STK9060 - Time Series
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1 Problem 1

To build an ARIMA model for the given data, I first inspect the time plot of
the data, which given on top of Figure 1. Obviously, the data is not stationary.
There is an obvious trend, and the variance seems to increase over time. This
suggests that we should use log transformation to first stabilize the variance.
The time series after taking log is shown in the middle of Figure 1. We see that
the variance is now more stable, but the trend is still there. Therefore, I try
to differentiate the log of mkts to detrend the series. The result is shown in
the bottom of Figure 1. I now refer to the differentiated log of mkts time series
as mkgr, which is the quarterly growth rate of mkts. The time plot of mkgr
shows that the time series is now relatively stable without any obvious trend.
However, we can see that there seems to be different periods where the variances
are significantly different. The time series variance is very large during 1980 to
around 1992. From 1993 to around 2007, the time series variance becomes very
small. After 2008, the variance seems to increase again. This suggests that
there could be a hidden state switching between those periods. I ignore those
facts for now, and try to build an ARIMA model for the mkgr time series.

We now look at the sample ACF and PACF to identify appropriate autore-
gressive order p and the moving average order q for the ARIMA model. The
plot is given in Figure 2. We see that the PACF is cutting off at lag 3, and the
ACF is tailing off without any strong seasonal pattern. This suggests an AR(3)
model. I also try an ARIMA(3,0,1) to see if an extra moving average term can
help in this case. After fitting the two models, I get the result that ARIMA(3,
0, 0) has a smaller BIC score. However, the AIC and AICc both prefer the
ARIMA(3, 0, 1) model. In this case, I choose the model ARIMA(3, 0, 0) to
work on, since a simple model is probably more easily to be interpreted.

The estimated ARIMA(3, 0, 0) model is:

$$x_t = .0154_{(.0091)} + -.6277_{(.0766)}x_{t-1} - .4534_{(.0840)} - .3261_{(.0760)} + \hat{w}_t \quad (1)$$
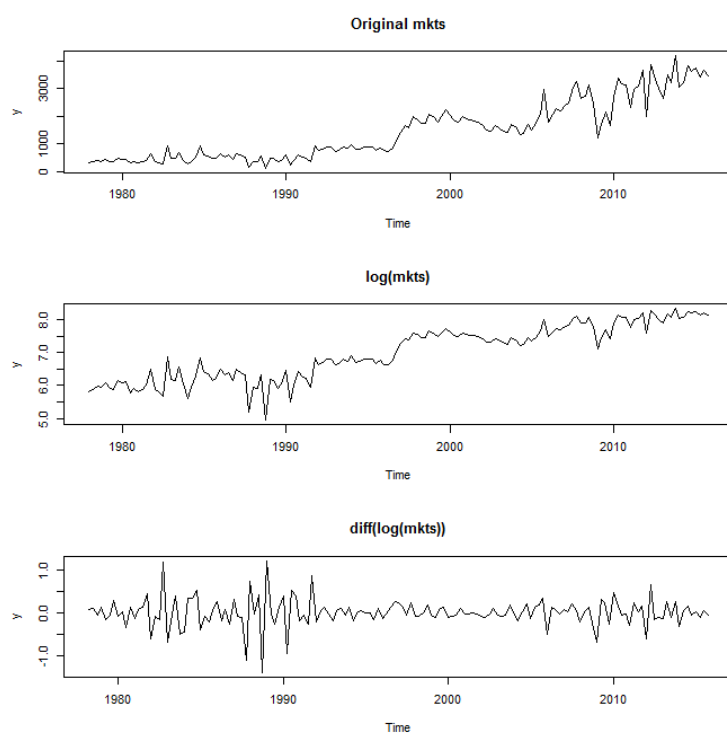
where $\hat{\sigma}_w = 0.0714$

Figure 1: *Top:* time plot of the original mkts data. *Middle:* mkts after taking log. *Bottom:* mkts after differencing log
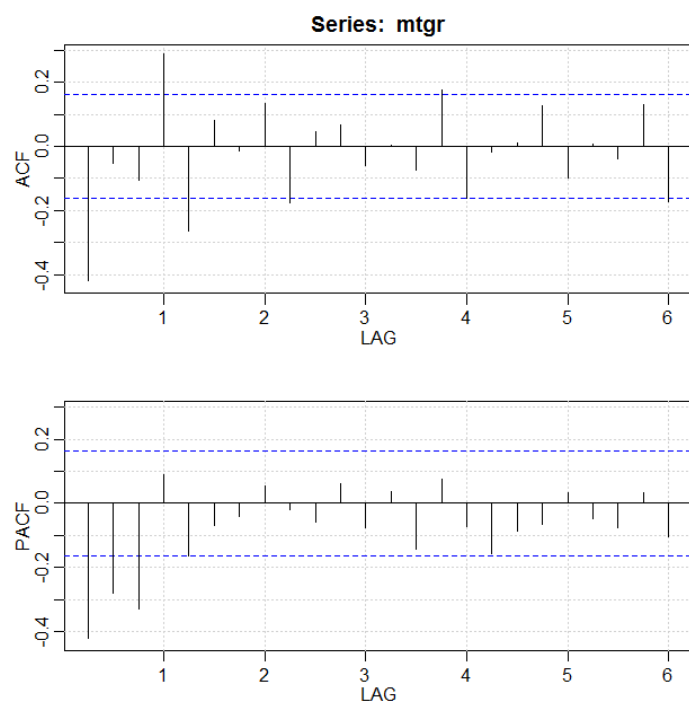
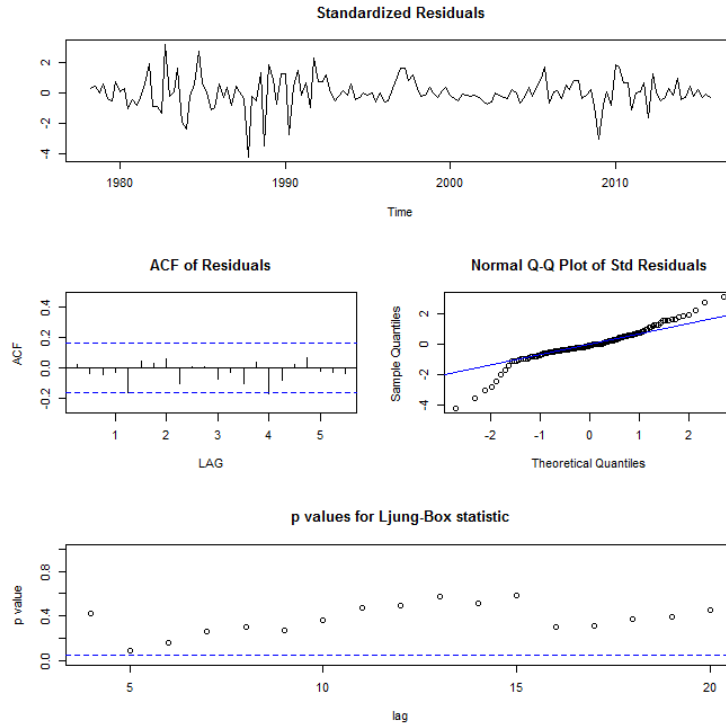Figure 2: The sample ACF and PACF of the mkgr series. Lag is in terms of years.

Figure 3: Diagnostics of the residuals from ARMA(3,0,0) fit on mkgr

1.1 Diagnostics for the fitted ARIMA(3, 0, 0)

The plot of the ARIMA(3, 0, 0) residuals, its sample autocorrelation function, Q-Q plot, and Q-statistic are given in Figure 3. From the time plot of the standardized residuals, we see that there are 3 periods that have significant difference in variance, as indicated previously. Other than that, there is no obvious patterns. The residuals' ACF plot confirms this argument, as there is not significant correlation in the residuals time series. The Q-statistic tells the same story, which is never significant in the plot. The Q-Q plot shows that the residuals departure from normality at the tails due to outliers during the high variance periods. In general, the model fit the data pretty well, with the only problem is the difference in variance during different periods, which leads to fat tails of residuals distribution.

