

1 Inversions and contractions — practical implementation on the lattice

1.1 Adjoint currents

We the following rule for complex conjugation of a product of Grassmann numbers z_1, z_2

$$(z_1 z_2)^* = z_2^* z_1^* . \quad (1)$$

With this convention, the quark-bilinear $\bar{q}q$ (e.g. part of the mass term) is a real quantity

$$(\bar{q}q)^* = \left(q^\dagger \gamma_0 q \right)^* = q^\dagger \gamma_0^\dagger q = \bar{q}q .$$

Thus for a baryon interpolating field Q and its adjoint \bar{Q} we have

$$Q_\alpha^c = \epsilon_{abc} \left(q_1^{aT} \Gamma q_2^b \right) q_{3\alpha}^c = \epsilon_{abc} \left(q_1^a \Gamma_{\gamma\delta} q_2^b \right) q_{3\alpha}^c \quad (2)$$

$$\begin{aligned} \bar{Q}_\beta^{c'} &= \left[\epsilon_{a'b'c'} \left(q_1^{a'} \Gamma_{\gamma\delta} q_2^{b'} \right) q_{3\beta}^{c'} \right]^* (\gamma_0)_{\rho\beta} = \epsilon_{a'b'c'} \left(q_2^{b'*} \Gamma_{\delta\gamma}^\dagger q_1^{a'*} \right) q_{3\rho}^{c'*} (\gamma_0)_{\rho\beta} \\ &= -\epsilon_{c'a'b'} \bar{q}_{3\beta}^{c'} \left(\bar{q}_2^{b'} (\gamma_0)_{\kappa\delta} \Gamma_{\delta\gamma}^\dagger (\gamma_0)_{\gamma\lambda} \bar{q}_1^{a'} \right) = -\epsilon_{c'a'b'} \bar{q}_{3\beta}^{c'} \left(\bar{q}_2^{a'} \tilde{\Gamma} \bar{q}_1^{b'T} \right) , \end{aligned} \quad (3)$$

where

$$\tilde{\Gamma} = \gamma_0 \Gamma^\dagger \gamma_0 = \sigma_{\Gamma^{0\dagger}} \Gamma$$

and we used the property $\gamma_0^T = \gamma_0$.

For a meson interpolating field, this entails

$$(\bar{q}_1 \Gamma q_2)^* = q_2^\dagger \Gamma^\dagger \gamma_0 q_1 = \bar{q}_2 \tilde{\Gamma} q_1 .$$

In particular for the charged pions we have $\pi^+ = \bar{d} i \gamma_5 u$ with $\pi^{+*} = \pi^-$.

The values of $\sigma_{\Gamma^{0\dagger}}$ for Γ on of the 16 basis matrices is given by

Γ	γ_0	γ_i	$\mathbb{1}$	γ_5	$\gamma_0 \gamma_5$	$\gamma_i \gamma_5$	$\gamma_0 \gamma_i$	$\gamma_i \gamma_j$
$\sigma_{\Gamma^{0\dagger}}$	+1	-1	+1	-1	+1	-1	+1	-1

Note: I do not include factors of imaginary unit i in the interpolating fields in the contractions. They can be added afterwards as a complex phase to the correlation matrix. This holds for $C = i \gamma_0 \gamma_2$ as well, which means in the contraction part I only consider the $\gamma_0 \gamma_2$.

Convention for contraction code For simplicity, I always contract bare quark-field and γ combinations, *without* the minus sign in (3). This means a correlator C from *contract_baryon* must receive a sign

$$C_{Q_1 - \bar{Q}_2} \rightarrow -\sigma_{\Gamma_{Q_2}^{0\dagger}} C_{Q_1 \bar{Q}_2} \quad (4)$$

$$C_{Q_1 - M_2^\dagger \bar{Q}_2} \rightarrow -\sigma_{\Gamma_{Q_2}^{0\dagger}} \sigma_{\Gamma_{M_2}^{0\dagger}} C_{Q_1 - M_2^\dagger \bar{Q}_2} \quad (5)$$

$$C_{M_1 Q_1 - M_2^\dagger \bar{Q}_2} \rightarrow -\sigma_{\Gamma_{Q_2}^{0\dagger}} \sigma_{\Gamma_{M_2}^{0\dagger}} C_{M_1 Q_1 - M_2^\dagger \bar{Q}_2} \quad (6)$$

The overall sign from the adjoint operator at source can be added e.g. via the *comp_list_sign*.

1.2 Δ^{++} to $\pi^+ N^+$ 3-point function

Motivated by the considerations in the previous section ?? we continue with the practical implementation of the estimation of the matrix elements in lattice QCD. We start from

$$\langle J_{\Delta^{++}}(x_f) J_M^\dagger(x_{i_2}) \bar{J}_N(x_{i_1}) \rangle = -\sigma_{\Gamma_N^{0\dagger}} \sigma_{\Gamma_M^{0\dagger}} \langle \underbrace{J_{\Delta^{++}}^\alpha}_{(u^T C \Gamma_\Delta u) u}(t_f, \vec{x}) \underbrace{J_{\pi^+}^\dagger}_{\bar{u} \gamma_5 d}(t_i, \vec{y}) \underbrace{\bar{J}_N^\beta}_{(\bar{d} C \Gamma_N \bar{u}^T) \bar{u}}(t_i, \vec{z}) \rangle \quad (7)$$

$$\langle \left[u_\gamma^a(x_f) (C \Gamma_\Delta)_{\gamma\delta} u_\delta^b(x_f) u_\alpha^c(x_f) \right] \left[\bar{u}_\sigma^d(x_{i_2}) (\Gamma_M)_{\sigma\tau} d_\tau^d(x_{i_2}) \right] \left[\bar{d}_\kappa^l(x_{i_1}) (C \Gamma_N)_{\kappa\lambda} \bar{u}_\lambda^m(x_{i_1}) \bar{u}_\beta^n(x_{i_1}) \right] \rangle_f = \quad (8)$$

$$\begin{aligned} & - T(x_f, x_{i_1}) C \Gamma_N (C \Gamma_\Delta U(x_f, x_{i_1}))^t U(x_f, x_{i_1}) \\ & - T(x_f, x_{i_1}) C \Gamma_N U(x_f, x_{i_1})^t C \Gamma_\Delta U(x_f, x_{i_1}) \\ & - U(x_f, x_{i_1}) (C \Gamma_\Delta T(x_f, x_{i_1}) C \Gamma_N)^t U(x_f, x_{i_1}) \\ & - U(x_f, x_{i_1}) (T(x_f, x_{i_1}) C \Gamma_N)^t C \Gamma_\Delta U(x_f, x_{i_1}) \\ & - U(x_f, x_{i_1}) \text{Tr} (T(x_f, x_{i_1}) C \Gamma_N U(x_f, x_{i_1})^t C \Gamma_\Delta) \\ & - U(x_f, x_{i_1}) \text{Tr} (T(x_f, x_{i_1}) C \Gamma_N (C \Gamma_\Delta U(x_f, x_{i_1}))^t) \\ & = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 \end{aligned} \quad (9)$$

