Nombre: Nicolas Toledo Parra, 2200017
Gabriela Sánchez Ariza, 2200816

C3C
· Nicolás Toledo Parra, 2200017
· Gabriela Sánchez Ariza, 2200816

Ejercicio 2 sección 3.3.5

Dados los tensores

$$R^i_j = \begin{bmatrix} 1/2 & 1 & 3/2 \\ 2 & 5/2 & 3 \\ 7/2 & 4 & 9/2 \end{bmatrix}, \quad T^i = \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}, \quad g^{ij} = g_{ji} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Encuentre la parte simétrica S^i_j y antisimétrica A^i_j de R^i_j

$$S^i_j = \frac{1}{2}(R^i_j + R^j_i) = \begin{bmatrix} 1/4 & 1/2 & 3/4 \\ 1 & 5/4 & 3/2 \\ 7/4 & 2 & 9/4 \end{bmatrix} + \begin{bmatrix} 1/4 & 1 & 7/4 \\ 1/2 & 5/4 & 2 \\ 3/4 & 3/2 & 9/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 3/2 & 5/2 \\ 3/2 & 5/2 & 7/2 \\ 5/2 & 7/2 & 9/2 \end{bmatrix}$$ respuesta

$$A^i_j = \frac{1}{2}(R^i_j - R^j_i) = \begin{bmatrix} 1/4 & 1/2 & 3/4 \\ 1 & 5/4 & 3/2 \\ 7/4 & 2 & 9/4 \end{bmatrix} - \begin{bmatrix} 1/4 & 1 & 7/4 \\ 1/2 & 5/4 & 2 \\ 3/4 & 3/2 & 9/4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1/2 & -1 \\ 1/2 & 0 & -1/2 \\ 1 & 1/2 & 0 \end{bmatrix}$$ respuesta

De forma que $R^i_j = S^i_j + A^i_j$

$$\begin{bmatrix} 1/2 & 1 & 3/2 \\ 2 & 5/2 & 3 \\ 7/2 & 4 & 9/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 & 5/2 \\ 3/2 & 5/2 & 7/2 \\ 5/2 & 7/2 & 9/2 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & -1 \\ 1/2 & 0 & -1/2 \\ 1 & 1/2 & 0 \end{bmatrix}$$

b) $R_{kj} = g_{ik} R^i_j$, $R^{ki} = g^{jk} R^i_j$, $T_j = g_{ij} T^i$ ¿Qué se concluye?

① $R_{kj} = g_{ik} R^i_j$ como se está realizando un producto entre matrices, se transpone g_{ik} para conseguir la notación:

$$(g_{ik})^T R^i_j \equiv g_{ki} R^i_j \equiv g_{ki} R^i_j$$

entonces:

$$(g_{ik})^T R^i_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1 & 3/2 \\ 2 & 5/2 & 3 \\ 7/2 & 4 & 9/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1 & 3/2 \\ -2 & -5/2 & -3 \\ 7/2 & 4 & 9/2 \end{bmatrix}$$

$$\therefore \quad R_{kj} = \begin{bmatrix} 1/2 & 1 & 3/2 \\ -2 & -5/2 & -3 \\ 7/2 & 4 & 9/2 \end{bmatrix} \quad \text{respuesta}$$

② $R^{ki} = g^{jk} R^i_j$ debe haber un producto matricial, por lo tanto,

$$R^{ki} = g^{jk} R^i_j \equiv R^i_j g^{jk}$$

$$R^i_j g^{jk} = \begin{bmatrix} 1/2 & 1 & 3/2 \\ 2 & 5/2 & 3 \\ 7/2 & 4 & 9/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1 & 3/2 \\ 2 & -5/2 & 3 \\ 7/2 & -4 & 9/2 \end{bmatrix} = R^{ik}$$

$$R^{ik} \equiv (R^{ik})^T \Rightarrow \begin{bmatrix} 1/2 & 2 & 7/2 \\ -1 & -5/2 & -4 \\ 3/2 & 3 & 9/2 \end{bmatrix} \quad \text{respuesta}$$

