Linear Programming and the Simplex method

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Mathematical Programming models

$$
\begin{aligned}
\min(\max) \quad & f(x) \\
\text{s.t.} \quad & g_i(x) = b_i \quad (i = 1 \dots k) \\
& g_i(x) \leq b_i \quad (i = k+1 \dots k') \\
& g_i(x) \geq b_i \quad (i = k'+1 \dots m) \\
& x \in \mathbb{R}^n
\end{aligned}
$$

- $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a vector (column) of n **REAL** variables;
- f e g_i are functions $\mathbb{R}^n \rightarrow \mathbb{R}$
- $b_i \in \mathbb{R}$

Linear Programming (LP) models

f e g_i are **linear** functions of x

$$
\begin{array}{llll}
\min(\max) & c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\
\text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n & = b_i & (i = 1 \ldots k) \\
& a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n & \leq b_i & (i = k+1 \ldots k') \\
& a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n & \geq b_i & (i = k'+1 \ldots m) \\
& x_i \in \mathbb{R} & & (i = 1 \ldots n)
\end{array}
$$

Notice: for the moment, just **CONTINUOUS variables are considered**!!!

We need different methods for models with integer or binary variables.

Resolution of an LP model

- *Feasible solution*: $x \in \mathbb{R}^n$ satisfying all the constraints
- *Feasible region*: set of all the feasible solutions x
- *Optimal solution* x^* [min]: $c^T x^* \leq c^T x, \forall x \in \mathbb{R}^n, x$ feasible.

Solving a LP model is determining if it:

- is unfeasible
- is unlimited
- has a (finite) optimal solution

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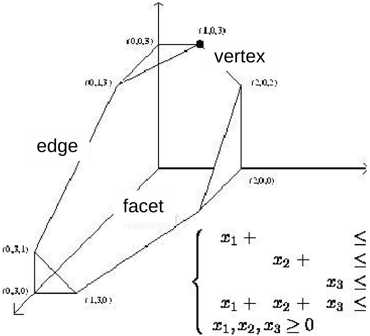
Solving a LP model is determining if it:

- is unfeasible i.e. there's no feasible solution in the feasible region: the constraints are contradictory
- is unlimited i.e. we have an infinite solution because the feasible region is unlimited in the direction to optimize
- has a (finite) optimal solution

Geometry of LP

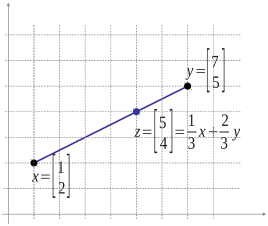
The feasible region is a **polyhedron** (intersection of a finite number of closed half-spaces and hyperplanes in $\mathbb{R}^n$)



LP problem: $\min(\max)\{c^T x : x \in P\}$, P is a polyhedron in $\mathbb{R}^n$.

Vertex of a polyhedron: definition

- $z \in \mathbb{R}^n$ is a **convex combination** of two points x and y if $\exists\ \lambda \in [0,1]$: $z = \lambda x + (1-\lambda)y$



- $z \in \mathbb{R}^n$ is a **strict convex combination** of two points x and y if $\exists\ \lambda \in < (0,1) > : z = \lambda x + (1-\lambda)y$.

- $v \in P$ is **vertex of a polyhedron** P if it **is not** a **strict convex combination** of two *distinct* points of P: $\nexists\ x, y \in P, \lambda \in (0,1) : x \neq y, v = \lambda x + (1-\lambda)y$

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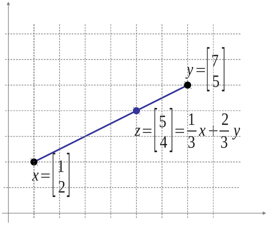

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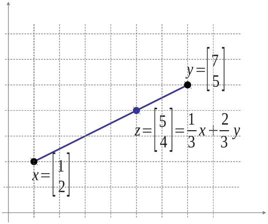

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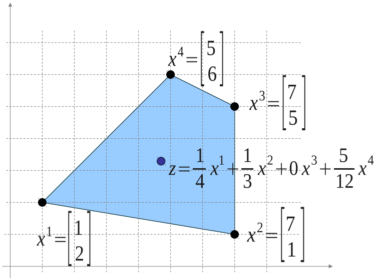
Representation of polyhedra

$z \in \mathbb{R}^n$ is **convex combination** of $x^1, x^2 \dots x^k$ if $\exists\ \lambda_1, \lambda_2 \dots \lambda_k \geq 0$:
$\sum_{i=1}^{k} \lambda_i = 1$ and $z = \sum_{i=1}^{k} \lambda_i x^i$



Theorem: representation of polyhedra [Minkowski-Weyl] - case limited

Polydron *limited* $P \subseteq \mathbb{R}^n$, $v^1, v^2, ..., v^k$ ($v^i \in \mathbb{R}^n$) vertices of P

if $x \in P$ then $x = \sum_{i=1}^{k} \lambda_i v^i$ with $\lambda_i \geq 0, \forall i = 1..k$ and $\sum_{i=1}^{k} \lambda_i = 1$

(x is convex combination of the vertices of P)

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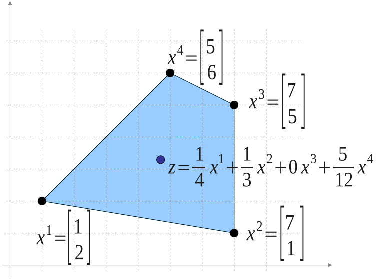

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Optimal vertex: from graphical intuition to proof

Theorem: optimal vertex(fix *min* objective function)

LP problem $\min\{c^T x : x \in P\}$, P non empty and limited

- LP ha optimal solution
- **one of the optimal solution of LP is a vertex of** P

Proof:

$V = \{v^1, v^2 \dots v^k\} \qquad v^* = \arg\min_{v \in V} c^T v$

$$c^T x = c^T \sum_{i=1}^{k} \lambda_i v^i = \sum_{i=1}^{k} \lambda_i c^T v^i \geq \sum_{i=1}^{k} \lambda_i c^T v^* = c^T v^* \sum_{i=1}^{k} \lambda_i = c^T v^*$$

Summarizing: $\forall x \in P, \;\; c^T v^* \leq c^T x$ ■

We can limit the search of an optimal solution to the vertices of P!

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Vertex comes from intersection of generating hyperplanes

$$
\begin{array}{rrcrcrl}
\max & 13x_1 & + & 10x_2 & & & \\
s.t. & 3x_1 & + & 4x_2 & \le & 24 & \text{(e1)} \\
 & x_1 & + & 4x_2 & \le & 20 & \text{(e2)} \\
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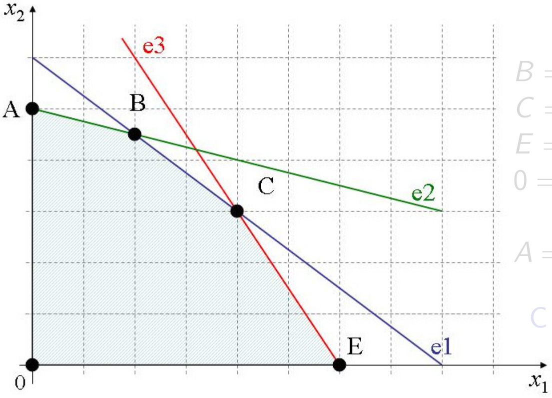

$B = e1 \cap e2$	$(2, 9/2)$	71
$C = e1 \cap e3$	$(4, 3)$	82
$E = e3 \cap (x_2 = 0)$	$(6, 0)$	78
$0 = (x_1 = 0) \cap (x_2 = 0)$	$(0, 0)$	0
$A = e2 \cap (x_1 = 0)$	$(0, 5)$	50

C optimum!