Eq. (8) defines the triangle diagrams T_1, \dots, T_6 .

We use the notation

$$T(x_f, x_i) = T_{\alpha\beta}^{f_1 f_2 ab}(x_f; t, \vec{q}; x_i) = \sum_{\vec{y}} \left(S_{f_1}(x_f; t, \vec{y}) \Gamma_M e^{i\vec{q}\vec{y}} S_{f_2}(t, \vec{y}; x_i) \right)_{\alpha\beta}^{ab} \quad (10)$$

for the sequential propagator with

- flavors “ f_1 after f_2 ”;
- sequential source timeslice t ;
- sequential source momentum \vec{q} ;
- sequential source Dirac structure Γ_M .

In particular we shall use the notation

$$T_{fii} = T(x_f; t_i, \vec{q}; x_i) \quad (11)$$

$$T_{ff i} = T(x_f; t_f, \vec{q}; x_i) \quad (12)$$

for 1-step sequential propagators.

Quantum numbers for Delta, pion and nucleon:

	Δ^{++}	π^+	$N^+ = \text{Proton}$
J	$\frac{3}{2}$	0	$\frac{1}{2}$
I	$\frac{3}{2}$	1	$\frac{1}{2}$
I_3	$+\frac{3}{2}$	+1	$+\frac{1}{2}$
P	+1	-1	+1

1.3 Δ^{++} to Δ^{++}

$$\langle J_\Delta(x_f) \bar{J}_\Delta(x_i) \rangle_f = -\sigma_{\Gamma_i^{0\dagger}} \langle (u^T C\Gamma_f u) u(x_f) (\bar{u}^T C\Gamma_i \bar{u}) \bar{u}(x_i) \rangle \quad (13)$$

$$\langle \left[\epsilon_{abc} u_\gamma^a(x_f) (C\Gamma_f)_{\gamma\delta} u_\delta^b(x_f) u_\alpha^c(x_f) \right] \left[\epsilon_{lmn} \bar{u}_\kappa^l(x_i) (C\Gamma_i)_{\kappa\lambda} \bar{u}_\lambda^m(x_i) \bar{u}_\beta^n(x_i) \right] \rangle_f = \quad (14)$$

$$\begin{aligned} & \epsilon_{abc} \epsilon_{lmn} (C\Gamma_f)_{\gamma\delta} (C\Gamma_i)_{\kappa\lambda} \left\{ \right. \\ & \quad + U_{\alpha\beta}^{cn} \left(U_{\delta\kappa}^{bl} U_{\gamma\lambda}^{am} - U_{\delta\lambda}^{bm} U_{\gamma\kappa}^{al} \right) \\ & \quad - U_{\alpha\lambda}^{cm} \left(U_{\delta\kappa}^{bl} U_{\gamma\beta}^{an} - U_{\delta\beta}^{bn} U_{\gamma\kappa}^{al} \right) \\ & \quad \left. + U_{\alpha\kappa}^{cl} \left(U_{\delta\lambda}^{bm} U_{\gamma\beta}^{an} - U_{\delta\beta}^{bn} U_{\gamma\lambda}^{am} \right) \right\} = \\ & - U(x_f, x_i) C\Gamma_i (C\Gamma_f U(x_f, x_i))^t U(x_f, x_i) \\ & - U(x_f, x_i) C\Gamma_i U(x_f, x_i)^t C\Gamma_f U(x_f, x_i) \\ & - U(x_f, x_i) (C\Gamma_f U(x_f, x_i) C\Gamma_i)^t U(x_f, x_i) \\ & - U(x_f, x_i) (U(x_f, x_i) C\Gamma_i)^t C\Gamma_f U(x_f, x_i) \\ & - U(x_f, x_i) \text{Tr} (C\Gamma_f U(x_f, x_i) C\Gamma_i U(x_f, x_i)^t) \\ & - U(x_f, x_i) \text{Tr} (C\Gamma_f U(x_f, x_i) (U(x_f, x_i) C\Gamma_i)^t) \\ & = D_1 + D_2 + D_3 + D_4 + D_5 + D_6. \end{aligned}$$

This define the $I = 3/2$ diagrams D_1, \dots, D_6 .

Adjoint correlator Using γ_5 -Hermiticity, parity and time reversal we expect, that

$$\begin{aligned} C_{\mu\nu}^{\alpha\beta}(x, y) &= \langle J_{\Delta\mu}^\alpha(x) \bar{J}_{\Delta\nu}^\beta(y) \rangle \\ C_{\mu\nu}(x, y) &= \sigma_\mu^{02} \sigma_\mu^{02} C_{\mu\nu}^{\tilde{\dagger}} \\ \sigma_\mu^{02} &= \begin{cases} +1 & \mu = 0, 2 \\ -1 & \mu = 1, 3 \end{cases}, \end{aligned}$$

where $\tilde{\dagger}$ denotes the conjugate with respect to the spinor indices. This relation should hold exactly in the free case (gauge field $U = 1$) and at the level of the gauge average in the non-free case.

$$\begin{aligned} (t_x, t_y) &\sim t_x - t_y \xrightarrow{\gamma_5\text{-Hermiticity}} (t_y, t_x) \sim t_y - t_x \\ &\xrightarrow{\mathcal{T}} (T - t_y, T - t_x) \sim (T - t_y) - (T - t_x) = t_x - t_y. \end{aligned}$$