③ $T_j = g_{ij} T^i$ para que se pueda realizar el producto matricial entonces

$$T_j = g_{ij} T^i = (T^i)^T g_{ij}$$

$$T_j = (T^i)^T g_{ij} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix} \quad \text{respuesta}$$

- podemos concluir que uno puede alterar el orden y aplicar propiedades de los tensores de tal manera que se cumplan productos matriciales para facilidad de calculo también, un tensor columna puede ser fila usando la metrica.

c) $R^i_j T_i$, $R^i_j T^j$, $R^i_j T_i T^j$

① $R^i_j T_i$ $\quad R^i_j T_i = A_j = T_i R^i_j = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1 & 3/2 \\ 2 & 5/2 & 3 \\ 7/2 & 4 & 9/2 \end{bmatrix}$

$$= \begin{bmatrix} \frac{7}{3} & \frac{8}{3} & 3 \end{bmatrix} \quad \text{Respuesta}$$

② $R^i_j T^j$ $\quad B^i = R^i_j T^j = \begin{bmatrix} 1/2 & 1 & 3/2 \\ 2 & 5/2 & 3 \\ 7/2 & 4 & 9/2 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 16/3 \\ 25/3 \end{bmatrix}$ Respuesta

③ $R^i_j T_i T^j$ $\quad R^i_j T_i T^j = (T_i R^i_j) T^j = C_j T^j = \beta$

$$\begin{bmatrix} \frac{7}{3} & \frac{8}{3} & 3 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix} = \frac{7}{9} + \frac{16}{9} + 3 = \boxed{\frac{50}{9}}$$ Respuesta

d) $R^i_j S^j_i$, $R^i_j A^j_i$, $A^j_i T^i$, $A^j_i T^i T_j$

① $R^i_j S^j_i$ $\Rightarrow R^1_j S^j_1 + R^2_j S^j_2 + R^3_j S^j_3$

$$= \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ 3/2 \\ 5/2 \end{bmatrix} + \begin{bmatrix} 2 & \frac{5}{2} & 3 \end{bmatrix} \begin{bmatrix} 3/2 \\ 5/2 \\ 7/2 \end{bmatrix} +$$

$$\begin{bmatrix} \frac{7}{2} & 4 & \frac{9}{2} \end{bmatrix} \begin{bmatrix} 5/2 \\ 7/2 \\ 9/2 \end{bmatrix} = \left(\frac{1}{4} + \frac{3}{2} + \frac{15}{4}\right) + \left(3 + \frac{25}{4} + \frac{21}{2}\right)$$

$$+ \left(\frac{35}{4} + \frac{28}{2} + \frac{81}{4}\right) = \frac{277}{4} = \boxed{69,25}$$ Respuesta

② $R^i_j A^j_i$ $\Rightarrow R^1_j A^j_1 + R^2_j A^j_2 + R^3_j A^j_3$

$$= \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & \frac{5}{2} & 3 \end{bmatrix} \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} \frac{7}{2} & 4 & \frac{9}{2} \end{bmatrix} \begin{bmatrix} -1 \\ -1/2 \\ 0 \end{bmatrix}$$

$$= \left(0 + \frac{1}{2} + \frac{3}{2}\right) + \left(-1 + 0 + \frac{3}{2}\right) + \left(-\frac{7}{2} - 2 + 0\right)$$

$$= \boxed{-3}$$ Respuesta

③ $A^j_i T^i$ $\Rightarrow D^j = \begin{bmatrix} 0 & -1/2 & -1 \\ 1/2 & 0 & -1/2 \\ 1 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4/3 \\ -1/3 \\ 2/3 \end{bmatrix}$

④ $A^j_i T^i T_j$ $\Rightarrow T_j (A^j_i T^i) = T_j D^j = \beta \quad \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} -4/3 \\ -1/3 \\ 2/3 \end{bmatrix} = \frac{4}{9}$

e) $R^i_j - 2\delta^i_j R^l_l$, $(R^i_j - 2\delta^i_j R^l_l) T_i$, $(R^i_j - 2\delta^i_j R^l_l) T_i T^j$

