The Discrete-Time Fourier Transform

1. Representation of Aperiodic Signals: the Discrete-Time Fourier Transform

the discrete-time Fourier transform pair

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega$$

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\mathrm{e}^{-\mathrm{j}\omega n}$$

The function $X(\mathrm{j}\omega)$ is referred to as the **discrete-time Fourier transform**, and will often be referred to as the **spectrum** of $x[n]$.
$X(\mathrm{j}\omega)$ is periodic with period 2π

If $\sum_{n=-\infty}^{+\infty}|x[n]| < \infty$ or $\sum_{n=-\infty}^{+\infty}|x[n]|^2 < \infty$, then $X(\mathrm{j}\omega)$ will converge.

2. The Fourier Transform For Periodic Signals

consider a periodic sequence $x[n]$ with period Nand with the Fourier series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k \mathrm{e}^{\mathrm{j}k\frac{2\pi}{N}n}$$

the Fourier transform is

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})$$

3. Properties of the Discrete-Time Fourier Transform

it will be convenient to adopt notation $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}\omega})$
Periodicity of the Discrete-Time Fourier Transform $X(\mathrm{e}^{\mathrm{j}(\omega+2\pi)}) = X(\mathrm{e}^{\mathrm{j}\omega})$
Linearity of the Fourier Transform $ax_1[n] + bx_2[n] \overset{\mathcal{F}}{\longleftrightarrow} aX_1(\mathrm{e}^{\mathrm{j}\omega}) + bX_2(\mathrm{e}^{\mathrm{j}\omega})$

Time Shifting and Frequency Shifting $x[n-n_0] \stackrel{\mathcal{F}}{\longleftrightarrow} \mathrm{e}^{-\mathrm{j}\omega n_0}X(\mathrm{e}^{\mathrm{j}\omega})$ and $\mathrm{e}^{\mathrm{j}\omega_0 n}x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}(\omega-\omega_0)})$

Conjugation and Conjugate Symmetry $x^*[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(\mathrm{e}^{-\mathrm{j}\omega})$

if $x[n]$ is real valued, then $X(\mathrm{e}^{\mathrm{j}\omega}) = X^*(\mathrm{e}^{-\mathrm{j}\omega})$

Differencing and Accumulation $x[n]-x[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} (1-\mathrm{e}^{-\mathrm{j}\omega})X(\mathrm{e}^{\mathrm{j}\omega})$ and $\sum_{m=-\infty}^{+\infty} x[m] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\mathrm{e}^{-\mathrm{j}\omega}}X(\mathrm{e}^{\mathrm{j}\omega}) + \pi X(\mathrm{e}^{\mathrm{j}0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$

Time Reversal $x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{-\mathrm{j}\omega})$

Time Expansion let k be a positive integer, and define the signal

$$x_{(k)}[n] = \begin{cases} x\left[\dfrac{n}{k}\right], & \text{if } n \text{ is a multiple of } k, \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases}$$

then $x_{(k)}[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}k\omega})$

Differentiation in Frequency $nx[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \mathrm{j}\frac{\mathrm{d}X(\mathrm{e}^{\mathrm{j}\omega})}{\mathrm{d}\omega}$

Parseval's Relation $\sum_{n=-\infty}^{+\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{2\pi}|X(\mathrm{e}^{\mathrm{j}\omega})|^2\,\mathrm{d}\omega$

$|X(\mathrm{e}^{\mathrm{j}\omega})|^2$ is referred to as the **energy-density spectrum** of the signal $x[n]$.

4. The Convolution Property

$$y[n] = x[n] * h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Y(\mathrm{e}^{\mathrm{j}\omega}) = X(\mathrm{e}^{\mathrm{j}\omega})H(\mathrm{e}^{\mathrm{j}\omega})$$

As in continuous time, it maps the convolution of two signals to the simple algebraic operation of multiplying their Fourier transforms.

5. The Multiplication Property

$$y[n] = x_1[n]x_2[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Y(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(\mathrm{e}^{\mathrm{j}\theta})X_2(\mathrm{e}^{\mathrm{j}(\omega-\theta)})\,\mathrm{d}\theta$$

It corresponds to a **periodic convolution** of $X_1(\mathrm{e}^{\mathrm{j}\omega})$ and $X_2(\mathrm{e}^{\mathrm{j}\omega})$.

6. Basic Discrete-Time Fourier Transform Pairs

$$\mathrm{e}^{\mathrm{j}\omega_0 n} \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi l)$$

$$\cos(\omega_0 n) \stackrel{\mathcal{F}}{\longleftrightarrow} \pi \sum_{l=-\infty}^{+\infty} [\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)]$$

$$\sin(\omega_0 n) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{\mathrm{j}} \sum_{l=-\infty}^{+\infty} [\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)]$$

$$\delta[n] \stackrel{\mathcal{F}}{\longleftrightarrow} 1$$

$$\sum_{k=-\infty}^{+\infty} \delta[n - kN] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta(\omega - \frac{2\pi k}{N})$$

$$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right), \qquad 0 < W < \pi \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) = \begin{cases} 1, & 0 \le |\omega| \le W, \\ 0, & W < |\omega| \le \pi, \end{cases}$$

$$u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \mathrm{e}^{-\mathrm{j}\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$$

$$a^n u[n], \quad |a| < 1 \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}}$$

7. Duality

Duality in the Discrete-Time Fourier Series
if $f[m] = \frac{1}{N} \sum_{r=\langle N \rangle} g[r] \mathrm{e}^{-\mathrm{j}r\frac{2\pi}{N}m}$, then $g[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} f[k]$ and $f[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} \frac{1}{N} g[-k]$

Duality between the Discrete-Time Fourier Transform and the Continuous-Time Fourier Series

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\mathrm{e}^{\mathrm{j}\omega}) \mathrm{e}^{\mathrm{j}\omega n} \, \mathrm{d}\omega \longleftrightarrow a_k = \frac{1}{T} \int_T x(t) \mathrm{e}^{-\mathrm{j}k\omega_0 t} \, \mathrm{d}t$$

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=-\infty}^{+\infty} x[n] \mathrm{e}^{-\mathrm{j}\omega n} \longleftrightarrow x(t) = \sum_{k=-\infty}^{+\infty} a_k \mathrm{e}^{\mathrm{j}k\omega_0 t}$$

8. Systems Characterized by Linear Constant-Coefficient Difference Equations

A general linear constant-coefficient difference equation for an LTI system with input $x[n]$ and output $y[n]$ is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

we have

$$H(\mathrm{e}^{\mathrm{j}\omega}) = \frac{Y(\mathrm{e}^{\mathrm{j}\omega})}{X(\mathrm{e}^{\mathrm{j}\omega})} = \frac{\sum_{k=0}^{M} b_k \mathrm{e}^{-\mathrm{j}k\omega}}{\sum_{k=0}^{N} a_k \mathrm{e}^{-\mathrm{j}k\omega}}$$