1.4 N^+ to N^+

$$\langle J_N(x_f) \bar{J}_N(x_i) \rangle_f = -\sigma_{\Gamma_i^{0\dagger}} \langle (u^T C\Gamma_f d) u(x_f) (\bar{d}^T C\Gamma_i \bar{u}) \bar{u}(x_i) \rangle \quad (15)$$

$$\begin{aligned} & \langle \left[\epsilon_{abc} u_\gamma^a(x_f) (C\Gamma_f)_{\gamma\delta} d_\delta^b(x_f) u_\alpha^c(x_f) \right] \left[\epsilon_{lmn} \bar{d}_\kappa^l(x_i) (C\Gamma_i)_{\kappa\lambda} \bar{u}_\lambda^m(x_i) \bar{u}_\beta^n(x_i) \right] \rangle_f = \quad (16) \\ & - U(x_f, x_i) (C\Gamma_f D(x_f, x_i) C\Gamma_i)^t U(x_f, x_i) \\ & - U(x_f, x_i) \text{Tr} \left((C\Gamma_f D(x_f, x_i) C\Gamma_i)^t U(x_f, x_i) \right) \\ & = N_1 + N_2. \end{aligned}$$

This defines the diagrams N_1, N_2 .

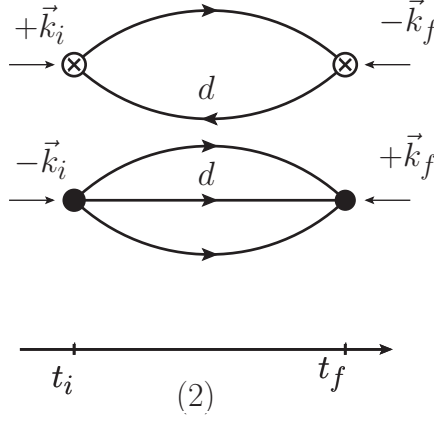


Figure 1: Graphical representation of the quark-disconnected contribution to the 4-pt. function $\pi N \rightarrow \pi N$

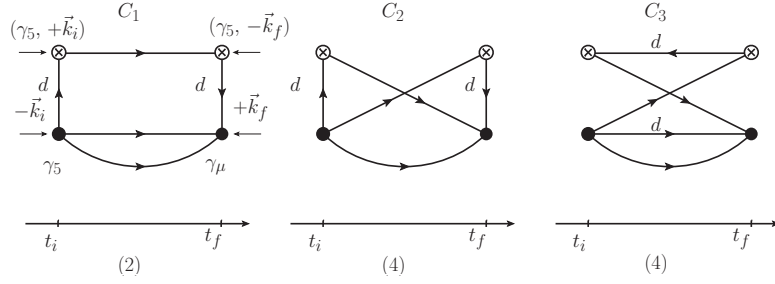


Figure 2: Graphical representation of the quark-connected contribution to the 4-pt. function $\pi N \rightarrow \pi N$ at zero total 3-momentum, $\vec{Q} = 0$

1.5 $\pi^+ N^+$ to $\pi^+ N^+$

In total these sum up to 12 contributions; we can check that is the right number 2 combinations of down quarks \times 3! combinations of up quarks

We introduce some notation to write out the necessary contractions, the 2-step sequential propagators P

$$P_{\alpha\beta}^{f_1 f_2 f_1 ab}(x_f; t_1, \vec{q}_1; t_2, \vec{q}_2; x_i) = \sum_{\vec{z}_1, \vec{z}_2} \left(S_{f_1}(x_f; t_1, \vec{z}_1) \Gamma_1 S_{f_2}(t_1, \vec{z}_1; t_2, \vec{z}_2) \Gamma_2 S_{f_1}(t_2, \vec{z}_2; x_i) \right)_{\alpha\beta}^{ab} e^{i(\vec{q}_1 \vec{z}_1 + \vec{q}_2 \vec{z}_2)} \quad (17)$$

where $f_{1/2} \in \{u, d\}$ and $f_1 \neq f_2$. In particular we shall use

$$P_{f_i f_i} = P^{udu}(x_{f_1}; t_i, \vec{q}_{i_2}; t_f, \vec{q}_{f_2}; x_{i_1}) \quad (18)$$

$$P_{f f_{ii}} = P^{dud}(x_{f_1}; t_f, \vec{q}_{f_2}; t_i, \vec{q}_{i_2}; x_{i_1}) \quad (19)$$

with $t_{f_1} = t_{f_2} = t_f$ and $t_{i_1} = t_{i_2} = t_i$.

With these generalized propagators we can write the contractions in a short way.

$$\begin{aligned} & \langle J_{\pi^+ N^+}(x_{f_1}; x_{f_2}) \bar{J}_{\pi^+ N^+}(x_{i_1}; x_{i_2}) \rangle_f \\ &= -\sigma_{\Gamma_{N_i}^{0\dagger}} \sigma_{\Gamma_{M_i}^{0\dagger}} \langle (u^t C \Gamma_{N_f} d) u(x_{f_1}) \bar{d} \Gamma_{M_f} u(x_{f_2}) \bar{u} \Gamma_{M_i} d(x_{i_2}) (\bar{d} C \Gamma_{N_i} u) u(x_{i_1}) \rangle \end{aligned} \quad (20)$$

$$\begin{aligned} & \langle \left[\epsilon_{abc} u_\gamma^a(x_{f_1}) (C \Gamma_{N_f})_{\gamma\delta} d_\delta^b(x_{f_1}) u_\alpha^c(x_{f_1}) \right] \left[\bar{d}_\sigma^d(x_{f_2}) (\Gamma_{M_f})_{\sigma\tau} u_\tau^d(x_{f_2}) \right] \times \\ & \quad \left[\bar{u}_\mu^e(x_{i_2}) (\Gamma_{M_i})_{\mu\nu} d_\nu^e(x_{i_2}) \right] \left[\epsilon_{lmn} \bar{d}_\kappa^l(x_{i_1}) (C \Gamma_{N_i})_{\kappa\lambda} \bar{u}_\lambda^m(x_{i_1}) \bar{u}_\beta^n(x_{i_1}) \right] \rangle \\ &= C_B + C_W + C_Z + C_{\text{disconnected}} \end{aligned} \quad (21)$$

