ELECTROMAGNETICS

EX 503

Lecture : 3 year : II

Tutorial : 1 Part : I

Practical : 3/2

Course Objectives:

To provide basic understanding of the fundamentals of Electromagnetics

1. Introduction (3 hours)

1.1 Co-ordinate system.

1.2 Scalar and vector fields.

1.3 Operations on scalar and vector fields.

2. Electric field (11 hours)

2.1 Coulomb's law.

2.2 Electric field intensity.

2.3 Electric flux density.

2.4 Gauss's law and applications.

2.5 Physical significance of divergence, Divergence theorem.

2.6 Electric potential, potential gradient.

2.7 Energy density in electrostatic field.

2.8 Electric properties of material medium.

2.9 Free and bound charges, polarization, relative permittivity, electric dipole.

2.10 Electric Boundary conditions.

2.11 Current, current density, conservation of charge, continuity equation, relaxation time.

2.12 Boundary value problems, Laplace and Poisson equations and their solutions, uniqueness theorem.

2.13 Graphical field plotting, numerical integration.

3. Magnetic field (9 hours)

3.1 Biot-Savart's law.

3.2 Magnetic field intensity.

3.3 Ampere's circuital law and its application.

3.4 Magnetic flux density.

3.5 Physical significance of curl, Stoke's theorem.

3.6 Scalar and magnetic vector potential.

3.7 Magnetic properties of material medium.

3.8 Magnetic force, magnetic torque, magnetic moment, magnetic dipole, magnetization.

3.9 Magnetic boundary condition.

4. Wave equation and wave propagation (12 hours)

4.1 Faraday's law, transformer emf, motional emf.

4.2 Displacement current.

4.3 Maxwell's equations in integral and point forms.

4.4 Wave propagation in lossless and lossy dielectric.

4.5 Plane waves in free space, lossless dielectric, good conductor.

4.6 Power and pointing vector.

4.7 Reflection of plane wave at normal and oblique incidence.

5. Transmission lines (5 hours)

5.1 Transmission line equations.
5.2 Input impedance, reflection coefficient, standing wave ratio.
5.3 Impedance matching, quarter wave transformer, single stub matching, double stub matching.

6. Wave guides (4 hours)

6.1 Rectangular wave guide.
6.2 Transverse electric mode, transverse magnetic mode.

7. Antennas (1 hour)

7.1 Introduction to antenna, antenna types and properties.

Practical:

1. Teledeltos (electro-conductive) paper mapping of electrostatic fields.
2. Determination of dielectric constant, display of a magnetic Hysteresis loop
3. Studies of wave propagation on a lumped parameter transmission line
4. Microwave sources, detectors, transmission lines
5. Standing wave patterns on transmission lines, reflections, power patterns on transmission lines, reflections, power measurement.
6. Magnetic field measurements in a static magnetic circuit, inductance, and leakage flux.

References:

1. W. H. Hayt, "Engineering Electromagnetics", McGraw-Hill Book Company.
2. J. D. Kraus, "Electromagnetics", McGraw-Hill Book Company.
3. N. N. Rao, "Elements of Engineering Electromagnetics", Prentice Hall.
4. Devid K. Cheng, "Field and Wave Electromagnetics", Addison-Wesley.
5. M. N. O. Sadiku, "Elements of Electromagnetics", Oxford University Press.

Evaluation Scheme

The questions will cover all the chapters of the syllabus. The evaluation scheme will be as indicated in the table below:

Chapter	Hours	Marks Distribution
1	3	5
2	11	20
3	9	16
4	12	21
5,6,7	10	16
Total	45	80

*There could be minor deviation in mark distribution.

Chapter 1: Introduction

Scalar and Vector

The quantities that can be represented completely by the magnitude only are called scalars. Example: temperature, mass, density, electric potential and so on.

The quantities that require both magnitude as well as direction for their complete significance are called vectors. Example: velocity, electric field intensity, force and so on.

Vector Algebra

Assume that A and B are two vector quantities and r and s are two constants.

1. Addition:
A + B = B + A
A + (B + C) = (A + B) + C

2. Subtraction:
A + (- B) = A - B

3. Multiplication:
(r + s) (A + B) = r A + r B + s A + s B

4. Division:
(1/r) (A + B) = A / r + B / r

Scalar Field and Vector Field

Scalar field is the field in which each point in space can be represented by scalar quantities. For example : temperature field i.e. T = e ^ (xyz / 20)

Vector field is the field in which each point in space can be represented by the vector quantities. For example : velocity field.

Dot Product

- Let us consider A and B be two vectors, then the dot product of these two vectors is represented as:
A . B = |A| * |B| * cosθ
where, θ represents the smaller angle between A and B.

- It is considered that: A . B = B . A

- Example of dot product is:
1. Work done by a force is given by:
W = F . L = F * L * cosθ = ∫ F . dL

- For A = A(x) a(x) + A(y) a(y) + A(z) a(z), where a(x), a(y) and a(z) are the unit vector components of A vector and A(x), A(y) and A(z) are constants. Similarly, for B = B(x) b(x) + B(y) b(y) + B(z) b(z)
A . B = A(x) B(x) + A(y) B(y) + A(z) B(z)

- Dot product of similar unit vectors is 1, while of dissimilar unit vectors is 0.
i.e. a(x) . a(x) = a(y) . a(y) = a(z) . a(z) = 1
a(x) . a(y) = a(y) . a(z) = a(z) . a(x) = 0

Cross Product

- For the two vectors A and B, the cross product of these two vectors is given by:
A × B = |A| * |B| * sinθ * n^ Where, n^ is unit vector which is perpendicular to the plane containing a and b, with direction such that the ordered set (a, b, n) is positively oriented

- It is perpendicular to both the vectors.
- a(x) × a(y) = a(z); a(y) × a(z) = a(x); a(z) × a(x) = a(y)
- a(y) × a(x) = -a(z); a(z) × a(y) = -a(x); a(x) × a(z) = -a(y)
- a(x) × a(x) = a(y) × a(y) = a(z) × a(z) = 0

- A × B != B × A

A × B = | a(x) a(y) a(z) |

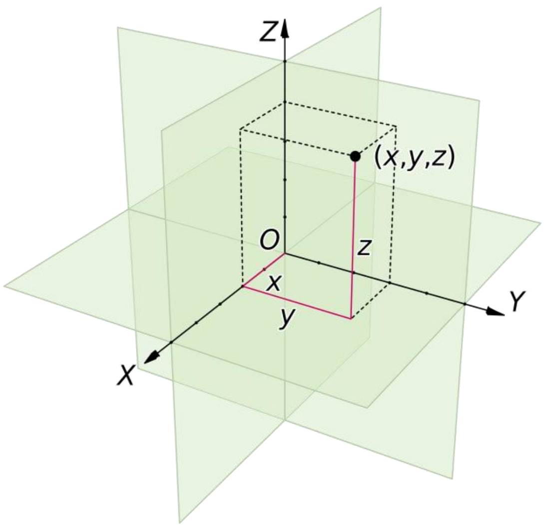
| A(x) A(y) A(z) |

| B(x) B(y) B(z) |

Coordinate System

Rectangular Coordinate System

- It consists of three coordinate axes mutually perpendicular to each other.
- The axes are x, y and z axes.
- A point is located by its x, y and z coordinates.
- Any vector in rectangular or cartesian coordinate system can be represented by:
B = B(x) a(x) + B(y) a(y) + B(z) a(z)
where, a(x), a(y) and a(z) are the unit vectors along x, y and z axes respectively.

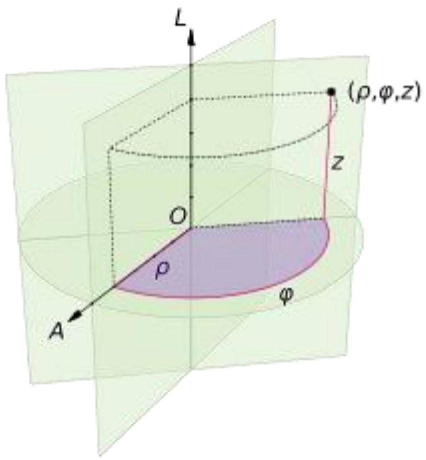


Circular Cylindrical Component System

- It is the 3 dimensional version of polar coordinate system.
- It consists of three planes; circular cylinder (p = constant), planes (z = constant and φ = constant)
- The unit vectors a(p), a(z) and a(φ) are mutually perpendicular to each other.
- a(p) is normal to cylindrical surface, a(φ) is normal to plane φ = constant and a(z) is normal to plane z = constant.

- For the two cylinders of radius p and p + dp, two radial planes at angles φ and φ + dφ, two horizontal planes at elevation z and z + dφ, can enclose a small volume.
The sides are of length dp, dz and pdφ.

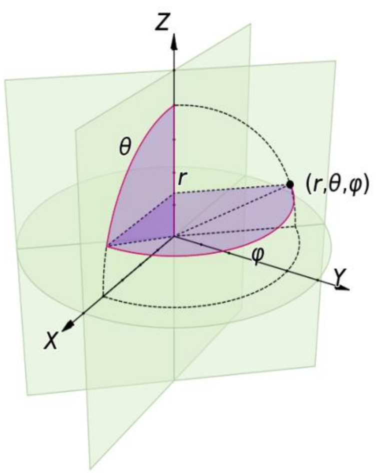
So, surface areas are pdφdp, pdφdz and dpdz.
The volume is pdpdφdz.



Spherical Coordinate System

- It consists of three mutually perpendicular surfaces (sphere; r = constant, cone; θ = constant, and plane; φ = constant)
- a(r), a(θ) and a(φ) are unit vectors.

- Differential Elements:
The lengths are dr, rdθ and rsinθdφ



Rectangular and Cylindrical Coordinate Conversion

We know that:

$p = \sqrt{x^2 + y^2}$

$\varphi = \tan^{-1}(y / x)$

Also,

$x = p \cos\varphi$

$y = p \sin\varphi$

The conversion table is shown in given table:

	a(p)	a(φ)	a(z)
a(x)	cosφ	- sinφ	0
a(y)	sinφ	cosφ	0
a(z)	0	0	1

Rectangular and Spherical Coordinate Conversion

We know:

$r = \sqrt{x^2 + y^2 + z^2}$

$\theta = \cos^{-1}(z / (\sqrt{x^2 + y^2 + z^2}))$

$\varphi = \tan^{-1}(y / x)$

Also,

$x = r \sin\theta \cos\varphi$

$y = r \sin\theta \sin\varphi$

$z = r \cos\theta$

The conversion table is shown in given table:

	a(r)	a(θ)	a(φ)
a(x)	sinθ cosφ	cosθ cosφ	- sinφ
a(y)	sinθ sinφ	cosθ sinφ	cosφ
a(z)	cosθ	-sinθ	0

Numerical Problems of Coordinate System

1. Convert B = y a(x) - x a(y) + z a(z) into cylindrical coordinate system.

We have, B = y a(x) - x a(y) + z a(z)

In cylindrical system, let B = B(p) a(p) + B(φ) a(φ) + B(z) a(z)

Now, from the rectangular and cylindrical conversion table,

1. $B(p) = B.a(p) = [y\, a(x) - x\, a(y) + z\, a(z)] . a(p)$
$= y\, a(x) . a(p) - x\, a(y) . a(p) + z\, a(z) . a(p)$
$= y \cos\varphi - x \sin\varphi$

2. $B(\varphi) = B.a(\varphi) = [y\, a(x) - x\, a(y) + z\, a(z)] . a(\varphi)$
$= y\, a(x) . a(\varphi) - x\, a(y) . a(\varphi) + z\, a(z) . a(\varphi)$
$= - y \sin\varphi - x \cos\varphi$

3. $B(z) = B.a(z) = z$

So,
$B = (y \cos\varphi - x \sin\varphi)\, a(p) - (y \sin\varphi + x \cos\varphi)\, a(\varphi) + z\, a(z)$

Substituting value of x and y, we get:
$B = (p \sin\varphi \cos\varphi - p \cos\varphi \sin\varphi)\, a(p) - (p (\cos\varphi)^2 + p (\sin\varphi)^2)\, a(\varphi) + z\, a(z)$
$= -pa(\varphi) + za(z)$

2. Convert $G = (xz / y)\, a(x)$ into spherical coordinate system.

Let in spherical coordinate,

$G = G(r)\, a(r) + G(\theta)\, a(\theta) + G(\varphi)\, a(\varphi)$

Now,
$G(r) = G.\, a(r) = (xz / y)\, a(x) . a(r) = xz/y \sin\theta \cos\varphi$
$G(\theta) = G.\, a(\theta) = (xz / y)\, a(x) . a(\theta) = xz/y \cos\theta \cos\varphi$
$G(\varphi) = G.\, a(\varphi) = (xz / y)\, a(x) . a(\varphi) = - xz/y \sin\varphi$

So,
$G = xz/y \sin\theta \cos\varphi\, a(r) + xz/y \cos\theta \cos\varphi\, a(\theta) - xz/y \sin\varphi\, a(\varphi)$

Substituting for the value of x, y and z;
$G = (r \cos\theta \cos\varphi / \sin\varphi) * \{ \sin\theta \cos\varphi\, a(r) + \cos\theta \cos\varphi\, a(\theta) - \sin\varphi\, a(\varphi) \}$

3. Transform to cylindrical coordinates: $F = 10\, a(x) - 8\, a(y) + 6\, a(z)$ at point $P(10, -8, 6)$.

Let us consider in cylindrical coordinates:

$F = F(p)\, a(p) + F(\varphi)\, a(\varphi) + F(z)\, a(z)$

So,
$F(p) = [10\, a(x) - 8\, a(y) + 6\, a(z)] . a(p) = 10\, a(x) . a(p) - 8\, a(y) . a(p) + 6\, a(z) . a(p) = 10 \cos\varphi - 8 \sin\varphi$
$F(\varphi) = 10\, a(x) . a(\varphi) - 8\, a(y) . a(\varphi) + 6\, a(z) . a(\varphi) = - 10 \sin\varphi - 8 \cos\varphi$
$F(z) = 6$

Hence, $F = (10 \cos\varphi - 8 \sin\varphi)\, a(p) - (10 \sin\varphi + 8 \cos\varphi)\, a(\varphi) + 6\, a(z)$

At point P(10, -8, 6):
$p = \sqrt{x^2 + y^2} = 12.81$
$\varphi = \tan^{-1}(y / x) = -38.66$ degree

Now,
$F = 12.81\, a(p) + 6\, a(z)$

4. At point P(-3, -4, 5), express that vector that extends from P to Q(2, 0, -1) in spherical coordinates.

The vector that extends from P to Q can be expressed as vector PQ given by:

$PQ = 5\, a(x) + 4\, a(y) - 6\, a(z)$

In terms of spherical coordinates:
$A(r) = 5\, a(x) \cdot a(r) + 4\, a(y) \cdot a(r) - 6\, a(z) \cdot a(r) = 5 \sin\theta \cos\varphi + 4 \sin\theta \sin\varphi - 6 \cos\theta$
$A(\theta) = 5\, a(x) \cdot a(\theta) + 4\, a(y) \cdot a(\theta) - 6\, a(z) \cdot a(\theta) = 5 \cos\theta \cos\varphi + 4 \cos\theta \sin\varphi + 6 \sin\theta$
$A(\varphi) = 5\, a(x) \cdot a(\varphi) + 4\, a(y) \cdot a(\varphi) - 6\, a(z) \cdot a(\varphi) = -5 \sin\varphi + 4 \cos\varphi$

Hence,
$PQ = (5 \sin\theta \cos\varphi + 4 \sin\theta \sin\varphi - 6 \cos\theta)\, a(r) + (5 \cos\theta \cos\varphi + 4 \cos\theta \sin\varphi + 6 \sin\theta)\, a(\theta) + (-5 \sin\varphi + 4 \cos\varphi)\, a(\varphi)$

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chapter 2 Electric field.

2.1 Coulomb's law :-

The force between two very small objects separated in a vacuum or free space by a distance, which is large compared to their size, is proportional to the charge on each and inversely proportional to the square of the distance between them, or

$$F = \frac{K Q_1 Q_2}{R^2} \longrightarrow (1)$$

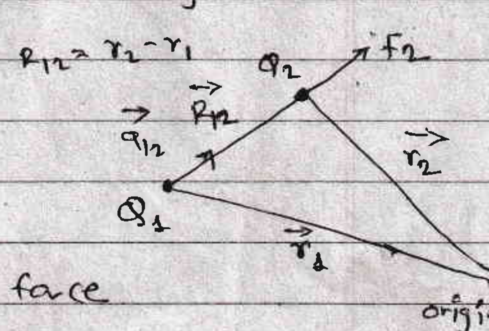
where, Q_1 and Q_2 are the positive or negative quantities of charge, R is the separation, and K is a proportionality constant given by $K = \frac{1}{4\pi \varepsilon_0}$.

Here, ε_0 is called the permittivity of free space and has magnitude, measured in farads per meter (F/m).

$$\varepsilon_0 = 8.854 \times 10^{-12} = \frac{1}{36\pi} 10^{-9} \text{ F/m}.$$

$\therefore$ Eq^n (1) can be written as $F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 R^2} \longrightarrow (2)$

To express the eq^n (2) in vector form, let us assume that the force acts along the line joining the two charges and is repulsive if the charges are alike in sign or attractive if they are of opposite sign.



let vector $\vec{r_1}$ locate Q_1, whereas $\vec{r_2}$ locates Q_2. Then the vector $\vec{R}_{12} = \vec{r_2} - \vec{r_1}$ represents the directed line segment from Q_1 to Q_2. $\vec{F_2}$ is the force on Q_2 when Q_1 and Q_2 have same sign.

$$\therefore \vec{F_2} = \frac{Q_1 Q_2}{4\pi \varepsilon_0 R_{12}^2} \vec{a}_{12} \longrightarrow (3)$$

where, $\vec{a}_{12}$ is a unit vector in the direction of $\vec{R}_{12}$

$$\therefore \vec{a}_{12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\vec{R}_{12}}{R_{12}} = \frac{\vec{r}_2 - \vec{r}_1}{|r_2 - r_1|} \rightarrow (4)$$

2.2 Electric field Intensity :-

If one charge is fixed and second charge moves then, there exists everywhere a force on the second charge which is given by

$$\vec{F}_t = \frac{Q_1 Q_t}{4\pi \varepsilon_0 R_{1t}^2} \vec{a}_{1t}$$

If the force is expressed as force per unit charge then electric field intensity arises from Q_1.

$$\therefore \vec{E}_1 = \frac{\vec{F}_t}{Q_1} = \frac{Q_1}{4\pi \varepsilon_0 R_{1t}^2} \vec{a}_{1t} \rightarrow (1)$$

$\vec{E}_1$ is interpreted as the vector force, arising from charge Q_1, that acts on a unit positive test charge.

$$\vec{E} = \frac{\vec{F}_t}{Q_t} \rightarrow (2)$$

If a single point charge is taken then electric field becomes

$$\vec{E} = \frac{Q}{4\pi \varepsilon_0 R^2} \vec{a}_R$$

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Field of a continuous volume charge :-

Let a region of space be filled with charges separated by small distances. Now, the volume charge density is given by ρ_v. If we consider a small volume Δv on the space then, the small amount of charge ΔQ is given by

$$\Delta Q = \rho_v \Delta V \longrightarrow (1)$$

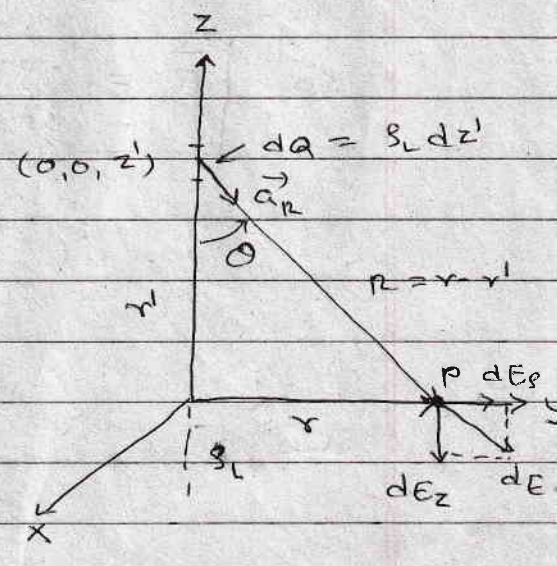
where, $$\rho_v = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V} \rightarrow (2)$$

The total charge can be obtained as:

$$Q = \int_{vol} \rho_v \, dv. \rightarrow (3)$$

Field of a line charge :-

Let us consider a straight line charge extending along the z axis in a cylindrical coordinate system from $-\infty$ to ∞. We desire the electric field intensity $\vec{E}$ at any and every point resulting from a uniform line charge density ρ_L



Let us choose a point $P(0, y, 0)$ on the y axis at which we need to find the field. The incremental field at P due to the incremental charge $dQ = \rho_L dz'$ given the field.

so, $$d\vec{E} = \frac{\rho_L dz' (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3}$$

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where,

$\vec{r} = y\vec{a_y} = \rho\vec{a_\rho}$

$r' = z'\vec{a_z}$

and $\vec{r} - \vec{r'} = \rho\vec{a_\rho} - z'\vec{a_z}$

Therefore,

$$d\vec{E} = \frac{\rho_L\, dz'\,(\rho\vec{a_\rho} - z'\vec{a_z})}{4\pi\varepsilon_0\,(\rho^2 + z'^2)^{3/2}}$$

since $\vec{E_\rho}$ component is present so the eqn can be simplified as:

$$dE_\rho = \frac{\rho_L\,\rho\, dz'}{4\pi\varepsilon_0\,(\rho^2 + z'^2)^{3/2}}$$

and

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L\,\rho\, dz'}{4\pi\varepsilon_0\,(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{\rho_L}{2\pi\varepsilon_0\rho}$$ [$\because$ Integrate considering $z' = \rho\cot\theta$]

$$\therefore E_\rho = \frac{\rho_L}{2\pi\varepsilon_0\rho}$$

So, $\boxed{\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\,\vec{a_\rho}}$

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Field of a sheet of charge :-

Let us consider a infinite sheet of charge having uniform density of ρ_s C/m².
Let us place the sheet of charge in yz plane and consider symmetrical.

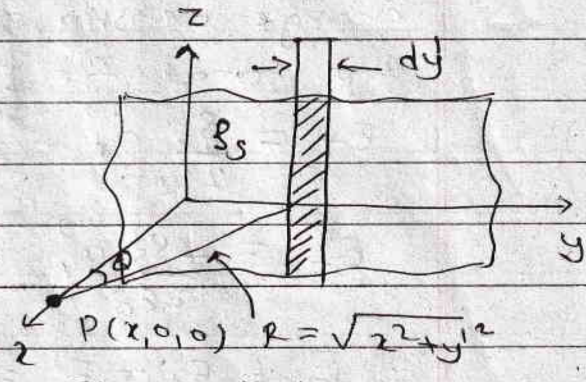


Fig: Infinite sheet of charge in the yz plane

Since the y and z components arising from differential elements of charge symmetrically located with respect to the point far field will cancel so only E_x is present.
Let us divide the infinite sheet into differential width strips as shown in figure. The line charge density is

$$\rho_L = \rho_s dy'$$

and distance from this line charge to our general point P on the x axis is

$$R = \sqrt{x^2+y'^2}.$$

Now, E_x at P due to differential width strip is

$$dE_x = \frac{\rho_s dy'}{2\pi\epsilon_0\sqrt{x^2+y'^2}}\cos\theta = \frac{\rho_s}{2\pi\epsilon_0}\frac{x\,dy'}{x^2+y^2}$$

Considering whole strip

$$E_x = \frac{\rho_s}{2\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{x\,dy'}{x^2+y'^2} = \left[\frac{\rho_s}{2\pi\epsilon_0}\tan^{-1}\frac{y'}{x}\right]_{-\infty}^{\infty} = \frac{\rho_s}{2\epsilon_0}$$

If the point P is choosen on the negative x axis then $E_x = -\frac{\rho_s}{2\epsilon_0}$

$$\therefore \boxed{\vec{E} = \frac{\rho_s}{2\epsilon_0}\vec{a_N}}$$

where, $\vec{a_N}$ is unit vector which is normal to the sheet and directed outwards.

If a second infinite sheet of charge having negative charge density $-\rho_s$, is located in the plane $x = a$, the total field is given by

$$\vec{E_+} = \frac{\rho_s}{2\varepsilon_0} \vec{a_x}$$

$$\vec{E_-} = \frac{-\rho_s}{2\varepsilon_0} \vec{a_x}$$

$\therefore \vec{E} = E_+ + E_- = 0$ for the region $x > a$.

for $x < 0$,

$$\vec{E_+} = \frac{-\rho_s}{2\varepsilon_0} \vec{a_x}$$

$$\vec{E_-} = \frac{\rho_s}{2\varepsilon_0} \vec{a_x}$$

$\therefore \vec{E} = E_+ + E_- = 0$

and when, $0 < x < a$,

$$\vec{E_+} = \frac{\rho_s}{2\varepsilon_0} \vec{a_x}$$

$$\vec{E_-} = \frac{\rho_s}{2\varepsilon_0} \vec{a_x}$$

$$\therefore \vec{E} = \vec{E_+} + \vec{E_-} = \frac{\rho_s}{\varepsilon_0} \vec{a_x}$$

Streamlines :-

The field of a line charge is given by

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0 \rho} \vec{a_\rho}.$$

Let us show the direction of $\vec{E}$ by drawing continuous lines, which are tangent to $\vec{E}$ everywhere from the charge. A symmetrical distribution of lines indicates azimuthal symmetry and arrowheads should be used to show direction.
These lines are so called streamlines.

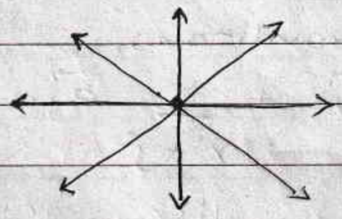


Fig : usual form of a streamline sketch.

Considering the two dimensional field and if we need to sketch field of point charge, let us consider $E_z = 0$. The streamlines are thus confined to planes for which z is constant and the sketch is same for any such plane. From the geometry the E_x and E_y components is given by

$$\frac{E_y}{E_x} = \frac{dy}{dx} \longrightarrow (1)$$

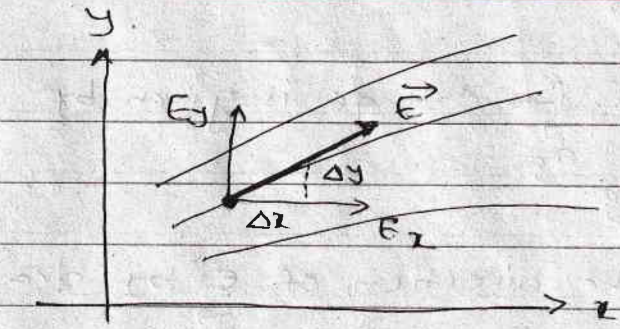


Fig: equation of stream line is obtained by solving differential eqn $E_y/E_x = dy/dx$.

Let us consider the field of uniform line charge with $\rho_L = 2\pi\epsilon_0$.

$$\therefore \vec{E} = \frac{1}{\rho}\vec{a_\rho}$$

In rectangular coordinates,

$$\vec{E} = \frac{x}{x^2+y^2}\vec{a_x} + \frac{y}{x^2+y^2}\vec{a_y}$$

Thus we form the differential eqn

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{y}{x}$$

$$\text{or, } \frac{dy}{y} = \frac{dx}{x}.$$

$$\text{or, } \ln y = \ln x + C_1$$

$$\text{or, } \ln y = \ln x + \ln c.$$

which gives the equations of steamlines.

$$y = cx.$$

If we want to find the equation of one particular streamline passing through point $P(-2, 7, 10)$, we substitute the coordinates of that point into our equation and evaluate c.

Here, $7 = c(-2)$

$\therefore c = -3.5$.

$\therefore y = -3.5x$.

2.3 Electric flux density :-

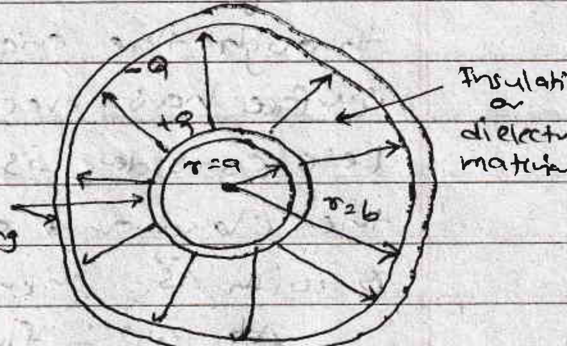
It is the amount of flux passing through a defined area that is perpendicular to the direction of the flux. Electric flux is the property of an electric field that may be thought of as the number of electric lines of force that intersect a given area.

