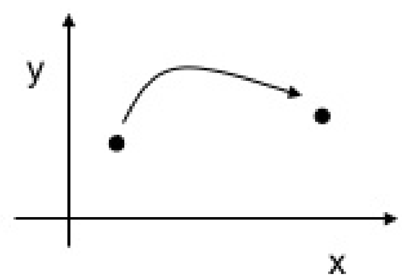
Chapter 3

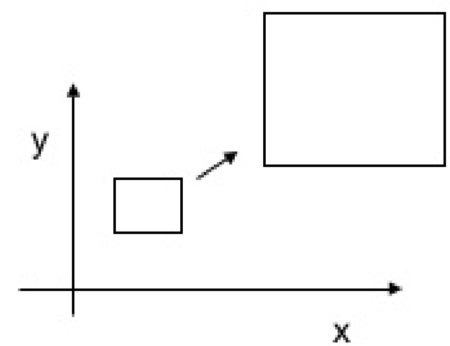
2D and 3D Coordinate Systems and Viewing Transformations

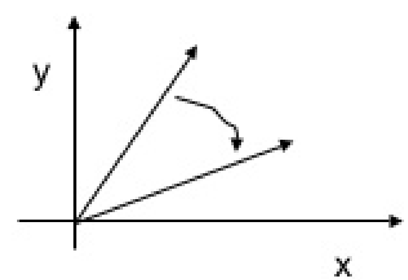
Two Dimensional Graphics

Basic Transformation

- Transformation means changing the object by changing position, orientation or size of original object by applying certain rules.
- The basic transformations are
  - Translation/Shifting
  - Scaling
  - Rotation







Translation

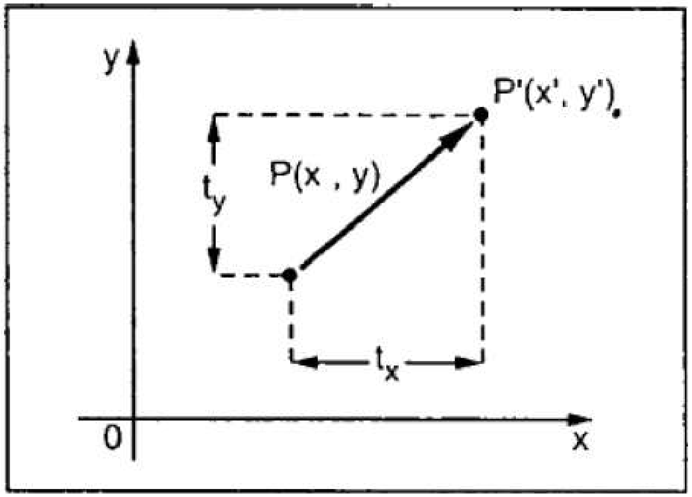
- Translation means repositioning an object along a straight line path from one coordinate location to another
- We add translational distance t_x ,t_y to original coordinate position (x,y) to move the point to a new position (x',y')

$$x' = x + t_x$$

$$y' = y + t_y$$

where the pair (tx , ty) is called the *translation vector.*

- We can write equation as a single matrix equation by using column vectors to represent coordinate points and translation vectors. i.e.



$$\mathbf{P} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} \mathbf{t_x} \\ \mathbf{t_y} \end{bmatrix} \qquad \mathbf{P'} = \begin{bmatrix} \mathbf{x'} \\ \mathbf{y'} \end{bmatrix}$$

- So, we can write

$$\mathbf{P'} = \mathbf{P} + \mathbf{T}$$

$$\begin{bmatrix} \mathbf{x'} \\ \mathbf{y'} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} \mathbf{t_x} \\ \mathbf{t_y} \end{bmatrix}$$

Scaling

- Scaling Transformation alters the size of an object i.e we can magnify and reduce the size of an object
- In case of polygons scaling is done by multiplying coordinate values (x, y) of each vertex by scaling factors s_x, s_y to produce the final transformed coordinates (x', y').
- s_x scales object in 'x' direction and s_y scales object in 'y' direction
- We can represent this in equation as

$$x' = x \cdot s_x \quad \text{and} \quad y' = y \cdot s_y$$

- We can also represent in matrix for as

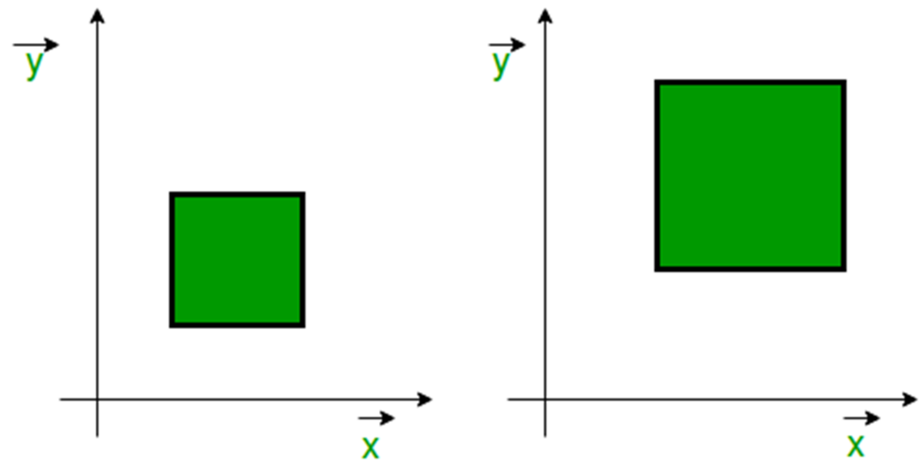
$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad T = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$

or

$$P' = S . P$$



- Values greater than 1 for s_x, s_y produce *enlargement*
- Values smaller than 1 for s_x, s_y *reduce* size of object
- $s_x = s_y = 1$ leaves the size of the object *unchanged*
- If $s_x = s_y$ then a *Uniform Scaling* is produced else *Differential Scaling* is produced

Rotation

- Rotation repositions an object along a circular path in xy plane
- To generate a rotation, we specify a rotation angle θ and the position (x_r, y_r) of rotation point about which the object is to be rotated.
- + value for 'θ' define *counter-clockwise* rotation about a point
- - value for 'θ' define *clockwise* rotation about a point
- If (x,y) is the original point 'r' the constant distance from origin, 'Φ' the original angular displacement from x-axis.
- Now the point (x,y) is rotated through angle 'θ' in a counter clock wise direction
- Express the transformed coordinates in terms of 'Φ' and 'θ' as

$$x' = r\cos(\Phi + \theta) = r\cos\Phi . \cos\theta - r\sin\Phi . \sin\theta \quad ....(i)$$

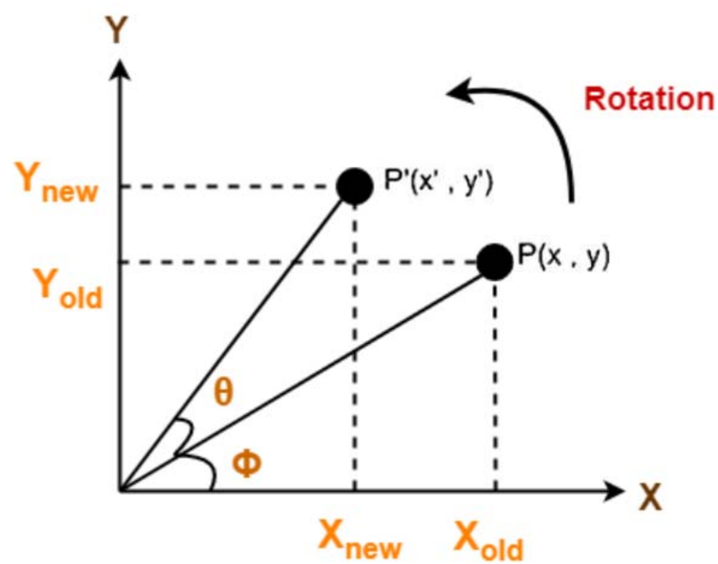
$$y' = r\sin(\Phi + \theta) = r\cos\Phi . \sin\theta + r\sin\Phi . \cos\theta \quad ....(ii)$$

- We know that original coordinates of point in polar coordinates are

$$x = r\cos\Phi \text{ and } y = r\sin\Phi$$

- substituting these values in (i) and (ii)
- we get,

$$x' = x\cos\theta - y\sin\theta \qquad \text{and} \qquad y' = x\sin\theta + y\cos\theta$$



Rotation

- So, using column vector representation for coordinate points the matrix form would be

$$P' = R \,.\, P$$

where

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

So,

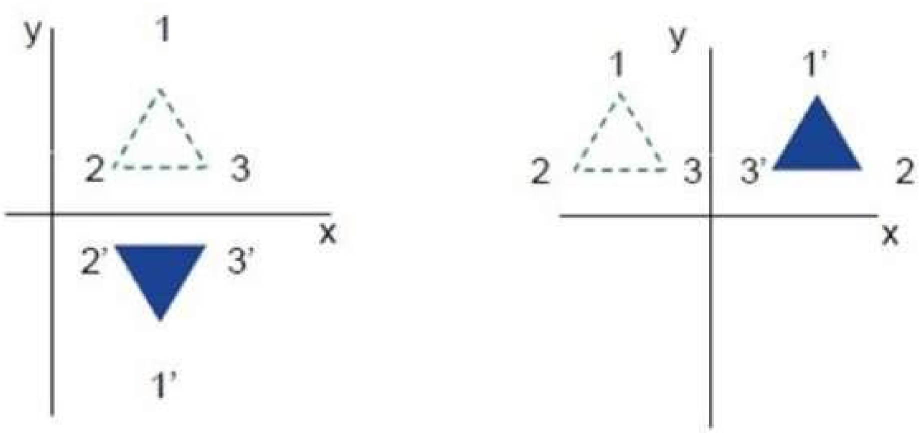
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

If rotation is in clockwise direction, we take negative angle

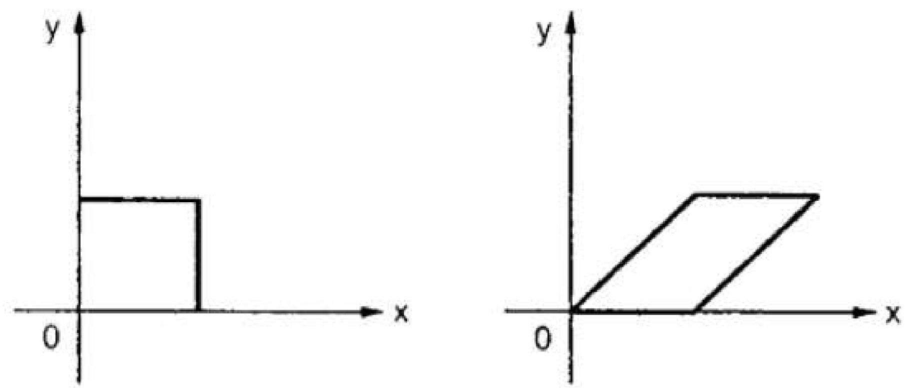
So, $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **and** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

Other Transformation

- Besides basic transformations, we have other two transformation. i.e
  - Reflection
  - Shearing



(a) (b)



(a) Original object (b) Object after x shear

Reflection

- Reflection is a transformation that produces a mirror image of an object
- Mirror image for 2D reflection is generated relative to an axis of reflection by rotating the object 180 degree about the reflection axis.

Reflection about x-axis or about line y=0

- Keeps 'x' value same but flips y value of coordinate points

So $x' = x$ and $y' = -y$

i.e.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

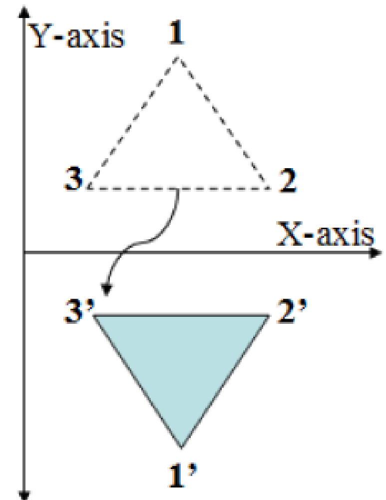


Fig. Reflection of an object about the x axis.

Reflection

Reflection about y-axis or about line x=0

- Keeps y value same but flips x value of coordinate points

So $x' = -x$ and $y' = y$

i.e.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

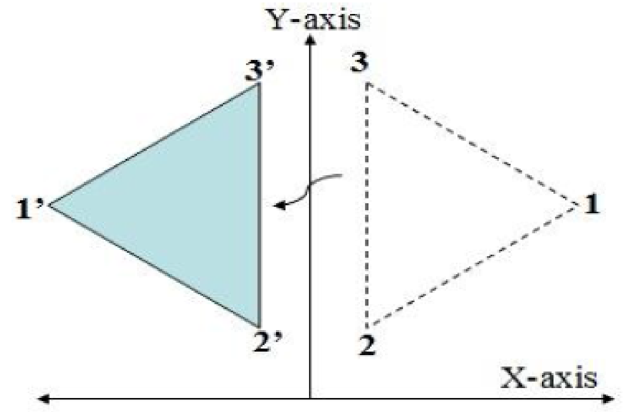
Reflection about origin

- Flip both x and y vale

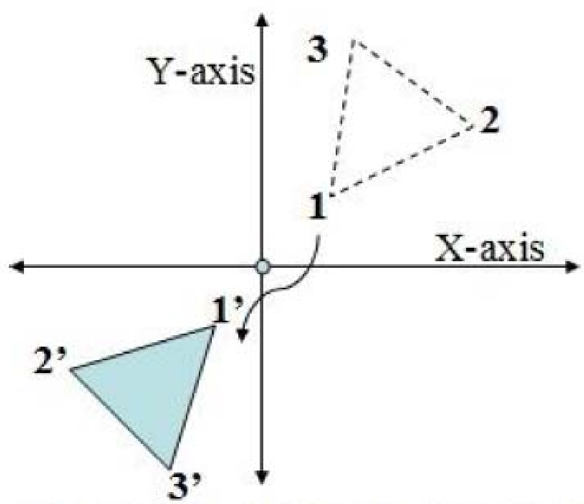
So $x' = -x$ and $y' = -y$

i.e.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Reflection of an object about the Y axis (X=0).