1.2 Forecasting

The forecast for the next two year of the mkts series using the above model is given in Figure 4. Note that the figure is given in log scale.
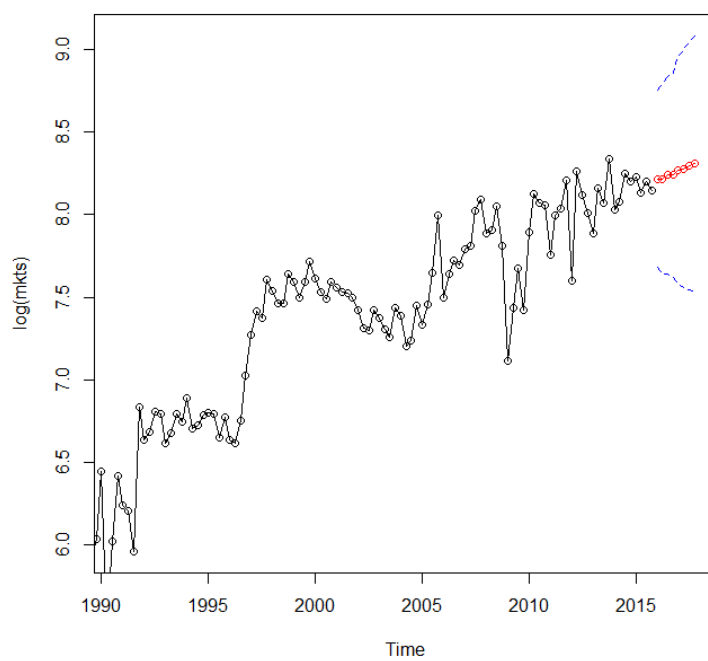
4

Figure 4: Forecast for the next two year of log of the mkts series

2 Problem 2

Given the weakly stationary ARMA(2,1) time series defined by:

$$x_t - 0.75x_{t-1} + 0.125x_{t-2} = w_t - 0.2w_{t-1} \qquad w_t \sim N(0, \sigma_w^2) \qquad (2)$$

2.1 Prove the model is causal and invertible

The AR polynomial of the model is:

$$\Phi(z) = 1 - 0.75z + 0.125z^2 \qquad (3)$$

Which has two roots $z_1 = 2$ and $z_2 = 4$ which are all outside the unit circle. Therefore the model is *causal*.

The MA polynomial of the model is:

$$\Theta(z) = 1 - 0.2z \qquad (4)$$

Which has one root $z = 5 > 1$. Therefore the model is invertible.

2.2 Find the ψ-weights

Since x_t is causal, it can we represented by $x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$.

Let the polynomial of ψ-weights be $\Psi(z)$. Then the ARMA(2,1) and its linear process representation must match. In other words, the $\Phi(z)$, $\Theta(z)$ and $\Psi(z)$ must satisfy:

$$\Phi(z)\Psi(z) = \Theta(z) \qquad (5)$$

Which can be expanded into:

$$(1 - 0.75z + 0.125z^2)(\psi_0 + \psi_1 z + \psi_2 z^2 ...) = 1 - 0.2z \qquad (6)$$

By matching the coefficients, we can find the difference equation with initial conditions for ψ as follow:

$$\psi_0 = 1 \qquad \text{matching constant}$$
$$\psi_1 - 0.75\psi_0 = -0.2 \qquad \text{matching z}$$
$$\psi_j - 0.75\psi_{j-1} + 0.125\psi_{j-2} = 0 \qquad \text{matching } z^j, (j \geq 2)$$

From the first two equations, we get the initial conditions $\psi_0 = 1$ and $\psi_1 = 0.55$. The difference equation ψ-weights given in the last equation can be solved by finding the roots of its characteristic polynomial, which is

$$1 - 0.75z + 0.125z^2 = 0 \qquad (7)$$

The above quadratic equation has two real distinct roots $z_1 = 2$ and $z_2 = 4$. Therefore, the general solution for ψ is:

$$\psi_j = c_1 2^{-j} + c_2 4^{-j} \qquad (8)$$

We can find c_1 and c_2 by solving for the initial conditions:

$$\psi_0 = 1 = c_1 + c_2$$
$$\psi_1 = 0.55 = \frac{c_1}{2} + \frac{c_2}{4} \qquad (9)$$

Solving that, we get $c_1 = 1.2$ and $c_2 = -0.2$. Therefore, the ψ-weights are fully described by:

$$\psi_j = 1.2 \cdot 2^{-j} - 0.2 \cdot 4^{-j} \qquad (10)$$

2.3 Find the $\gamma(h)$

From the ARMA(2,1) equation:

$$x_t - 0.75x_{t-1} + 0.125x_{t-2} = w_t - 0.2w_{t-1} \tag{11}$$

By multiplying both sides by x_{t-h} with $h \geq 0$ and take expectation, we get:

$$E(x_t x_{t-h}) - 0.75E(x_{t-1}x_{t-h}) + 0.125E(x_{t-2}x_{t-h}) = E(w_t x_{t-h}) - 0.2E(w_{t-1}x_{t-h}) \tag{12}$$

Using the fact that the ARMA(2,1) is causal, or $E(w_t x_s) = 0$ when $t > s$, we can get the difference equation for $\gamma(h)$ and its initial conditions by evaluating the above equation at $h = 0$, $h = 1$, $h = 2$ and $h \geq 2$ respectively:

$$
\begin{aligned}
(h = 0) \quad & \gamma(0) - 0.75\gamma(1) + 0.125\gamma(2) = \sigma_w^2 - 0.2(0.75 - 0.2)\sigma_w^2 = 0.89\sigma_w^2 \\
(h = 1) \quad & \gamma(1) - 0.75\gamma(0) + 0.125\gamma(1) = -0.2\sigma_w^2 \\
(h = 2) \quad & \gamma(2) - 0.75\gamma(1) + 0.125\gamma(0) = 0 \\
(h \geq 2) \quad & \gamma(h) - 0.75\gamma(h-1) + 0.125\gamma(h-2) = 0
\end{aligned}
\tag{13}
$$

(Note that we use the property $\gamma(h) = \gamma(-h)$)

Solving the first three equations, we get the initial conditions: $\gamma(0) = 1.414\sigma_w^2$, $\gamma(1) = 0.765\sigma_w^2$ and $\gamma(2) = 0.397\sigma_w^2$. The fourth equation is the difference equation for $\gamma(h)$, which has the following characteristic polynomial:

$$1 - 0.75z + 0.125z^2 = 0 \tag{14}$$

This quadratic equation has two real distinct roots $z_1 = 2$ and $z_2 = 4$. Therefore, the general solution for $\gamma(h)$ is $\gamma(h) = c_1 2^{-h} + c_2 4^{-h}$ for $h \geq 1$. To find c_1 and c_2, the general solution must satisfy the initial conditions, which leads to:

$$
\begin{aligned}
\gamma(1) = 0.765\sigma_w^2 &= \frac{c_1}{2} + \frac{c_2}{4} \\
\gamma(2) = 0.397\sigma_w^2 &= \frac{c_1}{4} + \frac{c_2}{16}
\end{aligned}
\tag{15}
$$

