C3C "operadores"

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Sección 4.1.4

Ejercicio 4
Suponga que $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$ demuestre que:

a) $(\mathbb{A}+\mathbb{B})^2 = \mathbb{A}^2 + 2\mathbb{A}\mathbb{B} + \mathbb{B}^2$

$$(\mathbb{A}+\mathbb{B})^2 = (\mathbb{A}+\mathbb{B})(\mathbb{A}+\mathbb{B}) = (\mathbb{A}\mathbb{A}) + (\mathbb{A}\mathbb{B}) + (\mathbb{B}\mathbb{A}) + (\mathbb{B}\mathbb{B})$$
$$= \mathbb{A}^2 + (\mathbb{A}\mathbb{B}) + (\mathbb{A}\mathbb{B}) + \mathbb{B}^2$$
$$= \mathbb{A}^2 + 2(\mathbb{A}\mathbb{B}) + \mathbb{B}^2$$

b) $(\mathbb{A}+\mathbb{B})^3 = \mathbb{A}^3 + 3\mathbb{A}^2\mathbb{B} + 3\mathbb{A}\mathbb{B}^2 + \mathbb{B}^3$

$$(\mathbb{A}+\mathbb{B})^3 = (\mathbb{A}+\mathbb{B})^2(\mathbb{A}+\mathbb{B}) = (\mathbb{A}^2 + 2(\mathbb{A}\mathbb{B}) + \mathbb{B}^2)(\mathbb{A}+\mathbb{B})$$
$$= \mathbb{A}^3 + 2(\mathbb{A}^2\mathbb{B}) + (\mathbb{A}\mathbb{B}^2) + (\mathbb{A}^2\mathbb{B}) + 2(\mathbb{A}\mathbb{B}^2) + \mathbb{B}^3$$
$$= \mathbb{A}^3 + 3(\mathbb{A}^2\mathbb{B}) + 3(\mathbb{A}\mathbb{B}^2) + \mathbb{B}^3$$

Ejercicio 5

Suponga que un operador $\mathbb{L}$ puede ser escrito como la composición de otros dos operadores $\mathbb{L} = \mathbb{L}_- \mathbb{L}_+$ con $[\mathbb{L}_-, \mathbb{L}_+] = \mathbb{I}$. Demostrar que:

conmuta $= L_- L_+ - L_+ L_- = \mathbb{I}$

Si $\mathbb{L}|x\rangle = \lambda |x\rangle$ y $|y\rangle = \mathbb{L}_+ |x\rangle$ entonces $\mathbb{L}|y\rangle = (\lambda + 1)|y\rangle$

$$L|y\rangle = (L_- L_+)(L_+ |x\rangle) = (\mathbb{I} + L_+ L_-)(L_+ |x\rangle)$$
$$= (L_+ |x\rangle) + (L_+ L |x\rangle) = |y\rangle + (L_+ (\lambda |x\rangle)) = |y\rangle + \lambda |y\rangle$$
$$= (1 + \lambda)|y\rangle$$

Si $\mathbb{L}|x\rangle = \lambda |x\rangle$ y $|z\rangle = \mathbb{L}_- |x\rangle$ entonces $\mathbb{L}|z\rangle = (\lambda - 1)|z\rangle$

$$L|z\rangle - (L_- L_+)(L_- |x\rangle) = L_- (L_+ L_-)|x\rangle = L_- L|x\rangle = \lambda L_- |x\rangle$$
$$= \lambda |z\rangle$$

$$L_+ L_- |z\rangle = L_+ L_- L_- |x\rangle = (L_- L_+ - \mathbb{I}) L_- |x\rangle$$
$$= L_- (L_+ L_- |x\rangle) - \mathbb{I} L_- |x\rangle = L_- L |x\rangle - L_- |x\rangle$$
$$= L_- (\lambda |x\rangle) - |z\rangle = \lambda (L_- |x\rangle) - |z\rangle$$
$$= \lambda |z\rangle - |z\rangle = (\lambda - 1)|z\rangle$$

Ejercicio 2 sección 4.3.8

a) considere el polinomio $|f\rangle_t \leftrightarrow f(t) = 5t + 3t^2 + 4t^3$ encuentre la expresión de este polinomio en término de las bases mencionadas. ¿cuál es la matriz de transformación?