Electric flux density is represented by D and the direction of $\vec{D}$ at a point is the direction of the flux lines at that point and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

It is given by

$$\vec{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\vec{a_r} \quad \text{(inner sphere)}$$

$$\vec{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\vec{a_r} \quad \text{(outer sphere)}$$



and at a radial distance r, where $a \le r \le b$,

$$\vec{D} = \frac{Q}{4\pi r^2}\vec{a_r} \rightarrow (1)$$

Fig: Electric flux in the region between a pair of charged concentric spheres

and electric field intensity of a point charge in free space is given by,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r} \rightarrow (2)$$

In free space,

$$\vec{D} = \varepsilon_0\vec{E} \rightarrow (3)$$

for a general volume charge distribution in free space,

$$\vec{E} = \int_{vol} \frac{\rho_v\, dv}{4\pi\varepsilon_0 R^2}\vec{a_R} \rightarrow (4)$$

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which leads to

$$\vec{D} = \int_{vol} \frac{\rho_v \, dv}{4\pi R^2} \vec{a}_R \rightarrow (5)$$

2.4 Gauss's law :-

It states that "The electric flux passing through any closed surface is equal to the total charge enclosed by that surface".

Let us consider a distribution of charge shown as a cloud of point charges as shown in figure. If the total charge is Q, then Q coulombs of electric flux will pass through the enclosing surface. Every point on the surface has electric flux density D_s.

Lets consider ΔS be an incremental element area in the cloud and at any point P, $\vec{D_s}$ makes an angle θ with $\vec{\Delta S}$. Then flux crossing ΔS is given by

$$\Delta\psi = \text{flux crossing } \Delta S$$
$$= D_{s,norm} \Delta S$$
$$= D_s \cos\theta \, \Delta S$$
$$= \vec{D_s} \cdot \vec{\Delta S}.$$

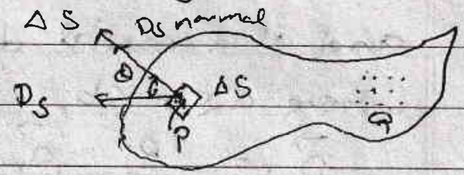

Fig: Electric flux density D_s at P arising from charge Q.

Now, total flux passing through closed surface is given by,

$$\psi = \int d\psi = \oint_{\substack{closed \\ surface}} \vec{D_s} \cdot \vec{ds}.$$

This closed surface is called gaussian surface and the flux is given by

$$\psi = \oint_s \vec{D_s} \cdot \vec{ds} = \text{charge enclosed} = Q.$$

The charge enclosed might be several point charges, so,

$$Q = \Sigma Q_n$$

or a line charge,

$$Q = \int \rho_L \, dL$$

or a surface charge.

$$Q = \int_S \rho_S \, dS$$

or a volume charge distribution,

$$Q = \int_{vol} \rho_v \, dv.$$

so, the Gauss's law may be written in terms of charge distribution as

$$\oint_S \vec{D_S} \cdot \vec{dS} = \int_{vol} \rho_v \, dv.$$

2.4.1 Application of Gauss's law; some symmetrical charge distributions :-

The charge on a closed surface is given by

$$Q = \oint_S D_S \cdot dS. \rightarrow (1)$$

To obtain D_S, the closed surface should satisfy two conditions :-

1) $\vec{D_S}$ is everywhere either normal or tangential to the closed surface so that $\vec{D_S} \cdot \vec{dS}$ becomes either $D_S \, dS$ or zero.

2) on that portion of the closed surface for which $\vec{D_S} \cdot \vec{dS}$ is not zero, D_S = constant.

Case I: Considering Sphere

let us consider a point charge Q at the origin of a spherical coordinate system where surface is a spherical surface, centered at the origin and of any radius 'r'. $\vec{D_S}$ is everywhere normal to the surface; D_S has the

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same value at all points on the surface.

Then,

$$\vec{Q} = \oint_S \vec{D_S} \cdot \vec{dS}$$

$$= D_S \oint_{sph} dS$$

$$= D_S \int_{\phi=0}^{\phi=2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= 4\pi r^2 D_S.$$

and hence

$$D_S = \frac{Q}{4\pi r^2}.$$

because r may have any value and because D_S is directed radially outward.

$$\vec{D} = \frac{Q}{4\pi r^2} \vec{a_r}$$

$$\vec{E} = \frac{Q}{4\pi \varepsilon_0 r^2} \vec{a_r}.$$

Also, we know that

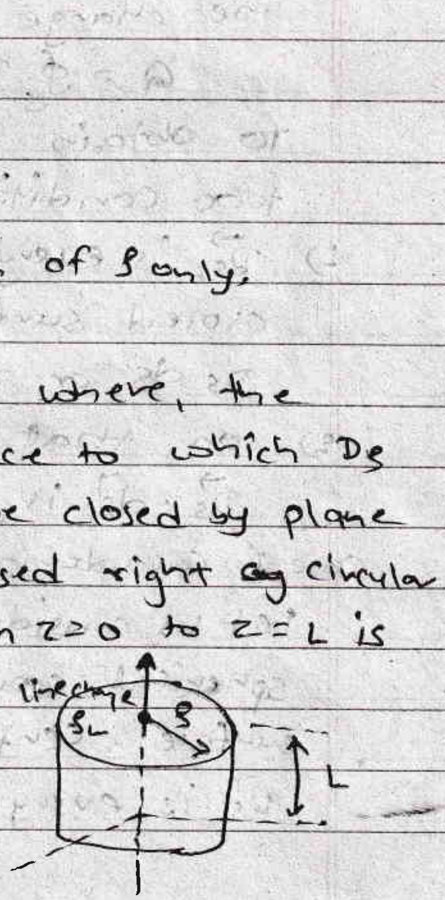
$$\vec{D} = D_\rho \vec{a_\rho}$$

and this component is a function of ρ only.

$$\therefore D_\rho = f(\rho)$$

case 2: Cylinder considered:

If the cylindrical surface is observed where, the cylindrical surface is the only surface to which D_ρ is everywhere normal, and it may be closed by plane surfaces normal to the z-axis. A closed right ~~cy~~ circular cylinder of radius ρ extending from $z=0$ to $z=L$ is shown below:



Applying Gauss's Law,

$$Q = \oint_{cyl} \vec{D_s} \cdot \vec{ds} = D_s \int_{sides} ds + 0\int_{top} ds + 0\int_{bottom} ds$$

$$= D_s \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi dz = D_s 2\pi \rho L$$

and,

$$D_s = D_\rho = \frac{Q}{2\pi \rho L}$$

In terms of the charge density ρ_L, the total charge enclosed is $Q = \rho_L L$.

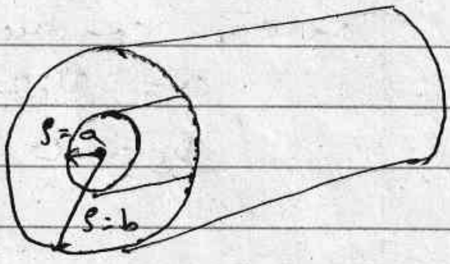
$$\therefore D_\rho = \frac{\rho_L}{2\pi \rho}$$

$$\text{or, } E_\rho = \frac{\rho_L}{2\pi \varepsilon_0 \rho}$$

If a right circular cylinder of length L and radius ρ is considered where $a<\rho<b$, then charge is

$$Q = D_s 2\pi \rho L$$

The total charge on a length L of the inner conductor is



$$Q = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_s a d\phi dz$$

$$= 2\pi a L \rho_s$$

from which we have,

$$D_s = \frac{a\rho_s}{\rho}$$

$$\vec{D} = \frac{a\rho_s}{\rho} \vec{a}_\rho \quad (a<\rho<b)$$

If we express in terms of charge per unit length,

$\rho_L = 2\pi a \rho_s$.

$$\therefore \vec{D} = \frac{\rho_L}{2\pi\rho} \vec{a}_\rho .$$

Since, every line of electric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of the outer cylinder, the total charge on that surface must be

$$Q_{outer\,cyl} = -2\pi a L \rho_{s,\,inner\,cyl}$$

and the surface charge on the outer cylinder is

$$2\pi b L \rho_{s,\,outer\,cyl} = -2\pi a L \rho_{s,\,inner\,cyl}$$

$$\therefore \rho_{s,\,outer\,cyl} = -\frac{a}{b} \rho_{s,\,inner\,cyl}$$

If we use a cylinder of radius ρ, $\rho > b$ for the gaussian surface, the total charge enclosed would then be zero, for there are equal and opposite charges on each conducting cylinder. Hence,

$$0 = D_s 2\pi \rho L \quad (\rho > b)$$

$$D_s = 0. \quad (\rho > b)$$

2.4.2 Application of Gauss's law: Differential volume Element :-

let us consider a nonsymmetrical surface. let us choose a very small closed surface that $\vec{D}$ is almost constant over the surface and the small charge in $\vec{D}$ may be adequately represented by using the first two terms of the Taylor's series expansion for $\vec{D}$.

let us Consider any point P as shown in figure located by a rectangular coordinate system. The value of $\vec{D}$ at the point P may be expressed in rectangular components

$$\vec{D}_0 = D_{x0}\vec{a_x} + D_{y0}\vec{a_y} + D_{z0}\vec{a_z}.$$

we choose as our closed surface the small rectangular box, centered at P, having sides of lengths Δx, Δy, and Δz and apply Gauss's law.

$$\oint_S \vec{D}\cdot\vec{ds} = Q$$

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \vec{D}\cdot\vec{ds} = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom}$$

Here,

$$\int_{front} = \vec{D}_{front}\cdot\vec{\Delta S}_{front}$$

$$= \vec{D}_{front}\cdot\Delta y\,\Delta z\,\vec{a_x}$$

$$= D_{x,front}\,\Delta y\,\Delta z$$

where we have only to approximate the value of $\vec{D_x}$ at this front face. The front face is at a distance of $\Delta x/2$ from P, and hence

$$D_{x,front} = D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x$$
$$= D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$

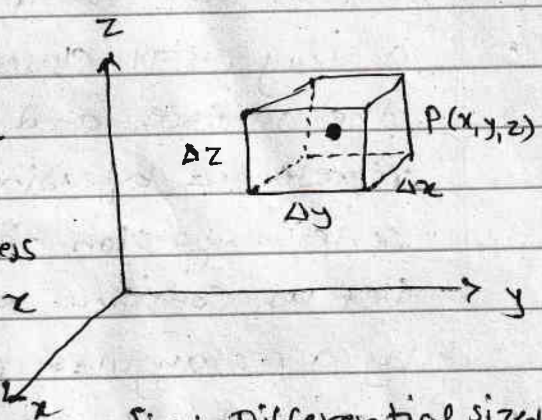


fig: Differential sized gaussian surface about the point P

where, D_{x0} is the value of D_x at P and where a partial derivative must be used to express the rate of change of D_x with x as D_x in general also varies with y and z.
considering constant term and the term involving the first derivative in Taylor's series expansion for D_x in neighborhood of P.
we now have,

$$\int_{front} = \left(D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

consider integral over back surface,

$$\int_{back} = \vec{D}_{back} \cdot \Delta \vec{S}_{back}$$
$$= \vec{D}_{back} \cdot (-\Delta y \Delta z \vec{a}_x)$$
$$= -D_{x,back} \Delta y \Delta z$$

and,

$$D_{x,back} = D_{x0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$

giving $\int_{back} = \left(-D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$

A

If we combine these two integrals, we have

$$\int_{front} + \int_{back} = \frac{\partial D_x}{\partial x} \Delta x \Delta y \Delta z$$

Similarly, $\int_{right} + \int_{left} = \frac{\partial D_y}{\partial y} \Delta x \Delta y \Delta z$

and $\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z} \Delta x \Delta y \Delta z$

and these results may be collected to yield

$$\oint_S \vec{D} \cdot d\vec{S} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \Delta y \Delta z$$

or, $\oint_S \vec{D} \cdot d\vec{S} = Q = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta V \rightarrow (1)$

so, charge enclosed in volume $\Delta V = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times$ volume Δ
$\rightarrow (2$

2.5 Divergence (physical significance of divergence):-

Let us shrink the volume element ΔV to zero.

So, from previous eqn (1)

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \lim_{\Delta V \to 0} \frac{\oint_S \vec{D} \cdot d\vec{S}}{\Delta V} = \lim_{\Delta V \to 0} \frac{Q}{\Delta V} =$$
$\rightarrow ($

in which the charge density ρ_v is identified in the second equality.

B If any vector $\vec{A}$ is used to find $\oint_S \vec{A} \cdot d\vec{S}$ for a small closed surface, leading to

$$\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) = \lim_{\Delta V \to 0} \frac{\oint_S \vec{A} \cdot d\vec{S}}{\Delta V}$$

where, $\vec{A}$ could represent velocity, temperature gradient or any other vector field.

The divergence of $\vec{A}$ is defined as

$$\text{Divergence of } \vec{A} = \text{div } \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v} \longrightarrow (A)$$

So, the divergence of the vector flux density $\vec{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

physical interpretation :-

Let us consider the divergence of the velocity of water in a bathtub after the drain has been opened. The net outflow of water through any closed surface lying entirely within the water must be zero, for water is essentially incompressible, and the water entering and leaving different regions of the closed surface must be equal. Hence the divergence of this velocity is zero.

If we consider the velocity of the air in a tyre that has just been punctured by a nail, we realize that the air is expanding as the pressure drops and that consequently there is a net outflow from any closed surface lying within the tire. The divergence of this velocity is therefore greater than zero.

positive divergence indicates source

Negative divergence indicates sink.

$$\therefore \text{div } \vec{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \text{(rectangular)} \longrightarrow (B)$$

If a differential volume unit $\rho \, d\rho \, d\phi \, dz$ in cylindrical coordinates or $r^2 \sin\theta \, dr \, d\theta \, d\phi$ in spherical coordinates is choosen,

$$\text{div } \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \cdot D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \quad \text{(cylindrical)} \rightarrow$$

$$\text{div } \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} \quad \text{(spher} \rightarrow$$

considering eqn (3) and (5)

$$\text{div } \vec{D} = \rho_v \rightarrow (8)$$

Eqn (8) represent Maxwell's eqn. and is apparantly called the point form of Gauss's law.

2.5.1 Divergence theorem :-

Let us consider an del operator ∇ as a vector operat

$$\nabla = \frac{\partial}{\partial x}\vec{a_x} + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}\vec{a_z} \rightarrow (1)$$

we know, $\vec{D} = D_x \vec{a_x} + D_y \vec{a_y} + D_z \vec{a_z} \rightarrow (2)$

Now,

$$\nabla \cdot \vec{D} = \left(\frac{\partial}{\partial x}\vec{a_x} + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}\vec{a_z}\right) \cdot \left(D_x \vec{a_x} + D_y \vec{a_y} + D_z \vec{a_z}\right)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \text{div }(\vec{D})$$

If we are considering $\vec{D}$ in cylindrical coordinates, then $\nabla \cdot \vec{D}$ is given by

$$\nabla \cdot \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \rightarrow (3)$$

from Gauss's law, we have,

$$\oint_S \vec{D} \cdot d\vec{S} = Q = \int_{vol} \rho_v \, dv = \int_{vol} \nabla \cdot \vec{D} \, dv$$

The first and last expressions constitute the divergence theorem.

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$$\oint_s \vec{D} \cdot d\vec{s} = \int_{vol} \nabla \cdot \vec{D} \, dv \rightarrow (4)$$

which may be stated as:
The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

2.6 Electric potential :-
It is the amount of work needed to move a unit charge from a reference point to a specific point against an electric field.
Let us suppose a charge Q be moved a distance dL in an electric field E. The force on Q arising from the electric field is

$$\vec{F}_E = Q\vec{E}.$$

where the subscript reminds us this force arises from the field. The component of this force in the direction dL which we must overcome is

$$F_{EL} = \vec{F} \cdot \vec{a}_L = Q\vec{E} \cdot \vec{a}_L .$$

where, $\vec{a}_L$ = a unit vector in the direction of $\vec{dL}$.
The force which we must apply is equal and opposite to the force associated with the field,

$$F_{appl} = -Q\vec{E} \cdot \vec{a}_L .$$

Now, differential work done by external source moving

$$dW = -Q\vec{E} \cdot \vec{a}_L \, dL$$

is $-Q\vec{E} \cdot \vec{dL}$,

or, $dW = -Q\vec{E} \cdot \vec{dL}$.

So, the work required to move the charge a finite distance must be determined by integrating "work done by electric field in moving a charge."

$$W = -Q\int_{init}^{final} \vec{E}\cdot\vec{dL}.$$

2.6.1 Potential Gradient :-

The potential difference 'V' is the work done in moving a unit positive charge from one point to another in an electri field.

$$\therefore \vec{V} = -\int_{init}^{final} \vec{E}\cdot\vec{dL} \longrightarrow (1)$$

eqn (1) may be applied to a very short element of length ΔL along which $\vec{E}$ is essentially constant, leading to an incremen potential difference ΔV,

$$\Delta V = -\vec{E}\cdot\vec{dL}.$$

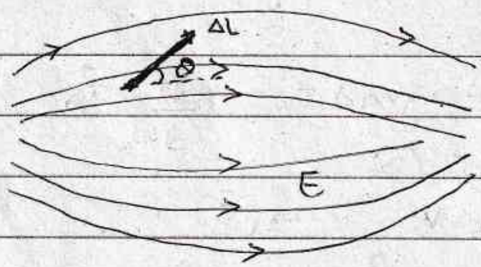

Fig: A vector increm element of length shown making an a of θ with an E fi indicated by its strea

Let us consider a region of space where $\vec{E}$ and V both cha when moving from point a to b. If an incremental vector element of length $\vec{\Delta L} = \Delta L \vec{a}_L$ be taken and we multiply its magnitude by the component of $\vec{E}$ in the direction of $\vec{a}_L$ to obtain the small potential difference between the final and initial points of $\vec{\Delta L}$ and also le us consider angle between $\vec{\Delta L}$ and $\vec{E}$ as θ then,

$$\Delta V = -E\Delta L\cos\theta.$$

$$\text{or, } \frac{\Delta V}{\Delta L} = -E\cos\theta.$$

If we limit the values and consider the derivative dV/

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$$\frac{dv}{dL} = -E\cos\theta.$$

let us consider $\vec{a_N}$ be a unit vector normal to the equipotential surface and directed towards the higher potential then the electric field intensity is expressed in terms of the potential as:

$$\vec{E} = -\frac{dv}{dL}\bigg|_{max} \vec{a_N} \rightarrow (2)$$

also, $\frac{dv}{dL}\bigg|_{max} = \frac{dv}{dN}$ [∵ $\Delta \vec{L}$ is in the direction of $\vec{a_N}$]

or, $$\vec{E} = -\frac{dv}{dN}\vec{a_N}. \rightarrow (3)$$

The operation on v by which $-\vec{E}$ is obtained is known as the gradient, and the gradient of a scalar field T is defined as

$$\text{Gradient of } T = \text{grad } T = \frac{dT}{dN}\vec{a_N} \rightarrow (4)$$

using eqn (4) in eqn (3)

$$\vec{E} = -\text{grad } V \rightarrow (5)$$

Since V is a unique function of x, y and z, we may take its ~~tot~~ al differential

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \rightarrow (6)$$

but we also have, $dv = -\vec{E}\cdot d\vec{L} = -E_x dx - E_y dy - E_z dz \rightarrow (7)$

~~Sin~~ Comparing (6) & (7)

$$E_x = -\frac{\partial v}{\partial x}, \quad E_y = -\frac{\partial v}{\partial y}, \quad E_z = -\frac{\partial v}{\partial z}$$

$$\vec{E} = -\left(\frac{\partial v}{\partial x}\vec{a_x} + \frac{\partial v}{\partial y}\vec{a_y} + \frac{\partial v}{\partial z}\vec{a_z}\right)$$

Using eqn (5)

$$\text{grad}\, V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}$$

we know, the vector operator is given by

$$\nabla = \frac{\partial}{\partial x}\vec{a_x} + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}\vec{a_z}$$

so,
$$\nabla V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}$$

Or, $\nabla V = \text{grad}\, V \rightarrow$ (7)

from eqn (5) and (7)

$\vec{E} = -\nabla V \rightarrow$ (8)

Note:

The gradient may be expressed as:

$$\nabla V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z} \quad \text{(rectangular)}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\vec{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a_\phi} + \frac{\partial V}{\partial z}\vec{a_z} \quad \text{(cylindrical)}$$

$$\nabla V = \frac{\partial V}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a_\phi} \quad \text{(spherical)}$$

2.7 Energy density in electrostatic field :-

Let us consider an empty universe. Let us bring a charge Q_1 from infinity to any position which requir no work since there is no field present. The positioning of Q_2 at a point in the field of Q_1 requires an amount of work given by the product of charge Q_2 and the poten at that point due to Q_1 $(V_{2,1})$

Then, work to position $Q_2 = Q_2 V_{2,1}$.

If additional charge is taken then,

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work to position $Q_3 = Q_3 V_{3,1} + Q_3 V_{3,2}$

work to position $Q_4 = Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$

Now, total work is

$$W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3} + \cdots$$

also,

$$Q_3 V_{3,1} = Q_3 \frac{Q_1}{4\pi\varepsilon_0 R_{13}} = Q_1 \frac{Q_3}{4\pi\varepsilon_0 R_{31}}$$

where R_{13} and R_{31} each represent the scalar distance between Q_1 and Q_3

If each term of the total energy expression is replaced by its equal then,

$$W_E = Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_2 V_{2,4} + Q_3 V_{3,4} + \cdots$$

Adding (1) & (2)

$$2W_E = Q_1(V_{1,2} + V_{1,3} + V_{1,4} + \cdots) + Q_2(V_{2,1} + V_{2,3} + V_{2,4} + \cdots) + Q_3(V_{3,1} + V_{3,2} + V_{3,4} + \cdots)$$

we know that

$$V_{1,2} + V_{1,3} + V_{1,4} + \cdots = V_1$$

or, $W_E = \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \cdots)$

$$= \frac{1}{2} \sum_{m=1}^{m=N} Q_m V_m$$

If contineous charge distribution is taken, $Q = \rho_v\, dv$

$$W_E = \frac{1}{2}\int_{vol} \rho_v V\, dv \rightarrow (3)$$

Now let us replace ρ_v by its equivalent $\nabla\cdot\vec{D}$ from Maxwell's first equation,

$$W_E = \frac{1}{2}\int_{vol} (\nabla\cdot\vec{D}) V\, dv$$

$$= \frac{1}{2}\int_{vol} [\nabla\cdot(V\vec{D}) - \vec{D}\cdot(\nabla V)]\, dv$$

$$= \frac{1}{2}\oint_S (V\vec{D})\cdot d\vec{S} - \frac{1}{2}\int_{vol} \vec{D}\cdot(\nabla V)\, dv$$

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Since surface integral is zero due to closed surface surrounding the universe.

$$\text{or, } w_E = \frac{1}{2}\oint_S \cancel{(V\vec{D})\cdot d\vec{s}}^{\,0} - \frac{1}{2}\int_{vol} \vec{D}\cdot(\nabla V)\, dv$$

$$\text{or, } w_E = \frac{1}{2}\int_{vol} \vec{D}\cdot\vec{E}\, dv \qquad [\because \vec{E} = -\nabla V]$$

$$\therefore w_E = \frac{1}{2}\int_{vol} \varepsilon_0 \vec{E}^2\, dv \rightarrow (4)$$

This eqn represents the energy density in electrostatic

2.8 Electric properties of material medium :-

2.11 current, current density, conservation of charge, continuity equation, relaxation time :-

2.11.1 Current :-

Electric charges in motion constitute a current.
The unit of current is in Ampere (A), defined as a rate of movement of charge passing a given reference point of one coulomb per second. Current is symbolized and is expressed as:

$$I = \frac{dQ}{dt} \rightarrow (1)$$

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2.8 Electric properties of Material Medium :-

a) conductivity :-
The property of a material that allows it to conduct electric current. Materials with high conductivity, like metals are used widely in electrical applications.

b) Resistance :-
The property defines a material's ability to resist the flow of electrical current. Materials with high resistance, such as glass, plastic and silicone are used as insulators

c) permittivity :-
A property that measures how much an electric field affects and is affected by a dielectric medium, essentially, determining how much electric charge a material can store in an electric field.

d) Magnetism :-
Denotes a material's response to a magnetic field, depends on manner in which electrons are aligned in the material. Certain materials, such as iron, nickel and cobalt show strong magnetic properties.

e) superconductivity :-
An exceptional phenomenon occurring at extremely low temperatures, which allows a material to conduct an electric current with zero resistance, providing massive efficiency gains.

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f) Composite materials :-
These are materials made from two or more components with differing properties that produce a material with unique characteristics. Examples include fibre reinforced plastics and concrete.

g) factors affecting insulation :-
The electrical properties of insulating materials depend on both intrinsic and extrinsic factors, including crystal structure, density, temperature, moisture content, purity, pressure, humidity and the frequency and strength of the applied electric field.

h) optical properties :-
These properties determine how a material interacts with light, with one key feature being refractive index. The interplay between electrical and optical properties such as photoconductivity is instrumental for applications like photo detectors and solar cells.

i) Conducting Materials :-
These are materials that allows electric charge to flow freely, due to the presence of charge carriers. Their characterization largely depends on their electric conductivity.

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2.9.1 Free and bound charges :-

Free charges are changes that can move around freely within a material. These changes are not bound to any specific atom or molecule and can easily move in response to an external electric field. For example, in a metal wire, the free electrons are considered free changes because they are not tightly bound to any particular atom and can move through the wire when a voltage is applied.

Bound changes on the other hand are changes that are bound to specific atoms or molecules within a material. These changes are not free to move around like free change when an external electric field is applied, bound charges do not move as freely as free changes. Bound changes can be found in dielectric materials like insulators. In a dielectric material, the positive and negative changes within the atoms or molecules get slightly separated when an external electric field is applied creating bound changes.

2.9.2 polarization :-

Polarization occurs when an electric field distorts the negative cloud of electrons around positive atomic nuclei in a direction opposite the field. This slight separation of change makes one side of the atom somewhat positive and the opposite side somewhat negative. In some materials whose molecules are permanently polarized by chemical forces, such as water molecule some of the polarization is caused by molecules rotating into the same alignment under the influence of the electric field.
one of the measures of polarization is electric dipole moment, which equals the distance between the slightly shifted

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centres of positive and negative charge multiplied by the amount of one of the charges. polarization 'P' in its quantitative meaning is the amount of dipole moment p per unit volume 'V' of a polarized material,

$P = p/V$.

2.9.3. Relative permittivity:-

It indicates how easily a material can become polarized by imposition of an electric field on an insulator. Relative permittivity is "the ratio of the permittivity of a substance to the permittivity of space or vacuum".

Relative permittivity can be expressed as

$\varepsilon_r = \varepsilon/\varepsilon_0 \longrightarrow (1)$

where, ε_r = relative permittivity or dielectric constant

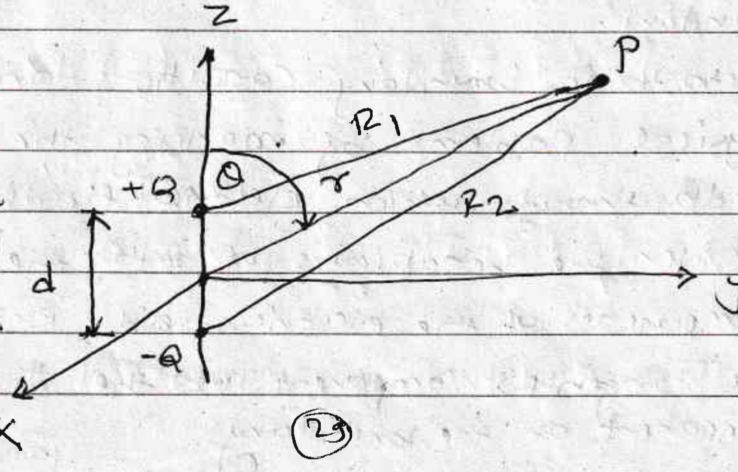
ε = permittivity of substance

ε_0 = permittivity of vacuum or free space $(8.854 \times 10^{-12}\ C^2/$

2.9.4 Electric Dipole:-

An electric dipole is the name given to two point charges of equal magnitude and opposite sign, separated by a distance which is small compared to the distance to the point 'P' at which we want to know the electric and potential fields.