Solving that we get $c_1 = 1.646\sigma_w^2$ and $c_2 = -0.232\sigma_w^2$. Replace that to the general solution of $\gamma(h)$ we get:

$$\gamma(h) = \sigma_w^2(1.646 \cdot 2^{-h} - 0.232 \cdot 4^{-h}) \tag{16}$$

2.4 Find $\gamma(h)$ of a general linear process

Given a time series:

$$x_t = \sum_{j=-\infty}^{\infty} \psi_j w_{t-j} \tag{17}$$

By definition, its autocovariance function $\gamma(h)$ is:

$$\gamma(h) = E(x_t x_{t+h})$$

$$= E(\sum_{j=-\infty}^{\infty} \psi_j w_{t-j} \sum_{k=-\infty}^{\infty} \psi_k w_{t-k+h})$$

$$= E(\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j w_{t-j} \psi_k w_{t-k+h}) \tag{18}$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k E(w_{t-j} w_{t-k+h})$$

Since w_t is independent and identical distributed $N(0, \sigma_w^2)$, we have:

$$\begin{cases} E(w_{t-j} w_{t-k+h}) = 0 & (k \neq j+h) \\ E(w_{t-j} w_{t-k+h}) = \sigma_w^2 & (k = j+h) \end{cases} \tag{19}$$

Apply that to (18) we get:

$$\gamma(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \tag{20}$$

Which is what we are looking for.

2.5   Find $\gamma(h)$ from linear process representation

The linear process representation of ARMA(2, 1) was derived in (10) as:

$$\psi_j = 1.2 \cdot 2^{-j} - 0.2 \cdot 4^{-j} \tag{21}$$

From this representation, we can easily derive the $\gamma(h)$ function using the formula derived in the previous question:

$$\gamma(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}$$

$$= \sigma_w^2 \sum_{j=0}^{\infty} (1.2 \cdot 2^{-j} - 0.2 \cdot 4^{-j})(1.2 \cdot 2^{-(j+h)} - 0.2 \cdot 4^{-(j+h)})$$

$$= \sigma_w^2 \sum_{j=0}^{\infty} (1.2^2 \cdot 2^{-2j} - 1.2 \cdot 0.2 \cdot 2^{-3j})2^{-h} + (0.2^2 \cdot 2^{-4j} - 1.2 \cdot 0.2 \cdot 2^{-3j})4^{-h}$$

$$\tag{22}$$

We have that $\sum_{j=0}^{\infty} a^j = \frac{1}{1-a}$ if $a < 1$. Apply that to the above equation, we get:

$$\gamma(h) = \sigma_w^2 [(\frac{1.2^2}{1 - 2^{-2}} - \frac{1.2 \cdot 0.2}{1 - 2^{-3}})2^{-h} + (\frac{0.2^2}{1 - 2^{-4}} - \frac{1.2 \cdot 0.2}{1 - 2^{-3}})4^{-h}]$$

$$= \sigma_w^2 (1.646 \cdot 2^{-h} - 0.232 \cdot 4^{-h}) \tag{23}$$

Which is identical to what we got in (16).

3   Problem 3

Let $x_1, ..., x_n$ be observations from a stationary time series $\{x_t\}$ with covariance function $\gamma_x(h)$. Its spectral density, discreet Fourier transform, and the periodogram are defined respectively as:

$$f_x(\omega) = \sum_{h=-\infty}^{\infty} \gamma_x(h)e^{-2\pi i \omega h}$$

$$d_f(\omega_j) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{-2\pi i \omega_j t} \tag{24}$$

$$I_f(\omega_j) = |d_f(\omega_j)|^2$$

With $\omega_j = j/n, j = 0, ..., n-1$ are Fourier coefficients.

3.1   Prove that periodogram $I_f(\omega_j)$ is a sample version of spectral density $f_x(\omega)$

First we rewrite the DFT of x_t to:

$$d_f(\omega_j) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (x_t - \bar{x})e^{-2\pi i \omega_j t} \qquad (j > 0) \tag{25}$$

With $\bar{x}$ is the mean of x. The result is derived by the fact that $\bar{x} \sum_{t=1}^{n} e^{-2\pi i \frac{j}{n} t} = 0$ when $j > 0$ [1].

By definition, for $j > 0$, the periodogram can be found by:

$$\begin{aligned} I_f(\omega_j) &= |d_f(\omega_j)|^2 \\ &= \frac{1}{n} \sum_{t=1}^{n} (x_t - \bar{x})e^{-2\pi i \omega_j t} \sum_{s=1}^{n} ((x_s - \bar{x})e^{-2\pi i \omega_j s})^* \\ &= \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} (x_t - \bar{x})(x_s - \bar{x})e^{-2\pi i \omega_j (t-s)} \end{aligned} \tag{26}$$

Note that conjugate of sum is sum of conjugate. Now by defining $h = t - s$, we can get:

$$\begin{aligned} I_f(\omega_j) &= \frac{1}{n} \sum_{t=1}^{n} \sum_{h=t-1}^{t-n} (x_t - \bar{x})(x_{t-h} - \bar{x})e^{-2\pi i \omega_j h} \\ &= \frac{1}{n} \sum_{h=-(n-1)}^{0} \sum_{t=1}^{n+h} (x_t - \bar{x})(x_{t-h} - \bar{x})e^{-2\pi i \omega_j h} \\ &+ \frac{1}{n} \sum_{h=1}^{n-1} \sum_{t=h+1}^{n} (x_t - \bar{x})(x_{t-h} - \bar{x})e^{-2\pi i \omega_j h} \end{aligned} \tag{27}$$

In the last equation, we split h into two cases $h \leq 0$ and $h > 0$, which allows us to change the summing order. Now for the first term, we apply the fact that

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[1] We can prove this easily by using the sum of geometric series

$h \leq 0$, and for the second term, we change the index to $t' = t - h$, which leads to.

$$I_f(\omega_j) = \frac{1}{n} \sum_{h=-(n-1)}^{0} \sum_{t=1}^{n-|h|} (x_t - \bar{x})(x_{t+|h|} - \bar{x})e^{-2\pi i\omega_j h}$$

$$+ \frac{1}{n} \sum_{h=1}^{n-1} \sum_{t'=1}^{n-h} (x_{t'+h} - \bar{x})(x_{t'} - \bar{x})e^{-2\pi i\omega_j h} \tag{28}$$

Now we can merge two terms together, using the fact that for the second term $h > 0$.

$$I_f(\omega_j) = \frac{1}{n} \sum_{h=-(n-1)}^{n-1} \sum_{t=1}^{n-|h|} (x_t - \bar{x})(x_{t+|h|} - \bar{x})e^{-2\pi i\omega_j h} \tag{29}$$

By definition, the estimate of the covariance is given by:

$$\hat{\gamma}_x(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_t - \bar{x})(x_{t+|h|} - \bar{x}) \tag{30}$$

Therefore, the periodogram is equal to:

$$I_f(\omega_j) = \sum_{h=-(n-1)}^{n-1} \hat{\gamma}_x(h)e^{-2\pi i\omega_j h}$$

$$= \sum_{|h|<n} \hat{\gamma}_x(h)e^{-2\pi i\omega_j h} \qquad (j > 0) \tag{31}$$

Which shows that periodogram is the sample version of the spectral density. This is also what we are looking for.

When $j = 0$, $d_f(0) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t$. Therefore:

$$I_f(0) = \left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t\right)^2 = n\bar{x}^2 \tag{32}$$

3.2   Find expectation of periodogram

Let μ be the population mean of x_t. Without changing anything, we can replace $\bar{x}$ by μ in the equation $(29)^2$. By taking the expectation on both sides, the equation (29) becomes:

$$E[I_f(\omega_j)] = \frac{1}{n} \sum_{h=-(n-1)}^{n-1} \sum_{t=1}^{n-|h|} E((x_t - \mu)(x_{t+|h|} - \mu))e^{-2\pi i\omega_j h}$$

$$= \frac{1}{n} \sum_{h=-(n-1)}^{n-1} \sum_{t=1}^{n-|h|} \gamma_x(h)e^{-2\pi i\omega_j h} \tag{33}$$

$$= \sum_{|h|<n} \left(1 - \frac{|h|}{n}\right)\gamma_x(h)e^{-2\pi i\omega_j h} \qquad (j > 0)$$

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[2]In fact, we can replace $\bar{x}$ by any constant

Which is what we are looking for.