Reflection of an object about origin

Reflection

Reflection about y=x

Steps required:

i. Rotate about origin in clockwise direction by 45 degree

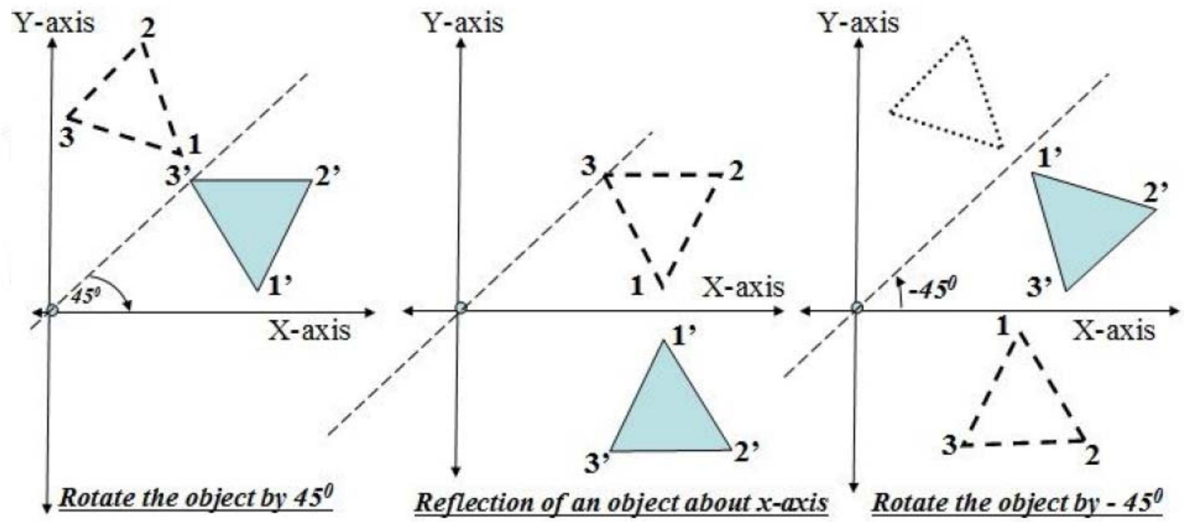
$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

ii. Take reflection against x-axis

$$R_f = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

iii. Rotate in anti-clockwise direction by same angle

$$R' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Reflection

Reflection about y=x

We have θ = 45

Solving all these steps, we get final result

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$R_f = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$R' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Reflection

Reflection about y=-x

Steps required:

i. Rotate about origin in clockwise direction by 45 degree

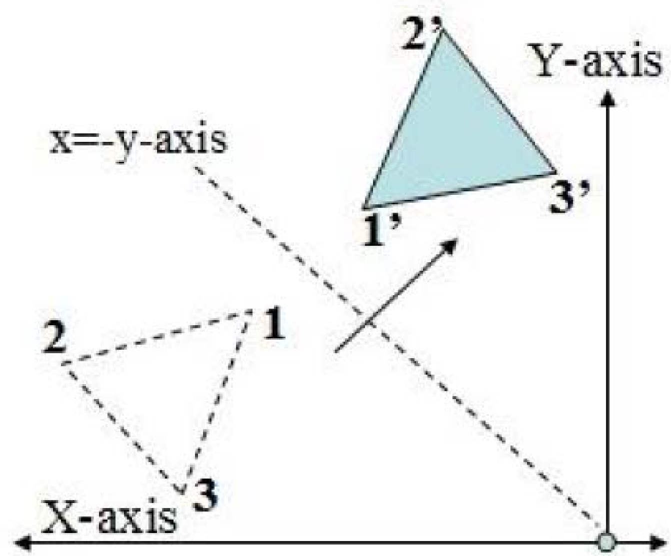
$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

ii. Take reflection against y-axis

$$R_f = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

iii. Rotate in anti-clockwise direction by same angle

$$R' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Reflection of an object about x= -y in clockwise direction

Reflection

Reflection about y=-x

We have $\theta = 45$

Solving all these steps, we get final result

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$R_f = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

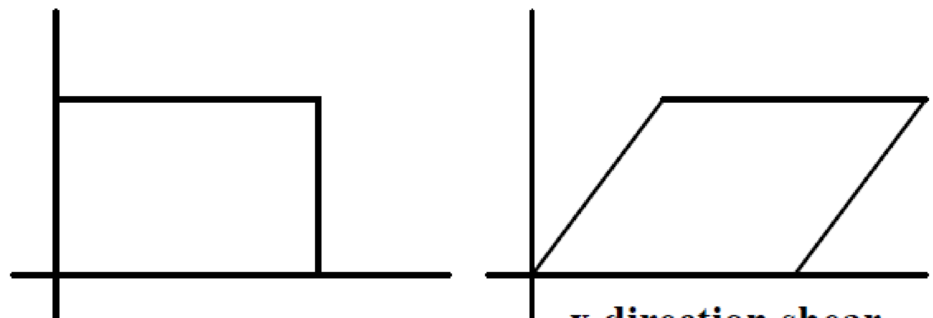
Shearing

- Shearing distorts the shape of an object in either x or y or both direction
- In case of single directional shearing (e.g. in ‘x’ direction can be viewed as an object made up of very thin layer and slid over each other with the *base* remaining where it is).

in ‘x’ direction,

$$x' = x + s_{hx} \cdot y$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & s_{hx} \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$

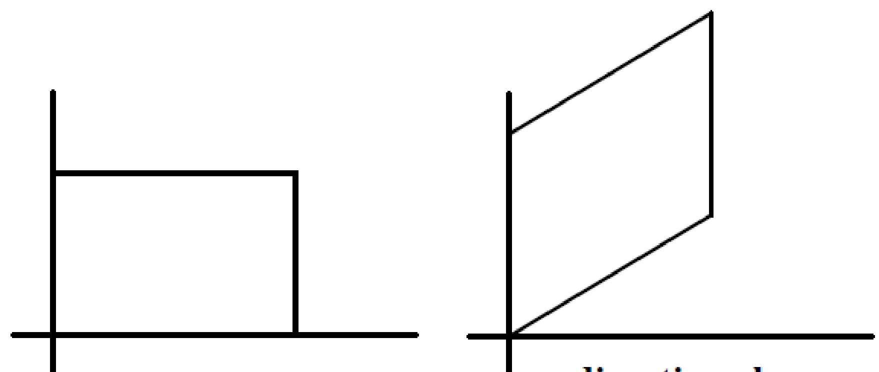


x direction shear

in ‘y’ direction,

$$x' = x$$
$$y' = y + s_{hy} \cdot x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s_{hy} & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$



y direction shear

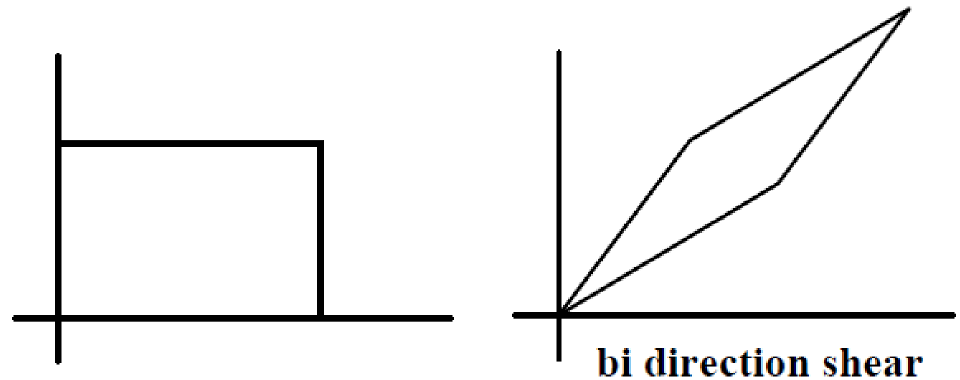
Shearing

in both directions,

$x' = x + s_{hx} \cdot y$

$y' = y + s_{hy} \cdot x$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & s_{hx} \\ s_{hy} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$



Rotation of a point about an arbitrary pivot position can be seen in the figure.

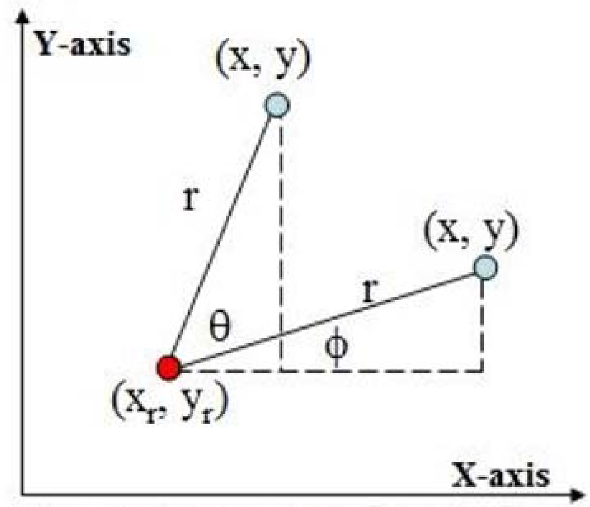


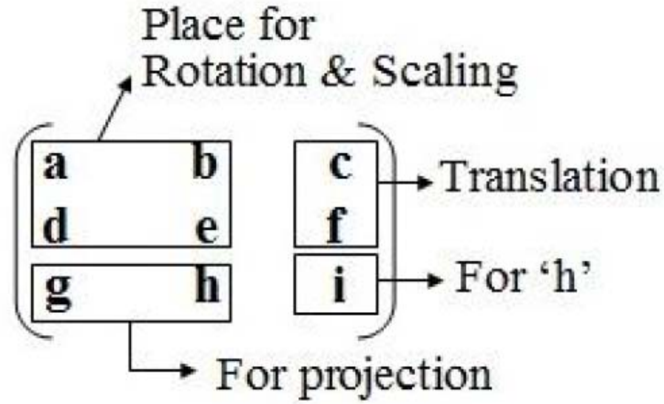
Fig. Rotating a point from position (x, y) to position (x', y') through an angle θ about rotation point (x_r, y_r).

Here,

$$x' = x_r + (x-x_r)\cos\theta \ - \ (y- y_r)\sin\theta$$
$$y' = y_r + (x- x_r)\sin\theta \ + (y- y_r)\cos\theta$$

Note: - This can also be achieved by translating the arbitrary point into the origin and then apply the rotation and finally perform the reverse translation.

Homogeneous Coordinate



- The matrix representations for translation, scaling and rotation are respectively:
  1. Translation: **P’ = T + P** (*Addition*)
  2. Scaling: **P’ = S . P** (*Multiplication*)
  3. Rotation: **P’ = R . P** (*Multiplication*)

- Since, the composite transformation include many sequence of translation, rotation etc and hence the many naturally different addition & multiplication sequence have to perform by the graphics allocation.

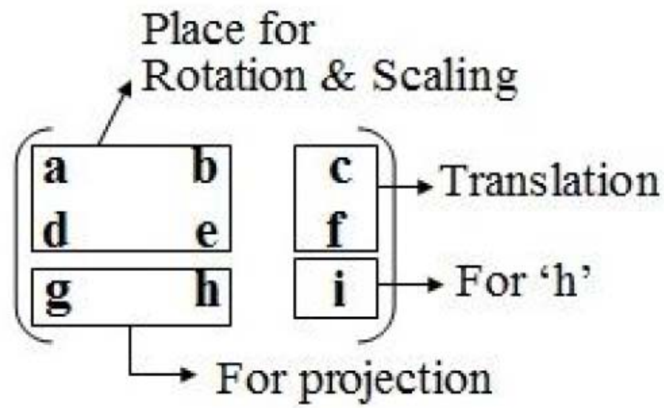
- Hence, the applications will take more time for rendering. Thus, we need to treat all three transformations in a consistent way so they can be combined easily & compute with one mathematical operation.

- If points are expressed in homogenous coordinates, all geometrical transformation equations can be represented as matrix multiplications.

- Here, in case of homogenous coordinates we add a third coordinate ‘*h*’ to a point (x, y) so that each point is represented by (*h*x, *h*y, *h*). The ‘*h*’ is normally set to 1. If the value of ‘h’ is more the one value then all the co-ordinate values are scaled by this value.

- Coordinates of a point are represented as three element column vectors, transformation operations are written as 3 x 3 matrices.