Fig: The geometry of the problem of an electric dipole. The dipole moment $p = Qd$ is in the a_z direction.



Total potential is

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\varepsilon_0} \frac{R_2 - R_1}{R_1 R_2}$$

Here, $R_2 - R_1 = d\cos\theta$

The final result is then

$$V = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\vec{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta)$$

also, $\vec{P} = Q\vec{d}$.

2.10 Electric Boundary Conditions :-

Electrostatic boundary conditions are rules that electric fields and potentials adhere to at the border of different materials. These conditions state that the perpendicular component of an electric field and the component of electric potential parallel to the boundary are continuous across the boundary.

An example of electrostatic boundary conditions is the behavior of electric fields at the boundary of two different media. The conditions state that the component of the electric field perpendicular to the boundary, and the parallel component of the displacement field are both continuous across the boundary.

Electrostatic boundary conditions are applied in solving physical problems by managing the interface of two different media through which electric fields are passing. They consider the behaviours of both the normal and tangential components of the electric field, ensuring the continuity of the tangential component and the discontinuity of the normal component at the boundary.

2.11.2 Current density :-

The Increment of current ΔI crossing an incremental surface ΔS normal to the current density is given by:

$$\Delta I = J_N \Delta S$$

So, $J_N = \frac{\Delta I}{\Delta S} \longrightarrow ①$

Here current density is a vector and represented by $\vec{J}$. In the case where the current density is not perpendicular to the surface,

$$\Delta I = \vec{J} \cdot \Delta S \longrightarrow ②$$

The total current is obtained by integrating eqⁿ ②

$$\therefore I = \int_S \vec{J} \cdot d\vec{S}$$

Current density may be related to the velocity of volume charge density at a point.

We know,

$$\Delta Q = \rho_v \Delta V$$
$$= \rho_v \Delta S \Delta L$$

Dividing both sides by Δt.

$$\frac{\Delta Q}{\Delta t} = \rho_v \frac{\Delta S \Delta L}{\Delta t}$$

or, $\Delta I = \rho_v \Delta S \Delta v$

or, $\frac{\Delta I}{\Delta S} = \rho_v \Delta v$

or, $\Delta J = \rho_v \Delta v$.

If we consider the total current density then,

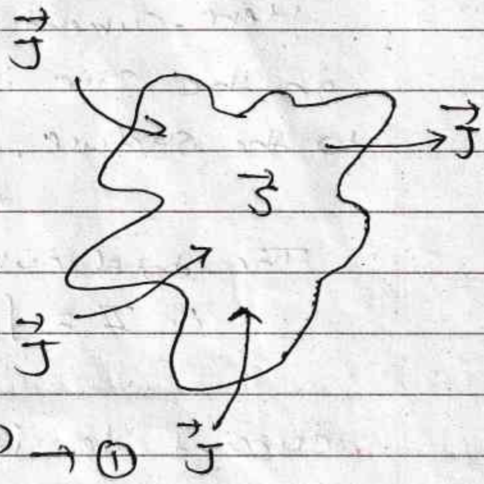
$$J = \rho_v v \longrightarrow ③$$

2.11.3 Conservation of charge :-

It is the principle that the total electric charge in an isolated system never changes. The net quantity of electric charge, the amount of positive charge minus the amount of negative charge in the universe is always conserved.

2.11.4 Continuity equation :-

Suppose the current flows through a closed surface: In this case, there must be charge piling up inside (or departing) the volume V enclosed by the surface S:



$$\oint_S \vec{J}\cdot d\vec{A} = -\frac{\partial}{\partial t} Q \text{ (contained by } S) \rightarrow (1)$$

As $d\vec{A}$ points out, $\vec{J}\cdot d\vec{A}$ is negative.

we know that

$$\oint_S \vec{J}\cdot d\vec{A} = \int_V (\vec{\nabla}\cdot\vec{J})\, dv, \rightarrow (2)$$

where, V is the volume enclosed by S. This is just Gauss's theorem. we also have

$$Q \text{ (enclosed by } S) = \int_V \rho\, dv. \rightarrow (3)$$

or, from (1), (2) & (3)

$$\int_V (\vec{\nabla}\cdot\vec{J})\, dv = -\frac{\partial}{\partial t}\int_V \rho\, dv$$

$$= -\int_V \frac{\partial \rho}{\partial t}\, dv$$

$$\text{or, } \int_V \left(\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t}\right) dv = 0$$

This equation is guaranteed to hold only if

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0.$$

This result is known as the continuity equation: it guara[ntees] conservation of electric charge in the presence of currents. It also tells us that when currents are steady - no time variation, so that $\partial\rho/\partial t = 0$

so we must have $\vec{\nabla}\cdot\vec{J} = 0$.

2.11.B Relaxation time :-

Relaxation time is the time gap between two successive electron collisions in a conductor. The relationship between the relaxati[on] time (τ) and drift velocity (V_d) is

$$V_d = -e\left(\frac{E\tau}{m}\right)$$

$$\text{or, } \tau = \left(\frac{V_d m}{e}\right)/E$$

where, V_d = drift velocity

e = charge of electron

E = field

m = mass of electron

τ = Relaxation time

Simply, relaxation time is denoted by $T = V/t$ where T is the relaxation time, v is the velocity and t is the time taken by particles in a system.

Relaxation time is defined by the formula $T = P/v$, where T is the relaxation time, p is the pressure and v is the volume of particles within a system.

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2.12.1 Boundary value problems:-

Let us set up a boundary between a conductor and free space as shown in figure showing tangential and normal ~~conditions~~ components of $\vec{D}$ and $\vec{E}$ on the free space side of the boundary. Both fields are zero in the conductor.

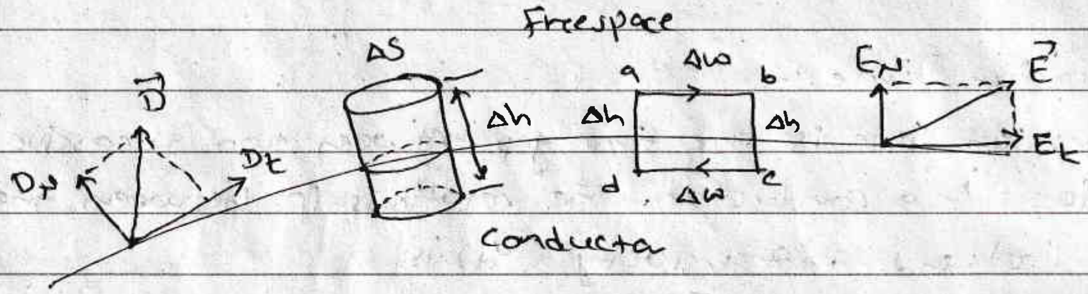

Fig: An appropriate closed path and gaussian surface are used to determine boundary conditions at a boundary between a conductor and free space; $E_t = 0$ and $D_N = \rho_S$

The tangential field may be determined by

$$\oint \vec{E} \cdot d\vec{L} = 0$$

around the small closed path abcda.

The integral must be broken up into four parts

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

Remembering that $\vec{E} = 0$ within the conductor, we let the length from a to b or c to d be Δw and from b to c or d to a be Δh and obtain

$$E_t \Delta w - E_{N,\text{at } b} \tfrac{1}{2}\Delta h + E_{N,\text{at } a} \tfrac{1}{2}\Delta h = 0$$

As we allow Δh to approach zero, keeping Δw small but finite, it makes no difference whether or not the normal fields are equal at a and b for Δh causes these products to become negligibly small. Hence

$$E_t \Delta w = 0$$

and therefore, $E_t = 0$

The condition on the normal field is found most readily by considering D_N rather than E_N and choosing a small cylinder as the gaussian surface. Let the height be Δh and the area of the top and bottom faces be ΔS. Again we shall let Δh approach zero. Using Gauss's law,

$$\oint_S \vec{D} \cdot ds = Q.$$

We integrate over the three distinct surfaces

$$\int_{top} + \int_{bottom} + \int_{sides} = Q.$$

and find that the last two are zero. Then

$$D_N \Delta S = Q = \rho_S \Delta S$$

or, $D_N = \rho_S$

These are the desired boundary conditions for the conductor to free space boundary in electrostatics,

$$D_t = E_t = 0 \rightarrow (1)$$

$$D_N = \epsilon_0 E_N = \rho_s \rightarrow (2)$$

2.12.2 Laplace and poissons equations :-

From point form of Gauss's law,

$$\nabla \cdot \vec{D} = \rho_v \rightarrow (1)$$

also, $\vec{D} = \epsilon \vec{E} \rightarrow (2)$

also, $\vec{E} = -\nabla V \rightarrow (3)$

using (2) & (3) in (1)

$$\nabla \cdot \vec{D} = \nabla \cdot (\epsilon \vec{E}) = -\nabla \cdot (\epsilon \nabla V) = \rho_v$$

or, $\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon} \rightarrow (4)$

This eqn is poisson's equation.

In rectangular coordinates,

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

or, $$\nabla V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}$$

and therefore

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

or, $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon} \rightarrow 5$$

If $\rho_v = 0$, indicating zero volume charge density, but allowing point charges, line charge and surface charge density to exist at singular locations as sources of the field then,

$$\nabla^2 V = 0 \rightarrow 6$$

which is Laplace's equation. The ∇^2 operation is called the Laplacian of V.

In rectangular coordinates Laplace's eqn is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(rectangular)}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \quad \text{(cylindrical)}$$

and $$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \quad \text{(spheri}$$

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2.12.3 Uniqueness theorem :-

Let us assume that we have two solutions of Laplace's equation, V_1 and V_2, both general functions of the coordinates used. Therefore

$$\nabla^2 V_1 = 0$$

and $$\nabla^2 V_2 = 0$$

from which $\nabla^2 (V_1 - V_2) = 0$

Each solution must also satisfy the boundary conditions and if we represent the given potential values on the boundaries by V_b, then the value of V_1 on the boundary V_{1b} and the value of V_2 on the boundary V_{2b} must both be identical to V_b,

$$V_{1b} = V_{2b} = V_b$$

or, $$V_{1b} - V_{2b} = 0$$

we know that

$$\nabla \cdot (V\vec{D}) = V(\nabla \cdot \vec{D}) + \vec{D} \cdot (\nabla V)$$

let us take $V_1 - V_2$ as scalar and $\nabla(V_1 - V_2)$ as vector,

$$\nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)] = (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)] + \nabla(V_1 - V_2) \cdot \nabla(V_1 - V_2)$$

we shall integrate throughout the volume enclosed by the boundary surfaces specified :

$$\int_{vol} \nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)]\, dv$$
$$= \int_{vol} (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)]\, dv + \int_{vol} [\nabla(V_1 - V_2)]^2\, dv$$

using divergence theorem,

$$\int_{vol} \nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)]\, dv = \oint_s [(V_{1b} - V_{2b})\nabla(V_{1b} - V_{2b})] \cdot ds$$

we know that $\nabla^2 (V_1 - V_2)$ is zero by hypothesis, so the remaining integral is zero.

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or, $\int_{vol} [\nabla(V_1 - V_2)]^2 \, dv = 0$

Since, $[\nabla(V_1-V_2)]^2$ cannot be negative. So,

$[\nabla(V_1-V_2)]^2 = 0$

and $\nabla(V_1-V_2) = 0$

Finally, if the gradient of V_1-V_2 is everywhere zero then V_1-V_2 cannot change with any coordinates, and V_1-V_2 = constant.

Here, $V_1-V_2 = V_{1b} - V_{2b} = 0$

and we see that the constant is indeed zero and therefore, $V_1 = V_2$.

giving two identical solutions.

2.13 Solving Laplace's equation through numerical iteration :-

Let us assume a 2 dimensional problem in which the potential does not vary with the z coordinate and divide the interior of a cross section of the region where the potential is desired into squares of length 'h' on a side. A portion of this region is shown in figure. The unknown values of the potential at five adjacent points are indicated as V_0, V_1, V_2, V_3 and V_4. The two-dimensional Laplace equation to be solved

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Approximate values for the partial derivatives may be obtained in terms of the assumed potentials as,

$$\left.\frac{\partial V}{\partial x}\right|_a = \frac{V_1 - V_0}{h}$$

and $$\left.\frac{\partial V}{\partial x}\right|_c = \frac{V_0 - V_3}{h}.$$

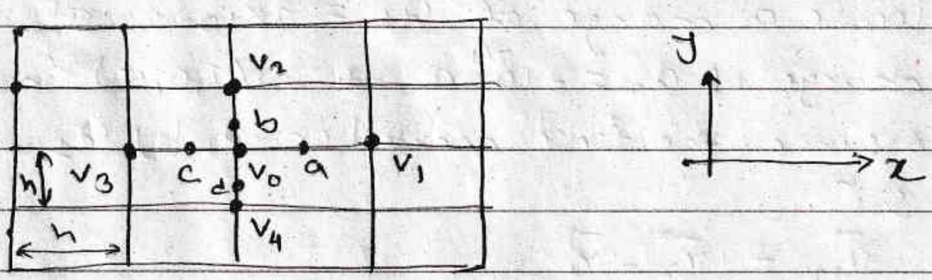


Fig: A portion of a region containing a 2D potential field, divided into squares of side h. The potential V_0 is approximately equal to the average of the potentials at the four neighboring

from which

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_0 = \frac{\left.\frac{\partial V}{\partial x}\right|_a - \left.\frac{\partial V}{\partial x}\right|_c}{h} = \frac{V_1 - V_0 - V_0 + V_3}{h^2}$$

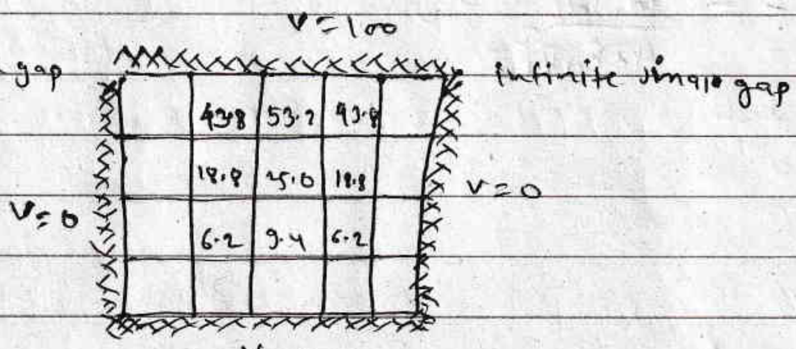
and similarly,

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_0 = \frac{V_2 - V_0 - V_0 + V_4}{h^2}$$

Combining, we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{V_1 + V_2 + V_3 + V_4 - 4V_0}{h^2} = 0$$

$$\therefore V_0 = \frac{1}{4}(V_1 + V_2 + V_3 + V_4)$$



Here, $V_0 = \frac{1}{4}(100 + 53.2 + 18.8 + 0) = 43.$

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(1) locate a charge of $Q_1 = 3\times10^{-4}$ c. at $M(1,2,3)$ and a charge of $Q_2 = -10^{-4}$ C at $N(2,0,5)$ in a vacuum. Assume the force exerted on Q_2 by Q_1.

Sol^n:

$$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1$$

$$= (2-1)\vec{a}_x + (0-2)\vec{a}_y + (5-3)\vec{a}_z$$

$$= \vec{a}_x - 2\vec{a}_y + 2\vec{a}_z$$

leading to

$|\vec{R}_{12}| = 3$ and the unit vector

$|\vec{a}_{12}| = \frac{1}{3}(\vec{a}_x - 2\vec{a}_y + 2\vec{a}_z)$ Thus,

$$\vec{F}_2 = \frac{3\times10^{-4}(-10^{-4})}{4\pi(1/36\pi)10^{-9}\times3^2}\left(\frac{\vec{a}_x - 2\vec{a}_y + 2\vec{a}_z}{3}\right)$$

$$= -30\left(\frac{\vec{a}_x - 2\vec{a}_y + 2\vec{a}_z}{3}\right) N$$

$$= -10\vec{a}_x + 20\vec{a}_y - 20\vec{a}_z.$$

$$\vec{F}_1 = -\vec{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\vec{a}_{21}$$

$$\therefore \vec{F}_1 = -\frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\vec{a}_{12}$$

27 Find $\vec{E}$ at $P(1,1,1)$ caused by four identical 3nC charges located at $P_1(1,1,0)$, $P_2(-1,1,0)$, $P_3(-1,-1,0)$ and $P_4(1,-1,0)$

Soln:-

Here, $\vec{r} = \vec{a_x} + \vec{a_y} + \vec{a_z}$

$\vec{r_1} = \vec{a_x} + \vec{a_y}$

$\vec{r_2} = -\vec{a_x} + \vec{a_y}$

$\vec{r_3} = -\vec{a_x} - \vec{a_y}$

$\vec{r_4} = \vec{a_x} - \vec{a_y}$

Now, $|\vec{r} - \vec{r_1}| = |\vec{a_z}| = 1$

$|\vec{r} - \vec{r_2}| = |2\vec{a_x} + \vec{a_z}| = \sqrt{5}$

$|\vec{r} - \vec{r_3}| = |2\vec{a_x} + 2\vec{a_y} + \vec{a_z}| = 3$

$|r - \vec{r_4}| = |2\vec{a_y} + \vec{a_z}| = \sqrt{5}$

we know that,

$$\vec{E}(r) = \frac{Q_1}{4\pi\varepsilon_0|\vec{r}-\vec{r_1}|^2}\vec{a_1} + \frac{Q_2}{4\pi\varepsilon_0|\vec{r}-\vec{r_2}|^2}\vec{a_2} + \dots + \frac{Q_n}{4\pi\varepsilon_0|\vec{r}-\vec{r_n}|^2}\vec{a_n}$$

$$= \frac{3\times10^{-9}}{4\pi\times8.854\times10^{-12}}\left[\frac{\vec{a_z}}{1}\cdot\frac{1}{1^2} + \frac{2\vec{a_x}+\vec{a_z}}{\sqrt{5}}\cdot\frac{1}{(\sqrt{5})^2} + \frac{2\vec{a_x}+2\vec{a_y}+\vec{a_z}}{3}\cdot\frac{1}{3^2} + \frac{2\vec{a_y}+\vec{a_z}}{\sqrt{5}}\cdot\frac{1}{(\sqrt{5})^2}\right]$$

$$= 26.96\left[\vec{a_z} + \frac{2\vec{a_x}+\vec{a_z}}{5\sqrt{5}} + \frac{2\vec{a_x}+2\vec{a_y}+\vec{a_z}}{27} + \frac{2\vec{a_y}+\vec{a_z}}{5\sqrt{5}}\right]$$

$$= 6.82\,\vec{a_x} + 6.82\,\vec{a_y} + 32.8\,\vec{a_z} \text{ V/m.}$$

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37 Derive the expression for the electric field intensity due to an infinitely long line charge with uniform charge density ρ_L by using Gauss law. A uniform line charge density of 20 nc/m is located at y = 3 and z = 5. Find $\vec{E}$ at P(5, 6, 1).

Soln:

$\rho_L = 20\,nc/m$ at $y = 3$ and $z = 5$

P(5, 6, 1)

Now,

$\vec{\rho} = (5,6,1) - (0,3,5)$

$= 5\vec{a_x} + 3\vec{a_y} - 4\vec{a_z}$

$|\vec{\rho}| = \rho = \sqrt{(5)^2 + (3)^2 + (-4)^2} = \sqrt{50}$

we know that,

$$\vec{E_L} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a_\rho}$$

or, $$\vec{E_P} = \frac{20\times10^{-9}}{2\pi\times8.854\times10^{-12}\times\sqrt{50}} \times \left(\frac{5\vec{a_x} + 3\vec{a_y} - 4\vec{a_z}}{\sqrt{50}}\right)$$

$\therefore \vec{E_P} = (36\vec{a_x} + 21.6\vec{a_y} - 28.8\vec{a_z})$ V/m.

47 Derive an expression to calculate the potential due to a dipole in terms of the dipole moment $(\vec{P})$. A dipole for which $\vec{P} = 3\vec{a_x} - 5\vec{a_y} + 10\vec{a_z}$ nc/m is located at the point (1, 2, -4). Find $\vec{E}$ at P.

Soln:

Given that

$\vec{P} = 3\vec{a_x} - 5\vec{a_y} + 10\vec{a_z}$ nc/m

~~let the point P be origin~~

Let r indicates the distance from origin to field point

then, $\vec{r} = -(0,0,0) + (1,2,-4)$

$= +\vec{a_x} + 2\vec{a_y} - 4\vec{a_z}$

The total potential is given by:

$$V_0 = \frac{\vec{p}\cdot\vec{a_r}}{4\pi\varepsilon_0 r^2}$$

$$= \frac{(3\vec{a_x} - 5\vec{a_y} + 10\vec{a_z})\times 10^{-9}}{4\pi \times 8.854\times 10^{-12} \times (\sqrt{(+1)^2+(+2)^2+(-4)^2})^2} \times \frac{(+\vec{a_x} + 2\vec{a_y} - 4\vec{a_z})}{\sqrt{(+1)^2+(+2)^2+(-4)^2}}$$

$$= \frac{(+3 - 10 - 40)\times 10^{-9}}{1.07\times 10^{-8}}$$

$$= -4.39\ V.$$

Now, $\vec{E_0} = -\nabla V_0$

$$= +\nabla(4.39)$$

$$= 0\ V/m.$$

57 An electric dipole located at the origin in free space has moment $\vec{P} = 3\vec{a_x} - 2\vec{a_y} + \vec{a_z}$ nCm (a) Find V at $P_A(2,3,4)$

Soln:

$$\vec{V} = \frac{\vec{p}\cdot\vec{a_r}}{4\pi\varepsilon_0 r^2}$$

$$= \frac{(3\vec{a_x} - 2\vec{a_y} + \vec{a_z})\times 10^{-9}}{4\pi\times 8.854\times 10^{-12}\times(\sqrt{(+2)^2+(+3)^2+(+4)^2})^2} \times \frac{(+2\vec{a_x} + 3\vec{a_y} + 4\vec{a_z})}{\sqrt{(+2)^2+(+3)^2+(+4)^2}}$$

$$= \frac{+6 - 6 + 4 \times 10^{-9}}{4\pi\times 8.854\times 10^{-12}\times 29\sqrt{29}}$$

$$= 0.23\ V$$

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67 A dipole having a moment $\vec{p} = 3\vec{a_x} - 5\vec{a_y} + 10\vec{a_z}$ ncm is located at Q(1,2,-4) in free space. find V at P(2,3,4).

Soln:

we use the general expression for the potential in the far field:

$$V = \frac{\vec{p} \cdot (\vec{r} - \vec{r'})}{4\pi\varepsilon_0 |\vec{r} - \vec{r'}|^3}$$

where, $\vec{r} - \vec{r'} = P - Q = (1,1,8)$ so,

$$V_p = \frac{(3\vec{a_x} - 5\vec{a_y} + 10\vec{a_z}) \cdot (\vec{a_x} + \vec{a_y} + 8\vec{a_z}) \times 10^{-9}}{4\pi\varepsilon_0 [1^2 + 1^2 + 8^2]^{1.5}}$$

$$= 1.31 \text{ V}.$$

77 Point charges of 120nC are located at A(0,0,1) and B(0,0,-1) in free space.

(a) find $\vec{E}$ at P(0.5,0,0)

(b) what single charge at the origin would provide the identical field strength?

Soln:

a) $$\vec{E_p} = \frac{Q}{4\pi\varepsilon_0 r^2} \vec{a_r}$$

$$= \frac{120 \times 10^{-9}}{4\pi\varepsilon_0} \times \left[\frac{\vec{R}_{AP}}{|\vec{R}_{AP}|^3} + \frac{\vec{R}_{BP}}{|\vec{R}_{BP}|^3} \right]$$

where, $\vec{R}_{AP} = 0.5\vec{a_x} - \vec{a_z}$

$\vec{R}_{BP} = 0.5\vec{a_x} + \vec{a_z}$

also, $|\vec{R}_{AP}| = |\vec{R}_{BP}| = \sqrt{(0.5)^2 + (1)^2} = \sqrt{1.25}$.

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Now, $\vec{E_p} = \dfrac{120\times10^{-9}}{4\pi\epsilon_0(1.25)^{1.5}}\vec{a_x}$

$= 771.73$ V/m.

b) The ~~ch~~ charge for identical field strength would be

$\dfrac{Q_0}{4\pi\epsilon_0(0.5)^2} = \dfrac{120\times10^{-9}\,Q_0}{4\pi\epsilon_0(0.5)^2} = 771.73$

or, $Q_0 = 771.73\times4\pi\epsilon_0(0.5)^2 = 21.4\times10^{-9} = 21.4$ n

87 A 2 μC point charge is located at A(4,3,5) in free spac
Find E_ρ, E_ϕ, and E_z at P(8,12,2).

Soln:

we know,

$\vec{E_p} = \dfrac{Q}{4\pi\epsilon_0}\dfrac{\vec{R_{AP}}}{|\vec{R_{AP}}|^3}$

$= \dfrac{2\times10^{-6}}{4\pi\epsilon_0}\left[\dfrac{(8-4)\vec{a_x}+(12-3)\vec{a_y}+(2-5)\vec{a_z}}{(4^2+9^2+(-3)^2)^{3/2}}\right]$

$= \dfrac{2\times10^{-6}}{4\pi\epsilon_0}\left[\dfrac{4\vec{a_x}+9\vec{a_y}-3\vec{a_z}}{(106)^{3/2}}\right]$

$= 65.9\vec{a_x}+148.3\vec{a_y}-49.4\vec{a_z}$

Then, at point P,

$\rho = \sqrt{8^2+12^2} = 14.4$

$\phi = \tan^{-1}(12/8) = 56.3^\circ$

and $z = 2$.

Now,

$E_\rho = \vec{E_p}\cdot\vec{a_\rho} = 65.9\cos(56.3^\circ)+148.3\sin(56.3^\circ) = 159.7$

$E_\phi = \vec{E_p}\cdot\vec{a_\phi} = 65.9(\vec{a_x}\cdot\vec{a_\phi})+148.3(\vec{a_y}\cdot\vec{a_\phi})$

$= -65.9\sin(56.3^\circ)+148.3\cos(56.3^\circ) = 27.4$

$E_z = -49.4$.

97 The cylindrical surface $\rho = 8$ cm contains the surface charge density, $\rho_s = 5e^{-20|z|}$ nC/m².

a) what is the total amount of charge present ?

b) How much flux leaves the surface $\rho = 8$ cm, 1 cm $< z < 5$ cm, $30° < \phi < 90°$? we just integrate the charge density on that surface to find the flux that leaves it.

Sol^n:

a) we integrate over the surface, so,

$$Q = 2\int_0^{\infty}\int_0^{2\pi} \rho\rho_s \, d\phi dz$$

$$= 2\int_0^{\infty}\int_0^{2\pi} 5e^{-20z}(0.08)\, d\phi\, dz \text{ nC}$$

$$= 20\pi(0.08)\left(-\frac{1}{20}\right)\left[e^{-20z}\right]_0^{\infty}$$

$$= 0.25 \text{ nC}.$$

b) we just integrate the charge density on that surface to find the flux that leaves it.

$$\phi = Q' = \int_{0.01}^{0.05}\int_{30°}^{90°} 5e^{-20z}(0.08)\, d\phi dz \text{ nC}$$

$$= \left(\frac{90-30}{360}\right)2\pi(5).(0.08)\left(\frac{-1}{20}\right)e^{-20z}\Big|_{0.01}^{0.05}$$

$$= 9.45 \times 10^{-3} \text{ nC}$$

$$= 9.45 \text{ pC}.$$

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10) volume charge density is located in free space as $\rho_v = 2e^{-1000r}$ nC/m³ for $0 < r < 1$ mm, and $\rho_v = 0$ elsewher

a) Find the total charge enclosed by the spherical surfa $r = 1$mm:

b) By using Gauss's law, calculate the value of D_r on the surface $r = 1$mm.

