Signals and Systems

1. Continuous-Time and Discrete-Time Signals

continuous-time signals $x(t)$
discrete-time signals $x[n]$

1.1 Signal Energy and Power

the total energy

$$E_\infty \triangleq \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2\,\mathrm{d}t=\int_{-\infty}^{\infty}|x(t)|^2\,\mathrm{d}t$$

$$E_\infty \triangleq \lim_{N\to\infty}\sum_{n=-N}^{+N}|x[n]|^2=\sum_{n=-\infty}^{+\infty}|x[n]|^2$$

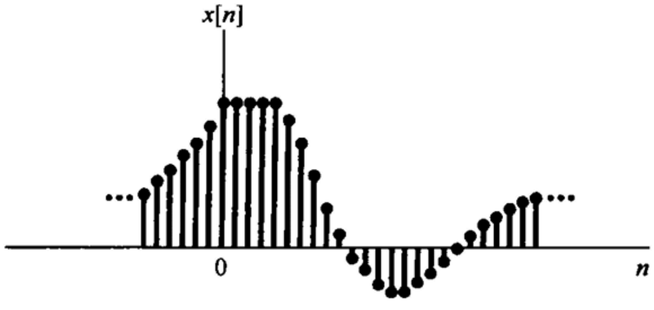
the time-averaged power

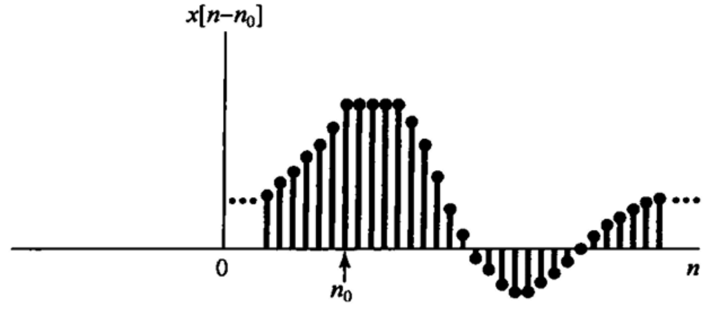
$$P_\infty \triangleq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\,\mathrm{d}t$$

$$P_\infty \triangleq \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{+N}|x[n]|^2$$

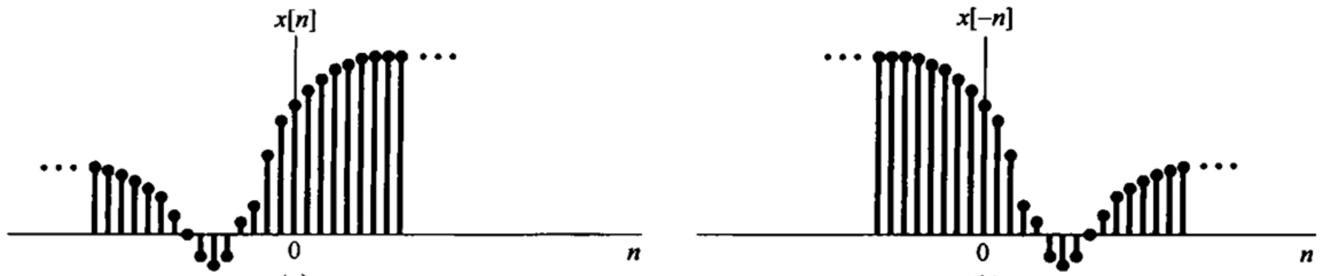
2. Transfermations of the Independent Variable

time shift $x[n]$ and $x[n-n_0]$



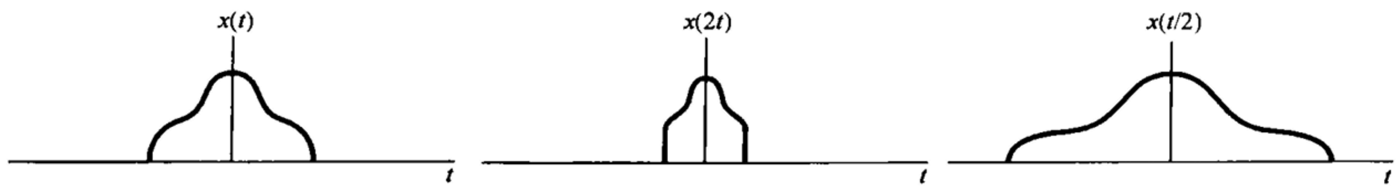


time reversal $x[-n]$



(a) (b)

time scaling $x(2t)$ and $x(\frac{t}{2})$



It is often of interest to determine the effect of transforming the independent variable of $x(t)$ to $x(\alpha t + \beta)$, where α and β are given numbers.

2.1 Periodic Signals

$x(t) = x(t+T)$
The **fundamental period** T_0 of $x(t)$ is the smallest positive value of T for which it holds.
$x[n] = x[n+N]$
The **fundamental period** N_0 of $x[n]$ is the smallest positive value of N for which it holds.
A signal that is not periodic will be referred to as an **aperiodic** signal.

