The Two-Body Problem

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June 4, 2025

This is one of the physics problems of all time; it is well-understood theoretically and has been investigated empirically for centuries. Most physics undergrads will have studied it in their mechanics courses by their third year, but we will give a little preview here to bring us up to speed. Empirically, massive bodies are seen to exert an influence over each other; this can be understood as the **gravitational force**, and can be understood within the framework established by Isaac Newtonian and his contemporaries.

Below we will review the Newtonian model of gravity and identify how we use it to understand the motion of two bodies influenced by their respective gravity. Our objective is to *determine* their motion. You might already be picturing plotting out their motion in space, such as on a Cartesian grid where the bodies move over time. But let's be careful to understand what "results" we'll be looking for. Do we want (x,y,z)-coordinates over time? Or are there other, more useful coordinates? To decide, let's first understand our model.

1 Theoretical Overview

Adapted from Section 3.2 of Gravity by Poisson and Will, Cambridge University Press.

The Newtonian model of gravity states that a body of inertial mass m_1 exerts a gravitational force $\vec{F}_{\rm g}$ on a body of mass m_2,

$$\vec{F}_{\rm g} = -\frac{G m_1 m_2}{r^2}\hat{r}, \tag{1}$$

where G is the gravitational constant, r the distance between the bodies, and $\hat{r} \equiv \vec{r}/|\vec{r}|$ the unit vector from body 2 to body 1; that is, if the position of the ith body is $\vec{r}_i$, then $\vec{r} = \vec{r}_1 - \vec{r}_2$ is their *relative separation*. Body 2 also exerts the same force on 1, just in the opposite direction. A *center of mass* (also known as the barycenter) can then be defined between the bodies,

$$\vec{R} = \frac{m_1}{m}\vec{r}_1 + \frac{m_2}{m}\vec{r}_2, \tag{2}$$

where $m = m_1 + m_2$ is the total mass. From Newton's second law it can be shown that $d^2\vec{R}/dt^2 = 0$, so we can set $\vec{R} = 0$ as the origin. Together, the definitions of the relative separation and the center of mass, along with Newton's second law, give the *relative acceleration*,

$$\ddot{\vec{r}} = -\frac{Gm}{r^2}\hat{r}. \tag{3}$$

The two-body problem has now been reduced to an effective one-body problem.

Let's refresh on where we want to be. We want a description of the motion of the two bodies; in terms of the relative separation, they are found from the above to be

$$\vec{r}_1 = \frac{m_2}{m}\vec{r}, \quad \vec{r}_2 = -\frac{m_1}{m}\vec{r}. \tag{4}$$

To obtain the values of these vectors, we then have to solve the differential equation for $\vec{r}$. In principle one could then solve for the Cartesian components of $\vec{r}$. Certain simplifications happen, however, if we use coordinates that better represent the symmetries of our problem. First, we notice that the acceleration is purely along the separation vector. This implies that the motion of the two bodies is constrained to a 2D plane of the 3D space we started with, and that it does not change over time. Setting up coordinates that reflect this, we have two polar basis vectors on the plane,

$$\hat{n} = [\cos\phi, \sin\phi, 0], \quad \hat{\lambda} = [-\sin\phi, \cos\phi, 0], \tag{5}$$

and a unit vector perpendicular to the plane, $\hat{z}$. We now have two degrees of freedom, r and ϕ, and can write the separation vector as $\vec{r} = r\hat{n}$. Noting that

$$\frac{d\hat{n}}{d\phi} = \hat{\lambda}, \quad \frac{d\hat{\lambda}}{d\phi} = -\hat{n}, \tag{6}$$

we can write the time derivatives of $\vec{r}$ in terms of time derivatives of r and ϕ. The relative acceleration becomes

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{n} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\phi})\hat{\lambda}. \tag{7}$$

We can match it to $-Gm/r^2\,\hat{r}$ to obtain the equations of motion (EOM). The symmetry we noticed earlier now appears explicitly: $r^2\dot{\phi}$ is the z-component of the (reduced) angular momentum vector $\vec{l} = \vec{r} \times \vec{v}$ and from the EOM is seen to be *conserved*, that is, is a constant in time. This constant l can be substituted into the radial equation to give

$$\ddot{r} - \frac{l^2}{r^3} = -\frac{Gm}{r^2}. \tag{8}$$

Along with $\dot{\phi} = l/r^2$, we now have the necessary equations of motion. We could try implementing these into an IVP solver, but it turns out the radial equation can be solved in terms of ϕ:

$$r(\phi) = \frac{p}{1 + e\cos(\phi - \omega)}. \tag{9}$$

Here, e and ω are constants of integration known respectively as the **eccentricity** and the **longitude of pericenter**, and $p \equiv l^2/(Gm)$ is known as the **semi-latus rectum**. All of these names allude to the fact that $r(\phi)$ describes a conic section, specifically an ellipse when $0 \leq e < 1$, as is the case for *bound orbits*. We note that the motion has a period in ϕ ($\Delta\phi = 2\pi$). Using this conic solution, we can substitute it into $\dot{\phi}$ to obtain

$$\dot{\phi} = \sqrt{\frac{Gm}{p^3}}(1 + e\cos(\phi - \omega))^2. \tag{10}$$

Let us now see how we can solve this equation.

