Appendix 1

FACTORIZATION OF PHYLOGENETIC LIKELIHOOD UNDER MULTIVARIATE BROWNIAN MOTION

$$|2\pi(LIL^T \otimes C^{-1}QC^{-T})|^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)^T(LIL^T \otimes C^{-1}QC^{-T})^{-1}(x-\mu)}$$

$$((2\pi)^{nm}|LIL^T|^m|C^{-1}QC^{-T}|^n)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)^T((L\otimes C^{-1})(I\otimes Q)(L^T\otimes C^{-T}))^{-1}(x-\mu)}$$

$$((2\pi)^{nm}|LIL^T|^m|C^{-1}QC^{-T}|^n)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)^T(L^T\otimes C^{-T})^{-1}(I\otimes Q)^{-1}(L\otimes C^{-1})^{-1}(x-\mu)}$$

$$((2\pi)^{nm}|LIL^T|^m|C^{-1}QC^{-T}|^n)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)^T(L^{-T}\otimes C^T)(I\otimes Q)^{-1}(L^{-1}\otimes C)(x-\mu)}$$

$$((2\pi)^{nm}|LIL^T|^m|C^{-1}QC^{-T}|^n)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)^T(L^{-1}\otimes C)^T(I\otimes Q)^{-1}(L^{-1}\otimes C)(x-\mu)}$$

$$((2\pi)^{nm}|LIL^T|^m|C^{-1}QC^{-T}|^n)^{-\frac{1}{2}} e^{-\frac{1}{2}((L^{-1}\otimes C)(x-\mu))^T(I\otimes Q)^{-1}(L^{-1}\otimes C)(x-\mu)}$$

$$((2\pi)^{nm}|LIL^T|^m|C^{-1}QC^{-T}|^n)^{-\frac{1}{2}} e^{-\frac{1}{2}((L^{-1}\otimes C)x-(L^{-1}\otimes C)\mu))^T(I\otimes Q)^{-1}((L^{-1}\otimes C)x-(L^{-1}\otimes C)\mu)}$$

Since

$$C\mu = 0$$

$$(L^{-1} \otimes C)\mu = 0$$

so

$$((2\pi)^{nm}|LIL^T|^m|C^{-1}QC^{-T}|^n)^{-\frac{1}{2}} e^{-\frac{1}{2}((L^{-1}\otimes C)x)^T(I\otimes Q)^{-1}((L^{-1}\otimes C)x)}$$