Cramer-Rao Lower Bound for QAM Phase and Frequency Estimation

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I. Introduction

The goal of this paper is to identify true CLRBs for estimating unknown phase offset & frequency component in QAM,PSK and PAM signals . The channels are assumed to be AWGN and no prior information is known (blind channel).The derived bounds are compared to existing true CLRBs of unmodulated carrier wave (CW),BPSK & QPSK . The paper also performs performance comparison on few existing QAM phase estimators.

II. Cramer-Rao Lower Bound

In estimation theory and statistics, the Cramér–Rao bound (CRB) expresses a lower bound on the variance of unbiased estimators of a deterministic (fixed, though unknown) parameter, stating that the variance of any such estimator is at least as high as the inverse of the Fisher information. Given a random variable $\mathbf{X}$, parameterised by ϕ , the CRLB is given by :

$$\operatorname{CRLB}(\hat{\phi})>=\frac{1}{-E\left[\frac{\partial^{2} \ln p(\mathbf{X} \mid \phi)}{\partial \phi^{2}}\right]} \tag{1}$$

A. Theorem I : CLRB for Unknown Phase

Given a QAM constellation , we can model received samples with phase offset ϕ as :

$$x_{k}=a_{k} e^{j \phi}+w_{k}, \quad k=0,1, \ldots, N-1 \tag{2}$$

where a_k are the transmitted QAM constellation & w_k is a complex Gaussian random variables. The symbol SNR is $E_s/N_o = 1/\sigma^2$. We will assume that we a distribution $\mathbf{X} = (X_0, X_1 \ldots X_{N-1})$, is a random vector and $E(.)$ denotes statistical expectation wrt to pdf $p_X(x|\phi)$. Here , we compute the joint distribution .

$$p(\mathbf{x} \mid \phi)=p_{\mathbf{X}}(\mathbf{X} \mid \phi)$$

=p$(x_0 \mid \phi)\, p(x_1 \mid \phi) \cdots p(x_{N-1} \mid \phi)$
=$\prod_k p(x_k \mid \phi)$
=$\prod_k \left[p_A(a) \sum_{a_i \in \mathbf{C}} \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left|x_k - a_i e^{j\phi}\right|^2}\right]$ (3)

We obtain the pdf , after substituting the a_i symbols of the 16 QAM constellation and symmetry properties as :

$$p(\mathbf{x} \mid \phi)=\prod_{k}\left\{\frac{p_{A}(a)}{\pi \sigma^{2}} e^{-\frac{\left|x_{k}\right|^{2}}{2 \sigma^{2}}} \sum_{a_{i} \in \mathbf{Q}_{1}} e^{-\frac{\left|a_{i}\right|^{2}}{2 \sigma^{2}}}\right.$$
$$\left.\times \sum_{r=1,-1} \cosh \left(\Re\left(\frac{|a_i|}{\sigma^2} x_k e^{-j\left(\phi+r \arctan \frac{\Im(a_j)}{\Re(a_i)}\right)}\right)\right)\right\} \tag{4}$$

Here , Q_1 consists of constellation points in first quadrant , and $\Re$ & $\Im$ are real and imaginary part of the complex number . After simplification , we can define a new term :

$$\xi\left(\phi, x_{k}, a_{i}, r\right)=\left|a_{i}\right| x_{k} e^{-j\left(\phi+r \arctan \frac{\Im\left(a_{i}\right)}{\Re\left(a_{i}\right)}\right)} \tag{5}$$

Simplifying the above , after using mathematical results of $cosh$ & $sinh$ function , we obtain :

$$E\left[\frac{\partial^{2} \ln p(\mathbf{x} \mid \phi)}{\partial \phi^{2}}\right]=\frac{1}{\sigma^{2}} E\left[\sum_{k} \Lambda\left(x_{k}\right)\right]$$

where $\Lambda(k)$ is dependant only on x_k. And the entire formulation , leads us to :

$$\begin{aligned} & E\left[\sum_{k} \Lambda\left(x_{k}\right)\right] \\ & =\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{k} \Lambda\left(x_{k}\right) p(\mathbf{x} \mid \phi) d|\mathbf{x}| \\ & =\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{k} \Lambda\left(x_{k}\right) p\left(x_{0} \mid \phi\right) \\ & \quad \times p\left(x_{1} \mid \phi\right) \ldots p\left(x_{N-1} \mid \phi\right) d\left|x_{0}\right| d\left|x_{1}\right| \ldots d\left|x_{N-1}\right| \end{aligned}$$