Homogeneous Coordinate



- **For Translation**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = T(t_x, t_y).P$$

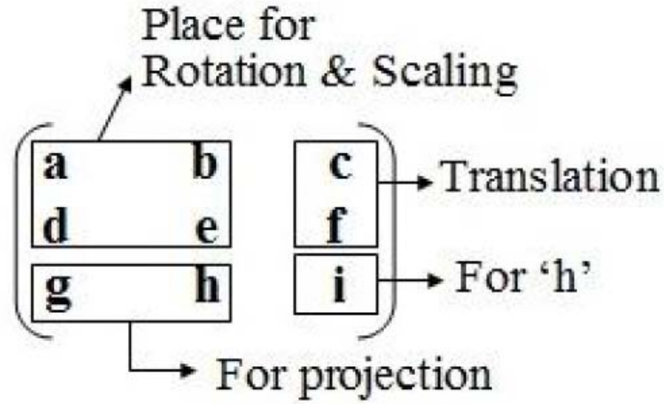
- **For rotation**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = R(\theta).P$$

- **For Scaling**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = S(s_x, s_y).P$$

Homogeneous Coordinate



- **For Reflection about X-axis**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **For Reflection about Y-axis**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

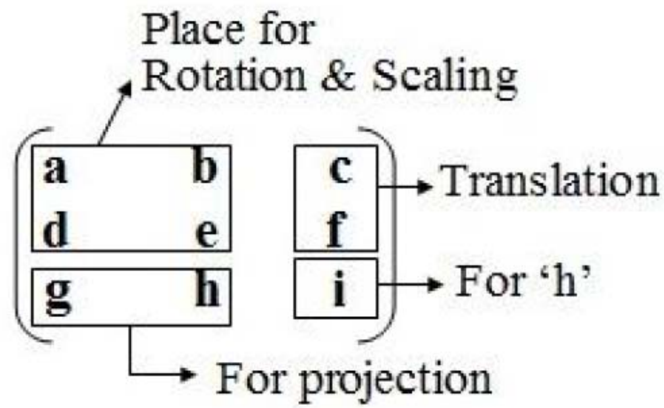
Homogeneous Coordinate

- **For Reflection about y=x**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **For Reflection about y=-x**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Composite Transformation

- With the matrix representation of transformation equations it is possible to setup a matrix for any sequence of transformations as a composite transformation matrix by calculating the matrix product of individual transformation.
- Forming products of transformation matrices is often referred to as a **concatenation,** or **composition,** of matrices.
- For column matrix representation of coordinate positions we form composite transformation by multiplying matrices in order from **right to left.**

i. Two Successive Translation are Additive

Let two successive translation vectors (t_{x1} , t_{y1}) and (t_{x2} , t_{y2}) are applied to a coordinate position P then
or, $P' = \{T(t_{x2}, t_{y2}) \cdot T(t_{x1}, t_{y1})\} \cdot P$
Here the composite transformation matrix for this sequence of translation is

$$\begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

or, $$T(t_{x2}, t_{y2}) \cdot T(t_{x1}, t_{y1}) = T(t_{x1} + t_{x2}, t_{y1} + t_{y2})$$

Composite Transformation

ii. Two successive Scaling operations are Multiplicative

Let (s_{x1} , s_{y1}) and (s_{x2} , s_{y2}) be two successive vectors applied to a coordinate position P then the composite scaling matrix thus produced is

or, $P' = \{S(s_{x2}, s_{y2}) \cdot S(s_{x1}, s_{y1})\} \cdot P$

Here the composite transformation matrix for this sequence of translation is

$$\begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{x2} \cdot s_{x1} & 0 & 0 \\ 0 & s_{y2} \cdot s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or, $S(s_{x2}, s_{y2}) \cdot S(s_{x1}, s_{y1}) = S(s_{x1} \cdot s_{x2}, s_{y1} \cdot s_{y2})$

Composite Transformation

iii. Two successive Rotation operations are Additive

Let $R(\theta_1)$ and $R(\theta_2)$ be two successive rotations applied to a coordinate position P then the composite scaling matrix thus produced is

or, $P' = \{R(\theta_2) \,.\, R(\theta_1)\} \,.\, P$

Here the composite transformation matrix for this sequence of translation is

or,
$$\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or,
$$\begin{bmatrix} \cos\theta_2 . \cos\theta_1 - \sin\theta_2 . \sin\theta_1 & -\cos\theta_2 . \sin\theta_1 - \sin\theta_2 . \cos\theta_1 & 0 \\ \sin\theta_2 . \cos\theta_1 + \sin\theta_1 . \cos\theta_2 & \cos\theta_2 . \cos\theta_1 - \sin\theta_2 . \sin\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or,
$$\begin{bmatrix} \cos(\theta_2 + \theta_1) & -\sin(\theta_2 + \theta_1) & 0 \\ \sin(\theta_2 + \theta_1) & \cos(\theta_2 + \theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or, $R(\theta_2) \,.\, R(\theta_1) = R(\theta_2 + \theta_1)$

Composite Transformation

Reflection about y=x

Steps required:

i. Rotate about origin in clockwise direction by 45 degree $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ii. Take reflection against x-axis $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

iii. Rotate in anti-clockwise direction by same angle $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Representing these steps in composite matrix

$$\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Step 3}} \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}}_{\text{Step 2}} \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Step 1}}$$

Composite Transformation

Reflection about y=mx+c

Steps required:

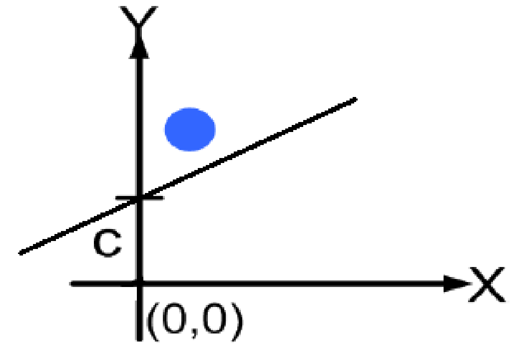
i. First translate the line so that it passes through the origin

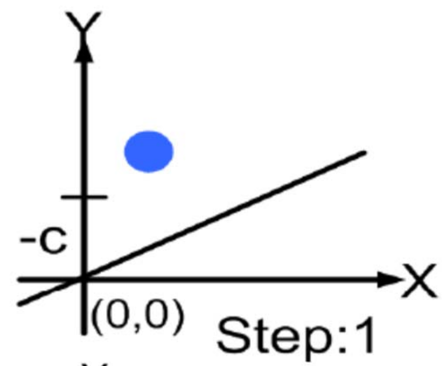
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

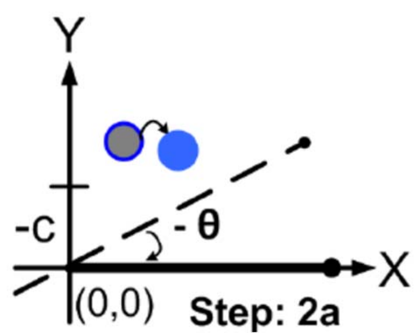
ii. Rotate the line onto one of the coordinate axes(say x-axis) and reflect about that axis (x-axis)

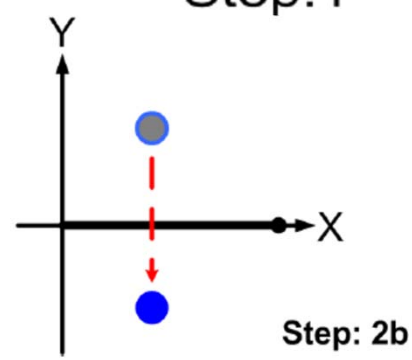
$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} --- \mathit{rotation}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} --- \mathit{reflection}$$









Composite Transformation

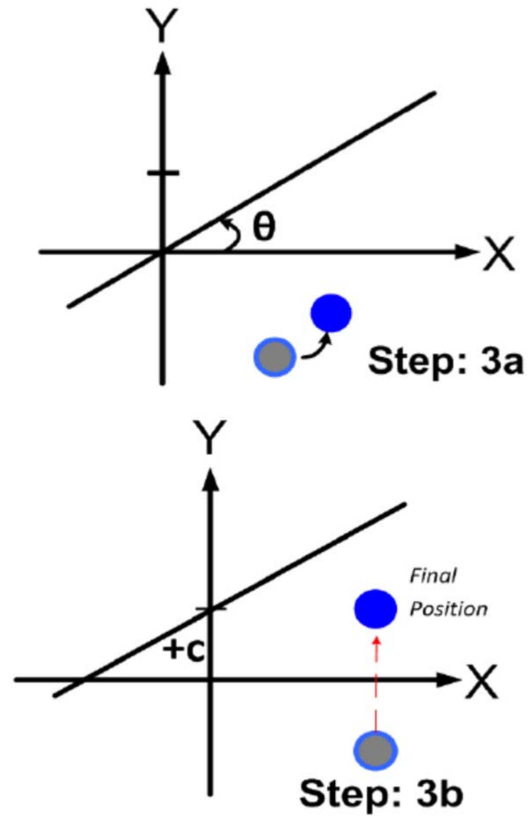
Reflection about y=mx+c

Steps required:

iii. Finally, restore the line to its original position with the inverse rotation and translation transformation.

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} --- rotation$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} --- translation$$



Composite Transformation

Reflection about y=mx+c

So, the multiplication sequence will be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

Solving all these we will get

$$\begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} & \frac{-2cm}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} & \frac{2c}{1+m^2} \\ 0 & 0 & 1 \end{bmatrix}$$

Since we have

$\tan\theta = m$

$\sec^2\theta = 1+\tan^2\theta$

$\sec\theta = \sqrt{1+\tan^2\theta}$

$\cos\theta = \frac{1}{\sec\theta}$

$= \frac{1}{\sqrt{1+m^2}}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta = \tan\theta . \cos\theta = \frac{m}{\sqrt{1+m^2}}$

Rotate the triangle (5,5), (7,3), (3,3) about fixed point (5,4) in counter clockwise by 90 degree.

Solution:

$$C.M. = T_{(5,4)}.R_{90}T_{(-5,-4)}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 90^0 & -\sin 90^0 & 0 \\ \sin 90^0 & \cos 90^0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 9 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $P' = C.M.* P$

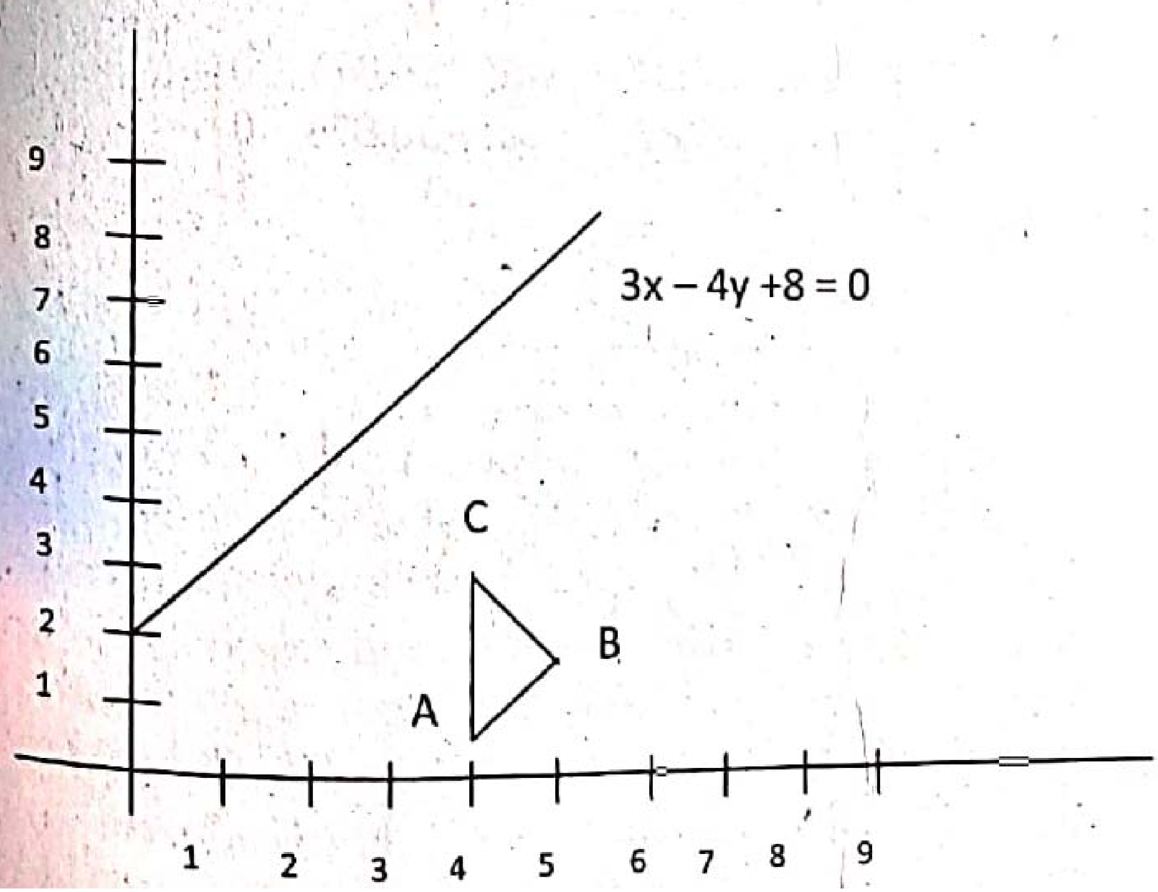
$$= \begin{bmatrix} 0 & -1 & 9 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 3 \\ 5 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -5+9 & -3+9 & -3+9 \\ 5-1 & 7-1 & 3-1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 6 \\ 4 & 6 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

∴ New co-ordinates are (4,4), (6,6) (6,2)

Reflect the triangle ABC about the line 3x -4y + 8 = 0 the position vector of coordinate ABC as A(4,1), B(5,2) and C(4,3)

Solution:



The arbitrary line about which the triangle ABC has to be reflected is $3x - 4y + 8 = 0$

i.e.; $y = \frac{3}{4}x + 2$

$m = \frac{3}{4}$

$c = 2$

$\theta = \tan^{-1}(m) = \tan^{-1}\frac{3}{4} = 36.8698°$

$\text{C.M.} = T^{-1}R^{-1}_{\theta}R_{fx}R_{\theta}T$

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{\theta} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(36.87) & \sin(36.87) & 0 \\ -\sin(36.87) & \cos(36.87) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{fx} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(-\theta) = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-36.87) & \sin(-36.87) & 0 \\ -\sin(-36.87) & \cos(-36.87) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{C.M.} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \text{C.M.} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Two Dimensional Viewing

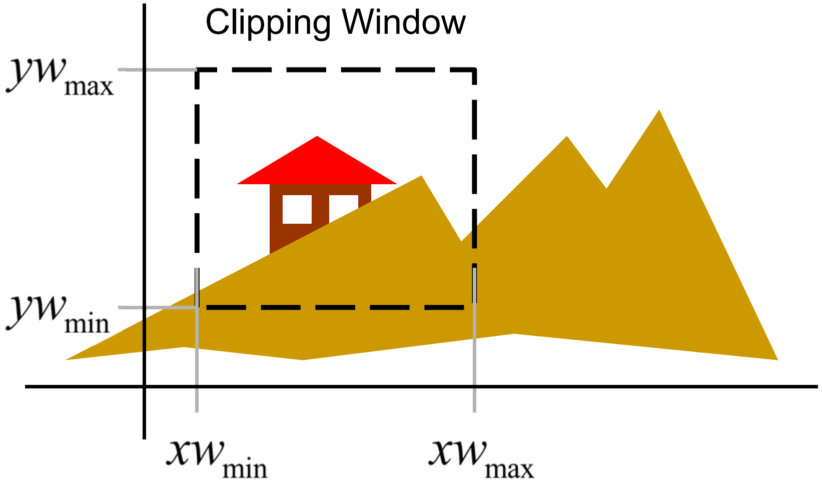
- **Window**
  - a world coordinate area selected for display
  - define what is to be viewed
- **View-port**
  - An area on a display device to which a window is mapped
  - Define where it is to be displayed
- **Windows and view-port**
  - Rectangle in standard positions, with rectangle edges parallel to coordinate axes
  - Other geometric takes longer to proceed
- **Viewing transformation**
  - the mapping of a world coordinate scene to device coordinates
- **Transformation pipeline**
  - Takes the object coordinates through several intermediate coordinates systems before finishing with device coordinates

Two Dimensional Viewing

Window in world coordinates.