$$\{1, t, t^2, t^3, t^4\} \longrightarrow f(t) = 5t + 3t^2 + 4t^3$$

$$\{P_0, P_1, P_2, P_3, P_4\} \equiv \left\{1, t, \frac{1}{2}(3t^2-1), \frac{1}{2}(5t^3-3t), \frac{1}{8}(35t^4 - 30t^2 + 3\right\}$$

$$5t + 3t^2 + 4t^3 = \alpha(1) + \beta(t) + \gamma\left(\frac{3}{2}t^2 - \frac{1}{2}\right) + \eta\left(\frac{5}{2}t^3 - \frac{3}{2}t\right)$$

$$= \alpha + \beta t + \frac{3}{2}\gamma t^2 - \frac{1}{2}\gamma + \frac{5}{2}\eta t^3 - \frac{3}{2}\eta t$$

$$5 = \beta - \frac{3}{2}\eta, \quad 3 = \frac{3}{2}\gamma, \quad 4 = \frac{5}{2}\eta, \quad 0 = \alpha - \frac{\gamma}{2}$$

$$\begin{array}{cccc} \alpha & \beta & \gamma & \eta \end{array}$$

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & -3/2 & 5 \\ 0 & 0 & 3/2 & 0 & 3 \\ 0 & 0 & 0 & 5/2 & 4 \\ 1 & 0 & -1/2 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 37/5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 8/5 \end{array}\right)$$

$$5t + 3t^2 + 4t^3 = 1|P_0\rangle + \frac{37}{5}|P_1\rangle + 2|P_2\rangle + \frac{8}{5}|P_3\rangle$$

$$5t+3t^2+4t^3 = 1|p_0\rangle + \frac{37}{5}|p_1\rangle + 2|p_2\rangle + \frac{8}{5}|p_3\rangle$$

para obtener la matriz de transformación se tendría en cuenta lo siguiente:

$$|p\rangle = (\alpha, \beta, \gamma, \eta, \omega) = \alpha|p_0\rangle + \beta|p_1\rangle + \gamma|p_2\rangle + \eta|p_3\rangle + \omega|p_4\rangle$$

$$= \alpha(1) + \beta(t) + \gamma\left(\frac{3}{2}t^2 - \frac{1}{2}\right) + \eta\left(\frac{5}{2}t^3 - \frac{3}{2}t\right)$$

$$+ \omega\left(\frac{35}{8}t^4 - \frac{30}{8}t^2 + \frac{3}{8}\right)$$

$$= \left(\alpha - \frac{1}{2}\gamma + \frac{3}{8}\omega\right)|1\rangle \left(\beta - \frac{3}{2}\eta\right)|t\rangle + \left(\frac{3}{2}\gamma - \frac{30}{8}\omega\right)|t^2\rangle$$

$$+ \left(\frac{5}{2}\eta\right)|t^3\rangle + \left(\frac{35}{8}\omega\right)|t^4\rangle \qquad = e^\alpha |E_\alpha\rangle$$

$$e^\alpha = \frac{\partial e^\alpha}{\partial p^\beta} p^\beta$$

$$\{|1\rangle, |t\rangle, |t^2\rangle, |t^3\rangle, |t^4\rangle\} = \{|E_\alpha\rangle\}$$

$$e_0 = \alpha - \frac{1}{2}\gamma + \frac{3}{8}\omega$$
$$e_1 = \beta - \frac{3}{2}\eta$$
$$e_2 = \frac{3}{2}\gamma - \frac{30}{8}\omega$$
$$e_3 = \frac{5}{2}\eta$$
$$e_4 = \frac{35}{8}\omega$$

$$\frac{\partial e^\alpha}{\partial p^\beta} = \begin{pmatrix} 1 & 0 & -1/2 & 0 & 3/8 \\ 0 & 1 & 0 & -3/2 & 0 \\ 0 & 0 & 3/2 & 0 & -30/8 \\ 0 & 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 0 & 35/8 \end{pmatrix}$$

la matriz que se obtuvo anteriormente es la matriz de cambio de base de la base de polinomios de Legendre a los polinomios de grado 4 $\{1, t, t^2, t^3, t^4\}$.

