1 Inversions and contractions — practical implementation on the lattice

1.1 Adjoint currents

We the following rule for complex conjugation of a product of Grassmann numbers z_1, z_2

$$(z_1\, z_2)^* = z_2^*\, z_1^*\,. \tag{1}$$

With this convention, the quark-bilinear $\bar{q}q$ (e.g. part of the mass term) is a real quantity

$$(\bar{q}q)^* = \left(q^\dagger\, \gamma_0\, q\right)^* = q^\dagger\, \gamma_0^\dagger\, q = \bar{q}q\,.$$

Thus for a baryon interpolating field Q and its adjoint $\bar{Q}$ we have

$$Q^c_\alpha = \epsilon_{abc}\, \left(q_1^{aT}\, \Gamma\, q_2^b\right)\, q^c_{3\,\alpha} = \epsilon_{abc}\, \left(q^a_{1\,\gamma}\, \Gamma_{\gamma\delta}\, q^b_{2\,\delta}\right)\, q^c_{3\,\alpha} \tag{2}$$

$$\begin{aligned}\bar{Q}^{c'}_\beta &= \left[\epsilon_{a'b'c'}\, \left(q^{a'}_{1\,\gamma}\, \Gamma_{\gamma\delta}\, q^{b'}_{2\,\delta}\right)\, q^{c'}_{3\,\rho}\right]^*\, (\gamma_0)_{\rho\beta} = \epsilon_{a'b'c'}\, \left(q^{b'*}_{2\,\delta}\, \Gamma^\dagger_{\delta\gamma}\, q^{a'*}_{1\,\gamma}\right)\, q^{c'*}_{3\,\rho}\, (\gamma_0)_{\rho\beta} \\ &= -\epsilon_{c'b'a'}\, \bar{q}^{c'}_{3\,\beta}\, \left(\bar{q}^{b'}_{2\,\kappa}\, (\gamma_0)_{\kappa\delta}\, \Gamma^\dagger_{\delta\gamma}\, (\gamma_0)_{\gamma\lambda}\, \bar{q}^{a'}_{1\,\lambda}\right) = -\epsilon_{c'a'b'}\, \bar{q}^{c'}_{3\,\beta}\, \left(\bar{q}^{a'}_2\, \tilde{\Gamma}\, \bar{q}^{b'T}_1\right)\,,\end{aligned} \tag{3}$$

where

$$\tilde{\Gamma} = \gamma_0\, \Gamma^\dagger\, \gamma_0 = \sigma_{\Gamma^{0\dagger}}\, \Gamma$$

and we used the property $\gamma_0^T = \gamma_0$.

For a meson interpolating field, this entails

$$(\bar{q}_1\, \Gamma\, q_2)^* = q_2^\dagger\, \Gamma^\dagger\, \gamma_0\, q_1 = \bar{q}_2\, \tilde{\Gamma}\, q_1\,.$$

In particular for the charged pions we have $\pi^+ = \bar{d}\, i\gamma_5\, u$ with $\pi^{+*} = \pi^-$.
The values of $\sigma_{\Gamma^{0\dagger}}$ for Γ on of the 16 basis matrices is given by

Γ	γ_0	γ_i	$\mathbb{1}$	γ_5	$\gamma_0\,\gamma_5$	$\gamma_i\,\gamma_5$	$\gamma_0\,\gamma_i$	$\gamma_i\,\gamma_j$
$\sigma_{\Gamma^{0\dagger}}$	+1	−1	+1	−1	+1	−1	+1	-1

Note: I do not include factors of imaginary unit i in the interpolating fields in the contractions. They can be added afterwards as a complex phase to the correlation matrix. This holds for $C = i\gamma_0\,\gamma_2$ as well, which means in the contraction part I only consider the $\gamma_0\,\gamma_2$.

Convention for contraction code For simplicity, I always contract bare quark-field and γ combinations, *without* the minus sign in (3). This means a correlator C from *contract_baryon* must receive a sign

$$C_{Q_1-\bar{Q}_2} \to -\sigma_{\Gamma^{0\dagger}_{Q_2}}\, C_{Q_1\bar{Q}_2} \tag{4}$$

$$C_{Q_1-M_2^\dagger\bar{Q}_2} \to -\sigma_{\Gamma^{0\dagger}_{Q_2}}\, \sigma_{\Gamma^{0\dagger}_{M_2}}\, C_{Q_1-M_2^\dagger\bar{Q}_2} \tag{5}$$

$$C_{M_1Q_1-M_2^\dagger\bar{Q}_2} \to -\sigma_{\Gamma^{0\dagger}_{Q_2}}\, \sigma_{\Gamma^{0\dagger}_{M_2}}\, C_{M_1Q_1-M_2^\dagger\bar{Q}_2} \tag{6}$$

The overall sign from the adjoint operator at source can be added e.g. via the *comp_list_sign*.

