1 Overview

1.1 Lambertian albedo

First note that any BRDF is nonnegative. This means $\rho/\pi \geq 0 \implies \rho \geq 0$. Now, consider the conservation of energy,

$$\begin{aligned}
\int_{\mathbf{\Omega}_{\text{out}}} f(\hat{\boldsymbol{\omega}}_{\text{in}}, \hat{\boldsymbol{\omega}}_{\text{out}}) \cos\theta_{\text{out}} \, \mathrm{d}\hat{\boldsymbol{\omega}}_{\text{out}} \leq 1 &\implies \int_{\mathbf{\Omega}_{\text{out}}} \frac{\rho}{\pi} \cos\theta_{\text{out}} \, \mathrm{d}\hat{\boldsymbol{\omega}}_{\text{out}} \leq 1 \\
&\implies \frac{\rho}{\pi} \int_{\mathbf{\Omega}_{\text{out}}} \cos\theta_{\text{out}} \, \mathrm{d}\hat{\boldsymbol{\omega}}_{\text{out}} \leq 1 \\
&\implies \frac{\rho}{\pi} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi/2} \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \leq 1 \\
&\implies \frac{\rho}{2\pi} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi/2} \sin 2\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \leq 1 \\
&\implies \frac{\rho}{2\pi} \int_{\phi=0}^{\phi=2\pi} \left[-\frac{1}{2} \cos 2\theta \right]_{\theta=0}^{\theta=\pi/2} \mathrm{d}\phi \leq 1 \\
&\implies \frac{\rho}{2\pi} \int_{\phi=0}^{\phi=2\pi} 1 \, \mathrm{d}\phi \leq 1 \\
&\implies \rho \leq 1.
\end{aligned}$$

1.2 Foreshortening

1. The solid angle subtended by $\mathrm{d}A$ at X_1 is

$$\frac{\mathrm{d}A}{D^2},$$

and that at X_2 is

$$\frac{\mathrm{d}A}{(D/\cos\alpha)^2} = \frac{\mathrm{d}A \cos^2\alpha}{D^2}.$$

2. First note that based on the definition of irradiance E and radiance L , we can relate them by

$$E = \int_{\Omega} L \cos\theta \, \mathrm{d}\omega.$$

Since we are considering only an infinitesimally small surface $\mathrm{d}A$, we have

$$\begin{aligned}
E(X_1) &= L \, \mathrm{d}\omega_1 = L \cdot \frac{\mathrm{d}A}{D^2} \\
E(X_2) &= L \, \mathrm{d}\omega_2 = L \cos^2\alpha \cdot \frac{\mathrm{d}A}{D^2}.
\end{aligned}$$

The ratio among them is thus

$$\frac{E(X_1)}{E(X_2)} = \frac{1}{\cos^2\alpha}.$$

1.3 Simple rendering

1. Quiver.

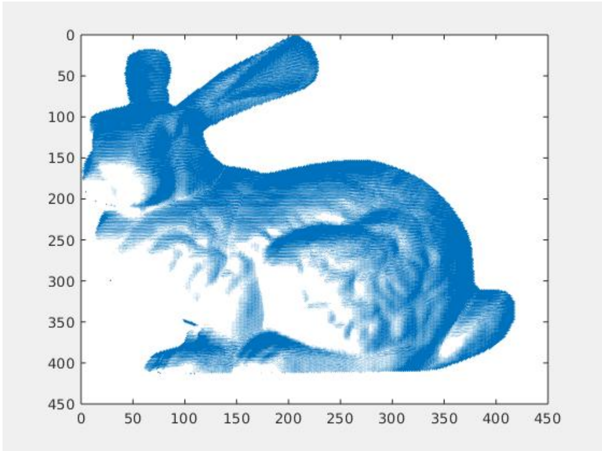


Figure 1: Quiver from `bunny.mat`.

2. Rendering with $\hat{\mathbf{s}} = (0, 0, 1)$ can be simply done by `imshow(N(:,:,3))` because the radiance of Lambertian surface is essentially directly proportional to $\cos\theta$, where θ is the angle between the light source and the surface normal. This is essentially $\hat{\mathbf{n}} \cdot \hat{\mathbf{s}}$ where $\hat{\mathbf{n}}$ is the surface normal. In the case of $\hat{\mathbf{s}} = (0, 0, 1)$, it can simply be done by accessing the z-values.