① $R^i_j - 2\delta^i_j R^l_l$ $\qquad R^l_l = R^1_1 + R^2_2 + R^3_3 = \frac{1}{2} + \frac{5}{2} + \frac{9}{2} = \frac{15}{2}$

$$R^i_j - 2\delta^i_j R^l_l = \begin{bmatrix} 1/2 & 1 & 3/2 \\ 2 & 5/2 & 3 \\ 7/2 & 4 & 9/2 \end{bmatrix} - 2\left(\frac{15}{2}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1 & 3/2 \\ 2 & 5/2 & 3 \\ 7/2 & 4 & 9/2 \end{bmatrix} + \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix} = \begin{bmatrix} -29/2 & 1 & 3/2 \\ 2 & -25/2 & 3 \\ 7/2 & 4 & -21/2 \end{bmatrix}$$ respuesta

② $(R^i_j - 2\delta^i_j R^l_l) T_i \Rightarrow T_i (R^i_j - 2\delta^i_j R^l_l)$

$$\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} -29/2 & 1 & 3/2 \\ 2 & -25/2 & 3 \\ 7/2 & 4 & -21/2 \end{bmatrix} = \begin{bmatrix} -8/3 & 38/3 & -12 \end{bmatrix}$$ respuesta

③ $(R^i_j - 2\delta^i_j R^l_l) T_i T^j$

$$\begin{bmatrix} -8/3 & 38/3 & -12 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix} = -\frac{8}{9} + \frac{76}{9} - 12 = \boxed{-\frac{40}{9}}$$ respuesta

Ejercicio 8.

Suponga un sistema de coordenadas ortogonales generalizadas (q^1, q^2, q^3)

$$q^1 = x + y, \quad q^2 = x - y, \quad q^3 = 2z.$$

a) (q^1, q^2, q^3) conforma un sistema de coordenadas ortogonales

$$\begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \\ 2z \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$q^1 \perp q^2 \perp q^3$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = 0$$

b) Vectores base para el sistema de coordenadas

Los vectores base para este sistema de coordenadas son:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$$

c) Tensor metrico $g^{ij} = \begin{pmatrix} q^1q^1 & q^1q^2 & q^1q^3 \\ q^2q^1 & q^2q^2 & q^2q^3 \\ q^3q^1 & q^3q^2 & q^3q^3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Elemento de volumen $dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j$

$q^1 = q^{1'} + q^{2'}$; $q^2 = q^{1'} - q^{2'}$, $q^3 = 2q^{3'} \Rightarrow \frac{\partial x^i}{\partial x^j} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$dx^i = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} dx' + dy' \\ dx' - dy' \\ 2dz' \end{pmatrix}$$

$dxdydz = (dx' + dy')(dx' - dy')(2dz') = ((dx')^2 - (dy')^2)\, 2dz'$

elemento de volumen

d) Expresión en el sistema (q^1, q^2, q^3) para los vectores

$\vec{A} = 2\hat{j}$, $\vec{B} = \hat{i} + 2\hat{j}$, $\vec{C} = \hat{i} + 7\hat{j} + 3\hat{k}$

$\vec{A} = 2\hat{j} \Rightarrow \left(\begin{array}{ccc|c} q_1 & q_2 & q_3 & \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$ $\vec{A} = 2\hat{j} = q^1 - q^2$

$\vec{B} = \hat{i} + 2\hat{j}$ $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \end{array}\right)$ $\vec{B} = \frac{3}{2} q^1 - \frac{1}{2} q^2$

$\vec{C} = \hat{i} + 7\hat{j} + 3\hat{k}$ $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 7 \\ 0 & 0 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3/2 \end{array}\right)$

$\vec{C} = \hat{i} + 7\hat{j} + 3\hat{k} = 4q^1 - 3q^2 + \frac{3}{2} q^3$

e) Encuentre el sistema (q^1, q^2, q^3) las expresiones para las siguientes relaciones.

$$\vec{A} \times \vec{B} \ , \ \vec{A} \cdot \vec{C} \ , \ (\vec{A} \times \vec{B}) \cdot \vec{C}$$