1.2 Δ^{++} to $\pi^+ N^+$ 3-point function

Motivated by the considerations in the previous section **??** we continue with the practical implementation of the estimation of the matrix elements in lattice QCD. We start from

$$\langle J_{\Delta^{++}}(x_f)\, J_M^\dagger(x_{i_2})\, \bar{J}_N(x_{i_1})\rangle = -\sigma_{\Gamma_N^{0\dagger}}\,\sigma_{\Gamma_M^{0\dagger}}\,\langle \underbrace{J^\alpha_{\Delta^{++}}}_{(u^T\,C\Gamma_\Delta\,u)\,u}(t_f,\,\vec{x})\,\underbrace{J^\dagger_{\pi^+}}_{\bar{u}\,\gamma_5\,d}(t_i,\,\vec{y})\,\underbrace{\bar{J}^\beta_N}_{(\bar{d}\,C\Gamma_N\,\bar{u}^T)\,\bar{u}}(t_i,\,\vec{z})\rangle \tag{7}$$

$$\langle\Big[u^a_\gamma(x_f)\,(C\Gamma_\Delta)_{\gamma\delta}\,u^b_\delta(x_f)\,u^c_\alpha(x_f)\Big]\,\Big[\bar{u}^d_\sigma(x_{i_2})\,(\Gamma_M)_{\sigma\tau}\,d^d_\tau(x_{i_2})\Big]\,\Big[\vec{d}^l_\kappa(x_{i_1})\,(C\Gamma_N)_{\kappa\lambda}\,\bar{u}^m_\lambda(x_{i_1})\,\bar{u}^n_\beta(x_{i_1})\Big]\,\rangle_f = \tag{8}$$

$$\begin{aligned}
&- T(x_f,x_{i_1})\,C\Gamma_N\,\left(C\Gamma_\Delta\,U(x_f,x_{i_1})\right)^t\,U(x_f,x_{i_1})\\
&- T(x_f,x_{i_1})\,C\Gamma_N\,U(x_f,x_{i_1})^t\,C\Gamma_\Delta\,U(x_f,x_{i_1})\\
&- U(x_f,x_{i_1})\,\left(C\Gamma_\Delta\,T(x_f,x_{i_1})\,C\Gamma_N\right)^t\,U(x_f,x_{i_1})\\
&- U(x_f,x_{i_1})\,\left(T(x_f,x_{i_1})\,C\Gamma_N\right)^t\,C\Gamma_\Delta\,U(x_f,x_{i_1})\\
&- U(x_f,x_{i_1})\,\mathrm{Tr}\left(T(x_f,x_{i_1})\,C\Gamma_N\,U(x_f,x_{i_1})^t\,C\Gamma_\Delta\right)\\
&- U(x_f,x_{i_1})\,\mathrm{Tr}\left(T(x_f,x_{i_1})\,C\Gamma_N\,\left(C\Gamma_\Delta\,U(x_f,x_{i_1})\right)^t\right)\\
&= T_1+T_2+T_3+T_4+T_5+T_6
\end{aligned} \tag{9}$$

Eq. (8) defines the triangle diagrams $T_1,\dots,T_6$.
We use the notation

$$T(x_f,x_i) = T^{f_1 f_2\,ab}_{\alpha\beta}(x_f;t,\vec{q};x_i) = \sum_{\vec{y}}\,\left(S_{f_1}(x_f;\,t,\vec{y})\,\Gamma_M\,\mathrm{e}^{i\vec{q}\vec{y}}\,S_{f_2}(t,\vec{y};\,x_i)\right)^{ab}_{\alpha\beta} \tag{10}$$

for the sequential propagator with

- flavors "f_1 after f_2";
- sequential source timeslice t;
- sequential source momentum $\vec{q}$;
- sequential source Dirac structure Γ_M.

In particular we shall use the notation

$$T_{fii} = T(x_f;t_i,\vec{q};x_i) \tag{11}$$
$$T_{ffi} = T(x_f;t_f,\vec{q};x_i) \tag{12}$$

for 1-step sequential propagators.

Quantum numbers for Delta, pion and nucleon:

	Δ^{++}	π^+	$N^+ =$ Proton
J	$\frac{3}{2}$	0	$\frac{1}{2}$
I	$\frac{3}{2}$	1	$\frac{1}{2}$
I_3	$+\frac{3}{2}$	+1	$+\frac{1}{2}$
P	+1	-1	+1

1.3 Δ^{++} to Δ^{++}

$$\langle J_\Delta(x_f)\,\bar{J}_\Delta(x_i)\rangle_f = -\sigma_{\Gamma_i^{0\dagger}}\,\langle \left(u^T\,C\Gamma_f\,u\right)\,u(x_f)\,\left(\bar{u}^T\,C\Gamma_i\,\bar{u}\right)\,\bar{u}(x_i)\rangle \tag{13}$$

