Assignment 5

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Download latex-tikz codes from

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https://github.com/Nik123-cpp/Assignment-5/blob
    /main/Assignment5.tex
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1 UGC /MATH /2019 Q:105

Consider a simple symmetric random walk on integers ,Where from every state i you to move to i-1 and i+1 with probability half each. Then which of the following are correct?

1) The random walk is aperiodic
2) The random walk is irreducible
3) The random walk is null recurrent
4) The random walk is positive recurrent

2 SOLUTION

This is a Markov Chain ,Where the state space consists of the integers ($i = 0, \pm1, \pm2, \pm3, ...$) and transition probability is given as

$$P_{i,i+1} = p = \frac{1}{2} \tag{2.0.1}$$

$$P_{i,i-1} = q = \frac{1}{2} \tag{2.0.2}$$

Let $P^n_{i,j}$ denotes the probability of being in state j after nth transition starting from state i.

1) In a Markov Chain for state j to be recurrent then it should satisfy following condition

$$\lim_{t\to\infty} \sum_{n=1}^{t} P^n_{j,j} = \infty \tag{2.0.3}$$

after n transitions to be at state j number of forward and backward steps must be same i.e if n is odd

$$P^n_{j,j} = 0 \tag{2.0.4}$$

if n is even,Let $m = \frac{n}{2}$. we have to select m forward steps from n steps and probablility becomes

$$P^{2m}_{j,j} = \binom{2m}{m} p^m q^m \tag{2.0.5}$$

$$= \frac{(2m)!}{m!.m!} p^m q^m \tag{2.0.6}$$

using Stirling approximation in equation (2.0.6)

$$P^{2m}_{j,j} = \frac{((2m)^{2m+\frac{1}{2}}).e^{-2m}.(2\pi)^{\frac{1}{2}}}{m^{m+\frac{1}{2}}.e^{-m}.m^{m+\frac{1}{2}}.e^{-m}.2\pi}.p^m q^m \tag{2.0.7}$$

$$= \frac{(4pq)^{2m}}{(m\pi)^{\frac{1}{2}}} \tag{2.0.8}$$

In this question $p = \frac{1}{2} = q$,then using (2.0.8)

$$\lim_{t\to\infty} \sum_{n=1}^{t} P^n_{j,j} = \sum_{n=2k,k=1}^{\infty} \frac{1}{(\frac{n}{2}\pi)^{\frac{1}{2}}} \tag{2.0.9}$$

Since $\frac{1}{n^{\frac{1}{2}}}$ is divergent,

$$\lim_{t\to\infty} \sum_{n=1}^{t} P^n_{j,j} = \infty \tag{2.0.10}$$

Therefore state j is recurrent ,as what we calculated is independent of j all states are recurrent .

The first-passage-time probability, $f_{i,j}(n)$, of a Markov chain is the probability,given as

$$f_{i,j}(n) = \Pr\left(X_n = j, X_{n-1} \neq j, X_{n-2} \neq j, \dots X_1 \neq j | X_0 = i\right) \tag{2.0.11}$$

The first-passage time $T_{j,j}$from a state j back to itself is of particular importance. It has the PMF $f_{j,j}(n)$ amd Distribution function $F_{j,j}(n)$

$$F_{j,j}(n) = \sum_{k=0}^{n} f_{j,j}(k) \tag{2.0.12}$$

We Know that all states are recurrent .Now i will find whether they are null recurrent or positive recurrent . For positive recurrent

$$\overline{T_{j,j}} < \infty \tag{2.0.13}$$

For null recurrent

$$\overline{T_{j,j}} = \infty \tag{2.0.14}$$

Where $\overline{T_{j,j}}$ is mean time to enter state j starting from j.

Now calculating $\overline{T_{j,j}}$ using below formula,

$$\overline{T_{j,j}} = 1 + \sum_{k=0}^{n} (1 - F_{j,j}(k)) \tag{2.0.15}$$

Using (2.0.15) and (2.0.12),We get

$$\overline{T_{j,j}} = \infty \tag{2.0.16}$$

Therefore all states are null recurrent.

2) We know that for state j in Markov chain to be **aperiodic** ,Then their exist k such that $P^n_{j,j} > 0$ for all $n \geq k$. but we saw that for all odd n, $P^n_{j,j} = 0$.So their cannot be any such k. therefore all states are **Periodic**.

Therefore correct options are **2,3**