Fourier Series Representation of Periodic Signals

1. The Response of LTI Systems to Complex Exponentials

A signal for which the system output is a constant times the input is referred to as an **eigenfunction** of the system, and the amplitude factor is referred to as the system's **eigenvalue**.

$$\text{continuous time:} \qquad e^{st} \longrightarrow H(s)e^{st}$$

$$\text{discrete time:} \qquad z^{n} \longrightarrow H(z)z^{n}$$

Complex exponentials are eigenfunctions of LTI systems.

If the input to a continuous-time LTI system is represented as a linear combination of complex exponentials

$$x(t) = \sum_{k} a_k \mathrm{e}^{s_k t}$$

then the output will be

$$y(t) = \sum_{k} a_k H(s_k) \mathrm{e}^{s_k t}$$

In an exactly analogous manner, we have

$$x[n] = \sum_{k} a_k z_k^n$$

and

$$y[n] = \sum_{k} a_k H(z_k) z_k^n$$

2. Fourier Series Representation of Continuous-Time Periodic Signals

2.1 Linear Combinations of Harmonically Related Complex Exponentials

the set of **harmonically related** complex exponentials

$$\phi_k(t) = \mathrm{e}^{\mathrm{j}k\omega_0 t} = \mathrm{e}^{\mathrm{j}k\frac{2\pi}{T}t} \qquad k = 0, \pm 1, \pm 2 \ldots$$

Thus, a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \mathrm{e}^{\mathrm{j}k\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k \mathrm{e}^{\mathrm{j}k\frac{2\pi}{T}t}$$

is also periodic with period T. This is referred to as the **Fourier series representation**.
the term for $k = 0$ is a constant
the terms for $k = +1$ and $k = -1$ are collectively referred to as the **fundamental components** or the **first harmonic components**.
the terms for $k = +N$ and $k = -N$ are referred to as the N**th harmonic components**.

suppose that $x(t)$ is real,then the Fourier series coefficients will be **conjugate symmetric** $a_k^* = a_{-k}$
we have

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

or

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos(k\omega_0 t) + C_k \sin(k\omega_0 t)]$$

2.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

the **synthesis** equation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \mathrm{e}^{\mathrm{j}k\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k \mathrm{e}^{\mathrm{j}k\frac{2\pi}{T}t}$$

the **analysis** equation

$$a_k = \frac{1}{T}\int_T x(t)\mathrm{e}^{-\mathrm{j}k\omega_0 t}\,\mathrm{d}t = \frac{1}{T}\int_T x(t)\mathrm{e}^{-\mathrm{j}k\frac{2\pi}{T}t}\,\mathrm{d}t$$

The set of coefficients $\{a_k\}$ are often called the **Fourier series coefficients** or the **spectral coefficients** of $x(t)$.

3. Convergence of the Fourier Series

The Dirichlet conditions are as follows:

1. Over any period, $x(t)$ must be **absolutely integrable** $\int_T |x(t)|\,\mathrm{d}t < \infty$
2. In any finite interval of time, $x(t)$ is of bounded variation
3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

A periodic signal can satisfy the Dirichlet conditions to guarantee that it can be represented by a Fourier series.

Gibbs phenomenon: as N increases, the ripples in the partial sums become compressed toward the discontinuity, but for any finite value of N, the peak amplitude of the ripples remains constant.

4. Properties of Continous-Time Fourier Series

We will use the notation $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ to signify the pairing of a periodic signal with its Fourier series coefficients.

Linearity $z(t) = Ax(t) + By(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k$

Time Shifting $x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} \mathrm{e}^{-\mathrm{j}k\omega_0 t_0}a_k = \mathrm{e}^{-\mathrm{j}k\frac{2\pi}{T}t_0}a_k$

When a periodic signal is shifted in time, the magnitudes of its Fourier series coefficients remain unaltered.

Time Reversal $x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}$

Time reversal applied to a continuous-time signal results in a time reversal of the corresponding sequence of Fourier series coefficients.

