Lecture 5

Linear Progamming :
Introduction to simplex method
and computational procedure for simplex method

5.1 Introduction

General Linear Programming Problem (GLPP)

Maximize / Minimize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots\ldots\ldots\ldots + c_nx_n$

Subject to constraints

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n$ ($\leq$ or $\geq$) b_1
$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n$ ($\leq$ or $\geq$) b_2
.
.
.
$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n$ ($\leq$ or $\geq$) b_m

and

$x_1 \geq 0,\ x_2 \geq 0, \ldots,\ x_n \geq 0$

Where constraints may be in the form of any inequality ($\leq$ or $\geq$) or even in the form of an equation (=) and finally satisfy the non-negativity restrictions.

5.2 Steps to convert GLPP to SLPP (Standard LPP)

Step 1 – Write the objective function in the maximization form. If the given objective function is of minimization form then multiply throughout by -1 and write Max $z' = \text{Min}(-z)$

Step 2 – Convert all inequalities as equations.

- If an equality of '$\leq$' appears then by adding a variable called **Slack variable.** We can convert it to an equation. For example $x_1 + 2x_2 \leq 12$, we can write as
  $x_1 + 2x_2 + s_1 = 12$.
- If the constraint is of '$\geq$' type, we subtract a variable called **Surplus variable** and convert it to an equation. For example
  $2x_1 + x_2 \geq 15$
  $2x_1 + x_2 - s_2 = 15$

Step 3 – The right side element of each constraint should be made non-negative

$2x_1 + x_2 - s_2 = -15$
$-2x_1 - x_2 + s_2 = 15$ (That is multiplying throughout by -1)

Step 4 – All variables must have non-negative values.

For example: $x_1 + x_2 \leq 3$
$x_1 > 0$, x_2 is unrestricted in sign

Then x_2 is written as $x_2 = x_2' - x_2''$ where $x_2', x_2'' \geq 0$
Therefore the inequality takes the form of equation as $x_1 + (x_2' - x_2'') + s_1 = 3$

Using the above steps, we can write the GLPP in the form of SLPP.

Write the Standard LPP (SLPP) of the following

Example 1

Maximize $Z = 3x_1 + x_2$

Subject to

$2\,x_1 + x_2 \leq 2$

$3\,x_1 + 4\,x_2 \geq 12$

and $x_1 \geq 0,\ x_2 \geq 0$

SLPP

Maximize $Z = 3x_1 + x_2$

Subject to

$2\,x_1 + x_2 + s_1 = 2$

$3\,x_1 + 4\,x_2 - s_2 = 12$

$x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0$

Example 2

Minimize $Z = 4x_1 + 2\,x_2$

Subject to

$3x_1 + x_2 \geq 2$

$x_1 + x_2 \geq 21$

$x_1 + 2x_2 \geq 30$

and $x_1 \geq 0,\ x_2 \geq 0$

SLPP

Maximize $Z' = -4x_1 - 2\,x_2$

Subject to

$3x_1 + x_2 - s_1 = 2$

$x_1 + x_2 - s_2 = 21$

$x_1 + 2x_2 - s_3 = 30$

$x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0$

Example 3

Minimize $Z = x_1 + 2\,x_2 + 3x_3$

Subject to

$2x_1 + 3x_2 + 3x_3 \geq -4$

$3x_1 + 5x_2 + 2x_3 \leq 7$

and $x_1 \geq 0,\ x_2 \geq 0,\ x_3$ is unrestricted in sign

SLPP

Maximize $Z' = -x_1 - 2\,x_2 - 3(x_3' - x_3'')$

Subject to

$-2x_1 - 3x_2 - 3(x_3' - x_3'') + s_1 = 4$

$3x_1 + 5x_2 + 2(x_3' - x_3'') + s_2 = 7$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3' \geq 0,\ x_3'' \geq 0,\ s_1 \geq 0,\ s_2 \geq 0$

5.3 Some Basic Definitions

Solution of LPP
Any set of variable (x_1, x_2… x_n) which satisfies the given constraint is called solution of LPP.