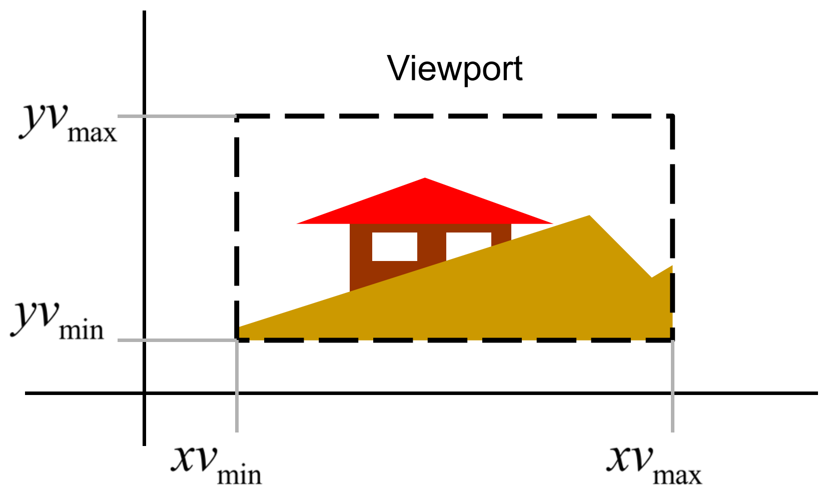
Viewport in
Device coordinates

Two Dimensional Viewing



World Coordinates

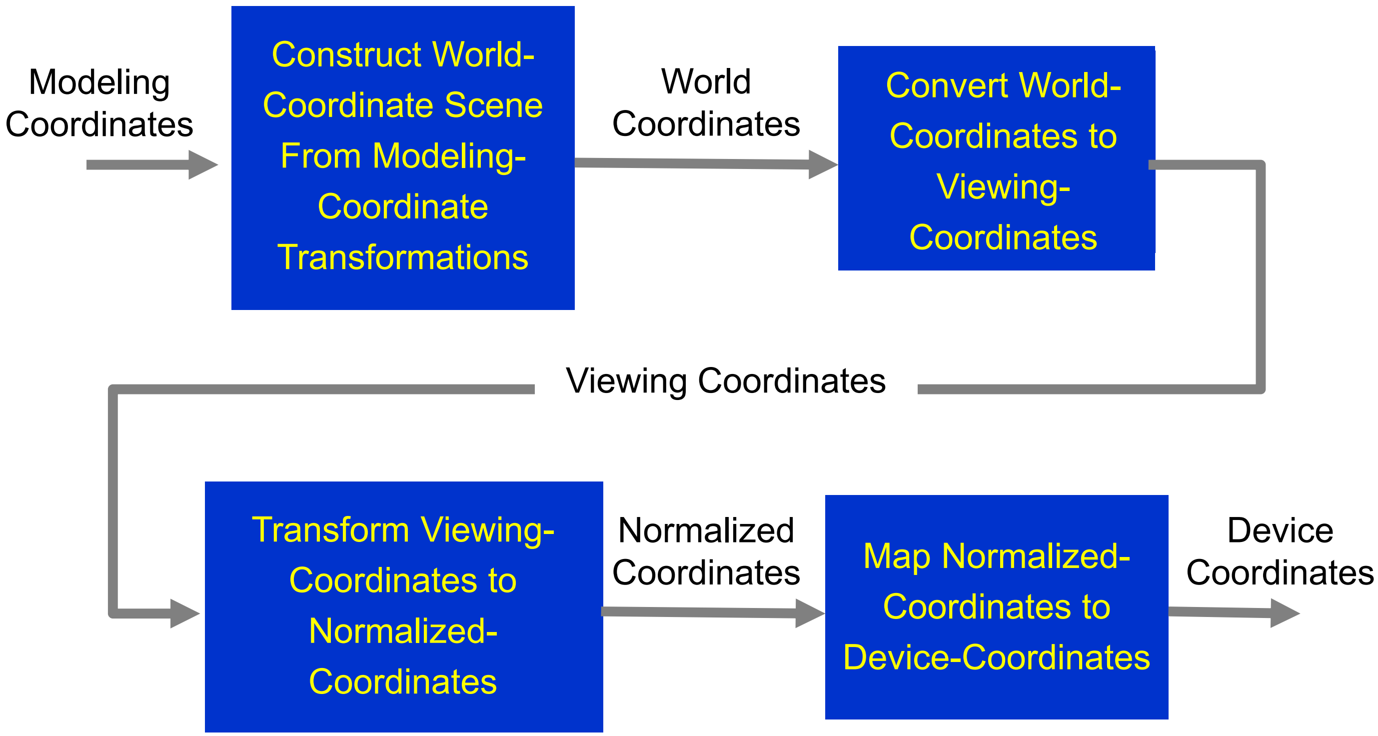
The clipping window is mapped into a viewport.



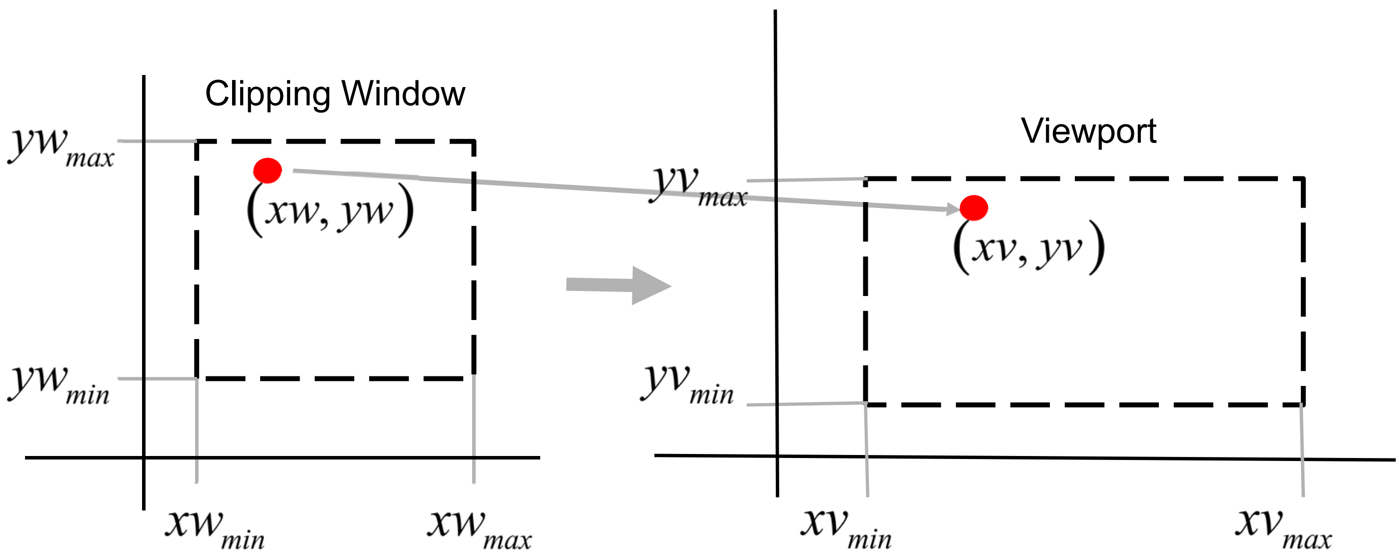
Viewing world has its own coordinates, which may be a non-uniform scaling of world coordinates.

Viewport Coordinates

Two Dimensional Viewing Transformation Pipeline



Window to Viewport Co-ordinate Transformation



Maintain relative size and position between clipping window and viewport.

$$\frac{xv - xv_{\min}}{xv_{\max} - xv_{\min}} = \frac{xw - xw_{\min}}{xw_{\max} - xw_{\min}} \qquad \frac{yv - yv_{\min}}{yv_{\max} - yv_{\min}} = \frac{yw - yw_{\min}}{yw_{\max} - yw_{\min}}$$

Window to Viewport Co-ordinate Transformation

$$\frac{xv - xv_{\min}}{xv_{\max} - xv_{\min}} = \frac{xw - xw_{\min}}{xw_{\max} - xw_{\min}} \; and \; \frac{yv - yv_{\min}}{yv_{\max} - yv_{\min}} = \frac{yw - yw_{\min}}{yw_{\max} - yw_{\min}}$$

Solving we get,

$$xv = xv_{\min} + (xw - xw_{\min})sx$$

$$yv = yv_{\min} + (yw - yw_{\min})sy$$

where scaling factors

$$sx = \frac{xv_{\max} - xv_{\min}}{xw_{\max} - xw_{\min}}$$

$$sy = \frac{yv_{\max} - yv_{\min}}{yw_{\max} - yw_{\min}}$$

Three Dimensional Graphics

Three Dimensional Transformation

- **Translation**

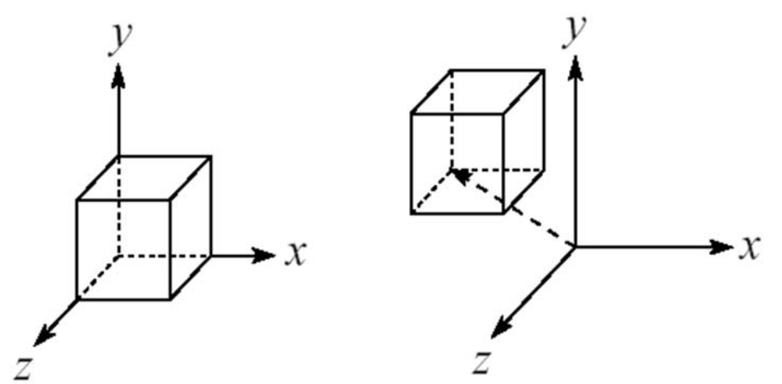
In a three dimensional homogenous coordinate representation, a point is translated from position P = (x,y,z) to position P' = (x',y',z') by following operation

$x' = x + t_x \qquad y' = y + t_y \qquad z' = z + t_z$

where the pair (t_x, t_y, t_z) is called the *translation vector.*

In matrix form,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$



$$P' = P \bullet T$$

Three Dimensional Transformation

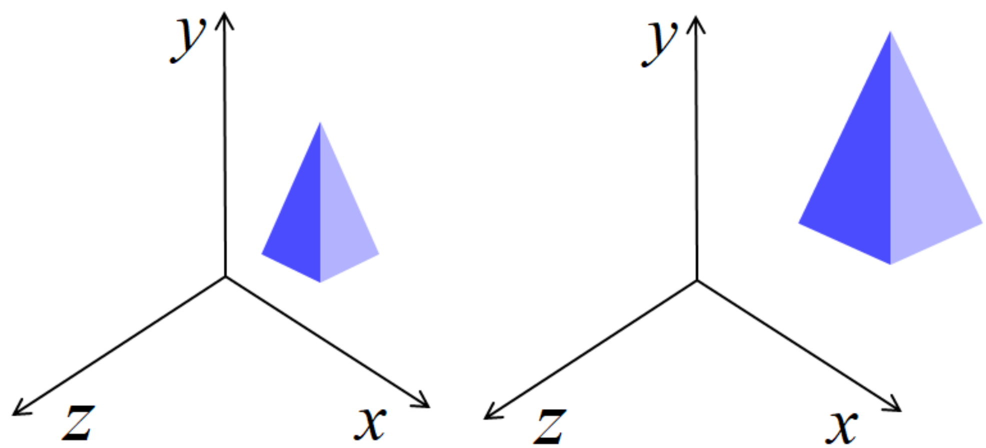
- **Scaling**

The three dimensional homogeneous coordinate representation of scaling about origin is

$x' = x.s_x$ $\qquad$ $y' = y.s_y$ $\qquad$ $z = z.s_z$

In Matrix form,

$$\mathbf{P'} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{S} \cdot \mathbf{P}$$

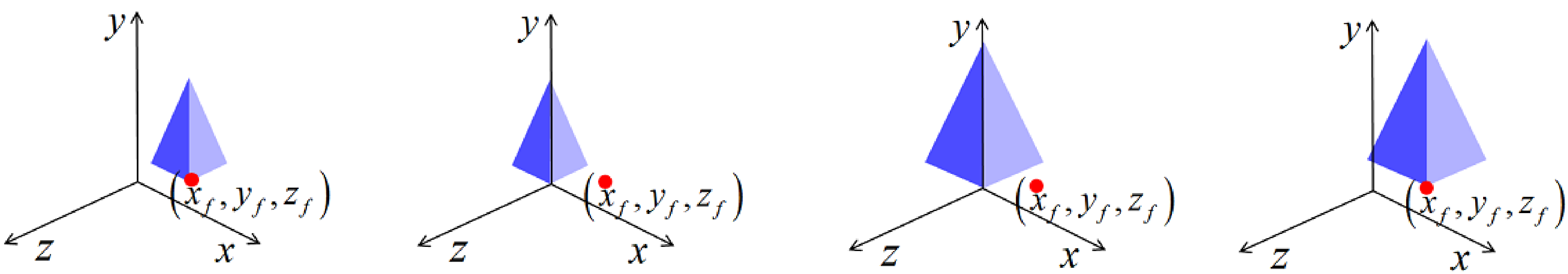


Three Dimensional Transformation

- **Scaling**

Scaling with respect to any fixed position (x_f, y_f, z_f) can be represented with following transformation sequence