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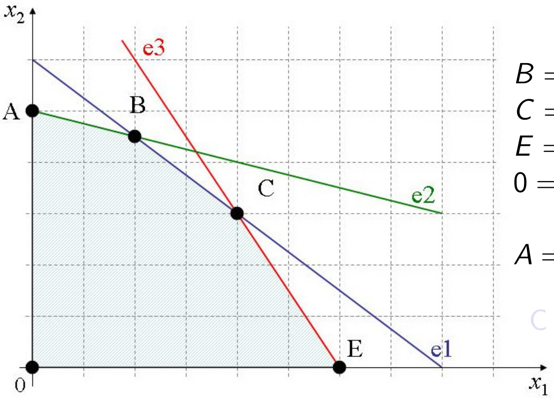

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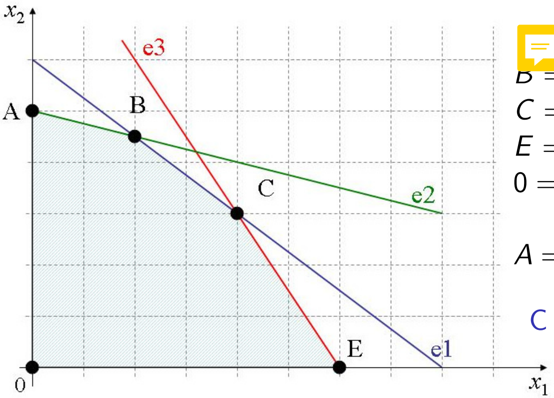

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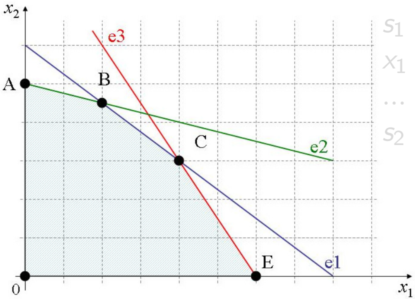
Algebraic representation of vertices

Write the constraints as **equations**

$$\begin{array}{rcrcrcrcrcl} 3x_1 & + & 4x_2 & + & s_1 & & & & & = & 24 \\ x_1 & + & 4x_2 & & & + & s_2 & & & = & 20 \\ 3x_1 & + & 2x_2 & & & & & + & s_3 & = & 18 \end{array}$$

$5 - 3 = 2$ degrees of freedom:

we can set (any) two variables to 0 and obtain a unique solution!



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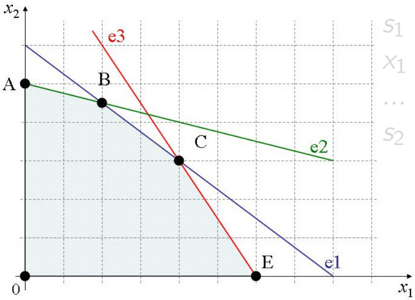

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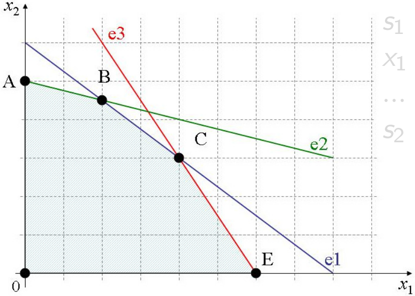

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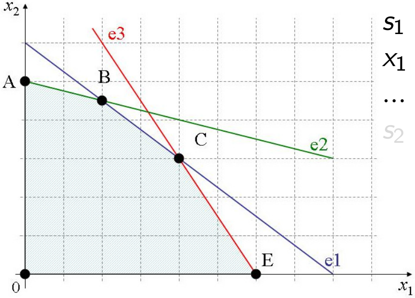

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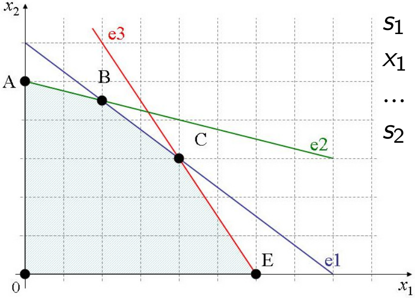

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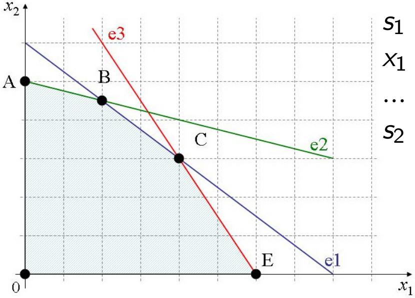

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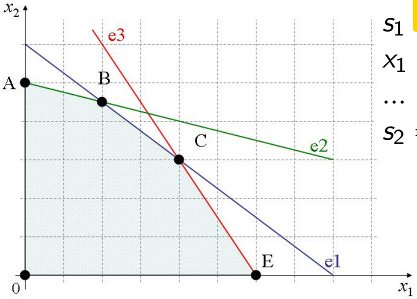

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Standard form for LP problems

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\begin{array}{lll}
\min & c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\
\text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \quad = b_i & (i = 1 \ldots m) \\
& x_i \in \mathbb{R}_+ & (i = 1 \ldots n)
\end{array}
$$

- **minimizing** objective function (if not, multiply by -1);
- variables ≥ 0; (if not, substitution)
- all constraints are equalities; ($+/-$ slack/surplus variables)
- $b_i \geq 0$. (if not, multiply by -1)

Standard form: example

$$
\begin{array}{ll}
\max & 5(-3x_1 + 5x_2 - 7x_3) + 34 \\
s.t. & -2x_1 + 7x_2 + 6x_3 - 2x_1 \leq 5 \\
 & -3x_1 + x_3 + 12 \geq 13 \\
 & x_1 + x_2 \leq -2 \\
 & x_1 \leq 0 \\
 & x_2 \geq 0
\end{array}
$$

$$
\begin{array}{ll}
\hat{x}_1 = -x_1 & (\hat{x}_1 \geq 0) \\
x_3 = x_3^+ - x_3^- & (x_3^+ \geq 0 \ , \ \ x_3^- \geq 0)
\end{array}
$$

$$
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 & 3\hat{x}_1 + x_3^+ - x_3^- - s_2 = 1 \\
 & \hat{x}_1 - x_2 - s_3 = 2 \\
 & \hat{x}_1 \geq 0 \ , \ x_2 \geq 0 \ , \ x_3^+ \geq 0 \ , \ x_3^- \geq 0 \ , \ s_1 \geq 0 \ , \ s_2 \geq 0 \ , \ s_3 \geq 0.
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 & \hat{x}_1 - x_2 - s_3 = 2 \\
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\end{array}
$$

Linear algebra: definitions

- column vector $v \in \mathbb{R}^{n\times 1}$: $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$
- row vector $v^T \in \mathbb{R}^{1\times n}$: $v^T = [v_1, v_2, ..., v_n]$
- matrix $A \in \mathbb{R}^{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$
- $v, w \in \mathbb{R}^n$, scalar product $v \cdot w = \sum_{i=1}^{n} v_i w_i = v^T w = vw^T$
- Rank of $A \in \mathbb{R}^{m\times n}$, $\rho(A)$, max linearly independent rows/columns
- $B \in \mathbb{R}^{m\times m}$ *invertible* $\iff \rho(B) = m \iff det(B) \neq 0$

Systems of linear equations

- *Systems of equations in matrix form*: a system of m equations in n variables can be written as:
  $Ax = b$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ e $x \in \mathbb{R}^n$.
- *Theorem of Rouché-Capelli*:
  $Ax = b$ has solutions $\iff \rho(A) = \rho(A|b) = r$ (∞^{n-r} solutions).
- *Elementary row operations*:
  - swap row i and row j;
  - multiply row i by a non-zero scalar;
  - substitute row i by row i plus α times row j ($\alpha \in \mathbb{R}$).