Para poder obtener la matriz de transformación de la base $\{1, t, t^2, t^3, t^4\}$ a la de polinomios de Legendre se aplica ortogonalidad, al multiplicar la matriz anterior por su inversa debe dar 1.

$$P^{\beta} = \frac{\partial P^{\beta}}{\partial e^{\alpha}} e^{\alpha} \qquad \frac{\partial P^{\beta}}{\partial e^{\alpha}} = \begin{pmatrix} 1 & 0 & -1/2 & 0 & 3/8 \\ 0 & 1 & 0 & -3/2 & 0 \\ 0 & 0 & 3/2 & 0 & 30/8 \\ 0 & 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 0 & 35/8 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 1/3 & 0 & 1/5 \\ 0 & 1 & 0 & 3/5 & 0 \\ 0 & 0 & 2/3 & 0 & 4/7 \\ 0 & 0 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 0 & 8/35 \end{pmatrix}$$

Segunda forma $|f\rangle_t \to f(t) = 5t + 3t^2 + 4t^3$

$$c_0 = \frac{\langle p|f\rangle}{\int_{-1}^{1} p^2(t)dt} = \frac{1}{2}\int_{-1}^{1} (5t + 3t^2 + 4t^3)(1)\,dt = \frac{1}{2}\left[\frac{5t^2}{2} + t^3 + t^4\right]_{-1}^{1} =$$

$$= \frac{1}{2}\left[\left[\frac{5}{2} + 1 + 1\right] - \left[\frac{5}{2} - 1 + 1\right]\right] = \frac{2}{2} = 1$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} (5t + 3t^2 + 4t^3)(t)\,dt = \frac{3}{2}\left[\frac{5t^3}{3} + \frac{3t^4}{4} + \frac{4t^5}{5}\right]_{-1}^{1}$$

$$= \frac{3}{2}\left[\left[\frac{5}{3} + \cancel{\frac{3}{4}} + \frac{4}{5}\right] - \left[-\frac{5}{3} + \cancel{\frac{3}{4}} - \frac{4}{5}\right]\right] = \frac{3}{2}\left[\frac{10}{3} + \frac{8}{5}\right] = \frac{3}{2}\left[\frac{74}{15}\right] = \frac{37}{5}$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} (5t + 3t^2 + 4t^3)\left(\frac{1}{2}(3t^2 - 1)\right)dt = \frac{5}{4}\left[\frac{15t^4}{4} + \frac{9t^5}{5} + \frac{12t^6}{6} - \right.$$

$$\left.\frac{5t^2}{2} - \frac{3t^3}{3} - \frac{4t^4}{4}\right]_{-1}^{1} = \frac{5}{4}\left[\frac{8}{5}\right] = 2$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} (5t + 3t^2 + 4t^3)\left(\frac{1}{2}(5t^3 - 3t)\right)dt = \frac{7}{4}\left[\frac{25t^5}{5} + \frac{15t^6}{6} + \right.$$

$$\left.\frac{20t^7}{7} - \frac{15t^3}{3} - \frac{9t^4}{4} - \frac{12t^5}{5}\right]_{-1}^{1} = \frac{8}{5}$$

$$c_4 = \frac{9}{2}\int_{-1}^{1} (5t + 3t^2 + 4t^3)\left(\frac{1}{8}(35t^4 - 30t^2 + 3)\right)dt = 0$$

b) construya un proyector sobre el subespacio $P(t)_2$, de polinomios de grado 2, y encuentre la proyección de $|f\rangle_t$ en ese subespacio discuta las diferencias y semejanzas

un proyector sobre el subespacio $P(t)_2$ es

$$\mathbb{P} = \frac{|P_0\rangle\langle P_0|}{\langle P_0|P_0\rangle} + \frac{|P_1\rangle\langle P_1|}{\langle P_1|P_1\rangle} + \frac{|P_2\rangle\langle P_2|}{\langle P_2|P_2\rangle}$$

$$\mathbb{P}|f\rangle_t = |P_0\rangle\frac{\langle P_0|f\rangle_t}{\langle P_0|P_0\rangle} + |P_1\rangle\frac{\langle P_1|f\rangle_t}{\langle P_1|P_1\rangle} + |P_2\rangle\frac{\langle P_2|f\rangle_t}{\langle P_2|P_2\rangle}$$