1.1 The Initial-Value Problem

The equations that we want to solve are

$$\tag{11}$$

$$\dot{\phi} = \sqrt{\frac{Gm}{p^3}}(1 + e\cos(\phi - \omega))^2, \tag{12}$$

$$r(\phi) = \frac{p}{1 + e\cos(\phi - \omega)}. \tag{13}$$

In general, we cannot write down an equation $r(t)$ or $\phi(t)$ unless $e = 0$; in this case then, $\dot{\phi}$ is constant and $r = p$ simply describes a circle. There are ways to solve for the motion approximately through an algebraic relation instead of through a differential equation (look up "Kepler's equation"), but that is not the purpose of this exercise. We want to solve this as an *initial-value problem* (IVP): given N differential equations, if we have N initial values for them, then it can be solved with generic algorithms. The C library `gsl` implements such algorithms, and so we will use it.

Let's see if our equations are a suitable IVP. There are a couple of parameters: that define our system. The main one is the total mass m, and the other is the semi-latus rectum $p = l^2/(Gm)$. For the latter, we will use a more practical definition in terms of the semi-major axis a, which is well-known for astrophysical systems:

$$p = a(1 - e^2). \tag{14}$$

We note we can find the **period** P of the orbit by integrating $\dot{\phi}$ from $\phi(t = 0) = 0$ to $\phi(t = P) = 2\pi$,

$$P = 2\pi\sqrt{\frac{a^3}{Gm}}. \tag{15}$$

We can define all the constants we need now, given a binary of total mass m, semi-major axis a, eccentricity e, and longitude ω, which are observable in practice. We note that ω is just the rotation off the $\phi = 0$ axis, and when $\phi = \omega$, the binary is at its minimum separation, or *pericenter*. Knowing values for these (such as can be found for the solar system in NASA's Planet Fact Sheets) allows us then to solve for $r(t)$ and $\phi(t)$ as an IVP; that is we can find the values of r and ϕ at specific points in t that we'd like.

1.2 Solving IVPs Numerically

To solve our IVP, we will use a common class of methods known as the *explicit Runge-Kutta* methods. The Runge-Kutta (RK) methods essentially uses the Taylor series of the function you are trying find to approximate its value at a time t_n by using points between the initial t_0 and t_n. We won't go into detail here but the web offers many descriptions of these methods, such as can be found on Wikipedia.

This section is not about these methods, but rather how to *use* these methods. Numerical methods work on taking steps from t_0 to t_n, and some, like the one we will use, try to adapt the step size to obtain good accuracy in the solution, which is measured as something related to the difference in that solution between taking a step of certain size and then taking more steps of a smaller size; if the difference is smaller than some threshold, then it is considered accurate and the method stops shrinking the step size. But how to choose this step size? How to choose the accuracy? The latter question is usually figured out by trial and error, but the first is one we can attempt to clarify.

Take a look again at our equations. We are trying to find ϕ at times $t \in [t_0, t_n]$; the step size must then be a fraction of $t_n - t_0$. But how small? A way to start this is by analyzing the units of our equations. r has units of length, while ϕ is in radians (and in a period only changes by 2π). We can thus try to get rid of these units and make our equations dimensionless. Radians are in a way not really units, so we can keep ϕ as is. How do we scale length? We could choose to scale the separation by a, but in our equations another quantity should be popping out: p. Thus, let's define the dimensionless separation $\bar{r} = r/p$. We are not done yet; we also have *time* to worry about. Nicely enough, we have a unit of time that pops out from the $\dot{\phi}$ equation: $t^* \equiv \sqrt{\frac{p^3}{Gm}}$, which for circular orbits would be the period. We thus define a dimensionless time variable $\bar{t} = t/t^*$. The differential equation thus reduces to

$$\frac{d\phi}{d\bar{t}} = (1 + e\cos(\phi - \omega))^2, \tag{16}$$

while the separation is simply $\bar{r} = (1 + e\cos(\phi - \omega))^{-1}$. To get our physical separation and time back, we just re-scale the output of the IVP solver. Fundamentally, then, you can see that a binary evolves without needing to know the actual mass or semi-latus rectum! These equations can be solved independently of those parameters, and serve only as scales.