$$\begin{aligned}
&\langle\Big[\,\epsilon_{abc}\,u^a_\gamma(x_f)\,\left(C\Gamma_f\right)_{\gamma\delta}\,u^b_\delta(x_f)\,u^c_\alpha(x_f)\Big]\,\Big[\epsilon_{lmn}\,\bar{u}^l_\kappa(x_i)\,\left(C\Gamma_i\right)_{\kappa\lambda}\,\bar{u}^m_\lambda(x_i)\,\bar{u}^n_\beta(x_i)\Big]\rangle_f = \\
&\quad \epsilon_{abc}\,\epsilon_{lmn}\,\left(C\Gamma_f\right)_{\gamma\delta}\,\left(C\Gamma_i\right)_{\kappa\lambda}\,\Big\{ \\
&\qquad + U^{cn}_{\alpha\beta}\,\left(U^{bl}_{\delta\kappa}\,U^{am}_{\gamma\lambda} - U^{bm}_{\delta\lambda}\,U^{al}_{\gamma\kappa}\right) \\
&\qquad - U^{cm}_{\alpha\lambda}\,\left(U^{bl}_{\delta\kappa}\,U^{an}_{\gamma\beta} - U^{bn}_{\delta\beta}\,U^{al}_{\gamma\kappa}\right) \\
&\qquad + U^{cl}_{\alpha\kappa}\,\left(U^{bm\;an}_{\delta\lambda\;\gamma\beta} - U^{bn}_{\delta\beta}\,U^{am}_{\gamma\lambda}\right) \\
&\Big\} = \\
&\quad - U(x_f,x_i)\,C\Gamma_i\,\left(C\Gamma_f\,U(x_f,x_i)\right)^t\,U(x_f,x_i) \\
&\quad - U(x_f,x_i)\,C\Gamma_i\,U(x_f,x_i)^t\,C\Gamma_f\,U(x_f,x_i) \\
&\quad - U(x_f,x_i)\,\left(C\Gamma_f\,U(x_f,x_i)\,C\Gamma_i\right)^t\,U(x_f,x_i) \\
&\quad - U(x_f,x_i)\,\left(U(x_f,x_i)\,C\Gamma_i\right)^t\,C\Gamma_f\,U(x_f,x_i) \\
&\quad - U(x_f,x_i)\,\mathrm{Tr}\left(C\Gamma_f\,U(x_f,x_i)\,C\Gamma_i\,U(x_f,x_i)^t\right) \\
&\quad - U(x_f,x_i)\,\mathrm{Tr}\left(C\Gamma_f\,U(x_f,x_i)\,\left(U(x_f,x_i)\,C\Gamma_i\right)^t\right) \\
&= D_1 + D_2 + D_3 + D_4 + D_5 + D_6\,.
\end{aligned} \tag{14}$$

This define the $I = 3/2$ diagrams $D_1, \dots, D_6$.

Adjoint correlator Using γ_5-Hermiticity, parity and time reversal we expect, that

$$\begin{aligned}
C^{\alpha\beta}_{\mu\nu}(x,y) &= \langle J^\alpha_{\Delta\mu}(x)\,\bar{J}^\beta_{\Delta\nu}(y)\rangle \\
C_{\mu\nu}(x,y) &= \sigma^{02}_\mu\,\sigma^{02}_\mu\,C^{\tilde{\dagger}}_{\mu\nu} \\
\sigma^{02}_\mu &= \begin{cases} +1 & \mu = 0,\,2 \\ -1 & \mu = 1,\,3 \end{cases},
\end{aligned}$$

where $\tilde{\dagger}$ denotes the conjugate with respect to the spinor indices. This relation should hold exactly in the free case (gauge field $U = \mathbb{1}$) and at the level of the gauge average in the non-free case.

$$\begin{aligned}
(t_x,\,t_y) \sim t_x - t_y &\xrightarrow{\gamma_5-\text{Hermiticity}} (t_y,\,t_x) \sim t_y - t_x \\
&\xrightarrow{\mathcal{T}} (T - t_y,\,T - t_x) \sim (T - t_y) - (T - t_x) = t_x - t_y\,.
\end{aligned}$$

1.4 N^+ to N^+

$$\langle J_N(x_f)\,\bar{J}_N(x_i)\rangle_f = -\sigma_{\Gamma_i^{0\dagger}}\,\langle\big(u^T\,C\Gamma_f\,d\big)\;u(x_f)\;\big(\bar{d}^T\,C\Gamma_i\,\bar{u}\big)\;\bar{u}(x_i)\rangle \tag{15}$$

$$\begin{aligned}
&\langle\Big[\,\epsilon_{abc}\,u^a_\gamma(x_f)\;(C\Gamma_f)_{\gamma\delta}\;d^b_\delta(x_f)\,u^c_\alpha(x_f)\Big]\;\Big[\epsilon_{lmn}\,\bar{d}^l_\kappa(x_i)\;(C\Gamma_i)_{\kappa\lambda}\;\bar{u}^m_\lambda(x_i)\,\bar{u}^n_\beta(x_i)\Big]\rangle_f = \\
&\quad -U(x_f,x_i)\;\big(C\Gamma_f\,D(x_f,x_i)\,C\Gamma_i\big)^t\;U(x_f,x_i) \\
&\quad -U(x_f,x_i)\,\mathrm{Tr}\,\Big(\big(C\Gamma_f\,D(x_f,x_i)\,C\Gamma_i\big)^t\;U(x_f,x_i)\Big) \\
&= N_1 + N_2\,.
\end{aligned} \tag{16}$$

This defines the diagrams N_1, N_2.