Quark-disconnected contribution — direct diagram

$$C_{\text{disconnected}} = -\text{Tr} \left(U(x_{f_2}, x_{i_2}) \Gamma_{M_i} D(x_{i_2}, x_{f_2}) \Gamma_{M_f} \right) \times (N_1 + N_2) . \quad (22)$$

Quark-connected contributions — B , W and Z diagrams The connected contractions $C_{B,W,Z}$ are

$$\begin{aligned}
C_B = & \quad (23) \\
& - U(x_{f_1}, x_{i_1}) (C\Gamma_{N_f} P_{ffii}(x_{f_1}, x_{i_1}) C\Gamma_{N_i})^t U(x_{f_1}, x_{i_1}) \\
& - U(x_{f_1}, x_{i_1}) \text{Tr} (C\Gamma_{N_f} P_{ffii}(x_{f_1}, x_{i_1}) C\Gamma_{N_i} U(x_{f_1}, x_{i_1})^t) \\
& = B_1 + B_2
\end{aligned}$$

$$\begin{aligned}
C_W = & \quad (24) \\
& - T_{fii}^{ud}(x_{f_1}, x_{i_1}) C\Gamma_{N_i} \left(C\Gamma_{N_f} T_{fffi}^{du}(x_{f_1}, x_{i_1}) \right)^t U(x_{f_1}, x_{i_1}) \\
& - T_{fii}^{ud}(x_{f_1}, x_{i_1}) C\Gamma_{N_i} U(x_{f_1}, x_{i_1})^t C\Gamma_{N_f} T_{fffi}^{du}(x_{f_1}, x_{i_1}) \\
& - U(x_{f_1}, x_{i_1}) \left(T_{fii}^{ud}(x_{f_1}, x_{i_1}) C\Gamma_{N_i} \right)^t C\Gamma_{N_f} T_{fffi}^{du}(x_{f_1}, x_{i_1}) \\
& - U(x_{f_1}, x_{i_1}) \text{Tr} \left(T_{fii}^{ud}(x_{f_1}, x_{i_1}) C\Gamma_{N_i} \left(C\Gamma_{N_f} T_{fffi}^{du}(x_{f_1}, x_{i_1}) \right)^t \right) \\
& = C_{W_1} + C_{W_2} + C_{W_3} + C_{W_4}
\end{aligned}$$

$$\begin{aligned}
C_Z = & \quad (25) \\
& - P_{fifi}^{udu}(x_{f_1}, x_{i_1}) (C\Gamma_{N_f} D(x_{f_1}, x_{i_1}) C\Gamma_{N_i})^t U(x_{f_1}, x_{i_1}) \\
& - P_{fifi}^{udu}(x_{f_1}, x_{i_1}) \text{Tr} \left((C\Gamma_{N_f} D(x_{f_1}, x_{i_1}) C\Gamma_{N_i})^t U(x_{f_1}, x_{i_1}) \right) \\
& - U(x_{f_1}, x_{i_1}) (C\Gamma_{N_f} D(x_{f_1}, x_{i_1}) C\Gamma_{N_i})^t P_{fifi}^{udu}(x_{f_1}, x_{i_1}) \\
& - U(x_{f_1}, x_{i_1}) \text{Tr} \left(P_{fifi}^{udu}(x_{f_1}, x_{i_1}) (C\Gamma_{N_f} D(x_{f_1}, x_{i_1}) C\Gamma_{N_i})^t \right) \\
& = C_{Z_1} + C_{Z_2} + C_{Z_3} + C_{Z_4}
\end{aligned}$$

Comments

- note that with degenerate up and down quarks we can use $T^{ud} = T^{du}$ and $P^{udu} = P^{dud}$. Moreover in the case at hand we will always have $\vec{k}_i = \vec{k}_f$
- the straightforward contractions using the *Seq²Propagators* from point sources would be demanding; for each momentum vector $\vec{k} = \vec{k}_f = \vec{k}_i$ we would need $N_s \times N_c \times (3N_{t_f} + 2)$ inversions to produce all required S, T, P fields. N_s, N_c, N_{t_f} are the numbers of spinor, color components and the number of sink timeslices
- but (1): from the investigations of nucleon/delta matrix elements with current insertions it is known that $N_{t_f} = \mathcal{O}(T/4/a)$, a quarter of the temporal extent of the lattice
- but (2): using momenta from the class represented by $\vec{k} = (0, 0, 1)$, could one save computing time by combining \vec{k} and $-\vec{k}$ using coherent sources?
- What about stochastic timeslice sources? Issue: we need spin and color dilution (do we?); would thus still require $\mathcal{O}(N_s N_c T/a)$ inversions, same order of magnitude; needs inversions for both sources with and without momentum
- timeslice sources may require \mathbb{Z}_5 sources
- would be nice if chosen method would be applicable to Δ^{++-} , N^+ -2-point functions and $\Delta^{++} \rightarrow \pi^+ N^+$ 3-point function as well

2 Post-poned spin-color reduction and factorization for $I = 3/2$, $I^3 = +3/2$

2.1 B -diagrams

$$P_{ffii}^{dud} = \phi^d(f_1) \xi(f_2)^\dagger \Gamma_{f_2}(\vec{p}_{f_2}) T^{ud}(f_2, i_2, i_1) \quad (26)$$

B_1

$$(B_1)_{\alpha\beta} = -b_{1\phi}(\vec{p}_{f_1}, \Gamma_{f_1})_{\beta\alpha\delta}^m (\Gamma_{i_1}(\vec{p}_{i_1}))_{\delta\gamma}^t b_{1\xi}(\vec{p}_{f_2}, \Gamma_{f_2})_{\gamma}^m b_{1\xi;\gamma}^m = \xi(f_2)^\dagger \Gamma_{f_2}(\vec{p}_{f_2}) T_{fii}(f_2, i_1)_{;\gamma}^m \quad (27)$$

$$b_{1\phi;\beta\alpha\delta}^m = \epsilon_{mnl} \left[\epsilon_{bca} \phi(f_1)_\kappa^b (C \Gamma_{f_1})_{\kappa\lambda}^t U(f_1, i_1)_{\lambda\beta}^{cn} \right] U(f_1, i_1)_{\alpha,\delta}^{al} e^{i\vec{p}_{f_1}} \quad (28)$$

Γ_{i_1} and \vec{p}_{i_1} remain open.