Soln:-

a) To find the charge we integrate

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{0.001} 2e^{-1000r} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{0.001} 2e^{-1000r} r^2 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= \int_0^{0.001} 2e^{-1000r} r^2 dr \cdot [-\cos\theta]_0^{\pi} \, [\phi]_0^{2\pi}$$

$$= \int_0^{0.001} 2e^{-1000r} r^2 dr \quad 2 \cdot 2\pi$$

$$= 8\pi \left[\left(\frac{-r^2 e^{-1000r}}{1000} \right)_0^{0.001} + \left(\frac{2}{1000} \frac{e^{-1000r}}{(1000)^2} (-1000r - 1) \right)_0^{0.001} \right]$$

$$= 4 \times 10^{-9} \text{ nC}.$$

b) By using Gauss's law:

The gaussian surface is a spherical shell of radius 1m
The enclosed charge is the result of part (a).
we thus write

$$4\pi r^2 D_r = Q$$

or, $$D_r = \frac{Q}{4\pi r^2} = \frac{4\times 10^{-9}}{4\pi (0.001)^2} = 3.2 \times 10^{-4} \text{ nC/m}^2.$$

117) Calculate the divergence of $\vec{D}$ at the point specified if

a) $\vec{D} = \left(\frac{1}{z^2}\right)\left[10xyz\,\vec{a_x} + 5x^2z\,\vec{a_y} + (2z^3 - 5x^2y)\vec{a_z}\right]$ at $p(-2,3,5)$.

b) $\vec{D} = 5z^2\vec{a_\rho} + 10\rho z\,\vec{a_z}$ at $p(3,-45°,5)$.

c) $\vec{D} = 2r\sin\theta\sin\phi\,\vec{a_r} + r\cos\theta\sin\phi\,\vec{a_\theta} + r\cos\phi\,\vec{a_\phi}$ at $p(3,45°,-45°)$.

Soln:

a) $\nabla\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$ so,

$$\nabla\cdot\vec{D} = \left[\frac{\partial}{\partial x}\left(\frac{10xyz}{z^2}\right) + \frac{\partial}{\partial y}\left(\frac{5x^2z}{z^2}\right) + \frac{\partial}{\partial z}\left(\frac{2z^3 - 5x^2y}{z^2}\right)\right]_{(-2,3,5)}$$

$$= \left[\frac{10y}{z} + 0 + 2 + \frac{10x^2y}{z^3}\right]_{(-2,3,5)}$$

$$= 8.96.$$

b) $\vec{D} = 5z^2\vec{a_\rho} + 10\rho z\,\vec{a_z}$ at $p(3,-45°,5)$

In cylindrical coordinates,

$$\nabla\cdot\vec{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}$$

$$= \left[\frac{5z^2}{\rho} + 0 + 10\rho\right]_{(3,-45°,5)}$$

$$= 71.67.$$

c) $\vec{D} = 2r\sin\theta\sin\phi\,\vec{a_r} + r\cos\theta\sin\phi\,\vec{a_\theta} + r\cos\phi\,\vec{a_\phi}$ at $p(3,45°,-45°)$

In spherical coordinates,

$$\nabla\cdot\vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

$$= \left[6\sin\theta\sin\phi + \frac{\cos2\theta\sin\phi}{\sin\theta} - \frac{\sin\phi}{\sin\theta}\right]_{(3,45°,-45°)}$$

$$= -2.$$

127 Inside the cylindrical shell, $3 < \rho < 4$ m, the electric flux density is given as

$$\vec{D} = 5(\rho - 3)^3 \vec{a}_\rho \ \ C/m^2$$

a) what is the volume charge density at $\rho = 4$ m?
b) what is the electric flux density at $\rho = 4$ m?

Solⁿ:

a) $$\rho_v = \nabla \cdot \vec{D} = \frac{1}{\rho}\frac{d}{d\rho}(\rho \vec{D}_\rho)$$

$$= \frac{1}{\rho}\frac{d}{d\rho}\left[5\rho(\rho-3)^3\right]$$

$$= \frac{1}{\rho}\left[5\rho \cdot 3 \cdot (\rho-3)^2 + (\rho-3)^3 5\right]$$

$$\cancel{= \frac{1}{\rho}\left[15\rho(\rho-3)^2 + 5(\rho-3)^3\right]}$$

$$\cancel{= 15(\rho-3)^2 + \frac{5}{\rho}(\rho-3)^3}$$

$$= \frac{1}{\rho}\left[5(\rho-3)^2\left[3\rho + \rho - 3\right]\right]$$

$$= \frac{5(\rho-3)^2}{\rho}\left[4\rho - 3\right] \ C/m^3$$

Evaluating at $\rho = 4$ m,

$$\rho_v(4) = \frac{5(4-3)^2}{4}\left[16-3\right] = \frac{5}{4} \times 13 = 16.25 \ C/m^3$$

b) we evaluate the given $\vec{D}$ at the given point to find

$$\vec{D}(4) = 5(\rho-3)^3 \vec{a}_\rho$$

$$= 5(4-3)^3 \vec{a}_\rho$$

$$= 5\vec{a}_\rho \ \ C/m^2.$$

137 within the spherical shell, $3<r<4m$, the electric flux density is given as

$$\vec{D} = 5(r-3)^3 \vec{a_r} \ C/m^2$$

a) what is the volume charge density at $r=4$? In this case we have

$$\rho_v = \nabla \cdot \vec{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r)$$

$$= \frac{5}{r}(r-3)^2(5r-6) \ C/m^3$$

which we evaluate at $r=4$ to find $\rho_v(r=4) = 17.5 \ C/m^3$.

b) what is the electric flux density at $r=4$? substitute $r=4$ into the given expression to find $\vec{D}(4) = 5\vec{a_r} \ C/m^2$

c) How much electric flux leaves the sphere $r=4$?

~~from Gauss's law, this will be the same as the outward flux, or again $Q = 320\pi C$.~~

using the result of part b, this will be $\Phi = 4\pi(4)^2(5)$

$$= 320\pi C.$$

d) How much charge is contained within the sphere, $r=4$? From Gauss's law, this will be the same as the outward flux or again $Q = 320\pi C$.

147 Given the field

$$\vec{D} = \frac{5\sin\theta\cos\phi}{r}\,\vec{a_r}\ C/m^2,$$

find:

a) the volume charge density

b) the total charge contained within the region $r<2m$.

c) The value of $\vec{D}$ at the surface $r=2$.

Soln:

a) The volume charge density is given as

$$\rho_v = \nabla\cdot\vec{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{5\sin\theta\cos\phi}{r^2}\ C/m^3$$

b) The total charge is given by

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{2}\frac{5\sin\theta\cos\phi}{r^2}\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

before plunging into this one notice that the ϕ integration is of $\cos\phi$ from zero to 2π. This yields a zero result, and so the total enclosed charge is $Q=0$.

c) substituting $r=2$ into the given field produces

$$\vec{D}(r=2) = \frac{5}{2}\sin\theta\cos\phi\,\vec{a_r}\ C/m^2.$$

157 Let $\vec{D} = 5r^2\vec{a_r}\ mC/m^2$ for $r\leq 0.08m$ and $\vec{D} = 0.205\,\vec{a_r}/r^2\ \mu C/m^2$ for $r\geq 0.08m$

a) find ρ_v for $r=0.06m$

b) find ρ_v for $r=0.1m$

c) what surface charge density could be located at $r=0.08m$ to cause $\vec{D}=0$ for $r>0.08m$?

Soln:-

a) This radius lies within the first region and so

$$\rho_v = \nabla\cdot\vec{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{1}{r^2}\frac{d}{dr}(5r^4) = 20r\ mC/m^3$$

which when evaluated at $r=0.06$ yields $\rho_v(r=0.06)=1.2\ mC/m^3$.

b) This is in the region where the second field expression is valid. The $1/r^2$ dependence of this field yields a zero divergence and so the volume charge density is zero at $0.1m$.

c) The total surface charge should be equal and opposite to the total volume charge. The latter is

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{0.08} 20r\,(mC/m^3)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 2.57 \times 10^{-3}\ mC$$

$$= 2.57\ \mu C.$$

So now,

$$\rho_s = -\left[\frac{2.57}{4\pi (0.08)^2}\right] = -32\ \mu C/m^2.$$

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167 The value of $\vec{E}$ at $P(\rho = 2, \phi = 40°, z = 3)$ is given as $\vec{E} = 100\vec{a}_\rho - 200\vec{a}_\phi + 300\vec{a}_z$ V/m. Determine the incremental work required to move a 20 μC charge a distance of 6 μm

a) In the direction of $\vec{a}_\rho$

b) In the direction of $\vec{a}_\phi$

c) In the direction of $\vec{a}_z$

d) In the direction of $\vec{E}$.

e) In the direction of $\vec{G} = 2\vec{a}_x - 3\vec{a}_y + 4\vec{a}_z$.

soln:-

a) The incremental work is given by $dw = -q\vec{E}\cdot d\vec{L}$ where in this case,

$$d\vec{L} = d\rho\,\vec{a}_\rho = 6\times10^{-6}\,\vec{a}_\rho. \quad \text{Thus}$$

$$dw = -q\vec{E}\cdot d\rho\,\vec{a}_\rho$$

$$= -(20\times10^{-6}\,\text{C})(100\,\text{V/m})(6\times10^{-6}\,\text{m})$$

$$= -12.0\times10^{-9}\,\text{J}$$

$$= -12\,\text{nJ}$$

b) In this case $d\vec{L} = 2d\phi\,\vec{a}_\phi = 6\times10^{-6}\,\vec{a}_\phi$ and so

$$dw = -(20\times10^{-6})(-200)(6\times10^{-6})$$

$$= 2.4\times10^{-8}\,\text{J}$$

$$= 24\,\text{nJ}$$

c) Here, $d\vec{L} = dz\,\vec{a}_z = 6\times10^{-6}\,\vec{a}_z$ and so,

$$dw = -(20\times10^{-6})(300)(6\times10^{-6})$$

$$= -3.6\times10^{-8}\,\text{J}$$

$$= -36\,\text{nJ}$$

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d) Here, $d\vec{L} = 6\times10^{-6}\,\vec{a}_E$, where

$$\vec{a}_E = \frac{100\vec{a}_\rho - 200\vec{a}_\phi + 300\vec{a}_z}{[100^2+200^2+300^2]^{1/2}}$$

$$= 0.267\vec{a}_\rho - 0.535\vec{a}_\phi + 0.802\vec{a}_z$$

Thus,

$$dw = -q\vec{E}\cdot\vec{a}_E\,dE$$

$$= -(20\times10^{-6})[100\vec{a}_\rho - 200\vec{a}_\phi + 300\vec{a}_z]\cdot[0.267\vec{a}_\rho - 0.535\vec{a}_\phi + 0.802\vec{a}_z](6\times10^{-6})$$

$$= -44.9\,nJ$$

e) Given,

$$\vec{G} = 2\vec{a}_x - 3\vec{a}_y + 4\vec{a}_z$$

Here, $d\vec{L} = 6\times10^{-6}\,\vec{a}_G$ where,

$$\vec{a}_G = \frac{2\vec{a}_x - 3\vec{a}_y + 4\vec{a}_z}{[2^2+3^2+4^2]^{1/2}}$$

$$= 0.371\vec{a}_x - 0.557\vec{a}_y + 0.743\vec{a}_z$$

so now,

$$dw = -(20\times10^{-6})[100\vec{a}_\rho - 200\vec{a}_\phi + 300\vec{a}_z]\cdot[0.371\vec{a}_x - 0.557\vec{a}_y + 0.743\vec{a}_z](6\times10^{-6})$$

$$= -(20\times10^{-6})[37.1(\vec{a}_\rho\cdot\vec{a}_x) - 55.7(\vec{a}_\rho\cdot\vec{a}_y) - 74.2(\vec{a}_\phi\cdot\vec{a}_x) + 111.4(\vec{a}_\phi\cdot\vec{a}_y) + 222.9](6\times10^{-6})$$

where, at ϕ $(\vec{a}_\rho\cdot\vec{a}_x) = (\vec{a}_\phi\cdot\vec{a}_y) = \cos(40^\circ) = 0.766$

$(\vec{a}_\rho\cdot\vec{a}_y) = \sin(40^\circ) = 0.643$ and

$(\vec{a}_\phi\cdot\vec{a}_x) = -\sin(40^\circ) = -0.643$.

substituting these results in

$$dw = -(20\times10^{-6})[28.4 - 35.8 + 47.7 + 85.3 + 222.9](6\times10^{-6})$$

$$= -41.8\,nJ.$$

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17) Find the amount of energy required to move a 6C charge from the origin to P(3,1,-1) in the field $\vec{E} = 2x\vec{a_x} - 3y^2\vec{a_y} + 4\vec{a_z}$ V/m along the straight line pa
$x = -3z,\ y = x + 2z$.

Soln:

We set up the computation as follows, and find the result does not depend on the path.

$$W = -q\int \vec{E}\cdot d\vec{L}$$

$$= -6\int (2x\vec{a_x} - 3y^2\vec{a_y} + 4\vec{a_z})\cdot(dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z})$$

$$= -6\int_0^3 2x\,dx + 6\int_0^1 3y^2 dy - 6\int_0^{-1} 4\,dz$$

$$= -6\left[x^2\right]_0^3 + 6\left[y^3\right]_0^1 - 6\left[4z\right]_0^{-1}$$

$$= -54 + 6 + 24$$

$$= -24\ J$$

18) The annular surface, $1\text{cm} < \rho < 3\text{cm}$, $z = 0$, carries the ~~information~~ non uniform surface charge density $\rho_s = 5\rho\ nC/m^2$. Find V at P(0,0,2 cm) if V=0 at infinit

Soln:

We use the superposition integral form

$$V_P = \iint \frac{\rho_s\, da}{4\pi\varepsilon_0|\vec{r} - \vec{r}'|}$$

where, $\vec{r} = z\vec{a_z}$ and $\vec{r}' = \rho\vec{a_\rho}$. We integrate over the surface of the annular region, with $da = \rho\, d\rho\, d\phi$

Substituting the given values, we find

$$V_P = \int_0^{2\pi}\int_{0.01}^{0.03} \frac{(5\times10^{-9})\rho^2\, d\rho\, d\phi}{4\pi\varepsilon_0\sqrt{\rho^2 + z^2}}$$

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substituting $z = 0.02$ and using tables, the integral evaluates as

$$V_P = \left[\frac{(5\times10^{-9})}{2\varepsilon_0}\right]\left[\frac{\rho}{2}\sqrt{\rho^2+(0.02)^2} - \frac{(0.02)^2}{2}\ln\left(\rho+\sqrt{\rho^2+(0.02)^2}\right)\right]_0$$

$$= 0.081\,V$$

197 Given the current density $\vec{J} = -10^4\left[\sin(2x)e^{-2y}\vec{a_x} + \cos(2x)e^{-2y}\vec{a_y}\right]$ KA/m^2:

a) Find the total current crossing the plane $y=1$ in the $\vec{a_y}$ direction in the region $0<x<1$, $0<z<2$.

b) Find the total current leaving the region $0<x, x<1, 2<z<3$ by integrating $\vec{J}\cdot\vec{ds}$ over the surface of the cube.

c) Repeat part b but use the divergence theorem.

Soln :-

a) $\vec{I} = \iint_S \vec{J}\cdot\vec{n}\,|_s\, da$

$$= \int_0^2\int_0^1 \vec{J}\cdot\vec{a_y}\,\Big|_{y=1} dx\,dz$$

$$= \int_0^2\int_0^1 -10^4\cos(2x)e^{-2}\,dx\,dz$$

$$= -10^4(2)\frac{1}{2}\sin(2x)\Big|_0^1 e^{-2}$$

$$= -1.23\,MA.$$

b) $$I = \int_2^3\int_0^1 \vec{J}\cdot(-\vec{a_y})\,\Big|_{y=0} dx\,dz + \int_2^3\int_0^1 \vec{J}\cdot(\vec{a_y})\,\Big|_{y=1} dx\,dz + \int_2^3\int_0^1 \vec{J}\cdot(\vec{a_x})\,\Big|_{x=1} dy\,dz$$

$$\text{or, } I = 10^4 \int_2^3 \int_0^1 \left[\cos(2x) e^{-0} - \cos(2x) e^{-2} \right] dx\,dz - 10^4 \int_2^3 \sin(2) e^{-2y} dy\,dz$$

$$= 10^4 \left(\tfrac{1}{2}\right) \sin(2x)\Big|_0^1 (3-2) \left[1 - e^{-2}\right] + 10^4 \left(\tfrac{1}{2}\right) \sin(2) e^{-2y}$$

$$(3-2) = 0$$

c) we find the net outward current through the surface of the cube by integrating the divergence of $\vec{J}$ over the cube volume. we have

$$\nabla \cdot \vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y}$$

$$= -10^{-4} \left[2\cos(2x) e^{-2y} - 2\cos(2x) e^{-2y} \right] -$$

$$= 0$$

20) let $\vec{J} = \dfrac{400 \sin\theta}{r^2 + 4} \vec{a_r}$ A/m²

a) Find the total current flowing through that portion of the spherical surface $r = 0.8$, bounded by $0.1\pi < \theta < 0.3\pi$, $0 < \phi < 2\pi$: This will be

$$I = \iint \vec{J} \cdot \vec{n} \Big|_s da$$

$$= \int_0^{2\pi} \int_{0.1\pi}^{0.3\pi} \frac{400 \sin\theta}{(0.8)^2 + 4} (0.8)^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{400 (0.8)^2 2\pi}{4.64} \int_{0.1\pi}^{0.3\pi} \sin^2 d\theta$$

$$= 346.5 \int_{0.1\pi}^{0.3\pi} \frac{1}{2} \left[1 - \cos(2\theta)\right] d\theta$$

$$= 77.4 \text{ A}.$$

b) Find the average value of $\vec{J}$ over the defined area. The area is

$$\text{area} = \int_0^{2\pi}\int_{0.1\pi}^{0.3\pi} (0.8)^2 \sin\theta \, d\theta \, d\phi$$

$$= 1.46 \, m^2$$

The average current density is thus

$$\vec{J}_{avg} = (77.4/1.46)\,\vec{a_r}$$

$$= 53\,\vec{a_r} \ \ A/m^2.$$

217 Let $\vec{J} = \frac{25}{\rho}\vec{a_\rho} - \frac{20}{\rho^2+0.01}\vec{a_z}$ A/m^2

a) Find the total current crossing the plane $z = 0.2$ in the $\vec{a_z}$ direction for $\rho < 0.4$.

b) Calculate $\partial\rho_v/\partial t$

c) Find the outward current crossing the closed surface defined by $\rho = 0.01$, $\rho = 0.4$, $z = 0$ and $z = 0.2$.

Soln:

a) $$I = \iint_S \vec{J}\cdot\vec{n}\Big|_{z=0.2} da$$

$$= \int_0^{2\pi}\int_0^{0.4} \frac{-20}{\rho^2+0.01}\,\rho\,d\rho\,d\phi$$

$$= -\left(\frac{1}{2}\right) 20 \ln(0.01+\rho^2)\Big|_0^{0.4} (2\pi)$$

$$= -20\pi \ln(17)$$

$$= -178 \ A.$$

b) calculate $\partial \rho_v / \partial t$

This is found using the equation of continuity

$$\frac{\partial \rho_v}{\partial t} = -\nabla \cdot \vec{J}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho J_\rho) + \frac{\partial J_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(25) + \frac{\partial}{\partial z}\left(\frac{-20}{\rho^2+0.01}\right) = 0$$

c) $$I = \int_0^{0.2}\int_0^{2\pi} \frac{25}{0.01}\vec{a}_\rho \cdot (-\vec{a}_\rho)(0.01)\, d\phi\, dz + \int_0^{0.2}\int_0^{2\pi} \frac{25}{0.4}\vec{a}_\rho (\vec{a}_\rho)(0 \ldots\, d\phi\, dz$$

$$+ \int_0^{2\pi}\int_0^{0.4} \frac{-20}{\rho^2+0.01}\vec{a}_z \cdot (-\vec{a}_z)\, \rho\, d\rho\, d\phi + \int_0^{2\pi}\int_0^{0.4} \frac{-20}{\rho^2+0.01}\vec{a}_z \cdot (\vec{a}_z)\, \rho\, d\rho\, d\phi$$

$= 0.$

chapter-3 Magnetic field

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3.1 Biot-Savart's law :-

It states that at any point P', the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point P at which the field is desired; also the magnitude of the magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point p.

i.e, $$d\vec{H} = \frac{I\, d\vec{L} \times \vec{a}_R}{4\pi R^2} = \frac{I\, d\vec{L} \times \vec{R}}{4\pi R^3}$$

The Biot-savarts law is sometimes called Ampere's law for the current element.

we know that,

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

therefore shows that $\nabla \cdot \vec{J} = 0$

or upon applying the divergence theorem,

$$\oint_S \vec{J} \cdot d\vec{S} = 0$$

The total current crossing any closed surface is zero, and this condition may be satisfied only by assuming a current flow around a closed path. It is this current flowing in a closed circuit that must be our experimental source, not the differential element.

It follows that only the integral form of the Biot Savart law can be verified experimentally.

$$\vec{H} = \oint \frac{I\,d\vec{L} \times \vec{a}_R}{4\pi R^2}$$

The Biot savart law may also be expressed in terms of distributed sources, such as current density $\vec{J}$ and surface current density $\vec{K}$. surface current flows in a sheet of vanishingly small thickness, and the current density $\vec{J}$, measured in amperes per square meter is therefore infinite. surface current density, however is measured in amperes per meter width and designated by $\vec{K}$. If the surface current density is uniform, the total current I in any width b is

$$I = Kb.$$

where we assume that the width b is measured perpendicularly to the direction in which the current is flowing.

$$I = \int K\,dN.$$

where, dN is a differential element of the path across which the current is flowing.

Thus the differential current element $I\,d\vec{L}$, where $d\vec{L}$ is in the direction of the current, may be expressed in terms of surface current density $\vec{K}$ or current density $\vec{J}$.

$$I\,d\vec{L} = K\,ds = \vec{J}\,dv$$

and alternate forms of the Biot Savart law obtained,

$$\vec{H} = \int_s \frac{\vec{K} \times \vec{a}_R\,ds}{4\pi R^2}$$

and $$\vec{H} = \int_{vol} \frac{\vec{J} \times \vec{a}_R\,dv}{4\pi R^2}$$

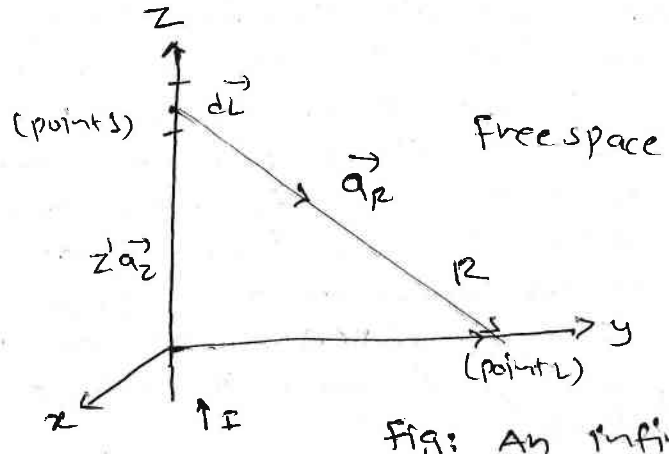

Fig: An infinitely long straight filament carrying a direct current I.

3.3 Ampere's circuital law and its application:

It states that the line integral of $\vec{H}$ about any closed path is exactly equal to the direct current enclosed by that path,

$$\oint \vec{H} \cdot \vec{dL} = I \longrightarrow (1)$$

3.3.1 Magnetic field intensity produced by an infinitely long filament carrying current:

Let us suppose the filament lies on the z axis in free space and the current flows in the direction given by $\vec{a_z}$.
Let us choose a path to any section of which $\vec{H}$ is either perpendicular or tangential and along which H is constant.

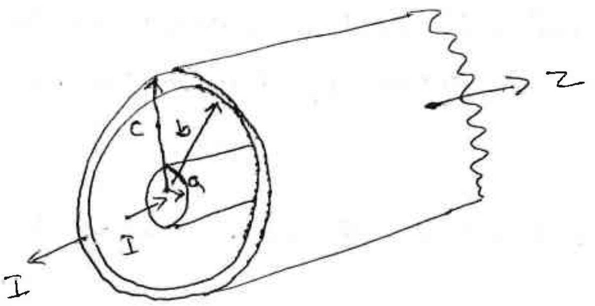

Fig 1: cross section of a coaxial cable carrying a uniformly distributed current I in the inner conductor and − I in the outer conductor.

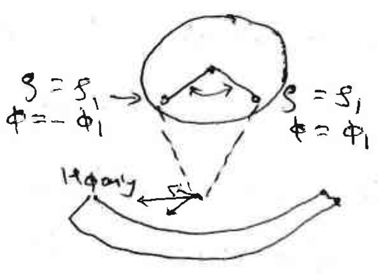

Fig 2: Current filaments at $\rho = \rho_1$, $\phi = \pm \phi_1$, produces H_ρ components which cancel. For the total field $\vec{H} = H_\phi \vec{a_\phi}$

If the path be a circle of radius ρ, and Ampere's circuital law becomes

$$\oint \vec{H} \cdot \vec{dL} = \int_0^{2\pi} H_\phi \rho \, d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi \rho = I$$

or, $H_\phi = \dfrac{I}{2\pi \rho}$.

As an application of the Ampere's circuital law, consider an infinitely long coaxial transmission line carrying a uniformly distributed total current I in the center conductor and − I in the outer conductor as shown in fig 1. Symmetry shows that H is not a function of ϕ or z. If solid conductors are composed of a large number of filaments and no filament has a z component of $\vec{H}$. Furthermore, the H_ρ component at $\phi = 0°$, produced by one filament located at $\rho = \rho_1$, $\phi = \phi_1$, is cancelled by the H_ρ component produced by a symmetrically located filament at $\rho = \rho_1$, $\phi = -\phi_1$. This symmetry is illustrated on fig 2.

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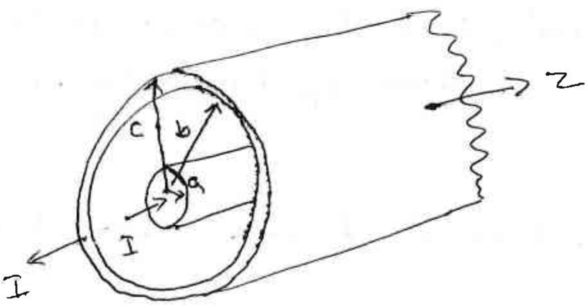
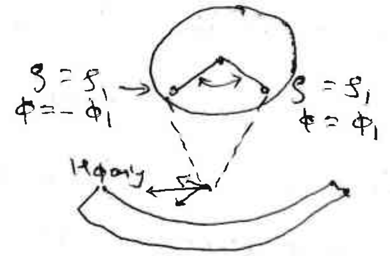


Fig 1: cross section of a coaxial cable carrying a uniformly distributed current I in the inner conductor and −I in the outer conductor.



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Again we find only an H_ϕ component which varies with ρ. A circular path of radius ρ, where ρ is larger than the radius of the inner conductor but less than the inner radi of the outer conductor, then leads immediately to

$$H_\phi = \frac{I}{2\pi\rho} \quad (a<\rho<b)$$

If we choose ρ smaller than the radius of the inner conducto the current enclosed is

$$I_{encl} = \frac{I\rho^2}{a^2}$$

$$\text{or, } 2\pi\rho H_\phi = \frac{I\rho^2}{a^2}$$

$$\text{or, } H_\phi = \frac{I\rho}{2\pi a^2} \quad (\rho<a)$$

If the radius ρ is larger than the outer radius of the outer conductor, no current is enclosed and

$$H_\phi = 0 \quad (\rho>c)$$

Finally, if the path lies within the outer conductor, we have,

$$2\pi\rho H_\phi = I - I\left(\frac{\rho^2-b^2}{c^2-b^2}\right)$$

$$\therefore \boxed{H_\phi = \frac{I}{2\pi\rho}\,\frac{c^2-\rho^2}{c^2-b^2}} \quad (b<\rho<c)$$

3.4 Magnetic flux and Magnetic flux density:

In free space, let us define the magnetic flux density B as

$$\vec{B} = \mu_0 \vec{H} \quad \text{(free space only)} \rightarrow (1)$$

where $\vec{B}$ is measured in webers per square meter (wb/
or in a newer unit adopted in the unit Tesla (T).

Here, $\mu_0 = 4\pi \times 10^{-7}$ H/m (permeability of free space)

Let us represent magnetic flux by Φ and define Φ as the flux passing through any designated area,

$$\Phi = \int_S \vec{B} \cdot ds \ \text{wb} \rightarrow (2)$$

we also know that the total flux passing through any closed surface is equal to the charge enclosed,

$$\psi = \oint_S \vec{D} \cdot ds = Q.$$

In the case of infinitely long straight filament carrying a direct current I, the $\vec{H}$ field formed concentric circles about the filament. since $\vec{B} = \mu_0 \vec{H}$, the $\vec{B}$ field is of the same form.