$$\langle P_0|f\rangle = \int_{-1}^{1}(1)(5t+3t^2+4t^3)dt = 2$$

$$\langle P_n|P_n\rangle = \frac{2}{2n+1}$$

$$\langle P_1|f\rangle = \int_{-1}^{1}(t)(5t+3t^2+4t^3)dt = \frac{74}{15}$$

$$\langle P_0|P_0\rangle = \frac{2}{2(0)+1} = 2$$

$$\langle P_2|f\rangle = \int_{-1}^{1}\left(\frac{3}{2}t^2-\frac{1}{2}\right)(5t^2+3t^2+4t^3)dt = \frac{4}{5}$$

$$\langle P_1|P_1\rangle = \frac{2}{2(1)+1} = \frac{2}{3}$$

$$\langle P_2|P_2\rangle = \frac{2}{2(2)+1} = \frac{2}{5}$$

$$\mathbb{P}|f\rangle = |P_0\rangle 1 + |P_1\rangle\frac{222}{30} + |P_2\rangle 2 = \begin{pmatrix}1\\37/5\\2\end{pmatrix}$$

$$\mathbb{P}|f\rangle = |1\rangle + \frac{37}{5}|t\rangle + 2\left(\frac{1}{2}(3|t^2\rangle - |1\rangle)\right)$$

$$= |1\rangle + \frac{37}{5}|t\rangle + 3|t^2\rangle - |1\rangle = \frac{37}{5}|t\rangle + 3|t^2\rangle = \begin{pmatrix}0\\37/5\\3\end{pmatrix}$$

c) para $P(t)_4$, construya el operador inverso $\mathbb{T}^{-1}$, el adjunto $\mathbb{T}^{\dagger}$ y precise si $\mathbb{T}$, es Hermitico o unitario

$$\mathbb{T} = e^{\mathbb{D}} \equiv \exp(\mathbb{D}) \quad \text{con } \mathbb{D} = \frac{d}{dt} \quad \text{con } e^{\mathbb{D}} = \sum_{n=0}^{\infty} \frac{\mathbb{D}^n}{n!}$$

$$T^{-1} = ?$$

por propiedades del operador tenemos

$$e^{\mathbb{D}} \cdot e^{-\mathbb{D}} = e^{\mathbb{D}-\mathbb{D}} \cdot \underbrace{e^{\frac{1}{2}[\mathbb{D},-\mathbb{D}]}}_{\text{conmutador}} = e^{0} e^{0} = \mathbb{I}$$

$$T^{-1} = e^{-\mathbb{D}} = \sum_{n=0}^{\infty} \frac{(-\mathbb{D})^n}{n!} = \frac{1}{0!}(-\mathbb{D})^0 + \frac{1}{1!}(-\mathbb{D})^1 + \frac{1}{2!}(-\mathbb{D})^2 + \frac{1}{3!}(-\mathbb{D})^3$$

$$+ \frac{1}{4!}(-\mathbb{D})^4 = \mathbb{I} - \mathbb{D} + \frac{1}{2}\mathbb{D}^2 - \frac{1}{6}\mathbb{D}^3 + \frac{\mathbb{D}^4}{24}$$

$$\mathbb{T}^{\dagger} = (e^{\mathbb{D}})^{\dagger} = \left(\sum_{n=0}^{\infty} \frac{\mathbb{D}^n}{n!}\right)^{\dagger} = \sum_{n=0}^{4} \frac{(\mathbb{D}^n)^{\dagger}}{n!}$$

$$(\mathbb{D}^n)^{\dagger} = (\underbrace{\mathbb{D}\,\mathbb{D}\,\mathbb{D} \cdots \mathbb{D}}_{n\text{-veces}})^{\dagger}$$

$$= \underbrace{\mathbb{D}^{\dagger}\mathbb{D}^{\dagger} \cdots \mathbb{D}^{\dagger}}_{n\text{-veces}} = (\mathbb{D}^{\dagger})^{n}$$

$$\mathbb{D}^{\dagger}|f\rangle_t = \frac{d^{\dagger}}{dt}|f\rangle_t = \frac{d}{dt}|f\rangle_t = \mathbb{D}|f\rangle_t$$

Definición de Hermítica.