B_2

$$(B_2)_{\alpha\beta} = -b_{2\phi}(\vec{p}_{f_1}, \Gamma_{f_1})_{\delta\alpha\beta}^m (\Gamma_{i_1}(\vec{p}_{i_1}))_{\delta\gamma}^t b_{2\xi}(\vec{p}_{f_2}, \Gamma_{f_2})_{\gamma}^m b_{2\xi;\gamma}^m = \xi(f_2)^\dagger \Gamma_{f_2}(\vec{p}_{f_2}) T_{fii}(f_2, i_1)_{;\gamma}^m = b_{1\xi;\gamma}^m \quad (29)$$

$$b_{2\phi;\delta\alpha\beta}^m = \epsilon_{mnl} \left[\epsilon_{bca} \phi(f_1)_\kappa^b (C \Gamma_{f_1})_{\kappa\lambda}^t U(f_1, i_1)_{\lambda\delta}^{cn} \right] U(f_1, i_1)_{\alpha,\beta}^{al} e^{i\vec{p}_{f_1}} = b_{1\phi;\delta\alpha\beta}^m \quad (30)$$

Γ_{i_1} and \vec{p}_{i_1} remain open.

2.2 W -diagrams

$$T_{ffii}^{du} = \phi^d(f_1) \xi(f_2)^\dagger \Gamma_{f_2}(\vec{p}_{f_2}) U(f_2, i_1). \quad (31)$$

Note, that we can use γ_5 -hermiticity and write

$$T_{ffii}^{du} = \gamma_5 \xi(f_1) \phi^u(f_2)^\dagger \gamma_5 \Gamma_{f_2}(\vec{p}_{f_2}) U(f_2, i_1), \quad (32)$$

where the Hermitean conjugation is with respect to spin-color indices. This means we can make corresponding replacements in $w_{n\xi}$ and $w_{n\phi}$ to separate contractions involving sequential propagators from those involving stochastic propagators (using $\phi^u = \phi^d$ in the case of Wilson-clover fermions). We can use $\gamma_5^t = \gamma_5$.

W_1

$$(W_1)_{\alpha\beta} = -w_{1\phi;\beta\alpha\delta}^m (C \Gamma_{i_1})_{\delta\gamma} w_{1\xi;\gamma}^m e^{i\vec{p}_{i_1}} w_{1\xi;\gamma}^m = \xi(f_2)^\dagger \Gamma_{f_2}(\vec{p}_{f_2}) U(f_2, i_1)_{;\gamma}^m \quad (33)$$

$$w_{1\phi;\beta\alpha\delta}^m = \epsilon_{mnl} \left[\epsilon_{bca} \phi(f_1)_\kappa^b (C \Gamma_{f_1})_{\kappa\lambda}^t U(f_1, i_1)_{\lambda\beta}^{cn} \right] T_{fii}(f_1, i_1)_{\alpha\delta}^{al} e^{i\vec{p}_{f_1}} \quad (34)$$

Γ_{i_1} and \vec{p}_{i_1} remain open.

W_2

$$(W_2)_{\alpha\beta} = -w_{2\phi;\gamma\alpha\delta}^n (C \Gamma_{i_1}(\vec{p}_{i_1}))_{\delta\gamma} w_{2\xi;\beta}^n$$

$$w_{2\xi;\beta}^n = \xi(f_2)^\dagger \Gamma_{f_2}(\vec{p}_{f_2}) U(f_2, i_1)_{\cdot,\beta}^n = w_{1\xi;\beta}^n \quad (35)$$

$$w_{2\phi;\gamma\alpha\delta}^n = \epsilon_{nml} \left[\epsilon_{cba} \phi(f_1)_\kappa^c (C \Gamma_{f_1})_{\kappa\lambda}^t U(f_1, i_1)_{\lambda\gamma}^{bm} \right] T_{fii}(f_1, i_1)_{\alpha\delta}^{al} e^{i\vec{p}_{f_1}}$$

$$= w_{1\phi;\beta\alpha\delta}^n \quad (36)$$

Γ_{i_1} and \vec{p}_{i_1} remain open.

W_3

$$(W_3)_{\alpha\beta} = -w_{3\phi;\gamma\alpha\delta}^n (C \Gamma_{i_1}(\vec{p}_{i_1}))_{\delta\gamma}^t w_{3\xi;\beta}^n$$

$$w_{3\xi;\beta}^n = \xi^\dagger \Gamma_{f_2}(\vec{p}_{f_2}) U(f_2, i_1)_{\cdot,\beta}^n = w_{1\xi;\beta}^n \quad (37)$$

$$w_{3\phi;\gamma\alpha\delta}^n = \epsilon_{nml} \left[\epsilon_{cba} \phi(f_1)_\lambda^c (C \Gamma_{f_1})_{\lambda\kappa}^t T_{fii}(f_1, i_1)_{\kappa\gamma}^{bm} \right] U(f_1, i_1)_{\alpha\delta}^{al} e^{i\vec{p}_{f_1}} \quad (38)$$

Γ_{i_1} and \vec{p}_{i_1} remain open.

W_4

$$(W_4)_{\alpha\beta} = -w_{4\phi;\gamma\alpha\beta}^n (C \Gamma_{i_1}(\vec{p}_{i_1}))_{\gamma\delta}^t w_{4\xi;\gamma}^n$$

$$w_{4\xi;\delta}^n = \xi^\dagger \Gamma_{f_2}(\vec{p}_{f_2}) U(f_2, i_1)_{\cdot,\delta}^n = w_{1\xi;\delta}^n \quad (39)$$

$$w_{4\phi;\gamma\alpha\beta}^n = \epsilon_{nml} \left[\epsilon_{cba} \phi(f_1)_\lambda^c (C \Gamma_{f_1})_{\lambda\kappa}^t T_{fii}(f_1, i_1)_{\kappa,\gamma}^{bm} \right] U(f_1, i_1)_{\alpha\beta}^{al} e^{i\vec{p}_{f_1}}$$

$$= w_{3\phi;\gamma\alpha\beta}^n \quad (40)$$

Γ_{i_1} and \vec{p}_{i_1} remain open.

