1 Inversions and contractions — practical implementation on the lattice

1.1 Adjoint currents

We the following rule for complex conjugation of a product of Grassmann numbers z_1, z_2

$$(z_1\, z_2)^* = z_2^*\, z_1^*\,. \tag{1}$$

With this convention, the quark-bilinear $\bar{q}q$ (e.g. part of the mass term) is a real quantity

$$(\bar{q}q)^* = \left(q^\dagger\, \gamma_0\, q\right)^* = q^\dagger\, \gamma_0^\dagger\, q = \bar{q}q\,.$$

Thus for a baryon interpolating field Q and its adjoint $\bar{Q}$ we have

$$Q^c_\alpha = \epsilon_{abc}\,\left(q_1^{aT}\,\Gamma\, q_2^b\right)\, q^c_{3\,\alpha} = \epsilon_{abc}\,\left(q^a_{1\,\gamma}\,\Gamma_{\gamma\delta}\, q^b_{2\,\delta}\right)\, q^c_{3\,\alpha} \tag{2}$$

$$\begin{aligned}\bar{Q}^{c'}_\beta &= \left[\epsilon_{a'b'c'}\,\left(q^{a'}_{1\,\gamma}\,\Gamma_{\gamma\delta}\, q^{b'}_{2\,\delta}\right)\, q^{c'}_{3\,\rho}\right]^*\,(\gamma_0)_{\rho\beta} = \epsilon_{a'b'c'}\,\left(q^{b'*}_{2\,\delta}\,\Gamma^\dagger_{\delta\gamma}\, q^{a'*}_{1\,\gamma}\right)\, q^{c'*}_{3\,\rho}\,(\gamma_0)_{\rho\beta}\\ &= -\epsilon_{c'b'a'}\,\bar{q}^{c'}_{3\,\beta}\,\left(\bar{q}^{b'}_{2\,\kappa}\,(\gamma_0)_{\kappa\delta}\,\Gamma^\dagger_{\delta\gamma}\,(\gamma_0)_{\gamma\lambda}\,\bar{q}^{a'}_{1\,\lambda}\right) = -\epsilon_{c'a'b'}\,\bar{q}^{c'}_{3\,\beta}\,\left(\bar{q}^{a'}_2\,\tilde{\Gamma}\,\bar{q}_1^{b'T}\right)\,,\end{aligned} \tag{3}$$

where

$$\tilde{\Gamma} = \gamma_0\,\Gamma^\dagger\,\gamma_0 = \sigma_\Gamma\,\Gamma$$

and we used the property $\gamma_0^T = \gamma_0$.

For a meson interpolating field, this entails

$$(\bar{q}_1\,\Gamma\, q_2)^* = q_2^\dagger\,\Gamma^\dagger\,\gamma_0\, q_1 = \bar{q}_2\,\tilde{\Gamma}\, q_1\,.$$

In particular for the charged pions we have $\pi^+ = \bar{d}\, i\gamma_5\, u$ with $\pi^{+*} = \pi^-$.
The values of σ_Γ for Γ on of the 16 basis matrices is given by

Γ	γ_0	γ_i	$\mathbb{1}$	γ_5	$\gamma_0\,\gamma_5$	$\gamma_i\,\gamma_5$	$\gamma_0\,\gamma_i$	$\gamma_i\,\gamma_j$
σ_Γ	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1

Note: I do not include factors of imaginary unit i in the interpolating fields in the contractions. They can be added afterwards as a complex phase to the correlation matrix. This holds for $C = i\gamma_0\,\gamma_2$ as well, which means in the contraction part I only consider the $\gamma_0\,\gamma_2$.