Despite , the integral seeming to have terms of $x(k)$, we can ignore all those as $\Lambda x(k)$ depends only on $x(k)$ and no other parameter.

$$E\left[\frac{\partial^{2} \ln p(\mathbf{x} \mid \phi)}{\partial \phi^{2}}\right]=\frac{N}{\sigma^{2}} E[\Lambda(x)]$$

We define :

$$F\left(\sigma^{2}\right)=-E[\Lambda(x)]$$

$$\operatorname{CRLB}(\hat{\phi})=\frac{\sigma^{2}}{N} \frac{1}{F\left(\sigma^{2}\right)}=\frac{1}{2 N \frac{E_{s}}{N_{o}}} \frac{1}{F\left(\frac{N_{o}}{2 E_{s}}\right)}$$

An important note , $1/(2N(E_s/N_o)$ term corresponds to CLRB for a carrier phase over N symbols . To evaluate $F_m((E_s/N_o))$, we substitute (1) in (8) :

$$\begin{aligned} \xi\left(\phi, n^{I}, n^{Q}, A_{k}, a_{i}, r\right)= & \left|a_{i}\right|\left(A_{k}+n^{I}+j n^{Q}\right) \\ & \times e^{-j r \arctan \frac{\operatorname{sia}\left(a_{i}\right)}{\Re\left(a_{i}\right)}} \end{aligned}$$

Essentially , we are trying to combine all the transmitted symbols A_k in all 4 quadrants and a_i belong only to first quadrant . Hence ,

$$F\left(\sigma^2\right)=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Lambda\left(\xi\left(\phi,n^I,n^Q,A_k,a_i,r\right)\right) \times p\left(n^I,n^Q\right)dn^I dn^Q \tag{6}$$

which is the general CLRB expression for symmetric signal constellations in the form of $A_k=\pm\Re\{a_i\}\pm j\Im\{a_i\}$.

B. Theorem II : CLRB for Frequency & Known Phase

Usually , note that frequency estimation assuming known carrier phase is usually unrealistic situation , since practical carrier frequency often occurs in presence of unknown carrier phase . To derive the following , we modify the aforementioned theorem to suit our case . Here , the received signal will be of the form :

$$x_k=a_k e^{j(k\omega+\phi_0)}+w_k,\quad k=-\frac{N-1}{2},\ldots,\frac{N-1}{2} \tag{7}$$

where ω is the frequency offset in radians per symbol period. Here , we replace ϕ with $k\omega$+ϕ_0 ,
resulting in :

$$p(\mathbf{x}\mid\omega)=\prod_k\left[\frac{p_A(a)}{\pi\sigma^2}e^{-\frac{|x_k|^2}{2\sigma^2}}\sum_{a_i\in\mathbf{Q}_1}e^{-\frac{|a_i|^2}{2\sigma^2}}*\right.$$

$$\left.\sum_{r=1,-1}\cosh\left(\Re\left(\frac{|a_i|}{\sigma^2}x_k e^{-j\left(k\omega+\phi_0+r\arctan\frac{\Im(a_i)}{\Re(a_i)}\right)}\right)\right)\right] \tag{8}$$

$$\frac{\partial^2\ln p(\mathbf{x}\mid\omega)}{\partial\omega^2}=\frac{1}{\sigma^2}\sum_k k^2\Lambda(x) \tag{9}$$

Given , that $E(\Lambda(x))$ is independent of k and for N odd , then :

$$\mathrm{CRLB}_{\phi_0}(\hat{\omega})^{-1}=2\frac{E_s}{N_o}\frac{N\left(N^2-1\right)}{12}F\left(\frac{N_o}{2E_s}\right) \tag{10}$$

C. Theorem III : Joint CLRB for Unknown Phase & Frequency

Here , we aim to estimate ϕ and ω , given the assumption that the received signal has perfect symbol timing and frequency offset is minimal . The modified signal becomes :

$$x_k=a_k e^{j(k\omega+\phi)}+w_k \tag{11}$$

We recompute the PDF pdf in terms of $\beta=(\omega,\phi)$ and known equiprobable symbols $\mathbf{a}=(a_0a_1\ldots a_{N-1})$ by :

$$p(\mathbf{x}\mid\beta)=\prod_k p\left(x_k\mid\beta\right)$$

$$=\prod_k p_{\mathbf{A}}(a)\sum_{a\in\mathbf{C}}\frac{1}{2\pi\sigma^2}e^{-\frac{1}{2\sigma^2}\left|x_k-ae^{j(k\omega+\phi)}\right|^2} \tag{12}$$