When $j = 0$, we have

$$E[I_f(0)] = E[d_f(0)^2] = E[(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t)^2] = n(\mu^2 + var(\bar{x})) \qquad (34)$$

3.3 Show a smooth version of periodogram is reasonable

First, we try to derive the variance of $I_f(\omega_j)$ when $n \to \infty$. Using the Appendix C.2 in the book, and the central limit theorem, we know that the distribution of cosine and sine transform of x_t converge to independent normal distributions with zero mean and variances approximately equal to one half of the theoretical spectrum:

$$d_c(\omega_j) \xrightarrow{d} AN(0, f_x(\omega)/2)$$
$$d_s(\omega_j) \xrightarrow{d} AN(0, f_x(\omega)/2) \qquad (35)$$

Since $I_f(\omega_j) = d_c(\omega_j)^2 + d_s(\omega_j)^2$, which is sum of squares of two independent normal random variables. Therefore, the asymptotic distribution of the periodogram is given by:

$$I_f(\omega_j) \xrightarrow{d} \frac{f_x(\omega)}{2} \chi_2^2 \qquad (36)$$

Therefore, the asymptotic variance of the periodogram is:

$$var(I_f(\omega_j)) \to 2f_x^2(\omega) \qquad (37)$$

Now, instead of directly using raw periodogram $I_f(\omega_j)$ to estimate the spectral density, we use the suggested estimator:

$$\hat{g}(\omega_j) = \sum_{|h|<n} (1 - \frac{|h|}{n}) \hat{\gamma}_x(h) e^{-2\pi i \omega_j h} \qquad (38)$$

We see that the above estimator follows the following lag window form:

$$\tilde{f}(\omega) = \sum_{|h| \leq r} w(\frac{h}{r}) \hat{\gamma}_x(h) e^{-2\pi i \omega h} \qquad (39)$$

In this case, $r = n$, and the weight function $w(\frac{h}{n}) = w(x) = 1 - |x|$.

This weight function satisfies that $w(0) = 1; |w(x)| \leq 1$ and $w(x) = w(-x)$. Therefore, we can apply the asymptotic theory for estimators in the lag window form (with $r = n$)

$$var(\hat{g}(\omega_j)) \to \frac{n}{n} f_x^2(\omega) \int_{-1}^{1} w^2(x)dx \qquad \omega \neq 0, 1/2$$
$$\to f_x^2(\omega) \int_{-1}^{1} (1 - |x|)^2 dx \qquad (40)$$
$$\to \frac{2}{3} f_x^2(\omega)$$

We see that the estimator $\hat{g}(\omega_j)$ has smaller asymptotic variance: $\frac{2}{3}f_x^2(\omega)$, comparing to the raw periodogram estimator $I_f(\omega_j)$, which has asymptotic variance of $2\, f_x^2(\omega_j)$. This makes $\hat{g}(\omega_j)$ a more reasonable estimator, comparing to the raw periodogram.

3.4 Show that expectation of periodogram converge to spectral density

From the expectation of periodogram derived in (33), we have:

$$E[I_f(\omega_j)] = \sum_{|h|<n} (1 - \frac{|h|}{n})\gamma_x(h)e^{-2\pi i\omega_j h}$$

$$= \sum_{|h|<n} \gamma_x(h)e^{-2\pi i\omega_j h} - \frac{\sum_{|h|<n} |h|\gamma_x(h)e^{-2\pi i\omega_j h}}{n} \qquad (j > 0) \tag{41}$$

When $n \to \infty$, we now try to prove that if we assume:

$$\sum_{h=-\infty}^{\infty} |h||\gamma_x(h)| = \theta < \infty \tag{42}$$

then the sum $\sum_{h=-\infty}^{h=\infty} |h|\gamma_x(h)e^{-2\pi i\omega_j h}$ is also bounded, which makes the second term in (41) vanishes when $n \to \infty$.

We have

$$|h|\gamma_x(h) \le |h||\gamma_x(h)| \le \sum_{h=-\infty}^{\infty} |h||\gamma_x(h)| = \theta \qquad \forall h \tag{43}$$

Therefore, we have:

$$\sum_{h=-\infty}^{h=\infty} |h|\gamma_x(h)e^{-2\pi i\omega_j h} \le \theta \sum_{h=-\infty}^{h=\infty} e^{-2\pi i\omega_j h} \tag{44}$$

Since $-1/2 < \omega_j < 1/2$ and $\omega_j \neq 0$, we have: $|e^{2\pi i\omega_j}| < 1$ and $|e^{-2\pi i\omega_j}| < 1$, which enough for us to bound the sum:

$$\sum_{h=-\infty}^{h=\infty} e^{-2\pi i\omega_j h} = \sum_{h=0}^{h=\infty} e^{-2\pi i\omega_j h} + \sum_{h=1}^{h=\infty} e^{2\pi i\omega_j h}$$

$$= \frac{1}{1 - e^{-2\pi i\omega_j}} + \frac{1}{1 - e^{2\pi i\omega_j}} - 1 < \infty \tag{45}$$

Therefore, we have proved that under the condition (42), the sum $\sum_{h=-\infty}^{h=\infty} |h|\gamma_x(h)e^{-2\pi i\omega_j h}$ is bounded. Apply this result to (41) and let $n \to \infty$, we have:

$$E[I_f(\omega_j)] \to \sum_{h=-\infty}^{\infty} \gamma_x(h)e^{-2\pi i\omega_j h} - 0$$

$$\to f_x(\omega) \qquad (\omega \neq 0) \tag{46}$$

With the assumed regularity condition in (42), which is:

$$\sum_{h=-\infty}^{\infty} |h||\gamma_x(h)| = \theta < \infty \qquad (47)$$

For the case $\omega = 0$, as shown in (34),

$$E[I_f(0)] - n\mu^2 = n \cdot var(\bar{x}) \to \sum_{h=-\infty}^{h=\infty} \gamma_x(h) = f_x(0) \qquad (48)$$

4 Problem 4

The Holt-Winters forecasting and recursive updating scheme are given by:

$$
\begin{aligned}
x_t^{t-1} &= m_{t-1} + b_{t-1} \\
m_t &= \lambda_0 x_t + (1 - \lambda_0)(m_{t-1} + b_{t-1}) \\
b_t &= \lambda_1(m_t - m_{t-1}) + (1 - \lambda_1)b_{t-1}
\end{aligned}
\tag{49}
$$

4.1 Why the updating formula of b_t is reasonable?

We see that x_t is forecasted as sum of the current estimate of the local level m_{t-1} and the local slope b_{t-1}. The new local level m_t is updated (i.e. estimated) as the weighted average of the prediction x_t^{t-1} and the actual value x_t. The new local linear trend b_t is then updated as the weighted average of the new trend (i.e. growth) $m_t - m_{t-1}$ and the last estimated trend b_{t-1}. Therefore, the updating formula of b_t in (49) estimates the local trend by smoothing the observed trend exponentially, which is reasonable.