$$\mathbb{T}^{\dagger} = \sum_{n=0}^{4} \frac{(\mathbb{D}^n)^{\dagger}}{n!} = \sum_{n=0}^{4} \frac{\mathbb{D}^n}{n!} = e^{\mathbb{D}} = \mathbb{T}$$

$$\mathbb{T}^{\dagger} = \mathbb{T}$$

d) ¿cuales son las representaciones matriciales de $\mathbb{T}$ en $P(t)_4$, para cada una de las bases mencionadas? ¿cómo transforman las representaciones? comprueba que la traza y el determinante de ambas coinciden

$$\mathbb{D} \rightarrow \frac{d}{dt}|1\rangle = 0, \quad \frac{d}{dt}|t\rangle = |1\rangle, \quad \frac{d}{dt}|t^2\rangle = 2|t\rangle, \quad \frac{d}{dt}|t^3\rangle = 3|t^2\rangle,$$

$$\frac{d}{dt}|t^4\rangle = 4|t^3\rangle \qquad \mathbb{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e^{\mathbb{D}} = \sum_{n=0}^{\infty} \frac{\mathbb{D}^n}{n!} = \mathbb{I} + \mathbb{D} + \frac{\mathbb{D}^2}{2} + \frac{\mathbb{D}^3}{6} + \frac{\mathbb{D}^4}{24} + \cdots$$

matrices de ceros

$$\mathbb{T} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{B} = T_{(B)}{}^{\alpha}_{\beta}$$

$\mathbb{D} \to \frac{d}{dt}|1\rangle = 0, \ \frac{d}{dt}|t\rangle = |1\rangle, \ \frac{d}{dt}|\frac{3}{2}t^2 - \frac{1}{2}\rangle = 3|t\rangle,$

$\frac{d}{dt}|\frac{5}{2}t^3 - \frac{3}{2}t\rangle = 5|\frac{3}{2}t^2 - 1\rangle + |1\rangle, \ \frac{d}{dt}|\frac{35}{8}t^4 - \frac{30}{8}t^2 + \frac{3}{8}\rangle$

$= 7|\frac{5}{2}t^3 - \frac{3}{2}t\rangle + 3|t\rangle \qquad \mathbb{D} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$$e^{tD} = \sum_{n=0}^{4} \frac{D^n}{n!} = \begin{pmatrix} 1 & 1 & 3/2 & 7/2 & 75/8 \\ 0 & 1 & 3 & 15/2 & 41/2 \\ 0 & 0 & 1 & 5 & 35/2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_R = \Pi = T_{(P)}{}^{\alpha}{}_{\beta}$$

$$\Rightarrow T_{(P)}{}^{\alpha}{}_{\beta} = \frac{\partial P^\alpha}{\partial e^\mu} \frac{\partial e^\nu}{\partial P^\beta} T_{(B)}{}^{\mu}{}_{\nu}$$

Traza:

- $Tr(T_{(B)}{}^{\alpha}{}_{\beta}) = T_{(B)}{}^{\alpha}{}_{\alpha} = T_{(B)}{}^{0}{}_{0} + T_{(B)}{}^{1}{}_{1} + T_{(B)}{}^{2}{}_{2} + T_{(B)}{}^{3}{}_{3} + T_{(B)}{}^{4}{}_{4} = 1 + 1 + 1 + 1 + 1 = 5$

- $Tr(T_{(P)}{}^{\alpha}{}_{\beta}) = T_{(P)}{}^{\alpha}{}_{\alpha} = T_{(P)}{}^{0}{}_{0} + T_{(P)}{}^{1}{}_{1} + T_{(P)}{}^{2}{}_{2} + T_{(P)}{}^{3}{}_{3} + T_{(P)}{}^{4}{}_{4} = 1 + 1 + 1 + 1 + 1 = 5$

determinante $\det(T_{(P)}) = \det(T_{(B)}) = 1$

Scribe 4.2.9

1 a) I) $(P^{\dagger})^{-1} = (P^{-1})^{\dagger}$

$$PP^{-1} = I$$

$$\langle x | I | y \rangle = \langle x | PP^{-1} | y \rangle = \langle y | \underbrace{(P^{-1})^{\dagger} P^{\dagger}}_{I} | x \rangle^* = \langle y | x \rangle^*$$

Entonces $\underbrace{(P^{-1})^{\dagger} P^{\dagger}}_{\text{Son inversos el uno del otro}} = I \Rightarrow \boxed{(P^{\dagger})^{-1} = (P^{-1})^{\dagger}}$

II)

$$(PQ)^{-1} = Q^{-1}P^{-1}$$

$$(PQ)^{-1}PQ = I \Rightarrow (PQ)^{-1}P = IQ^{-1} \Rightarrow \boxed{(PQ)^{-1} = Q^{-1}P^{-1}}$$