Time Scaling $x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k \mathrm{e}^{\mathrm{j}k(\alpha\omega_0)t_0}$

Multiplication $x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$

Conjugation $x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}$

Parseval's Relation for Continuous-Time Periodic Signals

$$\frac{1}{T}\int_T |a_k e^{jk\omega_0 t_0}|^2 \, dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

5. Fourier Series Representation of Discrete-Time Periodic Signals

5.1 linear Combinations of Harmonically Related Complex Exponentials

the set of **harmonically related** discrete-time complex exponentials

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N}n} \qquad k = 0, \pm 1, \pm 2 ...$$

there are only N distinct signals in the set $\phi_k[n] = \phi_{k+rN}[n]$
a linear combination has the form

$$x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n}$$

This is referred to as the **discrete-time Fourier series** and the coefficients a_k as the **Fourier series coefficients**.

5.2 Determination of the Fourier Series Representation of a Periodic Signal

The sum over one period of the values of a periodic complex exponential is zero, unless that complex exponential is a constant.

$$\sum_{n=\langle N \rangle} e^{jk\frac{2\pi}{N}n} = \begin{cases} N, & k = 0, \ \pm N, \ \pm 2N, \ \dots, \\ 0, & \text{otherwise.} \end{cases}$$

then we have the **discrete-time Fourier series pair**
the **synthesis** equation

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n}$$

the **analysis** equation

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \mathrm{e}^{-\mathrm{j}k\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \mathrm{e}^{-\mathrm{j}k\frac{2\pi}{N}n}$$

and $a_k = a_{k+N}$

6. Properties of Discrete-Time Fourier Series

There are strong similarities between the properties of discrete-time and continuous-time Fourier series.
if $x[n]$ is a periodic signal with period N and with Fourier series coefficients denoted by a_k then we will write $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$
Multiplication $x[n]y[n] \overset{\mathcal{FS}}{\longleftrightarrow} d_k = \sum_{l=\langle N \rangle} a_l b_{k-l}$
We refer to this type of operation as a **periodic convolution** between the two periodic sequences of Fourier coefficients.
First Difference $x[n] - x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} (1 - \mathrm{e}^{-\mathrm{j}k\frac{2\pi}{N}})a_k$
Parseval's Relation for Discrete-Time Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

The average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

7. Fourier Series and LTI Systems

we have

$$y(t) = H(s)\mathrm{e}^{st} \qquad H(s) = \int_{-\infty}^{+\infty} h(\tau)\mathrm{e}^{-s\tau}\,\mathrm{d}\tau$$

and

$$y[n] = H(z)z^n \qquad H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

When s or z are general complex numbers, $H(s)$ and $H(z)$ are referred to as the **system functions** of the corresponding systems.

For continuous-time signals and systems, the system function of the form $s = \mathrm{j}\omega$ is referred to as the **frequency response** of the system and is given by

$$H(\mathrm{j}\omega) = \int_{-\infty}^{+\infty} h(t)\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t$$

Similarly, for discrete-time signals and systems, the system function of the form $z = \mathrm{e}^{\mathrm{j}\omega}$ is referred to as the **frequency response** of the system and is given by

$$H(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=-\infty}^{+\infty} h[n]\mathrm{e}^{-\mathrm{j}\omega n}$$

Consider first the continuous-time case,

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \mathrm{e}^{\mathrm{j}k\omega_0 t}$$

then

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(\mathrm{j}k\omega_0)\mathrm{e}^{\mathrm{j}k\omega_0 t}$$

In discrete time,

$$x[n] = \sum_{k=\langle N\rangle} a_k \mathrm{e}^{\mathrm{j}k\frac{2\pi}{N}n}$$

then

$$y[n] = \sum_{k=\langle N\rangle} a_k H(\mathrm{e}^{\mathrm{j}\frac{2\pi k}{N}})\mathrm{e}^{\mathrm{j}k\frac{2\pi}{N}n}$$

The effect of the LTI system is to modify individually each of the Fourier coefficients of the input through multiplication by the value of the frequency response at the corresponding frequency.

8. Filtering

Linear time-invariant systems that change the shape of the spectrum are often referred to as **frequency-shaping filters**.

Systems that are designed to pass some frequencies essentially undistorted and significantly attenuate or eliminate others are referred to as **frequency-selective filters**.

For example, a continuous-time **ideal lowpass filter** with cutoff frequency ω_c