  Elementary operations on (augmented) matrix $[A|b]$ leave the same solutions as $Ax = b$.
- *Gauss-Jordan method* for solving $Ax = b$: make elementary row operations on $[A|b]$ so that A contains an identity matrix of dimension $\rho(A) = \rho(A|b)$.

Basic solutions

- **Assumptions**: system $Ax = b$, $A \in \mathbb{R}^{m \times n}$, $\rho(A) = m$, $m < n$
- **Basis of** A: square submatrix with maximum rank, $B \in \mathbb{R}^{m \times m}$
- $A = [B|N] \quad B \in \mathbb{R}^{m \times m}, det(B) \neq 0$
$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}, x_B \in \mathbb{R}^m, x_N \in \mathbb{R}^{n-m}$$
- $Ax = b \Longrightarrow [B|N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = Bx_B + Nx_N = b$
- $x_B = B^{-1}b - B^{-1}Nx_N$
- imposing $x_N = 0$, we obtain a so called **basic solution**:
$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$$
- many different basic solutions by choosing a **different basis** of A
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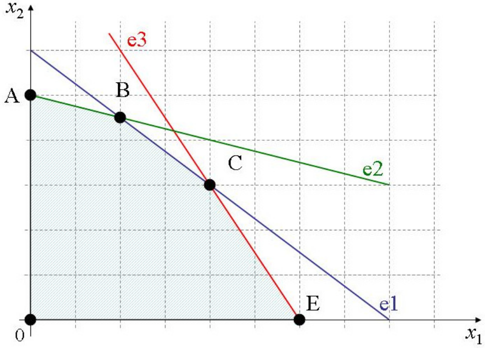
Basic solutions and LP in standard form

$$\begin{array}{llll} \min & c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\ \text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n & = b_i & (i = 1 \ldots m) \\ & x_i \in \mathbb{R}_+ & & (i = 1 \ldots n) \end{array} \qquad \begin{array}{lll} \min & c^T x & \\ \text{s.t.} & Ax & = b \\ & x & \geq 0 \end{array}$$

- basis B gives a **feasible basic solution** if $x_B = B^{-1}b \geq 0$

$$\begin{array}{llllll} 3x_1 & +4x_2 & +s_1 & & & = 24 \\ x_1 & +4x_2 & & +s_2 & & = 20 \\ 3x_1 & +2x_2 & & & +s_3 & = 18 \end{array} \qquad A = \begin{bmatrix} 3 & 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 24 \\ 20 \\ 18 \end{bmatrix}$$



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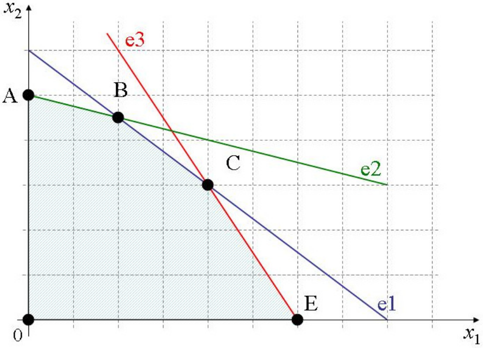

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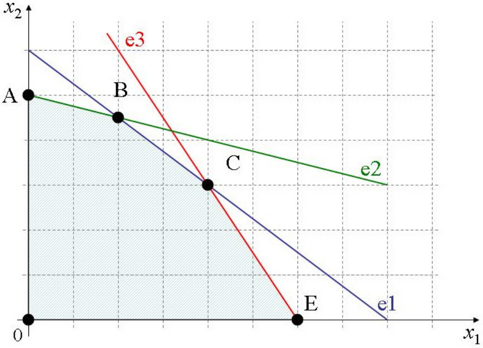

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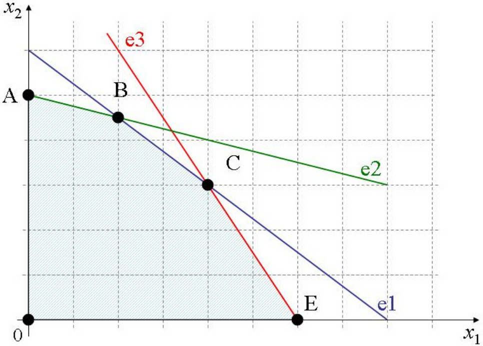

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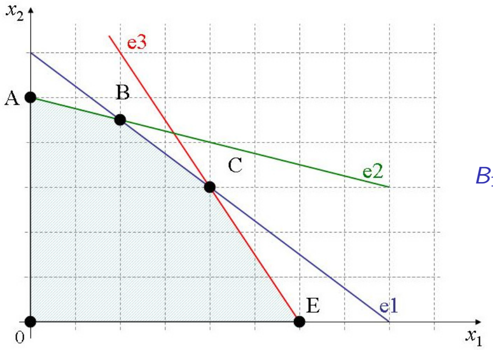

$$B_1 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

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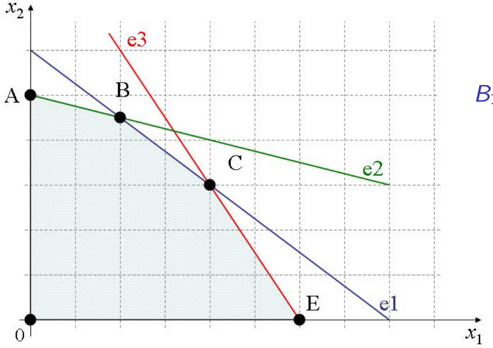

$$B_1 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = B_1^{-1}b = \begin{bmatrix} 2 \\ 4,5 \\ 3 \end{bmatrix}$$

$$x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Basic solutions and LP in standard form

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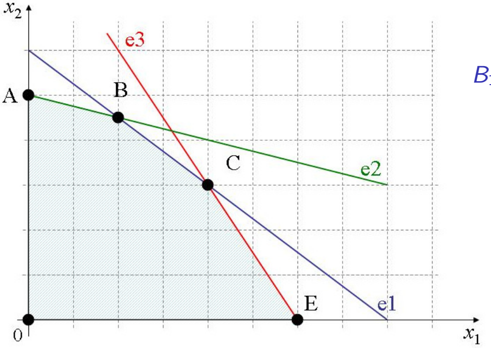