① $\vec{A} \times \vec{B}$: $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 0 \\ 1 & 2 & 0 \end{vmatrix} = (0)\hat{i} - (0)\hat{j} + (-2)\hat{k} = -q_3$

② $\vec{A} \cdot \vec{C} = (2\hat{j}) \cdot (\hat{i} + 7\hat{j} + 3\hat{k}) = 14$

$= (q^1 - q^2) \cdot (4q^1 - 3q^2 + \frac{3}{2} q^3)$

③ $(\vec{A} \times \vec{B}) \cdot \vec{C} = (-2\hat{k}) \cdot (\hat{i} + 7\hat{j} + 3\hat{k}) = -6$

$= -\frac{3}{2} \ \| q^3 \|^2$

f) considere los siguientes tensores y vectores en coordenadas cartesianas:

$$R^i_j = \begin{pmatrix} 1/2 & 1 & 3/2 \\ 2 & 5/2 & 3 \\ 7/2 & 4 & 9/2 \end{pmatrix} , \ T^i = \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix} , \ g^{ij} = g_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

y encuentre sus expresiones en el sistema de coordenadas (q^1, q^2, q^3)

① $R^{k'}_{m'} = \frac{\partial x^{k'}}{\partial x^i} \frac{\partial x^j}{\partial x^{m'}} R^i_j \qquad \frac{\partial x^j}{\partial x^{m'}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$q^1 = q^{1'} + q^{2'} \ , \ q^2 = q^{1'} - q^{2'} \ , \ q^3 = 2q^{3'}$

$q^{1'} = q^1 - q^{2'} = q^1 + (q^2 - q^{1'}) \qquad \underbrace{q^{1'} = \frac{q^1}{2} + \frac{q^2}{2}}$

$q^{2'} = q^1 - q^{1'} = q^1 - \frac{q^1}{2} - \frac{q^2}{2} = \underbrace{\frac{q^1}{2} - \frac{q^2}{2}}$

$\underbrace{q^{3'} = \frac{q^3}{2}} \qquad \frac{\partial x^{k'}}{\partial x^i} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$

$R^{k'}_{m'} = \frac{\partial x^{k'}}{\partial x^i} R^i_j \frac{\partial x^j}{\partial x^{m'}} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1 & 3/2 \\ 2 & 5/2 & 3 \\ 7/2 & 4 & 9/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$R^{k'}{}_{m'} = \begin{pmatrix} 3 & -\frac{1}{2} & \frac{9}{2} \\ -\frac{3}{2} & 0 & -\frac{3}{2} \\ \frac{15}{4} & -\frac{1}{4} & \frac{9}{2} \end{pmatrix}$$

$$g_{k'm'} = \frac{\partial x^i}{\partial x^{k'}} \frac{\partial x^j}{\partial x^{m'}} g_{ij} = \frac{\partial x^i}{\partial x^{k'}} g_{ij} \frac{\partial x^j}{\partial x^{m'}}$$

$$\hookrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$T^{k'} = \frac{\partial x^{k'}}{\partial x^i} T^i = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/6 \\ 1/2 \end{pmatrix}$$

⑥ Considerando al tensor de Maxwell:

$$F_{\mu\alpha} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -cB^z & cB^y \\ E^y & cB^z & 0 & -cB^x \\ E^z & -cB^y & cB^x & 0 \end{pmatrix}, \text{ y } \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

donde $\vec{E} = (E^x, E^y, E^z)$, $\vec{B} = (B^x, B^y, B^z)$ ⟹ Campos eléctricos y magnéticos para observador O

a) $\vec{E} = E^x \hat{i}$, $\vec{B} = v\hat{i}$

$$F_{\mu\alpha} = \begin{pmatrix} 0 & -E^x & 0 & 0 \\ E^x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

De modo que $F_{\mu\alpha}$ transforma como

$$F_{\mu'\alpha'} = \Lambda^{\nu}_{\mu'} \Lambda^{\beta}_{\alpha'} F_{\nu\beta}$$

