRLB with known phase

III. Joint Frequency & Phase Estimation CRLB

CRLB Analysis

$$F(\sigma^2) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Lambda(\xi(\phi, n^I, n^Q, A_k, a_i, r)) \times p(n^I, n^Q)\, dn^I\, dn^Q.$$



Phase Only Estimation :

- F(sigma^2) does not have analytical solution , the following is simply the plot of the 3D surface for 16QAM , 15dB .
- The execution time at low SNR favored the use of the numerical method , whilst at moderate to high SNR the MCE was more efficient (by a factor of 1000 at 10 dB).
- Lambda (16QAM) alone can be evaluated using Monte Carlo evaluation (MCE) as it is a relatively smooth function.
- MCE Evaluation of CRLB at different SNRs , it almost matches numerical integration results . But SNR <-9dB , MCE requires large number of trials.
- Note : At high SNR, all estimator variances approach CW performance.

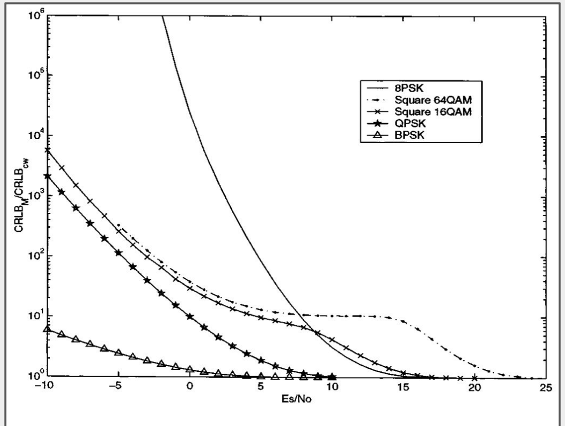


Fig. 2. Ratios F_m^{-1} of CRLB for various modulations to CRLB for carrier.

CRLB Analysis (Contd.)

Frequency CRLB

- In a practical receiver, the phase is usually unknown and uniformly distributed. Averaging over (phi + k*omega)) random variable makes it complex .

- The figure shows 16QAM, 64QAM, MPSK, and CW CRLBs for frequency estimation with an symbol block. The vertical axis is the standard deviation of frequency error normalized by symbol rate.

- Compared to 64 QAM , 16 QAM has less frequency error & std deviation .

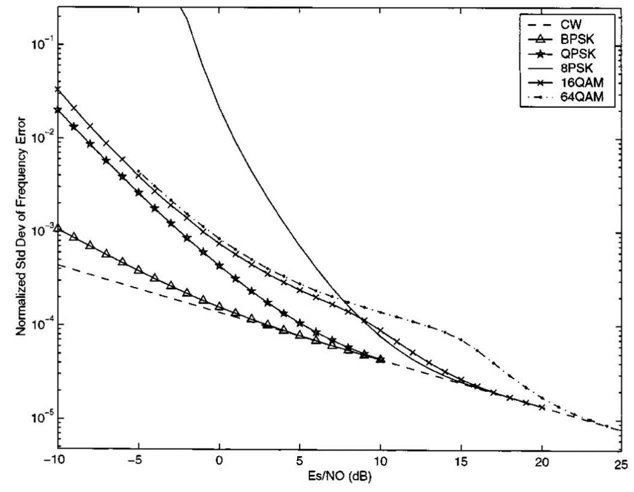


Fig. 3. Frequency offset CRLBs for various modulations with 200 symbols.

CRLB Analysis (Contd.)

Joint Phase & Frequency Distribution

- These results are the same as the unmodulated carrier (CW) case for equal average power, except for the factor (Fm(No/2Es) factor .
- Note that when the joint estimator is analyzed relative to the middle of the signal vector (i.e., symmetrically, in the range), then the FIM becomes diagonal and the frequency and phase estimators are decoupled .
- Maximum likelihood estimators phi and omega are asymptotically (for large N) independent.
- KN -> Set of received samples , KS-> sampling times are symmetrically located about zero .

$$\textbf{Joint CRLB}(\hat{\omega}) = \frac{12\sigma^2}{N(N^2-1)}\frac{1}{F(\sigma^2)} = \mathrm{CRLB}_{\mathrm{CW}}(\hat{\omega})\frac{1}{F\left(\frac{N_o}{2E_s}\right)}$$

$$\textbf{Joint CRLB}(\hat{\phi}) = \frac{\sigma^2}{N}\frac{2(2N-1)}{N+1}\frac{1}{F(\sigma^2)} = \mathrm{CRLB}_{\mathrm{CW}}(\hat{\phi})\frac{1}{F\left(\frac{N_o}{2E_s}\right)}.$$

$$\text{Joint CRLB}(\hat{\phi}) = \frac{1}{2N\frac{E_s}{N_o}}\frac{1}{F\left(\frac{N_o}{2E_s}\right)}\gamma$$

where

$$\gamma = \begin{cases} 1, & \mathcal{K}_S \text{ (decoupled)} \\ \frac{2(2N-1)}{N+1}, & \mathcal{K}_N \\ 4, & \mathcal{K}_N, \quad N \to \infty. \end{cases}$$