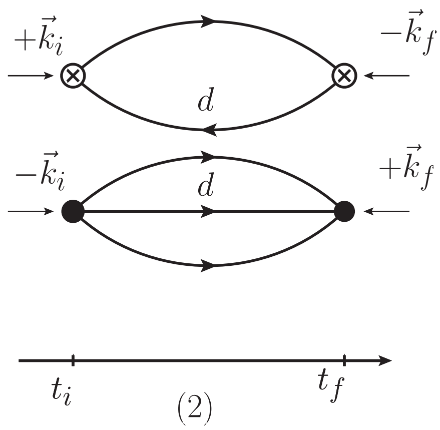

Figure 1: Graphical representation of the quark-disconnected contribution to the 4-pt. function $\pi\, N \to \pi\, N$

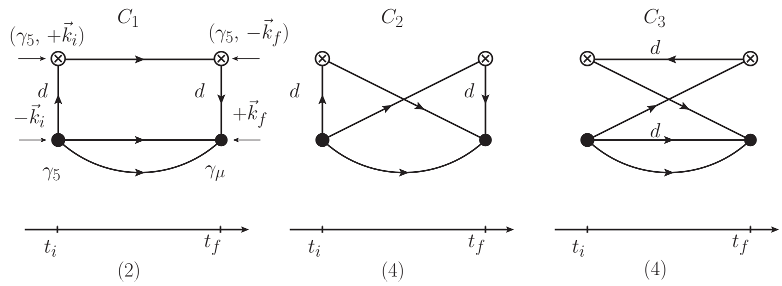

Figure 2: Graphical representation of the quark-connected contribution to the 4-pt. function $\pi\, N \to \pi\, N$ at zero total 3-momentum, $\vec{Q} = 0$

1.5 $\pi^+\, N^+$ to $\pi^+\, N^+$

In total these sum up to 12 contributions; we can check that is the right number 2 combinations of down quarks $\times$ 3! combinations of up quarks
We introduce some notation to write out the necessary contractions, the 2-step sequential propagators P

$$P^{f_1 f_2 f_1\, ab}_{\alpha\beta}(x_f;\, t_1, \vec{q}_1;\, t_2, \vec{q}_2;\, x_i) = \tag{17}$$
$$\sum_{\vec{z}_1, \vec{z}_2} \Big(S_{f_1}(x_f;\, t_1, \vec{z}_1)\, \Gamma_1\, S_{f_2}(t_1,\, \vec{z}_1;\, t_2, \vec{z}_2)\, \Gamma_2\, S_{f_1}(t_2, \vec{z}_2;\, x_i) \Big)^{ab}_{\alpha\beta}\, \mathrm{e}^{i(\vec{q}_1 \vec{z}_1 + \vec{q}_2 \vec{z}_2)}$$

where $f_{1/2} \in \{u,\, d\}$ and $f_1 \neq f_2$. In particular we shall use

$$P_{fifi} = P^{udu}(x_{f_1}; t_i, \vec{q}_{i_2}; t_f, \vec{q}_{f_2}; x_{i_1}) \tag{18}$$
$$P_{ffii} = P^{dud}(x_{f_1}; t_f, \vec{q}_{f_2}; t_i, \vec{q}_{i_2}; x_{i_1}) \tag{19}$$

with $t_{f_1} = t_{f_2} = t_f$ and $t_{i_1} = t_{i_2} = t_i$.

With these generalized propagators we can write the contractions in a short way.

$$\begin{aligned} &\langle J_{\pi^+ N^+}(x_{f_1}; x_{f_2})\, \bar{J}_{\pi^+ N^+}(x_{i_1}; x_{i_2})\rangle_f \\ &= -\sigma_{\Gamma^{0\dagger}_{N_i}}\, \sigma_{\Gamma^{0\dagger}_{M_i}}\, \langle \big(u^t\, C\Gamma_{N_f}\, d\big)\, u(x_{f_1})\, \bar{d}\, \Gamma_{M_f}\, u(x_{f_2})\, \bar{u}\, \Gamma_{M_i}\, d(x_{i_2})\, \big(\bar{d}\, C\Gamma_{N_i}\, u\big)\, u(x_{i_1})\rangle \end{aligned} \tag{20}$$

$$\begin{aligned} &\langle \Big[\epsilon_{abc}\, u^a_\gamma(x_{f_1})\, \big(C\Gamma_{N_f}\big)_{\gamma\delta}\, d^b_\delta(x_{f_1})\, u^c_\alpha(x_{f_1})\Big]\, \Big[\bar{d}^d_\sigma(x_{f_2})\, \big(\Gamma_{M_f}\big)_{\sigma\tau}\, u^d_\tau(x_{f_2})\Big] \times \\ &\quad \Big[\bar{u}^e_\mu(x_{i_2})\, (\Gamma_{M_i})_{\mu\nu}\, d^e_\nu(x_{i_2})\Big]\, \Big[\epsilon_{lmn}\, \bar{d}^l_\kappa(x_{i_1})\, (C\Gamma_{N_i})_{\kappa\lambda}\, \bar{u}^m_\lambda(x_{i_1})\, \bar{u}^n_\beta(x_{i_1})\Big]\rangle \\ &= C_B + C_W + C_Z + C_{\text{disconnected}} \end{aligned} \tag{21}$$

Quark-disconnected contribution — direct diagram

$$C_{\text{disconnected}} = -\mathrm{Tr}\,\big(U(x_{f_2}, x_{i_2})\, \Gamma_{M_i}\, D(x_{i_2}, x_{f_2})\, \Gamma_{M_f}\big) \times (N_1 + N_2)\ . \tag{22}$$