Computing the Fisher Matrix (I) , we get :

$$\mathbf{I}=\begin{pmatrix}-E\left[\frac{\partial^2\ln p(\mathbf{x}|\beta)}{\partial\omega^2}\right] & -E\left[\frac{\partial^2\ln p(\mathbf{x}|\beta)}{\partial\omega\partial\phi}\right]\\ -E\left[\frac{\partial^2\ln p(\mathbf{x}|\beta)}{\partial\omega\partial\phi}\right] & -E\left[\frac{\partial^2\ln p(\mathbf{x}|\beta)}{\partial\phi^2}\right]\end{pmatrix} \tag{13}$$

Again applying the condition of x_k and Λ function being independent of each other , we get the final Fisher Matrix as :

$$\mathbf{I}=\frac{1}{\sigma^2}\begin{pmatrix}\sum_k k^2 & \sum_k k\\ \sum_k k & \sum_k 1\end{pmatrix}F\left(\sigma^2\right) \tag{14}$$

Now , we try to represent the above equation in terms of the aforementioned CLRBs for unknown phase and frequency counterparts.

$$\begin{aligned}\text{Joint CRLB}(\hat{\omega})&=\frac{12\sigma^2}{N\left(N^2-1\right)}\frac{1}{F\left(\sigma^2\right)}\\&=\mathrm{CRLB}_{\mathrm{CW}}(\hat{\omega})\frac{1}{F\left(\frac{N_o}{2E_s}\right)}\end{aligned} \tag{15}$$

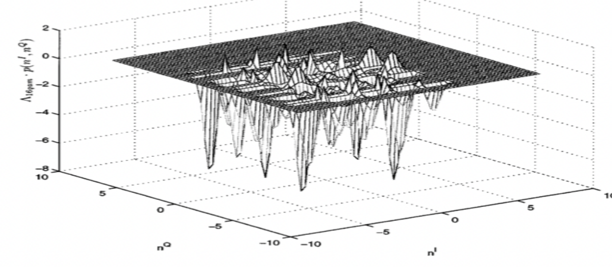
$$\begin{aligned}\text{Joint CRLB}(\hat{\phi})&=\frac{\sigma^2}{N}\frac{2(2N-1)}{N+1}\frac{1}{F\left(\sigma^2\right)}\\&=\mathrm{CRLB}_{\mathrm{CW}}(\hat{\phi})\frac{1}{F\left(\frac{N_o}{2E_s}\right)}\end{aligned} \tag{16}$$

III. Experiment Analysis of Theorems

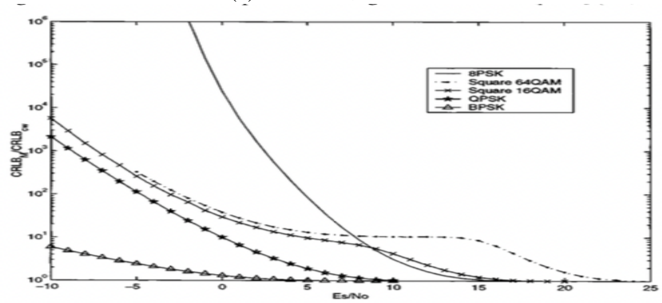
For all of the above estimators , the authors have tried to provide results either analytical or experimental for QAM,PSK,etc.

A. Unknown Phase Estimation

For low SNRs , the authors perform 3D plot of integrand for 16 QAM at SNR = 15 db . They use Monte Carlo Evaluation technique for estimating $\lambda_{16QAM}(x)$ at each SNR . As we lower the SNR value , more the number MCE trails required.It is observed that 16 QAM & 64 QAM bounds converge at lower SNR . In contrast , at high SNRs for all the signal sets considered it is the minimum distances between members , and all estimator variances approach CW performance .