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$x^T = (2 \;\; 9/2 \;\; 0 \;\; 0 \;\; 3)$ $\rightarrow$ **vertex** B

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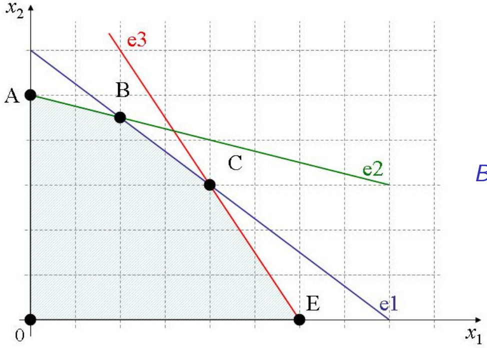

$$B_2 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

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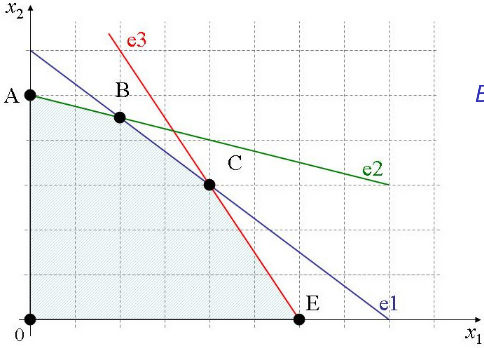

$$B_2 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_2 \end{bmatrix} = B_2^{-1}b = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

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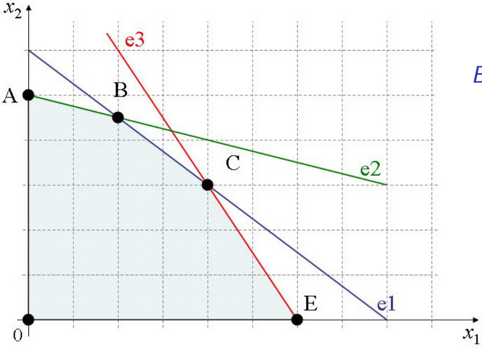

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$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_2 \end{bmatrix} = B_2^{-1}b = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$x_N = \begin{bmatrix} s_1 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x^T = (4\ 3\ 0\ 2\ 0)$ $\rightarrow$ **vertex** C

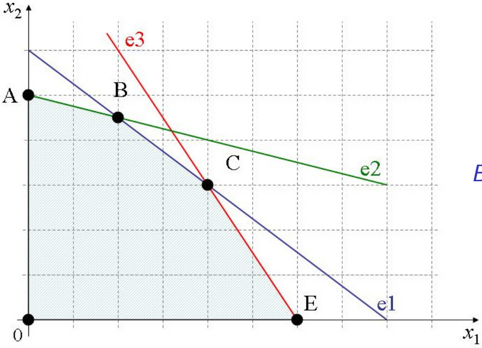
Basic solutions and LP in standard form

$$\begin{array}{lll} \min & c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\ \text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i & (i = 1 \ldots m) \\ & x_i \in \mathbb{R}_+ & (i = 1 \ldots n) \end{array} \qquad \begin{array}{lll} \min & c^T x & \\ \text{s.t.} & Ax & = b \\ & x & \geq 0 \end{array}$$

- basis B gives a **feasible basic solution** if $x_B = B^{-1}b \geq 0$

$$\begin{array}{llllll} 3x_1 & +4x_2 & +s_1 & & & = 24 \\ x_1 & +4x_2 & & +s_2 & & = 20 \\ 3x_1 & +2x_2 & & & +s_3 & = 18 \end{array}$$

$$A = \begin{bmatrix} 3 & 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 24 \\ 20 \\ 18 \end{bmatrix}$$



$$B_3 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

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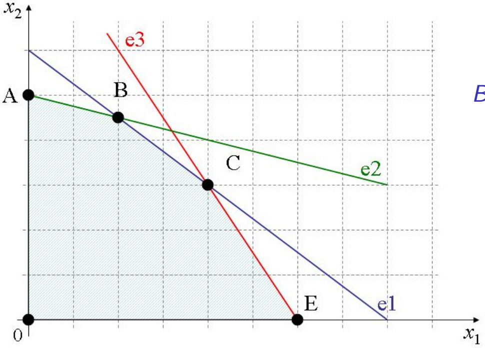

$$B_3 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ s_1 \\ s_2 \end{bmatrix} = B_3^{-1}b = \begin{bmatrix} 6 \\ 6 \\ 14 \end{bmatrix}$$

$$x_N = \begin{bmatrix} x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Basic solutions and LP in standard form

$$\begin{array}{llll} \min & c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\ \text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n & = b_i & (i = 1 \ldots m) \\ & x_i \in \mathbb{R}_+ & & (i = 1 \ldots n) \end{array} \qquad \begin{array}{lll} \min & c^T x & \\ \text{s.t.} & Ax & = b \\ & x & \geq 0 \end{array}$$

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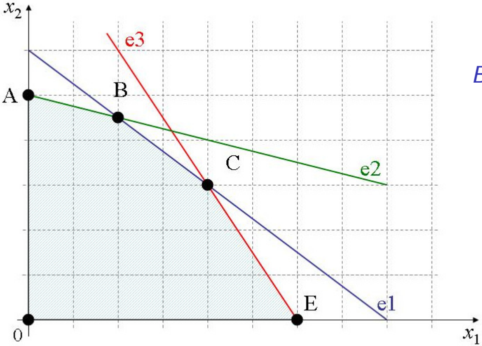

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Basic solutions and LP in standard form

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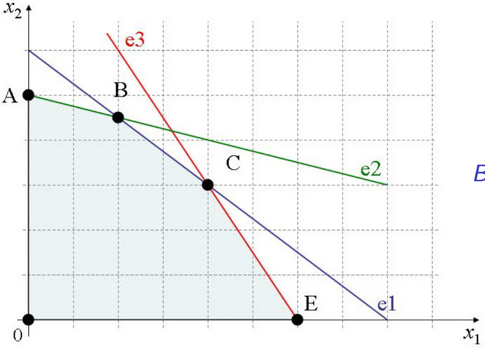

$$B_4 = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 4 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

Basic solutions and LP in standard form

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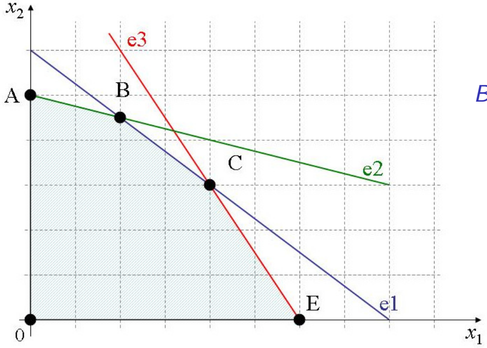

$$B_4 = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 4 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \end{bmatrix} = B_4^{-1}b = \begin{bmatrix} 18/5 \\ 21/5 \\ -18/5 \end{bmatrix}$$

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Basic solutions and LP in standard form

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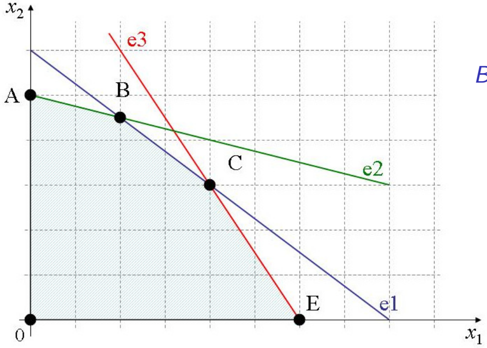

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$x^T = (18/5\ 21/5\ -18/5\ 0\ 0) \rightarrow$ **n.f.!**

Vertices and basic solution

Feasible basic solution $\rightsquigarrow$ $n-m$ variables are 0 $\rightsquigarrow$ intersection of the right number of hyperplanes $\rightsquigarrow$ vertex!