Exchanging $\phi^d \rightarrow \phi^u$ In all the W diagrams we exchange $\phi^d \xi^\dagger \rightarrow (\gamma_5 \xi) (\gamma_5 \phi^u)^\dagger$ and thereby achieve the separation of sequential propagators T_{fii} and stochastic propagators $\phi^{u/d}$ into separate diagrams. Thus the contractions can be split accordingly into

- one part using only forward and sequential propagators and stochastic sources;
- one part using only forward and stochastic propagators.

In the $SU(2)$ symmetric case, this comes at no additional cost or complexity, since $\phi^u = \phi^d$. We just change $\Gamma_{f_2} \rightarrow \gamma_5 \Gamma_{f_2}$ and $C \Gamma_{f_1} \rightarrow \gamma_5 C \Gamma_{f_1} = C \gamma_5 \Gamma_{f_1}$.

2.3 Z-diagrams

$$P_{fif_i}^{udu} = \phi^{u(\gamma)}(f_1) (\Gamma_{i_2}(\vec{p}_{i_2}) \gamma_5)_{\gamma\delta} \phi^{u(\delta)}(f_2)^\dagger \gamma_5 \Gamma_{f_2}(\vec{p}_{f_2}) U(f_2, i_1). \quad (41)$$

We omit the $(\Gamma_{i_2}(\vec{p}_{i_2}) \gamma_5)$ in the following.

Z_1

$$(Z_1)_{\alpha\beta} = -z_{1\xi;\gamma}^l (C\Gamma_{i_1}(\vec{p}_{i_1}))_{\gamma\delta}^t z_{1\phi;\alpha\delta\beta}^l$$

$$z_{1\xi;\gamma}^l = \phi(f_2)^\dagger \gamma_5 \Gamma_{f_2}(\vec{p}_{f_2}) U(f_2, i_1)_{\gamma}^l$$

$$z_{1\phi;\alpha\delta\beta}^l = \phi(f_1)_\alpha^a \left[\epsilon_{abc} \epsilon_{lmn} D(f_1, i_1)_{\delta,\kappa}^{bm}{}^t (C\Gamma_{f_1})_{\kappa\lambda}^t U(f_1, i_1)_{\lambda\beta}^{cn} \right] e^{i\vec{p}_{f_1}} \quad (42)$$

$$z_{1\phi;\alpha\delta\beta}^l = \phi(f_1)_\alpha^a \left[\epsilon_{abc} \epsilon_{lmn} D(f_1, i_1)_{\delta,\kappa}^{bm}{}^t (C\Gamma_{f_1})_{\kappa\lambda}^t U(f_1, i_1)_{\lambda\beta}^{cn} \right] e^{i\vec{p}_{f_1}} \quad (43)$$

Γ_{i_1} and \vec{p}_{i_1} remain open.

Z_2

$$(Z_2)_{\alpha\beta} = -z_{2\phi;\alpha\delta\gamma}^l (C\Gamma_{i_1}(\vec{p}_{i_1}))_{\gamma\delta}^t z_{2\xi;\beta}^l$$

$$z_{2\xi;\beta}^l = z_{1\xi;\beta}^l \quad (44)$$

$$z_{2\phi;\alpha\delta\gamma}^l = z_{1\phi;\alpha\delta\gamma}^l \quad (45)$$

Γ_{i_1} and \vec{p}_{i_1} remain open.

Z_3

$$(Z_3)_{\alpha\beta} = -z_{3\phi;\delta\alpha\gamma}^n (C\Gamma_{i_1}(\vec{p}_{i_1}))_{\gamma\delta}^t z_{3\xi;\beta}^n$$

$$z_{3\xi;\beta}^n = z_{1\xi;\beta}^n \quad (46)$$

$$z_{3\phi;\delta\alpha\gamma}^n = \epsilon_{nml} \left[\epsilon_{cba} \phi(f_1)_\kappa^c (C\Gamma_{f_1})_{\kappa\lambda} D(f_1, i_1)_{\lambda\delta}^{bm} \right] U(f_1, i_1)_{\alpha\gamma}^{al} e^{i\vec{p}_{f_1}} \quad (47)$$

Γ_{i_1} and \vec{p}_{i_1} remain open.

Z_4

$$(Z_4)_{\alpha\beta} = -z_{4\xi;\gamma}^n (C\Gamma_{i_1}(\vec{p}_{i_1}))_{\gamma\delta}^t z_{4\phi;\delta\alpha\beta}^n$$

$$z_{4\xi;\gamma}^n = z_{1\xi;\gamma}^n \quad (48)$$

$$z_{4\phi;\delta\alpha\beta}^n = \epsilon_{nml} \left[\epsilon_{cba} \phi(f_1)_\kappa^c (C\Gamma_{f_1})_{\kappa\lambda} D(f_1, i_1)_{\lambda\delta}^{bm} \right] U(f_1, i_1)_{\alpha\beta}^{al} e^{i\vec{p}_{f_1}}$$

$$= z_{3\phi;\delta\alpha\beta}^n \quad (49)$$

Γ_{i_1} and \vec{p}_{i_1} remain open.

2.4 Conclusion

For the B , W , Z -type diagrams we need

B : $b_{1\xi}$, $b_{1\phi}$;

W : $w_{1\xi}$, $w_{1\phi}$, $w_{3\phi}$;

Z : $z_{1\xi}$, $z_{1\phi}$, $z_{3\phi}$.

diagram type	object	function
B	$b_{1\xi}$	\mathcal{V}_3
	$b_{1\phi}$	\mathcal{V}_2
W	$w_{1\xi}$	\mathcal{V}_3
	$w_{1\phi}$	\mathcal{V}_2
	$w_{3\phi}$	\mathcal{V}_2
Z	$z_{1\xi}$	\mathcal{V}_3
	$z_{1\phi}$	\mathcal{V}_4
	$z_{3\phi}$	\mathcal{V}_2

Note: Γ_{f_2} covers the pion vertex at sink (2-point function diagrams) as well as the vector current vertex at the current insertion (3-point function diagrams).