$F_{\mu\alpha}$ → antisimétrico → $F_{\mu\alpha} = -F_{\mu\alpha}$

$F_{\mu'\alpha'} = \Lambda^0_{\mu'} \Lambda^1_{\alpha} F_{01} + \Lambda^1_{\mu'} \Lambda^0_{\alpha'} F_{10}$ (F_{10}, F_{01} componentes no nulos)

$\because F_{10} = -F_{01}$

$= \left(\Lambda^0_{\mu'} \Lambda^1_{\alpha} - \Lambda^1_{\mu'} \Lambda^0_{\alpha'} \right) F_{01}$

Teniendo en cuenta que $\Lambda^0_{0'} = \gamma$, $\Lambda^i_{0'} = \gamma v^i$, $\Lambda^0_{i'} = \gamma v_{i'}$,

$\Lambda^i_{j'} = \delta^i_{j'} + v^i v_{j'} \dfrac{\gamma - 1}{|v|^2}$ para $i, j = 1, 2, 3$

$v^i = (v, 0, 0)$

$\left(\Lambda^1_{1'} = \delta^1_{1'} + v^1 v_1 \dfrac{\gamma - 1}{v^2} = 1 + v^2 \dfrac{\gamma - 1}{v^2} = \gamma \right)$

$$\Lambda^{\alpha}_{\beta'} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

→ Componentes $\neq 0$:

$\Lambda^0_{0'}, \Lambda^1_{0'}, \Lambda^0_{1'}, \Lambda^1_{1'}, \Lambda^2_{2'}, \Lambda^3_{3'}$

$$F_{0'0'} = \left(\Lambda^0_{0'}\Lambda^1_{0'} - \Lambda^1_{0'}\Lambda^0_{0'}\right)F_{01} = \left(-\gamma^2 v + \gamma^2 v\right)F_{01} = \emptyset$$

$$F_{1'1'} = \left(\Lambda^0_{1'}\Lambda^1_{0'} - \Lambda^1_{0'}\Lambda^0_{1'}\right)F_{01} = \left(-\gamma^2 v^2 + \gamma^2 v^2\right)F_{01} = \emptyset$$

$$F_{1'0'} = \left(\Lambda^0_{1'}\Lambda^1_{0} - \Lambda^1_{1'}\Lambda^0_{0'}\right)(-E^x) = \left(\gamma^2 v^2 - \gamma^2\right)(-E^x)$$

De manera análoga, $F_{0'1'} = \left(\gamma^2 v^2 - \gamma^2\right)E^x$

Las demás componentes de $F_{\mu'\alpha'}$ tendrán necesariamente un elemento $\Lambda^{\alpha}_{\beta'}$ igual a cero, entonces

$$F_{\mu'\alpha'} = \begin{pmatrix} 0 & -E^{x'} & -E^{y'} & -E^{z'} \\ E^{x'} & 0 & cB^{z'} & -cB^{y'} \\ E^{y'} & -cB^{z'} & 0 & cB^{x'} \\ E^{z'} & cB^{y'} & -cB^{x'} & 0 \end{pmatrix} = \begin{pmatrix} 0 & (v^2-1)\gamma^2 E^x & 0 & 0 \\ -(v^2-1)\gamma^2 E^x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Entonces $\vec{E}' = (E^{x'}, E^{y'}, E^{z'}) = (-(v^2-1)\gamma^2 E^x, 0, 0)$ ✓ $\rightarrow$ solo percibe $\vec{E}''$

$\vec{B}' = (B^{x'}, B^{y'}, B^{z'}) = (0, 0, 0)$ ✓

donde $E^{x'} = -(v^2-1)\gamma^2 E^x = \dfrac{(1-v^2)}{c^2 - v^2} c^2 < 0 \rightarrow < 0$

b) Ec. Maxwell

Ampere-Maxwell: $\vec{\nabla}\times\vec{B} - \dfrac{1}{c^2}\dfrac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$, Gauss: $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$ (*)