The Gauss's law for the magnetic field is

$$\oint_S \vec{B} \cdot \vec{ds} = 0 \rightarrow (3)$$

and application of the divergence theorem shows that

$$\nabla \cdot \vec{B} = 0 \rightarrow (4)$$

we have for static electric fields and steady magnetic fields

$$\left.\begin{aligned} \nabla \cdot \vec{D} &= \rho_v \\ \nabla \times \vec{E} &= 0 \\ \nabla \times \vec{H} &= \vec{J} \\ \nabla \cdot \vec{B} &= 0 \end{aligned}\right\} \rightarrow (4)$$

To these equations, we may add the two expressions relating $\vec{D}$ to $\vec{E}$ and $\vec{B}$ to $\vec{H}$ in free space,

$$\vec{D} = \varepsilon_0 \vec{E} \rightarrow (5)$$

$$\vec{B} = \mu_0 \vec{H} \rightarrow (6)$$

Also, $\vec{E} = -\nabla V \rightarrow (7)$

From eqn (4), the equations specify the divergence and curl of an electric and magnetic field. The corresponding set of four integral equations that apply to static electric fields and steady magnetic fields is

$$\oint_s \vec{D}\cdot d\vec{s} = Q = \int_{vol} \rho_v \, dv$$

$$\left.\begin{aligned} &\oint \vec{E}\cdot d\vec{L} = 0 \\ &\oint \vec{H}\cdot d\vec{L} = I = \int_s \vec{J}\cdot d\vec{s} \\ &\oint_s \vec{B}\cdot d\vec{s} = 0 \end{aligned}\right\} \rightarrow (8)$$

If the flux between the conductors of the coaxial line has to be determined, the magnetic field intensity was found to be

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

and therefore, $\vec{B} = \mu_0 \vec{H} = \dfrac{\mu_0 I}{2\pi\rho} \vec{a}_\phi$

The magnetic flux contained between the conductors in a length d is the flux crossing any radial plane extending from $\rho = a$ to $\rho = b$ and from say $z = 0$ to $z = d$.

$$\phi = \int_s \vec{B}\cdot ds = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho} \vec{a}_\phi \cdot d\rho \, dz \, \vec{a}_\phi$$

$$\phi = \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a} \rightarrow (9)$$

3.5 Curl:

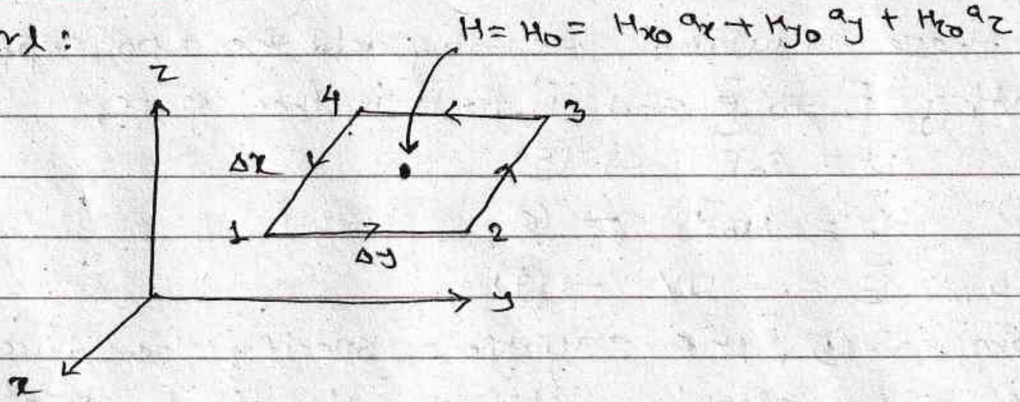


Fig: An incremental closed path in rectangular coordinates is selected for the application of Ampere's circuital law to determine the spatial rate of change of H.

lets choose incremental closed path of sides Δx and Δy. we assume that some current produces a reference value for $\vec{H}$ at the center of this small rectangle,

$$\vec{H_0} = H_{x0} a_x + H_{y0} a_y + H_{z0} a_z$$

The closed line integral of $\vec{H}$ about this path is then approximately the sum of the four values of $\vec{H} \cdot \vec{\Delta L}$ on each side. we choose the direction of traverse as 1-2-3-4-1 which corresponds to a current in the a_z direction, and the first contribution is therefore

$$(\vec{H} \cdot \vec{\Delta L})_{1-2} = H_{y,1-2} \Delta y$$

The value of H_y on this section of the path may be given in terms of the reference value H_{y0} at the center of the rectangle, the rate of change of H_y with x, and the distance $\Delta x/2$ from the center to the midpoint of side 1-2

$$H_{y,1-2} = H_{y0} + \frac{\partial H_y}{\partial x}\left(\frac{1}{2}\Delta x\right)$$

Thus $(\vec{H} \cdot \vec{\Delta L})_{1-2} = \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)\Delta y$

along the next section of the path we have

$(\vec{H}\cdot\Delta \vec{L})_{2-3} = H_{x,2-3}(-\Delta x) = -\left(H_{x0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)\Delta x$

Continuing for the remaining two segments and adding the results,

$$\oint \vec{H}\cdot d\vec{L} = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x \Delta y$$

By Ampere's circuital law, this result must be equal to the current enclosed by the path.

$\therefore\ \Delta I = J_z \Delta x \Delta y$ and

$$\oint \vec{H}\cdot d\vec{L} = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x \Delta y = J_z \Delta x \Delta y$$

or, $$\frac{\oint \vec{H}\cdot d\vec{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$

As we cause the closed path to shrink,

$$\lim_{\Delta x, \Delta y \to 0} \frac{\oint \vec{H}\cdot d\vec{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \quad \rightarrow (1)$$

similarly, $$\lim_{\Delta y, \Delta z \to 0} \frac{\oint \vec{H}\cdot d\vec{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \quad \rightarrow (2)$$

and $$\lim_{\Delta z, \Delta x \to 0} \frac{\oint \vec{H}\cdot d\vec{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \quad \rightarrow (3)$$

The mathematical form of the curl is

$$(\text{curl } \vec{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \vec{H}\cdot d\vec{L}}{\Delta S_N} \quad \rightarrow (4)$$

where ΔS_N is the planar area enclosed by the closed line integral. The N subscript indicates that the component of the curl is that component which is normal to the surface enclosed by the closed path.

$$\therefore\ \text{curl } \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)a_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)a_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)a_z \quad \rightarrow (5)$$

In terms of determinant,

$$\text{curl } \vec{H} = \begin{vmatrix} a_x & a_y & a_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} \quad \rightarrow (6)$$

which may be written in terms of the vector operator

$$\text{curl } \vec{H} = \nabla \times \vec{H} \quad \rightarrow (7)$$

The curl expansions in cylindrical and spherical coordinates that follow here is

$$\nabla \times \vec{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\vec{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\vec{a}_\phi + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\vec{a}_z \quad \text{(cylindrical)} \rightarrow (8)$$

$$\nabla \times \vec{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\vec{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)a_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)a_\phi \quad \text{(spherical)} \rightarrow (9)$$

3.5.1 Significance of curl :-

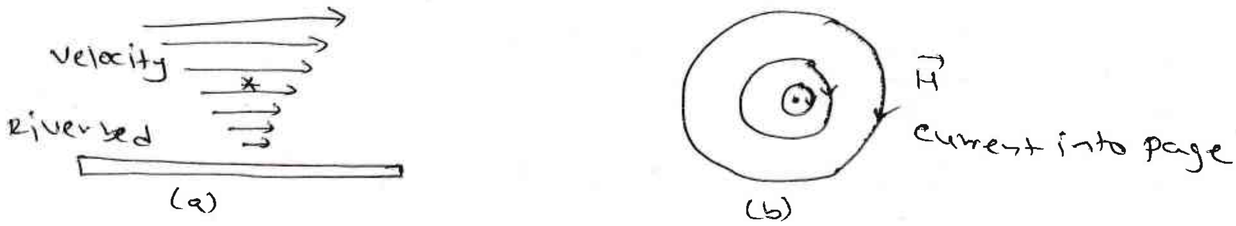


Fig: (a) Curl meter shows a component of the curl of the water velocity into the page
(b) Curl of magnetic field intensity about an infinitely long filament.

Consider the flow of water in a river. Figure shows the longitudinal section of a wide river taken at the middle of the river. The water velocity is zero at the bottom and increases linearly as the surface is approached.

A paddle wheel placed in the position shown, with its axis perpendicular to the paper, will turn in a clockwise direction, showing the presence of a component of curl in the direction of an inward normal to the surface of the page. If the velocity of water doesnot change as we go up or downstream and also shows no variation as we go across the river then this component is the only component present at the center of the stream, and the curl of the water velocity has a direction into the page.

3.5.2 Stoke's Theorem :-

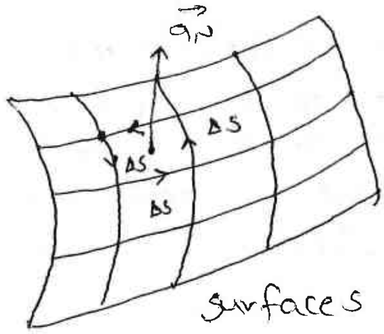


Fig: surface S broken in to incremental surfaces of area ΔS.

Let us consider a surface S as shown in figure which is broken up into incremental surfaces of area ΔS.

If we apply the definition of the curl to one of these incremental surfaces, then

$$\frac{\oint \vec{H}\cdot d\vec{L}_{\Delta S}}{\Delta S} = (\nabla \times \vec{H})_N$$

where the N subscript again indicates the right hand normal to the surface.

The subscript on $dL_{\Delta S}$ indicates that the closed path is the perimeter of an incremental area ΔS. This result may also be written

$$\frac{\oint \vec{H}\cdot dL_{\Delta S}}{\Delta S} = (\nabla \times \vec{H})\cdot \vec{a}_N$$

or, $\oint \vec{H}\cdot dL_{\Delta S} = (\nabla \times \vec{H})\cdot \vec{a}_N \Delta S = (\nabla \times \vec{H})\cdot \Delta S$

where $\vec{a}_N$ is a unit vector in the direction of the right hand normal to ΔS.

Now, let us determine this circulation for every ΔS comprising S and sum the results. As we evaluate the closed line integral for each ΔS, some cancellation will occur because every interior wall is covered once in each direction. The only boundaries on which cancellation cannot occur form the outside boundary, the path enclosing S. Therefore, we have

$$\oint \vec{H}\cdot d\vec{L} = \int_S (\nabla \times \vec{H})\cdot d\vec{S} \longrightarrow (1)$$

where $d\vec{L}$ is taken only on the perimeter of S.

Equation (1) is an identity, holding for any vector field, and is known as Stokes theorem.

3.6 Scalar and Magnetic vector potential :-

Magnetic scalar potential(ψ) is a quantity in classical electromagnetism analogous to electric potential. It is used to specify the magnetic $\vec{H}$ field in cases when there are no free currents. One important use of scalar potential is to determine the magnetic field due to permanent magnets when their magnetization is known.

where there is no free current,

$$\nabla \times \vec{H} = 0$$

so if this holds in simply connected domain we can define a magnetic scalar potential as

$$\vec{H} = -\nabla \varphi.$$

The magnetic vector potential ($\vec{A}$) is a vector field that serves as the potential for the magnetic field. The curl of the magnetic vector potential is the magnetic field.

$$\vec{B} = \nabla \times \vec{A}.$$

The magnetic vector potential contributed by a length $\vec{ds}$ with current I running through it is

$$d\vec{A} = \frac{\mu_0 I}{4\pi r} \vec{ds}.$$

3.7 Magnetic properties of Material Medium :-

Classification	Magnetic Moments	B values	Comments
Diamagnetic	$m_{orb} + m_{spin} = 0$	$B_{int} < B_{appl}$	$B_{int} = B_{appl}$
Paramagnetic	$m_{orb} + m_{spin} = \text{small}$	$B_{int} > B_{appl}$	$B_{int} = B_{appl}$
ferromagnetic	$\lvert m_{spin} \rvert \gg \lvert m_{orb} \rvert$	$B_{int} \gg B_{appl}$	Domains
Anti ferromagnetic	$\lvert m_{spin} \rvert \gg \lvert m_{orb} \rvert$	$B_{int} = B_{appl}$	Adjacent moments oppose
ferrimagnetic	$\lvert m_{spin} \rvert \gg \lvert m_{orb} \rvert$	$B_{int} > B_{appl}$	unequal adjacent moments oppose
Super paramagnetic	$\lvert m_{spin} \rvert \gg \lvert m_{orb} \rvert$	$B_{int} > B_{appl}$	non magnetic matrix. recording tapes.

3.8.1 Magnetic force :-

The force on a filamentary closed circuit is given by

$$\vec{F} = -I \oint \vec{B} \times d\vec{L}$$

Let us assume a uniform magnetic flux density, then $\vec{B}$ may be removed from the integral:

$$\vec{F} = -I B \times \oint d\vec{L}$$

In an electrostatic potential field that $\oint d\vec{L} = 0$ and therefore the force on a closed filamentary circuit in a uniform magnetic field is zero.

If the field is not uniform, the total force need not be zero. The circuit may contain surface currents or volume current density as well. If the total current is divided into filaments, the force on each one is zero and the total force is again zero.

Therefore any real closed circuit carrying direct currents experiences a total vector force of zero in a uniform magnetic field.

3.8.2 Torque :-

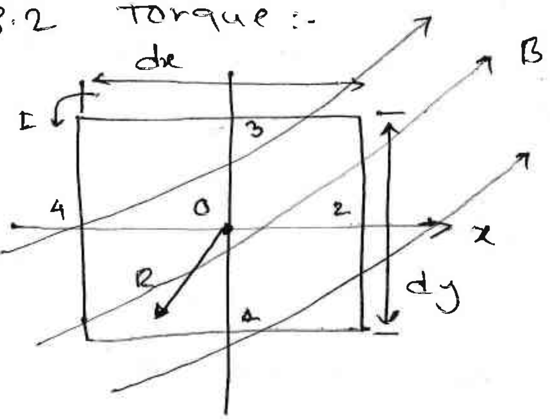
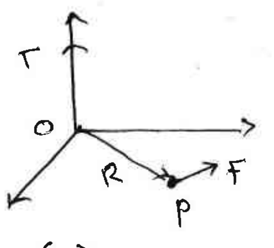


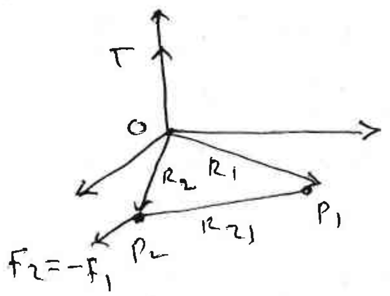
Fig 1: A differential current loop in a magnetic field B. The torque on the loop is $d\vec{T} = I(dx\ dy\ \vec{a}_z) \times \vec{B}_0 = I\, d\vec{s} \times \vec{B}$.

Let us consider both an origin at a about which the torque is to be calculated, and the point at which the force is applied. Let us apply a force $\vec{F}$ at point P, and we establish an origin at O with a rigid lever arm $\vec{R}$ extending from O to P. The torque about point O is a vector whose magnitude is the product of the magnitudes of $\vec{R}$ of $\vec{F}$, and of the sine of the angle between these two vectors.



(a)

Fig: (a) Lever arm $\vec{R}$ extending from an origin O to a point P where force $\vec{F}$ is applied, the torque about O is $\vec{T} = \vec{R} \times \vec{F}$



(b) If $F_2 = -F_1$ then the torque $T = R_{21} \times F_1$ is independent of the choice of origin for R_1 and R_2.

The direction of the vector torque $\vec{T}$ is normal to both the force $\vec{F}$ and the lever arm $\vec{R}$ and is in the direction of progress of a right handed screw as the lever arm is rotated into the force vector through the smaller angle. The torque is expressible as a cross product,

$$\vec{T} = \vec{R} \times \vec{F}$$

The torque about origin as shown in fig (b) is

$$\vec{T} = \vec{R_1} \times \vec{F_1} + \vec{R_2} \times \vec{F_2}$$

where, $\vec{F_1} + \vec{F_2} = 0$

and therefore, $\vec{T} = (\vec{R_1} - \vec{R_2}) \times \vec{F_1} = \vec{R_{21}} \times \vec{F_1}$

If we consider the fig d, a differential current loop in a magnetic field B_0 the torque could be obtained. Initially,

vector force on side 1 is

$$d\vec{F_1} = I\,dx\,\vec{a_x} \times \vec{B_0}$$

or, $d\vec{F_1} = I\,dx\,(B_{0y}\vec{a_z} - B_{0z}\vec{a_y})$

for this side, $\vec{R}$ extends from origin to the midpoint of the side, $\vec{R_1} = -\frac{1}{2}dy\,\vec{a_y}$ and contribution to the total torque is

$$d\vec{T_1} = \vec{R_1} \times d\vec{F_1}$$
$$= -\frac{1}{2}dy\,\vec{a_y} \times I\,dx\,(B_{0y}\vec{a_z} - B_{0z}\vec{a_y})$$
$$= -\frac{1}{2}dx\,dy\,I\,B_{0y}\vec{a_x}$$

The torque contribution on side 3 is found to be the same,

$$d\vec{T_3} = \vec{R_3} \times d\vec{F_3} = \frac{1}{2}dy\,\vec{a_y} \times (-I\,dx\,\vec{a_x} \times \vec{B_0})$$
$$= -\frac{1}{2}dx\,dy\,I\,B_{0y}\vec{a_x}$$
$$= d\vec{T_1}.$$

and $d\vec{T_1} + d\vec{T_3} = -dx\,dy\,I\,B_{0y}\vec{a_x}$

Evaluating torque on sides 2 and 4, we have

$$d\vec{T_2} + d\vec{T_4} = dx\,dy\,I\,B_{0x}\vec{a_y}$$

and total torque is

$$d\vec{T} = I\,dx\,dy\,(B_{0x}\vec{a_y} - B_{0y}\vec{a_x})$$

The quantity within the bracket can be represented by cross product.

$$\therefore d\vec{T} = I\,dx\,dy\,(\vec{a_z} \times \vec{B_0})$$

$$\therefore d\vec{T} = I\,d\vec{s} \times \vec{B} \longrightarrow (1)$$

where ds is vector area.

The differential magnetic dipole moment ($d\vec{m}$) is thus

$$d\vec{m} = I\, d\vec{s} \rightarrow (2)$$

and $\quad d\vec{T} = d\vec{m} \times \vec{B} \rightarrow (3)$

also, $\quad d\vec{T} = d\vec{p} \times \vec{E} \rightarrow (4)$

where, $\vec{p}$ = electric dipole and $\vec{E}$ = electric field.

Also,

$$\vec{T} = I\vec{S} \times \vec{B} = \vec{m} \times \vec{B} \rightarrow (5)$$

3.8.3 Magnetic moment, magnetic dipole, magnetization :-

Let the bound current I_b circulates about a path enclosing a differential area $d\vec{s}$ establishing a dipole moment (m).

$$\therefore \quad m = I_b\, d\vec{s} \rightarrow (1)$$

If there are 'n' magnetic dipoles per unit volume and we consider a volume ΔV then the total magnetic dipole moment is found by the vector sum

$$m_{total} = \sum_{i=1}^{n\Delta V} m_i \rightarrow (2)$$

Magnetization $\vec{M}$ is defined as the magnetic dipole moment per unit volume.

$$\vec{M} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_{i=1}^{n\Delta V} m_i .$$

3.9 Magnetic Boundary conditions :-

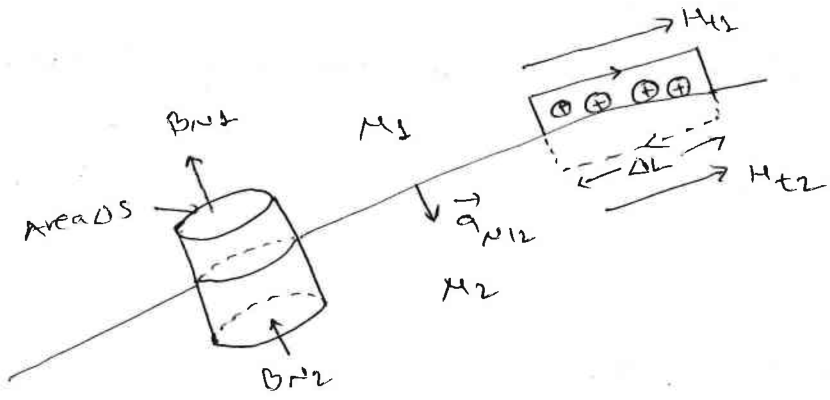


Fig 1: Gaussian surface and a closed path constructed at the boundary between media 1 and 2

Figure shows a boundary between two isotropic homogeneous linear materials with permeabilities μ_1 and μ_2. The boundary condition on the normal components is determined by allowing the surface to cut a small cylindrical gaussian surface.

Applying Gauss's law for the magnetic field

$$\oint_S \vec{B} \cdot d\vec{s} = 0$$

we find that

$$B_{N_1} \Delta S - B_{N_2} \Delta S = 0$$

or, $B_{N_2} = B_{N_1} \rightarrow (1)$

Thus, $H_{N_2} = \frac{\mu_1}{\mu_2} H_{N_1} \rightarrow (2)$

The normal component of $\vec{B}$ is contineous, but the normal component of $\vec{H}$ is discontinuous by the ratio μ_1/μ_2.

For linear magnetic materials,

$$M_{N_2} = \chi_{m2} \frac{\mu_1}{\mu_2} H_{N_1} = \frac{\chi_{m2} \mu_1}{\chi_{m1} \mu_2} M_{N_1} \rightarrow (3)$$

where, χ_m = magnetic susceptibility, μ = permeability, H = magnetic field intensity, M = magnetization.

Applying Ampere's circuital law $\oint \vec{H} \cdot d\vec{L} = I$ about a small closed path in a plane normal to the boundary surface clockwise,

$$H_{t_1} \Delta L - H_{t_2} \Delta L = K \Delta L$$

where K is surface current density so,

$$H_{t_1} - H_{t_2} = K$$

or, $(\vec{H}_1 - \vec{H}_2) \times \vec{a}_{N12} = \vec{K}$

where $\vec{a}_{N12}$ is the unit normal at the boundary directed from region 1 to region 2.

$$\therefore \vec{H}_{t_1} - \vec{H}_{t_2} = \vec{a}_{N12} \times \vec{K}$$

For tangential $\vec{B}$, we have

$$\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K \rightarrow (4)$$

The boundary condition on the tangential component of the magnetization for linear materials is therefore

$$M_{t2} = \frac{\chi_{m2}}{\chi_{m1}} M_{t1} - \chi_{m2} K \rightarrow (5)$$

The last three boundary conditions on the tangential components are much simpler of course, if the surface current density is zero. This is a free current density and it must be zero if neither material is a conductor.

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17 let $v = 2xy^2z^3$ and $\varepsilon = \epsilon_0$. Given point $P(1,2,-1)$, find:

a) V at P

b) $\vec{E}$ at P

c) ρ_v at P

d) The equation of the equipotential surface passing through

e) The equation of the streamline passing through P.

Solⁿ:

a) Given: $V = 2xy^2z^3$

substituting the points on eqⁿ

$V_P = 2 \times 1 \times (2)^2 \times (-1)^3$

$\therefore V_P = -8\ V$

b) using $\vec{E} = -\nabla V = \cancel{-2y^2z^3\vec{a_x}}$

$= -\frac{\partial v}{\partial x}\vec{a_x} + \frac{\partial v}{\partial y}\vec{a_y} + \frac{\partial v}{\partial z}\vec{a_z}$

$= -2y^2z^3\vec{a_x} - 4xyz^3\vec{a_y} - 6xy^2z^2\vec{a_z}$.

at point 'p'.

$\vec{E_p} = 8\vec{a_x} + 8\vec{a_y} - 24\vec{a_z}$ V/m

c) we know

$\rho_v = \nabla \cdot \vec{D} = -\epsilon_0 \nabla^2 V$

$= -\epsilon_0(4xz^3 + 12xy^2z)$ c/m³.

d) At P we know that

$V = -8V$

so the eqⁿ becomes

$2xy^2z^3 = -8$.

$\therefore xy^2z^3 = -4$.

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To find

e) The equation of the streamline;

$$\frac{E_y}{E_x} = \frac{dy}{dx} = \frac{4xyz^3}{2y^2z^3} = \frac{2x}{y}$$

Thus,

$$y\,dy = 2x\,dx$$

and so, $\frac{1}{2}y^2 = x^2 + C_1$

Evaluating at P, we find $C_1 = 1$.

Next

$$\frac{E_z}{E_x} = \frac{dz}{dx} = \frac{6xy^2z^2}{2y^2z^3} = \frac{3x}{z}$$

Thus, $3x\,dx = z\,dz$

and so $\frac{3}{2}x^2 = \frac{1}{2}z^2 + C_2$

Evaluating at P, we find $C_2 = 1$. The streamline is now specified by the equations:

$$y^2 - 2x^2 = 2$$

and $3x^2 - z^2 = 2$

27 Coaxial conducting cylinders are located at $\rho = 0.5$ cm and $\rho = 1.2$ cm. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100V and the outer at 0V, find:

a) the location of the 20V equipotential surface.

b) $E_{\rho\,max}$

c) ϵ_R if the charge per meter length on the inner cylinder is 20nC/m

Soln:

a) ~~From~~ 1

we know that

$$V(\rho) = V_{inn}\left(\frac{\ln \rho_2/\rho}{\ln(\rho_2/\rho_1)}\right) V$$

$$\text{or, } V(\rho) = 100\,\frac{\ln(0.012/\rho)}{\ln(0.012/0.005)}\ V$$

$$\text{or, } 20 = 100\,\frac{\ln(0.012/\rho)}{\ln(2.4)}$$

$$\text{or, } \rho = \frac{0.012}{(2.4)^{0.2}} = 1.01\ \text{cm}.$$

b) $E_{\rho\,max} = ?$

we know,

$$E_\rho = -\frac{\partial v}{\partial \rho} = -\frac{dv}{d\rho} = \frac{100}{\rho \ln(2.4)}$$

whose maximum value will occur at the inner cylinder or at $\rho = 0.5$ cm

$$E_{\rho max} = \frac{100}{0.005 \ln(2.4)} = 2.28 \times 10^4\ \text{V/m} = 22.8\ \text{kV/m}.$$

c) ϵ_R ?