Convention for contraction code For simplicity, I always contract bare quark-field and γ combinations, *without* the minus sign in (3). This means a correlator C from *contract_baryon* must receive a sign

$$C_{Q_1-\bar{Q}_2} \to -\sigma_{\Gamma_{Q_2}}\, C_{Q_1\bar{Q}_2} \tag{4}$$

$$C_{Q_1-M_2^\dagger\bar{Q}_2} \to -\sigma_{\Gamma_{Q_2}}\,\sigma_{\Gamma_{M_2}} C_{Q_1-M_2^\dagger\bar{Q}_2} \tag{5}$$

$$C_{M_1Q_1-M_2^\dagger\bar{Q}_2} \to -\sigma_{\Gamma_{Q_2}}\,\sigma_{\Gamma_{M_2}} C_{M_1Q_1-M_2^\dagger\bar{Q}_2} \tag{6}$$

The overall sign from the adjoint operator at source can be added e.g. via the *comp_list_sign*.

1.2 Δ^{++} to $\pi^+ N^+$ 3-point function

Motivated by the considerations in the previous section **??** we continue with the practical implementation of the estimation of the matrix elements in lattice QCD. We start from

$$\langle J_{\Delta^{++}}(x_f)\, J_M^\dagger(x_{i_2})\, \bar{J}_N(x_{i_1})\rangle = -\sigma_{\Gamma_N}\,\sigma_{\Gamma_M}\,\langle \underbrace{J^\alpha_{\Delta^{++}}}_{(u^T\,C\Gamma_\Delta\,u)\,u}(t_f,\,\vec{x})\;\underbrace{J^\dagger_{\pi^+}}_{\bar{u}\,\gamma_5\,d}(t_i,\,\vec{y})\;\underbrace{\bar{J}^\beta_N}_{(\bar{d}\,C\Gamma_N\,\bar{u}^T)\,\bar{u}}(t_i,\,\vec{z})\rangle \tag{7}$$

$$\langle\left[u^a_\gamma(x_f)\;(C\Gamma_\Delta)_{\gamma\delta}\;u^b_\delta(x_f)\,u^c_\alpha(x_f)\right]\;\left[\bar{u}^d_\sigma(x_{i_2})\;(\Gamma_M)_{\sigma\tau}\;d^d_\tau(x_{i_2})\right]\;\left[\vec{d}^l_\kappa(x_{i_1})\;(C\Gamma_N)_{\kappa\lambda}\;\bar{u}^m_\lambda(x_{i_1})\,\bar{u}^n_\beta(x_{i_1})\right]\rangle_f = \tag{8}$$

$$\begin{aligned}
&-T(x_f,x_{i_1})\,C\Gamma_N\;\left(C\Gamma_\Delta\,U(x_f,x_{i_1})\right)^t\;U(x_f,x_{i_1})\\
&-T(x_f,x_{i_1})\,C\Gamma_N\,U(x_f,x_{i_1})^t\,C\Gamma_\Delta\,U(x_f,x_{i_1})\\
&-U(x_f,x_{i_1})\;\left(C\Gamma_\Delta\,T(x_f,x_{i_1})\,C\Gamma_N\right)^t\;U(x_f,x_{i_1})\\
&-U(x_f,x_{i_1})\;\left(T(x_f,x_{i_1})\,C\Gamma_N\right)^t\;C\Gamma_\Delta\,U(x_f,x_{i_1})\\
&-U(x_f,x_{i_1})\,\mathrm{Tr}\left(T(x_f,x_{i_1})\,C\Gamma_N\,U(x_f,x_{i_1})^t\,C\Gamma_\Delta\right)\\
&-U(x_f,x_{i_1})\,\mathrm{Tr}\left(T(x_f,x_{i_1})\,C\Gamma_N\;\left(C\Gamma_\Delta\,U(x_f,x_{i_1})\right)^t\right)\\
&= T_1+T_2+T_3+T_4+T_5+T_6
\end{aligned} \tag{9}$$

Eq. (8) defines the triangle diagrams $T_1,\dots,T_6$.
We use the notation

$$T(x_f,x_i) = T^{f_1 f_2\,ab}_{\alpha\beta}(x_f;t,\vec{q};x_i) = \sum_{\vec{y}}\;\left(S_{f_1}(x_f;\,t,\vec{y})\,\Gamma_M\,\mathrm{e}^{i\vec{q}\vec{y}}\,S_{f_2}(t,\vec{y};\,x_i)\right)^{ab}_{\alpha\beta} \tag{10}$$

for the sequential propagator with

- flavors "f_1 after f_2";
- sequential source timeslice t;
- sequential source momentum $\vec{q}$;
- sequential source Dirac structure Γ_M.

