Neutrino Oscillations

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Abstract

Neutrino oscillations is the most reasonable explanation for the missing amount of neutrinos received from the sun and its discovery is an evidence on the neutrino mass. In this project the oscillation probabilities are calculated and it is shown that, in order for the probabilities to be non zero, neutrino must be a massive particle. In fact, it must also have different masses for each mass eigenstates. A conceptual description of the CP violation is given, along with an overview of the PMNS mixing matrix.

For a better physical understanding of this phenomenon, a system of two and three coupled pendula are illustrated. The normal coordinates of the coupled pendula can represent the mass eigenstates, and the oscillators can describe each of the flavour states. However, the analogy breaks for the case of three coupled pendula.

Contents

Chapter 1

Introduction

1.1 Solar neutrino problem

Neutrinos are elementary subatomic particles with half integer spins. They have zero electric charge and their mass is very small compared to other elementary particles. In fact for years, they were considered to be mass-less particles. Like other leptons, they do not participate in strong interactions and the effect of gravity on them is very small, so they can pass through matter without any trace. Thus, the only possible way of detecting them is through the weak force. Neutrinos have never been observed directly, they were first proposed by Wolfgang Pauli in 1930, to explain the missing momentum and energy in a beta decay [1].

There are three types of neutrino flavours, electron neutrino v_e, tau neutrino v_τ and muon neutrino v_μ, each corresponds to the charged lepton produced along with them in a weak interaction. The largest source of neutrinos on earth are due to the electron neutrinos produced from the nuclear fusions in the Sun. Different experimental techniques have been used to measure this flux of solar neutrinos. The earliest experiment, called the Homestake experiment, measured the flux of electron neutrinos to be only 1/3 of what was predicted [2]. This deficiency of electron neutrinos, unexplained for almost 30 years, became known as the solar neutrino problem [3].

In 1975, neutrino flavour oscillations was suggested by Bruno Pontecorvo, to provide a solution for the solar neutrino problem [4]. Neutrino oscillations is a quantum mechanical phenomenon which postulates, that a neutrino of a given flavour, can change to a different flavour as it propagates through space. According to quantum mechanics, there is a wave-function associated with a particle. The wave-function of a neutrino with a specific flavour, can be written as a linear superposition of the other flavours, which causes a non zero probability for the neutrino to change to a different flavour. However, this phenomenon is only possible if neutrinos are massive particles and have different masses for each mass eigenstates [2] (mass eigenstates will be explained further in section (2.2)).

Neutrino flavour oscillations were confirmed experimentally by the SNO experiment. This experiment was designed to measure, both the amount of electron neutrinos and the total neutrinos, received on earth from the sun [2]. The results from this experiment showed that, the total flux of neutrinos from the sun is in agreement with the theoretical prediction, but instead of only consisting v_e, it also consists the other flavours, v_τ and v_μ. Clearly, v_τ and v_μ cannot be produced by the nuclear fusions in the sun [2]. Therefore, this experiment illustrates that the electron neutrinos produced in the sun, went through flavour transitions on the course of their journey to the earth. This experiment gives a solid evidence on

the phenomenon of neutrino oscillations and the fact that neutrinos have non zero mass.

1.2 Some motivations

The helicity of a particle is the projection of its spin vector into the direction of motion. A particle is said to have right-handed helicity, if its spin is in the same direction as its motion. A particle is left handed, if its spin and motion are in opposite directions. In the case of massive particles, as they travel slower than the speed of light, it is always possible to move into a frame which is travelling faster than that particle. Therefore, its velocity will be in the opposite direction with respect to that frame, its spin is unchanged, so the particle to that frame will have the opposite helicity. Thus, all the massive particles must have both right-handed and left-handed helicity. Although neutrinos have been proved to have mass, but only left-handed neutrinos have been observed.

Dirac particles are all the spin 1/2 particles that can be described by the Dirac wave equation. By definition a Dirac particle has four types, left and right handed particles, left and right handed antiparticles [5]. In the case of neutral fermions, they can also be identified as a different type of particle, known as the Majorana particles, which are equal to their antiparticles. Clearly, charged fermions cannot be a Majorana, in fact the only fermion that is neutral and can be a Majorana particle is the neutrino [5].

If neutrino is a Majorana particle, it implies that neutrino and anti-neutrino are the same. It has been proposed that if this is true, one should be able to observe the neutrino-less double beta decay, $2n \rightarrow 2p + 2e^{-}$. This process starts with a neutron decaying to a proton, an electron and a neutrino, then the neutrino is absorbed by another neutron releasing a proton and an electron. This decay violates the lepton number conservation, and if observed, can provide a proof that neutrinos are in fact Majorana particles.

If neutrinos are Dirac particles, in order to explain the missing right-handed neutrinos, it has been suggested that a right-handed neutrino exists, which does not interact through the weak force. Hence, the only source of detecting them is through gravity. This is called the sterile neutrino [5]. Neutrino oscillations is a proof of its mass, and this requires the neutrino to be a Dirac or a Majorana particle, which is still an open question and is an active area of research.

Chapter 2

Theory of Neutrino Oscillations

2.1 Outline of the chapter

Neutrino oscillations is a quantum mechanical phenomenon which is described in terms of a theoretical relationship between the flavour and mass eigenstates. Mass eigenstates are the eigenvalues of the Hamiltonian describing in this case a free neutrino [2]. If a system is in an eigenstate of the Hamiltonian, then the energy of the system has a specific value which is precisely the eigenvalue of the Hamiltonian. However, if the system is described as a superposition of the eigenstates then simply it will not have a specific energy. There will be a certain probability of measuring each of the eigenstates. Which is exactly the case with neutrino oscillations, we write the mass eigenstates of a free neutrino as v_1, v_2, v_3.

Flavour eigenstates v_e, v_μ and v_τ correspond to the flavour of the charged lepton produced in a weak interaction along with a neutrino [2]. Therefore, an electron neutrino v_e is defined as the neutrino state that is produced in a weak interaction with an electron. In the next section it will be explained why it is important to distinguish between mass and flavour eigenstates and the relationship between them will be shown. Afterwards, two and three flavour oscillations will be described mathematically and a brief explanation on the mixing matrix (Pontecorvo-Maki-Nakagawa-Sakata matrix) will be given. In section (2.5) it will be explained why a wave-packet treatment is needed along with the limitations of the plane wave approximation. The calculations in section (2.3) and (2.4) follows closely the reference [2].

2.2 Mass and flavour eigenstates

It is important to be able to distinguish between the mass and flavour eigenstates. In the figure (2.1) it is shown that any of the three mass eigenstates can be produced in this weak interaction, and there is no way to know which mass eigenstate is being produced. So the system has to be described as a linear superposition of the eigenstates.

Mass and flavour states can be thought as coordinate basis and can be related by a unitary mixing matrix U (PMNS matrix):

$$\begin{pmatrix} v_e \\ v_\mu \\ v_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{2.1}$$

It can be seen that a neutrino propagates as a linear combination of mass eigenstates, when it interacts

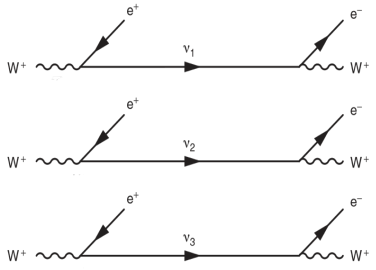

Figure 2.1: Possible weak interactions for a positron and an electron, in terms of mass eigenstates. [2]

weakly its wave-function collapses into a particular state. The equation below represents this statement:

$$|\psi\rangle = U_{e1}\,|v_1\rangle + U_{e2}\,|v_2\rangle + U_{e3}\,|v_3\rangle \tag{2.2}$$

If the masses v_1, v_2 and v_3 are not the same, there will be phase differences between their wave-functions and oscillations would occur. Hence, a neutrino produced with a specific flavour can change to a neutrino with a different flavour. For instance a neutrino produced along with an electron can interact to produce a muon [2].

2.3 Oscillations of two flavours

Let's consider the weak eigenstates v_e and v_μ which can be written as a coherent linear superposition of the mass eigenstates v_1 and v_2. It can be assumed that mass eigenstates propagate as plane waves:

$$|v_1(t)\rangle = |v_1\rangle\, e^{i(\vec{p_1}.\vec{x}-E_1 t)} = |v_1\rangle\, e^{-ip_1.x} = |v_1\rangle\, e^{-i\phi_1} \tag{2.3}$$

$$|v_2(t)\rangle = |v_2\rangle\, e^{i(\vec{p_2}.\vec{x}-E_2 t)} = |v_2\rangle\, e^{-ip_2.x} = |v_2\rangle\, e^{-i\phi_2}, \tag{2.4}$$

The phase of the wave functions can be written as:

$$\phi_i = p_i.x = E_i t - p_i x \quad \text{where} \quad i = 1, 2 \tag{2.5}$$

Where p_i is the momentum in three dimensions and E_i is the energy.

The Feynman diagrams below shows the decay of a proton into a neutron and the relation between mass and weak eigenstates.

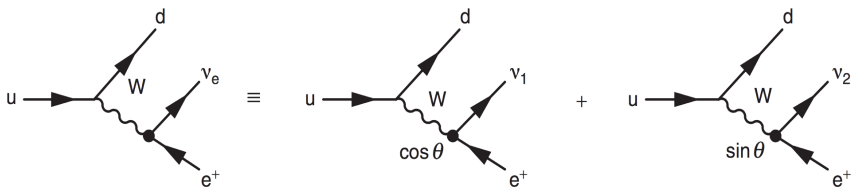

Figure 2.2: β decay, showing the relationship between weak and mass eigenstates [2].