PL $\min\{c^T x : Ax = b, x \geq 0\}$ $\qquad$ $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$

Theorem: vertices correspond to feasible basic solutions
(algebraic characterization of the vertices of a polyhedron)

x **feasible basic solution of** $Ax = b$ $\iff$ x **is a vertex of** P

Corollary: optimal basic solution

If P non empty and limited, then **there exists at least an optimal solution which is a basic feasible solution**

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Algorithm for LP (case limited): sketch

Consider **all** the feasible basic solutions:

1. put the LP in standard form $\min\{c^T x : Ax = b, x \geq 0\}$
2. $incumbent = +\infty$
3. **repeat**
4. generate a combination of m columns of A
5. let B be the corresponding submatrix of A
6. **if** $det(B) == 0$ **then continue else** compute $x_B = B^{-1}b$
7. **if** $x_B \geq 0$ **and** $c^T x_B < incumbent$ **then** update $incumbent$
8. **until**(no other column combinations)

Complexity: up to $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$ basic solution!!!

$\Rightarrow$ **Symplex method**: more efficient exploration of the basic solutions (only **feasible** and **improving**)

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Example

LP problem in **standard form**:

$$\begin{array}{lrcrcrcrcrcl} \min & z = -13x_1 & - & 10x_2 & & & & & & & & \\ s.t. & 3x_1 & + & 4x_2 & + & s_1 & & & & & = & 24 \\ & x_1 & + & 4x_2 & & & + & s_2 & & & = & 20 \\ & 3x_1 & + & 2x_2 & & & & & + & s_3 & = & 18 \\ & x_1 & , & x_2 & , & s_1 & , & s_2 & , & s_3 & \geq & 0 \end{array}$$

an initial **basic feasible solution** (vertex B):

- $B = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
- $x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 9/2 \\ 3 \end{bmatrix} \quad x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $z_B = c^T x = c_B^T x_B + c_N^T x_N = -71$

Example

Change basis: **New basic solution** $\Rightarrow$ one non-basic variable increases **affecting x_B and z_B**

$$
\begin{aligned}
x_B &= B^{-1}b - B^{-1}N\,x_N \\
z &= c^T x = c_B^T x_B + c_N^T x_N = c_B^T(B^{-1}b - B^{-1}N\,x_N) + c_N^T x_N \\
&\qquad = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N)\,x_N
\end{aligned}
$$

Write x_B and z as functions of only **non-basic** variables

For the sake of manual computation, use **Gauss-Jordan**:

$$Ax = b \quad \rightsquigarrow \quad [\,B\ N \mid b\,] \quad \rightsquigarrow \quad [\,B^{-1}B = I\ B^{-1}N = \bar{N} \mid B^{-1}b = \bar{b}\,]$$

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Example

	x_1	x_2	s_3	s_1	s_2	$\bar{b}$
	3	4	0	1	0	24
	1	4	0	0	1	20
	3	2	1	0	0	18
$(R_1/3)$	1	4/3	0	1/3	0	8
$(R_2 - R_1/3)$	0	8/3	0	$-1/3$	1	12
$(R_3 - R_1)$	0	-2	1	-1	0	-6
$(R_1 - 1/2\ R_2)$	1	0	0	1/2	$-1/2$	2
$(3/8\ R_2)$	0	1	0	$-1/8$	3/8	9/2
$(R_3 + 3/4\ R_2)$	0	0	1	$-5/4$	3/4	3

$$x_1 = 2 - 1/2\ s_1 + 1/2\ s_2$$
$$x_2 = 9/2 + 1/8\ s_1 - 3/8\ s_2$$
$$s_3 = 3 + 5/4\ s_1 - 3/4\ s_2$$

$$z = -13x_1 - 10x_2 = -71 + 21/4\ s_1 - 11/4\ s_2$$

Example

$$
\begin{array}{l|rrrrr|r}
 & x_1 & x_2 & s_3 & s_1 & s_2 & \bar{b} \\
\hline
 & 3 & 4 & 0 & 1 & 0 & 24 \\
 & 1 & 4 & 0 & 0 & 1 & 20 \\
 & 3 & 2 & 1 & 0 & 0 & 18 \\
\hline
(R_1/3) & 1 & 4/3 & 0 & 1/3 & 0 & 8 \\
(R_2 - R_1/3) & 0 & 8/3 & 0 & -1/3 & 1 & 12 \\
(R_3 - R_1) & 0 & -2 & 1 & -1 & 0 & -6 \\
\hline
(R_1 - 1/2\ R_2) & 1 & 0 & 0 & 1/2 & -1/2 & 2 \\
(3/8\ R_2) & 0 & 1 & 0 & -1/8 & 3/8 & 9/2 \\
(R_3 + 3/4\ R_2) & 0 & 0 & 1 & -5/4 & 3/4 & 3
\end{array}
$$

$$
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(R_3 + 3/4\ R_2) & 0 & 0 & 1 & -5/4 & 3/4 & 3
\end{array}$$

$$\begin{array}{lcrcrlcrl}
x_1 & = & 2 & - & 1/2 & s_1 & + & 1/2 & s_2 \\
x_2 & = & 9/2 & + & 1/8 & s_1 & - & 3/8 & s_2 \\
s_3 & = & 3 & + & 5/4 & s_1 & - & 3/4 & s_2 \\
z = -13x_1 - 10x_2 & = & -71 & + & 21/4 & s_1 & - & 11/4 & s_2
\end{array}$$

Example

$$z = -71 + 21/4 \; s_1 - 11/4 \; s_2$$

- In order to minimize, it is convenient to increase s_2 (and keep $s_1 = 0$)
- Equalities have to be always satisfied...:

$$\begin{array}{rclcl} x_1 & = & 2 & + & 1/2 \; s_2 \\ x_2 & = & 9/2 & - & 3/8 \; s_2 \\ s_3 & = & 3 & - & 3/4 \; s_2 \end{array}$$

- while preserving non-negativity:

$$\begin{array}{lclclcl} x_1 \geq 0 & \Rightarrow & 2 + 1/2 s_2 \geq 0 & \Rightarrow & s_2 & \geq & -4 \quad \textbf{always!} \\ x_2 \geq 0 & \Rightarrow & 9/2 - 3/8 s_2 \geq 0 & \Rightarrow & s_2 & \leq & 12 \\ s_3 \geq 0 & \Rightarrow & 3 - 3/4 s_2 \geq 0 & \Rightarrow & s_2 & \leq & 4 \end{array}$$

- New **feasible** and **better** solutions with $s_1 = 0$ and $0 \leq s_2 \leq 4$
- $s_2 = 4 \Rightarrow s_3 = 0$: new **basic**, **feasible** and **better** solution