In particular we shall use the notation

$$T_{fii} = T(x_f;t_i,\vec{q};x_i) \tag{11}$$
$$T_{ffi} = T(x_f;t_f,\vec{q};x_i) \tag{12}$$

for 1-step sequential propagators.

Quantum numbers for Delta, pion and nucleon:

	Δ^{++}	π^+	$N^+ =$ Proton
J	$\frac{3}{2}$	0	$\frac{1}{2}$
I	$\frac{3}{2}$	1	$\frac{1}{2}$
I_3	$+\frac{3}{2}$	+1	$+\frac{1}{2}$
P	+1	-1	+1

1.3 Δ^{++} to Δ^{++}

$$\langle J_\Delta(x_f)\,\bar{J}_\Delta(x_i)\rangle_f = -\sigma_{\Gamma_i}\,\langle \left(u^T\,C\Gamma_f\,u\right)\,u(x_f)\,\left(\bar{u}^T\,C\Gamma_i\,\bar{u}\right)\,\bar{u}(x_i)\rangle \tag{13}$$

$$\begin{aligned}
&\langle\Big[\,\epsilon_{abc}\,u^a_\gamma(x_f)\,\left(C\Gamma_f\right)_{\gamma\delta}\,u^b_\delta(x_f)\,u^c_\alpha(x_f)\Big]\,\Big[\epsilon_{lmn}\,\bar{u}^l_\kappa(x_i)\,\left(C\Gamma_i\right)_{\kappa\lambda}\,\bar{u}^m_\lambda(x_i)\,\bar{u}^n_\beta(x_i)\Big]\rangle_f = \qquad (14)\\
&\quad \epsilon_{abc}\,\epsilon_{lmn}\,\left(C\Gamma_f\right)_{\gamma\delta}\,\left(C\Gamma_i\right)_{\kappa\lambda}\,\Big\{\\
&\qquad + U^{cn}_{\alpha\beta}\,\left(U^{bl}_{\delta\kappa}\,U^{am}_{\gamma\lambda} - U^{bm}_{\delta\lambda}\,U^{al}_{\gamma\kappa}\right)\\
&\qquad - U^{cm}_{\alpha\lambda}\,\left(U^{bl}_{\delta\kappa}\,U^{an}_{\gamma\beta} - U^{bn}_{\delta\beta}\,U^{al}_{\gamma\kappa}\right)\\
&\qquad + U^{cl}_{\alpha\kappa}\,\left(U^{bm}_{\delta\lambda}\,{}^{an}_{\gamma\beta} - U^{bn}_{\delta\beta}\,U^{am}_{\gamma\lambda}\right)\\
&\Big\} =\\
&\\
&\quad - U(x_f,x_i)\,C\Gamma_i\,\left(C\Gamma_f\,U(x_f,x_i)\right)^t\,U(x_f,x_i)\\
&\quad - U(x_f,x_i)\,C\Gamma_i\,U(x_f,x_i)^t\,C\Gamma_f\,U(x_f,x_i)\\
&\quad - U(x_f,x_i)\,\left(C\Gamma_f\,U(x_f,x_i)\,C\Gamma_i\right)^t\,U(x_f,x_i)\\
&\quad - U(x_f,x_i)\,\left(U(x_f,x_i)\,C\Gamma_i\right)^t\,C\Gamma_f\,U(x_f,x_i)\\
&\quad - U(x_f,x_i)\,\mathrm{Tr}\left(C\Gamma_f\,U(x_f,x_i)\,C\Gamma_i\,U(x_f,x_i)^t\right)\\
&\quad - U(x_f,x_i)\,\mathrm{Tr}\left(C\Gamma_f\,U(x_f,x_i)\,\left(U(x_f,x_i)\,C\Gamma_i\right)^t\right)\\
&= D_1 + D_2 + D_3 + D_4 + D_5 + D_6\,.
\end{aligned}$$