The weak and mass eigenstates are related by a 2×2 unitary rotation matrix (which is a special PMNS matrix, will be explained in section (2.6)),

$$\begin{pmatrix} v_e \\ v_\mu \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.6}$$

Let's consider an v_e produced in a β^+ decay. According to equation (2.6) the propagation of its wave-function is dependent on the plane wave solutions of the mass states:

$$|\psi(\vec{x},t)\rangle = \cos(\theta)\,|v_1(t)\rangle + \sin(\theta)\,|v_2(t)\rangle = \cos(\theta)\,|v_1\rangle\,e^{-i\phi_1} + \sin(\theta)\,|v_2\rangle\,e^{-i\phi_2} \tag{2.7}$$

If the neutrino interacts at time T and distance L along its direction of propagation, equation (2.7) can be written in the following from:

$$|\psi(L,T)\rangle = \cos(\theta)\,|v_1\rangle\,e^{-i\phi_1} + \sin(\theta)\,|v_2\rangle\,e^{-i\phi_2} \tag{2.8}$$

Using the inverse of the equation (2.6) mass eigenstates can be written in terms of flavour states:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} v_e \\ v_\mu \end{pmatrix} \tag{2.9}$$

Therefore, $|v_1\rangle = \cos(\theta)\,|v_e\rangle - \sin(\theta)\,|v_\mu\rangle$ and $|v_2\rangle = \sin(\theta)\,|v_e\rangle + \cos(\theta)\,|v_\mu\rangle$, these expressions can be substituted in equation (2.8):

$$\begin{aligned} |\psi(L,T)\rangle =\ & \cos^2(\theta)\,|v_e\rangle\,e^{-i\phi_i} - \sin(\theta)\cos(\theta)\,|v_\mu\rangle\,e^{-i\phi_1} + \sin^2(\theta)\,|v_e\rangle\,e^{-i\phi_2} + \sin(\theta)\cos(\theta)\,|v_\mu\rangle\,e^{-i\phi_2} \\ &= (\cos^2(\theta)e^{-i\phi_1} + \sin^2(\theta)e^{-i\phi_2})\,|v_e\rangle + (-\sin(\theta)\cos(\theta)e^{-i\phi_1} + \sin(\theta)\cos(\theta)e^{-i\phi_2})\,|v_\mu\rangle \\ &= e^{-i\phi_1}\left[(\cos^2(\theta) + \sin^2(\theta)e^{i\triangle\phi_{12}})\,|v_e\rangle - \sin(\theta)\cos(\theta)(1 - e^{i\triangle\phi_{12}})\,|v_\mu\rangle\right] \end{aligned} \tag{2.10}$$

where:

$$\triangle\phi_{12} = \phi_1 - \phi_2 = T(E_1 - E_2) - (P_1 - P_2)L \tag{2.11}$$

It can be seen from equation (2.10), if the phase difference $\triangle_{12} = 0$ there will be no contribution from the v_μ part and neutrino will stay in the same electron neutrino state. However, if $\triangle_{12} \neq 0$ there will be a muon component in the wave-function, therefore there is a certain probability of observing electron neutrino undergoing oscillations [2]. In order to find this probability the equation (2.10) is written as:

$$|\psi(L,T)\rangle = C_e\,|v_e\rangle + C_\mu\,|v_\mu\rangle$$

Multiplying the above equation with $\langle v_e|$ from left and taking the square modulus would give the probability of the neutrino staying in the same electron neutrino state:

$$p(v_e \to v_e) = |\,\langle v_e|\psi\rangle\,|^2 = |C_e\,\langle v_e|v_e\rangle + C_\mu\,\langle v_e|v_\mu\rangle\,|^2 = |C_e|^2 \quad \text{where} \quad \langle v_e|v_e\rangle = 1 \quad \text{and} \quad \langle v_e|v_\mu\rangle = 0$$

as flavour states are orthonormal. Same operation can be done to find the probability of electron neutrino interacting to produce a muon neutrino:

$$p(v_e \to v_\mu) = |\,\langle v_\mu|\psi\rangle\,|^2 = |C_e\,\langle v_\mu|v_e\rangle + C_\mu\,\langle v_\mu|v_\mu\rangle\,|^2 = |C_\mu|^2 \quad \text{where} \quad \langle v_\mu|v_e\rangle = 0 \quad \text{and} \quad \langle v_\mu|v_\mu\rangle = 1$$

Where,

$$C_e = e^{-i\phi_1}(\cos^2(\theta) + \sin^2(\theta)e^{i\triangle\phi_{12}}) \quad \text{and} \quad C_\mu = -\sin(\theta)\cos(\theta)(1 - e^{i\triangle\phi_{12}})e^{-i\phi_1}$$

Hence, the probability of neutrino undergoing flavour oscillation will be:

$$\begin{aligned} p(v_e \to v_\mu) = C_\mu C_\mu^* &= \sin^2(\theta)\cos^2(\theta)(1 - e^{i\triangle\phi_{12}})(1 - e^{-i\triangle\phi_{12}}) \\ &= \frac{1}{4}\sin^2(2\theta)\big(2 - (e^{-i\triangle\phi_{12}} + e^{i\triangle\phi_{12})}\big) \\ &= \frac{1}{4}\sin^2(2\theta)\big(2 - (2\cos(\triangle\phi_{12})\big) \\ &= \frac{1}{2}\sin^2(2\theta)(1 - \cos(\triangle\phi_{12})) \\ &= \sin^2(2\theta)\sin^2\left(\frac{\triangle\phi_{12}}{2}\right) \end{aligned} \tag{2.12}$$

The probability of neutrino staying the same can either be obtained from $C_e C_e^*$ or using the fact that $p(v_e \to v_\mu) + p(v_e \to v_e) = 1$,

$$p(v_e \to v_e) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\triangle\phi_{12}}{2}\right) \tag{2.13}$$

At this point one can assume equal momenta or energies for the mass eigenstates. There is no justification on these assumptions other than that it simplifies the problem and will give the correct prediction for the probabilities. These assumptions have some unphysical consequences and in order to overcome these objections, a full wave-packet treatment is required. In section (2.5) these limitations are explained in detail.

Assuming equal momenta, p_1=p_2=p, and using the relativistic dispersion relation in natural units, $E^2 = m^2 + p^2$, will give the following expression for the phase difference of massive neutrinos (mass eigenstates):

$$\triangle\phi_{12} = (E_1 - E_2)T = \left[p\big(1 + \frac{m_1^2}{p^2}\big)^{1/2} - p\big(1 + \frac{m_2^2}{p^2}\big)^{1/2}\right] \tag{2.14}$$

Where, $E_1 = (m_1^2 + p^2)^{1/2} = p\big(1 + \frac{m_1^2}{p^2}\big)$ and $E_2 = (m_2^2 + p^2)^{1/2} = p\big(1 + \frac{m_2^2}{p^2}\big)$. As $p \gg m$, the expressions for the energies are approximately,

$$p\big(1 + \frac{m_2}{p^2}\big) \approx 1 + \frac{m^2}{2p^2} \tag{2.15}$$

Substituting this into (2.14) gives,

$$\triangle\phi_{12} = \big(\frac{m_1^2 - m_2^2}{2p}\big)L \tag{2.16}$$

Where it is assumed that, in natural units $T \approx L$.

Similarly, the same result can be obtained by assuming $E_1 = E_2$,

$$\begin{aligned}
\triangle\phi_{12} &= T(E_1 - E_2) - (P_1 - P_2)L \\
&= T(E_1 - E_2) - \left(\frac{(p_1 - p_2)(p_1 + p_2)}{p_1 + p_2}\right)L \\
&= T(E_1 - E_2) - \left(\frac{p_1^2 - p_2^2}{p_1 + p_2}\right)L \\
&= T(E_1 - E_2) - \left(\frac{E_1^2 - m_1^2 - E_2^2 + m_2^2}{p_1 + p_2}\right)L \\
&= T(E_1 - E_2) - \left(\frac{E_1^2 - E_2^2}{p_1 + p_2}\right)L - \left(\frac{m_2^2 - m_1^2}{p_1 + p_2}\right)L \\
&= T(E_1 - E_2) - \left(\frac{(E_1 - E_2)(E_1 + E_2)}{p_1 + p_2}\right)L - \left(\frac{m_2^2 - m_1^2}{p_1 + p_2}\right)L \\
&= (E_1 - E_2)\left[T - \frac{E_1 + E_2}{p_1 + p_2}\right] + \left(\frac{m_1^2 - m_2^2}{p_1 + p_2}\right)L \\
&= (\frac{m_1^2 - m_2^2}{p_1 + p_2})L
\end{aligned} \tag{2.17}$$

Clearly the first term on the right hand side is zero and the same expression as equation (2.16) is obtained. So although with a wave packet treatment there is no need of making any assumptions, it is still a simple way of correctly calculating the phase difference. Combining (2.16) or (2.17) with the expression (2.12) gives the probability of neutrino undergoing flavour oscillation,

$$p(v_e \to v_\mu) = \sin^2(2\theta)\sin^2\left(\frac{(m_1^2 - m_2^2)L}{4p}\right) \tag{2.18}$$

Hence, the survival probability for the electron neutrino is,

$$p(v_e \to v_e) = 1 - \sin^2(2\theta)\sin^2\left(\frac{(m_1^2 - m_2^2)L}{4p}\right) \tag{2.19}$$

Therefore, in order for the flavour oscillation to occur, neutrino must be a massive particle and $\triangle m^2 \neq 0$. Also, if the difference between the mass eigenstates is small, flavour oscillations develop over very large distances.

