Mathematical Preliminaries

1. Sets

usually designated by enumeration of its elements between braces
such as $\{2, 4, 6, 8\}$
or denoted by $\{x \mid P(x)\}$, for example $\{n \mid n \text{ is even and } 1 < n < 9\}$

a set with a single element is called a **singleton**

subset $B \subset A$
proper subset $B \subset A$ and $B \neq A$
empty set is denoted by $\varnothing$, such as $\{a \mid a \neq a\}$

union $A \cup B$
intersection $A \cap B$
complement $\sim A$ and $A \sim B \equiv \{a \mid a \in A \text{ and } a \notin B\}$

universal set In any application of set theory there is an underlying universal set X whose subsets are the objects of study

Cartesian product $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
It is the set of **ordered pairs** (a, b)
for example $A^n = \{(a_1, a_2, ..., a_n) \mid a_i \in A\}$

1.1 Equivalence Relations

A **relation** on A is a comparison test between members of ordered pairs of elements of A
denoted by $a \triangleright b$
An **equivalence relation** on A is a relation that has the following properties:

$$a \triangleright a \quad \forall a \in A \quad \text{(reflexivity)}$$

$$a \triangleright b \Rightarrow b \triangleright a \quad a, b \in A \quad \text{(symmetry)}$$

$$a \triangleright b,\ b \triangleright c \Rightarrow a \triangleright c \quad a, b, c \in A \quad \text{(transivity)}$$

equivalence class $[\![a]\!] = \{b \in A \mid b \triangleright a\}$
If $\triangleright$ is an equivalence relation on A and $a, b \in A$,then either $[\![a]\!] \cap [\![b]\!] = \varnothing$ or $[\![a]\!] = [\![b]\!]$

partition of a set B_α are disjoint and $\bigcup_\alpha B_\alpha = A$
quotient set $A/\bowtie$ is the collection of all equivalence classes of A
clearly,it is a partition of A

2. Maps

map $f : X \longrightarrow Y$ or $X \xrightarrow{f} Y$
and $y = f(x)$ or $x \mapsto f(x)$ or $x \overset{f}{\mapsto} y$
the set X is called the **domain**, and Y the **codomain**
A map whose codomain is the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$ is commonly called a **function**.

identity map $\mathrm{id}_{\mathrm{A}}(a) = a \quad \forall a \in A$
graph of a map $\mathit{\Gamma}_f = \{(a, f(a)) \mid a \in A\} \subset A \times B$

If A is a subset of X, we call $f(A) = \{f(x) \mid x \in A\}$ the **image** of A.
Similarly, if $B \subset f(X)$, we call $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ the **preimage**.
The subset $f(X)$ of the codomain of a map f is called the **range** of f.
composition $h(x) = g(f(x))$ or $h = g \circ f$

injective or **one-to-one** $f(x_1) = f(x_2)$ implies that $x_1 = x_2$
surjective or **onto** $f(X) = Y$
bijective or **one-to-one correspondence** a map that is both injective and surjective
inverse of a map $f^{-1}(y) = x$, f is a bijection from X onto Y

define an equivalence relation $\bowtie$ on X by saying $x_1 \bowtie x_2$ if $f(x_1) = f(x_2)$, then there is a map $\tilde{f} : X/\bowtie \longrightarrow Y$,called **quotient map**, given by $\tilde{f}[\![x]\!] = f(x)$, which is bijective

binary operation $f : X \times X \longrightarrow X$

3. Metric Spaces

A **metric space** is a set X together with a real-valued function $d : X \times X \longrightarrow \mathbb{R}$ such that

$$d(x, y) \geq 0 \quad \forall x, y, \text{ and } d(x, y) = 0 \text{ iff } x = y$$

$$d(x, y) = d(y, x) \quad \text{(symmetry)}$$

$$d(x, y) \leq d(x, z) + d(z, y) \quad \text{(the triangle inequality)}$$

sequence $s : \mathbb{N} \longrightarrow X$, X is a metric space
the sequence $\{x_n\}_{n=1}^{\infty}$ **converges** to x if there exists $N \in \mathbb{N}$ such that $\forall \epsilon \in \mathbb{R}^+, \quad d(x_n, x) < \epsilon$ whenever $n > N$
write it $\lim_{n \to \infty} d(x_n, x) = 0$ or simply $x_n \to x$
Cauchy sequence $\lim_{m,n \to \infty} d(x_m, x_n) = 0$
complete metric space every Cauchy sequence converges

4. Cardinality

If two sets are in one-to-one correspondence, they are said to have the same **cardinality**.
A is said to be **countably infinite** if there exists a bijection between A and $\mathbb{N}$.
Sets that are neither finite nor countably infinite are said to be **uncountable**.

5. Mathematical Induction

S_n is true for every positive integer provided the following two conditions hold:

1. S_1 is true.
2. If S_m is true for some given positive integerm,then S_{m+1} is also true.