This define the $I = 3/2$ diagrams $D_1, \ldots, D_6$.

Adjoint correlator Using γ_5-Hermiticity, parity and time reversal we expect, that

$$\begin{aligned}
C^{\alpha\beta}_{\mu\nu}(x,y) &= \langle J^\alpha_{\Delta\mu}(x)\,\bar{J}^\beta_{\Delta\nu}(y)\rangle\\
C_{\mu\nu}(x,y) &= \sigma^{02}_\mu\,\sigma^{02}_\mu\,C^{\tilde{\dagger}}_{\mu\nu}\\
\sigma^{02}_\mu &= \begin{cases} +1 & \mu = 0,\,2\\ -1 & \mu = 1,\,3\end{cases}\,,
\end{aligned}$$

where $\tilde{\dagger}$ denotes the conjugate with respect to the spinor indices. This relation should hold exactly in the free case (gauge field $U = \mathbb{1}$) and at the level of the gauge average in the non-free case.

$$\begin{aligned}
(t_x,\,t_y) \sim t_x - t_y &\xrightarrow{\gamma_5-\text{Hermiticity}} (t_y,\,t_x) \sim t_y - t_x\\
&\xrightarrow{\mathcal{T}} (T - t_y,\,T - t_x) \sim (T - t_y) - (T - t_x) = t_x - t_y\,.
\end{aligned}$$

1.4 N^+ to N^+

$$\langle J_N(x_f)\,\bar{J}_N(x_i)\rangle_f = -\sigma_{\Gamma_i}\,\langle \big(u^T\,C\Gamma_f\,d\big)\;u(x_f)\;\big(\bar{d}^T\,C\Gamma_i\,\bar{u}\big)\;\bar{u}(x_i)\rangle \tag{15}$$

$$\begin{aligned}
&\langle\Big[\,\epsilon_{abc}\,u^a_\gamma(x_f)\;(C\Gamma_f)_{\gamma\delta}\;d^b_\delta(x_f)\,u^c_\alpha(x_f)\Big]\;\Big[\epsilon_{lmn}\,\bar{d}^l_\kappa(x_i)\;(C\Gamma_i)_{\kappa\lambda}\;\bar{u}^m_\lambda(x_i)\,\bar{u}^n_\beta(x_i)\Big]\rangle_f = \\
&\quad -U(x_f,x_i)\;\big(C\Gamma_f\,D(x_f,x_i)\,C\Gamma_i\big)^t\;U(x_f,x_i) \\
&\quad -U(x_f,x_i)\,\mathrm{Tr}\,\Big(\big(C\Gamma_f\,D(x_f,x_i)\,C\Gamma_i\big)^t\;U(x_f,x_i)\Big) \\
&= N_1 + N_2\,.
\end{aligned} \tag{16}$$

This defines the diagrams N_1, N_2.