2.2 Even and Odd Signals

even signals $x(-t) = x(t)$ and $x[-n] = x[n]$
odd signals $x(-t) = -x(t)$ and $x[-n] = -x[n]$

any signal can be broken into a sum of two signals, one of which is even and one of which is odd
the **even part** of $x(t)$: $\mathcal{E}_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$
the **odd part** of $x(t)$: $\mathcal{O}_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$

3. Exponential and Sinusoidal Signals

3. 1 Continuous-Time Complex Exponential and Sinusoidal Signals

The continuous-time complex exponential signal is of the form $x(t) = C\mathrm{e}^{at}$.
real exponential signals C and a are real
periodic complex exponential signals $x(t) = \mathrm{e}^{\mathrm{j}\omega_0 t}$
it is periodic, $T_0 = \frac{2\pi}{\omega_0}$
sinusoidal signals $x(t) = A\cos(\omega_0 t + \phi)$
Euler's relation $\mathrm{e}^{\mathrm{j}\omega_0 t} = \cos(\omega_0 t) + \mathrm{j}\sin(\omega_0 t)$

fundamental frequency $|\omega_0|$
harmonically related complex exponential $\phi_k(t) = \mathrm{e}^{\mathrm{j}k\omega_0 t} \quad k = 0, \pm 1, \pm 2, ...$

general complex exponential signals
C expressed in polar form and a in rectangular form
we have $C = |C|\mathrm{e}^{\mathrm{j}\theta}$ and $a = r + \mathrm{j}\omega_0$
then $C\mathrm{e}^{at} = |C|\mathrm{e}^{rt}\mathrm{e}^{\mathrm{j}(\omega_0 t + \theta)}$
Sinusoidal signals multiplied by decaying exponentials are commonly referred to as **damped sinusoids**.

3.2 Discrete-Time Complex Exponential and Sinusoidal Signals

The discrete-time complex exponential signal is of the form $x[n] = C\alpha^n$.
real exponential signals C and α are real
sinusoidal signals $|\alpha| = 1$
then we have $x[n] = A\cos(\omega_0 n + \phi)$
general complex exponential signals
we have $C = |C|\mathrm{e}^{\mathrm{j}\theta}$ and $\alpha = |\alpha|\mathrm{e}^{\mathrm{j}\omega_0}$
then $C\alpha^n = |C||\alpha|^n\cos(\omega_0 n + \theta) + \mathrm{j}|C||\alpha|^n\sin(\omega_0 n + \theta)$

$\mathrm{e}^{\mathrm{j}(\omega_0 + 2\pi)n} = \mathrm{e}^{\mathrm{j}\omega_0 n}$
The exponential at frequency $\omega_0 + 2\pi$ is the same as that at frequency ω_0,therefore,we need only consider a frequency interval of length 2π in which to choose ω_0.
The low-frequency discrete-time exponentials have values of ω_0 near $2k\pi \quad k \in \mathbb{Z}$, while the high frequencies are located near $\omega_0 = (2k+1)\pi \quad k \in \mathbb{Z}$

The signal $\mathrm{e}^{\mathrm{j}\omega_0 n}$ is periodic if $\frac{\omega_0}{2\pi} = \frac{m}{N}$ is a rational number and is not periodic otherwise.
its fundamental frequency is $\frac{2\pi}{N} = \frac{\omega_0}{m}$

harmonically related complex exponential $\phi_k[n] = \mathrm{e}^{\mathrm{j}k\frac{2\pi}{N}n} \quad k = 0, \pm 1, \pm 2, \ldots$
and $\phi_{k+N}[n] = \phi_k[n]$, which implies that there are only N distinct periodic exponentials in the set

4. The Unit Impulse and Unit Step Functions

4.1 The Discrete-Time Unit Impulse and Unit Step Sequences

unit impulse or **unit sample**

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

unit step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

There is a close relationship between the discrete-time unit impulse and unit step.

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] = \sum_{k=0}^{n} \delta[n-k]$$

and

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

4.2 The Continuous-Time Unit Step and Unit Impulse Functions

unit step function

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

unit impluse function

$$\delta(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t}$$

similarly,we have

$$u(t)=\int_{-\infty}^{t} \delta(t)\,\mathrm{d}t=\int_{0}^{\infty} \delta(t-\sigma)\,\mathrm{d}\sigma$$

$$x(t)\delta(t-t_0)=x(t_0)\delta(t-t_0)$$

5. Continuous-Time and Discrete-Time Systems

continuous-time system $x(t)\rightarrow y(t)$
discrete-time systems $x[n]\rightarrow y[n]$

6. Basic System Properities

A system is said to be **memoryless** if its output for each value of the independent variable at a given time is dependent only on the input at that same time.
for example $y(t)=x(t)$
An example of a discrete-time system **with memory** is $y[n]=x[n-1]$.

A system is said to be **invertible** if distinct inputs lead to distinct outputs.
for example $y(t)=2x(t)$, for which the inverse system is $w(t)=\frac{1}{2}y(t)$
Examples of **noninvertible** systems are $y[n]=0$

A system is **causal** if the output at any time depends only on values of the input at the present time and in the past.

A system is **stable** if small inputs lead to responses that do not diverge.

A system is **time invariant** if the behavior and characteristics of the system are fixed over time.

A system is **linear** if:
1.The response to $x_1(t)+x_2(t)$ is $y_1(t)+y_2(t)$. (additivity)
2.The response to $ax_1(t)$ is $ay_1(t)$, where a is any complex constant. (homogeneity)

superposition property

$$x[n]=\sum_k a_k x_k[n]\rightarrow y[n]=\sum_k a_k y_k[n]$$