- Mostrar que estas ecuaciones se pueden expresar como

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = F^{\mu\nu}_{\ \ ,\nu} = c\mu_0 J^\mu$$

donde $J^\mu = (c\rho, \vec{J})$ $\quad \vec{E} = (E^x, E^y, E^z)$

$\vec{J} = (J^1, J^2, J^3)$ $\quad \vec{B} = (B^x, B^y, B^z)$

$x^\nu = (ct, x, y, z)$

Ahora,
$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} = \eta^{\mu\alpha}F_{\alpha\beta}\eta^{\beta\nu}$$

$$F^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & cB^z & -cB^y \\ E^y & -cB^z & 0 & cB^x \\ E^z & cB^y & -cB^x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & cB^z & -cB^y \\ -E^y & -cB^z & 0 & cB^x \\ -E^z & cB^y & -cB^x & 0 \end{pmatrix}$$

Así
$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = F^{\mu\nu}_{\ \ ,\nu} = F^{\mu 0}_{\ \ ,0} + F^{\mu 1}_{\ \ ,1} + F^{\mu 2}_{\ \ ,2} + F^{\mu 3}_{\ \ ,3}$$

1) $\rightarrow F^{0\nu}_{\ \ ,\nu} = F^{00}_{\ \ ,0} + F^{01}_{\ \ ,1} + F^{02}_{\ \ ,2} + F^{03}_{\ \ ,3} = 0 + \frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z}$

$$= \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = \frac{c\rho}{c\varepsilon_0} = \frac{J^0}{c\frac{1}{\mu_0 c^2}} = \frac{J^0}{\frac{1}{\mu_0 c}} = c\mu_0 J^0$$

Ley Gauss

2) $F^{1\nu}_{\ \ ,\nu} = F^{10}_{\ \ ,0} + F^{11}_{\ \ ,1} + F^{12}_{\ \ ,2} + F^{13}_{\ \ ,3} = -\frac{1}{c}\frac{\partial E^x}{\partial t} + c\frac{\partial B^z}{\partial y} - c\frac{\partial B^y}{\partial z}$

Ley Ampere-Maxwell

$$= c\left(-\frac{1}{c^2}\frac{\partial E^x}{\partial t} + \frac{\partial B^z}{\partial y} - \frac{\partial B^y}{\partial z}\right) = c\left(-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_x = c\mu_0 J^1$$

donde
$$\mu_0 J^1 = \varepsilon^{1jk}\partial_j B_k - \frac{1}{c^2}\frac{\partial E^1}{\partial t} = (\partial_2 B_3 - \partial_3 B_2) - \frac{1}{c^2}\frac{\partial E^1}{\partial t}$$

$$= \frac{\partial B^z}{\partial y} - \frac{\partial B^y}{\partial z} - \frac{1}{c^2}\frac{\partial E^x}{\partial t} = \left(-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_x \text{ (componente } x\text{)}$$

$$F^{2\nu}{}_{,\nu} = F^{20}{}_{,0} + F^{21}{}_{,1} + F^{22}{}_{,2} + F^{23}{}_{,3} = -\frac{1}{c}\frac{\partial E^y}{\partial t} - c\frac{\partial B^z}{\partial x} + c\frac{\partial B^x}{\partial z}$$

$$= c\left(-\frac{1}{c^2}\frac{\partial E^y}{\partial t} - \frac{\partial B^z}{\partial x} + \frac{\partial B^x}{\partial z}\right) = c\left(\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_y = c\mu_0 J^2,$$

$$\mu_0 J^2 = \epsilon^{2jk}\partial_j B_k - \frac{1}{c^2}\frac{\partial E^2}{\partial t} = (\partial_3 B_1 - \partial_1 B_3) - \frac{1}{c^2}\frac{\partial E^2}{\partial t}$$

$$= \frac{\partial B^x}{\partial z} - \frac{\partial B^z}{\partial x} - \frac{1}{c^2}\frac{\partial E^y}{\partial t}$$

$$F^{3\nu}{}_{,\nu} = F^{30}{}_{,0} + F^{31}{}_{,1} + F^{32}{}_{,2} + F^{33}{}_{,3} = -\frac{1}{c}\frac{\partial E^z}{\partial t} + c\frac{\partial B^y}{\partial x} - c\frac{\partial B^x}{\partial y}$$

$$= c\left(-\frac{1}{c^2}\frac{\partial E^z}{\partial t} + \frac{\partial B^y}{\partial x} - \frac{\partial B^x}{\partial y}\right) = c\mu_0 J^3$$

donde $\mu_0 J^3 = \epsilon^{3jk}\partial_j B_k - \frac{1}{c^2}\frac{\partial E^3}{\partial t} = (\partial_1 B_2 - \partial_2 B_1) - \frac{1}{c^2}\frac{\partial E^3}{\partial t}$