2.5 Reduction operations

2.5.1 Fermion - Propagator scalar products

These are of the form

$$V_3(x, y)_\beta^b = \mathcal{V}_3(\xi, S) = \sum_{a, \alpha} \sum_{\vec{x}} \xi(t_x, \vec{x})_\alpha^{a*} \Gamma e^{i\vec{p}\vec{x}} S(t_x, \vec{x}; t_y, \vec{y})_{\alpha\beta}^{ab} \quad (50)$$

This is the same we use for the meson-meson contractions.

2.5.2 Propagator - Propagator ϵ products

$$A(x, y)_{\alpha_{i_1}, \alpha_{i_2}}^{al} = \sum_{b, c} \sum_{m, n} \sum_{\alpha_{i_3}, \alpha_{i_4}} \epsilon_{abc} \epsilon_{lmn} S(x, y)_{\alpha_1 \alpha_2}^{bm} S(x, y)_{\alpha_3 \alpha_4}^{cn} \delta_{\alpha_{i_3}, \alpha_{i_4}} \quad (51)$$

with $i_1, i_2, i_3, i_4 \in \{1, 2, 3, 4\}$ and $i_k \neq i_l \quad \forall k \neq l$ and $i_1 < i_2, i_3 < i_4$.

This is *qcd.quarkContractAB* or *-fp-eps-contractAB-fp*. With the special choice of $A = 1, B = 3$ we get \mathcal{V}_4 .

$$V_4(x, y)_{\alpha\beta\gamma}^l = \mathcal{V}_4(\phi, S, S) = \sum_a \phi(x)_\alpha^a \text{contract13}(S(x, y), S(x, y))_{\beta\gamma}^{al}. \quad (52)$$

2.5.3 Fermion - Propagator ϵ product

This reduction will be useful for a fermion ϕ and a point-to-all propagator S .

$$V_1(x, y)_{\alpha_2}^{am} = \mathcal{V}_1(\phi, S) = \sum_{\alpha_1} \sum_{c, b} \epsilon_{cba} \phi(x)_{\alpha_1}^c S(x, y)_{\alpha_1 \alpha_2}^{bm}. \quad (53)$$

The V_1 s is a mixed type with 2 color indices and 1 Dirac index. (Can be done per spin-color component (α_2, m) if need be).

The final reduction step will then be of the form

$$V_2(x, y)_{\alpha_1 \alpha_2 \alpha_3}^n = \mathcal{V}_2(V_1, S) = \sum_a \sum_{l, m} \epsilon_{nml} V_1(x, y)_{\alpha_1}^{am} S(x, y)_{\alpha_2 \alpha_3}^{al}, \quad (54)$$

which gives again a mixed type with 1 color index and 3 Dirac indices.

V_2 will be Fourier transformed and stored to disk.

We will thus need the functions $\mathcal{V}_1, \mathcal{V}_2$ in eqs. (53), (54) above.

In *QLUA* The function \mathcal{V}_1 and \mathcal{V}_2 are implemented as functions called by

\mathcal{V}_1 : ColorVector[36] $v1 = qcd.contractV1(\text{DiracFermion } F, \text{DiracPropagator } P)$
indexing $v1(a, b, \alpha) = v1[4 \cdot (3 \cdot a + b) + \alpha]$

\mathcal{V}_2 : ColorVector[192] $v2 = qcd.contractV2(\text{ColorVector}[36] v1, \text{DiracPropagator } P)$
indexing $v2(\alpha, \beta, \gamma, a) = v2[3 \cdot (4 \cdot (4 \cdot \alpha + \beta) + \gamma) + a]$

\mathcal{V}_3 $qcd.contractV3$

\mathcal{V}_4 use existing $qcd.quarkContractAB$

To do If need be, generalized contraction functions for B_k, W_k, Z_k

2.6 Adding the ρ -channel, $I = 1, I_3 = 0$

We add the diagrams for $\pi\pi$ -scattering in the ρ -channel with $I = 1$ and $I_3 = 0$. The interpolating fields of interest are given by

$$O_{\pi\pi}(t, \vec{p}_1, \vec{p}_2) = \pi^+(t, \vec{p}_1) \pi^-(t, \vec{p}_2) - \pi^-(t, \vec{p}_1) \pi^+(t, \vec{p}_2) \quad (55)$$

$$\pi^\pm(t, \vec{p}) = \sum_{\vec{x}} \bar{\psi}(t, \vec{x}) \gamma_5 \tau^\pm \psi(t, \vec{x}) e^{i\vec{p}\vec{x}}$$

$$O_\rho(t, \vec{P}) = \bar{\psi}(t, \vec{x}) \Gamma \tau^3 \psi(t, \vec{x}) e^{i\vec{P}\vec{x}} \quad (56)$$

2.6.1 Contractions in position space

$\pi\pi - \pi\pi$ We have two types contributions for $I = 1, I_3 = 0$, the box diagram and the direct diagram.¹

¹Wick contractions also give a diagram with a charged pion propagating within the source and sink timeslice, respective. This diagram vanishes for $I = 1, I_3 = 0$ for any momentum combination at source and sink.

From the *box diagram* we get

$$C_{\pi\pi-\pi\pi}^{\text{box}} \quad (57)$$

$$\begin{aligned} &= -\text{tr} (U(x_{i_1}, x_{i_2}) \gamma_5 D(x_{i_2}, x_{f_2}) \gamma_5 U(x_{f_2}, x_{f_1}) \gamma_5 D(x_{f_1}, x_{i_1}) \gamma_5) \\ &\quad - \text{tr} (U(x_{i_1}, x_{f_1}) \gamma_5 D(x_{f_1}, x_{f_2}) \gamma_5 U(x_{f_2}, x_{i_2}) \gamma_5 D(x_{i_2}, x_{i_1}) \gamma_5) \\ &= -2 \text{Re} (\text{tr} (U(x_{i_1}, x_{f_1}) \gamma_5 D(x_{f_1}, x_{f_2}) \gamma_5 U(x_{f_2}, x_{i_2}) \gamma_5 D(x_{i_2}, x_{i_1}) \gamma_5)) \\ &= C_1^{\text{box}} + C_2^{\text{box}} \end{aligned} \quad (58)$$