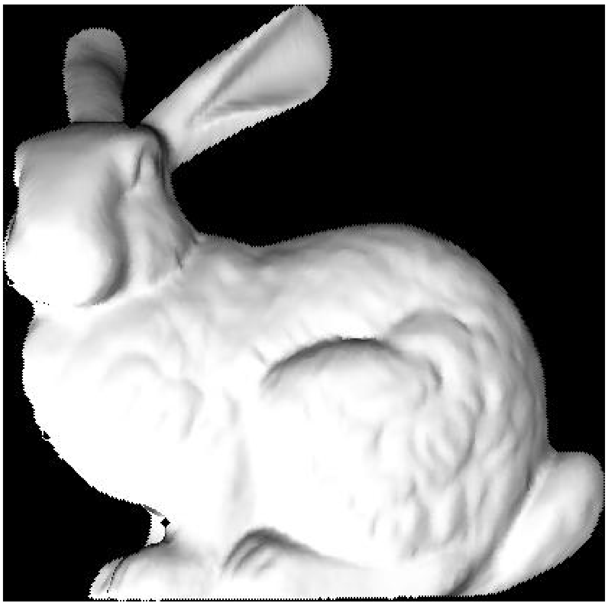


Figure 2: Simple rendering from $\hat{\mathbf{s}} = (0, 0, 1)$.

3. Renders.

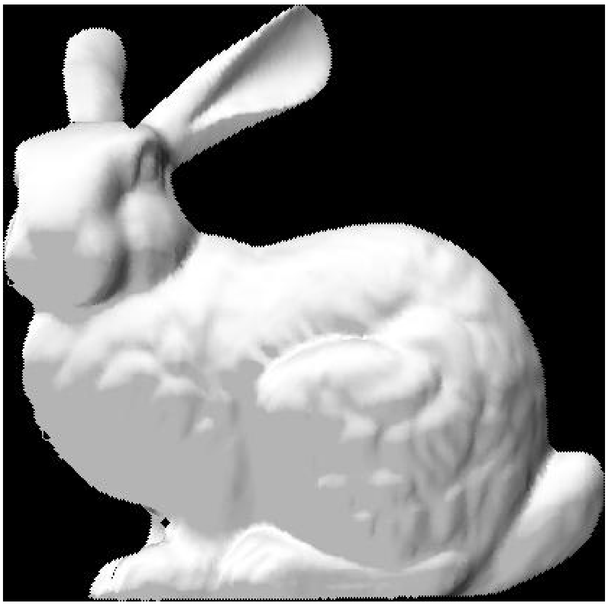


Figure 3: Simple rendering from $\hat{\mathbf{s}}$ being rotated 45° upwards from $(0, 0, 1)$.

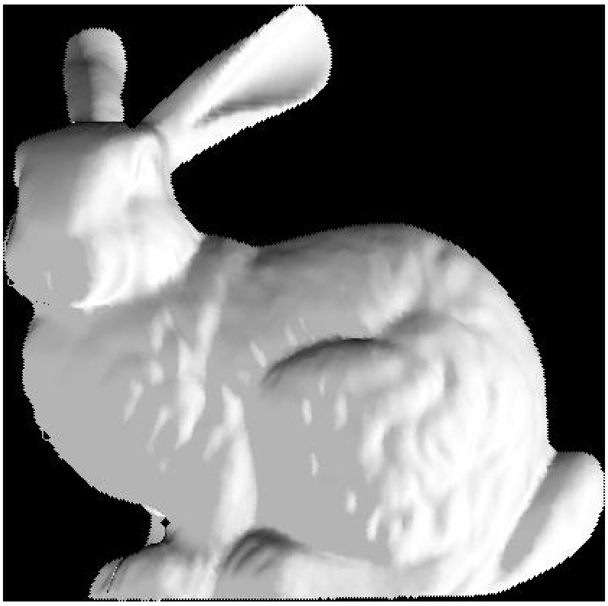


Figure 4: Simple rendering from $\hat{\mathbf{s}}$ being rotated 45° rightwards from $(0, 0, 1)$.