Example

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$$\begin{array}{lcrcrcrl} x_1 \geq 0 & \Rightarrow & 2 + 1/2 s_2 \geq 0 & \Rightarrow & s_2 & \geq & -4 & \textbf{always!} \\ x_2 \geq 0 & \Rightarrow & 9/2 - 3/8 s_2 \geq 0 & \Rightarrow & s_2 & \leq & 12 & \\ s_3 \geq 0 & \Rightarrow & 3 - 3/4 s_2 \geq 0 & \Rightarrow & s_2 & \leq & 4 & \end{array}$$

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Example

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- while preserving non-negativity:

$$\begin{array}{lcrcccrl} x_1 \geq 0 & \Rightarrow & 2 + 1/2 s_2 \geq 0 & \Rightarrow & s_2 & \geq & -4 & \textbf{always!} \\ x_2 \geq 0 & \Rightarrow & 9/2 - 3/8 s_2 \geq 0 & \Rightarrow & s_2 & \leq & 12 & \\ s_3 \geq 0 & \Rightarrow & 3 - 3/4 s_2 \geq 0 & \Rightarrow & s_2 & \leq & 4 & \end{array}$$

- New **feasible** and **better** solutions with $s_1 = 0$ and $0 \leq s_2 \leq 4$
- $s_2 = 4 \Rightarrow s_3 = 0$: new **basic**, **feasible** and **better** solution

Example

$$z = -71 + 21/4\ s_1 - 11/4\ s_2$$

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New basic solution! s_2 (now > 0) takes the place of s_3 (now $= 0$):

$$B = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 1 \\ 3 & 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$x_N = \begin{bmatrix} s_1 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad z_B = c^T x = c_B^T x_B + c_N^T x_N = -82$$

Same arguments as before: x_B and z **as a function of** x_N:

$$\begin{array}{rcrcrcrcrc}
x_1 & = & 4 & + & 1/3 & s_1 & - & 2/3 & s_3 \\
x_2 & = & 3 & - & 1/2 & s_1 & - & 1/2 & s_3 \\
s_3 & = & 4 & + & 5/3 & s_1 & - & 4/3 & s_3 \\
\\
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\end{array}$$

Optimal solution! Visited 2 out of $\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10$ possible basis

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LP in *canonical* form

PL $\min\{z = c^T x : Ax = b, x \geq 0\}$ is in **canonical form with respect to basis** B if all basic variables and the objective are explicitly written as functions of **non-basic variables only**:

$$
\begin{array}{rcl}
z & = & \bar{z}_B + \bar{c}_{N_1} \, x_{N_1} + \bar{c}_{N_2} \, x_{N_2} + \ldots + \bar{c}_{N_{(n-m)}} \, x_{N_{(n-m)}} \\
x_{B_i} & = & \bar{b}_i - \bar{a}_{iN_1} \, x_{N_1} - \bar{a}_{iN_2} \, x_{N_2} - \ldots - \bar{a}_{iN_{(n-m)}} \, x_{N_{(n-m)}} \quad (i = 1 \ldots m)
\end{array}
$$

- $\bar{z}_B$ scalar (objective function value for the corresponding basic solution)
- $\bar{b}_i$ scalar (value of basic variable i)
- B_i index of the i-th basic variable ($i = 1 \ldots m$)
- N_j index of the j-th non-basic variable ($j = 1 \ldots n - m$)
- $\bar{c}_{N_j}$ coefficient of the j-th non-basic variable in the objective function (**reduced cost of the variable with respect to basis** B)
- $-\bar{a}_{iN_j}$ coefficient of the j-th non-basic variable in the constraints that makes explicit the i-th basic variable

Simplex method: optimality check

- **Reduced cost** of a variable: marginal unit increment of the objective function
- The reduced cost of a basis variable is $\bar{c}_{B_i} = 0$

Theorem: Sufficient optimality conditions

Given an LP and a feasible basis B, if all the reduced costs with respect to B are ≥ 0, then B is an optimal basis

$$\bar{c}_j \geq 0, \ \forall\, j = 1 \dots n \quad \Rightarrow \quad B \text{ optimal}$$

- Notice: the inverse is not true! [there may be optimal basic solutions with negative reduced costs]

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Simplex method: basis change

- From feasible basis B, obtain a $\tilde{B}$ **adjacent, feasible, improving**
- **One** column ($\approx$ variable) enters and one variable leaves the basis

- **Entering** variable (improvement): *any* $x_h : \bar{c}_h < 0$

$$z = \bar{z}_B + \bar{c}_h x_h = \bar{z}_{\tilde{B}} \leq \bar{z}_B$$

- **Leaving** variable (feasibility): [min ratio rule]

$$x_{B_i} \geq 0 \quad \Rightarrow \quad b_i - \bar{a}_{ih}\, x_h \geq 0,\ \forall\ i \quad \Rightarrow \quad x_h \leq \frac{\bar{b}_i}{\bar{a}_{ih}},\ \forall\ i : \bar{a}_{ih} > 0$$

$$t = \arg \min_{i=1\dots m} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$$

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Simplex method: check for unlimited LP

- Let x_h: $\bar{c}_h < 0$.

$$
\begin{aligned}
z &= \bar{z}_B + \bar{c}_h \; x_h \\
x_{B_i} &= \bar{b}_i - \bar{a}_{ih} \; x_h \quad (i = 1 \dots m)
\end{aligned}
$$

- If $a_{ih} \leq 0$, $\forall\, i = 1 \dots m$, feasible solution with $x_h \to +\infty$

Condition of unlimited LP

There exists a basis such that

$$
\exists\, x_h : \; (\bar{c}_h \; < \; 0) \; \wedge \; (\bar{a}_{ih} \; \leq \; 0, \; \forall \; i = 1 \dots m)
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Simplex method: summary

Init: PL in standard form $\min\{c^T x : Ax = b, x \geq 0\}$, and an initial feasible basis B

repeat

write the LP in canonical form with respect to B

$$\begin{aligned} z &= \bar{z}_B + \bar{c}_{N_1} x_{N_1} + \bar{c}_{N_2} x_{N_2} + \ldots + \bar{c}_{N_{(n-m)}} x_{N_{(n-m)}} \\ x_{B_i} &= \bar{b}_i - \bar{a}_{iN_1} x_{N_1} - \bar{a}_{iN_2} x_{N_2} - \ldots - \bar{a}_{iN_{(n-m)}} x_{N_{(n-m)}} \quad (i = 1 \ldots m) \end{aligned}$$

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$B \leftarrow B \oplus A_h \ominus A_{B_t}$ [basis change]

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Simplex tableau

- Represent the canonical form, can be used to operate Gauss-Jordan
- **Objective function as a constraint** (imposing the value of a new variable z):

  $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \rightsquigarrow \quad c_1x_1 + c_2x_2 + \ldots + c_nx_n - z = 0$

	x_{B_1} … x_{B_m}	x_{N_1} … $x_{N_{n-m}}$	z	$\bar{b}$
riga 0	c_B^T	c_N^T	-1	0
riga 1			0	
$\vdots$	B	N	$\vdots$	b
riga m			0	

- Elementary row (z included) operations: up to reading x_B (and z) as functions of x_N