To get the initial values b_2 and m_2, assume that we have observed x_1 and x_2, and the innovations (i.e. prediction errors) $\epsilon_0 = \epsilon_1 = 0$, we then have:

$$
\begin{aligned}
x_1 &= m_0 + b_0 \\
x_2 &= m_1 + b_1 \\
m_2 &= \lambda_0 x_2 + (1 - \lambda_0)(m_1 + b_1) = x_2 \\
m_1 &= \lambda_0 x_1 + (1 - \lambda_0)(m_0 + b_0) = x_1 \\
b_1 &= x_2 - m_1 = x_2 - x_1 \\
b_2 &= \lambda_1(m_2 - m_1) + (1 - \lambda_1)b_1 = x_2 - x_1
\end{aligned}
\tag{50}
$$

Therefore, reasonable starting values are $m_2 = x_2$ and $b_2 = x_2 - x_1$, which makes sense since m_2 is the local level and b_2 is local linear trend at $t = 2$.

4.2 Rearrange the updating formulas

Having $x_t^{t-1} = m_{t-1} + b_{t-1}$, we can rearrange the updating formula for m_t as following:

$$
\begin{aligned}
& m_t = \lambda_0 x_t + (1 - \lambda_0)(m_{t-1} + b_{t-1}) \\
\Leftrightarrow\ & m_t = \lambda_0 x_t + m_{t-1} + b_{t-1} - \lambda_0(m_{t-1} + b_{t-1}) \\
\Leftrightarrow\ & m_t = m_{t-1} + b_{t-1} + \lambda_0(x_t - x_t^{t-1})
\end{aligned}
\tag{51}
$$

This leads to the following updating formula for b_t:

$$
\begin{aligned}
& b_t = \lambda_1(m_t - m_{t-1}) + (1 - \lambda_1)b_{t-1} \\
\Leftrightarrow\ & b_t = \lambda_1(b_{t-1} + \lambda_0(x_t - x_t^{t-1})) + (1 - \lambda_1)b_{t-1} \\
\Leftrightarrow\ & b_t = b_{t-1} + \lambda_0\lambda_1(x_t - x_t^{t-1})
\end{aligned}
\tag{52}
$$

I now define the innovation series as $e_t = x_t - x_t^{t-1}$, which simplifies the updating formulas and is useful for later questions:

$$
\begin{aligned}
e_t &= x_t - x_t^{t-1} \\
m_t &= m_{t-1} + b_{t-1} + \lambda_0 e_t \\
b_t &= b_{t-1} + \lambda_0\lambda_1 e_t
\end{aligned}
\tag{53}
$$

4.3 Express the local linear trend model on state-space form

The local linear trend model is given by:

$$
\begin{aligned}
y_t &= \mu_t + \epsilon_t \\
\mu_t &= \mu_{t-1} + \beta_{t-1} + \eta_t \\
\beta_t &= \beta_{t-1} + \zeta_t
\end{aligned}
\tag{54}
$$

where $\epsilon_t, \eta_t, \zeta_t$ are independent with $\epsilon_t \sim N(0,1), \eta_t \sim N(0, \sigma_\eta^2), \zeta_t \sim N(0, \sigma_\zeta^2)$.

We can express the local linear trend model in (54) in the matrix form:

$$
\begin{aligned}
y_t &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix} + \epsilon_t \\
\begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{t-1} \\ \beta_{t-1} \end{pmatrix} + \begin{pmatrix} \eta_t \\ \zeta_t \end{pmatrix}
\end{aligned}
\tag{55}
$$

Therefore, if we define $x_t = \begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix}$ as the state vector, we can then express the model by the two following transition and measurement equations:

$$
\begin{aligned}
y_t &= \begin{pmatrix} 1 & 0 \end{pmatrix} x_t + \epsilon_t & \epsilon_t &\sim N(0,1) \\
x_t &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x_{t-1} + \begin{pmatrix} \eta_t \\ \zeta_t \end{pmatrix} & \begin{pmatrix} \eta_t \\ \zeta_t \end{pmatrix} &\sim N(0, \begin{pmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix})
\end{aligned}
$$

Therefore, the above local linear trend model can be fully described in state-space form using the following parameters:

$$
\begin{aligned}
\Phi &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
A &= \begin{pmatrix} 1 & 0 \end{pmatrix} \\
Q &= \begin{pmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix} \\
R &= 1
\end{aligned}
\tag{56}
$$

4.4 Kalman filter

With the parameters given in (56), and assume that the starting values of the state is $x_0 \sim N(\begin{pmatrix} \mu_0 \\ \beta_0 \end{pmatrix}, P_0^0)$, the Kalman filter for x_t is given as following.

Assume that we have observed data until $t-1$, first we make prediction for the next state: x_t^{t-1}, its variance matrix P_t^{t-1}, and prediction of the observable data y_t^{t-1}

$$
\begin{aligned}
x_t^{t-1} &= \Phi x_{t-1}^{t-1} \\
P_t^{t-1} &= \Phi P_{t-1}^{t-1} \Phi' + Q \\
y_t^{t-1} &= A x_t^{t-1}
\end{aligned}
\tag{57}
$$

Then after observing y_t, we calculate the innovation e_t (prediction errors), and use that to update the state vector, using the fact that the innovation has correlation with the state. The optimal updating scheme is given by the following Kalman filter:

$$
\begin{aligned}
e_t &= y_t - y_t^{t-1} = y_t - Ax_t^{t-1} \\
\Sigma_t &= var(e_t) = AP_t^{t-1}A' + R \\
K_t &= P_t^{t-1}A'\Sigma_t^{-1} \\
P_t^t &= [I - K_t A]P_t^{t-1} \\
x_t^t &= x_t^{t-1} + K_t e_t
\end{aligned}
\tag{58}
$$

We see that the new estimate of the state x_t^t is the combination of its prediction x_t^{t-1} and the prediction error e_t, under the Kalman gain K_t. This is quite similar to the updating scheme of Holt-Winters model that we obtained in (53). I will exploit this fact in the later question.

4.5 Kalman smoother

After observing y_t for $t = 1, ..., n$, we can estimate the state using all these observed data, which is call smoothing. To run the Kalman smoother, we must first run the Kalman filter forward (from $t = 1$ to $t = n$), and then run the following Kalman smoother backward (from $t = n - 1$ to $t = 1$) with the initial value x_n^n and P_n^n:

$$
\begin{aligned}
x_{t-1}^n &= x_{t-1}^{t-1} + J_{t-1}(x_t^n - x_t^{t-1}) \\
P_{t-1}^n &= P_{t-1}^{t-1} + J_{t-1}(P_t^n - P_t^{t-1})J_{t-1}' \\
J_{t-1} &= P_{t-1}^{t-1}\Phi'[P_t^{t-1}]^{-1}
\end{aligned}
\tag{59}
$$

4.6 Steady state of the Kalman filter

I first assume that the regularity conditions are satisfied by our local linear trend model, so that its Kalman filter can reach the steady state after some finite number of time steps. At this steady state, the state error covariance matrix P_t^{t-1} must converge to a time invariant matrix, or $P_t^{t-1} = P$

Following the Kalman filter given in (58), we have:

$$
\begin{aligned}
& P_t^{t+1} = \Phi P_{t-1}^{t-1}\Phi' + Q \\
\Leftrightarrow\ & P_t^{t+1} = \Phi[I - K_t A]P_t^{t-1}\Phi' + Q \\
\Leftrightarrow\ & P_t^{t+1} = \Phi[I - P_t^{t-1}A'\Sigma_t^{-1}A]P_t^{t-1}\Phi' + Q \\
\Leftrightarrow\ & P_t^{t+1} = \Phi[I - P_t^{t-1}A'(AP_t^{t-1}A' + R)^{-1}A]P_t^{t-1}\Phi' + Q
\end{aligned}
\tag{60}
$$

Using the fact that the P_t^{t-1} matrix has already converged at the steady state, we have $P_t^{t+1} = P_t^{t-1} = P$. Replace that into the above formula, we have:

$$
\begin{aligned}
& P_t^{t+1} = \Phi[I - P_t^{t-1}A'(AP_t^{t-1}A' + R)^{-1}A]P_t^{t-1}\Phi' + Q \\
\Leftrightarrow & P = \Phi[I - PA'(APA' + R)^{-1}A]P\Phi' + Q \\
\Leftrightarrow & P - \Phi P\Phi' + \Phi PA'(APA' + R)^{-1}AP\Phi' - Q = 0
\end{aligned}
\tag{61}
$$

Which is the Ricatti equation.