$*Q^{-1}$ $\quad *Q^{-1}$ $\quad *P^{-1}$ $\quad *P^{-1}$

III)

Si $[P, Q] = 0$, entonces $P(Q)^{-1} = (Q)^{-1}P$

$$PQ - QP = 0, \quad PQ = QP$$

$$PQQ^{-1} = QPQ^{-1} \rightarrow P = QPQ^{-1} \rightarrow \boxed{Q^{-1}P = PQ^{-1}}$$

Q^{-1} $\quad Q^{-1}$

IV) $(e^{P})^{\dagger} = e^{P^{\dagger}}$

$$\left[\sum_{n=0}^{\infty} \frac{P^n}{n!}\right]^{\dagger} = \left[I + P + \frac{P^2}{2!} + \dots + \frac{P^n}{n!} + \dots\right]^{\dagger}$$

$$= \left[I^{\dagger} + P^{\dagger} + \frac{(P^{\dagger})^2}{2!} + \dots + \frac{(P^{\dagger})^n}{n!} + \dots\right] = \left[\sum_{n=0}^{\infty} \frac{(P^{\dagger})^n}{n!}\right] = e^{P^{\dagger}}$$

v) $\underbrace{P e^{Q} P^{-1}}$ = $e^{PQP^{-1}}$

$$P\left[I + Q + \frac{Q^2}{2!} + \dots + \frac{Q^n}{n!} + \dots\right]P^{-1} = \left[I + PQP^{-1} + \frac{PQ^2P^{-1}}{2!} + \dots + \frac{PQ^nP^{-1}}{n!} + \dots\right]$$

$$= \sum_{n=0}^{\infty} \frac{PQ^nP^{-1}}{n!} = e^{PQP^{-1}} = \left[I + PQP^{-1} + \frac{PQ\overbrace{P^{-1} \cdot P}^{I}QP^{-1}}{2!} + \dots\right]$$

(b) Si A es hermítico, entonces $\tilde{A} = U^{-1}AU$ también es hermítico

$$(\tilde{A})^{\dagger} = (U^{-1}AU)^{\dagger} = U^{\dagger}A^{\dagger}(U^{-1})^{\dagger} = U^{-1}A(U^{-1})^{-1} = U^{-1}AU = \tilde{A}$$

$$\Rightarrow (\tilde{A})^{\dagger} = \tilde{A}$$

(c) Si A es hermítico, entonces e^{iA} es unitario $(e^{iA})^{\dagger} = (e^{iA})^{-1}$

- $(iI)^{\dagger} = i^{*} I = -iI$
- $[iA, -iA] = A^2 - (-i)AiA = 0$

$$(e^{iA})^{\dagger} \cdot e^{iA} = e^{A^{\dagger}(iI)^{\dagger}} \cdot e^{iA} = e^{-Ai} \cdot e^{iA} = e^{-Ai} e^{Ai} = e^{-Ai + Ai}$$

$$= e^{\frac{1}{2}[-Ai, Ai]} = I$$, entonces $\Rightarrow (e^{iA})^{-1} = (e^{iA})^{\dagger}$

(d) Si $K = -K^{\dagger}$, entonces $\tilde{K} = U^{-1}KU$ en particular será $\tilde{K} = iA$

$$-\tilde{K}^{\dagger} = -[U^{-1}KU]^{\dagger} = -U^{\dagger}K^{\dagger}(U^{-1})^{\dagger} = -U^{-1}(-K)U = U^{-1}KU = \tilde{K}$$

Por ende $\tilde{K} = -\tilde{K}^{\dagger}$

(e) A, B hermíticos, su composición AB será hermítica si A y B conmutan

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} = BA$$

$$(AB)^{\dagger} = AB$$

$$[A,B] = AB - BA = 0$$

$$\underbrace{AB = BA}_{\text{conmutar}}$$

(F) Si S es real y antisimétrico, I el operador unidad; probar.

