Final project STK4060/STK9060-sp16 - Time Series

This is the problem set for the project part of the finals in STK4060-sp16/STK9060-sp16. The reports shall be individually written. You may discuss the solutions in general with your fellow students, but the intention is that the final formulations shall be done individually. Thus you are not supposed to share your report with others, or copy their solutions.

The deadline for turning in the reports is

Tuesday May 3'rd at 3 pm.

Two copies marked with your candidate number shall be delivered in the expedition at the seventh floor in N. H. Abel's house. Handwritten reports are acceptable. Enclose the parts of the computer outputs which are necessary for the answering the questions. The other parts can be collected in appendices. When you refer to material in these, be careful to indicate explicitly where.

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Problem 1

Consider the series import of marketing and technical services from the Norwegian national account the period 1978:1 to 2015:4. The series is quarterly and describes the import in million Norwegian crowns, NOK. The series can be found on the course web page as $\texttt{mkts}_t$.

a) Use the techniques for ARIMA and seasonal ARIMA modeling to determine suitable models for the series. Estimate the unknown coefficients. Remember that a log transform is often applied for this kind of series.

b) Plot the residuals, and describe how you evaluate the fitted model from the properties of the residuals.

c) What are the forecast for the next two years based on the model you have found?

The purpose with this problem is not that you find a waterproof model, but rather that you explain the choices you do in the modeling process. Relate your model search to the strategy described on pages 159-160 in Shumway and Stoffer: Time Series Analysis and its Applications. Two pages with explanations for each series are sufficient in addition to the plots and R-output.

Problem 2

Consider the weakly stationary ARMA(2,1) time series defined by

$$x_t - 0.75x_{t-1} + 0.125x_{t-2} = w_t - 0.2w_{t-1}$$

where $\{w_t\}$ is white noise with mean zero and variance σ_w^2.

a) Explain why the series is causal and invertible.

b) Find the difference equation that determines the coefficients ψ_j in the representation $x_t = \sum_{j=0}^{\infty} \psi_i w_{t-j}$, and find these coefficients.

c) What is the difference equation satisfied by the auto covariances $\gamma(h)$? State this equation and determine the solution.

d) Show formula 1.30 in the textbook by Shumway and Stoffer, i.e for a time series of the form $x_t = \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$ where $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, the auto covariance function, $\gamma(h)$, is $\gamma(h) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}$.

e) Verify the identity found in part d) for the auto covariances and coefficients you found in part b) and c).

Problem 3

Let $x_1, \ldots, x_n$ be observations from a stationarytime series $\{x_t\}$ with covariance function $\gamma_x(h)$ and spectral density $f_x(\omega) = \sum_{h=-\infty}^{\infty} \gamma_x(h) e^{-2\pi i \omega h}$. Then the discrete Fourier transform is $d_f(\omega_j) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{-2\pi i \omega_j t}$ where $\omega_j = j/n,\ j = 0, \ldots, n-1$ are the Fourier coefficient and periodogram $I_f(\omega_j) = |d_f(\omega_j)|^2$.

a) Show that $I_f(\omega_j) = \sum_{|h|<n} \widehat{\gamma}_x(h) e^{-2\pi i \omega_j h}$, where $\widehat{\gamma}_x(h)$ is the usual estimate of the covariance. What is $I_f(0)$?

b) Show that $E[I_f(\omega_j)] = \sum_{|h|<n} (1 - \frac{|h|}{n}) \gamma_x(h) e^{-2\pi i \omega_j h},\ j = 1, \ldots, n-1$. What happens when $j = 0$?

c) Explain why $\widehat{g}(\omega_j) = \sum_{|h|<n} (1 - \frac{|h|}{n}) \widehat{\gamma}_x(h) e^{-2\pi i \omega_j h} = \sum_{|h|<n} w(\frac{|h|}{n}) \widehat{\gamma}_x(h) e^{-2\pi i \omega_j h}$ may be a more reasonable estimator for the spectral density than the raw periodogram discussed in part a).

d) Let, as $n \to \infty$, $\omega_n = j_n/n \to \omega$ where $-1/2 < \omega < 1/2$. Show that under appropriate regularity conditions $E[I_f(\omega_n)] \to f_x(\omega)$. Explain which regularity conditions are needed.