Quark-connected contributions — B, W and Z diagrams The connected contractions $C_{B,W,Z}$ are

$$\begin{aligned}
C_B =& \\
&- U(x_{f_1}, x_{i_1}) \left(C\Gamma_{N_f}\, P_{ffii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, U(x_{f_1}, x_{i_1}) \\
&- U(x_{f_1}, x_{i_1})\, \mathrm{Tr}\left(C\Gamma_{N_f}\, P_{ffii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\, U(x_{f_1}, x_{i_1})^t\right) \\
=& B_1 + B_2
\end{aligned} \tag{23}$$

$$\begin{aligned}
C_W =& \\
&- T^{ud}_{fii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i} \left(C\Gamma_{N_f}\, T^{du}_{ffi}(x_{f_1}, x_{i_1})\right)^t\, U(x_{f_1}, x_{i_1}) \\
&- T^{ud}_{fii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\, U(x_{f_1}, x_{i_1})^t\, C\Gamma_{N_f}\, T^{du}_{ffi}(x_{f_1}, x_{i_1}) \\
&- U(x_{f_1}, x_{i_1}) \left(T^{ud}_{fii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, C\Gamma_{N_f}\, T^{du}_{ffi}(x_{f_1}, x_{i_1}) \\
&- U(x_{f_1}, x_{i_1})\, \mathrm{Tr}\left(T^{ud}_{fii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i} \left(C\Gamma_{N_f}\, T^{du}_{ffi}(x_{f_1}, x_{i_1})\right)^t\right) \\
=& C_{W_1} + C_{W_2} + C_{W_3} + C_{W_4}
\end{aligned} \tag{24}$$

$$\begin{aligned}
C_Z =& \\
&- P^{udu}_{fifi}(x_{f_1}, x_{i_1}) \left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, U(x_{f_1}, x_{i_1}) \\
&- P^{udu}_{fifi}(x_{f_1}, x_{i_1})\, \mathrm{Tr}\left(\left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, U(x_{f_1}, x_{i_1})\right) \\
&- U(x_{f_1}, x_{i_1}) \left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, P^{udu}_{fifi}(x_{f_1}, x_{i_1}) \\
&- U(x_{f_1}, x_{i_1})\, \mathrm{Tr}\left(P^{udu}_{fifi}(x_{f_1}, x_{i_1}) \left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\right) \\
=& C_{Z_1} + C_{Z_2} + C_{Z_3} + C_{Z_4}
\end{aligned} \tag{25}$$

Comments

- note that with degenerate up and down quarks we can use $T^{ud} = T^{du}$ and $P^{udu} = P^{dud}$. Moreover in the case at hand we will always have $\vec{k}_i = \vec{k}_f$
- the straightforward contractions using the *Seq*^{2}*Propagators* from point sources would be demanding; for each momentum vector $\vec{k} = \vec{k}_f = \vec{k}_i$ we would need $N_s \times N_c \times (3N_{t_f} + 2)$ inversions to produce all required S, T, P fields. N_s, N_c, N_{t_f} are the numbers of spinor, color components and the number of sink timeslices
- but (1): from the investigations of nucleon/delta matrix elements with current insertions it is know that $N_{t_f} = \mathcal{O}(T/4/a)$, a quarter of the temporal extent of the lattice
- but (2): using momenta from the class represented by $\vec{k} = (0, 0, 1)$, could one save computing time by combining $\vec{k}$ and $-\vec{k}$ using coherent sources ?
- What about stochastic timeslice sources? Issue: we need spin and color dilution (do we?); would thus still require $\mathcal{O}(N_s\, N_c\, T/a)$ inversions, same order of magnitude; needs inversions for both sources with and without momentum
- timeslice sources may require $\mathbb{Z}_5$ sources
- would be nice if chosen method would be applicable to Δ^{++}-, N^{+}-2-point functions and $\Delta^{++} \to \pi^{+} N^{+}$ 3-point function as well

2 Post-poned spin-color reduction and factorization for $I = 3/2,\ I^3 = +3/2$

2.1 B-diagrams

$$P^{dud}_{ffii} = \phi^d(f_1)\,\xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,T^{ud}(f_2, i_2, i_1) \tag{26}$$

B_1

$$\begin{aligned}
(B_1)_{\alpha\beta} &= -b_{1\phi}\,(\vec{p}_{f_1}, \Gamma_{f_1})^m_{\beta\alpha\delta}\,\left(\Gamma_{i_1}(\vec{p}_{i_1})\right)^t_{\delta\gamma}\,b_{1\xi}\,(\vec{p}_{f_2}, \Gamma_{f_2})^m_{\gamma} \\
b^m_{1\xi;\,\gamma} &= \xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,T_{fii}(f_2, i_1)^{;,m}_{;,\gamma} \qquad (27)\\
b^m_{1\phi;\,\beta\alpha\delta} &= \epsilon_{mnl}\,\left[\epsilon_{bca}\,\phi(f_1)^b_\kappa\,\left(C\,\Gamma_{f_1}\right)^t_{\kappa\lambda}\,U(f_1, i_1)^{cn}_{\lambda,\beta}\right]\,U(f_1, i_1)^{al}_{\alpha,\delta}\,\mathrm{e}^{i\vec{p}_{f_1}} \qquad (28)
\end{aligned}$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