Additional Phase Estimators

I . Power Law Estimator

- For generic rotationally symmetric constellation diagrams, in moderate to high SNR regions. (M=4 QPSK/QAM)
- PLE Simulation & PLE Approximation are added to the graphs .

$$\hat{\theta} = \frac{1}{M} \arg\left(E\left[a_k^{*M} \right] \sum_{k=0}^{N-1} x_k^M \right)$$

$$\mathrm{Var}[\hat{\theta} - \theta] \simeq \frac{1}{K} B_1 \left(\frac{N_o}{2E_s} \right) + \frac{1}{K} B_2 \qquad (14)$$

with

$$B_1 = \frac{E[|c_k^2|]E[|c_k^2|^{N-1}]}{|E[c_k^N]|^2} \geq 1 \qquad (15)$$

$$B_2 = \frac{2|E[c_k^N]|^2 E[|c_k^2|^N] - E^2[c_k^N]E[c_k^{*2N}] - E^2[c_k^{*N}]E[c_k^{2N}]}{4N^2|E[c_k^N]|^4}. \qquad (16)$$

II . Histogram Algorithm

- A simple algorithm for blind carrier phase acquisition that can be used for square, star & cross constellations.
- A histogram of all quantized phi(k) is generated and the largest bin entry is selected. This bin corresponds to the estimated phase.
- Modified histogram algorithm (MHA) which improved the performance significantly at high SNR.

$$L(\phi) = \sum_{k=0}^{N-1} \ln\left[\sum_a \exp\left(-\frac{1}{2\sigma^2} \left| |x_k| - |a| \times e^{j(\phi + \psi - \theta_k)} \right|^2 \right) \right] \qquad (31)$$

$$\hat{\phi}_k = (\theta_k - \alpha_k) \bmod \left(\frac{\pi}{2} \right).$$

Two-Stage-Conjugate Algorithm (By authors)

- Applicable to any square constellation .
- The signals close to the inner and outer rings are called Class I and the others close to the middle ring are called Class II .
- Viterbi and Viterbi phase estimation (VVPE) is applied to the Class I signal treated as QPSK symbols to obtain the estimated coarse phase offset .

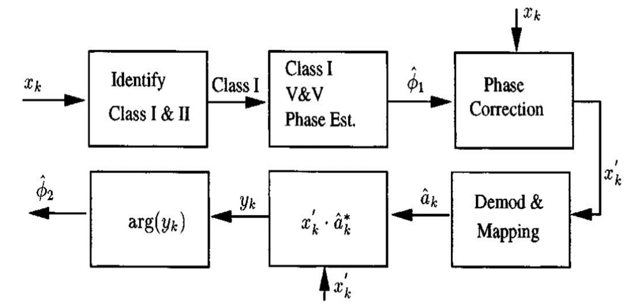


Fig. 5. Two-stage-conjugate phase estimation scheme.

$$x_k \in \begin{cases} \mathbf{C_1}, & \text{if } |x_k| \le T_1, \quad \text{or} \quad |x_k| \ge T_2, \\ \mathbf{C_2}, & \text{if } T_1 < |x_k| < T_2 \end{cases}$$

$$T_1 = \sqrt{2} + (\sqrt{2}/2) \text{ and } T_2 = (\sqrt{10} + \sqrt{18})/2,$$

$$\hat{\phi}_1 = \frac{1}{4} \tan^{-1} \sum_{x_k \in \mathbf{C_1}} \frac{\Im\left(\frac{x_k^4}{|x_k|^3}\right)}{\Re\left(\frac{x_k^4}{|x_k|^3}\right)}.$$

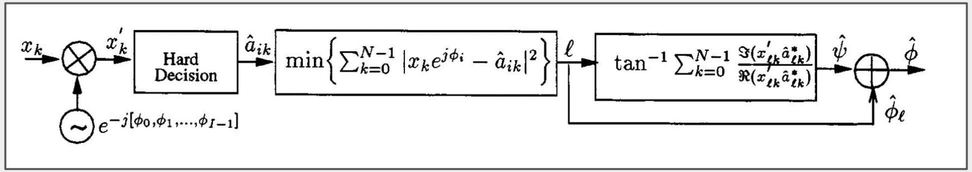
$$y_k = x_k' \hat{a}_k^*, \quad k = 0, 1, \ldots, N-1$$

$$\hat{\phi}_2 = \tan^{-1} \sum_{k=0}^{N-1} \frac{\Im(y_k)}{\Re(y_k)}.$$

$$\hat{\phi} = \hat{\phi}_1 + \hat{\phi}_2.$$

Minimum Distance Estimator

- Applicable for QAM constellation good performance but increase computation cost .
- Rotate the rec signal by phi set , in range of [- 45* , 45 *].
- Perform Hard decisions on the estimated (aik)
- Choose arg min index of the vector.