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	x_{B_1}	$\ldots$	x_{B_m}	x_{N_1}	$\ldots$	$x_{N_{n-m}}$	z	$\bar{b}$
riga 0	0	$\ldots$	0	$\square$	$\ldots$	$\square$	-1	$\square$
riga 1	1		0	$\square$	$\ldots$	$\square$	0	$\square$
$\vdots$		$\ddots$		$\square$	$\ldots$	$\square$	$\vdots$	$\square$
riga m	0		1	$\square$	$\ldots$	$\square$	0	$\square$

Tableau in canonical form

- Elementary row (z included) operations: up to reading x_B (and z) as functions of x_N

Tableau and canonical form

	x_{B_1}	$\ldots$	x_{B_m}	x_{N_1}	$\ldots$	$x_{N_{n-m}}$	z	$\bar{b}$
$-z$	0	$\ldots$	0	$\square$	$\ldots$	$\square$	-1	$\square$
x_{B_1}	1		0	$\square$	$\ldots$	$\square$	0	$\square$
x_{B_i}		$\ddots$		$\square$	$\ldots$	$\square$	$\vdots$	$\square$
x_{B_m}	0		1	$\square$	$\ldots$	$\square$	0	$\square$

$$z = \bar{z}_B + \bar{c}_{N_1}\, x_{N_1} + \bar{c}_{N_2}\, x_{N_2} + \ldots + \bar{c}_{N_{(n-m)}}\, x_{N_{(n-m)}}$$

$$x_{B_i} = \bar{b}_i - \bar{a}_{iN_1}\, x_{N_1} - \bar{a}_{iN_2}\, x_{N_2} - \ldots - \bar{a}_{iN_{(n-m)}}\, x_{N_{(n-m)}} \quad (i = 1 \ldots m)$$

Tableau and canonical form

	x_{B_1}	$\dots$	x_{B_m}	x_{N_1}	$\dots$	$x_{N_{n-m}}$	z	$\bar{b}$
$-z$	0	$\dots$	0	$\bar{c}_{N_1}$	$\dots$	$\bar{c}_{N_{n-m}}$	-1	$-\bar{z}_B$
x_{B_1}	1		0	$\bar{a}_{1\,N_1}$	$\dots$	$\bar{a}_{1\,N_{n-m}}$	0	$\bar{b}_1$
x_{B_i}		$\ddots$		$\bar{a}_{i\,N_1}$	$\dots$	$\bar{a}_{i\,N_{n-m}}$	$\vdots$	$\bar{b}_i$
x_{B_m}	0		1	$\bar{a}_{m\,N_1}$	$\dots$	$\bar{a}_{m\,N_{n-m}}$	0	$\bar{b}_m$

$$z = \bar{z}_B + \bar{c}_{N_1}\, x_{N_1} + \bar{c}_{N_2}\, x_{N_2} + \dots + \bar{c}_{N_{(n-m)}}\, x_{N_{(n-m)}}$$

$$x_{B_i} = \bar{b}_i - \bar{a}_{iN_1}\, x_{N_1} - \bar{a}_{iN_2}\, x_{N_2} - \dots - \bar{a}_{iN_{(n-m)}}\, x_{N_{(n-m)}} \quad (i = 1 \dots m)$$

Retrieving an intial feasible basis: **two-phases method**

- **Phase I**: solve an *artificial problem*

$$
\begin{aligned}
w^* = \quad \min w = \quad & 1^T y = y_1 + y_2 + \cdots + y_m \\
s.t. \quad & Ax + Iy = b \\
& x, y \geq 0
\end{aligned}
\qquad
y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m_+
$$

  If $w^* > 0$, the original problem is unfeasible, stop!

  If $w^* = 0$, then $y = 0$

  - if some y in the (degenarate) basis, change basis to put all y out, thus obtaining an x_B feasible for the original problem!

- **Phase II**: solve the problem starting from the provided basis B

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\end{aligned}
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- **Phase II**: solve the problem starting from the provided basis B

Simplex algorithm with matrix operations: revised simplex

$$\begin{aligned} \min z = \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

standard form

$$\begin{aligned} \min z = \quad & c_B^T x_B + c_N^T x_N \\ \text{s.t.} \quad & B\,x_B + N\,x_N = b \\ & x_B, x_N \geq 0 \end{aligned}$$

with (feasible) basis

$$\begin{array}{rcrcl} -z & + & \bar{c}_N^T x_N & = & -z_B \\ I\,x_B & + & \bar{N}\,x_N & = & \bar{b} \end{array}$$

canonical form

- $\bar{b} = B^{-1} b$
- $z_B = c_B^T \bar{b}$
- $\bar{N} = B^{-1} N$
- $\bar{c}_N^T = c_N^T - c_B^T B^{-1} N$

The (revised) simplex algorithm

1. Let $\beta[1], \ldots, \beta[m]$ be the column indexes of the **initial basis**
2. Let $B = \left[A_{\beta[1]}|\ldots|A_{\beta[m]}\right]$ and compute B^{-1} e $u^T = c_B^T B^{-1}$
3. compute **reduced costs**: $\bar{c}_h = c_h - u^T A_h$ for non-basic variables x_h
4. If $\bar{c}_h \geq 0$ for all non-basic variables x_h, **STOP**: B is **optimal**
5. Choose any x_h having $\bar{c}_h < 0$
6. Compute $\bar{b} = B^{-1}b = \left[\ \bar{b}_i\ \right]_{i=1}^{m}$ e $\bar{A}_h = B^{-1}A_h = \left[\ \bar{a}_{ih}\ \right]_{i=1}^{m}$
7. If $\bar{a}_{ih} \leq 0, \forall i = 1 \ldots m$, **STOP**: **unlimited**
8. Determine $t = \arg\min_{i=1\ldots m}\left\{\bar{b}_i/\bar{a}_{ih}, \bar{a}_{ih} > 0\right\}$
9. Change basis: $\beta[t] \leftarrow h$.
10. Iterate from Step 2

Example

Solve:

$$
\begin{array}{lrcrcrcl}
\max & 3x_1 & + & x_2 & - & 3x_3 & & \\
\text{s.t.} & 2x_1 & + & x_2 & - & x_3 & \leq & 2 \\
 & x_1 & + & 2x_2 & - & 3x_3 & \leq & 5 \\
 & 2x_1 & + & 2x_2 & - & x_3 & \leq & 6 \\
\end{array}
$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , \quad x_3 \leq 0$$

Standard form

$$
\begin{array}{lrcrcrcrcrcrcl}
\min & -3x_1 & - & x_2 & - & 3\hat{x}_3 & & & & & & & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & \hat{x}_3 & + & x_4 & & & & & = & 2 \\
 & x_1 & + & 2x_2 & + & 3\hat{x}_3 & & & + & x_5 & & & = & 5 \\
 & 2x_1 & + & 2x_2 & + & \hat{x}_3 & & & & & + & x_5 & = & 6 \\
 & x_1 & , & x_2 & , & \hat{x}_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}
$$