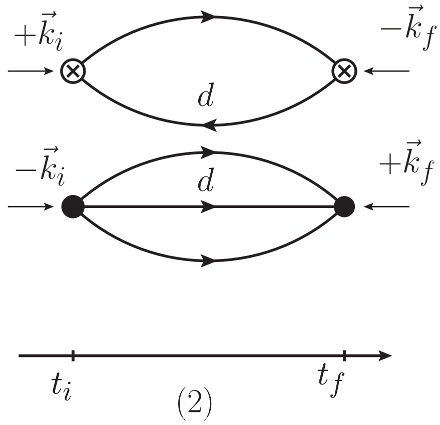

Figure 1: Graphical representation of the quark-disconnected contribution to the 4-pt. function $\pi\, N \to \pi\, N$

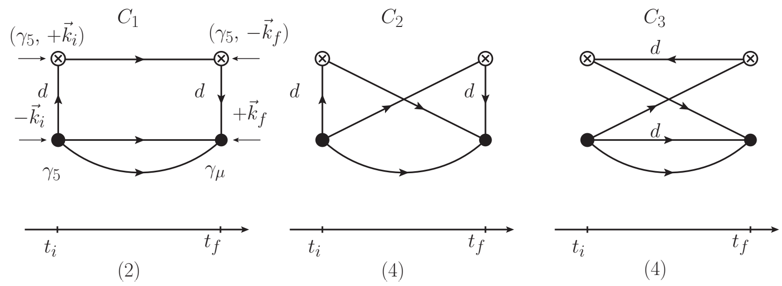

Figure 2: Graphical representation of the quark-connected contribution to the 4-pt. function $\pi\, N \to \pi\, N$ at zero total 3-momentum, $\vec{Q} = 0$

1.5 $\pi^+\, N^+$ to $\pi^+\, N^+$

In total these sum up to 12 contributions; we can check that is the right number 2 combinations of down quarks $\times$ 3! combinations of up quarks
We introduce some notation to write out the necessary contractions, the 2-step sequential propagators P

$$P^{f_1 f_2 f_1\, ab}_{\alpha\beta}(x_f;\, t_1, \vec{q}_1;\, t_2, \vec{q}_2;\, x_i) = \tag{17}$$
$$\sum_{\vec{z}_1, \vec{z}_2} \Big(S_{f_1}(x_f;\, t_1, \vec{z}_1)\, \Gamma_1\, S_{f_2}(t_1,\, \vec{z}_1;\, t_2, \vec{z}_2)\, \Gamma_2\, S_{f_1}(t_2, \vec{z}_2;\, x_i) \Big)^{ab}_{\alpha\beta} \mathrm{e}^{i(\vec{q}_1 \vec{z}_1 + \vec{q}_2 \vec{z}_2)}$$

where $f_{1/2} \in \{u,\, d\}$ and $f_1 \neq f_2$. In particular we shall use

$$P_{fifi} = P^{udu}(x_{f_1}; t_i, \vec{q}_{i_2}; t_f, \vec{q}_{f_2}; x_{i_1}) \tag{18}$$
$$P_{ffii} = P^{dud}(x_{f_1}; t_f, \vec{q}_{f_2}; t_i, \vec{q}_{i_2}; x_{i_1}) \tag{19}$$

with $t_{f_1} = t_{f_2} = t_f$ and $t_{i_1} = t_{i_2} = t_i$.

With these generalized propagators we can write the contractions in a short way.

$$\begin{aligned}&\langle J_{\pi^+N^+}(x_{f_1}; x_{f_2})\, \bar{J}_{\pi^+N^+}(x_{i_1}; x_{i_2})\rangle_f \\ &= -\sigma_{\Gamma_{N_i}}\, \sigma_{\Gamma_{M_i}}\, \langle \big(u^t\, C\Gamma_{N_f}\, d\big)\, u(x_{f_1})\, \bar{d}\, \Gamma_{M_f}\, u(x_{f_2})\, \bar{u}\, \Gamma_{M_i}\, d(x_{i_2})\, \big(\bar{d}\, C\Gamma_{N_i}\, u\big)\, u(x_{i_1})\rangle \end{aligned} \tag{20}$$