B_2

$$\begin{aligned}
(B_2)_{\alpha\beta} &= -b_{2\phi}\,(\vec{p}_{f_1}, \Gamma_{f_1})^m_{\delta\alpha\beta}\,\left(\Gamma_{i_1}(\vec{p}_{i_1})\right)^t_{\delta\gamma}\,b_{2\xi}\,(\vec{p}_{f_2}, \Gamma_{f_2})^m_{\gamma} \\
b^m_{2\xi;\,\gamma} &= \xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,T_{fii}(f_2, i_1)^{;,m}_{;,\gamma} = b_{1\xi;\,\gamma} \qquad (29)\\
b^m_{2\phi;\,\delta\alpha\beta} &= \epsilon_{mnl}\,\left[\epsilon_{bca}\,\phi(f_1)^b_\kappa\,\left(C\,\Gamma_{f_1}\right)^t_{\kappa\lambda}\,U(f_1, i_1)^{cn}_{\lambda\delta}\right]\,U(f_1, i_1)^{al}_{\alpha,\beta}\,\mathrm{e}^{i\vec{p}_{f_1}} \qquad (30)\\
&= b^m_{1\phi;\,\delta\alpha\beta}
\end{aligned}$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

2.2 W-diagrams

$$T^{du}_{ffi} = \phi^d(f_1)\,\xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2, i_1)\,. \tag{31}$$

Note, that we can use γ_5-hermiticity and write

$$T^{du}_{ffi} = \gamma_5\,\xi(f_1)\,\phi^u(f_2)^\dagger\,\gamma_5\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2, i_1)\,, \tag{32}$$

where the Hermitean conjugation is with respect to spin-color indices. This means we can make corresponding replacements in $w_{n\xi}$ and $w_{n\phi}$ to separate contractions involving sequential propagators from those involving stochastic propagators (using $\phi^u = \phi^d$ in the case of Wilson-clover fermions). We can use $\gamma_5^t = \gamma_5$.

W_1

$$\begin{aligned}
(W_1)_{\alpha\beta} &= -w^m_{1\phi;\,\beta\alpha\delta}\,\left(C\,\Gamma_{i_1}\right)_{\delta\gamma}\,w^m_{1\xi;\,\gamma}\,\mathrm{e}^{i\vec{p}_{i_1}} \\
w^m_{1\xi;\,\gamma} &= \xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2, i_1)^{;,m}_{;,\gamma} \qquad (33)\\
w^m_{1\phi;\,\beta\alpha\delta} &= \epsilon_{mnl}\,\left[\epsilon_{bca}\,\phi(f_1)^b_\kappa\,\left(C\,\Gamma_{f_1}\right)^t_{\kappa\lambda}\,U(f_1, i_1)^{cn}_{\lambda\beta}\right]\,T_{fii}(f_1, i_1)^{al}_{\alpha\delta}\,\mathrm{e}^{i\vec{p}_{f_1}} \qquad (34)
\end{aligned}$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

W_2

$$
\begin{aligned}
(W_2)_{\alpha\beta} &= -w^n_{2\phi;\,\gamma\alpha\delta}\ (C\,\Gamma_{i_1}(\vec{p}_{i_1}))_{\delta\gamma}\ w^n_{2\xi;\beta} \\
w^n_{2\xi;\beta} &= \xi(f_2)^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)^{\cdot;n}_{\cdot,\beta} = w^n_{1\xi;\beta} && (35)\\
w^n_{2\phi;\,\gamma\alpha\delta} &= \epsilon_{nml}\ \left[\epsilon_{cba}\,\phi(f_1)^c_\kappa\ (C\,\Gamma_{f_1})^t_{\kappa\lambda}\ U(f_1,i_1)^{bm}_{\lambda\gamma}\right]\ T_{fii}(f_1,i_1)^{al}_{\alpha\delta}\,\mathrm{e}^{i\vec{p}_{f_1}} && (36)\\
&= w^n_{1\phi;\,\beta\alpha\delta}
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

W_3

$$
\begin{aligned}
(W_3)_{\alpha\beta} &= -w^n_{3\phi;\,\gamma\alpha\delta}\ (C\,\Gamma_{i_1}(\vec{p}_{i_1}))^t_{\delta\gamma}\ w^n_{3\xi;\beta} \\
w^n_{3\xi;\beta} &= \xi^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)^{\cdot;n}_{\cdot,\beta} = w^n_{1\xi;\beta} && (37)\\
w^n_{3\phi;\,\gamma\alpha\delta} &= \epsilon_{nml}\ \left[\epsilon_{cba}\,\phi(f_1)^c_\lambda\ (C\,\Gamma_{f_1})^t_{\lambda\kappa}\,T_{fii}(f_1,i_1)^{bm}_{\kappa\gamma}\right]\ U(f_1,i_1)^{al}_{\alpha\delta}\,\mathrm{e}^{i\vec{p}_{f_1}} && (38)
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