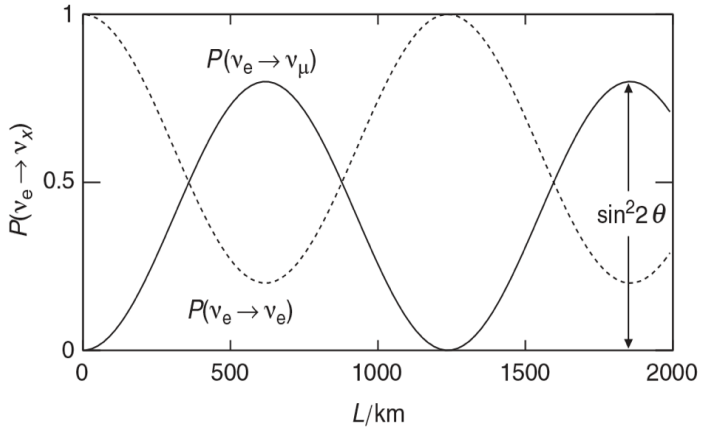

Figure 2.3: The two oscillation probabilities plotted as a function of distance L [2].

2.4 Oscillations of three flavours

The derivation of three flavour oscillation follows closely the mathematics shown in the previous section, however it has to be done with care as the algebra is more involved. This derivation does not add to physics and it will be shown again that oscillations are a result of phase difference between the wave-functions of massive neutrinos.

As shown in equation (2.1), the three weak states are related to the three mass states by the unitary PMNS matrix. Using the unitarity of this matrix, $U^{-1} = U^{\dagger}$ mass states can be written in terms of flavour states,

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} U_{e1}^* & U_{\mu 1}^* & U_{\tau 1}^* \\ U_{e2}^* & U_{\mu 2}^* & U_{\tau 2}^* \\ U_{e3}^* & U_{\mu 3}^* & U_{\tau 3}^* \end{pmatrix} \begin{pmatrix} v_e \\ v_\mu \\ v_\tau \end{pmatrix} \tag{2.20}$$

As the matrix U is unitary, $UU^{\dagger} = I$, hence,

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} U_{e1}^* & U_{\mu 1}^* & U_{\tau 1}^* \\ U_{e2}^* & U_{\mu 2}^* & U_{\tau 2}^* \\ U_{e3}^* & U_{\mu 3}^* & U_{\tau 3}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.21}$$

Which gives nine relations between the elements of the matrix, three of them which will be used in the calculations are shown below:

$$\begin{aligned} U_{e1}U_{e1}^* + U_{e2}U_{e2}^* + U_{e3}U_{e3}^* &= 1 \\ U_{e1}U_{\mu_1}^* + U_{e2}U_{\mu_2}^* + U_{e3}U_{\mu_3}^* &= 0 \\ U_{e1}U_{\tau_1}^* + U_{e2}U_{\tau_2}^* + U_{e3}U_{\tau_3}^* &= 0 \end{aligned} \tag{2.22}$$

Let's consider an electron neutrino produced in a weak interaction along with an electron, the time evolution of its wavefunction is dependent on the plane wave solutions of the mass eigenstates, using equation (2.1) gives,

$$\begin{aligned} |\psi(\vec{x},t)\rangle = |v_e\rangle &= U_{e1}\,|v_1(t)\rangle + U_{e2}\,|v_2(t)\rangle + U_{e3}\,|v_3(t)\rangle \\ &= U_{e1}\,|v_1\rangle\, e^{-ip_1x_1} + U_{e2}\,|v_2\rangle\, e^{-ip_2x_2} + U_{e3}\,|v_3\rangle\, e^{-ip_3x_3} \\ &= U_{e1}\,|v_1\rangle\, e^{-i\phi_1} + U_{e2}\,|v_2\rangle\, e^{-i\phi_2} + U_{e3}\,|v_3\rangle\, e^{-i\phi_3} \end{aligned} \tag{2.23}$$

where ϕ_i is the same as equation (2.5) and represents the phase of the plane wave corresponding to each mass eigenstates. It should be mentioned that, in most of the literature that has been looked at during the course of this project, the complex conjugate of the elements of the PMNS matrix are used to write the above equation. The reason behind this could not be well understood, moreover using either of them, complex conjugate or not, is going to result in the same form for the probabilities. Therefore, in this project, in order for the calculations to flow the normal elements of the PMNS matrix are used.

From expression (2.20) each of the massive states can be written in terms of the elements of the matrix and weak states,

$$\begin{aligned} |v_1\rangle &= U_{e_1}^*\,|v_e\rangle + U_{\mu_1}^*\,|v_\mu\rangle + U_{\tau_1}^*\,|v_\tau\rangle \\ |v_2\rangle &= U_{e_2}^*\,|v_e\rangle + U_{\mu_2}^*\,|v_\mu\rangle + U_{\tau_2}^*\,|v_\tau\rangle \\ |v_3\rangle &= U_{e_3}^*\,|v_e\rangle + U_{\mu_3}^*\,|v_\mu\rangle + U_{\tau_3}^*\,|v_\tau\rangle \end{aligned} \tag{2.24}$$

Therefore, expression (2.23) can be written as,

$$\begin{aligned}|\psi(\vec{x},t)\rangle = U_{e_1}\big(U^*_{e_1}|v_e\rangle + U^*_{\mu_1}|v_\mu\rangle + U^*_{\tau_1}|v_\tau\rangle\big)e^{-i\phi_1}\\ + U_{e_2}\big(U^*_{e_2}|v_e\rangle + U^*_{\mu_2}|v_\mu\rangle + U^*_{\tau_2}|v_\tau\rangle\big)e^{-i\phi_2}\\ + U_{e_3}\big(U^*_{e_3}|v_e\rangle + U^*_{\mu_3}|v_\mu\rangle + U^*_{\tau_3}|v_\tau\rangle\big)e^{-i\phi_3}\end{aligned} \tag{2.25}$$

Which can be rearranged to give,

$$\begin{aligned}|\psi(\vec{x},t)\rangle &= \big(U^*_{e_1}U_{e_1}e^{-i\phi_1} + U^*_{e_2}U_{e_2}e^{-i\phi_2} + U^*_{e_3}U_{e_3}e^{-i\phi_3}\big)|v_e\rangle\\ &+ \big(U_{e_1}U^*_{\mu_1}e^{-i\phi_1} + U_{e_2}U^*_{\mu_2}e^{-i\phi_2} + U_{e_3}U^*_{\mu_3}e^{-i\phi_3}\big)|v_\mu\rangle\\ &+ \big(U_{e_1}U^*_{\tau_1}e^{-i\phi_1} + U_{e_2}U^*_{\tau_2}e^{-i\phi_2} + U_{e_3}U^*_{\tau_3}e^{-i\phi_3}\big)|v_\tau\rangle\\ &= C_e|v_e\rangle + C_\mu|v_\mu\rangle + C_\tau|v_\tau\rangle\end{aligned} \tag{2.26}$$

Now, let's first find the probability that this electron neutrino changes to a muon neutrino,

$$p(v_e \to v_\mu) = |\langle v_\mu|\psi(\vec{x},t)\rangle|^2 = |C_\mu|^2 = |U_{e_1}U^*_{\mu_1}e^{-i\phi_1} + U_{e_2}U^*_{\mu_2}e^{-i\phi_2} + U_{e_3}U^*_{\mu_3}e^{-i\phi_3}|^2 \tag{2.27}$$

As shown in expression (2.22), $U_{e_1}U^*_{\mu_1} + U_{e_2}U^*_{\mu_2} + U_{e_3}U^*_{\mu_3} = 0$, then if the phase differences ϕ_i were all the same, would imply that $p(v_e \to v_\mu) = 0$ and no oscillation would occur. Therefore, as explained before neutrino oscillations is only possible if $\triangle_{\phi_i} \neq 0$, so neutrinos must have mass and the masses are all different.

Using the identity of complex numbers, $|z_1+z_2+z_3|^2 \equiv |z_1|^2+|z_2|^2+|z_3|^2+2\Re\{z_1z_2^*+z_1z_3^*+z_2z_3^*\}$, equation (2.27) can be simplified,

$$\begin{aligned}p(v_e \to v_\mu) = |U_{e_1}U^*_{\mu_1}|^2 + |U_{e_2}U^*_{\mu_2}|^2 + |U_{e_3}U^*_{\mu_3}|^2\\ + 2\Re\big\{\big(U_{e1}U^*_{\mu_1}U^*_{e_2}U_{\mu_2}\big)e^{-i(\phi_1-\phi_2)}\big\}\\ + 2\Re\big\{\big(U_{e1}U^*_{\mu_1}U^*_{e_3}U_{\mu_3}\big)e^{-i(\phi_1-\phi_3)}\big\}\\ + 2\Re\big\{\big(U_{e_2}U^*_{\mu_2}U^*_{e_3}U_{\mu_3}\big)e^{-i(\phi_2-\phi_3)}\big\}\end{aligned} \tag{2.28}$$

This expression can be simplified further by applying the complex numbers identity to the second unitary relation in expression (2.22), $|U_{e_1}U^*_{\mu_1} + U_{e_2}U^*_{\mu_2} + U_{e_3}U^*_{\mu_3}|^2 = 0$,

$$|U_{e_1}U^*_{\mu_1}|^2 + |U_{e_2}U^*_{\mu_2}|^2 + |U_{e_3}U^*_{\mu_3}|^2 = -2\Re\{U_{e_1}U^*_{\mu_1}U^*_{e_2}U_{\mu_2}\} - 2\Re\{U_{e_1}U^*_{\mu_1}U^*_{e_3}U_{\mu_3}\} - 2\Re\{U_{e_2}U^*_{\mu_2}U^*_{e_3}U_{\mu_3}\}$$

Substituting this into equation (2.28) gives,

$$\begin{aligned}p(v_e \to v_\mu) = 2\Re\{U_{e_1}U^*_{\mu_1}U^*_{e_2}U_{\mu_2}\big(e^{-i(\phi_1-\phi_2)} - 1\big)\}\\ +2\Re\{U_{e_1}U^*_{\mu_1}U^*_{e_3}U_{\mu_3}\big(e^{-i(\phi_1-\phi_3)} - 1\big)\}\\ +2\Re\{U_{e_2}U^*_{\mu_2}U^*_{e_3}U_{\mu_3}\big(e^{-i(\phi_2-\phi_3)} - 1\big)\}\end{aligned} \tag{2.29}$$