Problem 4

The so-called Holt-Winters recursions define a much used forecasting scheme. It consists of defining the forecast function as a sum of a level and a slope so

$$\widehat{x}_{n+l}^{n} = m_n + b_n l.$$

Two smoothing constants λ_0 and λ_1 where $0 < \lambda_0, \lambda_1 < 1$ are used for defining a recursion discounting previous observations. The one-step ahead forecast is $\widehat{x}_n^{n-1} = m_{n-1} + b_{n-1}$. The estimate for the level at time t, $t = 3, \ldots, n$ is a linear combination of $\widehat{x}_t^{t-1}$ and x_t, i.e.

$$m_t = \lambda_0 x_t + (1-\lambda_0)\widehat{x}_t^{t-1} = \lambda_0 x_t + (1-\lambda_0)(m_{t-1} + b_{t-1}). \tag{1}$$

a) Why is then
$$b_t = \lambda_1(m_t - m_{t-1}) + (1-\lambda_1)b_{t-1}. \tag{2}$$
a reasonable update of b_t? What are the reasonable starting values for m_2 and b_2?

b) Show how (1) and (2) can be rearranged as

$$\begin{aligned} m_t &= \quad m_{t-1} + \quad b_{t-1} + \lambda_0(x_t - \widehat{x}_t^{t-1}) \\ b_t &= \qquad\qquad\quad b_{t-1} + \lambda_0\lambda_1(x_t - \widehat{x}_t^{t-1}). \end{aligned}$$

We shall now consider the local linear trend model

$$\begin{aligned} y_t &= \mu_t && +\epsilon_t \\ \mu_t &= \mu_{t-1} + \beta_{t-1} && +\eta_t \\ \beta_t &= \beta_{t-1} && +\zeta_t \end{aligned}$$

where ϵ_t, η_t and $\zeta_t, t = 1, \ldots, n$ are independent $\epsilon_t \sim N(0,1)$, $\eta_t \sim N(0, \sigma_\eta^2)$, $\zeta_t \sim N(0, \sigma_\zeta^2)$.

c) Express the model on state-space form, and identify the transition equation and measurement equation.

d) What do the Kalman filter look like in this case?

e) And the Kalman smoother?

It can be shown that under some regularity conditions a steady state exists for this model where the error covariance matrix is time invariant so that $P_t^{t-1} = P$.

f) Show that P then must satisfy the so-called algebraic Ricatti equation

$$P - \Phi P \Phi' + \Phi P A'(APA' + R)^{-1} AP\Phi' - Q = 0.$$

g) Show how the Kalman filter in this case can be expressed as the Holt-Winter recursions from part a) and b).

h) How do the smoothing constants λ_0 and λ_1 look like expressed by the elements of the matrix $P = \{p_{ij}\}_{i,j=1,2}$? Remember that $p_{12} = p_{21}$.

i) Use the result from part h) to show that $\sigma_\zeta^2 = (\lambda_0^2 \lambda_1^2)/(1-\lambda_0)$ and $\sigma_\eta^2 = (\lambda_0^2 + \lambda_0^2\lambda_1 - 2\lambda_0\lambda_1)/(1-\lambda_0)$.

[Hint: From the Ricatti equation in part f) three equations can be obtained which can be solved for σ_η^2 and σ_ζ^2.]

j) How must the results above be modified if instead of assuming that ϵ_t is standard normal we assume $\epsilon_t \sim N(0, \sigma_\epsilon^2)$?

Problem 5

In this problem you shall consider the state space model with state and measurement equation

$$x_t = \phi x_{t-1} + w_t \text{ and } y_t = x_t + v_t \tag{3}$$

where $w_t \sim N(0, \sigma_w^2)$ and $v_t \sim N(0, \sigma_v^2)$ are independent random variables. The initial distribution is given by $x_0 \sim N(0, \sigma_w^2/(1-\phi^2)$.

a) Explain how this process can expressed as an ARMA(1,1) process with the same autocorrelation function for suitable choice of parameters.

b) Let $\phi = 0.5$ and assume that $\sigma_w^2 = 1$. What are the corresponding invertible ARMA processes in the two cases (i) $\sigma_v^2 = 2$ and (i) σ_v^2=20?

c) Plot in the same figure the two spectra of the two processes you found in part b). Comment on the differences.

d) Simulate 200 observations from (3) corresponding to the two cases in part b) and estimate the spectra by smoothing the periodogram.