$= \frac{\partial B^y}{\partial x} - \frac{\partial B^x}{\partial y} - \frac{1}{c^2}\frac{\partial E^z}{\partial t}$. Se concluye que $F^{\mu\nu}{}_{,\nu} = c\mu_0 J^\mu$

y las leyes de Maxwell pedidas

c) Identidad Bianchi:

$$F_{\mu\nu,\gamma} + F_{\nu\gamma,\mu} + F_{\gamma\mu,\nu} = 0$$

Demostrar $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0$ están contenidas en $F^{\mu\nu}{}_{,\nu} = c\mu_0 J^\mu$

$$\to F_{13,2} + F_{32,1} + F_{21,3} = 0$$

$$-c\frac{\partial B^y}{\partial y} - c\frac{\partial B^x}{\partial x} - c\frac{\partial B^z}{\partial z} = 0$$

$$\frac{\partial B^x}{\partial x} + \frac{\partial B^y}{\partial y} + \frac{\partial B^z}{\partial z} = 0 \longrightarrow \vec{\nabla}\cdot\vec{B} = 0$$

De modo que: $F_{\mu\nu,0} + F_{\nu 0,\mu} + F_{0\mu,\nu} = 0$

$\rightarrow \mu = 2, \nu = 3$: $F_{23,0} + F_{30,2} + F_{02,3} = 0$

$$\frac{c}{c}\frac{\partial B^x}{\partial t} + \frac{\partial E^z}{\partial y} - \frac{\partial E^y}{\partial z} = 0 \rightarrow (\dot{\vec{B}} + \vec{\nabla}\times\vec{E})_x = 0$$

$$\cdot\ (\vec{\nabla}\times\vec{E})_x = \epsilon^{1jk}\partial_j E_k = \partial_2 E_3 - \partial_3 E_2 = \frac{\partial E^z}{\partial y} - \frac{\partial E^y}{\partial z}$$

$\rightarrow \mu = 1, \nu = 3$ $F_{13,0} + F_{30,1} + F_{01,3} = 0$

$$-\frac{c}{c}\frac{\partial B^y}{\partial t} + \frac{\partial E^z}{\partial x} - \frac{\partial E^x}{\partial z} = 0 \rightarrow (-\dot{\vec{B}} - \vec{\nabla}\times\vec{E})_y = 0$$

donde: $(-\vec{\nabla}\times\vec{E})_y = -\epsilon^{2jk}\partial_j E_k = -(\partial_3 E_1 - \partial_1 E_3) = -\frac{\partial E^x}{\partial z} + \frac{\partial E^z}{\partial x}$

$\rightarrow \mu = 2, \nu = 1$: $F_{21,0} + F_{10,2} + F_{02,1} = 0$

$$-\frac{c}{c}\frac{\partial B^z}{\partial z} + \frac{\partial E^x}{\partial y} - \frac{\partial E^y}{\partial x} = 0 \rightarrow (-\dot{\vec{B}} - \vec{\nabla}\times\vec{E})_z = 0$$

donde $(-\vec{\nabla}\times\vec{E})_z = -\epsilon^{3jk}\partial_j E_k = -(\partial_1 E_2 - \partial_2 E_1) = -\frac{\partial E^y}{\partial x} + \frac{\partial E^x}{\partial y}$

· Podemos concluir que la identidad de Bianchi contiene a las ecuaciones pedidas

$$\vec{\nabla}\cdot\vec{B} = 0,\quad \dot{\vec{B}} + \vec{\nabla}\times\vec{E} = 0$$