The probability of the electron neutrino changing to a tau neutrino can be calculated in a similar manner,

$$p(v_e \to v_\tau) = |\langle v_\tau|\psi(\vec{x},t)\rangle|^2 = |C_\tau|^2 = |U_{e_1}U^*_{\tau_1}e^{-i\phi_1} + U_{e_2}U^*_{\tau_2}e^{-i\phi_2} + U_{e_3}U^*_{\tau_3}e^{-i\phi_3}|^2 \tag{2.30}$$

Using the complex numbers identity gives,

$$\begin{aligned}p(v_e \to v_\tau) = |U_{e_1}U^*_{\tau_1}|^2 + |U_{e_2}U^*_{\tau_2}|^2 + |U_{e_3}U^*_{\tau_3}|^2\\ + 2\Re\big\{\big(U_{e1}U^*_{\tau_1}U^*_{e_2}U_{\tau_2}\big)e^{-i(\phi_1-\phi_2)}\big\}\\ + 2\Re\big\{\big(U_{e1}U^*_{\tau_1}U^*_{e_3}U_{\tau_3}\big)e^{-i(\phi_1-\phi_3)}\big\}\\ + 2\Re\big\{\big(U_{e_2}U^*_{\tau_2}U^*_{e_3}U_{\tau_3}\big)e^{-i(\phi_2-\phi_3)}\big\}\end{aligned} \tag{2.31}$$

Applying the complex numbers identity to the third unitary relation in expression (2.22), $|U_{e_1}U^*_{\tau_1} + U_{e_2}U^*_{\tau_2} + U_{e_3}U^*_{\tau_3}|^2 = 0$ gives,

$$|U_{e_1}U^*_{\tau_1}|^2 + |U_{e_2}U^*_{\tau_2}|^2 + |U_{e_3}U^*_{\tau_3}|^2 = -2\Re\{U_{e_1}U^*_{\tau_1}U^*_{e_2}U_{\tau_2}\} - 2\Re\{U_{e_1}U^*_{\tau_1}U^*_{e_3}U_{\tau_3}\} - 2\Re\{U_{e_2}U^*_{\tau_2}U^*_{e_3}U_{\tau_3}\}$$

The probability is calculated by substituting the above equation into (2.31),

$$\begin{aligned} p(v_e \to v_\tau) = {} & 2\Re\{U_{e_1}U^*_{\tau_1}U^*_{e_2}U_{\tau_2}\big(e^{-i(\phi_1-\phi_2)} - 1\big)\} \\ & +2\Re\{U_{e_1}U^*_{\tau_1}U^*_{e_3}U_{\tau_3}\big(e^{-i(\phi_1-\phi_3)} - 1\big)\} \\ & +2\Re\{U_{e_2}U^*_{\tau_2}U^*_{e_3}U_{\tau_3}\big(e^{-i(\phi_2-\phi_3)} - 1\big)\} \end{aligned} \tag{2.32}$$

Similarly, the survival probability $p(v_e \to v_e)$ can be found. Complex numbers identity is applied to $|C_e|^2$ and the first unitary relation in expression (2.22), then the expression found for the unitary relation is substituted for the probability. In this case each element of the matrix is multiplied by its complex conjugate which gives real numbers for the elements of the matrix, leading to the expression below for the electron neutrino staying the same,

$$\begin{aligned} p(v_e \to v_e) = {} & 1 + 2|U_{e_1}|^2|U_{e_2}|^2\Re\{e^{-i(\phi_1-\phi_2)} - 1\} \\ & +2|U_{e_1}|^2|U_{e_3}|^2\Re\{e^{-i(\phi_1-\phi_3)} - 1\} \\ & +2|U_{e_2}|^2|U_{e_3}|^2\Re\{e^{-i(\phi_2-\phi_3)} - 1\} \end{aligned} \tag{2.33}$$

The above equation can be simplified further by writing,

$$\Re\{e^{-i(\phi_i-\phi_j)} - 1\} = \cos(\phi_j - \phi_i) - 1 = -2\sin^2\big(\frac{\phi_j - \phi_i}{2}\big) = -2\sin^2 \triangle_{ji}$$

by assuming equal momenta $\triangle_{ji}$ is defined to be,

$$\triangle_{ji} = \frac{\phi_j - \phi_i}{2} = \big(\frac{m_j^2 - m_i^2}{2p}\big)L \tag{2.34}$$

Therefore, the survival probability (2.33) can be written as,

$$\begin{aligned} p(v_e \to v_e) = {} & 1 - 4|U_{e_1}|^2|U_{e_2}|^2\sin^2(\triangle_{21}) \\ & - 4|U_{e_1}|^2|U_{e_3}|^2\sin^2(\triangle_{31}) \\ & - 4|U_{e_2}|^2|U_{e_3}|^2\sin^2(\triangle_{32}) \end{aligned} \tag{2.35}$$

As it can be seen from the above equation, the neutrino survival probability is dependent on the differences of squared masses. It can be concluded, neutrino oscillations only occur if neutrinos are massive particles and the mass eigenstates are different.

2.5 Physical consequences of the assumptions

In the previous sections, the theory of neutrino oscillations was described and over the course of these derivations several assumptions were made. However, some of these assumptions would lead to unphysical consequences.

The first problem is arising from the assumption of equal momenta. The mass eigenstates have different masses, it means that in order for them to have the same momentum, their velocities must be different. Hence, they will not cover the distance L in the same amount of time [2]. Another problem

will be due to the Heisenberg Uncertainty Principle ($\triangle x \triangle p \geqslant \hbar/2$) , considering the mass eigenstates propagating as plane waves, requires their wave-functions to have a well defined momentum, which means zero uncertainty on their momentum. This will lead the plane waves to have infinite uncertainty on their position, thus losing completely its locality, which is not true in the case of neutrinos as they have a specific position in space [6].

Similarly, assuming equal energies indicates that the energy is measured precisely, which will lead to infinite uncertainty on time. One has to wait an unknown amount of time to observe such an energy. These objections can be overcome with a full wave-packet treatment and no such assumptions are needed to calculate the probabilities.

In the wave-packet treatment, the mass eigenstates are described by a wave-packet, which its momentum will be a Gaussian function in the momentum space [7] [6]. Therefore, the momentum does not have a precise value and there is a certain amount of uncertainty associated with its measurement, this can be seen in the figure below (2.4). The peak of the function shows the average momentum p_0 and the uncertainty is the width of the function at half maximum, σ_p.

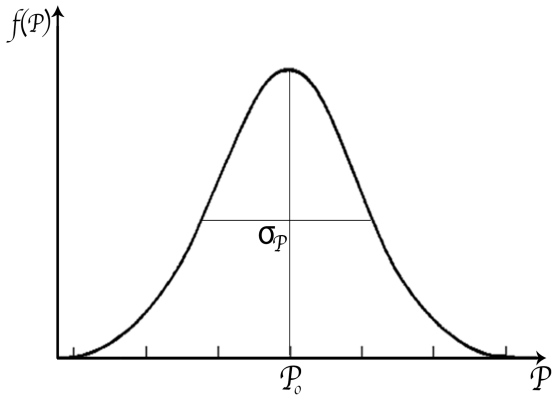

Figure 2.4: A Gaussian function, which is a representation of the momentum of a wave-packet in momentum space, centred at the mean momentum p_o, with an uncertainty equal to the width at half maximum σ_p.

2.6 CP violation and the Pontecorvo-Maki-Nakagawa-Sakata matrix

It is believed that in the early universe there were equal amounts of matter and antimatter, one would imagine if this was still true and the quantities of matter and antimatter were equal, the two would annihilate and thus no life would exist. Clearly, this is not the case and the observed quantities of antiparticles is far less than those of particles and there is no evidence of primordial antiparticles [8]. Therefore, there is a huge asymmetry between matter and antimatter in the universe, a candidate explanation for this asymmetry is the violation of charge and parity symmetries (CP symmetry).

Charge conjugation $\hat{C}$ is an operator, which reverses the sign of electric charge of a particle. The effect of the parity operator is to spatially invert the wavefunction, for instance $\hat{P}\psi(r) = \psi(-r)$, and applying parity and charge conjugation together is going to make sure a particle is replaced with its antiparticle [9]. For example applying $\hat{C}\hat{P}$ to a right handed electron is going to give a left handed positron. The reason is

quite clear, $\hat{C}$ is going to reverse the charge and parity is going to invert the direction of the momentum, therefore helicity is also going to change.

In the Standard Model of particle physics, Quantum Electrodynamics and Quantum Chromodynamics interactions conserve C and P separately therefore they conserve CP together [2], it means applying $\hat{C}$ to an interaction and then applying $\hat{P}$ gives back the initial interaction. However, in contrast to electromagnetic and strong force, weak interactions maximally violate CP symmetries and this is due to the terms that appear in the vertex of charged-current weak interaction (A detailed explanation on this reasoning is beyond the scope of this project and is given in [9] [2]). Therefore, the only place that CP violation can be observed is in the weak sector of the Standard Model.

The flavour oscillation of neutrinos requires at least two unique values for its mass eigenstates, which contradicts with the mass-less neutrinos predicted by the Standard Model. This leads to the possibility of CP violation in the neutrino sector. If CP is violated in an interaction involving neutrinos, the rate that a neutrino decays to a lepton and an anti-lepton will be different, hence generating a asymmetry between matter and antimatter [8]. This phenomenon supports leptogenesis, which is a hypothetical process that caused a asymmetry between lepton and anti-lepton through the decay of right handed neutrinos [3] (the analogous process for baryons is called baryogenesis).

It is believed that CPT, charge, parity and time reversal is an exact symmetry of the universe, and all the interactions must obey such a symmetry [2]. Time reversal is also a discrete symmetry operation that has the following effect on a physical process, $\hat{T}\psi(t) = \psi(-t)$ [9]. If a process is invariant under the combined operation of C and P, then it must also be unchanged under the application of T. Similarly, violation of CP symmetry implies that T is also violated [2].