(a) MSE vs N , SNR = 100



(b) MSE vs SNR , N = 20

Figure 1: Unknown Phase Estimation

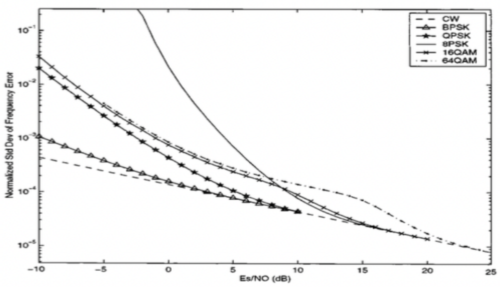


Figure 2: Frequency Modulations for 200 symbols

B. Unknown Frequency Estimation

In order to treat phase as an unknown entity , averaging over random variables must be performed , which leads to complex expression . In the following figure , the authors have tried to derive CLRBs for phase frequency offset .

$$x_k = a_k e^{j(k\omega+\phi_0)} + w_k, \quad k = -\frac{N-1}{2}, \ldots, \frac{N-1}{2} \quad (17)$$

where ω is the frequency offset in radians per symbol and k is selected to be at symmetric center . An important note is that frequency estimation assuming known carrier phase is not usually a realistic situation, since practical carrier frequency estimation often occurs in the presence of an unknown carrier phase.

Assumed that the frequency offset is relatively small so that intersymbol interference may be neglected. The probability density is similar to previous results , but we replace : ϕ with k*ω+ϕ_0. So, the mathematical derivative changes to :

$$x_k = a_k e^{j(k\omega+\phi_0)} + w_k, \quad k = -\frac{N-1}{2}, \ldots, \frac{N-1}{2} \quad (18)$$

, and assuming N is odd , we get :

$$\mathrm{CRLB}_{\phi_0}(\hat{\omega})^{-1} = 2\frac{E_s}{N_o}\frac{N\left(N^2-1\right)}{12}F\left(\frac{N_o}{2E_s}\right). \quad (19)$$

C. Unknown Phase & Frequency Estimation

For finite N, we deduce the following relationship between the frequency CRLB's for known phases, joint estimation, and unknown phase cases, respectively:

$$\mathrm{CRLB}_{\mathcal{K}_\mathcal{N}}\left(\hat{\omega}_{\phi_0}\right) < \mathrm{CRLB}_{\mathcal{K}_\mathcal{S}}\left(\hat{\omega}_{\phi_0}\right)$$

=Joint $\mathrm{CRLB}(\hat{\omega})$
$\leq$ CRLB with unknown phase. (20)

Asymptotically, the right-hand side becomes an equality, and $\mathrm{CRLB}_{\mathcal{K}_\mathcal{N}}\left(\hat{\omega}_{\phi_0}\right)$ becomes a quarter of $\mathrm{CRLB}_{\mathcal{K}_\mathcal{S}}\left(\hat{\omega}_{\phi_0}\right)$. For the phase case

$$\text{Joint } \mathrm{CRLB}(\hat{\phi}) = \frac{1}{2N\frac{E_s s}{N_o}}\frac{1}{F\left(\frac{N_o}{2E_s}\right)}\gamma$$

where

$$\gamma = \begin{cases} 1, & \mathcal{K}_\mathcal{S} \text{ (decoupled)} \\ \frac{2(2N-1)}{N+1}, & \mathcal{K}_\mathcal{N} \\ 4, & \mathcal{K}_\mathcal{N}, \quad N \to \infty \end{cases}$$

These results apply to any QAM, PSK, and PAM modulation, including the previously known CW case.

IV. Other Phase Estimators

In this section , we look at existing phase estimators for QAM and their performance w.r.t the derived true CLRBs for 16 QAM and 64 QAM .

A. Power Law Estimator

For rotationally symmetric signal constellations , such as MPSK and QAM , the phase estimate can be given by :

$$\hat{\theta} = \frac{1}{M}\arg\left(E\left[a_k^{*M}\right]\sum_{k=0}^{N-1} x_k^M\right) \quad (21)$$

B. Histogram Algorithm

This algorithm is very close to Maximum Likelihood Estimation (MLE) , and can be applied to different kinds of constellation.

$$L(\phi) = \sum_{k=0}^{N-1}\ln\left[\sum_a \exp\left(-\frac{1}{2\sigma^2}\left||x_k| - |a\right| \times e^{j(\phi+\psi-\theta_k)}\right|^2\right)\right] \quad (22)$$

where ψ is the angle of QAM symbol , a,θ_k is the angle of x_k . The amplitude of x_k which is close to that of a has the most significant contribution. The estimated ϕ_k with corresponding α_k is mapped as follows :

$$\hat{\phi}_k = (\theta_k - \alpha_k) \bmod \left(\frac{\pi}{2}\right) \quad (23)$$

where a_k is the transmitted symbol and M=2 for BPSK & M=4 for QPSK & QAM .