The capacitance per meter length is

$$C = \frac{2\pi\epsilon_0\epsilon_R}{\ln(2.4)} = \frac{Q}{V_0}$$

$$\text{or, } \epsilon_R = \frac{Q\ln(2.4)}{2\pi\epsilon_0 V_0} = \frac{(20\times10^{-9})\ln(2.4)}{2\pi\epsilon_0\,(100)} = 3.15$$

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37 Find $\vec{H}$ in cartesian components at $p(2,3,4)$ if there is a current filament on the z axis carrying 8 mA in the $\vec{a_z}$ direction:

soln:

Applying Biot Savart law,

$$\vec{H_a} = \int_{-\infty}^{\infty} \frac{I\,d\vec{L} \times \vec{a_R}}{4\pi R^2} = \int_{-\infty}^{\infty} \frac{I\,dz\,\vec{a_z} \times [2\vec{a_x} + 3\vec{a_y} + (4-z)\vec{a_z}]}{4\pi(z^2 - 8z + 29)^{3/2}}$$

$$= \int_{-\infty}^{\infty} \frac{I\,dz\,[2\vec{a_y} - 3\vec{a_x}]}{4\pi(z^2 - 8z + 29)^{3/2}}$$

$R = 2^2 + 3^2 + (4-z)^2$
$= 4 + 9 + 16 - 8z + z^2$
$= 29 - 8z + z^2$

$$= \frac{I}{4\pi}\left[\frac{2(2z-8)(2\vec{a_y} - 3\vec{a_x})}{52(z^2 - 8z + 29)^{1/2}}\right]_{-\infty}^{\infty}$$

$$= \frac{I}{26\pi}(2\vec{a_y} - 3\vec{a_x})$$

Then with $I = 8\,mA$, we finally obtain

$$\vec{H_a} = -294\,\vec{a_x} + 196\,\vec{a_y} \;\mu A/m.$$

47 A current sheet $\vec{K} = 8\,\vec{a_x}$ A/m flows in the region $-2 < y < 2$ in the plane $z = 0$. calculate H at $p(0,0,3)$

soln:

Using Biot-Savart law, we write

$$\vec{H_p} = \iint \frac{\vec{K} \times \vec{a_R}\,dx\,dy}{4\pi R^2}$$

$$= \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8\,\vec{a_x} \times (-x\vec{a_x} - y\vec{a_y} + 3\vec{a_z})\,dx\,dy}{4\pi(x^2 + y^2 + 9)^{3/2}}$$

Taking the cross product gives:

$$\vec{H_p} = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8(-y\vec{a_z} - 3\vec{a_y})\,dx\,dy}{4\pi(x^2 + y^2 + 9)^{3/2}}$$

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we note that the z component is antisymmetric in y about the origin (odd parity). since the limits are symmetric, the integral of the z component over y is zero. we are left with

$$\vec{H}_p = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{-24\,\vec{a_y}\,dx\,dy}{4\pi(x^2+y^2+9)^{3/2}}$$

$$= -\frac{6}{\pi}\,\vec{a_y}\int_{-2}^{2} \frac{x}{(y^2+9)\sqrt{x^2+y^2+9}}\Bigg|_{-\infty}^{\infty} dy$$

$$= -\frac{6}{\pi}\,\vec{a_y}\int_{-2}^{2} \frac{2}{y^2+9}\,dy$$

$$= -\frac{12}{\pi}\,\vec{a_y}\,\frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right)\Bigg|_{-2}^{2}$$

$$= -\frac{4}{\pi}(2)(0.59)\,\vec{a_y}$$

$$= -1.5\,\vec{a_y}\ \text{A/m}.$$

57 Let a filamentary current of 5 mA be directed from infi- to the origin on the positive z axis and then bac out to infinity on the positive x axis. Find $\vec{H}$ at P(0,1

Sol^n:

The Biot Savart law is applied to the two wire segments using the following set up:

$$\vec{H}_p = \int \frac{I\,d\vec{L}\times\vec{a_R}}{4\pi R^2} = \int_0^{\infty} \frac{-I\,dz\,\vec{a_z}\times(-z\vec{a_z}+\vec{a_y})}{4\pi(z^2+1)^{3/2}} + \int_0^{\infty} \frac{I\,dx\,a_x\times(-x}{4\pi(x^2+1)^{3/2}}$$

$$= \int_0^{\infty} \frac{I\,dz\,\vec{a_x}}{4\pi(z^2+1)^{3/2}} + \int_0^{\infty} \frac{I\,dx\,\vec{a_z}}{4\pi(x^2+1)^{3/2}}$$

$$= \frac{I}{4\pi}\left[\frac{z\,\vec{a_x}}{\sqrt{z^2+1}}\Bigg|_0^{\infty} + \frac{x\,\vec{a_z}}{\sqrt{x^2+1}}\Bigg|_0^{\infty}\right] = \frac{I}{4\pi}(\vec{a_x}+\vec{a_z}) = 0.4(\vec{a_x}+\vec{a}$$

mA/m

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6) Let $\vec{A} = (3y-z)\vec{a_x} + 2xz\vec{a_y}$ wb/m in a certain region of free space.

a) Show that $\nabla \cdot \vec{A} = 0$

b) At $P(2,-1,3)$, find $\vec{A}, \vec{B}, \vec{H}$ and $\vec{J}$.

Soln:

a) $\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(3y-z) + \frac{\partial}{\partial y} 2xz = 0$

b) At $P(2,-1,3)$, find $\vec{A}, \vec{B}, \vec{H}$ and $\vec{J}$

First $\vec{A_p} = -6\vec{a_x} + 12\vec{a_y}$

Then, using the curl formula in cartesian coordinates,

$\vec{B} = \nabla \times \vec{A}$

$= -2x\vec{a_x} - \vec{a_y} + (2z-3)\vec{a_z}$

Now;

$\vec{B_p} = -4\vec{a_x} - \vec{a_y} + 3\vec{a_z}$ wb/m²

Now,

$\vec{H_p} = (1/\mu_0)\vec{B_p}$

$= -3.2\times10^6\,\vec{a_x} - 8\times10^5\,\vec{a_y} + 2.4\times10^6\,\vec{a_z}$ A/m

then,

$\vec{J} = \nabla \times \vec{H}$

$= \left(\frac{1}{\mu_0}\right)\nabla \times \vec{B}$

$= 0$

as the curl formula in cartesian coordinates shows.

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7) The solenoid contains 400 turns, carries a current $I = 5A$, has a length of 8cm, and a radius $a = 1.2$ cm.

a) find $\vec{H}$ within the solenoid. Assuming the current flows in the $\vec{a}_\phi$ direction, $\vec{H}$ will then be along the positive z direct and will be given by

$$\vec{H} = \frac{NI}{d}\vec{a_z} = \frac{(400)(5)}{0.08}\vec{a_z} = 2.5\times10^4 \text{ A/m}$$

b) If $V_m = 0$ at the origin, specify $V_m(\rho, \phi, z)$ inside the solenoid:

Since $\vec{H}$ is only in the z direction, V_m should vary with z c use

$$\vec{H} = -\nabla V_m = -\frac{dV_m}{dz}\vec{a_z}$$

$$\Rightarrow V_m = -H_z Z + C$$

at $z=0$, $V_m = 0$, so $C = 0$. Therefore $V_m(z) = -2.5\times10^4 z$ A.

c) let $\vec{A} = 0$ at the origin, and specify $\vec{A}(\rho, \phi, z)$ inside the solenoid if the medium is free space. $\vec{A}$ should be in the same direction as the current, and so would have a ϕ component only. Furthermore, since $\nabla\times\vec{A} = \vec{B}$, the curl will be z-directed only. Therefore

$$\nabla\times\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)\vec{a_z} = \mu_0 H_z \vec{a_z}.$$

Then,

$$\frac{\partial}{\partial\rho}(\rho A_\phi) = \mu_0 H_z \rho$$

$$\Rightarrow A_\phi = \frac{\mu_0 H_z \rho}{2} + C.$$

$A_\phi = 0$ at the origin, so $C = 0$. Finally,

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$$\vec{A} = \frac{(4\pi \times 10^{-7})(2.5 \times 10^{4})\rho}{2}\,\vec{a_\phi} = 15.7\,a_\phi \text{ mWb/m}$$

8) planar current sheets of $\vec{K} = 30\vec{a_z}$ A/m and $-30\vec{a_z}$ A/m are located in free space at $x = 0.2$ and $x = -0.2$ respectively. For the region $-0.2 < x < 0.2$

a) Find $\vec{H}$

since we have parallel current sheets carrying equal and opposite currents, we use eq(12), $\vec{H} = \vec{K} \times \vec{a_N}$, where $\vec{a_N}$ is the unit normal directed into the region between currents and where either one of the two currents are used. choosing the sheet at $x = 0.2$, we find

$$\vec{H} = 30\vec{a_z} \times -\vec{a_x} = -30\vec{a_y} \text{ A/m.}$$

b) Obtain and expression for V_m if $V_m = 0$ at $P(0.1, 0.2, 0.3)$: use

$$\vec{H} = -30\vec{a_y} = -\nabla V_m = -\frac{dV_m}{dy}\vec{a_y}$$

so, $\frac{dV_m}{dy} = 30 \Rightarrow V_m = 30y + c_1$

Then, $0 = 30(0.2) + c_2$

$\Rightarrow c_1 = -6$

$\Rightarrow V_m = 30y - 6$ A.

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9) let $\vec{A} = (3y^2 - 2z)\vec{a_x} - 2x^2z\vec{a_y} + (x+2y)\vec{a_z}$ Wb/m in free space. Find $\nabla \times \nabla \times \vec{A}$ at $P(-2, 3, -1)$:

Soln:

First $\nabla \times \vec{A} =$

$$\left(\frac{\partial(x+2y)}{\partial y} - \frac{\partial(-2x^2z)}{\partial z}\right)\vec{a_x} + \left(\frac{\partial(3y^2-2z)}{\partial z} - \frac{\partial(x+2y)}{\partial x}\right)\vec{a_y} +$$

$$\left(\frac{\partial(-2x^2z)}{\partial x} - \frac{\partial(3y^2-2z)}{\partial y}\right)\vec{a_z} = (2+2x^2)\vec{a_x} - 3\vec{a_y} - (4xz+6y)\vec{a_z}$$

Then

$$\nabla \times \nabla \times \vec{A} = \frac{\partial(4xz+6y)}{\partial x}\vec{a_y} - \frac{\partial(4xz+6y)}{\partial y}\vec{a_x}$$

$$= -6\vec{a_x} + 4z\vec{a_y}$$

at P this becomes

$$\nabla \times \nabla \times \vec{A}\big|_P = -6\vec{a_x} - 4\vec{a_y} \text{ Wb/m}^3.$$

Chapter 4

wave equation and wave propagation

4.1 Faraday's Law :-

It states that a time varying magnetic field produces an electromotive force which may establish a current in a suitable closed circuit. An electromotive force is merely a voltage that arises from conductors moving in a magnetic field or from changing magnetic fields.

i.e, $emf = -\frac{d\phi}{dt}$ v. $\rightarrow$ (1)

The negative sign indicates that the emf is in such a direction as to produce a current whose flux, if added to the original flux would reduce the magnitude of the emf. If closed path is made by N turns then

$$emf = -N\frac{d\phi}{dt} \rightarrow (2) \quad \cancel{\text{(transformer emf)}}$$

we also know that,

$$emf = \oint \vec{E}\cdot d\vec{L} \rightarrow (3)$$

~~from~~ Replacing eqn (1) for ϕ with surface integral of $\vec{B}$

$$emf = \oint \vec{E}\cdot d\vec{L} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} \rightarrow (4)$$

If we consider stationary path, the magnetic flux is the only time varying quantity on the right side of (4) and a partial derivative may be taken under the integral sign,

$$emf = \oint \vec{E}\cdot d\vec{L} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} \quad \text{(transformer emf)}$$

Applying Stokes theorem

$$\int_S (\nabla\times\vec{E})\cdot d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

$$\text{or, } (\nabla\times\vec{E})\cdot d\vec{s} = -\frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

$$\therefore \nabla\times\vec{E} = -\partial\vec{B}/\partial t \rightarrow (5)$$

Motional emf:

The force on a charge Q moving at a velocity 'v' in a magnet field $\vec{B}$ is

$$\vec{F} = Qv \times \vec{B}$$

or, $\frac{\vec{F}}{Q} = v \times \vec{B} \rightarrow (1)$

or, $E_m = v \times \vec{B}$ where E_m = motional Electric field intens

The motional emf produced by the moving conductor is

$$emf = \oint \vec{E_m} \cdot d\vec{L} = \oint (v \times \vec{B}) \cdot d\vec{L} \rightarrow (2)$$ motional emf.

or, $\oint (v \times \vec{B}) \cdot d\vec{L} = \int_d^0 vB\,dx = -BVd$

4.2 Displacement current:-

The total displacement current crossing any given surface i expressed by the surface Integral

$$I_d = \int_S \vec{J_d} \cdot d\vec{S} = \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}.$$

we may obtain the current in the loop as

$$I = -\omega C V_0 \sin \omega t$$

$$= -\omega \frac{\epsilon S}{d} V_0 \sin \omega t$$

So, $I_d = \frac{\partial D}{\partial t} S = -\omega \frac{\epsilon S}{d} V_0 \sin \omega t$

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4.3.1 Maxwell's eqn in point form:-

Maxwell's eqn for time varying fields is given as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow (1)$$

and $$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \rightarrow (2)$$

The remaining two equations are unchanged from their non time varying form:

$$\nabla \cdot \vec{D} = \rho_v \rightarrow (3)$$

$$\nabla \cdot \vec{B} = 0 \rightarrow (4)$$

The auxiliary equations relating $\vec{D}$ and $\vec{E}$ is

$$\vec{D} = \epsilon \vec{E} \rightarrow (5)$$

relating $\vec{B}$ and $\vec{H}$

$$\vec{B} = \mu \vec{H} \rightarrow (6)$$

Defining conduction current density,

$$\vec{J} = \sigma \vec{E} \rightarrow (7)$$

Defining convection current density in terms of the volume charge density ρ_v

$$\vec{J} = \rho_v \vec{v} \rightarrow (8)$$

are also required to define and relate the quantities appearing in Maxwell's eqns.

Replacing (5) and (6) with the relationships involving the polarization and magnetization fields

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \rightarrow (9)$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) \rightarrow (10)$$

for linear materials we may relate $\vec{P}$ to $\vec{E}$

$$\vec{P} = \chi_e \epsilon_0 \vec{E} \rightarrow (11)$$

and $\vec{M}$ to $\vec{H}$

$$\vec{M} = \chi_m \vec{H} \rightarrow (12)$$

Now the Lorentz force equation written in point form as the force per unit volume, $\vec{f} = \rho_v (\vec{E} + \vec{v} \times \vec{B}) \rightarrow (13)$

4.3.2 Maxwell's equations in Integral form :-

Integrating the eqn

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

over a surface and applying stokes theorem

$$\oint \vec{E} \cdot d\vec{L} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \rightarrow (1).$$

Integrating the eqn $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ over a surface and applying stokes theorem

$$\oint \vec{H} \cdot d\vec{L} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s} \rightarrow (2)$$

Gauss's law for the electric and magnetic fields are obtained by integrating the equating

$$\nabla \cdot \vec{D} = \rho_v$$

$$\nabla \cdot \vec{B} = 0$$

throughout a volume and using the divergence theorem

$$\oint_S \vec{D} \cdot d\vec{s} = \int_{vol} \rho_v \, dv \rightarrow (3)$$

$$\oint_S \vec{B} \cdot d\vec{s} = 0 \rightarrow (4)$$

4.4 wave propagation in dielectrics :-

Let us consider a uniform plane wave to propagate in a dielectric of permittivity ϵ and permeability μ. The medium is assumed to be homogeneous (having constant μ and ϵ) and isotropic (μ and ϵ are invariant with field orientatio

The helmholtz eqn is

$$\nabla^2 \vec{E_s} = -k^2 \vec{E_s} \rightarrow (1)$$

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where the wavenumber is a function of the material properties :

$$k = \omega\sqrt{\mu\varepsilon} = k_0\sqrt{\mu_r\varepsilon_r} \rightarrow (2)$$

for E_{xs} we have

$$\frac{d^2 E_{xs}}{dz^2} = -k^2 E_{xs} \rightarrow (3)$$

k can be complex valued and so can be represented as :

$$jk = \alpha + j\beta \rightarrow (4)$$

we also know that

$$E_{xs} = E_{x0} e^{-jkz} = E_{x0} e^{-\alpha z} e^{-j\beta z} \rightarrow (5)$$

multiplying by $e^{j\omega t}$ and taking the real part yields a form of the field that can be observed as:

$$E_x = E_{x0} e^{-\alpha z} \cos(\omega t - \beta z) \rightarrow (6)$$

when a material physical process ~~is~~ affects the wave electric field, it can be described through a complex permittivity.

$$\therefore \varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon_0(\varepsilon_r' - j\varepsilon_r'') \rightarrow (7)$$

Losses arising from the response of the medium to the magnetic field is described through complex permeability.

$$\mu = \mu' - j\mu'' = \mu_0(\mu_r' - j\mu_r''). \rightarrow (8)$$

substituting eq^n (7) in eq^n (2)

$$k = \omega\sqrt{\mu(\varepsilon' - j\varepsilon'')} = \omega\sqrt{\mu\varepsilon'}\sqrt{1 - j\frac{\varepsilon''}{\varepsilon'}} \rightarrow (9)$$

Here, α and β are found by taking the real and imaginary parts of jk from eq^n (9)

$$\therefore \alpha = Re\{jk\} = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left(\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right)^{1/2} \rightarrow (10)$$

$$\beta = Im\{jk\} = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left(\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right)^{1/2} \rightarrow (11)$$

Here, $\varepsilon''/\varepsilon'$ is called loss tangent.
we know that wave phase velocity is given by

$$v_p = \frac{\omega}{\beta} \rightarrow (12)$$

we also know that wavelength $(\lambda) = 2\pi/\beta \rightarrow (13)$
since we have a uniform plane wave, the magnetic field is found through

$$H_{ys} = \frac{E_{xo}}{\eta} e^{-\alpha z} e^{-j\beta z}$$

where, the intrinsic impedance is now a complex quantity

$$\eta = \sqrt{\frac{\mu}{\varepsilon' - j\varepsilon''}} = \sqrt{\frac{\mu}{\varepsilon'}} \frac{1}{\sqrt{1 - j(\varepsilon''/\varepsilon')}} \rightarrow (14)$$

case 1: on lossless or perfect dielectric, in which $\varepsilon'' = 0$, and so $\varepsilon = \varepsilon'$, from eqn (10)

$$\alpha = 0$$

and from eqn (11)

$$\beta = \omega\sqrt{\mu\varepsilon'}$$

with $\alpha = 0$, the real field assumes the form

$$\vec{E_x} = \vec{E_{xo}} \cos(\omega t - \beta z)$$

so, $v_p = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{\mu\varepsilon'}} = \dfrac{c}{\sqrt{\mu_r \varepsilon_r'}}$

and $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{\omega\sqrt{\mu\varepsilon'}} = \dfrac{1}{f\sqrt{\mu\varepsilon'}} = \dfrac{c}{f\sqrt{\mu_r\varepsilon_r'}} = \dfrac{\lambda_0}{\sqrt{\mu_r\varepsilon_r'}}$ (lossless medium)

and $\vec{H_y} = \dfrac{\vec{E_{xo}}}{\eta} \cos(\omega t - \beta z)$

where, the intrinsic impedance is

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \rightarrow (15)$$

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Case II: Lossy dielectric:

If we consider the case of conductive materials, in which currents are formed by the motion of free electrons or holes under the influence of an electric field. The relation is

$$\vec{J} = \sigma \vec{E}$$

where, σ is the material conductivity.

With finite conductivity, the wave loses power through resistive heating of the material.

considering Maxwell curl eqn using eqn (7)

$$\nabla \times \vec{H_s} = j\omega(\epsilon' - j\epsilon'')\vec{E_s} = \omega\epsilon''\vec{E_s} + j\omega\epsilon'\vec{E_s} \rightarrow (16)$$

If conduction current is included:

$$\nabla \times \vec{H_s} = \vec{J_s} + j\omega\epsilon\vec{E_s} \rightarrow (17)$$

also, $\vec{J_s} = \sigma\vec{E_s}$ and interpret eqn (17) as ϵ'.

$$\nabla \times \vec{H_s} = (\sigma + j\omega\epsilon')\vec{E_s} = \vec{J_{cs}} + \vec{J_{ds}} \rightarrow (18)$$

where, $\vec{J_{cs}} = \sigma\vec{E_s}$ and $\vec{J_{ds}} = j\omega\epsilon'\vec{E_s}$.

comparing eqn (16) and (18)

$$\epsilon'' = \sigma/\omega \rightarrow (19)$$

Also, the ratio of conduction current density to displacement current density is

$$\frac{J_{cs}}{J_{ds}} = \frac{\epsilon''}{j\epsilon'} = \frac{\sigma}{j\omega\epsilon'} \rightarrow (20)$$

also, loss tangent

$$\tan\theta = \frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon'} \rightarrow (21)$$

For a small loss tangent, $\epsilon''/\epsilon' \ll 1$, for a good dielectric and consider a conductive material for which

$$\epsilon'' = \sigma/\omega$$

then eqn (?) becomes,

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$$jk = j\omega\sqrt{\mu\varepsilon'}\sqrt{1-j\frac{\sigma}{\omega\varepsilon'}} \rightarrow (22)$$

we may expand the second radical using the binomial theorem

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

where $|x| \ll 1$. we identify x as $-j\sigma/\omega\varepsilon'$ and n as $1/2$ and thus,

$$jk = j\omega\sqrt{\mu\varepsilon'}\left[1-j\frac{\sigma}{2\omega\varepsilon'}+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon'}\right)^2+\cdots\right] = \alpha+j\beta$$

Now for a good dielectric

$$\alpha = Re(jk) = j\omega\sqrt{\mu\varepsilon'}\left(-j\frac{\sigma}{2\omega\varepsilon'}\right) = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon'}} \rightarrow (23)$$

and $\beta = Im(jk) = \omega\sqrt{\mu\varepsilon'}\left[1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon'}\right)^2\right] \rightarrow (24)$

comparing (23) & (24) with low loss conditions, we can obtain α and β for transmission lines.

Now, the second term of eqn (24) is very small so,

$$\beta = \omega\sqrt{\mu\varepsilon'} \rightarrow (25)$$

Now, $\eta = \sqrt{\frac{\mu}{\varepsilon'}}\left[1-\frac{3}{8}\left(\frac{\sigma}{\omega\varepsilon'}\right)^2+j\frac{\sigma}{2\omega\varepsilon'}\right] \rightarrow (45)$

$$\therefore \eta = \sqrt{\frac{\mu}{\varepsilon'}}\left(1+j\frac{\sigma}{2\omega\varepsilon'}\right) \rightarrow (26)$$

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4.5 propagation of plane waves in good conductor :-

The general expression for the propagation constant is

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1-j\frac{\sigma}{\omega\epsilon'}}$$

which we immediately simplify to obtain

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{-j\frac{\sigma}{\omega\epsilon'}}$$

or, $jk = j\sqrt{-j\omega\mu\sigma}$

but $-j = 1\angle -90^\circ$

and $\sqrt{1\angle -90^\circ} = 1\angle -45^\circ = \frac{1}{\sqrt{2}}(1-j)$

$$\therefore jk = j(1-j)\sqrt{\frac{\mu\omega\sigma}{2}} = (1+j)\sqrt{\pi f\mu\sigma} = \alpha + j\beta \rightarrow (1)$$

Hence, $\alpha = \beta = \sqrt{\pi f\mu\sigma} \rightarrow (2)$

If we again assume only an E_x component traveling in the $+z$ direction, then

$$E_x = E_{x0}\, e^{-z\sqrt{\pi f\mu\sigma}} \cos(\omega t - z\sqrt{\pi f\mu\sigma}) \rightarrow (3)$$

we may tie this field in the conductor to an external field at the conductor surface. we let the region $z>0$ be the good conductor and the region $z<0$ be a perfect dielectric. At the boundary surface $z=0$, eqn (3) becomes

$$E_x = E_{x0}\cos\omega t \quad (z=0)$$

since, displacement current is negligible,

$$\vec{J} = \sigma\vec{E}$$

Thus, $J_x = \sigma E_x = \sigma E_{x0}\, e^{-z\sqrt{\pi f\mu\sigma}} \cos(\omega t - z\sqrt{\pi f\mu\sigma}) \rightarrow$

If negative exponential term is considered,

$$\tau = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

This distance is denoted by δ and is termed the depth of penetration or the skin depth,

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha} = \frac{1}{\beta} \rightarrow (5)$$

or, $\alpha = \beta = \frac{1}{\delta} = \sqrt{\pi f \mu \sigma}$

we know, $\beta = 2\pi/\lambda$

so, $\lambda = 2\pi\delta$

and $v_p = \omega/\beta = \omega\delta \rightarrow (6)$

4.6. Poynting's theorem and wave power:-

Let us assume that the medium be conductive:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \rightarrow (1)$$

Next, taking scalar product on both sides with $\vec{E}$

$$\vec{E} \cdot \nabla \times \vec{H} = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \rightarrow (2)$$

also, $\nabla \cdot (\vec{E} \times \vec{H}) = -\vec{E} \cdot \nabla \times \vec{H} + \vec{H} \cdot \nabla \times \vec{E} \rightarrow (3)$

using (3) in (2)

$$\vec{H} \cdot \nabla \times \vec{E} - \nabla \cdot (\vec{E} \times \vec{H}) = \vec{J} \cdot \vec{E} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

also, we know that

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\therefore -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \nabla \cdot (\vec{E} \times \vec{H}) = \vec{J} \cdot \vec{E} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

or, $-\nabla \cdot (\vec{E} \times \vec{H}) = \vec{J} \cdot \vec{E} + \cancel{\vec{E} \cdot \frac{\partial \vec{D}}{\partial t}}\ \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} \rightarrow (4)$

The two time derivatives in eq^n (4) can be rearranged as follows:

$$\varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}\cdot\vec{E}\right) \rightarrow (5)$$

and $\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}\cdot\vec{H}\right) \rightarrow (6)$

with these, eq^n (4) becomes

$$-\nabla\cdot(\vec{E}\times\vec{H}) = \vec{J}\cdot\vec{E} + \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}\cdot\vec{E}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}\cdot\vec{H}\right) \rightarrow (7)$$

Finally, we integrate eq^n (7) throughout a volume.

$$-\int_{vol}\nabla\cdot(\vec{E}\times\vec{H})\,dv = \int_{vol}\vec{J}\cdot\vec{E}\,dv + \int_{vol}\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}\cdot\vec{E}\right)dv$$
$$+\int_{vol}\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}\cdot\vec{H}\right)dv$$

Applying divergence theorem to the left hand side, and converting the volume integral there into an integral over the surface that encloses the volume.

Thus,

$$-\oint_{area}(\vec{E}\times\vec{H})\cdot d\vec{s} = \int_{vol}\vec{J}\cdot\vec{E}\,dv + \frac{d}{dt}\int_{vol}\frac{1}{2}\vec{D}\cdot\vec{E}\,dv$$
$$+\frac{d}{dt}\int_{vol}\frac{1}{2}\vec{B}\cdot\vec{H}\,dv \rightarrow (8)$$

This equation is known as poynting's theorem.

4.7 Reflection of plane wave at normal and oblique incidence:

4.7.1 Reflection of plane wave at normal incidence :-

Let two regions are composed of two different materials. we again assume that we have only a single vector component of the electric field intensity.

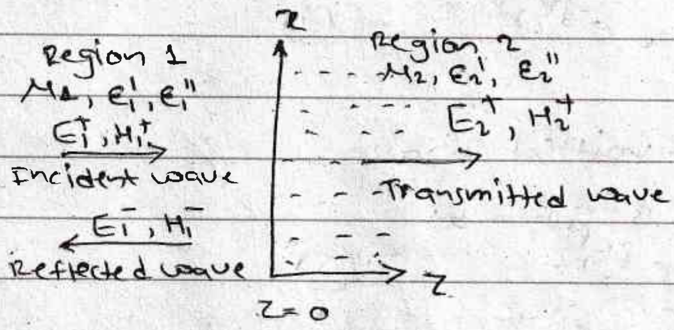

Fig: plane wave incident on a boundary establishes reflected and transmitted waves having the indicated propagation direction.

Refering to figure, we define region 1 (ε_1, μ_1) as the half-space for which $z<0$; region 2 (ε_2, μ_2) is the half-space for which $z>0$. Initially we establish a wave in region 1, traveling in the $+z$ direction, and linearly polarized along x.

$$\mathcal{E}_{x1}^{+}(z,t) = E_{x10}^{+} e^{-\alpha_1 z} \cos(\omega t - \beta_1 z)$$

In phasor form, this is

$$E_{xs1}^{+}(z) = E_{x10}^{+} e^{-jkz} \longrightarrow (1)$$

where we take E_{x10}^{+} as real. The subscript 1 identifies the region, and the superscript + indicates a positively traveling wave. Associated with $E_{xs1}^{+}(z)$ is a magnetic field in the y-direction,

$$H_{ys1}^{+}(z) = \frac{1}{\eta_1} E_{x10}^{+} e^{-jk_1 z} \rightarrow (2)$$

where, k_1 and η_1 are complex unless ϵ_1'' (or σ_1) is zero.

It is incident wave.

Energy is transmitted across the boundary surface at $z=0$ into region 2. The phasor electric and magnetic fields for this wave are

$$E^{+}_{xs2}(z) = E^{T}_{x20}\, e^{-jk_2 z} \rightarrow (3)$$

$$H^{+}_{ys2}(z) = \frac{1}{\eta_2} E^{T}_{x20}\, e^{-jk_2 z} \rightarrow (4)$$

This is called transmitted wave.

