The Continuous-Time Fourier Transform

1. Representation of Aperiodic Signals: the Continuous-Time Fourier Transform

Fourier transform pair:

Fourier Transform or **Fourier integral**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\, dt$$

inverse Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t}\, d\omega$$

The transform $X(j\omega)$ of an aperiodic signal $x(t)$ is commonly referred to as the **spectrum** of $x(t)$.

Convergence of Fourier Transforms: the Dirichlet conditions

1. $x(t)$ be absolutely integrable.
2. $x(t)$ have a finite number of maxima and minima within any finite interval.
3. $x(t)$ have a finite number of discontinuities within any finite interval. Futhermore, each of these discontinuities must be finite.

sine functions $\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$

2. The Fourier Transform for Periodic Signals

we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

The Fourier transform of a periodic signal with Fourier series coefficients a_k can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the kth harmonic frequency $k\omega_0$ is 2π times the kth Fourier series coefficient a_k.

3. Properties of the Continuous-Time Fourier Transform

We will refer to $x(t)$ and $X(j\omega)$ as a Fourier transform pair with the notation $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

Linearity $ax(t) + by(t) \overset{\mathcal{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega)$

Time Shifting $x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$

One consequence of the time-shift property is that a signal which is shifted in time does not have the magnitude of its Fourier transform altered.

Conjugation and Conjugate Symmetry $x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)$

and if $x(t)$ is real, then $X(-j\omega) = X^*(j\omega)$

If $x(t)$ is both real and even, then $X(j\omega)$ will also be real and even.

If $x(t)$ is a real and odd function of time, then $X(j\omega)$ is purely imaginary and odd.

Differentiation and Integration $\frac{dx}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$ and $\int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$

Time and Frequency Scaling $x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X(\frac{j\omega}{a})$

letting $a = -1$, then $x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-j\omega)$

Duality for any transform pair, there is a dual pair with the time and frequency variables interchanged.

we have $-jtx(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}$, $e^{j\omega_0 t} x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0))$ and $-\frac{1}{jt} x(t) + \pi x(0)\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)\, d\eta$

Parseval's Relation $\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2\, d\omega$

$|X(j\omega)|^2$ is often referred to as the **energy-density spectrum** of the signal $x(t)$.

4. The Convolution Property

$$y(t) = h(t) * x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

The Fourier transform maps the convolution of two signals into the product of their Fourier transforms.

5. The Multiplication Property

$$r(t) = s(t)p(t) \xleftrightarrow{\mathcal{F}} R(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(j\theta)P(j(\omega - \theta)) \, d\theta$$

The multiplication of two signals is often referred to as **amplitude modulation**, then the equation is sometimes referred to as the **modulation property**.

6. Basic Fourier Transform Pairs

$$e^{j\omega t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0)$$

$$\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$

$$\sum_{n=-\infty}^{+\infty} \delta(t - nT) \xleftrightarrow{\mathcal{F}} \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$\frac{\sin Wt}{\pi t} \xleftrightarrow{\mathcal{F}} X(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

$$u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} + \pi\delta(\omega)$$

$$e^{-at}u(t), \ \mathcal{Re}\{a\} > 0 \xleftrightarrow{\mathcal{F}} \frac{1}{a + j\omega}$$