1. Translate the fixed point to the origin
2. Scale the object relative to the coordinate origin
3. Translate the fixed point back to its original position



Three Dimensional Transformation

- **Scaling**

$$T(x_f, y_f, z_f) \cdot S(s_x, s_y, s_z) \cdot T(-x_f, -y_f, -z_f) = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_f \\ 0 & 1 & 0 & y_f \\ 0 & 0 & 1 & z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_f \\ 0 & 1 & 0 & -y_f \\ 0 & 0 & 1 & -z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\mathbf{T} \cdot \mathbf{S} \cdot \mathbf{T}^{-1} = \begin{bmatrix} S_x & 0 & 0 & (1 - S_x) x_f \\ 0 & S_y & 0 & (1 - S_y) y_f \\ 0 & 0 & S_z & (1 - S_z) z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Three Dimensional Transformation

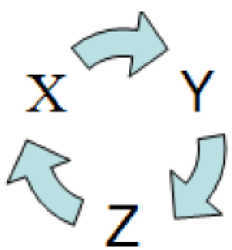
- **Rotation**

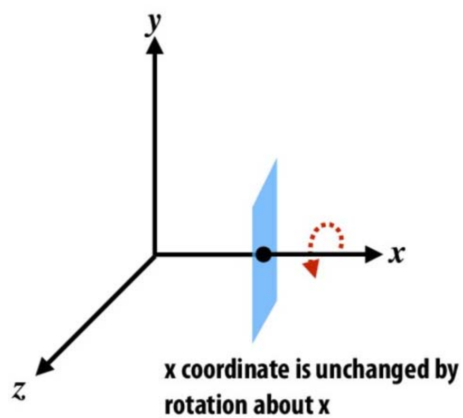
➢ To generate a rotation transformation for an object in 3D space, we require the following:
  - Angle of rotation.
  - Pivot point
  - Direction of rotation
  - Axis of rotation

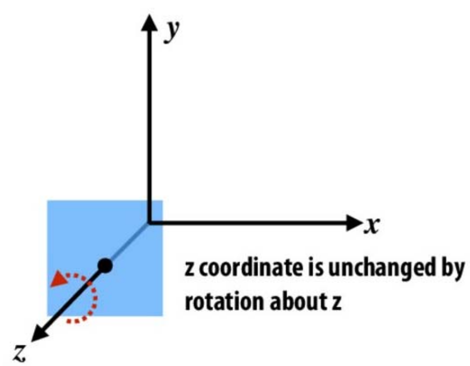
➢ Axes that are parallel to the coordinate axes are easy to handle.

➢ Cyclic permutation of the coordinate parameters x, y and z are used to get transformation equations for rotations about the coordinates

$x \rightarrow y \rightarrow z$







Three Dimensional Transformation

- **Rotation**

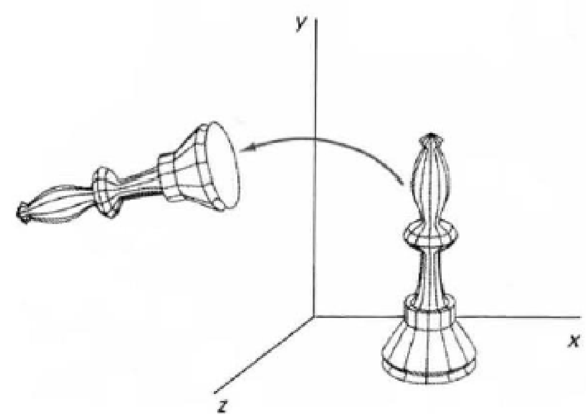
➢ Taking origin as the centre of rotation, when a point P(x, y, z) is rotated through an angle θ about any one of the axes to get the transformed point P'(x', y', z'), we have the following equation for each.

➢ 3D z-axis rotation equations are expressed in homogenous coordinate form as

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

➢ In Matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



P' = Rz (θ) . P

Three Dimensional Transformation

- **Rotation**

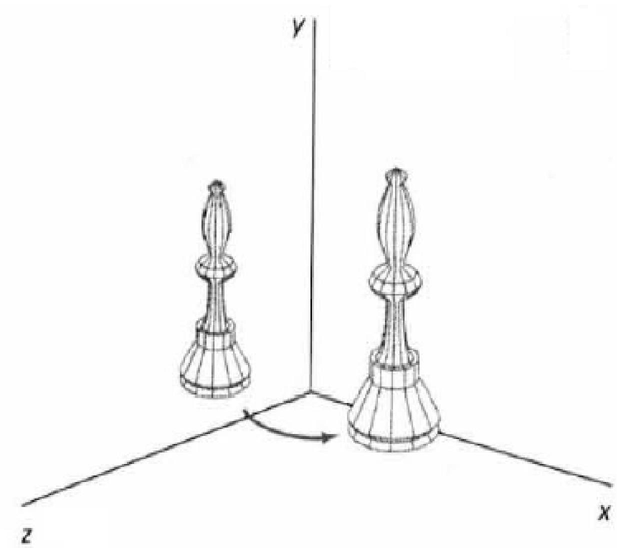
➢ 3D y-axis rotation equations are expressed in homogenous coordinate form as

$$x' = z\sin\theta + x\cos\theta$$
$$y' = y$$
$$z' = z\cos\theta - x\sin\theta$$

➢ In Matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

P' = Ry (θ) . P



Three Dimensional Transformation

• Rotation

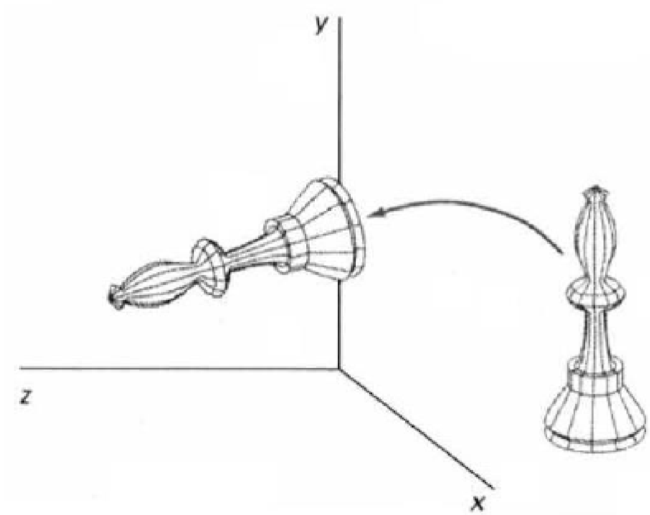
➢ 3D x-axis rotation equations are expressed in homogenous coordinate form as

$x' = x$
$y' = y\cos\theta - z\sin\theta$
$z' = y\sin\theta + z\cos\theta$

➢ In Matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

P' = Rx (θ) . P



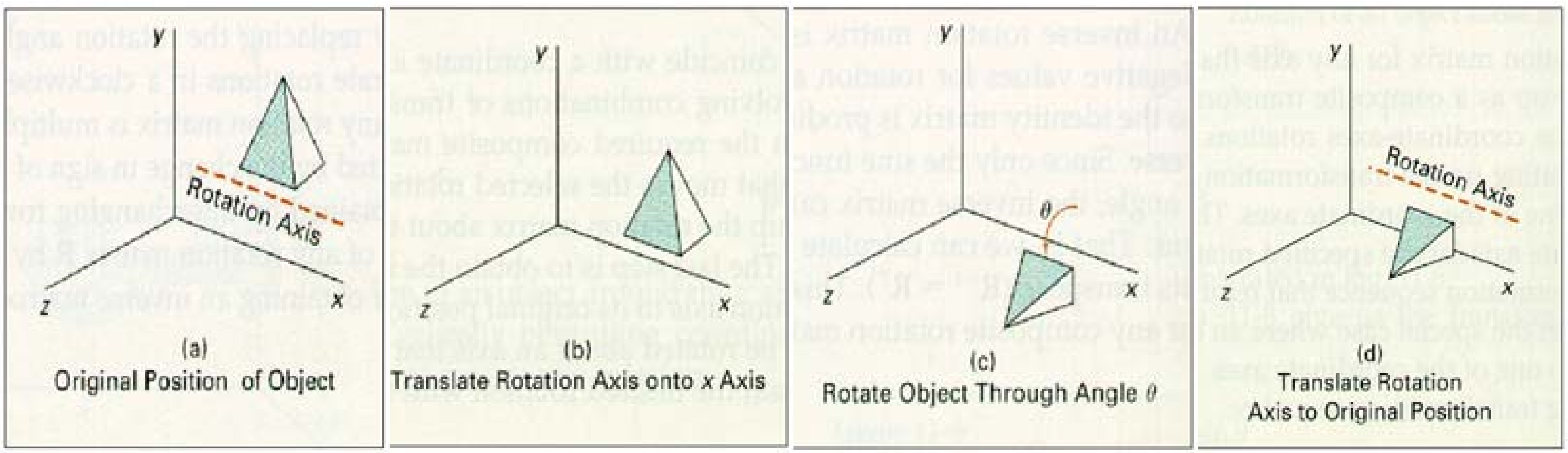
Three Dimensional Transformation

- **Rotation about an axis parallel to one of the coordinate axes:**

➢ Steps:

- Translate object so that rotation axis coincides with the parallel coordinate axis.
- Perform specified rotation about that axis
- Translate object back to its original position.

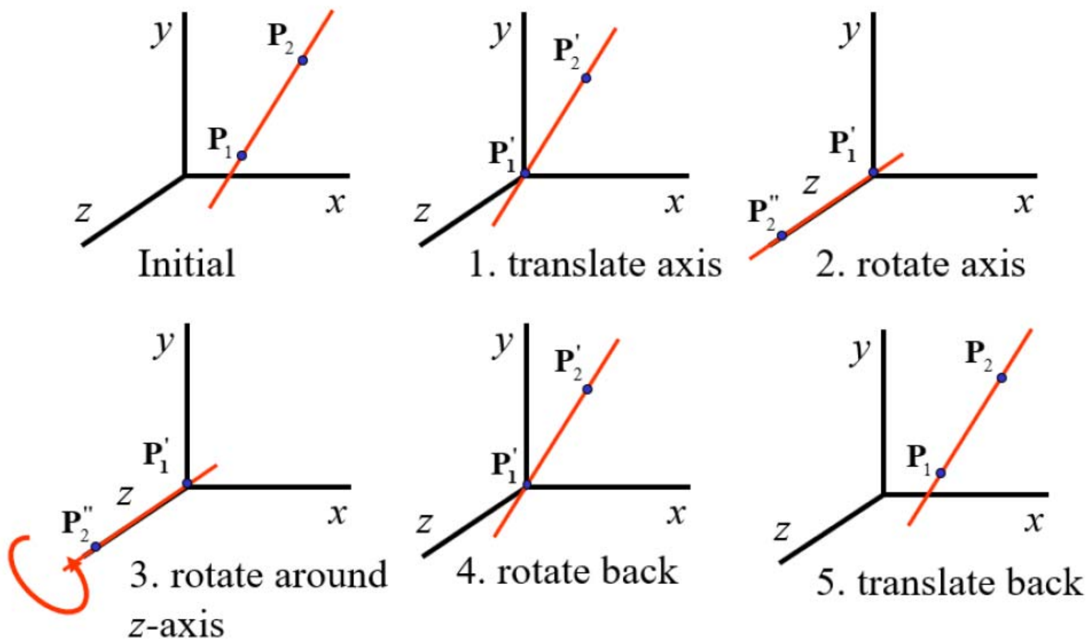
i.e. $\mathbf{P' = [\ T^{-1} .\ R_x(\theta)\ .\ T\ ]\ .\ P}$



Three Dimensional Transformation

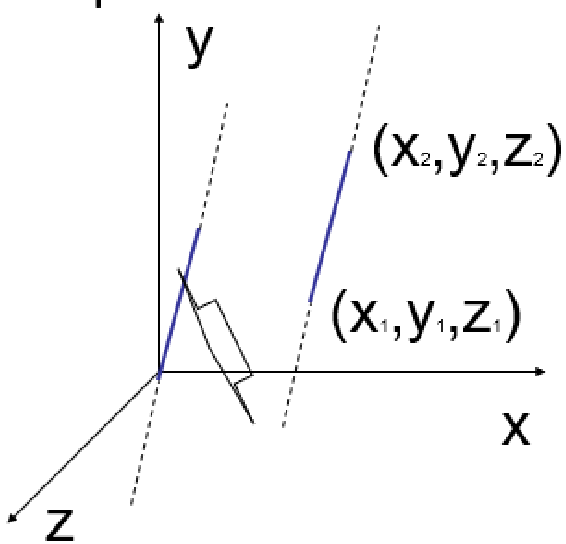
- **Rotation about any arbitrary axis in 3D Space**

- Steps:
    - ➢ Translate the object such that rotation axis passes through the origin.
    - ➢ Rotate the object such that rotation axis coincides with one of Cartesian axes.
    - ➢ Perform specified rotation about the Cartesian axis.
    - ➢ Apply inverse rotation to return rotation axis to original direction.
    - ➢ Apply inverse translation to return rotation axis to original position.