W_4

$$
\begin{aligned}
(W_4)_{\alpha\beta} &= -w^n_{4\phi;\,\gamma\alpha\beta}\ (C\,\Gamma_{i_1}(\vec{p}_{i_1}))^t_{\gamma\delta}\ w^n_{4\xi;\gamma} \\
w^n_{4\xi;\delta} &= \xi^\dagger\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)^{\cdot;n}_{\cdot,\delta} = w^n_{1\xi;\delta} && (39)\\
w^n_{4\phi;\,\gamma\alpha\beta} &= \epsilon_{nml}\ \left[\epsilon_{cba}\,\phi(f_1)^c_\lambda\ (C\,\Gamma_{f_1})^t_{\lambda\kappa}\,T_{fii}(f_1,i_1)^{bm}_{\kappa,\gamma}\right]\ U(f_1,i_1)^{al}_{\alpha\beta}\,\mathrm{e}^{i\vec{p}_{f_1}} && (40)\\
&= w^n_{3\phi;\,\gamma\alpha\beta}
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

Exchanging $\phi^d \to \phi^u$ In all the W diagrams we exchange $\phi^d\,\xi^\dagger \to (\gamma_5\,\xi)\ (\gamma_5\,\phi^u)^\dagger$ and thereby achieve the separation of sequential propagators T_{fii} and stochastic propagators $\phi^{u/d}$ into separate diagrams. Thus the contractions can be split accordingly into

- one part using only forward and sequential propagators and stochastic sources;
- one part using only forward and stochastic propagators.

In the $SU(2)$ symmetric case, this comes at no additional cost or complexity, since $\phi^u = \phi^d$. We just change $\Gamma_{f_2} \to \gamma_5\,\Gamma_{f_2}$ and $C\,\Gamma_{f_1} \to \gamma_5\,C\,\Gamma_{f_1} = C\,\gamma_5\,\Gamma_{f_1}$.

2.3 Z-diagrams

$$
P^{udu}_{fifi} = \phi^{u(\gamma)}(f_1)\ (\Gamma_{i_2}(\vec{p}_{i_2})\,\gamma_5)_{\gamma\delta}\ \phi^{u(\delta)}(f_2)^\dagger\,\gamma_5\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)\,. \tag{41}
$$

We omit the $(\Gamma_{i_2}(\vec{p}_{i_2})\,\gamma_5)$ in the following.

Z_1

$$
\begin{aligned}
(Z_1)_{\alpha\beta} &= -z^{l}_{1\xi;\,\gamma}\; (C\,\Gamma_{i_1}(\vec{p}_{i_1}))^{t}_{\gamma\delta}\; z^{l}_{1\phi;\,\alpha\delta\beta} \\
z^{l}_{1\xi;\,\gamma} &= \phi(f_2)^{\dagger}\,\gamma_5\,\Gamma_{f_2}(\vec{p}_{f_2})\,U(f_2,i_1)^{\cdot;l}_{\cdot;\gamma} && (42)\\
z^{l}_{1\phi;\,\alpha\delta\beta} &= \phi(f_1)^{a}_{\alpha}\;\Big[\epsilon_{abc}\,\epsilon_{lmn}\,D(f_1,i_1)^{bm\,t}_{\delta,\kappa}\;(C\,\Gamma_{f_1})^{t}_{\kappa\lambda}\;U(f_1,i_1)^{cn}_{\lambda\beta}\Big]\;\mathrm{e}^{i\vec{p}_{f_1}} && (43)
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

Z_2

$$
\begin{aligned}
(Z_2)_{\alpha\beta} &= -z^{l}_{2\phi;\,\alpha\delta\gamma}\; (C\,\Gamma_{i_1}(\vec{p}_{i_1}))^{t}_{\gamma\delta}\; z^{l}_{2\xi;\,\beta} \\
z^{l}_{2\xi;\,\beta} &= z^{l}_{1\xi;\,\beta} && (44)\\
z^{l}_{2\phi;\,\alpha\delta\gamma} &= z^{l}_{1\phi;\,\alpha\delta\gamma} && (45)
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

Z_3

$$
\begin{aligned}
(Z_3)_{\alpha\beta} &= -z^{n}_{3\phi;\,\delta\alpha\gamma}\; (C\,\Gamma_{i_1}(\vec{p}_{i_1}))^{t}_{\gamma\delta}\; z^{n}_{3\xi;\,\beta} \\
z^{n}_{3\xi;\,\beta} &= z^{n}_{1\xi;\,\beta} && (46)\\
z^{n}_{3\phi;\,\delta\alpha\gamma} &= \epsilon_{nml}\;\Big[\epsilon_{cba}\,\phi(f_1)^{c}_{\kappa}\;(C\,\Gamma_{f_1})_{\kappa\lambda}\;D(f_1,i_1)^{bm}_{\lambda\delta}\Big]\;U(f_1,i_1)^{al}_{\alpha\gamma}\,\mathrm{e}^{i\vec{p}_{f_1}} && (47)
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

Z_4

$$
\begin{aligned}
(Z_4)_{\alpha\beta} &= -z^{n}_{4\xi;\,\gamma}\; (C\,\Gamma_{i_1}(\vec{p}_{i_1}))^{t}_{\gamma\delta}\; z^{n}_{4\phi;\,\delta\alpha\beta} \\
z^{n}_{4\xi;\,\gamma} &= z^{n}_{1\xi;\,\gamma} && (48)\\
z^{n}_{4\phi;\,\delta\alpha\beta} &= \epsilon_{nml}\;\Big[\epsilon_{cba}\,\phi(f_1)^{c}_{\kappa}\;(C\,\Gamma_{f_1})_{\kappa\lambda}\;D(f_1,i_1)^{bm}_{\lambda\delta}\Big]\;U(f_1,i_1)^{al}_{\alpha\beta}\,\mathrm{e}^{i\vec{p}_{f_1}} && (49)\\
&= z^{n}_{3\phi;\,\delta\alpha\beta}
\end{aligned}
$$

Γ_{i_1} and $\vec{p}_{i_1}$ remain open.