It is going to be shown that in order to have violation of CP symmetry in the case of neutrino oscillations the elements of the PMNS matrix must be complex. The violation of time reversal symmetry implies, CP violation, therefore if the following oscillation probabilities are not equal, this can suggest the possibility of CP violation,

$$\hat{T}p(v_e \to v_\mu) = p(v_\mu \to v_e)$$

where the probability $p(v_e \to v_\mu)$ is given by equation (2.29),

$$p(v_e \to v_\mu) = 2\Re\{U_{e_1}U^*_{\mu_1}U^*_{e_2}U_{\mu_2}\big(e^{-i(\phi_1-\phi_2)} - 1\big)\} + ...$$

and, $p(v_\mu \to v_e)$ can be obtained by swapping e and μ,

$$p(v_\mu \to v_e) = 2\Re\{U_{\mu_1}U^*_{e_1}U^*_{\mu_2}U_{e_2}\big(e^{-i(\phi_1-\phi_2)} - 1\big)\} + ...$$

It can be seen that the elements of the PMNS matrix in $p(v_\mu \to v_e)$ are just the complex conjugate of those in $p(v_e \to v_\mu)$. Hence, these probabilities are equal unless all the elements are complex, $U^*_{e_i} \neq U_{e_i}$ and $U^*_{\mu_i} \neq U_{\mu_i}$.

CP operation transforms the flavour oscillation $v_e \to v_\mu$ in the following way,

$$v_e \to v_\mu \quad \xrightarrow{\hat{C}\hat{P}} \quad \bar{v}_e \to \bar{v}_\mu$$

where P changes left handed neutrinos into right handed ones and C reverses the charge. The probability $p(\bar{v}_e \to \bar{v}_\mu)$ can be written by swapping U with U^* and vice versa,

$$p(\bar{v}_e \to \bar{v}_\mu) = 2\Re\{U^*_{e_1}U_{\mu_1}U_{e_2}U^*_{\mu_2}\big(e^{-i(\phi_1-\phi_2)} - 1\big)\} + ...$$

Again, in order for the two probabilities to be different, $p(\bar{v}_e \to \bar{v}_\mu) \neq p(v_e \to v_\mu)$, the elements of the mixing matrix must be complex. Now let's consider the combined effect of CPT,

$$v_e \to v_\mu \quad \overset{\hat{C}\hat{P}\hat{T}}{\longrightarrow} \quad \bar{v}_\mu \to \bar{v}_e$$

T is going to swap e with μ in the probabilities and CP changes each of the elements of the matrix to its complex conjugate,

$$p(\bar{v}_\mu \to \bar{v}_e) = 2\Re\{U^*_{\mu_1} U_{e_1} U_{\mu_2} U^*_{e_2} \left(e^{-i(\phi_1 - \phi_2)} - 1\right)\} + ... = p(v_e \to v_\mu)$$

As expected, the probabilities are equal under the combined operation of CPT, which is believed to be true for any physical process. It can be concluded that the imaginary parts of the PMNS matrix can provide a source for CP violation in the Standard Model [2].

Now, it is going to be explained briefly how the PMNS matrix is described in the Standard Model. In general a 3×3 unitary matrix can be parametrised as a product of three unitary rotation matrices in the complex (1,2), (2,3) and (3,1) planes [10],

$$R = \begin{pmatrix} e^{i\gamma_1} & 0 & 0 \\ 0 & c_{23}e^{i\alpha_1} & s_{23}e^{-i\beta_1} \\ 0 & -s_{23}e^{i\beta_1} & c_{23}e^{-i\alpha_1} \end{pmatrix} \times \begin{pmatrix} c_{13}e^{i\alpha_2} & 0 & s_{13}e^{-i\beta_2} \\ 0 & e^{i\gamma_2} & 0 \\ -s_{13}e^{i\beta_2} & 0 & c_{13}e^{-i\alpha_2} \end{pmatrix} \times \begin{pmatrix} c_{12}e^{i\alpha_3} & s_{12}e^{-i\beta_3} & 0 \\ -s_{12}e^{i\beta_3} & c_{12}e^{-i\alpha_3} & 0 \\ 0 & 0 & e^{i\gamma_3} \end{pmatrix} \tag{2.36}$$

where $s_{ij} = \sin(\theta_{ij})$ and $c_{ij} = \cos(\theta_{ij})$ and θ_{ij} is the angle of rotation between the planes. The above matrices, contain three Euler angles and nine complex phases. It turns out five of the complex phases can be absorbed into definitions of the particles [2]. In order to see this the possible leptonic charged-current weak interaction vertices has to be written. Also the remaining four phases can be defined relative to one common phase, hence leaving out only one complex phase in the matrix. This is a vague explanation on how this parametrisation is done, a detailed explanation is beyond the scope of this project, and is given in references [2] and [10]. Therefore, the PMNS matrix can be written as,

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \times \begin{pmatrix} c_{13}e & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.37}$$

It should be noted that the unitary mixing matrix, in the case of two flavour oscillations (2.6) has four parameters, consisting of one rotation angle and three complex phases. All the complex phases can be absorbed into the definitions of the particles, hence the mixing matrix is entirely real. Therefore, it cannot accommodate the violation of CP symmetry, thus the CP violation which is considered to originate from the PMNS matrix, occurs only for the oscillation of three flavours [2].

Chapter 3

Classical Analogue of Neutrino Oscillations

3.1 Outline of the chapter

A system of coupled pendula can help with having a physical picture of neutrino oscillations in mind. First a two coupled pendula is considered in section (3.2), representing the oscillation of two flavours. Newton's second law is used to find the equations of motions and an eigenvalue problem is solved to find the frequencies of normal modes. Also normal coordinates is introduced to decouple equations of motions and eventually build amplitudes of oscillations. By the use of relativistic dispersion relation an expression for the frequencies is found, which led to drawing the analogy between the amplitudes of oscillations and the probability of neutrino oscillations. In section (3.3) the analogous problem involving three masses is solved. With the help of a computer program, the coupled differential equations are also solved numerically, which gave, as the analytical solutions, a wave-packet behaviour for the pendula in their normal modes of oscillations.

3.2 Normal modes of two coupled pendula

A system of two coupled pendula, consisting of two equal masses, equal lengths for the strings and connected by a spring of spring constant k is arranged as shown in figure (3.1).

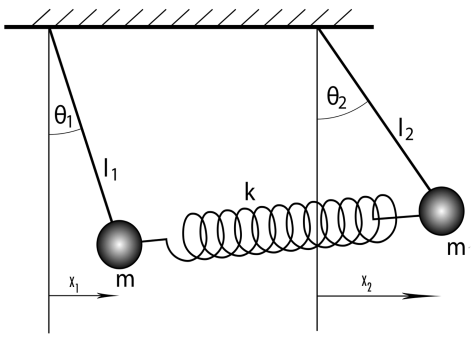

Figure 3.1: Two coupled pendula with masses $m_1 = m_2 = m$, suspended by strings of equal lengths $l_1 = l_2 = l$, with displacements x_1 and x_2 from their equilibrium positions. [11]

Let x_1 and x_2 measure the displacements of the mass m_1 and the mass m_2 from their respective equilibrium positions. The forces acting on the mass m_1, are $F_1, s = k(x_2 - x_1)$ and $F_1, g = -m_1 g \sin(\theta_1)$, where F_1, s and F_1, g are the forces due to the spring and gravity respectively. Similarly, the forces acting on mass m_2, are $F_2, s = k(x_1 - x_2)$ and $F_2, g = -m_2 g \sin(\theta_2)$ [12] [13]. Using Newton's second law of motion, $F = ma$ equations of motions of the two masses are written in the following form,

$$\ddot{x}_1 = k/m(x_2 - x_1) - (g/l)x_1 \tag{3.1}$$

$$\ddot{x}_2 = k/m(x_1 - x_2) - (g/l)x_2 \tag{3.2}$$

The calculations are done by making small angle approximation, therefore $x_1(t) = l\theta_1(t)$ and $x_2(t) = l\theta_2(t)$. The above equations form a coupled differential equation so they cannot be solved independently. The simple case of normal modes of oscillations can be considered, where the masses oscillate with the same frequency. The general solutions can be built later by considering any motion of oscillation as a superposition of the normal modes [13]. As there are two normal modes for this system, as shown in figure (3.2), there are two possible solutions for the frequencies,

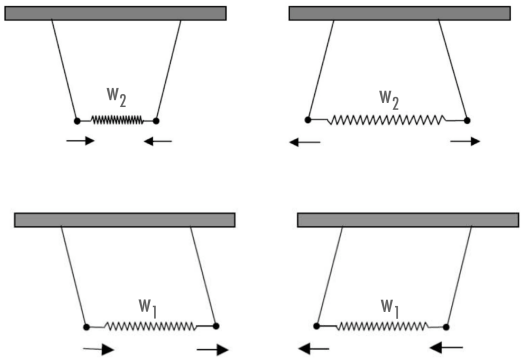

Figure 3.2: Normal modes of oscillations for a system of two coupled pendula, each of the masses are oscillating with the same frequency in their normal mode, for this system there are two possible normal frequencies, w_1 and w_2 [14].