Three Dimensional Transformation

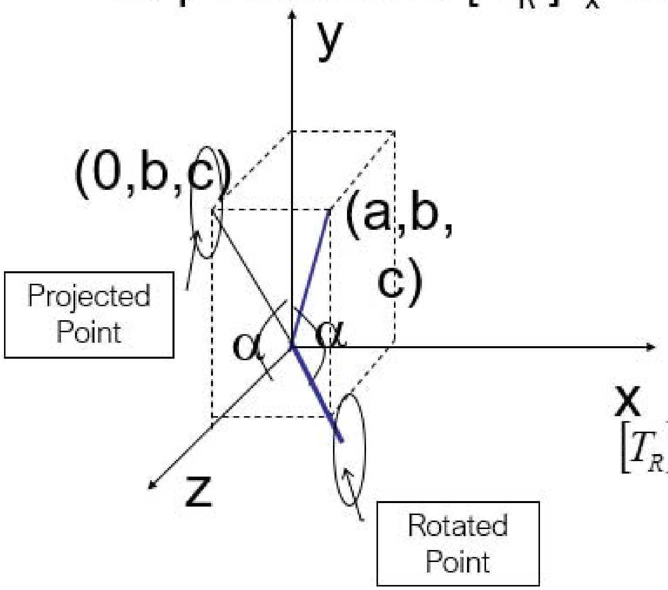
- **Rotation about any arbitrary axis in 3D Space**
    - Step 1. Translation



$$T_{TR} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Three Dimensional Transformation

- **Rotation about any arbitrary axis in 3D Space**
  - Step 2. Establish $[\ T_R\ ]^{\alpha}_{x}$ *x* axis



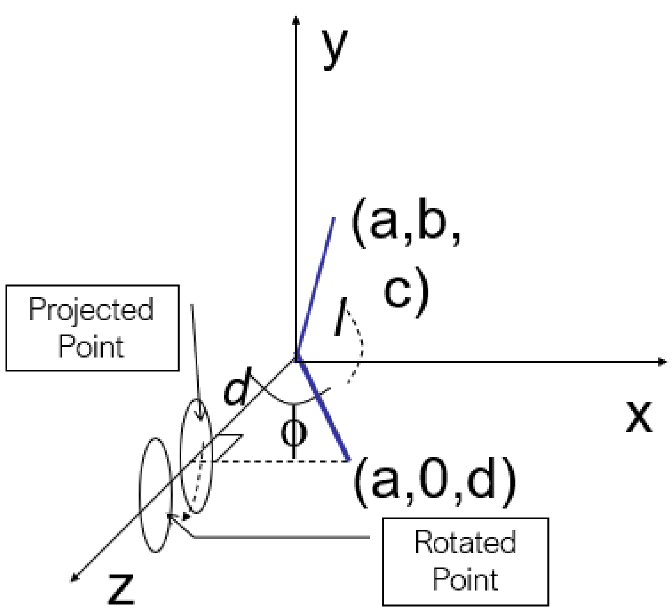
$$\sin\alpha = \frac{b}{\sqrt{b^2+c^2}} = \frac{b}{d}$$

$$\cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}$$

$$[T_R]^{\alpha}_{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Three Dimensional Transformation

- **Rotation about any arbitrary axis in 3D Space**
  - Step 3. Rotate about *y* axis by ϕ



$$\sin\phi = \frac{a}{l}, \qquad \cos\phi = \frac{d}{l}$$

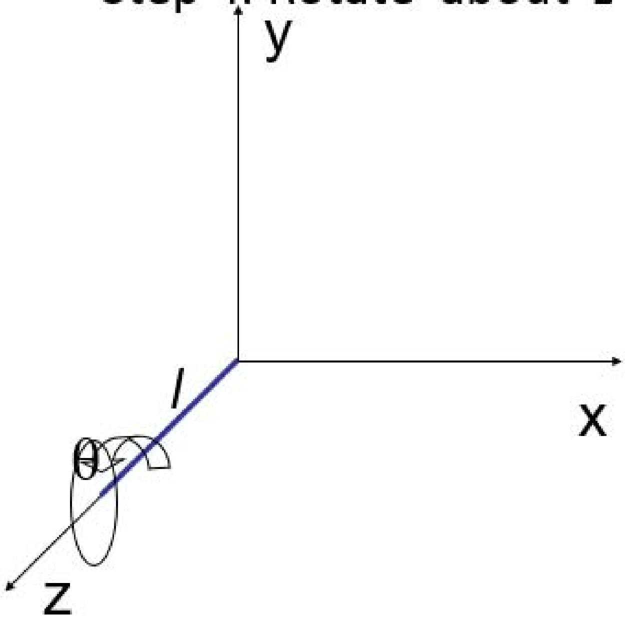
$$l^2 = a^2 + b^2 + c^2 = a^2 + d^2$$

$$d = \sqrt{b^2 + c^2}$$

$$[T_R]_y^{\phi} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d/l & 0 & -a/l & 0 \\ 0 & 1 & 0 & 0 \\ a/l & 0 & d/l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Three Dimensional Transformation

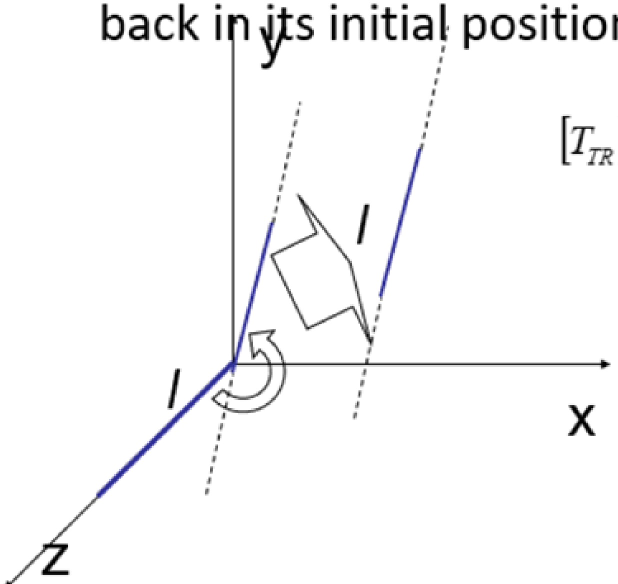
- **Rotation about any arbitrary axis in 3D Space**
  - Step 4. Rotate about *z* axis by the desired angle θ



$$[T_R]_z^\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Three Dimensional Transformation

- **Rotation about any arbitrary axis in 3D Space**
  - Step 5. Apply the reverse transformation to place the axis back in its initial position

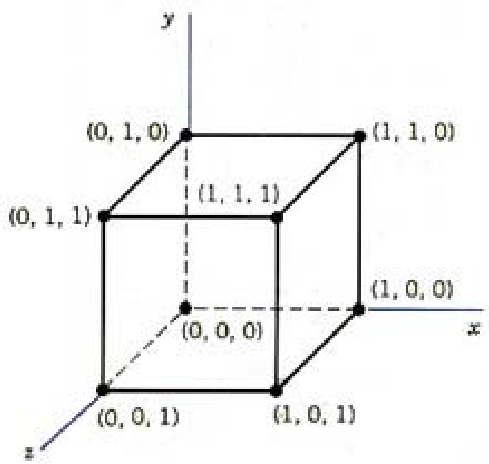


$$[T_{TR}]^{-1}[T_R]_x^{-\alpha}[T_R]_y^{-\phi} = \begin{bmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T_R]_{ARB} = [T_{TR}]^{-1}[T_R]_x^{-\alpha}[T_R]_y^{-\phi}[T_R]_z^{\theta}[T_R]_y^{\phi}[T_R]_x^{\alpha}[T_{TR}]$$

Example

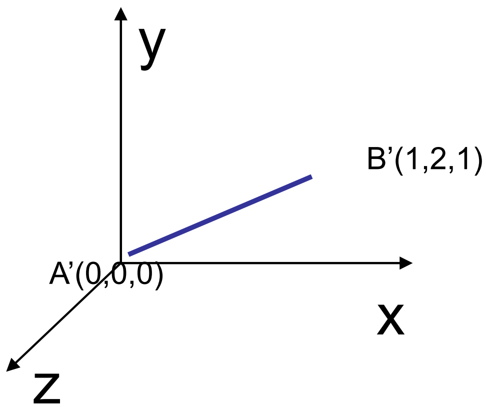
Find the new coordinates of a unit cube 90º-rotated about an axis defined by its endpoints A(2,1,0) and B(3,3,1).



A Unit Cube

Example

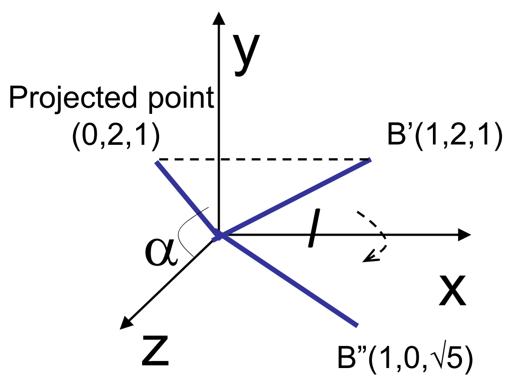
- Step1. Translate point A to the origin



$$[T_{TR}] = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Step 2. Rotate axis *A'B'* about the x axis by an angle α, until it lies on the *xz* plane.



$$\sin\alpha = \frac{2}{\sqrt{2^2+1^2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\cos\alpha = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\sin\alpha = \frac{b}{\sqrt{b^2+c^2}} = \frac{b}{d}$$

$$\cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}$$

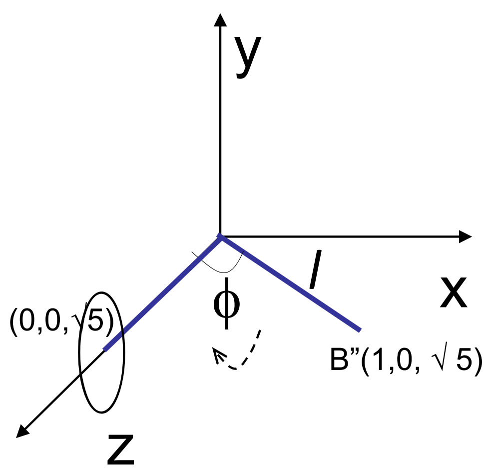
$$l = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$[T_R]_x^\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} & 0 \\ 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Step 3. Rotate axis *A'B''* about the y axis by and angle ϕ, until it coincides with the z axis.



$$\sin\phi = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

$$\cos\phi = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$$

$$\sin\phi = \frac{a}{l}, \qquad \cos\phi = \frac{d}{l}$$

$$l^2 = a^2 + b^2 + c^2 = a^2 + d^2$$

$$d = \sqrt{b^2 + c^2}$$

$$[T_R]_y^{\phi} = \begin{bmatrix} \frac{\sqrt{30}}{6} & 0 & -\frac{\sqrt{6}}{6} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Step 4. Rotate the cube 90° about the z axis

$$[T_R]_z^{90^\circ} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, the concatenated rotation matrix about the arbitrary axis AB becomes,

$$[T_R]_{ARB} = [T_{TR}]^{-1}[T_R]_x^{-\alpha}[T_R]_y^{-\phi}[T_R]_z^{90^\circ}[T_R]_y^{\phi}[T_R]_x^{\alpha}[T_{TR}]$$

Example

$$[T_R]_{ARB} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} & 0 \\ 0 & -\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{30}}{6} & 0 & \frac{\sqrt{6}}{6} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{30}}{6} & 0 & -\frac{\sqrt{6}}{6} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} & 0 \\ 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.166 & -0.075 & 0.983 & 1.742 \\ 0.742 & 0.667 & 0.075 & -1.151 \\ -0.650 & 0.741 & 0.167 & 0.560 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- Multiplying $[T_R]_{AB}$ by the point matrix of the original cube

$$[P^*]=[T_R]_{ARB}.[P]$$

$$[P^*]=\begin{bmatrix} 0.166 & -0.075 & 0.983 & 1.742 \\ 0.742 & 0.667 & 0.075 & -1.151 \\ -0.650 & 0.741 & 0.167 & 0.560 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 2.650 & 1.667 & 1.834 & 2.816 & 2.725 & 1.742 & 1.909 & 2.891 \\ -0.558 & -0.484 & 0.258 & 0.184 & -1.225 & -1.151 & -0.409 & -0.483 \\ 1.467 & 1.301 & 0.650 & 0.817 & 0.726 & 0.560 & -0.091 & 0.076 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Three Dimensional Transformation

- **Reflection**

➢ A three-dimensional reflection can be performed relative to a selected reflection axis or with respect to a selected reflection plane.

➢ The matrix representation for the reflection of a point relative to XY-plane is given by

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\boxed{\begin{array}{c} x' = x \\ y' = y \\ z' = -z \end{array}}$$

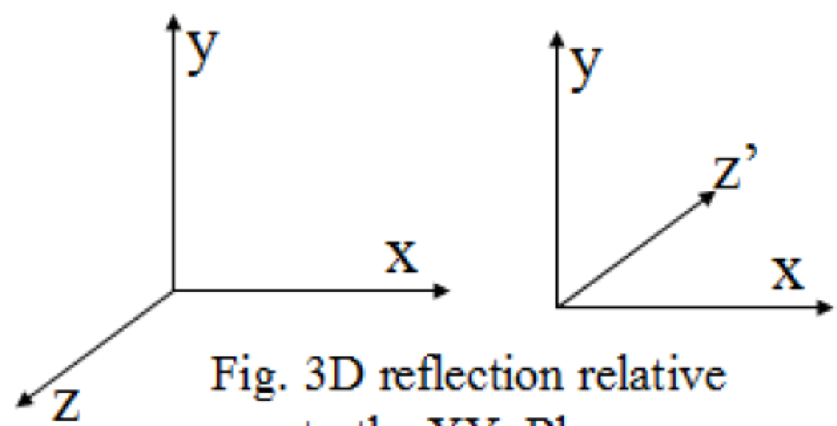


Fig. 3D reflection relative to the XY- Plane

Three Dimensional Transformation

- **Reflection**

➢ The matrix representation for the reflection of a point relative to YZ-plane is given by

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} x' &= -x \\ y' &= y \\ z' &= z \end{aligned}}$$

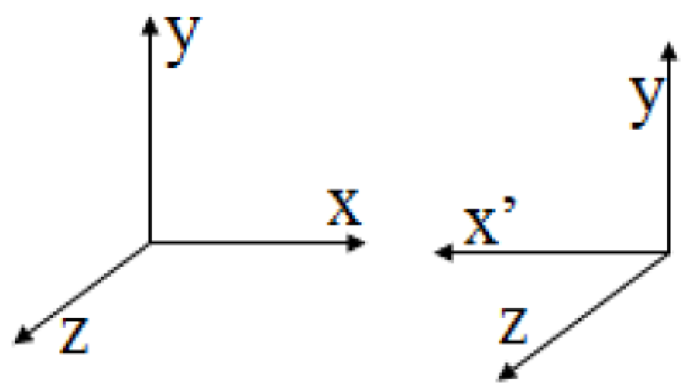


Fig. 3D reflection relative to the YZ- Plane

Three Dimensional Transformation

- **Reflection**

➢ The matrix representation for the reflection of a point relative to ZX-plane is given by

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\boxed{\begin{matrix} x' = x \\ y' = -y \\ z' = z \end{matrix}}$$

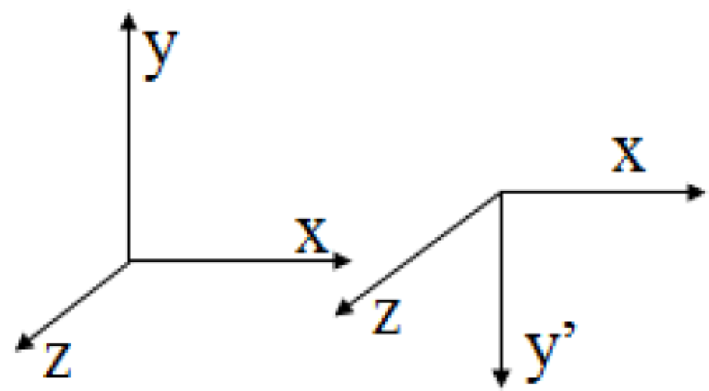


Fig. 3D reflection relative to the ZX- Plane

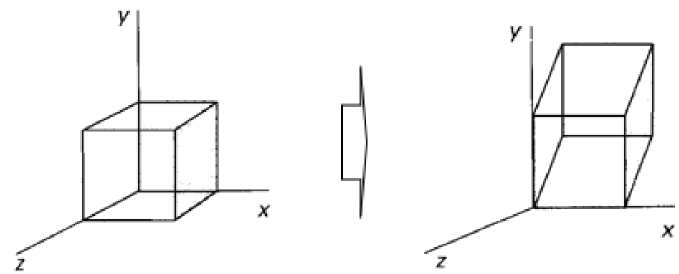
Three Dimensional Transformation

- **Shear**

➢ Shearing transformations are used to modify object shapes.

