The Continuous-Time Fourier Transform

1. Representation of Aperiodic Signals: the Continuous-Time Fourier Transform

Fourier transform pair:

Fourier Transform or **Fourier integral**

$$X(\mathrm{j}\omega) = \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t$$

inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega$$

The transform $X(\mathrm{j}\omega)$ of an aperiodic signal $x(t)$ is commonly referred to as the **spectrum** of $x(t)$.

Convergence of Fourier Transforms: the Dirichlet conditions

1. $x(t)$ be absolutely integrable.
2. $x(t)$ have a finite number of maxima and minima within any finite interval.
3. $x(t)$ have a finite number of discontinuities within any finite interval. Futhermore, each of these discontinuities must be finite.

sine functions $\operatorname{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$

2. The Fourier Transform for Periodic Signals

we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \mathrm{e}^{\mathrm{j}k\omega_0 t}$$

$$X(\mathrm{j}\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

The Fourier transform of a periodic signal with Fourier series coefficients a_k can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the kth harmonic frequency $k\omega_0$ is 2π times the kth Fourier series coefficient a_k.

3. Properties of the Continuous-Time Fourier Transform

We will refer to $x(t)$ and $X(\mathrm{j}\omega)$ as a Fourier transform pair with the notation $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\mathrm{j}\omega)$
Linearity $ax(t)+by(t) \overset{\mathcal{F}}{\longleftrightarrow} aX(\mathrm{j}\omega)+bY(\mathrm{j}\omega)$
Time Shifting $x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{e}^{-\mathrm{j}\omega t_0}X(\mathrm{j}\omega)$
One consequence of the time-shift property is that a signal which is shifted in time does not have the magnitude of its Fourier transform altered.
Conjugation and Conjugate Symmetry $x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-\mathrm{j}\omega)$
and if $x(t)$ is real, then $X(-\mathrm{j}\omega)=X^*(\mathrm{j}\omega)$
If $x(t)$ is both real and even, then $X(\mathrm{j}\omega)$ will also be real and even.
If $x(t)$ is a real and odd function of time, then $X(\mathrm{j}\omega)$ is purely imaginary and odd.
Differentiation and Integration $\frac{\mathrm{d}x}{\mathrm{d}t} \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{j}\omega X(\mathrm{j}\omega)$ and $\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{\mathrm{j}\omega}X(\mathrm{j}\omega)+\pi X(0)\delta(\omega)$
Time and Frequency Scaling $x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X(\frac{\mathrm{j}\omega}{a})$
letting $a=-1$, then $x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-\mathrm{j}\omega)$
Duality for any transform pair, there is a dual pair with the time and frequency variables interchanged.
we have $-\mathrm{j}tx(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\mathrm{d}X(\mathrm{j}\omega)}{\mathrm{d}\omega}$, $\mathrm{e}^{\mathrm{j}\omega_0 t}x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\mathrm{j}(\omega-\omega_0))$ and $-\frac{1}{\mathrm{j}t}x(t)+\pi x(0)\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)\,\mathrm{d}\eta$
Parseval's Relation $\int_{-\infty}^{+\infty} |x(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\mathrm{j}\omega)|^2\,\mathrm{d}\omega$
$|X(\mathrm{j}\omega)|^2$ is often referred to as the **energy-density spectrum** of the signal $x(t)$.

4. The Convolution Property

$$y(t)=h(t)*x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\mathrm{j}\omega)=H(\mathrm{j}\omega)X(\mathrm{j}\omega)$$

The Fourier transform maps the convolution of two signals into the product of their Fourier transforms.

5. The Multiplication Property

$$r(t) = s(t)p(t) \stackrel{\mathcal{F}}{\longleftrightarrow} R(\mathrm{j}\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\mathrm{j}\theta)P(\mathrm{j}(\omega-\theta))\,\mathrm{d}\theta$$

The multiplication of two signals is often referred to as **amplitude modulation**, then the equation is sometimes referred to as the **modulation property**.

6. Basic Fourier Transform Pairs

$$\mathrm{e}^{\mathrm{j}\omega t} \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)$$

$$\cos(\omega_0 t) \stackrel{\mathcal{F}}{\longleftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

$$\sin(\omega_0 t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{\mathrm{j}}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$$

$$\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 1$$

$$\sum_{n=-\infty}^{+\infty}\delta(t-nT) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega-\frac{2\pi k}{T})$$

$$\frac{\sin Wt}{\pi t} \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{j}\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

$$u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{\mathrm{j}\omega} + \pi\delta(\omega)$$

$$\mathrm{e}^{-at}u(t),\ \mathcal{Re}\{a\} > 0 \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+\mathrm{j}\omega}$$

7. Characterized by Linear Constant-Cofficient Differential Equations

for the input and output satisfy a linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}$$

the frequency response of such an LTI system is:

$$H(\mathrm{j}\omega) = \frac{Y(\mathrm{j}\omega)}{X(\mathrm{j}\omega)} = \frac{\sum_{k=0}^{M} b_k(\mathrm{j}\omega)^k}{\sum_{k=0}^{N} a_k(\mathrm{j}\omega)^k}$$