Basic solution
Is a solution obtained by setting any 'n' variable equal to zero and solving remaining 'm' variables. Such 'm' variables are called **Basic variables** and 'n' variables are called **Non-basic variables**.

Basic feasible solution
A basic solution that is feasible (all basic variables are non negative) is called basic feasible solution. There are two types of basic feasible solution.

1. **Degenerate basic feasible solution**
   If any of the basic variable of a basic feasible solution is zero than it is said to be degenerate basic feasible solution.
2. **Non-degenerate basic feasible solution**
   It is a basic feasible solution which has exactly 'm' positive x_i, where i=1, 2, … m. In other words all 'm' basic variables are positive and remaining 'n' variables are zero.

Optimum basic feasible solution
A basic feasible solution is said to be optimum if it optimizes (max / min) the objective function.

5.4 Introduction to Simplex Method

It was developed by G. Danztig in 1947. The simplex method provides an algorithm (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the fundamental theorem of linear programming.

The Simplex algorithm is an iterative procedure for solving LP problems in a finite number of steps. It consists of

- Having a trial basic feasible solution to constraint-equations
- Testing whether it is an optimal solution
- Improving the first trial solution by a set of rules and repeating the process till an optimal solution is obtained

Advantages

- Simple to solve the problems
- The solution of LPP of more than two variables can be obtained.

5.5 Computational Procedure of Simplex Method

Consider an example

Maximize $Z = 3x_1 + 2x_2$

Subject to

$x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Step 1 – Write the given GLPP in the form of SLPP

Maximize $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$

Subject to

$x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

Step 2 – Present the constraints in the matrix form

$x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Step 3 – Construct the starting simplex table using the notations

		$C_j \rightarrow$	3	2	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	4	1	1	1	0	
s_2	0	2	1	-1	0	1	
	$Z = C_B X_B$		Δ_j				

Step 4 – Calculation of Z and Δ_j and test the basic feasible solution for optimality by the rules given.

$Z = C_B X_B$

$= 0 * 4 + 0 * 2 = 0$

$\Delta_j = Z_j - C_j$

$= C_B X_j - C_j$

$\Delta_1 = C_B X_1 - C_j = 0 * 1 + 0 * 1 - 3 = -3$

$\Delta_2 = C_B X_2 - C_j = 0 * 1 + 0 * -1 - 2 = -2$

$\Delta_3 = C_B X_3 - C_j = 0 * 1 + 0 * 0 - 0 = 0$

$\Delta_4 = C_B X_4 - C_j = 0 * 0 + 0 * 1 - 0 = 0$

Procedure to test the basic feasible solution for optimality by the rules given

Rule 1 – If all $\Delta_j \geq 0$, the solution under the test will be **optimal**. Alternate optimal solution will exist if any non-basic Δ_j is also zero.

Rule 2 – If atleast one Δ_j is negative, the solution is not optimal and then proceeds to improve the solution in the next step.

Rule 3 – If corresponding to any negative Δ_j, all elements of the column X_j are negative or zero, then the solution under test will be **unbounded**.

In this problem it is observed that Δ_1 and Δ_2 are negative. Hence proceed to improve this solution

Step 5 – To improve the basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined.

- **Incoming vector**
  The incoming vector X_k is always selected corresponding to the most negative value of Δ_j. It is indicated by (↑).

- **Outgoing vector**
  The outgoing vector is selected corresponding to the least positive value of minimum ratio. It is indicated by (→).

Step 6 – Mark the key element or pivot element by '□'.The element at the intersection of outgoing vector and incoming vector is the pivot element.

		Cj →	3	2	0	0	
Basic Variables	C_B	X_B	X_1 (X_k)	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	4	1	1	1	0	4 / 1 = 4
s_2	0	2	$\boxed{1}$	-1	0	1	2 / 1 = 2 → outgoing
	$Z = C_B X_B = 0$		↑incoming $\Delta_1 = -3$	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	

- If the number in the marked position is other than unity, divide all the elements of that row by the key element.
- Then subtract appropriate multiples of this new row from the remaining rows, so as to obtain zeroes in the remaining position of the column X_k.