4.7 Steady state Kalman filter vs. Holt-Winter updating scheme

At the steady state, the Kalman gain also converges to $K_t = K$. Let $K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$, the Kalman filter can be written as:

$$
\begin{aligned}
& x_t^t = x_t^{t-1} + K e_t \\
\Leftrightarrow \ & x_t^t = \Phi x_{t-1}^{t-1} + K e_t \\
\Leftrightarrow \ & \begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{t-1} \\ \beta_{t-1} \end{pmatrix} + \begin{pmatrix} K_1 e_t \\ K_2 e_t \end{pmatrix} \\
\Leftrightarrow \ & \begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix} = \begin{pmatrix} \mu_{t-1} + \beta_{t-1} \\ \beta_{t-1} \end{pmatrix} + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} e_t
\end{aligned} \tag{62}
$$

Now looking at the Holt-Winters updating scheme derived in (53):

$$
\begin{aligned}
m_t &= m_{t-1} + b_{t-1} + \lambda_0 e_t \\
b_t &= b_{t-1} + \lambda_0 \lambda_1 e_t
\end{aligned} \tag{63}
$$

We see that from the above Kalman filter at the steady state, if we replace μ_t by m_t and β_t by b_t, and set $K_1 = \lambda_0$ and $K_2 = \lambda_0 \lambda_1$, we will get the exact Holt-Winters updating formulas.

4.8 Find P given λ_0 and λ_1

At the steady state, let $P_t = P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$ (since P is symmetric). At the steady state, K_t and Σ_t will also converge to $K_t = K$ and $\Sigma_t = \Sigma$. From the Kalman filter formulas, we have:

$$
\begin{aligned}
\Sigma &= APA' + R = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 = p_{11} + 1 \\
K &= PA'\Sigma^{-1} = \frac{\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{p_{11} + 1} = \frac{\begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix}}{p_{11} + 1}
\end{aligned} \tag{64}
$$

Therefore, we have $K_1 = \frac{p_{11}}{p_{11}+1}$ and $K_2 = \frac{p_{12}}{p_{11}+1}$

From the previous question, if we want our Kalman filter converges to the Holt-Winter updating scheme at its steady state, we must have $K_1 = \lambda_0$ and $K_2 = \lambda_0 \lambda_1$. This leads to the following conditions on p_{11} and p_{12}:

$$
\begin{aligned}
K_1 &= \frac{p_{11}}{p_{11}+1} = \lambda_0 \\
K_2 &= \frac{p_{12}}{p_{11}+1} = \lambda_0 \lambda_1
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
p_{11} &= \frac{\lambda_0}{1 - \lambda_0} \\
p_{12} &= \frac{\lambda_0 \lambda_1}{1 - \lambda_0}
\end{aligned} \tag{65}
$$

4.9 Relationship between σ_η^2, σ_ζ^2 and λ_0, λ_1

The Ricatti equation derived in (61):

$$
P - \Phi P \Phi' + \Phi PA'(APA' + R)^{-1}AP\Phi' - Q = 0 \tag{66}
$$

can be expanded by the following facts:

$$\Phi P\Phi' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} p_{11} + 2p_{12} + p_{22} & p_{12} + p_{22} \\ p_{12} + p_{22} & p_{22} \end{pmatrix}$$

$$\Phi P A' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_{11} + p_{12} \\ p_{12} \end{pmatrix}$$

$$APA' + R = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 = p_{11} + 1$$

$$AP\Phi' = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} p_{11} + p_{12} & p_{12} \end{pmatrix}$$

$$(67)$$

The Ricatti equation becomes:

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} - \begin{pmatrix} p_{11} + 2p_{12} + p_{22} & p_{12} + p_{22} \\ p_{12} + p_{22} & p_{22} \end{pmatrix} + \frac{\begin{pmatrix} p_{11} + p_{12} \\ p_{12} \end{pmatrix} \begin{pmatrix} p_{11} + p_{12} & p_{12} \end{pmatrix}}{p_{11} + 1} = \begin{pmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix}$$

$$(68)$$

$$\Leftrightarrow - \begin{pmatrix} -2p_{12} - p_{22} & -p_{22} \\ -p_{22} & 0 \end{pmatrix} + \frac{\begin{pmatrix} (p_{11} + p_{12})^2 & p_{12}(p_{11} + p_{12}) \\ p_{12}(p_{11} + p_{12}) & p_{12}^2 \end{pmatrix}}{p_{11} + 1} = \begin{pmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix}$$

$$(69)$$

This leads to the following three equations:

$$\sigma_\eta^2 = \frac{(p_{11} + p_{12})^2}{1 + p_{11}} - 2p_{11} - p_{22}$$

$$0 = \frac{p_{12}(p_{11} + p_{12})}{1 + p_{11}} - p_{22} \qquad (70)$$

$$\sigma_\zeta^2 = \frac{p_{12}^2}{1 + p_{11}}$$

Replace the p_{22} in the middle equation to the first, we have:

$$\sigma_\eta^2 = \frac{p_{11}^2 - p_{11}p_{12} - 2p_{12}}{1 + p_{11}}$$

$$\sigma_\zeta^2 = \frac{p_{12}^2}{1 + p_{11}} \qquad (71)$$

Using the conditions of p_{11} and p_{12} derived at (65), we have:

$$\sigma_\eta^2 = \frac{\lambda_0^2 + \lambda_0^2\lambda_1 - 2\lambda_0\lambda_1}{1 - \lambda_0}$$

$$\sigma_\zeta^2 = \frac{\lambda_0^2\lambda_1^2}{1 - \lambda_0} \qquad (72)$$

This is what we are looking for. (The result could be quickly obtained by the fact that $p_{11} + 1 = \frac{1}{1 - \lambda_0}$ and $\frac{p_{12}}{1 + p_{11}} = \lambda_0\lambda_1$).

18

4.10 What happens when $R = \sigma_\epsilon^2$

Instead of setting $R = 1$, we now let it as it is and try to obtain the formulas for σ_η^2 and σ_ζ^2.