C. Two-Stage-Conjugate Algorithm

This is an algorithm very close to Two Pass Algorithm where the 16QAM constellation points can be partitioned into two groups. The signals close to the inner and outer rings are called Class I and the others close to the middle ring are called Class II , based on their amplitude .

$$x_k \in \begin{cases} \mathbf{C}_1, & \text{if } |x_k| \leq T_1, \quad \text{or} \quad |x_k| \geq T_2, \\ \mathbf{C}_2, & \text{if } T_1 < |x_k| < T_2 \end{cases}$$

On the basis of Class I/II or Class III / IV , we perform corrections accordingly . There are two stages of estimation when it comes ϕ , and the final result is addition of both the estimations . The 2SC algorithm reaches very close to CLRB for 64 QAM .

$$\hat{\phi}_1 = \frac{1}{4}\tan^{-1}\sum_{x_k\in\mathbf{C}_1}\frac{\Im\left(\frac{x_k^4}{|x_k|^3}\right)}{\Re\left(\frac{x_k^4}{|x_k|^3}\right)}.$$

$$\hat{\phi}_2 = \tan^{-1}\sum_{k=0}^{N-1}\frac{\Im\left(y_k\right)}{\Re\left(y_k\right)}$$

$$\hat{\phi} = \hat{\phi}_1 + \hat{\phi}_2$$

D. Minimum Distance Estimator

In the cost of increasing computational complexity , we can give a good performance via MDE . The received signals are rotated by ϕ in the range [-45,45] to obtain a series of I hypothesis sets .

$$D_i = \sum_{k=0}^{N-1} \left| x_k e^{j\phi_i} - \hat{a}_{ik} \right|^2 \tag{24}$$

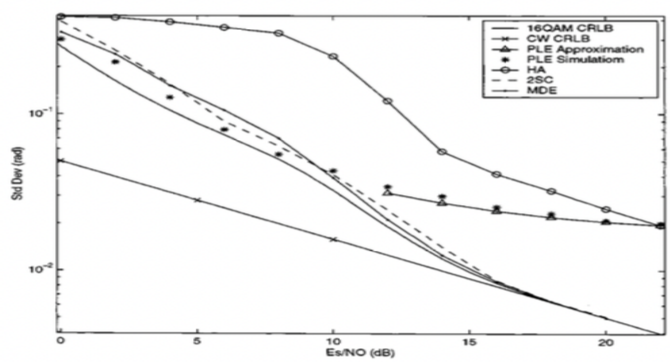
whose minimum $D_l = min(D_0, D_1, ..D_{l-1})$.

$$\hat{\psi} = \tan^{-1} \sum_{k=0}^{N-1} \frac{\Im\left(x'_{\ell k} \hat{a}^*_{\ell k}\right)}{\Re\left(x'_{\ell k} \hat{a}^*_{\ell k}\right)} \tag{25}$$

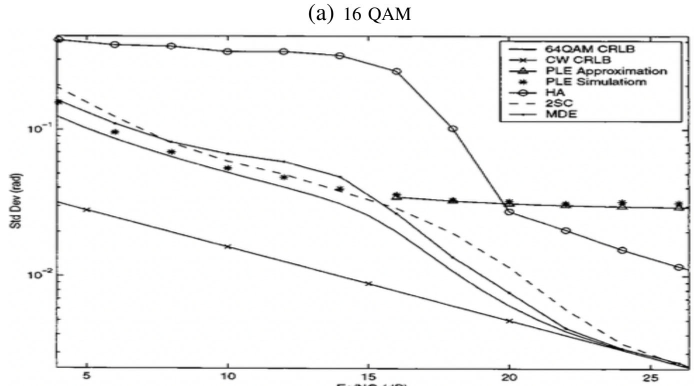
$$\hat{\phi} = \hat{\phi}_\ell + \hat{\psi} \tag{26}$$

V. Conclusion

In the final MCE simulation results for HA , 2SC & MDE, we can conclude that for various ranges and choices of SNR , we can see performance variance . PLE stands out among all the low SNR given scenarios , and HA is relatively poor performing . Suprsingly, PLE's performance drops as SNR increases.MDE is the best estimator and follows QAM's CRLB in both cases .This can be very clearly observed from the following graphs , as well :



(a) 16 QAM



(b) 64 QAM

Figure 3: Comparison Charts

VI. References

- Cramér–Rao Lower Bounds for QAM Phase and Frequency Estimation Official Paper
- Aruba Blog https://blogs.arubanetworks.com/solutions/mobility/what-is-qam/
- Cramer–Rao Bounds of SNR Estimates for BPSK and QPSK Modulated Signals Paper.
- Stephen Kay Textbook for Estimation (Volume I)