$$\begin{aligned}&\langle \Big[\epsilon_{abc}\, u^a_\gamma(x_{f_1})\, \big(C\Gamma_{N_f}\big)_{\gamma\delta}\, d^b_\delta(x_{f_1})\, u^c_\alpha(x_{f_1})\Big]\, \Big[\bar{d}^d_\sigma(x_{f_2})\, \big(\Gamma_{M_f}\big)_{\sigma\tau}\, u^d_\tau(x_{f_2})\Big] \times \\ &\quad \Big[\bar{u}^e_\mu(x_{i_2})\, (\Gamma_{M_i})_{\mu\nu}\, d^e_\nu(x_{i_2})\Big]\, \Big[\epsilon_{lmn}\, \bar{d}^l_\kappa(x_{i_1})\, (C\Gamma_{N_i})_{\kappa\lambda}\, \bar{u}^m_\lambda(x_{i_1})\, \bar{u}^n_\beta(x_{i_1})\Big]\rangle \\ &= C_B + C_W + C_Z + C_{\text{disconnected}}\end{aligned} \tag{21}$$

Quark-disconnected contribution — direct diagram

$$C_{\text{disconnected}} = -\mathrm{Tr}\,\big(U(x_{f_2}, x_{i_2})\, \Gamma_{M_i}\, D(x_{i_2}, x_{f_2})\, \Gamma_{M_f}\big) \times (N_1 + N_2)\ . \tag{22}$$

Quark-connected contributions — B, W and Z diagrams The connected contractions $C_{B,W,Z}$ are

$$\begin{aligned} C_B =& \\ &- U(x_{f_1}, x_{i_1})\, \big(C\Gamma_{N_f}\, P^t_{ffii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\big)^t\, U \\ &- U(x_{f_1}, x_{i_1})\, \mathrm{Tr}\,\big(C\Gamma_{N_f}\, P_{ffii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\, U(x_{f_1}, x_{i_1})^t\big) \\ =& B_1 + B_2 \end{aligned} \tag{23}$$

$$\begin{aligned} C_W =& \\ &- T^{ud}_{fii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\, \Big(C\Gamma_{N_f}\, T^{du}_{ffi}(x_{f_1}, x_{i_1})\Big)^t\, U(x_{f_1}, x_{i_1}) \\ &- T^{ud}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\, U(x_{f_1}, x_{i_1})^t\, C\Gamma_{N_f}\, T^{du}_{ffi}(x_{f_1}, x_{i_1}) \\ &- U(x_{f_1}, x_{i_1})\, \Big(T^{ud}_{fii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\Big)^t\, C\Gamma_{N_f}\, T^{du}_{fii}(x_{f_1}, x_{i_1}) \\ &- U(x_{f_1}, x_{i_1})\, \mathrm{Tr}\left(T^{ud}_{fii}(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\, \Big(C\Gamma_{N_f}\, T^{du}_{ffi}(x_{f_1}, x_{i_1})\Big)^t\right) \\ =& C_{W_1} + C_{W_2} + C_{W_3} + C_{W_4} \end{aligned} \tag{24}$$

$$
\begin{aligned}
C_Z = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (25) \\
& - P^{udu}_{fifi}(x_{f_1}, x_{i_1}) \left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, U(x_{f_1}, x_{i_1}) \\
& - P^{udu}_{fifi}(x_{f_1}, x_{i_1})\, \mathrm{Tr}\left(\left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, U(x_{f_1}, x_{i_1})\right) \\
& - U(x_{f_1}, x_{i_1}) \left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\, P^{udu}_{fifi}(x_{f_1}, x_{i_1}) \\
& - U(x_{f_1}, x_{i_1})\, \mathrm{Tr}\left(P^{udu}_{fifi}(x_{f_1}, x_{i_1}) \left(C\Gamma_{N_f}\, D(x_{f_1}, x_{i_1})\, C\Gamma_{N_i}\right)^t\right) \\
= & \; C_{Z_1} + C_{Z_2} + C_{Z_3} + C_{Z_4}
\end{aligned}
$$