The reflected wave is given by

$$E^{-}_{xs1}(z) = E^{-}_{x10}\, e^{jk_1 z} \rightarrow (5)$$

$$H^{-}_{ys1}(z) = -\frac{E^{-}_{x10}}{\eta_1}\, e^{jk_1 z} \rightarrow (6)$$

Here, $E_{xs1} = E_{xs2} \quad (z=0)$

or, $E^{+}_{xs1} + E^{-}_{xs1} = E^{+}_{xs2} \quad (z=0)$

$$\therefore\ E^{+}_{x10} + E^{-}_{x10} = E^{T}_{x20} \rightarrow (7)$$

Furthermore,

$$H_{ys1} = H_{ys2} \quad (z=0)$$

or, $H^{+}_{ys1} + H^{-}_{ys1} = H^{+}_{ys2} \quad (z=0)$

$$\therefore\ \frac{E^{+}_{x10}}{\eta_1} - \frac{E^{-}_{x10}}{\eta_1} = \frac{E^{+}_{x20}}{\eta_2} \rightarrow (8)$$

Solving (8) for E^{+}_{x20} and substituting into (7) we find

$$E^{+}_{x10} + E^{-}_{x10} = \frac{\eta_2}{\eta_1} E^{+}_{x10} - \frac{\eta_2}{\eta_1} E^{-}_{x10}$$

or, $E^{-}_{x10} = E^{+}_{x10}\, \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

The ratio of the amplitudes of the reflected and incident electric fields defines the reflection coefficient given by

$$\Gamma = \frac{E^{-}_{x10}}{E^{+}_{x10}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = |\Gamma|\, e^{j\phi}. \rightarrow (9)$$

The transmission coefficient τ is given by

$$\tau = \frac{E^+_{x20}}{E^+_{x10}} = \frac{2\eta_2}{\eta_1+\eta_2} = 1+\Gamma = |\tau| e^{j\phi_i}.$$

47.2 plane wave reflection at oblique incidence angles :-

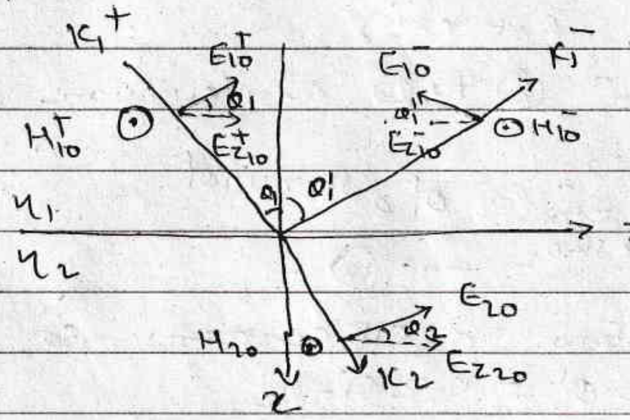

Fig: p-polarization with $\vec{E}$ in the plane of Incidence.

The incident, reflected and transmitted fields in phasor form are :

$$\vec{E}^+_{s1} = \vec{E}^+_{10} e^{-j\vec{k}_1^+ \cdot r} \rightarrow (1)$$

$$\vec{E}^-_{s1} = \vec{E}^-_{10} e^{-j\vec{k}_1^- r} \rightarrow (2)$$

$$\vec{E}_{s2} = \vec{E}_{20} e^{-jk_2 r} \rightarrow (3)$$

where,

$$\vec{k}_1^+ = k_1(\cos\theta_1 \vec{a}_x + \sin\theta_1 \vec{a}_z) \rightarrow (4)$$

$$\vec{k}_1^- = k_1(-\cos\theta_1' \vec{a}_x + \sin\theta_1' \vec{a}_z) \rightarrow (5)$$

$$\vec{k}_2 = k_2(\cos\theta_2 \vec{a}_x + \sin\theta_2 \vec{a}_z) \rightarrow (6)$$

and where

$$\vec{r} = x\vec{a}_x + z\vec{a}_z \rightarrow (7)$$

The wavevector magnitudes are $k_1 = \omega\sqrt{\epsilon_{r1}}/c$ and $k_2 = \omega\sqrt{\epsilon_{r2}}/c = n_2\omega/c$.

Now, to evaluate the boundary condition that require continuous tangential electric field, we need to find the components of the electric fields (z components) that are parallel to the interface. projecting all $\vec{E}$ field in the z direction, and using eqn (1) to (7) we find

$$E_{zs1}^{+} = E_{z10}^{+} e^{-jk_1^{+}\cdot r} = E_{10}^{+} \cos\theta_1 e^{-jk_1(x\cos\theta_1 + z\sin\theta_1)} \rightarrow$$

$$E_{zs1}^{-} = E_{z10}^{-} e^{-jk_1^{-}\cdot r} = E_{10}^{-} \cos\theta_1' e^{jk_1(x\cos\theta_1' - z\sin\theta_1')} \rightarrow$$

$$E_{zs2} = E_{z20} e^{-jk_2\cdot r} = E_{20} \cos\theta_2 e^{-jk_2(x\cos\theta_2 + z\sin\theta_2)} \rightarrow$$

$$E_{zs1}^{+} + E_{zs1}^{-} = E_{zs2} \quad (\text{at } x=0)$$

we now substitute eqn (8) to (10) and evaluate the result at $x=0$ to obtain

$$E_{10}^{+} \cos\theta_1 e^{-jk_1 z\sin\theta_1} + E_{10}^{-} \cos\theta_1' e^{-jk_1 z\sin\theta_1'}$$
$$= E_{20} \cos\theta_2 e^{-jk_2 z\sin\theta_2} \rightarrow (11)$$

Here, E_{10}^{+}, E_{10}^{-} and E_{20} are all constants so,

$$k_1 z\sin\theta_1 = k_1 z\sin\theta_1' = k_2 z\sin\theta_2$$

also, $\theta_1' = \theta_1$ so,

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 \rightarrow (12)$$

It is known as snell's law of refraction.

Also, $k = n\omega/c$ so,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \rightarrow (13)$$

At the boundary, the field amplitudes are related through $H_{10}^{+} + H_{10}^{-} = H_{20} \rightarrow (14)$

using fact $\theta_1' = \theta_1$ and eqn (11) becomes

$$E_{10}^{+} \cos\theta_1 + E_{10}^{-} \cos\theta_1 = E_{20} \cos\theta_2 \rightarrow (15)$$

using the medium intrinsic impedances,

$E_{10}^{+}/H_{10}^{+} = \eta_1$ and $E_{20}^{+}/H_{20}^{+} = \eta_2$.

so eqn (14) becomes

$$\frac{E_{10}^{+}\cos\theta_1}{\eta_{1p}} - \frac{E_{10}^{-}\cos\theta_1}{\eta_{1p}} = \frac{E_{20}^{+}\cos\theta_2}{\eta_{2p}} \rightarrow (16)$$

Now effective impedances can be written as

$$\eta_{1p} = \eta_1 \cos\theta_1 \rightarrow (17)$$

$$\eta_{2p} = \eta_2 \cos\theta_2 \rightarrow (18)$$

using eq^{ns} (15) & (16) we can find

$$\Gamma_p = \frac{E_{10}^-}{E_{10}^+} = \frac{\eta_{2p} - \eta_{1p}}{\eta_{2p} + \eta_{1p}} \rightarrow (19)$$

$$\tau_p = \frac{E_{20}}{E_{10}^+} = \frac{2\eta_{2p}}{\eta_{2p} + \eta_{1p}} \left(\frac{\cos\theta_1}{\cos\theta_2} \right) \rightarrow (20)$$

similarly for S- polarization,

$$\Gamma_s = \frac{E_{y10}^-}{E_{y10}^+} = \frac{\eta_{2s} - \eta_{1s}}{\eta_{2s} + \eta_{1s}} \rightarrow (21)$$

$$\tau_s = \frac{E_{y20}}{E_{y10}^+} = \frac{2\eta_{2s}}{\eta_{2s} + \eta_{1s}} \rightarrow (22)$$

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Numericals :-

17 A point charge, $Q = -0.3\mu C$ and $m = 3\times10^{-16}$ kg is moving through the field $\vec{E} = 30\,\vec{a}_z$ V/m. use eq^n ① and newton's laws to develop the appropriate differential equations and solve them, subject to the initial conditions at $t=0$: $\vec{v} = 3\times10^5\,\vec{a}_x$ m/s at the origin. At $t = 3\mu s$, find:

a) the position $P(x,y,z)$ of the charge

b) the velocity v

c) the kinetic energy of the charge.

Sol^n;

a) The force on the charge is given by $\vec{F} = q\vec{E}$. and newton's second law becomes:

$$\vec{F} = m\vec{a}$$

$$= m\frac{d^2\vec{z}}{dt^2}$$

$$= q\vec{E}$$

or, $\vec{F} = (-0.3\times10^{-6})(30\,\vec{a}_z)$

describing motion of the charge in the z direction. The initial velocity in x is constant and so no force is applied in that direction. we integrate once:

$$\frac{dz}{dt} = v_z = \frac{qE}{m}t + C_1$$

The initial velocity along z, $v_z(0)$ is zero, and so $C_1 = 0$. Integrating a second time yields the z coordinate

$$z = \frac{qE}{2m}t^2 + C_2$$

The charge lies at the origin at $t=0$ and so $C_2 = 0$. Introducing the given values, we find

$$z = \frac{(-0.3\times10^{-6})(30)}{2\times3\times10^{-16}}t^2 = -1.5\times10^{10}\,t^2 \text{ m}.$$

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At $t = 3\,\mu s$, $z = -(1.5\times10^{10})(3\times10^{-6})^2 = -0.135$ cm.

Now, considering the initial constant velocity in x, the charge in $3\mu s$ attains an x coordinate of $x = vt$

or, $x = vt = (3\times10^5)(3\times10^{-6}) = 0.9$ m

In summary, at $t = 3\,\mu s$ we have $p(x,y,z) = (0.9, 0, -0$

b) we know,

$$V_z = \frac{qE}{m}t = -(3\times10^{10})(3\times10^{-6}) = -9\times10^4 \text{ m/s}$$

Including the initial x-directed velocity, we finally obtain

$$\vec{v} = 3\times10^5 \vec{a_x} - 9\times10^4 \vec{a_z} \text{ m/s}.$$

c) the kinetic energy of the charge.

Here,

$$K.E = \frac{1}{2}m|v|^2 = \frac{1}{2}(3\times10^{-16})(1.13\times10^5)^2 = 1.5\times10^{-5} \text{ J}.$$

27 A point charge for which $Q = 2\times10^{-16}$ C and $m = 5\times10^{-26}$ kg is moving in the combined fields $\vec{E} = 100\vec{a_x} - 200\vec{a_y} + 300\vec{a_z}$ and $\vec{B} = -3\vec{a_x} + 2\vec{a_y} - \vec{a_z}$ mT. If the charge velocity at $t=0$ $\vec{v}(0) = (2\vec{a_x} - 3\vec{a_y} - 4\vec{a_z})\times10^5$ m/s.

a) give the unit vector showing the direction in which the charge is accelerating at $t=0$.

b) find the kinetic energy of the charge at $t=0$

Soln:-

a) $\vec{F}(t=0) = q[\vec{E} + (v(0)\times\vec{B})]$

$$\vec{v}(0)\times\vec{B} = (2\vec{a_x} - 3\vec{a_y} - 4\vec{a_z})10^5 \times (-3\vec{a_x} + 2\vec{a_y} - \vec{a_z})10^{-3}$$

$$= 1100\vec{a_x} + 1400\vec{a_y} - 500\vec{a_z}.$$

So, the force in newtons becomes

$$\vec{F}(0) = (2\times10^{-16})\left[(100+1100)\vec{a_x} + (1400-200)\vec{a_y} + (300-500)\right.$$

$$= 4\times10^{-14}\left[6\vec{a_x} + 6\vec{a_y} - \vec{a_z}\right]$$

The unit vector that gives the acceleration direction is found from the force to be

$$\vec{a_F} = \frac{6\vec{a_x} + 6\vec{a_y} - \vec{a_z}}{\sqrt{73}}$$

$$= 0.7\vec{a_x} + 0.7\vec{a_y} - 0.12\vec{a_z}$$

b) find the kinetic energy of the charge at $t=0$:

$$K.E = \frac{1}{2} m|\vec{v}(0)|^2$$

$$= \frac{1}{2}(5\times10^{-26}\,kg)(5.39\times10^{5}\,m/s)^2$$

$$= 7.25\times10^{-15}\,J$$

$$= 7.25\,fJ.$$

37 A solid conducting filament extends from $x=-b$ to $x=b$ along the line $y=2, z=0$. This filament carries a current of 3A in the $\vec{a_x}$ direction. An infinite filament on the z axis carries 5A in the $\vec{a_z}$ direction. obtain an expression for the torque exerted on the finite conductor about an origin located at $(0,2,0)$: ~~The~~

Sol^n;

The differential force on the wire segment arising from the field from the infinite wire is

$$d\vec{F} = 3dx\,\vec{a_x} \times \frac{5\mu_0}{2\pi\rho}\vec{a_\phi}$$

$$= -\frac{15\mu_0\cos\phi\,dx}{2\pi\sqrt{x^2+4}}\vec{a_z} = -\frac{15\mu_0 x\,dx}{2\pi(x^2+4)}\vec{a_z}$$

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so now the differential torque about the (0,2,0) origin is

$$d\vec{T} = \vec{R_T} \times d\vec{F} = x\vec{a_x} \times \left(-\frac{15\mu_0 x\, dx}{2\pi(x^2+4)}\, \vec{a_z}\right)$$

$$= \frac{15\mu_0 x^2\, dx}{2\pi(x^2+4)}\, \vec{a_y}$$

The torque is then

$$\vec{T} = \int_{-b}^{b} \frac{15\mu_0 x^2\, dx}{2\pi(x^2+4)}\, \vec{a_y}$$

$$= \frac{15\mu_0}{2\pi}\, \vec{a_y} \left[x - 2\tan^{-1}\left(\frac{x}{2}\right)\right]_{-b}^{b}$$

$$= (6\times10^{-6})\left[b - 2\tan^{-1}\left(\frac{b}{2}\right)\right] \vec{a_y} \text{ N·m.}$$

47 A current of 6A flows from M(2,0,5) to N (5,0,5) in a straight solid conductor in free space. An infinite current filament lies along the z axis and carries 50 A in the $\vec{a_z}$ direction. Compute the vector torque on the wire segment using

a) an origin at (0,0,5)
b) an origin at (0,0,0)
c) an origin at (3,0,0)

Soln:

The $\vec{B}$ field from the long wire at the short wire is

$\vec{B} = (\mu_0 I_z \vec{a_y})/(2\pi x)$ T.

Then the force acting on a differential length of the wire segment is

$$d\vec{F} = I_w d\vec{L} \times \vec{B} = I_w\, dx\, \vec{a_x} \times \frac{\mu_0 I_z}{2\pi x}\, \vec{a_y} = \frac{\mu_0 I_w I_z}{2\pi x}\, dx\, \vec{a_z}$$

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Now the differential torque about (0,0,5) will be

$$d\vec{T} = \vec{R_T} \times d\vec{F}$$

$$= x\vec{a_x} \times \frac{\mu_0 I_w I_z}{2\pi x} dx\,\vec{a_z}$$

$$= -\frac{\mu_0 I_w I_z}{2\pi} dx\,\vec{a_y}$$

The net torque is now found by integrating the differential torque over the length of the wire segment:

$$\vec{T} = \int_2^5 -\frac{\mu_0 I_w I_z}{2\pi} dx\,\vec{a_y}$$

$$= -3\frac{\mu_0 (6)(50)}{2\pi}\vec{a_y}$$

$$= -1.8 \times 10^{-4}\,\vec{a_y}\ \text{N·m}$$

b) Here, the only modification is in $\vec{R_T}$, which is now $\vec{R_T} = x\vec{a_x} + 5\vec{a_z}$ so now,

$$d\vec{T} = \vec{R_T} \times d\vec{F}$$

$$= [x\vec{a_x} + 5\vec{a_z}] \times \frac{\mu_0 I_w I_z}{2\pi x} dx\,\vec{a_z}$$

$$= -\frac{\mu_0 I_w I_z}{2\pi} dx\,\vec{a_y}$$

Everything from here is the same as in part a, so again

$$\vec{T} = -1.8 \times 10^{-4}\,\vec{a_y}\ \text{N·m}.$$

c) In this case,

$\vec{R_T} = (x-3)\vec{a_x} + 5\vec{a_z}$ and the differential torque

$$d\vec{T} = [(x-3)\vec{a_x} + 5\vec{a_z}] \times \frac{\mu_0 I_w I_z}{2\pi x} dx\,\vec{a_z}$$

or, $d\vec{T} = -\dfrac{\mu_0 I_1 I_2 (x-3)}{2\pi x}\, dx\, \vec{a_y}$

Thus,

$\vec{T} = \int_2^5 -\dfrac{\mu_0 I_1 I_2 (x-3)}{2\pi x}\, dx\, \vec{a_y}$

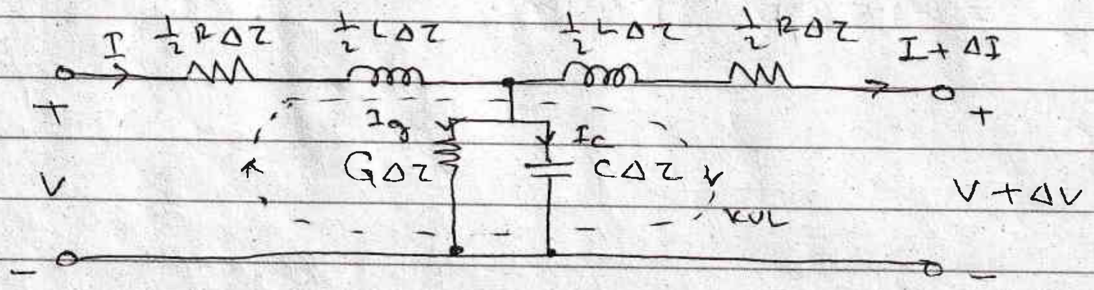
$= -6\times10^{-5}\left[3 - 3\ln\left(\frac{5}{2}\right)\right]\vec{a_y}$

$= -1.5\times10^{-5}\,\vec{a_y}$ N.m.

Chapter-5 Transmission Lines

5.1. Transmission line equations:-

Let us consider an incremental length of line having inductance 'L', capacitance 'C', shunt conductance 'G' and series resistance 'R'. We assume propagation in the a_z direction. The line section of length Δz has resistance $R\Delta z$, inductance $L\Delta z$, conductance $G\Delta z$ and capacitance $C\Delta z$.



Using KVL to the loop

$$V = \frac{1}{2}RI\Delta z + \frac{1}{2}L\frac{\partial I}{\partial t}\Delta z + \frac{1}{2}L\left(\frac{\partial I}{\partial t} + \frac{\partial \Delta I}{\partial t}\right)\Delta z$$

$$+\frac{1}{2}R(I+\Delta I)\Delta z + (V+\Delta V) \longrightarrow (1)$$

or, $$\frac{\Delta V}{\Delta z} = -\left(RI + L\frac{\partial I}{\partial t} + \frac{1}{2}L\frac{\partial \Delta I}{\partial t} + \frac{1}{2}R\Delta I\right) \rightarrow (2)$$

Next, $\Delta I = \frac{\partial I}{\partial z}\Delta z$ and $\Delta V = \frac{\partial V}{\partial z}\Delta z \rightarrow (3)$

which are then substituted into (2) to result in

$$\frac{\partial V}{\partial z} = -\left(1 + \frac{\Delta z}{2}\frac{\partial}{\partial z}\right)\left(RI + L\frac{\partial I}{\partial t}\right) \rightarrow (4)$$

Now, in the limit as Δz approaches zero eqn (4) simplifies to the final form:

$$\frac{\partial V}{\partial z} = -\left(RI + L\frac{\partial I}{\partial t}\right) \rightarrow (5)$$

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eqn (5) is the first of the two equations that we are looking for. To find second equation, we apply KCL to the upper central node in the circuit, noting from the symmetry that the voltage at the node will be $V + \Delta V/$

$$I = I_g + I_c + (I + \Delta I) = G\Delta z \left(V + \frac{\Delta V}{2}\right) + C\Delta z \frac{\partial}{\partial t}\left(V + \frac{\Delta V}{2}\right) + I + \Delta I$$

Then, using (3) and simplifying, we obtain

$$\frac{\partial I}{\partial z} = -\left(1 + \frac{\Delta z}{2}\frac{\partial}{\partial z}\right)\left(GV + C\frac{\partial V}{\partial t}\right) \rightarrow (7)$$

When Δz is negligible.

$$\frac{\partial I}{\partial z} = -\left(GV + C\frac{\partial V}{\partial t}\right) \rightarrow (8)$$

Differentiating eqn (5) with respect to z and eqn (8) with respect to t, obtaining

$$\frac{\partial^2 V}{\partial z^2} = -R\frac{\partial I}{\partial z} - L\frac{\partial^2 I}{\partial t \partial z} \rightarrow (9)$$

and $$\frac{\partial I}{\partial z \partial t} = -G\frac{\partial V}{\partial t} - C\frac{\partial^2 V}{\partial t^2} \rightarrow (10)$$

substituting eqn (8) and (10) into (9)

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2} + (LG + RC)\frac{\partial V}{\partial t} + RGV \rightarrow (11)$$

The eqns for currents are:

$$\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2} + (LG + RC)\frac{\partial I}{\partial t} + RGI \rightarrow (12)$$

Eqns (11) and (12) are the general wave equations for the transmission line.

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Transmission line equations and their solutions in phasor form :-

The equation of transmission line is

$$\frac{\partial^2 v}{\partial z^2} = LC\frac{\partial^2 v}{\partial t^2} + (LG+RC)\frac{\partial v}{\partial t} + RGV \rightarrow (1)$$

The fractor $\partial/\partial t = j\omega$ and $v = V_s$ so,

$$\frac{d^2 V_s}{dz^2} = -\omega^2 LC V_s + j\omega(LG+RC)V_s + RGV_s \rightarrow (2)$$

or, $$\frac{d^2 V_s}{dz^2} = \underbrace{(R+j\omega L)}_{z}\underbrace{(G+j\omega C)}_{Y}V_s = \gamma^2 V_s \rightarrow (3)$$

where z and Y as indicated are respectively the net series impedance and the net shunt admittance in the transmission line.

The propagation constant in the line is defined as

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{zY} = \alpha + j\beta \rightarrow (4)$$

The eq^n (3) can be written as

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \rightarrow (5)$$

The phasor current can be expressed as:

$$I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \rightarrow (6)$$

from eq^n (3)

$$\frac{dV_s}{\partial z} = -(R+j\omega L)I_s = -zI_s \rightarrow (6)$$

and $$\frac{dI_s}{dz} = -(G+j\omega C)V_s = -YV_s \rightarrow (7)$$

Substituting eq^n (5) and (6) into (6) & (7)

$$-\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} = -z(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}) \rightarrow (8)$$

equating like terms

$$z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{z}{\gamma} = \frac{z}{\sqrt{zY}} = \sqrt{\frac{z}{Y}} \rightarrow (9)$$

$$\therefore Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = |Z_0| e^{j\theta} \rightarrow (10)$$

5.2.1 Input impedance :-

The total voltage in the transmission line is

$$V_{sT}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

where V_0^+ and V_0^- are complex amplitudes.

The total current is given by

$$I_{sT}(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z}$$

Now, wave impedance $Z_\omega(z)$ is

$$Z_\omega(z) = \frac{V_{sT}(z)}{I_{sT}(z)} = \frac{V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}}{I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}} \rightarrow (1)$$

we know that

$$V_0^- = \Gamma V_0^+, \quad I_0^+ = V_0^+/Z_0 \text{ and } I_0^- = -V_0^-/Z_0.$$

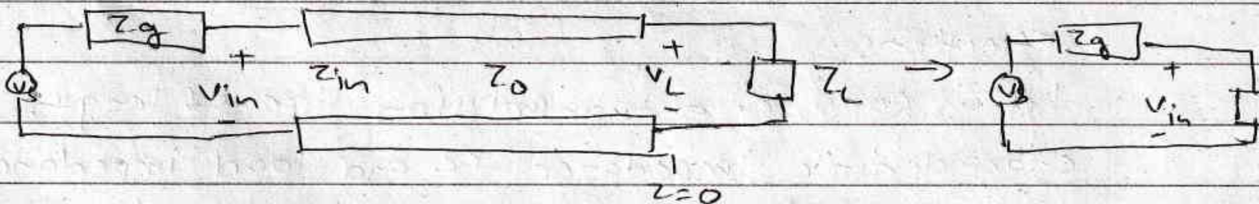

fig: Finite length transmission line configuration and its equivalent circuit.

using $V_0^- = \Gamma V_0^+$, $I_0^+ = V_0^+/Z_0$ and $I_0^- = -V_0^-/Z_0$.

$$Z_\omega(z) = Z_0 \left[\frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{-j\beta z} - \Gamma e^{j\beta z}}\right] \rightarrow (2)$$

using Euler's identity and substituting $\Gamma = \frac{(Z_L - Z_0)}{(Z_L + Z_0)}$

eqn (2) becomes

$$Z_\omega(z) = Z_0 \left[\frac{Z_L \cos(\beta z) - j Z_0 \sin(\beta z)}{Z_0 \cos(\beta z) - j Z_L \sin(\beta z)}\right] \rightarrow (3)$$

The wave impedance at the line input is found by placing $z = -l$.

$$\therefore Z_{in} = Z_0 \left[\frac{Z_L \cos(\beta l) + j Z_0 \sin(\beta l)}{Z_0 \cos(\beta l) + j Z_L \sin(\beta l)}\right] \rightarrow (4)$$

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5.2.2 Reflection coefficient :-

The reflection coefficient is defined as the ratio of the reflected voltage amplitude to the incident voltage amplitude.

$$\Gamma = \frac{V_{or}}{V_{oi}} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi_r} \longrightarrow (1)$$

5.2.3 Transmission coefficient :-

It is defined as the ratio of the load voltage amplitude to the incident voltage amplitude.

$$\tau = \frac{V_L}{V_{oi}} = 1 + \Gamma = \frac{2Z_L}{Z_0 + Z_L} = |\tau| e^{j\phi_t} \longrightarrow (2)$$

5.3.1 Impedance matching :-

It is performed in transmission lines to prevent reflections.

Let us consider a transmission line of length 'L' with characteristic impedance Z_c and load impedance Z_L as shown in figure.

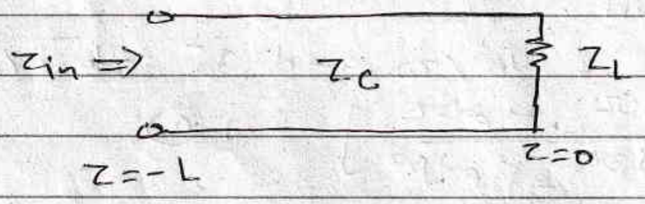


fig: Input impedance of a transmission line with load

Here, $Z_{in}(L) = Z(z = -L)$

The input impedance is given by

$$Z(z) = Z_c \cdot \frac{Z_L \cdot 2\cos(kz) - Z_c \cdot 2j\sin(kz)}{Z_c \cdot 2\cos(kz) - Z_L \cdot 2j\sin(kz)}$$

$$\text{or, } Z(z) = Z_c \cdot \frac{Z_L - jZ_c\tan(kz)}{Z_c - jZ_L\tan(kz)}$$

Here, $Z_{in}(L) = Z(z=-L)$

Inserting $z=-L$ then,

$$Z_{in}(L) = Z_C \cdot \frac{Z_L + jZ_C \tan(kL)}{Z_C + jZ_L \tan(kL)} \longrightarrow (1)$$

~~For an impedance matched transmission line, the input impedance is equal to the characteristic impedance~~

If transmission line is impedance matched ($Z_L = Z_C$) then,
i.e, $Z_L = Z_C$ then,

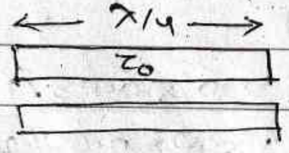
$$Z_{in}(L) = Z_C \cdot \frac{Z_C + jZ_C \tan(kL)}{Z_C + jZ_C \tan(kL)}$$

or, $Z_{in}(L) = Z_C \longrightarrow (2)$

i.e, for an impedance matched transmission line, the input impedance is equal to the characteristic impedance.

5.3.2 Quarter wave transformer :-

A quarter wave transmission line has a characteristic impedance of Z_0, which matches the impedance of the source and the load. The length of the quarter wave line is chosen so that the impedance seen at the source end of the line matches the desired impedance.

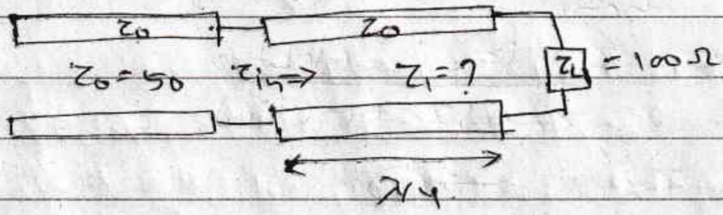


The impedance transformation of a quarter wave transmission line is based on the principle of electrical resonance. At a specific frequency the impedance of the quarter wave line is equal to the load impedance, which causes a maximum transfer of power from the source to the load. The quarter wave line acts as an inductor or a capacitor depending on the load impedance.