Basic Variables	C_B	X_B	X_1	X_2 (X_k)	S_1	S_2	Min ratio X_B / X_k
s_1	0	2	($R_1 = R_1 - R_2$) 0	$\boxed{2}$	1	-1	2 / 2 = 1 ⟶ outgoing
x_1	3	2	1	-1	0	1	2 / -1 = -2 (neglect in case of negative)
	Z=0*2+3*2= 6		Δ_1=0	↑incoming Δ_2= -5	Δ_3=0	Δ_4=3	

Step 7 – Now repeat step 4 through step 6 until an optimal solution is obtained.

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
x_2	2	1	($R_1 = R_1 / 2$) 0	1	1/2	-1/2	
x_1	3	3	($R_2 = R_2 + R_1$) 1	0	1/2	1/2	
	Z = 11		Δ_1=0	Δ_2=0	Δ_3=5/2	Δ_4=1/2	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 11, $x_1 = 3$ and $x_2 = 1$

5.6 Worked Examples

Solve by simplex method

Example 1

Maximize $Z = 80x_1 + 55x_2$

Subject to

$4x_1 + 2x_2 \leq 40$

$2x_1 + 4x_2 \leq 32$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Maximize $Z = 80x_1 + 55x_2 + 0s_1 + 0s_2$
Subject to

$4x_1 + 2x_2 + s_1 = 40$
$2x_1 + 4x_2 + s_2 = 32$
$x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0$

			$C_j \rightarrow$ 80	55	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	40	$\boxed{4}$	2	1	0	$40 / 4 = 10 \longrightarrow$ outgoing
s_2	0	32	2	4	0	1	$32 / 2 = 16$
	$Z = C_B X_B = 0$		$\uparrow$incoming $\Delta_1 = -80$	$\Delta_2 = -55$	$\Delta_3 = 0$	$\Delta_4 = 0$	
x_1	80	10	$(R_1 = R_1 / 4)$ 1	1/2	1/4	0	$10/1/2 = 20$
s_2	0	12	$(R_2 = R_2 - 2R_1)$ 0	$\boxed{3}$	-1/2	1	$12/3 = 4 \longrightarrow$ outgoing
	$Z = 800$		$\Delta_1 = 0$	$\uparrow$incoming $\Delta_2 = -15$	$\Delta_3 = 40$	$\Delta_4 = 0$	
x_1	80	8	$(R_1 = R_1 - 1/2R_2)$ 1	0	1/3	-1/6	
x_2	55	4	$(R_2 = R_2 / 3)$ 0	1	-1/6	1/3	
	$Z = 860$		$\Delta_1 = 0$	$\Delta_2 = 0$	$\Delta_3 = 35/2$	$\Delta_4 = 5$	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = 860$, $x_1 = 8$ and $x_2 = 4$

Example 2

Maximize $Z = 5x_1 + 7x_2$

Subject to

$x_1 + x_2 \leq 4$

$3x_1 - 8x_2 \leq 24$

$10x_1 + 7x_2 \leq 35$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Maximize $Z = 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to

$x_1 + x_2 + s_1 = 4$

$3x_1 - 8x_2 + s_2 = 24$

$10x_1 + 7x_2 + s_3 = 35$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

		$C_j \rightarrow$	5	7	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
s_1	0	4	1	$\boxed{1}$	1	0	0	$4/1 = 4 \rightarrow$ outgoing
s_2	0	24	3	-8	0	1	0	–
s_3	0	35	10	7	0	0	1	$35/7 = 5$
	$Z = C_B X_B = 0$		-5	-7 $\uparrow$ incoming	0	0	0	$\leftarrow \Delta_j$
x_2	7	4	1	1	1	0	0	
s_2	0	56	($R_2 = R_2 + 8R_1$) 11	0	8	1	0	
s_3	0	7	($R_3 = R_3 - 7R_1$) 3	0	-7	0	1	
	$Z = 28$		2	0	7	0	0	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = 28$, $x_1 = 0$ and $x_2 = 4$