Matrices and initial basis

$$\begin{array}{lrcrcrcrcrcrcl}
\min & -3x_1 & - & x_2 & - & 3\hat{x}_3 & & & & & & & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & \hat{x}_3 & + & x_4 & & & & & = & 2 \\
& x_1 & + & 2x_2 & + & 3\hat{x}_3 & & & + & x_5 & & & = & 5 \\
& 2x_1 & + & 2x_2 & + & \hat{x}_3 & & & & & + & x_6 & = & 6 \\
& x_1 & , & x_2 & , & \hat{x}_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}$$

$$A = \left[\begin{array}{c|c|c|c|c|c} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array}\right] = \left[\begin{array}{c|c|c|c|c|c} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \qquad b = \left[\begin{array}{c} 2 \\ 5 \\ 6 \end{array}\right]$$

$$x^T = \left[\begin{array}{cccccc} x_1 & x_2 & \hat{x}_3 & x_4 & x_5 & x_6 \end{array}\right] \qquad c^T = \left[\begin{array}{cccccc} -3 & -1 & -3 & 0 & 0 & 0 \end{array}\right]$$

Feasible initial basis (suppose given): $B = [A_4|A_5|A_6]$

$$\beta[1] = 4 \qquad \beta[2] = 5 \qquad \beta[3] = 6$$

Iteration 1: steps 2–5

$$x_B^T = \begin{bmatrix} x_4 & x_5 & x_6 \end{bmatrix} \qquad c_B^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = -3 - 0 = -3$$

$$\bar{c}_2 = c_2 - u^T A_2 = -1 - \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -1 - 0 = -1 \qquad h = 2 \ (x_2 \text{ enters})$$

$$\bar{c}_3 = c_3 - u^T A_3 = -3 - \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = -3 - 0 = -3$$

Iteration 1: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \end{array} \quad \begin{array}{c} x_4 \\ x_5 \\ x_6 \end{array}$$

$$\bar{A}_h = B^{-1}A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$t = \arg\min \left\{ \tfrac{2}{1} \quad \tfrac{5}{2} \quad \tfrac{6}{2} \right\} = \arg\left(\frac{2}{1}\right) = 1 \qquad \rightsquigarrow x_4 \text{ leaves}$$

$\beta[1] = 2$ (column 2 replaces $\beta[1]$ that was 4)

Iteration 2: steps 2–5

$$x_B^T = \left[\begin{array}{ccc} x_2 & x_5 & x_6 \end{array}\right] \qquad c_B^T = \left[\begin{array}{ccc} -1 & 0 & 0 \end{array}\right]$$

$$B = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{array}\right] \qquad B^{-1} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{array}\right]$$

$$u^T = c_B^T B^{-1} = \left[\begin{array}{ccc} -1 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc} -1 & 0 & 0 \end{array}\right]$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - \left[\begin{array}{ccc} -1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} 2 \\ 1 \\ 2 \end{array}\right] = -3 - (-2) = -1$$

$$\bar{c}_3 = c_3 - u^T A_3 = -3 - \left[\begin{array}{ccc} -1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} 1 \\ 3 \\ 1 \end{array}\right] = -3 + 1 = -2 \qquad h = 3$$

($\hat{x}_3$ enters)

$$\bar{c}_4 = c_4 - u^T A_4 = \quad 0 - \left[\begin{array}{ccc} -1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] = 0 - (-1) = 1$$

Iteration 2: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \begin{array}{c} x_2 \\ x_5 \\ x_6 \end{array}$$

$$\bar{A}_h = B^{-1}A_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$t = \arg\min \left\{ \tfrac{2}{1} \quad \tfrac{1}{1} \quad \text{X} \right\} = \arg\left(\frac{1}{1}\right) = 2 \qquad \rightsquigarrow x_5 \text{ leaves}$$

$\beta[2] = 3$ (column 3 replaces column $\beta[2]$ that was 5)

Iteration 3: steps 2–5

$$x_B^T = \left[\begin{array}{ccc} x_2 & \hat{x}_3 & x_6 \end{array}\right] \qquad c_B^T = \left[\begin{array}{ccc} -1 & -3 & 0 \end{array}\right]$$

$$B = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 2 & 1 & 1 \end{array}\right] \qquad B^{-1} = \left[\begin{array}{rrr} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{array}\right]$$

$$u^T = c_B^T B^{-1} = \left[\begin{array}{ccc} -1 & -3 & 0 \end{array}\right] \left[\begin{array}{rrr} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc} 3 & -2 & 0 \end{array}\right]$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - \left[\begin{array}{ccc} 3 & -2 & 0 \end{array}\right] \left[\begin{array}{c} 2 \\ 1 \\ 2 \end{array}\right] = -3 - (4) = -7 \qquad h = 1$$

(x_1 enters)

It is not necessary to compute all reduced costs, stop as soon **one of them** is negative!

Iteration 3: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \quad \begin{array}{c} x_2 \\ \hat{x}_3 \\ x_6 \end{array}$$

$$\bar{A}_h = B^{-1}A_1 = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$$

$$t = \arg\min \left\{ \ \tfrac{1}{5} \quad \mathrm{X} \quad \mathrm{X} \ \right\} = \arg\left(\frac{1}{5}\right) = 1 \qquad \rightsquigarrow x_2 \text{ leaves}$$

$\beta[1] = 1$ (column 1 replaces column $\beta[1]$ that was 2)

Iteration 4

$$x_B^T = \left[\begin{array}{ccc} x_1 & \hat{x}_3 & x_6 \end{array}\right] \qquad c_B^T = \left[\begin{array}{ccc} -3 & -3 & 0 \end{array}\right]$$

$$B = \left[\begin{array}{ccc} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 1 \end{array}\right] \qquad B^{-1} = \left[\begin{array}{rrr} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{array}\right]$$

$$u^T = c_B^T B^{-1} = \left[\begin{array}{ccc} -3 & -3 & 0 \end{array}\right] \left[\begin{array}{rrr} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc} -6/5 & -3/5 & 0 \end{array}\right]$$

$$\bar{c}_2 = c_2 - u^T A_2 = -1 - \left[\begin{array}{ccc} -6/5 & -3/5 & 0 \end{array}\right] \left[\begin{array}{c} 1 \\ 2 \\ 2 \end{array}\right] = -1 - (12/5) = 7/5$$

$$\bar{c}_4 = c_4 - u^T A_4 = 0 - \left[\begin{array}{ccc} -6/5 & -3/5 & 0 \end{array}\right] \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] = 0 - (6/5) = 6/5$$

$$\bar{c}_5 = c_5 - u^T A_5 = 0 - \left[\begin{array}{ccc} -6/5 & -3/5 & 0 \end{array}\right] \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right] = 0 - (3/5) = 3/5$$

Optimal solution

Standard form (the one we solved by simplex method):

- $x_B^* \begin{bmatrix} x_1 \\ \hat{x}_3 \\ x_6 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 8/5 \\ 4 \end{bmatrix}$
- $x_1^* = 1/5$; $x_2^* = 0$; $\hat{x}_3^* = 8/5$; $x_4^* = 0$; $x_5^* = 0$; $x_6^* = 4$
- $z_{MIN}^* = c^T x^* = c_B^T x_B^T = \begin{bmatrix} -3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1/5 \\ 8/5 \\ 4 \end{bmatrix} = -27/5$

Optimal solution for the initial problem:

- $x_1^* = 1/5$
- $x_2^* = 0$
- $x_3^* = -\hat{x}_3^* = -8/5$
- first constraint satisfied with equality (since $x_4^* = 0$)
- second constraint satisfied with equality (since $x_5^* = 0$)
- third constraint satisfied with a slack of 4 (since $x_6^* = 4$)
- $z_{MAX}^* = -z_{MIN}^* = 27/5.$