A quarter wave transmission line is a type of transmission line that is designed to be one quarter of the wavelength of the signal it is carrying. The length of the transmission line is given by $L = \lambda/4$, where λ is the wavelength of the signal. It is used for impedance matching.

Example :-

Suppose we have a load impedance Z_L of 100 ohms and we want to match this to a transmission line with a characteristic impedance Z_0 of 50 ohms.



lets calculate length of quarte wave transmission line.

$$\lambda = v/f = \frac{2\times10^8}{1\times10^9} = 0.2\,m$$

$$L = \lambda/4 = 0.2/4 = 0.05\,m.$$

$Z_1 = ?$

$$Z_1 = \sqrt{(50\times100)} = \sqrt{Z_0 Z_L} = 70.7\,\Omega.$$

Reflection coefficient $(\Gamma_{in}) = \dfrac{Z_{in}-Z_0}{Z_{in}+Z_0}$

Now, $Z_{in} = Z_0 \times \dfrac{(Z_1 + jZ_0 \tan(2\pi fL/v))}{(Z_0 + jZ_1 \tan(2\pi fL/v))}$

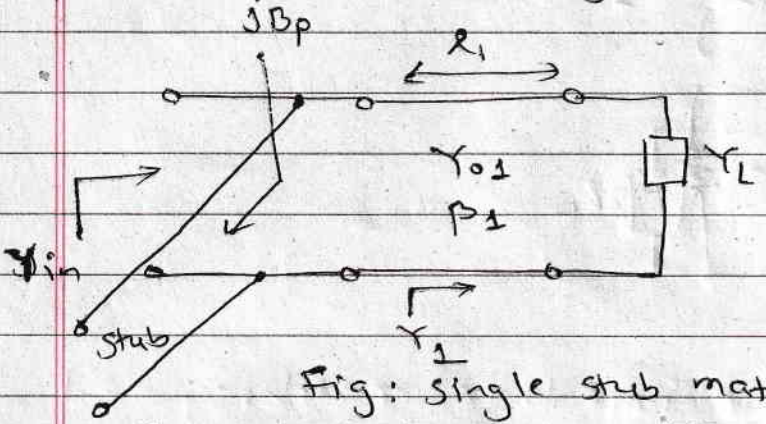
$$= 50 \times \frac{(70.7 + j50\tan(2\pi\times1\times10^9\times0.05/2\times10^8))}{(50 + j70.7\tan(2\pi\times1\times10^9\times0.05/2\times10^8))}$$

$$= \cancel{50\times(70.7+j50}$$

$$= \infty$$

$$\Gamma_{in} = 1.$$

5.3.3 Single stub matching :-



stub characteristics: Y_{o2} β_2

Fig: single stub matching.

figure: single stub matching.

This scheme is usually implemented using the parallel reactance approach. Although a series reactance scheme is also possible in principle, it is usually not as convenient. This is because most transmission lines use one of their two conductors as a local datum. As shown in figure parallel attached stub and the transmission line to which it is attached both have one terminal at ground.

Here the parallel reactance is implemented using a short or open circuited stub as opposed a discrete inductor or capacitor.

→ in parallel reactance matching

Here, $Z_{in}(l) = jZ_0 \tan\beta l$ and $Z_{in}(l) = -jZ_0 \cot\beta l$.

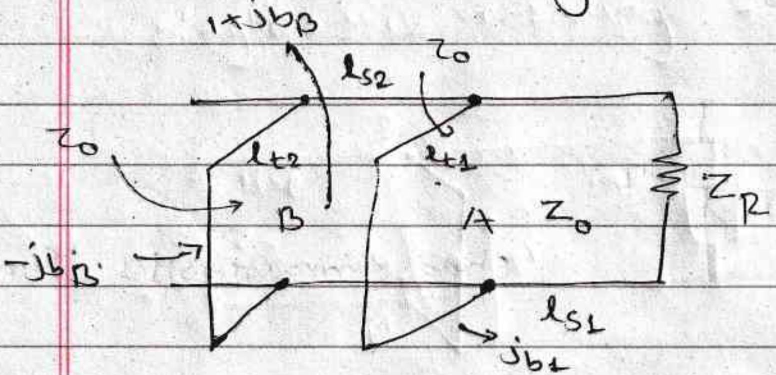
incase of input impedance of an open and short circuited stub.

$B_p = -Y_{o2} \cot(\beta_2 l_2)$ short circuited stub

$B_p = Y_{o2} \tan(\beta_2 l_2)$ open circuited stub.

and characteristic impedance $Z_{o2} = 1/Y_{o2}$ is an independent variable.

5.3.4 Double stub matching :-



i) The locations of two stubs are fixed where as the lengths of the stubs can be changed independently.

$$Z_{in} = Z_0\left[\frac{Z_R + jZ_0 \tan\beta l}{Z_0 + jZ_R \tan\beta l}\right]$$

where, Z_{in} = input impedance of lossless line

or, $z_{in} = \dfrac{z_R + j\tan\beta l}{1 + jz_R \tan\beta l}$ [z = normalized impeda

or, $\dfrac{1}{y_{in}} = \dfrac{1/y_r + j\tan\beta l}{1 + j\dfrac{1}{y_r}\tan\beta l}$

or, $\dfrac{1}{y_{in}} = \dfrac{1 + jy_r \tan\beta l}{y_r + j\tan\beta l}$

or, $y_{in} = \dfrac{y_r + j\tan\beta l}{1 + jy_r \tan\beta l} \times \dfrac{1 - jy_r \tan\beta l}{1 - jy_r \tan\beta l}$

or, $y_{in} = \dfrac{y_r + y_r^2 \tan\beta l}{1 + y_r^2 \tan^2\beta l} + \dfrac{j(\tan\beta l - y_r^2 \tan\beta l)}{1 + y_r^2 \tan^2\beta l}$

or, $g + jb =$ "

at location (A)

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$$y_A = g_A + jb_A = \frac{y_r + y_r^2 \tan\beta l_{s1}}{1 + y_r^2 \tan^2\beta l_{s1}} + \frac{j(\tan\beta l_{s1} - y_r^2 \tan\beta}{1 + y_r^2 \tan^2\beta l_{s1}}$$

1^{st} stub changes only susceptance

$$y_A' = g_A + jb_A'$$

2^{nd} stub is located in such a way that

$$y_B = 1 + jb_B$$

After adding 2^{nd} stub

$$y_B' = 1 + jb_B - jb_B$$

$$\Rightarrow y_B' = 1.$$

The purpose is to transfer maximum power from source to load.

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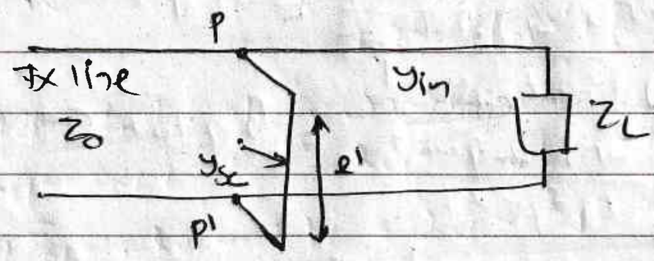
Single stub matching :-



- It is used to provide impedance matching and to redue reflection of signal
- Stub should be positioned at input admittance to be $1+jx$
- length of short circuited stub should be such that admittance of stub should be $-jx$.
- So total admittance at PP'

$$y = y_1 + y_2$$
$$= 1 + jx + (-jx)$$
$$= 1$$

Input impedance of tx line

$$Z_{in} = Z_0\left[\frac{Z_L + jZ_0\tan\beta l}{Z_0 + jZ_L\tan\beta l}\right]$$

Normalized impedance, $z_{in} = \frac{Z_{in}}{Z_0}$, $z_L = \frac{Z_L}{Z_0}$

$$\Rightarrow \left(\frac{Z_{in}}{Z_0}\right) = \left[\frac{(Z_L/Z_0) + j\tan\beta l}{1 + j(Z_L/Z_0)\tan\beta l}\right]$$

$$\Rightarrow z_{in} = \frac{z_L + j\tan\beta l}{1 + jz_L\tan\beta l}$$

$\rightarrow$ Admittance $= 1/$impedance

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$$\Rightarrow \left(\frac{1}{y_{in}}\right) = \frac{(1/y_2) + j\tan\beta l}{1 + j(1/y_2)\tan\beta l}$$

$$\text{or, } y_{in} = \frac{y_2 + j\tan\beta l}{1 + jy_2\tan\beta l} \times \frac{1 - jy_2\tan\beta l}{1 - jy_2\tan\beta l}$$

$$= \frac{y_2 + y_2\tan^2\beta l + j(\tan\beta l - y_2^2\tan\beta l)}{1 + y_2^2\tan^2\beta l}$$

$$= \frac{y_2(1+\tan^2\beta l)}{1 + y_2^2\tan^2\beta l} + j\frac{(1-y_2^2)\tan\beta l}{1 + y_2^2\tan^2\beta l}$$

Here, real part = 1

$$\Rightarrow \frac{y_2(1+\tan^2\beta l)}{1 + y_2^2\tan^2\beta l} = 1$$

$$\Rightarrow y_2 + y_2\tan^2\beta l = 1 + y_2^2\tan^2\beta l$$

$$\Rightarrow \tan^2\beta l\,(y_2 - y_2^2) = 1 - y_2$$

$$\Rightarrow \tan^2\beta l = \frac{1}{y_2}$$

$$\Rightarrow \tan\beta l = \sqrt{\frac{1}{y_2}}$$

$$\Rightarrow \tan\beta l = \sqrt{z_L}$$

$$\Rightarrow l = \frac{1}{\beta}\tan^{-1}\sqrt{\frac{Z_L}{Z_0}}$$

$$\Rightarrow l = \frac{\lambda}{2\pi}\tan^{-1}\sqrt{\frac{Z_L}{Z_0}}$$

Here our stub is short circuited stub

$$\Rightarrow Z_{SC} = +jZ_0\tan\beta l'$$

$$\Rightarrow \frac{Z_{SC}}{Z_0} = j\tan\beta l'$$

$$\Rightarrow z_{SC} = j\tan\beta l'$$

$$\Rightarrow y_{sc} = -j\cot\beta l'$$

when,

$$y_{sc} = -jx' \quad \text{or,} \quad -j\cot\beta l' = \frac{j\tan\beta l(1-y_2^2)}{1+y_2^2\tan^2\beta l}$$

$$\Rightarrow \cot\beta l' = \frac{\tan\beta l(1-y_2^2)}{1+y_2^2\tan^2\beta l}$$

$$\Rightarrow \cot\beta l' = \frac{\tan\beta l(1-y_2^2)}{1+y_2^2(1/y_2)}$$

$$= \frac{\tan\beta l(1-y_2^2)}{1+y_2}$$

$$\Rightarrow \cot\beta l' = \tan\beta l(1-y_2)$$

$$\Rightarrow \cot\beta l' = \frac{1}{\sqrt{y_2}}(1-y_2)$$

$$\Rightarrow \cot\beta l' = \sqrt{z_L}\left(1-\frac{1}{z_L}\right)$$

$$= \frac{z_L-1}{\sqrt{z_L}}$$

$$= \frac{z_L/z_0 - 1}{\sqrt{z_L/z_0}}$$

$$= \frac{z_L-z_0}{\sqrt{z_L z_0}}$$

$$\Rightarrow \tan\beta l' = \frac{\sqrt{z_L z_0}}{z_L-z_0} \quad \text{or,} \quad \beta l' = \tan^{-1}\left(\frac{\sqrt{z_L z_0}}{z_L-z_0}\right)$$

$$\therefore \quad l' = \frac{\lambda}{2\pi}\tan^{-1}\left(\frac{\sqrt{z_L z_0}}{z_L-z_0}\right)$$

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17 A lossless transmission line is 80 cm long and operates at a frequency of 600 MHz. The line parameters are $L = 0.25 \mu$ and $C = 100 pF/m$. Find the characteristic impedance, the phase constant and the phase velocity.

Solⁿ:

Since the line is lossless, both R and G are zero. The characteristic impedance is

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.25\times10^{-6}}{100\times10^{-12}}} = 50\Omega$$

because $\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$

$$= j\omega\sqrt{LC}$$

$$\beta = \omega\sqrt{LC}$$
$$= 2\pi(600\times10^{6})\sqrt{(0.25\times10^{-6})(100\times10^{-12})}$$
$$= 18.85 \text{ rad/m}$$

also,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi(600\times10^{6})}{18.85} = 2\times10^{8} \text{ m/s.}$$

27 A 50Ω lossless transmission line is terminated by a load impedance, $Z_L = 50 - j75\,\Omega$. If the incident power is 100 mW find the power dissipated by the load.

Solⁿ: The reflection coefficient is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - j75 - 50}{50 - j75 + 50} = 0.36 - j0.48 = 0.6\,e^{-j0.93}$$

then, $\langle P_t \rangle = (1 - |\Gamma|^2)\langle P_i \rangle$

$$= [(1 - (0.6)^2](100)$$
$$= 64 \text{ mW}.$$

3> Two lossy lines are to be joined end to end. The first line is 10m long and has a loss rating of 0.2 dB/m. The second line is 15m long and has a loss rating of 0.1 dB/m. The reflection coefficient at the junction (line 1 to line 2) is $\Gamma = 0.3$. The input power (to line 1) is 100mW. (a) Determine the total loss of the combination in dB. (b) Determine the power transmitted to the output end of line 2.

Solⁿ:

(a) The dB loss of the joint is

$$L_j(dB) = 10\log_{10}\left(\frac{1}{1-|\Gamma|^2}\right) = 10\log_{10}\left(\frac{1}{1-0.09}\right) = 0.41\,dB$$

The total loss of the link in dB is now

$$L_t(dB) = (0.2)(10) + 0.41 + (0.1)(15) = 3.91\,dB$$

(b) The output power will be $P_{out} = 100 \times 10^{-0.391} = 41\,mW.$

$$= 100 \times 10^{-(dB/10)}$$

4> A lossless transmission line having $Z_0 = 120\Omega$ is operating at $\omega = 5\times10^8$ rad/s. If the velocity on the line is 2.4×10^8 m/s, find:

a) L b) C c) Let Z_L be represented by an inductance of 0.6 µH in series with a 100Ω resistance. Find Γ and s.

Solⁿ:

a) with $Z_0 = \sqrt{L/C}$ and $v = 1/\sqrt{LC}$

we find

$$L = Z_0/v = 120/(2.4\times10^8) = 0.5\,\mu H/m$$

b) using $Z_0 v = \sqrt{L/C}\,/\sqrt{LC}$

$$\text{or,}\quad C = \frac{1}{(Z_0 v)} = (120\times2.4\times10^8)^{-1} = 35\,pf/m$$

c) The inductive impedance is

$j\omega L = j(5\times10^{8})(0.6\times10^{-6})$

$= j300.$

so, load impedance is

$Z_L = 100 + j300\,\Omega$

Now,

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 + j300 - 120}{100 + j300 + 120} = 0.62 + j0.52 = 0.808\angle 4$$

Then,

$$S = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.808}{1-0.808} = 9.4$$

57 Two characteristics of a certain lossless transmission line are $Z_0 = 50\,\Omega$ and $\gamma = 0 + j0.2\pi\ m^{-1}$ at $f = 60$ MHz.

a) find L and C for the line.

b) A load, $Z_L = 60 + j80\,\Omega$ is located at $z=0$. what is the shortest distance from the load to a point at which $Z_{in} = R_{in} + j0$?

Soln:

a) we have,

$\beta = 0.2\pi = \omega\sqrt{LC}$ and $Z_0 = 50 = \sqrt{L/C}$. Thus,

$$\frac{\beta}{Z_0} = \omega C \Rightarrow C = \frac{\beta}{\omega Z_0} = \frac{0.2\pi}{(2\pi\times60\times10^{6})(50)} = \frac{1}{3}\times10^{10} = 33.3\,\text{pF/m}$$

Then, $L = CZ_0^2 = (33.3\times10^{-12})(50)^2$

$= 8.33\times10^{-8}$ H/m

$= 83.3$ nH/m.

b) we know that

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \right]$$

lets normalize the impedances w.r.to Z_0 then

$$z_{in} = \frac{Z_{in}}{Z_0} = \left[\frac{z_L + j\tan(\beta l)}{1 + jz_L \tan(\beta l)} \right]$$

where, $z_L = (60 + j80)/50 = 1.2 + j1.6.$

also, ~~we know that~~ let us assume, $x = \tan(\beta l)$

$$\text{or,} \quad z_{in} = \left[\frac{1.2 + j(1.6 + x)}{(1 - 1.6x) + j1.2x} \right] \left[\frac{(1 - 1.6x) - j1.2x}{(1 - 1.6x) - j1.2x} \right]$$

$$= \frac{1.2(1 - 1.6x) + 1.2x(1.6 + x) - j[(1.2)^2 x - (1.6 + x)(1 - 1.6x)}{(1 - 1.6x)^2 + (1.2)^2 x^2}$$

since, imaginary part is zero,

$$(1.2)^2 x - (1.6 + x)(1 - 1.6x) = 0.$$

$$\Rightarrow 1.6x^2 + 3x - 1.6 = 0$$

$$\text{so,} \quad x = \tan(\beta l) = \frac{-3 \pm \sqrt{9 + 4(1.6)^2}}{2(1.6)} = (0.433, -2.31)$$

Now,

$$\tan\beta l = 0.433$$

$$\text{or,} \quad \beta l = \tan^{-1}(0.433) = 0.409$$

$$\Rightarrow l = \frac{0.409}{0.2\pi} = 0.65\,m = 65\,cm$$

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6> The parameters of a certain transmission line operating at 6×10^{8} rad/s are $L = 0.4\ \mu H/m$, $C = 40\ pF/m$, $G = 80\ mS/m$, and $R = 20\ \Omega/m$.

a) Find $\gamma, \alpha, \beta, \lambda$ and Z_0.

Solⁿ:

$$\gamma = \sqrt{ZY} = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$= \sqrt{[20 + j(6\times10^{8})(0.4\times10^{-6})][80\times10^{-3} + j(6\times10^{8})(40\times10^{-12})]}$$

$$= 2.8 + j3.5\ m^{-1}$$

$$= \alpha + j\beta.$$

Therefore, $\alpha = 2.8$ Np/m, $\beta = 3.5$ rad/m and $\lambda = 2\pi/\beta =$

Finally,

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$= \sqrt{\frac{20 + j2.4\times10^{2}}{80\times10^{-3} + j2.4\times10^{-2}}}$$

$$= 44 + j30\ \Omega.$$

$\sqrt{1055.04 + j2683.4}$

~~44 + j30~~

$\sqrt{2883.43 \angle 68.53}$

~~20.1 ∠ 10.939~~ $53 \angle 34.265$

~~448 ∠ 101.935~~ $43.801 + j29.8$

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chapter-6 waveguides

6.1 Rectangular waveguide:-

It is a structure that is usually used in the microwave region of the electromagnetic spectrum.
It guides waves by restricting the transmission of energy to one direction. There are different types of waveg for different types of waves. It is generally a hollow conductive metal pipe which is used to carry high frequency radio waves particularly microwave.

Examples:

- optical fibers transmit light and signals for long distances with low attenuation and a wide usable range of wavelengths.
- In a microwave oven a waveguide transfers power from the magnetron where waves are formed to the cooking chamber
- In a radar, a waveguide transfers radio frequency energy to and from the antenna, where the impedance needs to be matched for efficient power transmission.
- Rectangular and circular waveguides are commonl used to connect feeds of parabolic dishes to their electronics either low noise receivers a power amplifier/transmitters.

$F_m(x) = A_m \cos(K_m x) + B_m \sin(K_m x) \rightarrow (8)$

$G_p(y) = C_p \cos(K_p y) + D_p \sin(K_p y) \rightarrow (9)$

using eqⁿ (8) & (9) with eqⁿ (2), the general solution for z component of $\vec{E_s}$ for a single TM mode can be constructed.

$$E_{zs} = [A_m \cos(K_m x) + B_m \sin(K_m x)][C_p \cos(K_p y) + D_p \sin(K_p y)] \exp(-j\beta_{mp} z) \rightarrow (10)$$

The boundary conditions for evaluating constants of eqⁿ (10) are

$E_{zs} = 0$ at $x=0$, $y=0$, $x=a$, and $y=b$

obtaining zero field at $x=0$ and $y=0$ is accomplished by dropping the cosine terms setting $A_m = C_p = 0$.
The values of K_m and K_p that appear in the remaining sine terms are then set to the following, in order to assure zero field at $x=a$ and $y=b$.

$$K_m = \frac{m\pi}{a} \rightarrow (11)$$

$$K_p = \frac{p\pi}{b} \rightarrow (12)$$

Defining $B = B_m D_p$ eqⁿ (10) becomes

$$E_{zs} = B \sin(K_m x) \sin(K_p y) \exp(-j\beta_{mp} z) \rightarrow (13)$$

Now to find the remaining (transverse) field components we substitute eqⁿ (13) into eqⁿ below,

$$E_{xs} = -\frac{j}{K^2}\left[\beta \frac{\partial E_{zs}}{\partial x} + \omega\mu \frac{\partial H_{zs}}{\partial y}\right]$$

$$E_{ys} = -\frac{j}{K^2}\left[\beta \frac{\partial E_{zs}}{\partial y} - \omega\mu \frac{\partial H_{zs}}{\partial x}\right]$$

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$$E_{xs} = -j\beta_{mp} \frac{k_m}{k_{mp}^2} B\cos(k_m x)\sin(k_p y)\exp(-j\beta_{mp} z)$$

$$E_{ys} = -j\beta_{mp} \frac{k_p}{k_{mp}^2} B\sin(k_m x)\cos(k_p y)\exp(-j\beta_{mp} z)$$

$$H_{xs} = j\omega\varepsilon \frac{k_p}{k_{mp}^2} B\sin(k_m x)\cos(k_p y)\exp(-j\beta_{mp} z)$$

$$H_{ys} = -j\omega\varepsilon \frac{k_m}{k_{mp}^2} B\cos(k_m x)\sin(k_p y)\exp(-j\beta_{mp} z)$$

Q2.2 TE modes :-

TO obtain the TE mode fields, we solve the wave equation for the z component of $\vec{H}$ and use the eqn

$$\vec{H}_{xs} = -\frac{j}{k^2}\left[\beta \frac{\partial \vec{H}_{zs}}{\partial x} - \omega\varepsilon \frac{\partial \vec{E}_{zs}}{\partial y}\right]$$

The wave eqn is now,

$$\frac{\partial^2 \vec{H}_{zs}}{\partial x^2} + \frac{\partial^2 \vec{H}_{zs}}{\partial y^2} + (k^2 - \beta_{mp}^2)\vec{H}_{zs} = 0 \rightarrow (1)$$

and the solution is of the form:

$$\vec{H}_{zs}(x,y,z) = \sum_{m,p} F_m'(x)\, G_p'(y)\exp(-j\beta_{mp} z) \rightarrow (2)$$

The general solution is

$$H_{zs} = [A_m' \cos(k_m x) + B_m' \sin(k_m x)][C_p' \cos(k_p y) + D_p' \sin(k_p y)]\exp(-j\beta_{mp} z) \rightarrow (3)$$

If electric field is related to magnetic field derivatives, we get

$$E_{xs}\big|_{y=0,b} = 0 \Rightarrow \frac{\partial H_{zs}}{\partial y}\Big|_{y=0,b} = 0$$

$$E_{ys}\big|_{x=0,a} = 0 \Rightarrow \frac{\partial H_{zs}}{\partial x}\Big|_{x=0,a} = 0$$

The boundary conditions are now applied to eq^n (3)

$$\frac{\partial H_{zs}}{\partial y} = [A'_m \cos(k_m x) + B'_m \sin(k_m x)]$$
$$\times [-k_p C'_p \sin(k_p y) + k_p D'_p \cos(k_p y)] \exp(-j\beta_{mp} z)$$

Similarly,

$$\frac{\partial H_{zs}}{\partial x} = [-k_m A'_m \sin(k_m x) + k_m B'_m \cos(k_m x)]$$
$$\times [C'_p \cos(k_p y) + D'_p \sin(k_p y)] \exp(-j\beta_{mp} z)$$

After applying boundary conditions,

$$H_{zs} = A \cos(k_m x) \cos(k_p y) \exp(-j\beta_{mp} z)$$

where we define $A = A'_m C'_p$

$$\therefore H_{xs} = j\beta_{mp} \frac{k_m}{k_{mp}^2} A \sin(k_m x) \cos(k_p y) \exp(-j\beta_{mp} z)$$

$$H_{ys} = j\beta_{mp} \frac{k_p}{k_{mp}^2} A \cos(k_m x) \sin(k_p y) \exp(-j\beta_{mp} z)$$

$$E_{xs} = j\omega\mu \frac{k_p}{k_{mp}^2} A \cos(k_m x) \sin(k_p y) \exp(-j\beta_{mp} z)$$

$$E_{ys} = -j\omega\mu \frac{k_m}{k_{mp}^2} A \sin(k_m x) \cos(k_p y) \exp(-j\beta_{mp} z)$$

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Numericals :-

17 A rectangular waveguide has dimensions a = 6 cm and b = 4 c

a) over what range of frequencies will the guide operate sing mode? The cutoff frequency for mode mp is:

$$f_{c,mn} = \frac{c}{2n}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{p}{b}\right)^2}$$

where n is the refractive index of the guide interior. we require that the frequency lie between the cutoff frequenci of the TE_{10} and TE_{01} modes. These will be:

$$f_{c10} = \frac{c}{2na} = \frac{3\times10^8}{2n(0.06)} = \frac{2.5\times10^9}{n}$$

$$f_{c01} = \frac{c}{2nb} = \frac{3\times10^8}{2n(0.04)} = \frac{3.75\times10^9}{n}$$

Thus, the range of frequencies over which single mode operation will occur is

$$\frac{2.5}{n}\ \text{GHz} < f < \frac{3.75}{n}\ \text{GHz}$$

b) over what frequency range will the guide support both TE_{10} and TE_{01} modes and no others? we note first that f must be greater than f_{c01} to support both modes, but must be less than the cutoff frequency for the next higher order mode. This will be f_{c11}, given by

$$f_{c11} = \frac{c}{2n}\sqrt{\left(\frac{1}{0.06}\right)^2 + \left(\frac{1}{0.04}\right)^2} = \frac{30c}{2n} = \frac{4.5\times10^9}{n}$$

The allowed frequency range is then

$$\frac{3.75}{n}\ \text{GHz} < f < \frac{4.5}{n}\ \text{GHz}$$

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27 An air filled rectangular waveguide is to be constructed for single-mode operation at 15 GHz. Specify the guide dimensions a and b, such that the design frequency is 10% while being 10% lower than the cutoff frequency for the next higher order mode:

For an air filled guide, we have

$$f_{c,mp} = \sqrt{\left(\frac{mc}{2a}\right)^2 + \left(\frac{pc}{2b}\right)^2}$$

For TE_{10} we have $f_{c10} = c/2a$, while for the next mode (TE_{01}), $f_{c01} = c/2b$. our requirements state that $f = 1.1\ f_{c10} = 0.9\ f_{c01}$. So, $f_{c10} = 15/1.1 = 13.6$ GHz and $f_{c01} = 15/0.9 = 16.7$ GHz. The guide dimensions will be

$$a = \frac{c}{2f_{c10}} = \frac{3\times10^{10}}{2(13.6\times10^9)} = 1.1 \text{ cm}$$

$$b = \frac{c}{2f_{c01}} = \frac{3\times10^{10}}{2(16.7\times10^9)} = 0.9 \text{ cm}$$

37 A symmetric slab waveguide is known to support only a single pair of TE and TM modes at wavelength $\lambda = 1.55\ \mu m$. If the slab thickness is $5\ \mu m$, what is the maximum value of n_1 if $n_2 = 3.3$ (assume 3.3)?

Soln:

we have,

$$\frac{2\pi d}{\lambda}\sqrt{n_1^2 - n_2^2} < \pi$$

or, $$n_1 < \sqrt{\frac{\lambda}{2d} + n_2^2}$$

$$= \sqrt{\frac{1.55}{2(5)} + (3.3)^2}$$

$$= 3.32$$