2.4 Conclusion

For the B, W, Z-type diagrams we need

B: $b_{1\xi}$, $b_{1\phi}$;

W: $w_{1\xi}$, $w_{1\phi}$, $w_{3\phi}$;

Z: $z_{1\xi}$, $z_{1\phi}$, $z_{3\phi}$.

diagram type	object	function
B	$b_{1\xi}$	$\mathcal{V}_3$
	$b_{1\phi}$	$\mathcal{V}_2$
W	$w_{1\xi}$	$\mathcal{V}_3$
	$w_{1\phi}$	$\mathcal{V}_2$
	$w_{3\phi}$	$\mathcal{V}_2$
Z	$z_{1\xi}$	$\mathcal{V}_3$
	$z_{1\phi}$	$\mathcal{V}_4$
	$z_{3\phi}$	$\mathcal{V}_2$

Note: Γ_{f_2} covers the pion vertex at sink (2-point function diagrams) as well as the vector current vertex at the current insertion (3-point function diagrams).

2.5 Reduction operations

2.5.1 Fermion - Propagator scalar products

These are of the form

$$V_3(x,y)^b_\beta = \mathcal{V}_3(\xi,\,S) = \sum_{a,\alpha}\sum_{\vec{x}} \xi(t_x,\vec{x})^{a*}_\alpha\,\Gamma\,\mathrm{e}^{i\vec{p}\vec{x}}\,S(t_x,\vec{x};t_y,\vec{y})^{ab}_{\alpha\beta} \tag{50}$$

This is the same we use for the meson-meson contractions.

2.5.2 Propagator - Propagator ϵ products

$$A(x,y)^{al}_{\alpha_{i_1},\alpha_{i_2}} = \sum_{b,c}\sum_{m,n}\sum_{\alpha_{i_3},\alpha_{i_4}} \epsilon_{abc}\,\epsilon_{lmn}\,S(x,y)^{bm}_{\alpha_1\alpha_2}\,S(x,y)^{cn}_{\alpha_3\alpha_4}\,\delta_{\alpha_{i_3},\alpha_{i_4}} \tag{51}$$

with $i_1,i_2,i_3,i_4 \in \{1,2,3,4\}$ and $i_k \neq i_l \quad \forall k \neq l$ and $i_1 < i_2,\, i_3 < i_4$.

This is *qcd.quarkContractAB* or *_fp_eq_fp_eps_contractAB_fp*. With the special choice of $A = 1,\ B = 3$ we get $\mathcal{V}_4$.

$$V_4(x,y)^l_{\alpha\beta\gamma} = \mathcal{V}_4\,(\phi,\,S,\,S) = \sum_a \phi(x)^a_\alpha\ \ \mathrm{contract13}\,(S(x,y),\,S(x,y))^{al}_{\beta\gamma}\ . \tag{52}$$

2.5.3 Fermion - Propagator ϵ product

This reduction will be useful for a fermion ϕ and a point-to-all propagator S.

$$V_1(x,y)^{am}_{\alpha_2} = \mathcal{V}_1\,(\phi,\,S) = \sum_{\alpha_1}\sum_{c,b} \epsilon_{cba}\,\phi(x)^c_{\alpha_1}\,S(x,y)^{bm}_{\alpha_1\alpha_2}\,. \tag{53}$$

The V_1s is a mixed type with 2 color indices and 1 Dirac index. (Can be done per spin-color component $(\alpha_2,\, m)$ if need be).

The final reduction step will then be of the form

$$V_2(x,y)^n_{\alpha_1\alpha_2\alpha_3} = \mathcal{V}_2\left(V_1,\, S\right) = \sum_a \sum_{l,m} \epsilon_{nml}\, V_1(x,y)^{am}_{\alpha_1}\, S(x,y)^{al}_{\alpha_2\,\alpha_3}\,, \tag{54}$$

which gives again a mixed type with 1 color index and 3 Dirac indices.
V_2 will be Fourier transformed and stored to disk.
We will thus need the functios $\mathcal{V}_1$, $\mathcal{V}_2$ in eqs. (53), (54) above.

In *QLUA* The function $\mathcal{V}_1$ and $\mathcal{V}_2$ are implemented as functions called by

$\mathcal{V}_1$: ColorVector[36] $v1 = qcd.contractV1$ (DiracFermion F , DiracPropagator P)
indexing $v1(a,b,\alpha) = v1[4\cdot(3\cdot a+b)+\alpha]$

$\mathcal{V}_2$: ColorVector[192] $v2 = qcd.contractV2$ (ColorVector[36] $v1$, DiracPropagator P)
indexing $v2(\alpha,\beta,\gamma,a) = v2[3\cdot(4\cdot(4\cdot\alpha+\beta)+\gamma)+a]$

$\mathcal{V}_3$ $qcd.contractV3$

$\mathcal{V}_4$ use existing $qcd.quarkContractAB$

To do If need be, generalized contraction functions for B_k, W_k, Z_k