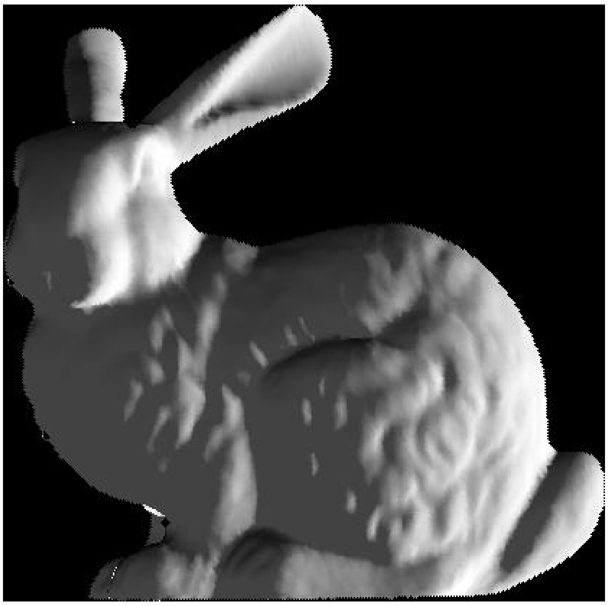


Figure 5: Simple rendering from $\hat{\mathbf{s}}$ being rotated 75° rightwards from $(0, 0, 1)$.

Some errors can be seen from this simple rendering. Some places of the bunny should be blocked out of light but they still show up in our rendering. In another words, shadow is not accounted for.

1.4 Photometric stereo

1. Estimated albedo and surface normal.

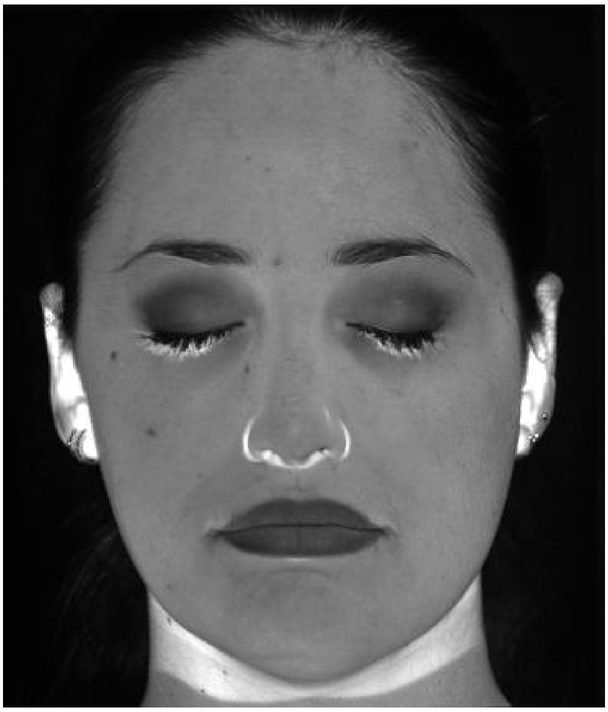


Figure 6: Estimated albedo.

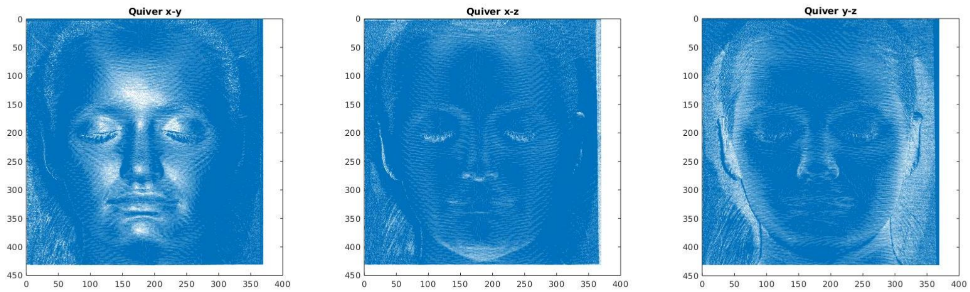


Figure 7: Estimated surface normal.

2. The poor estimation of albedo near the nasal area is because of shadow. In this simple rendering, we assumed of no shadows, yet near the nasal area, shadows are often there. We might be able to improve this by only estimating the albedo based on the images where the nasal area is not blocked by shadows.