Ansatz can be made for the solutions of equations (3.1) and (3.2), in the following form: $x_1 = C_1 e^{iwt}$, $x_2 = C_2 e^{iwt}$. Substituting these back into the equations gives the following expressions,

$$-w^2 x_1 = (k/m)x_2 - (k/m + g/l)x_1 \tag{3.3}$$

$$-w^2 x_2 = (k/m)x_1 - (k/m + g/l)x_2 \tag{3.4}$$

Above equations can be written in a matrix form,

$$\begin{pmatrix} k/m + g/l - w^2 & -k/m \\ -k/m & k/m + g/l - w^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.5}$$

From above it can be seen that the frequencies of the normal modes w^2 can be found as the eigenvalues of the matrix below,

$$M = \begin{pmatrix} k/m + g/l & -k/m \\ -k/m & k/m + g/l \end{pmatrix}$$

The following results are obtained for this eigenvalue problem:

$$w_1^2 = g/l \quad \text{and} \quad w_2^2 = 2k/m + g/l$$

By addition and subtraction of the equations (3.1) and (3.2) normal coordinates can be introduced to decouple the differential equations [13],

$$\ddot{q_1} = \ddot{x}_1 + \ddot{x}_2 = -g/l(x_1 + x_2) = -w_1^2(x_1 + x_2) = -w_1^2 q_1 \tag{3.6}$$

$$\ddot{q_2} = \ddot{x}_2 - \ddot{x}_1 = -(g/l + 2k/m)(x_2 - x_1) = -w_2^2(x_2 - x_1) = -w_2^2 q_2 \tag{3.7}$$

It can be seen that for any motion of the system the quantity q_1=x_1+x_2 oscillates with the frequency w_1 and the quantity q_2=x_2-x_1 oscillates with the frequency w_2. The combination of these coordinates are the normal coordinates of this system. Again, ansatz can be made for the solutions of (3.6) and (3.7) in the form of plane waves, $q_1 = Ae^{iw_1 t}$ and $q_2 = Ae^{iw_2 t}$. Therefore, the displacements x_1 and x_2 can be written in terms of q_1 and q_2:

$$x_1 = 1/2(q_1 - q_2) = 1/2A((\cos(w_1 t) - \cos(w_2 t)) \tag{3.8}$$

$$x_2 = 1/2(q_1 + q_2) = 1/2A((\cos(w_1 t) + \cos(w_2 t)) \tag{3.9}$$

It should be noted that in the above equations only the real parts of the normal coordinates are considered as the imaginary parts are unphysical.

The trigonometric identities below are used to simplify equations (3.8) and (3.9),

$$\cos a + \cos b = 2\cos\big(\frac{a+b}{2}\big)\cos\big(\frac{a-b}{2}\big) \quad \text{and} \quad \cos a - \cos b = -2\sin\big(\frac{a+b}{2}\big)\cos\big(\frac{a-b}{2}\big)$$

Hence, equations (3.8) and (3.9) can be written as,

$$x_1 = -A\sin\left(\frac{(w_1 - w_2)t}{2}\right)\sin\left(\frac{(w_1 + w_2)t}{2}\right) \tag{3.10}$$

$$x_2 = A\cos\left(\frac{(w_1 - w_2)t}{2}\right)\cos\left(\frac{(w_1 + w_2)t}{2}\right) \tag{3.11}$$

It is clear that equations (3.10) and (3.11) represent the phenomenon of beating. The oscillators have different frequencies and due to the principle of superposition of waves an envelope is being built, as shown in figure (3.3). The frequency of the envelope is equal to half the difference between the frequencies of the two original waves and the the frequency of the modulation is the average of the two frequencies. Therefore, the equations showing the time evolution of the wave-packets have the following form,

$$A_{x1} = A\sin\left(\frac{(w_1 - w_2)t}{2}\right) \quad \text{and} \quad A_{x2} = A\cos\left(\frac{(w_1 - w_2)t}{2}\right)$$

The coupled differential equation is also solved numerically using a computer program written in python, which gives the same wave packet prediction for the motion of the oscillators as the analytical solution. The figure (3.3) represents the data calculated by the computer program.

It is convenient to square the amplitudes at this step to be able to draw the analogy to the probability of neutrino oscillations calculated in chapter (2),

$$A_{x1}^2 = A^2\sin^2\left(\frac{(w_1 - w_2)t}{2}\right) \quad \text{and} \quad A_{x2}^2 = A^2\cos^2\left(\frac{(w_1 - w_2)t}{2}\right) \tag{3.12}$$

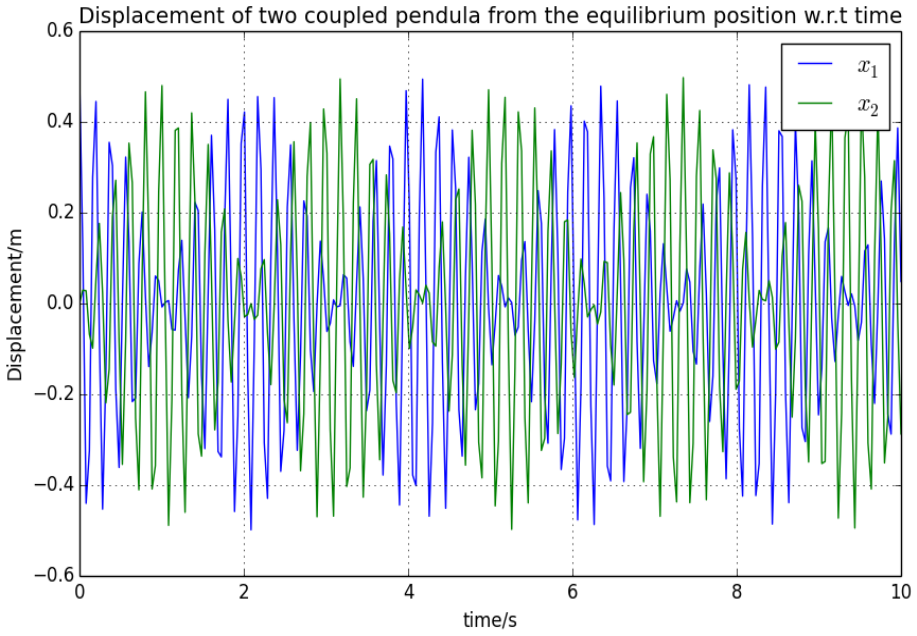

Figure 3.3: Displacement of the oscillators in a system of two coupled pendula, plotted with respect to time. This graph is drawn from the data gathered by a computer program, which solved the equations of motion numerically.

Now, as the purpose of these calculations is to show the analogy between coupled pendula and neutrino oscillations, relativistic dispersion relation, $E^2 = m^2 + p^2$, is used to find expressions for the frequencies [15]. These expressions can later be substituted into the equations of amplitudes. First, the dispersion relation can be taylor expanded,

$$E = (p^2 + m^2)^{1/2} = p(1 + \frac{m^2}{p^2})^{1/2} \approx p(1 + \frac{m^2}{2p^2}) = p + \frac{m^2}{2p}$$

For the purpose of these calculations it is reasonable to make the assumption that the momentum of the oscillators are the same $p_1 = p_2 = p$. By using the Planck-Einstein relation, $E = \hbar w$, the subtraction of the frequencies, $w_1 - w_2$ in equation (3.12) becomes,

$$w_1 - w_2 = \frac{1}{\hbar}(E_1 - E_2) = \frac{1}{\hbar}\left[\left(p + \frac{m_1^2}{2p}\right) - \left(p + \frac{m_2^2}{2p}\right)\right] = \frac{1}{\hbar}\left(\frac{m_1^2 - m_2^2}{2p}\right) \tag{3.13}$$

Substituting the above equation, (3.13) into the equation of amplitudes, (3.12) gives,

$$A_{x1}^2 = A^2 \sin^2\left(\frac{(m_1^2 - m_2^2)t}{4\hbar p}\right) \quad \text{and} \quad A_{x2}^2 = A^2 \cos^2\left(\frac{(m_1^2 - m_2^2)t}{4\hbar p}\right)$$

In natural units, $\hbar = c = 1$ and by writing time as $t = \frac{L}{c} = L$ the above equations become,

$$A_{x1}^2 = A^2 \sin^2\left(\frac{(m_1^2 - m_2^2)L}{4p}\right) \quad \text{and} \quad A_{x2}^2 = A^2 \cos^2\left(\frac{(m_1^2 - m_2^2)L}{4p}\right) \tag{3.14}$$

The obtained amplitudes bear a close resemblance to the probabilities calculated in section (2.3). Comparing the two results it can be seen that the amplitude A in equation (3.14) is just the $\sin(2\theta)$ found before. This quantity in both cases is the amplitude of the oscillation and is responsible for the strength of

the coupling, where a maximum coupling is achieved if the mixing angle θ is 45 degrees. In this case, as it was assumed that the length of the two strings are equal, led to maximum coupling and hence, $\sin(2\theta) = 1$ and the amplitudes of the wave-packets, $A = 1$.

From the first part of the equations (3.8) and (3.9) it can be seen that x_1 and x_2 can be related to the normal coordinates by a rotation matrix,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{3.15}$$

Where in this case the mixing angle, θ is 45 degrees clockwise. Comparing this relation with equation (2.9), one can say that the normal coordinates in a system of coupled oscillators represent the mass eigenstates, whereas individual oscillators describe the flavour states [6]. The figure below illustrates this relation as coordinates transformation.

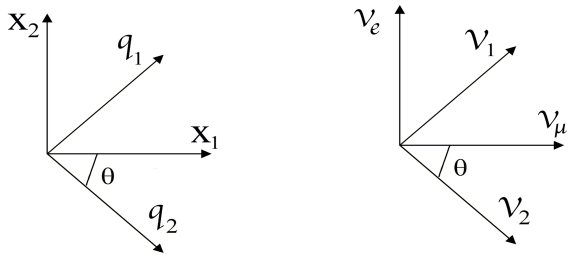

Figure 3.4: An analogy to neutrino mixing, normal coordinates q_1 and q_2 represent mass eigenstates and individual oscillators x_1 and x_2 describes the flavour of the neutrino. [15] [6]

3.3 Normal modes of three coupled pendula

Consider a system of three coupled pendula of equal masses coupled by three identical springs, as shown in figure (3.5). Although, it is not possible physically to have a coupling between the first and third pendulum, however in these calculations the third coupling is considered mathematically. Also, it is assumed that the length of the strings l are equal.