$$\phi = (\phi_0, \phi_1, \ldots \phi_{I-1})$$

$$D_i = \sum_{k=0}^{N-1} |x_k e^{j\phi_i} - \hat{a}_{ik}|^2$$

$$\text{s } D_\ell = \min(D_0, D_1, \ldots, D_{I-1}).$$

$$\hat{\psi} = \tan^{-1} \sum_{k=0}^{N-1} \frac{\Im(x'_{\ell k}\hat{a}^*_{\ell k})}{\Re(x'_{\ell k}\hat{a}^*_{\ell k})}.$$

$$\hat{\phi} = \hat{\phi}_\ell + \hat{\psi}.$$

Performance Comparison

- In order to determine the performance of the HA, 2SC and MDE phase estimators were simulated using a Monte Carlo technique.
- The 16QAM and 64QAM signals are randomly generated. Then the signal is multiplied by e^(j*phi) where phi is a static phase offset taken from a random uniform distribution. Noise is generated by using a complex Gaussian noise generator & scaled appropriately .
- The number of symbols   in the simulation is 200, with a minimum of 1000 trials MCE to ensure accuracy.
- The approximate analytic expression for the power low estimator (PLE) is also plotted for comparison.
- The vertical scale indicates the standard deviation of phase error in radians & horizontal scale - SNR .

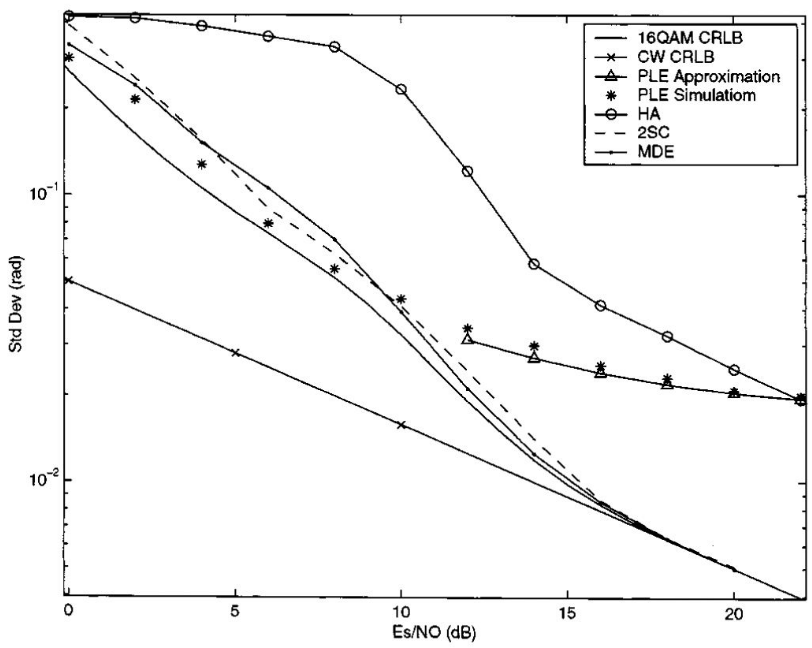


Fig. 7. Comparison of the phase estimator performance and the true 16QAM CRLB for 200 symbols.

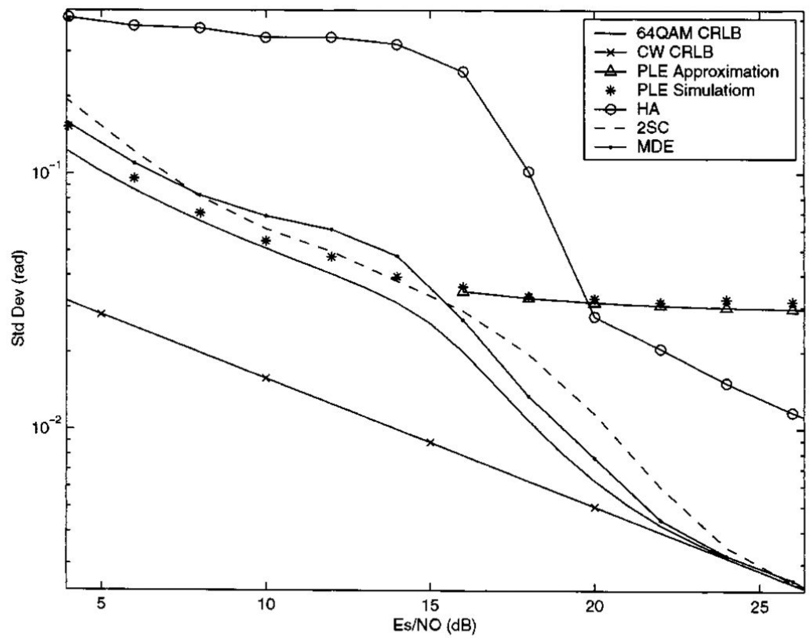


Fig. 8. Comparison of the phase estimator performance and the true 64QAM CRLB for 200 symbols.

Thank You