➢ shearing relative to the z axis:

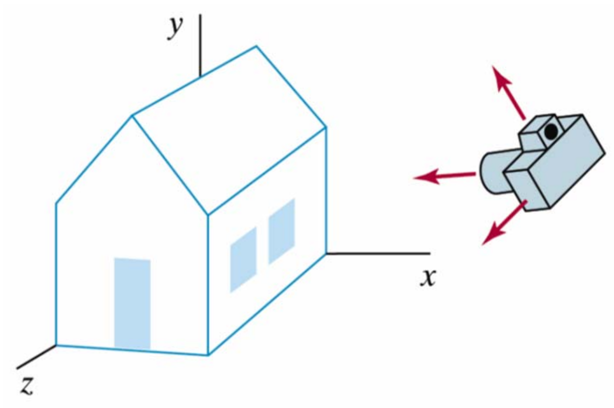
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



x' = x + az	a=shear factor for x
y' = y + bz	b=shear factor for y
z' = z	

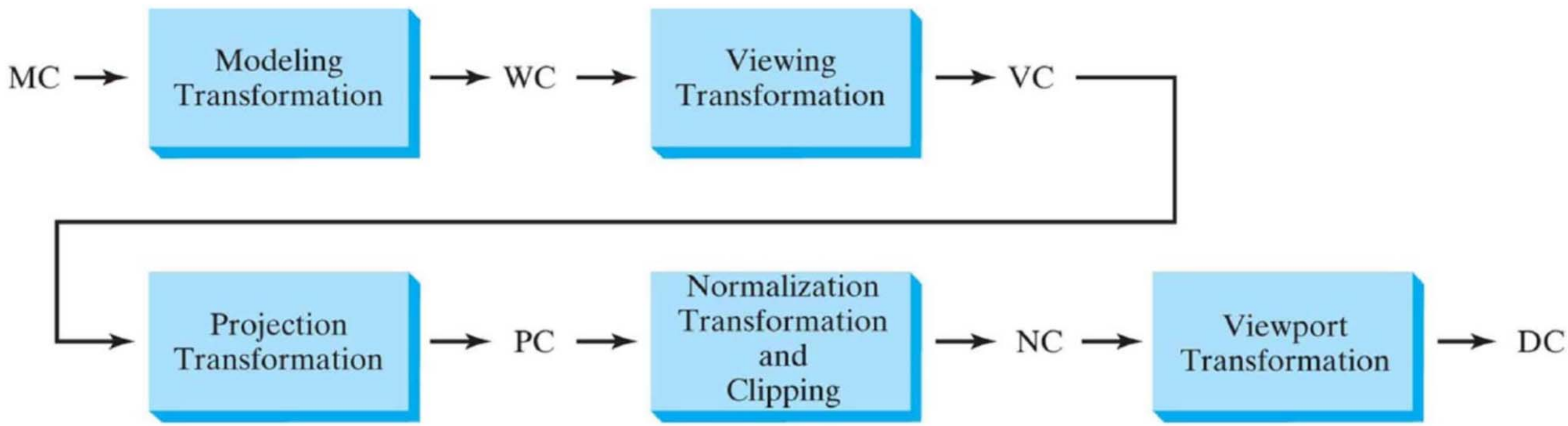
Three Dimensional Viewing Pipeline

- Three dimensional viewing process is inherently more complex than two dimensional viewing process because of the added dimension and the fact that even though object is three dimensional, the display device are only two dimensional
- The mismatch between 3D object and 2D display is compensated by introducing projection. The projection transform 3D objects into a 2D projection plane.
- The steps for computer generation of a view of a three dimensional scene are somewhat analogous to the processes involved in taking a photograph.
- It is the general processing steps for modeling and converting a world coordinate description of a scene to device coordinates



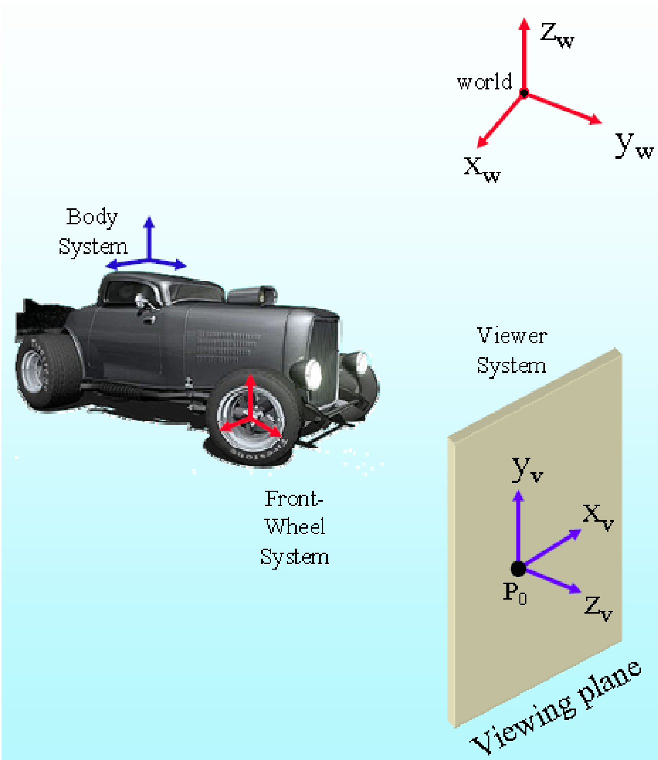
Three Dimensional Viewing Pipeline

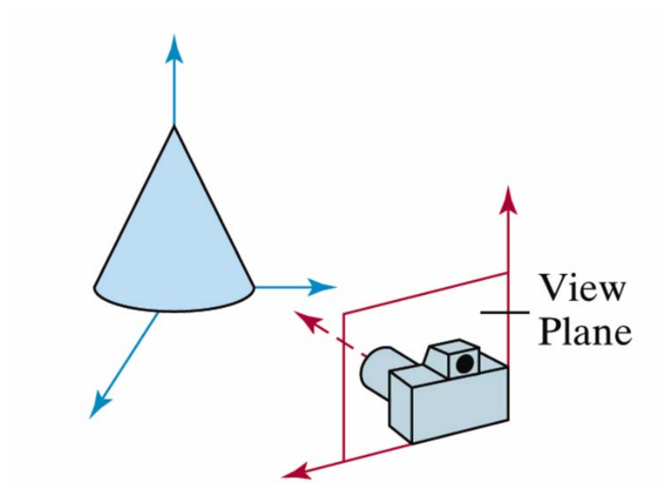
- Steps
  - Construct the shape of individual objects in a scene within modeling coordinate, and place the objects into appropriate positions within the scene (world coordinate).
  - World coordinate positions are converted to viewing coordinates.
  - Convert the viewing coordinate description of the scene to coordinate positions on the projection plane.
  - Positions on the projection plane, will then mapped to the Normalized coordinate and output device.



Viewing Coordinate

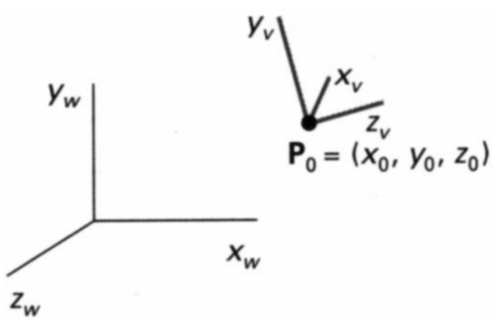
- Viewing coordinates system describe 3D objects with respect to a viewer.
- A Viewing (Projector) plane is set up perpendicular to z_v and aligned with (x_v, y_v).

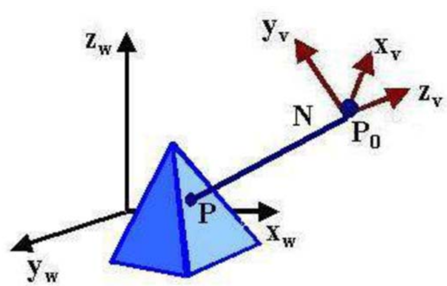


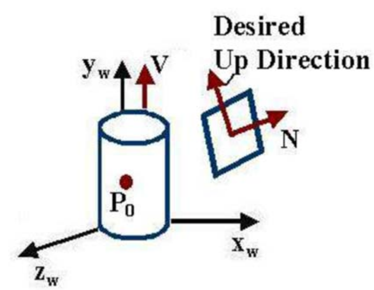


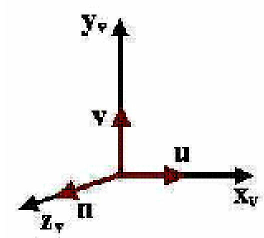
Viewing Coordinate

- We first pick a world coordinate position called **view reference point** (origin of our viewing coordinate system).
- View reference point (P_0) is a point where a camera or eye is located.
- Next, we select the positive direction for the viewing $\mathbf{z_v}$ axis, by specifying the **view plane normal vector**, **N**.
- The direction of **N**, is from the **look at point** (P) to the view reference point (P_0).
- Finally, we choose the ***up direction*** for the view by specifying a vector ***V***, called the ***view up vector***.
- This vector is used to establish the positive direction for the $\mathbf{y_v}$ axis.
- ***V*** is projected into a plane that is perpendicular to the normal vector.
- Using vectors **N** and **V**, the graphics package computer can compute a third vector **U**, perpendicular to both **N** and **V**, to define the direction for the $\mathbf{x_v}$ axis.







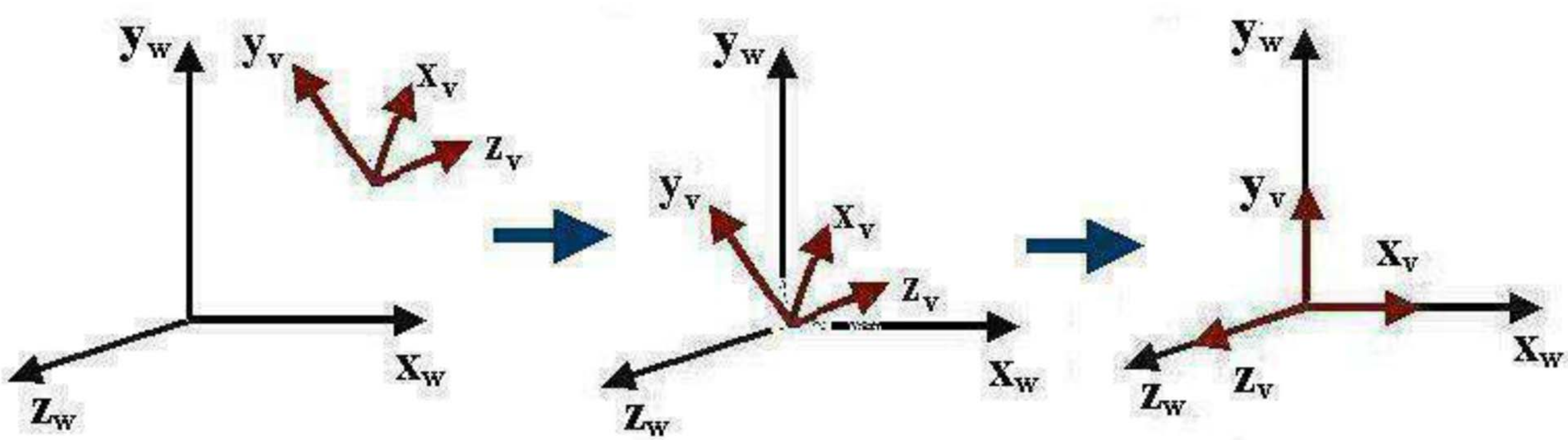


Transforming World to Viewing Coordinate

- Transforming world to viewing coordinate include following sequences:

1. Translate the view reference point to the origin of the world coordinate system
2. Apply the rotation to align the x_v, y_v and z_v axes with the world x_w, y_w and z_w axes respectively.