From (64), we now have $K_1 = \frac{p_{11}}{p_{11}+R}$ and $K_2 = \frac{p_{12}}{p_{11}+R}$. Letting them equal to λ_0 and $\lambda_0\lambda_1$ respectively, we get the following conditions for p_{11} and p_{12}

$$p_{11} = \frac{R\lambda_0}{1 - \lambda_0}$$
$$p_{12} = \frac{R\lambda_0\lambda_1}{1 - \lambda_0}$$

(73)

From Ricatti formula, we see that the only affected term is $APA' + R$ which become $p_{11} + R$ instead of $p_{11} + 1$. Updating the equation (70) accordingly and following the similar procedure, we now have:

$$\sigma_\eta^2 = \frac{p_{11}^2 - p_{11}p_{12} - 2Rp_{12}}{R + p_{11}}$$
$$\sigma_\zeta^2 = \frac{p_{12}^2}{R + p_{11}}$$

(74)

Deviding both side by R, we obtain:

$$\frac{\sigma_\eta^2}{R} = \frac{(\frac{p_{11}}{R})^2 - \frac{p_{11}}{R}\frac{p_{12}}{R} - 2\frac{p_{12}}{R}}{1 + \frac{p_{11}}{R}}$$
$$\frac{\sigma_\zeta^2}{R} = \frac{(\frac{p_{12}}{R})^2}{1 + \frac{p_{11}}{R}}$$

(75)

Which is very similar to (71). Since we have $\frac{p_{11}}{R}$ and $\frac{p_{12}}{R}$ equal to p_{11} and p_{12} respectively when $R = 1$, we then get the same result as in (72) for the right hand side of the above formula. Replacing $R = \sigma_\epsilon^2$, we get:

$$\frac{\sigma_\eta^2}{\sigma_\epsilon^2} = \frac{\lambda_0^2 + \lambda_0^2\lambda_1 - 2\lambda_0\lambda_1}{1 - \lambda_0}$$
$$\frac{\sigma_\zeta^2}{\sigma_\epsilon^2} = \frac{\lambda_0^2\lambda_1^2}{1 - \lambda_0}$$

(76)

Which is what we are looking for.

5 Problem 5

Given the following state space model:

$$x_t = \phi x_{t-1} + w_t \qquad\qquad w_t \sim N(0, \sigma_w^2)$$
$$y_t = x_t + v_t \qquad\qquad v_t \sim N(0, \sigma_v^2)$$

With the initial distribution is given by $x_0 \sim N(0, \sigma_w^2/(1-\phi^2))$.

5.1 Matching the autocorrelation function to an ARMA(1,1)

First we find the autocovariance function $\gamma'(h)$ of a general causal ARMA(1,1) model:

$$y_t = \phi' y_{t-1} + \theta' w'_{t-1} + w'_t \qquad w'_t \sim N(0, \sigma_{w'}^2) \tag{77}$$

Here I use the symbols $\gamma'(h)$, ϕ', θ' and w'_t to distinguish with the similar symbols using by the state space model. Multiplying both sides of the ARMA(1,1) equation by y_{t-h} and take the expectation, we can derive the difference equation for $\gamma'(h)$:

$$E(y_t y_{t-h}) = \phi' E(y_{t-1} y_{t-h}) + \theta' E(w_{t-1} y_{t-h}) + E(w'_t y_{t-h}) \tag{78}$$

$$\Leftrightarrow \gamma'(h) = \phi' \gamma'(h-1) + \theta' E(w_{t-1} y_{t-h}) + E(w'_t y_{t-h}) \tag{79}$$

Using the causality assumption: $E(w'_t y_s) = 0$ when $t > s$, we can easily derive the difference equation of $\gamma'(h)$:

$$\gamma'(h) = \phi' \gamma'(h-1) \qquad\qquad (h \geq 2) \tag{80}$$

with initial conditions:

$$\gamma'(0) = \phi' \gamma'(1) + (1 + \theta' \phi' + \theta'^2)\sigma_{w'}^2 \qquad\qquad (h = 0)$$
$$\gamma'(1) = \phi' \gamma'(0) + \theta' \sigma_{w'}^2 \qquad\qquad (h = 1)$$

which can be easily solved to:

$$\gamma'(0) = \sigma_{w'}^2 \frac{(1 + \theta' \phi')(\phi' + \theta')}{1 - \phi'^2}$$
$$\gamma'(1) = \sigma_{w'}^2 \frac{1 + 2\theta' \phi' + \theta'^2}{1 - \phi'^2} \tag{81}$$

Since the difference equation (80) has one real root $z_0 = \frac{1}{\phi'}$, the general solution must be in the form $\gamma'(h) = c\phi'^h$ for $h \geq 1$. Therefore $c = \frac{\gamma'(1)}{\phi'}$

Using the initial conditions, we now have the following equations that can fully describe the autocorrelation function of the ARMA(1, 1) model:

$$\gamma'(h) = \sigma_{w'}^2 \frac{(1 + \theta' \phi')(\phi' + \theta')}{1 - \phi'^2} \phi'^{h-1} \qquad (h \geq 1)$$
$$\gamma'(0) = \sigma_{w'}^2 \frac{1 + 2\theta' \phi' + \theta'^2}{1 - \phi'^2} \tag{82}$$

Now we derive the autocorrelation function $\gamma(h)$ for the given state space model.

$$
\begin{aligned}
& y_t = x_t + v_t \\
\Leftrightarrow\ & E(y_t y_{t-h}) = E(x_t y_{t-h}) + E(v_t y_{t-h}) \\
\Leftrightarrow\ & \gamma(h) = E(x_t(x_{t-h} + v_{t-h})) + E(v_t y_{t-h}) \\
\Leftrightarrow\ & \gamma(h) = \gamma_x(h) + E(v_t y_{t-h})
\end{aligned}
\tag{83}
$$

First we must find $\gamma_x(h)$, which is the autocovariance function of x_t. Since x_t is a causal $AR(1)$ process, the difference equation of its autocovariance function can be derived as:

$$
\begin{aligned}
& x_t = \phi x_{t-1} + w_t \\
\Leftrightarrow\ & E(x_t x_{t-h}) = \phi E(x_{t-1} x_{t-h}) + E(w_t x_{t-h}) \\
\Leftrightarrow\ & \gamma_x(h) = \phi \gamma_x(h-1) + E(w_t x_{t-h})
\end{aligned}
\tag{84}
$$

Using the causality, we get the following difference equation with initial condition:

$$
\begin{aligned}
\gamma_x(h) &= \phi \gamma_x(h-1) & (h \geq 1) \\
\gamma_x(0) &= \phi \gamma_x(1) + \sigma_w^2 & (h = 0) \\
\gamma_x(1) &= \phi \gamma_x(0) & (h = 1)
\end{aligned}
$$

The initial condition can easily be solved to:

$$
\begin{aligned}
\gamma_x(0) &= \frac{\sigma_w^2}{1 - \phi^2} \\
\gamma_x(1) &= \frac{\phi \sigma_w^2}{1 - \phi^2}
\end{aligned}
\tag{85}
$$

Since the difference equation has only one real root $z_0 = \frac{1}{\phi}$, its general solution has the form $\gamma_x(h) = c\phi^h$ when $h \geq 1$. Therefore $c = \frac{\gamma_x(1)}{\phi} = \frac{\sigma_w^2}{1-\phi^2}$

This leads to the general autocovariance function of AR(1):

$$
\gamma_x(h) = \frac{\sigma_w^2}{1 - \phi^2} \phi^h
\tag{86}
$$

Replace this to (83), and use the causality property of y_t, we get the covariance function of the given state space model:

$$
\begin{aligned}
\gamma(h) &= \frac{\sigma_w^2}{1 - \phi^2} \phi^h & (h \geq 1) \\
\gamma(0) &= \frac{\sigma_w^2}{1 - \phi^2} + \sigma_v^2
\end{aligned}
\tag{87}
$$

By matching the above $\gamma(h)$ and $\gamma(0)$ to $\gamma'(h)$ and $\gamma'(0)$ given in (82), we can then express the given state space model in an ARMA(1,1) form. This leads

to the following conditions:

$$\frac{\sigma_w^2}{1 - \phi^2}\phi^h = \sigma_{w'}^2 \frac{(1 + \theta'\phi')(\phi' + \theta')}{1 - \phi'^2}\phi'^{h-1} \qquad (h \geq 1)$$

$$\frac{\sigma_w^2}{1 - \phi^2} + \sigma_v^2 = \sigma_{w'}^2 \frac{1 + 2\theta'\phi' + \theta'^2}{1 - \phi'^2}$$

(88)