Adding the momenta and using the sequential propagator and the stochastic source and propagator, we get

$$\begin{aligned} C_2^{\text{box}}(t_f, t_i; \vec{p}_{f_1} \vec{p}_{f_2}, \vec{p}_{i_2}) &= \\ \text{tr} (U(x_{i_1}, x_{f_1}) \Gamma_{f_1}(\vec{p}_{f_1}) D(x_{f_1}, x_{f_2}) \Gamma_{f_2}(\vec{p}_{f_2}) U(x_{f_2}, x_{i_2}) \Gamma_{i_2}(\vec{p}_{i_2}) D(x_{i_2}, x_{i_1}) \Gamma_{i_1}(\vec{p}_{i_1})) \\ &= \text{tr} \left(D(x_{f_1}, x_{i_1})^{\dagger} \gamma_5 \Gamma_{f_1} e^{i\vec{p}_{f_1} \vec{x}_{f_1}} \bar{\phi}_{t_{f_2}}^r(x_{f_1}) \xi_{t_{f_2}}^{r\dagger}(x_{f_2}) \Gamma_{f_2} e^{i\vec{p}_{f_2} \vec{x}_{f_2}} T^{ud}(x_{f_2}; t_{i_2}, \vec{p}_{i_2}; x_{i_1}) \Gamma_{i_1} \gamma_5 \right) \end{aligned} \quad (59)$$

$$C_{\pi\pi-\pi\pi}^{\text{box}} = \left[C_2^{\text{box}}(t_f, t_i; \vec{p}_{f_1} \vec{p}_{f_2}, \vec{p}_{i_2}) + C_2^{\text{box}}(t_f, t_i; -\vec{p}_{f_1} - \vec{p}_{f_2}, -\vec{p}_{i_2})^* \prod_l \sigma_{\Gamma_l}^{5\dagger} \right] e^{i\vec{p}_{i_1} \vec{x}_{i_1}}, \quad (60)$$

with $\sigma_{\Gamma_l}^{5\dagger} = 1 \quad \forall l \in \{i_1, i_2, f_1, f_2\}$ in the present case.

Thus we can construct this contraction from

$$V_3 : \xi_{t_{f_2}}^{r\dagger}(x_{f_2}) \Gamma_{f_2} e^{i\vec{p}_{f_2} \vec{x}_{f_2}} T^{ud}(x_{f_2}; t_{i_2}, \vec{p}_{i_2}; x_{i_1}) \quad (61)$$

$$\begin{aligned} V_3^* : D(x_{f_1}, x_{i_1})^{\dagger} \gamma_5 \Gamma_{f_1} e^{i\vec{p}_{f_1} \vec{x}_{f_1}} \bar{\phi}_{t_{f_2}}^r(x_{f_1}) \\ = \left[\bar{\phi}_{t_{f_2}}^r(x_{f_1})^{\dagger} (\gamma_5 \Gamma_{f_1})^{\dagger} e^{-i\vec{p}_{f_1} \vec{x}_{f_1}} D(x_{f_1}, x_{i_1}) \right]^* \end{aligned} \quad (62)$$

Note that $\Gamma_{i_1} = \gamma_5 = \Gamma_{f_1}$, thus $\Gamma_{i_1} \gamma_5 = \mathbb{1}$ and $(\gamma_5 \Gamma_{f_1})^{\dagger} = \mathbb{1}$.

The *direct diagram* gives

$$\begin{aligned} C_{\pi\pi-\pi\pi}^{\text{direct}} \quad (63) \\ = \text{tr} (U(x_{i_1}, x_{f_1}) \Gamma_{f_1}(\vec{p}_{f_1}) D(x_{f_1}, x_{i_1}) \Gamma_{i_1}(\vec{p}_{i_1})) \text{tr} (U(x_{f_2}, x_{i_2}) \Gamma_{i_2}(\vec{p}_{i_2}) D(x_{i_2}, x_{f_2}) \Gamma_{f_2}(\vec{p}_{f_2})) \end{aligned}$$

The first factor on the right-hand side of (63) is obtained from point-source-propagator contractions, the second factor from oet-stochastic timeslice propagator contractions.

$\pi\pi - \rho$ We get

$$\begin{aligned}
C_{\pi\pi-\rho}^{\text{triangle}} & \\
&= \text{tr} (U(x_{i_1}, x_{i_2}) \Gamma_{i_2}(\vec{p}_{i_2}) D(x_{i_2}, x_f) \Gamma_f(\vec{p}_f) D(x_f, x_{i_1}) \Gamma_{i_1}(\vec{p}_{i_1})) \\
&\quad - \text{tr} (U(x_{i_1}, x_f) \Gamma_f(\vec{p}_f) U(x_f, x_{i_2}) \Gamma_{i_2}(\vec{p}_{i_2}) D(x_{i_2}, x_{i_1}) \Gamma_{i_1}(\vec{p}_{i_1})) \\
&= C_1^{\text{triangle}} + C_2^{\text{triangle}}
\end{aligned} \tag{64}$$

$$C_2^{\text{triangle}}(t_f, t_i; \vec{p}_f, \vec{p}_{i_2}) = -\text{tr} \left(D(x_f, x_{i_1})^{\dagger} \gamma_5 \Gamma_f(\vec{p}_f) T^{ud}(x_f; t_i, \vec{p}_{i_2}; x_{i_1}) \Gamma_{i_1}(\vec{p}_{i_1}) \gamma_5 \right) \tag{65}$$

$$\begin{aligned}
C_{\pi\pi-\rho}^{\text{triangle}} & \\
&= \left[C_2^{\text{triangle}}(t_f, t_i; \vec{p}_f, \vec{p}_{i_2}) - \sigma_{\Gamma_{i_1}}^{5\tilde{\dagger}} \sigma_{\Gamma_{i_2}}^{5\tilde{\dagger}} \sigma_{\Gamma_f}^{5\tilde{\dagger}} C_2^{\text{triangle}}(t_f, t_i; -\vec{p}_f, -\vec{p}_{i_2})^* \right] e^{i\vec{p}_{i_1} \vec{x}_{i_1}}
\end{aligned} \tag{66}$$

Note, that with $\Gamma_{i_1} = \gamma_5 = \Gamma_{i_2}$ we have $\sigma_{\Gamma_{i_{1,2}}}^{5\tilde{\dagger}} = 1$ and with $\Gamma_f = \gamma_j$, $\gamma_j \gamma_0$ we have $\sigma_{\Gamma_f}^{5\tilde{\dagger}} = -1$.