I) $(I-S)$ y $(I+S)$ conmutan

o sea, $\underbrace{(I-S)(I+S)}_{I-S^2} = \underbrace{(I+S)(I-S)}_{I-S^2}$

$$I - S^2 = I - S^2$$

Verdadero

II) • $(I-S)(I+S)$ es simétrico

•• $(I-S)(I+S)^{-1}$ es ortogonal

• $$[(I-S)(I+S)]^{\dagger} = (I+S)^{\dagger}(I-S)^{\dagger} = (I+S^{\dagger})(I-S^{\dagger}) = (I-S)(I+S)$$

Es simétrico ✓

•• $$[(I-S)(I+S)^{-1}]^{\dagger}[(I-S)(I+S)^{-1}] = (I+S^{\dagger})^{-1}(I-S^{\dagger})(I-S)(I+S)^{-1}$$

$$= (I-S)^{-1}(I+S)(I-S)(I+S)^{-1} = \underbrace{(I-S)^{-1}(I-S)}_{I}\underbrace{(I+S)(I+S)^{-1}}_{I} = I \checkmark$$

(g) $R = \begin{pmatrix} \text{Cos}\theta & \text{Sen}\theta \\ -\text{Sen}\theta & \text{Cos}\theta \end{pmatrix}$, encontrar expresión para S que reproduce

$$R = (I-S)(I+S)^{-1}$$

$$R(I+S) = (I-S)$$

$$\begin{pmatrix} \text{Cos}\theta & \text{Sen}\theta \\ -\text{Sen}\theta & \text{Cos}\theta \end{pmatrix}\begin{pmatrix} 1+S_{11} & S_{12} \\ S_{21} & 1+S_{22} \end{pmatrix} = \begin{pmatrix} 1-S_{11} & -S_{12} \\ -S_{21} & 1-S_{22} \end{pmatrix}$$

$$\begin{pmatrix} \text{Cos}\theta + \text{Cos}\theta S_{11} + \text{Sen}\theta S_{21} & \text{Cos}\theta S_{12} + \text{Sen}\theta + \text{Sen}\theta S_{22} \\ -\text{Sen}\theta - \text{Sen}\theta S_{11} + \text{Cos}\theta S_{21} & -\text{Sen}\theta S_{12} + \text{Cos}\theta + \text{Cos}\theta S_{22} \end{pmatrix}$$

$$\text{Cos}\theta + \text{Cos}\theta S_{11} + S_{11} - 1 + \text{Sen}\theta S_{21} = 0$$

(1) $S_{11}[\text{Cos}\theta + 1] + S_{21}[\text{Sen}\theta] = 1 - \text{Cos}\theta$

(2) $S_{12}[1+\text{Cos}\theta] + S_{22}\,\text{Sen}\theta = -\text{Sen}\theta$

(3) $-\text{Sen}\theta S_{11} + S_{21}[1+\text{Cos}\theta] = \text{Sen}\theta$

(4) $-\text{Sen}\theta S_{12} + S_{22}[1+\text{Cos}\theta] = 1 - \text{Cos}\theta$

$$\begin{pmatrix} \text{Cos}\theta+1 & 0 & \text{Sen}\theta & 0 \\ 0 & 1+\text{Cos}\theta & 0 & \text{Sen}\theta \\ -\text{Sen}\theta & 0 & 1+\text{Cos}\theta & 0 \\ 0 & -\text{Sen}\theta & 0 & 1+\text{Cos}\theta \end{pmatrix}\begin{pmatrix} S_{11} \\ S_{12} \\ S_{21} \\ S_{22} \end{pmatrix} = \begin{pmatrix} 1-\text{Cos}\theta \\ -\text{Sen}\theta \\ \text{Sen}\theta \\ 1-\text{Cos}\theta \end{pmatrix}$$

Por lo tanto $\rightarrow$ $S_{11} = 0$, $S_{22} = 0$, $S_{12} = -S_{21}$

$$S_{12} = \frac{-2\,\text{Sen}\theta}{\text{Sen}^2\theta + \text{Cos}^2\theta + 2\text{Cos}\theta + 1} = \frac{-\cancel{2}\,\text{Sen}\theta}{\cancel{2}(1+\text{Cos}\theta)} = \frac{-\text{Sen}\theta}{1+\text{Cos}\theta}$$

$$S_{21} = \frac{2\,\text{Sen}\theta}{\text{Sen}^2\theta + \text{Cos}^2\theta + 2\text{Cos}\theta + 1} = \frac{\cancel{2}\,\text{Sen}\theta}{\cancel{2}(1+\text{Cos}\theta)} = \frac{\text{Sen}\theta}{1+\text{Cos}\theta}$$