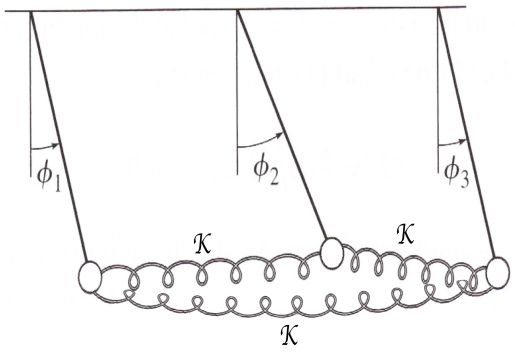

Figure 3.5: Three identical pendula of mass m and length l are coupled by three identical springs of spring constant k. [12]

In order to find the equation of motions in this case, instead of using Newton's second law the Lagrangian will be used. Then from the Euler-Lagrange equation, equation of motions can be written. The kinetic energy and the total gravitational energy of this system is,

$$T = \frac{1}{2}ml^2(\dot{\phi}_1 + \dot{\phi}_2 + \dot{\phi}_3), \qquad U_{grav} = \frac{1}{2}mgl(\phi_1^2 + \phi_2^2 + \phi_3^2) \tag{3.16}$$

Also the potential energy due to the springs has to be considered, however for arbitrary values of ϕ this is rather complicated. For this reason it is assumed that the angle of oscillation is small and the spring can only be stretched and compressed horizontally, so each pendulum moves a distance of $L\phi$. Thus, for instance the right spring is compressed about $L(\phi_3 - \phi_2)$ [12], and the total potential energy due to the springs can be written as,

$$U_s = \frac{1}{2}kl^2\big((\phi_2 - \phi_1)^2 + (\phi_3 - \phi_2)^2 + (\phi_3 - \phi_1)^2\big) \tag{3.17}$$

Therefore, the Lagrangian $L = T - V$ becomes,

$$L = \frac{1}{2}ml^2(\dot{\phi}_1+\dot{\phi}_2+\dot{\phi}_3)-\frac{1}{2}mgl(\phi_1^2+\phi_2^2+\phi_3^2)-\frac{1}{2}kl^2[2\phi_1^2+2\phi_2^2+2\phi_3^2-2\phi_1\phi_2-2\phi_3\phi_2-2\phi_1\phi_3] \tag{3.18}$$

The Lagrangian contains most of the dynamics of the system and the evolution of the system can be described using the Euler-Lagrange equation [12],

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \quad \text{for i=1,...,n} \tag{3.19}$$

Where q_i are generalised coordinates, in this case the generalised coordinates are the three angles ϕ_1, ϕ_2 and ϕ_3, and the equilibrium position is where $\phi_1 = \phi_2 = \phi_3 = 0$. Now, the Lagrangian (3.18) can be substituted into the equation (3.19) to write down the Lagrange equations of motion for ϕ_1, ϕ_2 and ϕ_3,

$$\frac{\partial L}{\partial \phi_1} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_1} \qquad \text{so,} \qquad \ddot{\phi}_1 = \phi_1(-g/l - 2k/m) + \phi_2(k/m) + \phi_3(k/m) \tag{3.20}$$

and,

$$\frac{\partial L}{\partial \phi_2} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_2} \qquad \text{so,} \qquad \ddot{\phi}_2 = \phi_1(k/m) + \phi_2(-g/l - 2k/m) + \phi_3(k/m) \tag{3.21}$$

and,

$$\frac{\partial L}{\partial \phi_3} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_3} \qquad \text{so,} \qquad \ddot{\phi}_3 = \phi_1(k/m) + \phi_2(k/m) + \phi_3(-g/l - 2k/m) \tag{3.22}$$

As the above equations form a system of coupled differential equations they cannot be solved independently, and the case of normal modes has to be considered to be able to solve these analytically. The general solutions can be built later by writing any motion of the system as a superposition of the normal modes [13]. At this point, ansatz can be made for the solutions of the the above equations of motion in their normal modes of oscillation, so assuming same frequency for each of the solutions,

$$\phi_1 = C_1e^{iwt} \quad \text{and} \quad \phi_2 = C_2e^{iwt} \quad \text{and} \quad \phi_3 = C_3e^{iwt} \tag{3.23}$$

Substituting these back into the equations of motion (3.20), (3.21) and (3.22) and replacing ϕ with $\frac{x}{l}$ (small angle approximation allows for the replacement of ϕ with $\frac{x}{l}$) gives the following relations,

$$\begin{aligned}
x_1\big(\frac{g}{l} + \frac{2k}{m} - w^2\big) + x_2\big(\frac{-k}{m}\big) + x_3\big(\frac{-k}{m}\big) &= 0 \\
x_1\big(\frac{-k}{m}\big) + x_2\big(\frac{g}{l} + \frac{2k}{m} - w^2\big) + x_3\big(\frac{-k}{m}\big) &= 0 \\
x_1\big(\frac{-k}{m}\big) + x_2\big(\frac{-k}{m}\big) + x_3\big(\frac{g}{l} + \frac{2k}{m} - w^2\big) &= 0
\end{aligned} \tag{3.24}$$

The relations (3.24) can be written in terms of a matrix equation,

$$\begin{pmatrix} \frac{g}{l}+\frac{2k}{m}-w^2 & \frac{-k}{m} & \frac{-k}{m} \\ \frac{-k}{m} & \frac{g}{l}+\frac{2k}{m}-w^2 & \frac{-k}{m} \\ \frac{-k}{m} & \frac{-k}{m} & \frac{g}{l}+\frac{2k}{m}-w^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.25}$$

The frequencies of the normal modes are just the eigenvalues of the above matrix, as expected there will be three solutions for this eigenvalue problem,

$$w_1 = \sqrt{g/l} \quad w_2 = w_3 = \sqrt{g/l + 3k/m} \tag{3.26}$$

and the corresponding eigenvectors are,

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \tag{3.27}$$

In this case as there are three equations of motion, it is not easy to see what the normal coordinates are, and therefore the equation below is used to write the solutions for the equations of motion in terms of the normal coordinates [12],

$$\vec{x}(t) = \sum_{i=1}^{3} q_i(t)\vec{v}_i \tag{3.28}$$

Where $q_i(t)$ are the normal coordinates and $\vec{v}_i$ are the eigenvectors of the matrix constructed from the equations (3.24), (for a proof of this formula please see reference [12]) expanding this formula gives,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = q_1(t) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + q_2(t) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + q_3(t) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \tag{3.29}$$

Hence,

$$\begin{aligned} x_1(t) &= q_1(t) - q_2(t) - q_3(t) \\ x_2(t) &= q_1(t) + q_3(t) \\ x_2(t) &= q_1(t) + q_2(t) \end{aligned} \tag{3.30}$$

Using the above equations it can be checked that the normal coordinates oscillate with the normal frequencies found before in expression (3.26),

$$\ddot{q}_1 = -w_1^2 q_1, \quad \ddot{q}_2 = -w_2^2 q_1 \quad \text{and} \qquad q_3 = -w_3^2 q_1$$

As before, we can guess plane wave solutions for the above equations,

$$q_1 = Ae^{iw_1 t}, \quad q_2 = Ae^{iw_2 t} \quad \text{and} \quad q_3 = Ae^{iw_3 t} \tag{3.31}$$

The general equations, for any motion of the coupled pendula can be written by substituting the above relations into the expressions of (3.30), it should be noted that only real parts of the ansatz solutions are considered as the imaginary parts would be unphysical,

$$x_1(t) = A\big(\cos(w_1 t) - \cos(w_2 t) - \cos(w_3 t)\big) \tag{3.32}$$

$$x_2(t) = A\big(\cos(w_1 t) + \cos(w_3 t)\big) \\ = 2A\cos\big(t/2(w_1 + w_3)\big)\cos\big(t/2(w_1 - w_3)\big) \tag{3.33}$$

$$x_3(t) = A\big(\cos(w_1 t) + \cos(w_2 t)\big) \\ = 2A\cos\big(t/2(w_1 + w_2)\big)\cos\big(t/2(w_1 - w_2)\big) \tag{3.34}$$

As the normal frequencies w_2 and w_3 are equal, the motion of the second and the third pendula are the same. It is a consequence of including the coupling between the first and the third pendulum. Clearly, x_2 and x_3 have the same form as the equation (3.10) and (3.11) and thus they oscillate as a beating phenomenon. Hence, it is expected to see an envelope for the plot of their motion with respect to time. In the case of x_1, a wave-packet equation cannot be derived using the trigonometric identities. However, an envelope is seen when solving the coupled differential equations numerically using the computer program, shown in the figure below.

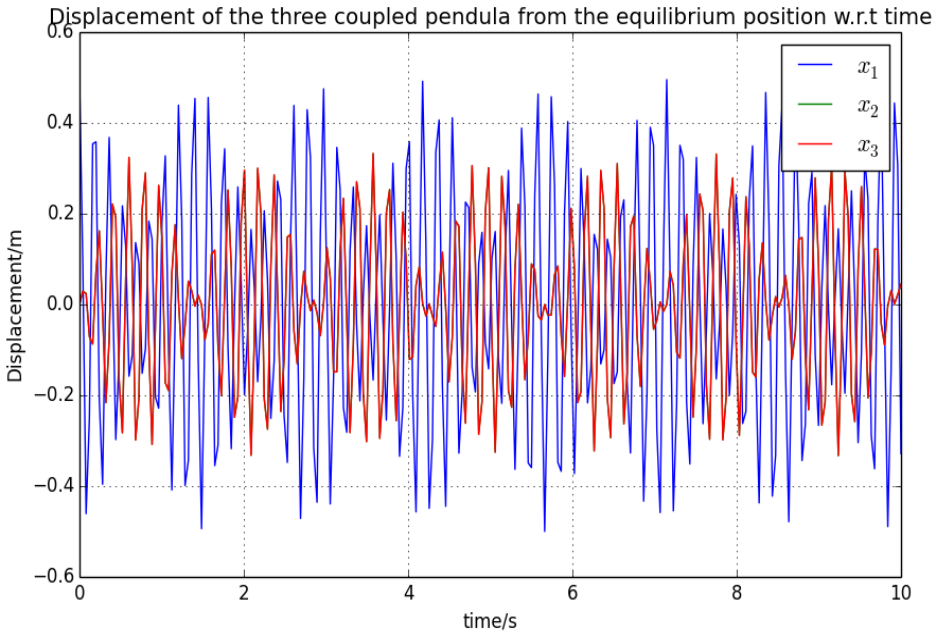

Figure 3.6: Beating phenomenon predicted for the motion of a system of three coupled pendula, by the numerical solution of the computer program.