3. Simple renderings.

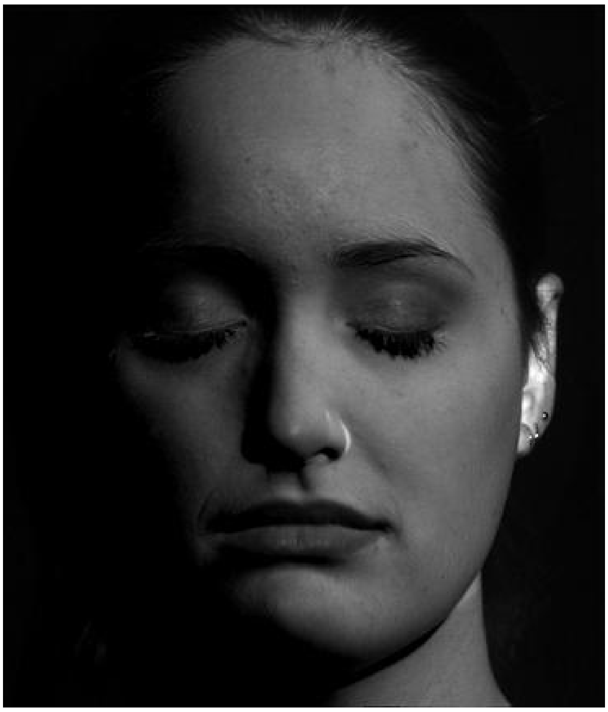


Figure 8: Simple rendering from $\hat{\mathbf{s}} = (0.58, -0.58, -0.58)$.

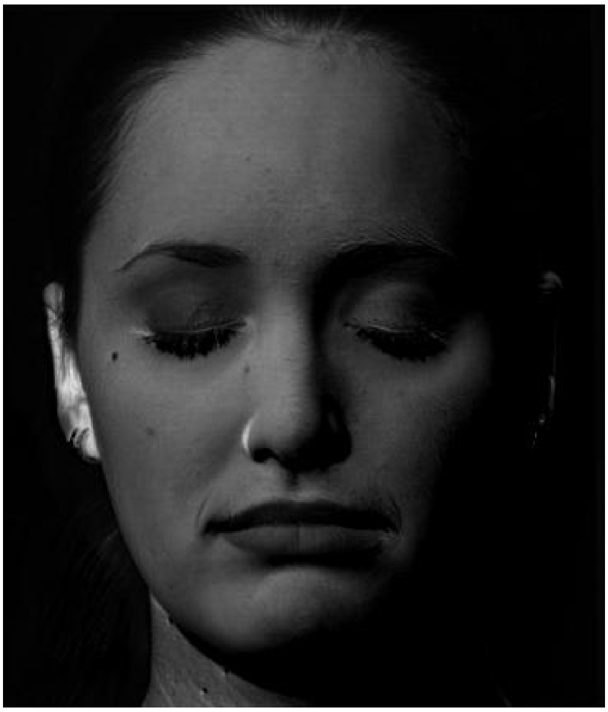
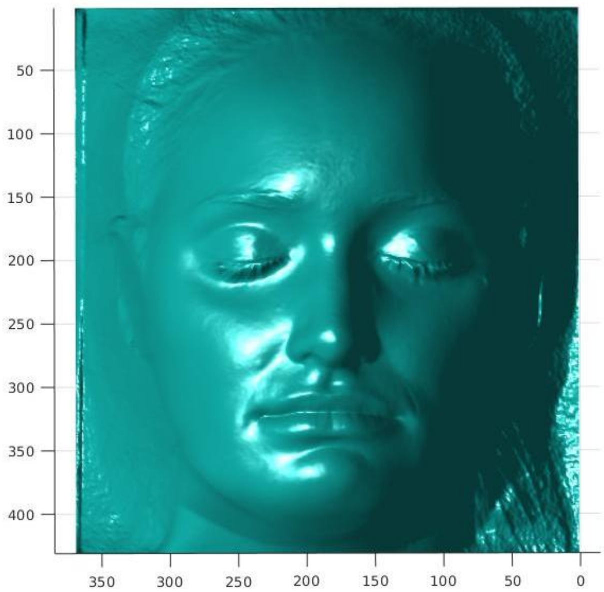
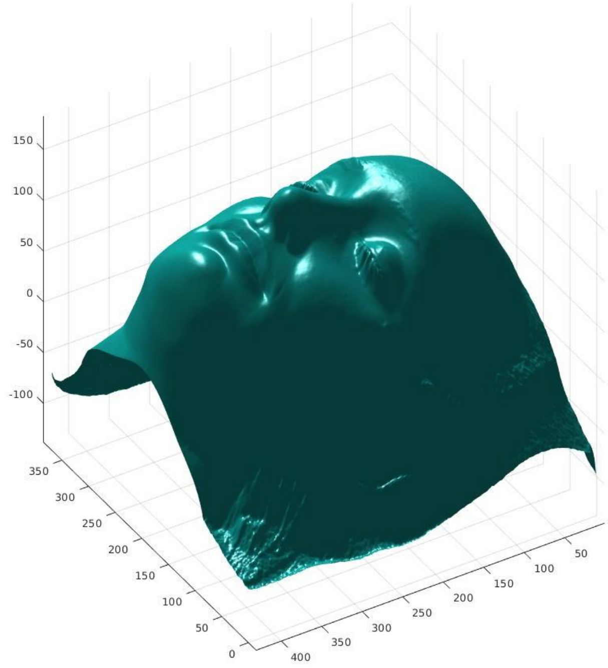