We can also see that $\phi' = \phi$, otherwise the first condition can not be satisfied for all $h \geq 1$. We can then simplify the condition to the following form:

$$\sigma_{w'}^2(1 + \theta'\phi)(\phi + \theta') = \phi\sigma_w^2$$

$$\sigma_{w'}^2(1 + 2\theta'\phi + \theta'^2) = \sigma_w^2 + (1 - \phi^2)\sigma_v^2$$

(89)

5.2 Find the parameters for invertible ARMA(1,1) model

5.2.1 Case 1: $\phi = 0.5, \sigma_w^2 = 1, \sigma_v^2 = 2$

To find the parameter of the corresponding invertible ARMA(1,1), we replace ϕ, σ and σ_v into (89):

$$\sigma_{w'}^2(1 + 0.5\theta')(0.5 + \theta') = 0.5$$

$$\sigma_{w'}^2(1 + \theta' + \theta'^2) = 2.5$$

(90)

$$\Leftrightarrow \theta'^2 + 3.5\theta' + 1 = 0$$

$$\sigma_{w'}^2(1 + \theta' + \theta'^2) = 2.5$$

(91)

Since the ARMA(1,1) is invertible, we must have $|\theta'| < 1$, which leads to the following solution:

$$\theta' \approx -0.314$$

$$\sigma_{w'}^2 \approx 3.186$$

(92)

5.2.2 Case 2: $\phi = 0.5, \sigma_w^2 = 1, \sigma_v^2 = 20$

Replacing these assumptions into (89), we have:

$$\sigma_{w'}^2(1 + 0.5\theta')(0.5 + \theta') = 0.5$$

$$\sigma_{w'}^2(1 + \theta' + \theta'^2) = 16$$

(93)

$$\Leftrightarrow \theta'^2 + 2.6\theta' + 1 = 0$$

$$\sigma_{w'}^2(1 + \theta' + \theta'^2) = 16$$

(94)

Using the condition that $|\theta'| < 1$, we get the following solution:

$$\theta' \approx -0.469$$

$$\sigma_{w'}^2 \approx 21.307$$

(95)

Case 1: phi=0.5, theta=-0.314,var.noise = 3.186

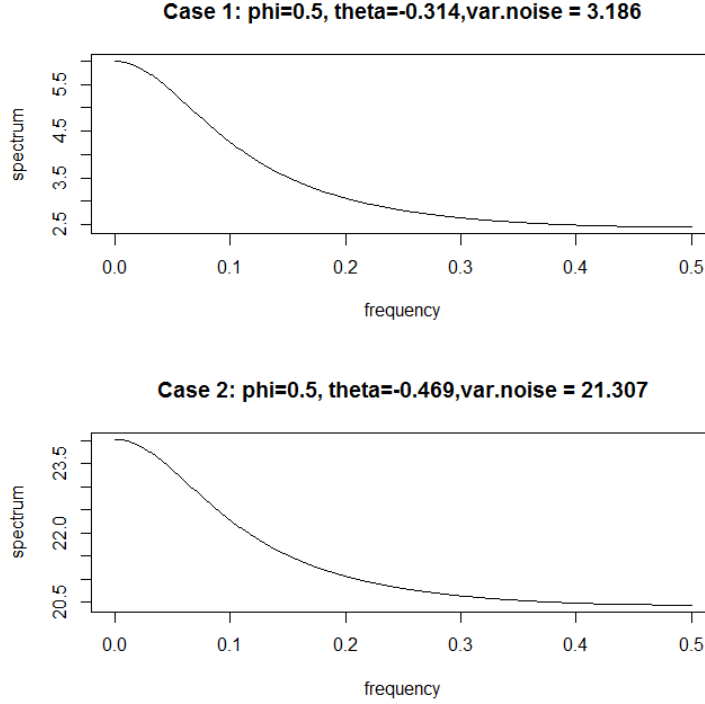

Case 2: phi=0.5, theta=-0.469,var.noise = 21.307

Figure 5: Theoretical spectral density of the two ARMA(1,1) models

5.3 Spectral density of the two ARMA(1, 1)

We see that in Figure 5, the shape of the spectral density of the two ARMA(1,1) models are almost the same. However, the spectral density of the second case is higher than the first case by a constant factor equal to 18 which is the difference in σ_v^2 of the two cases. This shows that, the observation noise in the state space model contributes uniformly to its spectral density.

This effect is obvious when we look at the autocovariance function $\gamma(h)$ of the state space model given in (87). We see that σ_v^2 only contribute to $\gamma(0)$, which will become an addictive constant when we apply the transform $f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h)e^{-2\pi i \omega h}$.

5.4 Simulating and Estimating the Spectra of two cases

The simulation of 200 observations of the two ARMA(1,1) models is given in Figure 6. Their estimated spectra is given in Figure 7. I used the averaged periodogram as the smoothing method, with $L = 9$.
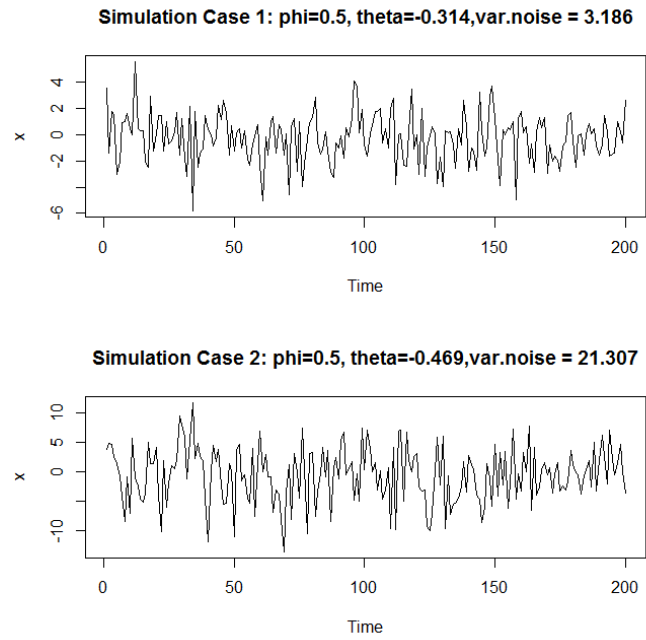
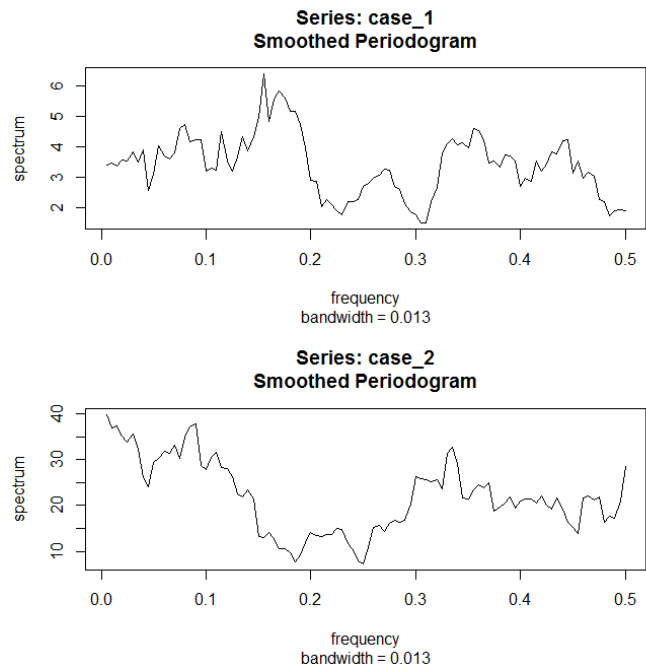Figure 6: Simulation of 200 observations of the two ARMA(1,1) models

Figure 7: Estimated spectra of the two ARMA(1,1) models, using averaged periodograms with $L = 9$