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Transforming World to Viewing Coordinate

- Background

$$\mathbf{i}\times\mathbf{j}=\mathbf{k} \qquad \mathbf{j}\times\mathbf{i}=-\mathbf{k}$$
$$\mathbf{j}\times\mathbf{k}=\mathbf{i} \qquad \mathbf{k}\times\mathbf{j}=-\mathbf{i} \qquad \mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$$
$$\mathbf{k}\times\mathbf{i}=\mathbf{j} \qquad \mathbf{i}\times\mathbf{k}=-\mathbf{j}$$

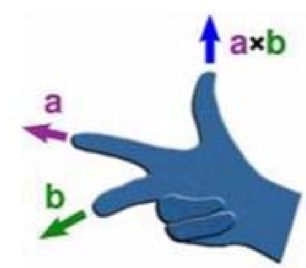
$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

- Given vectors N and V, calculating the unit vector

$$\mathbf{n}=\frac{\mathbf{N}}{|\mathbf{N}|}=(n_x,n_y,n_z)$$
$$\mathbf{u}=\frac{\mathbf{V}\times\mathbf{N}}{|\mathbf{V}\times\mathbf{N}|}=(u_x,u_y,u_z)$$
$$\mathbf{v}=\mathbf{n}\times\mathbf{u}=(v_x,v_y,v_z)$$

Rotation Matrix

$$R=\begin{bmatrix}u_x & u_y & u_z & 0\\ v_x & v_y & v_z & 0\\ n_x & n_y & n_z & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

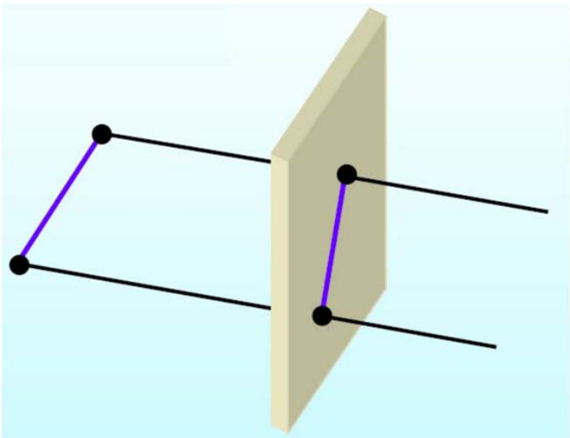


- The complete world to viewing coordinate transformation matrix is

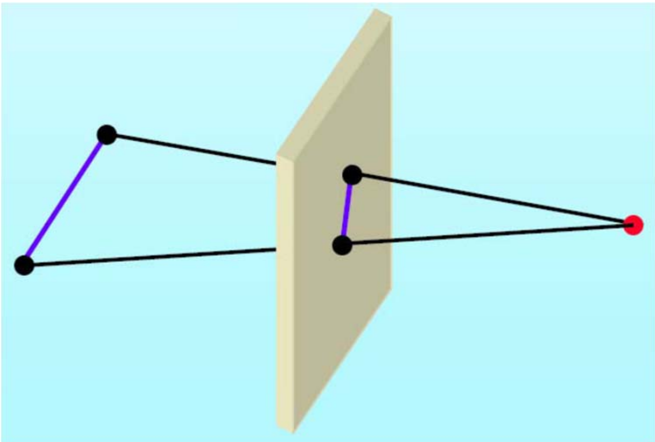
$$M_{wc\text{-}vc}=R\,.\,T$$

Projection

- Projection is the process of representing a three dimensional object or scene into two dimensional medium
- Types of projection
  1. **Parallel Projection**
  - ➢ Coordinate position are transformed to the view plane along **parallel lines**.
  2. **Perspective Projection**
  - ➢ Object positions are transformed to the view plane along lines that converge to the **projection reference (center) point**.



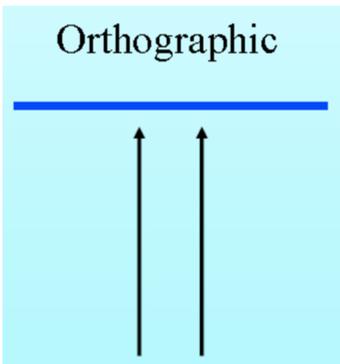
Parallel Projection

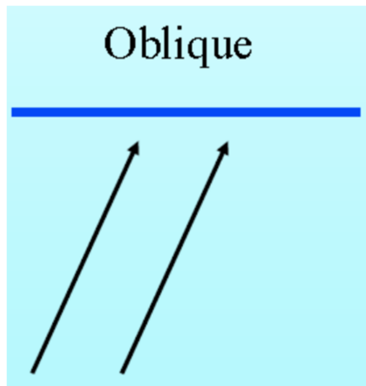


Perspective Projection

Projection

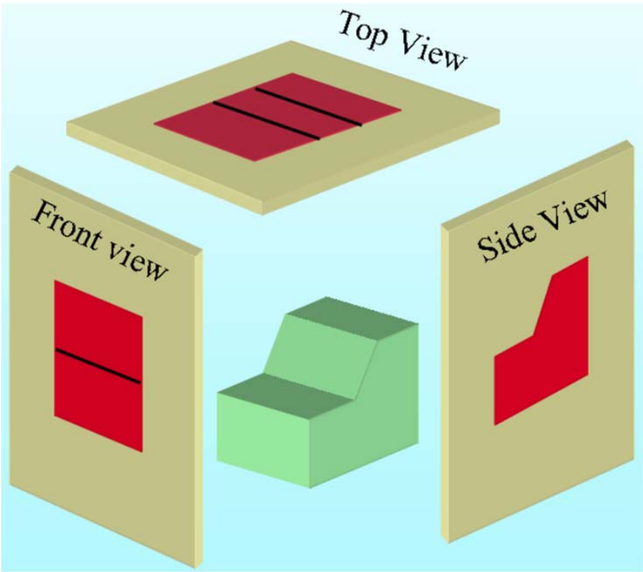
- **Parallel Projection Types**
- **Orthographic-** when the projection is perpendicular to the view plane. Used to produce Front, Side and Top view of an object. Most commonly used projection
- **Oblique** – when the projection is not perpendicular to the view plane. Not commonly used





Projection

- **Orthographic Projection**
  - when the projection is perpendicular to the view plane.
  - Most often used to produce front, side and top view of an object
  - Orthographic projection can display more than one face of an object. Such views are called axonometric orthographic projection
  - The most commonly used axonometric projection is isometric projection

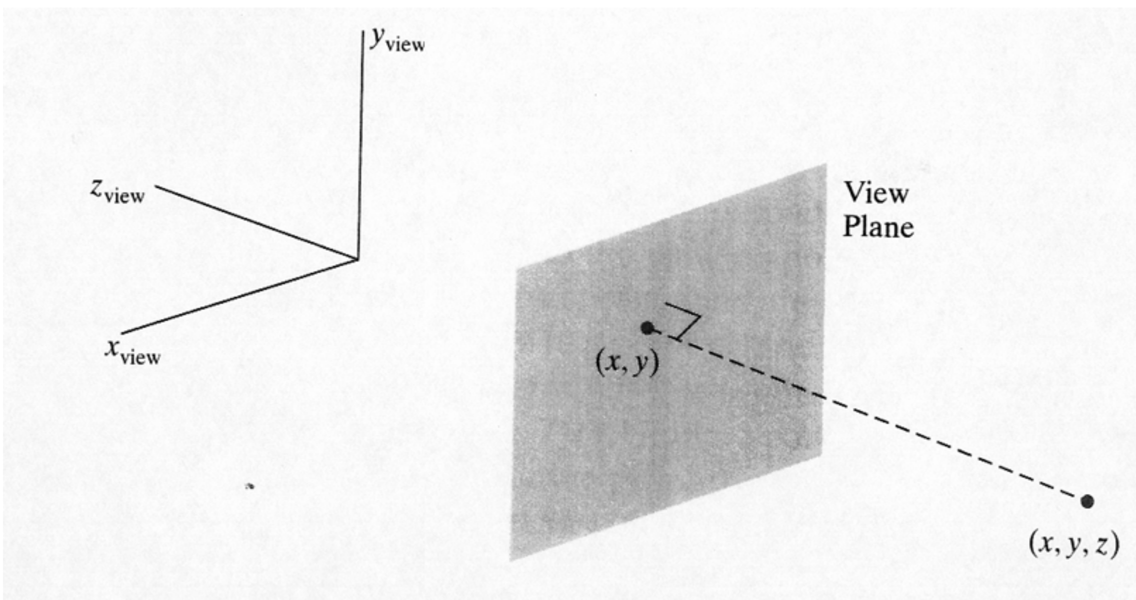


Projection

- **Orthographic Projection**
- If the view plane is placed at position z_{vp} along the z_v axis. Then any point (x,y,z) in viewing coordinates is transformed to projection coordinates as:

  $x_p = x,\ y_p = y$

  where the original z-coordinate is kept for depth information

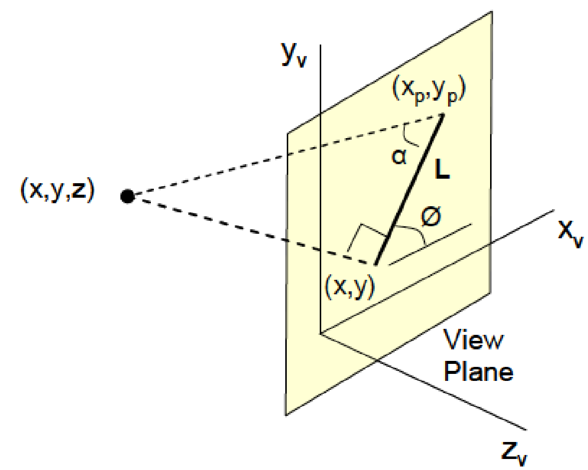


Projection

- **Oblique Projection**
  - when the projection is not perpendicular to the view plane
  - A vector direction is defining the projection lines
  - Can improve the view of an object
  - Point (x,y,z) is projected to position (x_p,y_p) on the view plane.
  - Projector (oblique) from (x,y,z) to (x_p,y_p) makes an angle α with the line (L) on the projection plane that joins (x_p,y_p) and (x,y).
  - Line L is at an angle ϕ with the horizontal direction in the projection plane.
  - Expressing projection coordinates in terms of x, y, L and ϕ:

$$x_p = x + L.\cos\varnothing$$
$$y_p = y + L.\sin\varnothing$$

  - Length L depends on the angle α and z-coordinate of the line to be projected



$$\tan\alpha = \frac{Z}{L} \qquad , L = \frac{Z}{\tan\alpha}, L = ZL_1 \qquad \text{where,} \quad L_1 = \frac{1}{\tan\alpha}$$

$$x_p = x + z(L_1\cos\varnothing)$$
$$y_p = y + z(L_1\sin\varnothing)$$

Projection

- **Oblique Projection**
- The transformation matrix for parallel projection onto $x_v y_v$-plane can be written as,

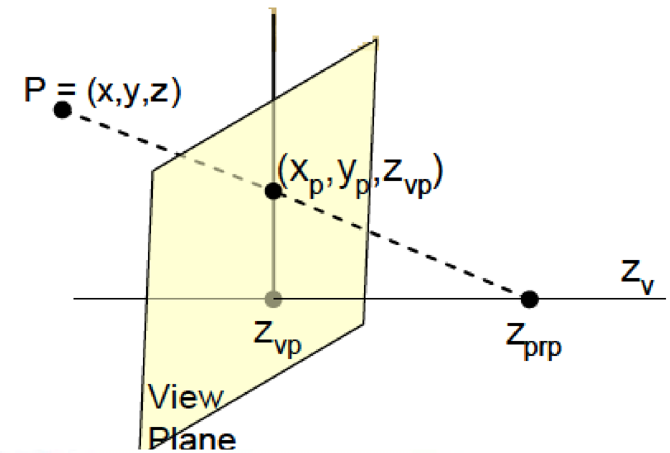
$$M_{parallel} = \begin{bmatrix} 1 & 0 & L_1\cos\phi & 0 \\ 0 & 1 & L_1\sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- If $L_1 = 0$, orthographic projection is obtained

- Common choices for angle ϕ are 30 degree and 45 degree
- Two commonly used values for α are those for which tanα = 1 and tanα = 2
- If tanα = 1, α=45degree, the views obtained are called cavalier projections
- If tanα = 2, α=63.4 degree (approx.), the views is obtained are called cabinet projection
- Cabinet projection appear more realistic than cavalier projection

Projection

- **Perspective Projection**
- For a perspective projection, object positions are transformed to the view plane along lines that converge to a point called projection reference point or center of projection.
- Suppose, we set the projection reference point at position z_{prp} along z_v axis, and we place the view plane at z_{vp}



- Parametric equation of perspective projection line to describe coordinate positions

$$x' = x - xu$$
$$y' = y - yu$$
$$z' = z - (z - z_{prp})u$$

u takes value between 0 and 1

P'(x',y',z') represents any point along the projection line

- On view plane

$$z' = z_{vp}; \text{ therefore}$$
$$u = (z_{vp} - z)/(z_{prp} - z)$$

Projection

- **Perspective Projection**

So,

$$\left.\begin{aligned} x_p &= x\left(\frac{z_{prp}-z_{vp}}{z_{prp}-z}\right) = x\left(\frac{d_p}{z_{prp}-z}\right) \\ &\text{and} \\ y_p &= y\left(\frac{z_{prp}-z_{vp}}{z_{prp}-z}\right) = y\left(\frac{d_p}{z_{prp}-z}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

In a 3D homogeneous coordinate system representation

$$x_p = x_h/h \quad \text{and} \quad y_p = y_h/h \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now comparing eqn 1 and eqn 2, we get

$$x_h = x \quad \text{and} \quad y_h = y$$

and

$$h = (z_{prp} - z)/d_p$$

Projection

- **Perspective Projection**

We know that,

$$z_p = z_{vp}$$

So,

$$z_h = z_p \text{ x } h$$

$$= z_{vp} \text{ x } (z_{prp} - z)/d_p$$

$$z_h = - z.z_{vp}/d_p + z_{vp}.z_{prp}/d_p$$

Also,

$$h = - z/d_p + z_{prp}/d_p$$

Now, the perspective projection transformation matrix in homogeneous coordinate representation is

$$\begin{bmatrix} x_h \\ y_h \\ z_h \\ h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -z_{vp}/d_p & z_{vp}(z_{prp}/d_p) \\ 0 & 0 & -1/d_p & z_{prp}/d_p \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$