Figure 9: Simple rendering from $\hat{\mathbf{s}} = (-0.58, -0.58, -0.58)$.

4. Recovered surface.





1.5 Dichromatic reflectance

1. The pixel values and the BRDF may be related by

$$\mathbf{C}(\mathbf{u}) = \hat{\mathbf{n}}(\mathbf{u}) \cdot \hat{\mathbf{l}} \int_\lambda f(\lambda, \hat{\boldsymbol{\omega}}_i, \hat{\boldsymbol{\omega}}_o) I(\lambda) \begin{bmatrix} c_R(\lambda) \\ c_G(\lambda) \\ c_B(\lambda) \end{bmatrix} \, \mathrm{d}\lambda.$$

   Under the dichromatic reflectance model, we then have

$$\mathbf{d}(\mathbf{u}) = \int_\lambda f_d(\lambda) I(\lambda) \begin{bmatrix} c_R(\lambda) \\ c_G(\lambda) \\ c_B(\lambda) \end{bmatrix} \, \mathrm{d}\lambda,$$

   and

$$\mathbf{s} = \int_\lambda I(\lambda) \begin{bmatrix} c_R(\lambda) \\ c_G(\lambda) \\ c_B(\lambda) \end{bmatrix} \, \mathrm{d}\lambda.$$

2. For $i = 1, 2$, since $\hat{\mathbf{r}}_i \perp \mathbf{s}$, we have

$$\hat{\mathbf{r}_i} \cdot \mathbf{C}(\mathbf{u}) = (\hat{\mathbf{n}}(\mathbf{u}) \cdot \hat{\mathbf{l}})(\hat{\mathbf{r}}_i \cdot \mathbf{d}(\mathbf{u})),$$

   which is not dependent on g_s (and hence f_s), and is linearly dependent on $\hat{\mathbf{n}}(\mathbf{u})$.

3. Note that

$$J(\mathbf{u}) = \sqrt{(\hat{\mathbf{r}_1} \cdot \mathbf{C}(\mathbf{u}))^2 + (\hat{\mathbf{r}_2} \cdot \mathbf{C}(\mathbf{u}))^2} = (\hat{\mathbf{n}}(\mathbf{u}) \cdot \hat{\mathbf{l}})\sqrt{(\hat{\mathbf{r}_1} \cdot \mathbf{d}(\mathbf{u}))^2 + (\hat{\mathbf{r}_2} \cdot \mathbf{d}(\mathbf{u}))^2},$$

   which is also linearly dependent on $\hat{\mathbf{n}}(\mathbf{u})$ under the fact that only the case of $\hat{\mathbf{n}}(\mathbf{u}) \cdot \hat{\mathbf{l}} \geq 0$ is considered.

4. `imout` might be more useful to a computer vision system because the glares (i.e. the spectular reflections) may be obsucring or distracting the system into given an incorrect result.

Figure 10: Result of `makeLambertian`.

1.6 Color metamers (Extra Credit)

1. Suppose $\mathbf{f} \in \mathbb{R}^N$ is the discretization of $f(\lambda)$, $\boldsymbol{\ell} \in \mathbb{R}^N$ is the discretization of $l(\lambda)$ and let $C(\lambda)$ be the CIE XYZ color matching function, we have

$$\begin{aligned} \mathbf{C}_f &= \int_\lambda C(\lambda) f(\lambda) l(\lambda)\, \mathrm{d}\lambda \\ &\simeq \mathbf{R}(\mathbf{f} \odot \boldsymbol{\ell}) \\ &= \underbrace{\left(\begin{bmatrix} \mathbf{f} \\ \mathbf{f} \\ \mathbf{f} \end{bmatrix} \odot \mathbf{R} \right)}_{\mathbf{L}_f \in \mathbb{R}^{3\times n}} \boldsymbol{\ell}. \\ &= \mathbf{L}_f \boldsymbol{\ell}, \end{aligned}$$

where $\odot$ denotes the elementwise multiplication.