These calculations cannot show an analogy to the neutrino oscillations of three flavours, because of the PMNS matrix whose elements are complex. From equation (3.30) it can be seen that oscillators are related to the normal coordinates through a 3×3 non complex matrix. In fact, the mixing matrix is not even a rotation matrix, so oscillators and normal coordinates cannot be thought as coordinate basis, and cannot be related to the flavour and mass eigenstates of a neutrino.

3.4 The numerical solutions

As explained before, the equations of motions in the case of two pendula and three pendula, formed a system of coupled differential equations, so could not be solved directly. For this reason, a computer program is written that solves the coupled differential equations numerically. A module in python is a file which contains built in functions. In this program "scipy" module is imported which contains

"integrate.odeint" function, which is able to integrate a system of ordinary differential equations [16]. However, this function integrates only up to first order, so a new variable is defined such that it transforms the second order ODE to a first order ODE, for example,

$$y_1 = \dot{x}_1 \quad \text{so,} \quad \dot{y}_1 = \ddot{x}_1 = -(g/l + k/m)x_1 + (k/m)x_2$$

where y_1 is the new variable, which is just the velocity of the pendulum. This transformation which introduces a new variable has been done in the first program two-pendula.py (the programs are shown in the appendix). In the second program, two-pendula-solver.py, the "scipy" package is imported and "odeint" function is called. It also contains, the initial conditions, maximum error allowed, the time interval and the number of points the equations are solved for. The third program is written to plot the data from the file two-pendula.dat.

The exact same method is used to solve the coupled differential equations in the case of three coupled pendula, the only difference is that a third variable is added to take into account the motion of the third pendulum.

Appendix A

The computer program

A.1 Two coupled pendula programs

A template was used to write the following programs in python language to solve the coupled differential equations numerically [17]. The programs are given in the order which has to be compiled:

Defining the equations of motion, two-pendula.py program:

```
# two_pendula.py
#
"""
This module defines the vector field for a system of two coupled pendula
connected by a spring.
"""


def vectorfield(w, t, p):
    """
    Defines the differential equations for the two coupled pendula system.

    Arguments:
        w :  vector of the state variables:
                  w = [x1,y1,x2,y2]
        t :  time
        p :  vector of the parameters:
                  p = [m,k,L,g]
    """
    x1, y1, x2, y2 = w
    m, k, l, g = p

    # Create f = (x1',y1',x2',y2'):
    # The coupled ODE's are the following:
    f = [y1,
```

```
        (- (g/l + k/m)*x1 + (k/m)*x2),
        y2,
        (-(g/l + k/m)*x2 + (k/m)*x1)]
    return f
```

Solving the coupled differential equations, two-pendula-solevr.py program:

```
# two_pendula_solver.py
#
"""Use ODEINT to solve the differential equations defined by the vector field
in two_pendula.py.
"""
#import ODEINT from scipy.integrate package:
from scipy.integrate import odeint
import two_pendula

# Parameter values
# Masses:
m = 1.0
# Spring constants
k = 100.0
# Natural lengths
l = 0.01
g=10.0

# Initial conditions
# x1 and x2 are the initial displacements; y1 and y2 are the initial velocities
x1 = 0.005
y1 = 0.0
x2 = 0
y2 = 0.0

# ODE solver parameters:
#set the errors:
abserr = 1.0e-8
relerr = 1.0e-6
#set time interval:
stoptime = 10.0
#set the number of points:
numpoints = 250

# Create the time samples for the output of the ODE solver.
# I use a large number of points, only because I want to make
# a plot of the solution that looks nice.
```

```
t = [stoptime * float(i) / (numpoints - 1) for i in range(numpoints)]

# Pack up the parameters and initial conditions:
p = [m, k, l, g]
w0 = [x1, y1, x2, y2]

# Call the ODE solver.
wsol = odeint(two_pendula.vectorfield, w0, t, args=(p,),
              atol=abserr, rtol=relerr)

# Print the solution.
for t1, w1 in zip(t, wsol):
    print t1, w1[0], w1[1], w1[2], w1[3]
```

The program to plot the data, two-pendula-plot.py:

```
# two_pendula_plot.py
#
"""Plot the solution that was generated by two_pendula_solver.py."""

from numpy import loadtxt
from pylab import figure, plot, xlabel,ylabel , grid, hold, legend, title, sav
#package needed to plot a graph:
from matplotlib.font_manager import FontProperties

#reading the data from the file two_pendula.dat
t, x1, xy, x2, y2 = loadtxt('two_pendula.dat', unpack=True)
#set the figure size:
figure(1, figsize=(10, 6))
#set the label for x_axis:
xlabel('t/s')
#include the grid:
grid(True)
#plot the two displacements on the same graph:
hold(True)
#set the line width in the plot:
lw = 1
#set the label for the y_axis:
ylabel('Displacement/m')
plot(t, x1, 'g', linewidth=lw)
plot(t, x2, 'b', linewidth=lw)

legend((r'$x_1$', r'$x_2$'), prop=FontProperties(size=16))
```

```
title('Displacement of two coupled pendula from the equilibrium position')
savefig('two_pendula.png', dpi=100)
```

A.2 Three coupled pendula program

Same form of programs are used to generate data for the motion of a system of three coupled pendula.

Defining the equations of motion, three-pendula.py program:

```
#
# three_pendula.py
#
"""
This module defines the vector field for a system of three coupled pendula
connected by three springs.
"""



def vectorfield(w, t, p):
    """
    Defines the differential equations for the three coupled pendula system.

    Arguments:
        w :  vector of the state variables:
                  w = [x1,y1,x2,y2,x3,y3]
        t :  time
        p :  vector of the parameters:
                  p = [m,k,L,g]
    """
    x1, y1, x2, y2, x3, y3 = w
    m, k, l, g = p

    # Create f = (x1',y1',x2',y2',x3',y3'):
    # The coupled ODE's are the following:
    f = [y1,
         (- (g/l + 2*k/m)*x1 + (k/m)*x2 + (k/m)*x3),
         y2,
         (-(g/l + 2*k/m)*x2 + (k/m)*x1 + (k/m)*x3),
         y3,
         (k/m)*x1 + (k/m)*x2 + (- (g/l + 2*k/m)*x3)]
    return f
```

Solving the coupled differential equations, three-pendula-solevr.py program:

```
#
# three_pendula_solver.py
```

```
#
"""Use ODEINT to solve the differential equations defined by the vector field
in three_pendula.py.
"""
#import ODEINT from scipy.integrate package:
from scipy.integrate import odeint
import three_pendula

# Parameter values
# Masses:
m = 1.0
# Spring constants
k = 100.0
# Natural lengths
l = 0.01
g=10.0

# Initial conditions
# x1 and x2 are the initial displacements; y1 and y2 are the initial velocitie
x1 = 0.005
y1 = 0.0
x2 = 0
y2 = 0.0
x3 = 0.0
y3 = 0.0
# ODE solver parameters
#set the errors:
abserr = 1.0e-8
relerr = 1.0e-6
#set time interval:
stoptime = 10.0
#set the number of points:
numpoints = 250

# Create the time samples for the output of the ODE solver.
# I use a large number of points, only because I want to make
# a plot of the solution that looks nice.
t = [stoptime * float(i) / (numpoints - 1) for i in range(numpoints)]

# Pack up the parameters and initial conditions:
p = [m, k, l, g]
w0 = [x1, y1, x2, y2, x3, y3]
```

```
# Call the ODE solver.
wsol = odeint(three_pendula.vectorfield, w0, t, args=(p,),
              atol=abserr, rtol=relerr)

# Print the solution.
for t1, w1 in zip(t, wsol):
    print t1, w1[0], w1[1], w1[2], w1[3], w1[4], w1[5]
```

The program to plot the data, two-pendula-plot.py:

```
#
# three_pendula_plot.py
#
"""Plot the solution that was generated by three_pendula_solver.py."""

from numpy import loadtxt
from pylab import figure, plot, xlabel,ylabel , grid, hold, legend, title, savefig
#package needed to plot a graph:
from matplotlib.font_manager import FontProperties

#reading the data from the file three_pendula.dat
t, x1, xy, x2, y2, x3, y3 = loadtxt('three_pendula.dat', unpack=True)
#set the figure size:
figure(1, figsize=(10, 6))
#set the label for x_axis:
xlabel('time/s')
#include the grid:
grid(True)
#plot the two displacements on the same graph:
hold(True)
#set the line width in the plot:
lw = 1
#set the label for the y_axis:
ylabel('Displacement/m')
plot(t, x1, 'b', linewidth=lw)
plot(t, x2, 'g', linewidth=lw)
plot(t, x3, 'r', linewidth=lw)

legend((r'$x_1$', r'$x_2$',r'$x_3$'), prop=FontProperties(size=16))
title('Displacement of the three coupled pendula from the equilibrium position w.r
savefig('three_pendula.png', dpi=100)
```

A.3 The commands

The commands used in terminal to compile the programs are the following:

```
1-python two-pendula.py
2-python two-pendula-solver.py > two-pendula.dat
```

and,

```
1-python three-pendula.py
2-python three-pendula-solver.py > three-pendula.dat
```

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