2. The Euclidean distance would be $\|\mathbf{L}_f\boldsymbol{\ell} - \mathbf{L}_g\boldsymbol{\ell}\|_2$.

4. The optimal temperature found is 8282.1459K.

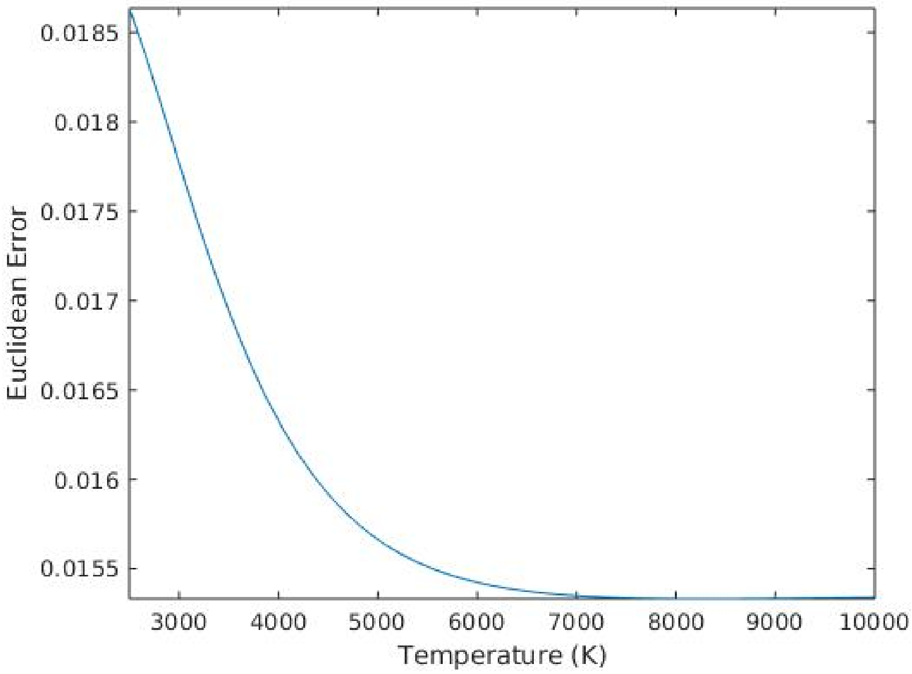


Figure 11: Error function in 1.6.2. against temperature T.

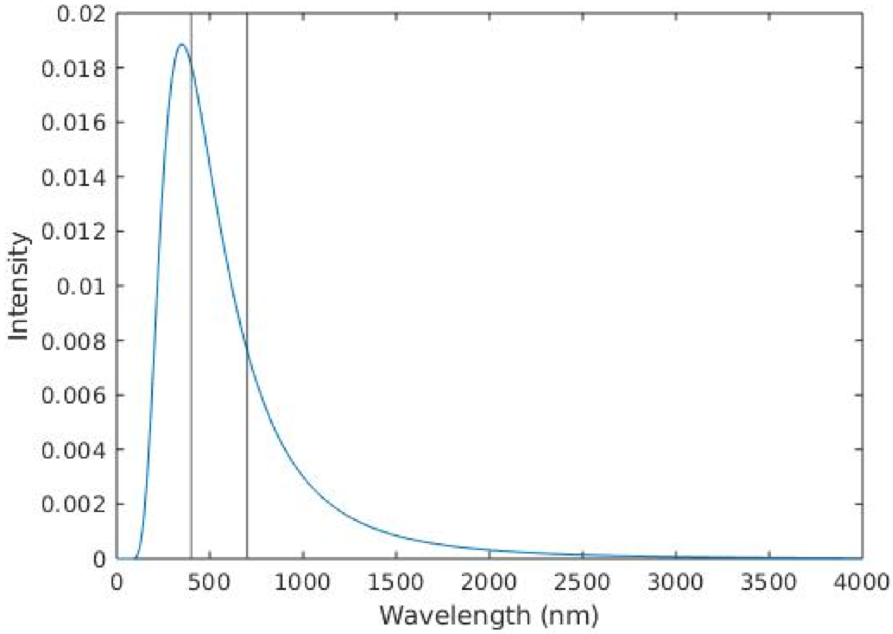


Figure 12: Blackbody radiation at $T = 8282.1459$K, with two vertical lines at 400nm and 700nm indicating the visible spectrum.