Chapter - 1

# TRANSIENTS IN ELECTRIC CIRCUIT

## 1.1 Network Analysis

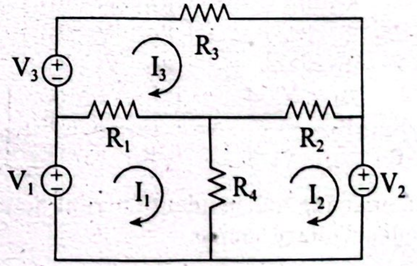
### 1.1.1 Mesh Analysis

*Mesh analysis* (or the mesh current method) is a method that is used to solve planar circuits for the currents (and indirectly the voltages) at any place in the electrical circuit. *Planar circuits* are circuits that can be drawn on a plane surface with no wires crossing each other. A more general technique, called *loop analysis* (with the corresponding network variables called loop currents) can be applied to any circuit, planar or not. Mesh analysis and loop analysis both make use of Kirchhoff's voltage_law to arrive at a set of equations guaranteed to be solvable if the circuit has a solution. Mesh analysis is usually easier to use when the circuit is planar, compared to loop analysis.

A *loop* is any closed path in a circuit, in which no node is encountered more than once. A *mesh* is a loop that has no other loops inside of it. A loop can be found by starting from a point and travelling through a path, to finish at the same point such that the same node is not traversed twice (except the starting point).

To solve a circuit by mesh analysis, we have to classify the circuit on the basis of source as follows:

**1. Circuit Containing only Independent Voltage Source**




Using KVL in loop 1,

$V_1 - I_1R_1 + I_3R_1 - I_1R_4 + I_2R_4 = 0$

or, $(R_1 + R_4)I_1 + (-R_4)I_2 + (-R_1)I_3 = V_1$ ...............(1)

or, $R_{11} I_1 + R_{12} I_2 + R_{13}I_3 = V_1$ ..................(1a)

Applying KVL in loop 2,

$-I_2R_2 + I_3R_2 - V_2 - I_2R_4 + I_1R_4 = 0$

or, $(-R_4)I_1 + (R_2 + R_4)I_2 + (-R_2)I_3 = -V_2$..........................(2)

or, $R_{21}I_1 + R_{22}I_2 + R_{23}I_3 = -V_2$ ..................(2a)

Applying KVL in loop 3,

$-I_3R_3 - I_3R_2 + I_2R_2 - I_3R_1 + I_1R_1 + V_3 = 0$

or, $(-R_1)I_1 + (-R_2)I_2 + (R_1 + R_2 + R_3)I_3 = V_3$ ............(3)

or, $R_{31}I_1 + R_{32}I_2 + R_{33}I_3 = V_3$....................(3a)

The equation for $n^{th}$ loop is given by

$R_{n1}I_1 + R_{n2}I_2 + ..............+ R_{nn}I_n = \Sigma V$

In matrix from,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \\ V_3 \end{bmatrix}$$

or, $[R]_{(m\times n)}\,[I]_{(n\times 1)} = [V]_{(n\times 1)}$

where $R_{ii}$ = self resistance of $i^{th}$ loop with +ve sign

$R_{ij} = R_{ji}$ = mutual or common resistance between $i^{th}$ and $j^{th}$ loop.

The sign will be -ve if the direction of the current in both loops are either both clockwise or both anticlockwise. The sign will be +ve if the direction of the current in one loop (say $i^{th}$ loop) is clockwise and that in other (say $j^{th}$ loop) is anticlockwise.

Applying Cramer's rule,

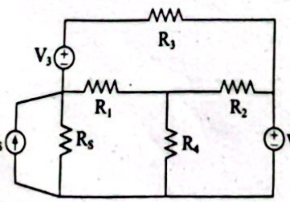
$$I_1 = \frac{\Delta_1}{\Delta},\ I_2 = \frac{\Delta_2}{\Delta},\ I_3 = \frac{\Delta_3}{\Delta}$$

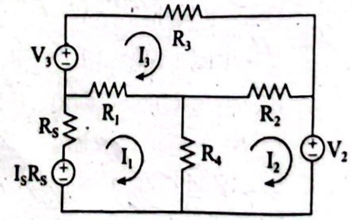
where $\Delta_1 = \begin{vmatrix} V_1 & R_{12} & R_{13} \\ -V_2 & R_{22} & R_{23} \\ V_3 & R_{32} & R_{33} \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} R_{11} & V_1 & R_{13} \\ R_{21} & -V_2 & R_{23} \\ R_{31} & V_3 & R_{33} \end{vmatrix}$,

$\Delta_3 = \begin{vmatrix} R_{11} & R_{12} & V_1 \\ R_{21} & R_{22} & -V_2 \\ R_{31} & R_{32} & V_3 \end{vmatrix}$, $\Delta = \begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix}$

**2. Circuit Containing Independent Current Source in Addition to Independent Voltage Source**

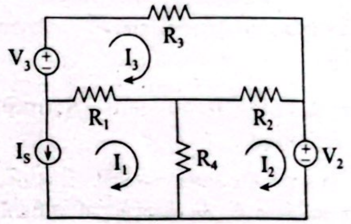
**i. Current source transformable into voltage source**




**ii. Current source not transformable into voltage source**

**a. Current source present in the perimeter of an individual loop**




Applying KVL in loop 2,

$R_{21}I_1 + R_{22}I_2 + R_{23}I_3 = -V_2$

or, $-R_4I_1 + (R_2 + R_4)I_2 - R_2I_3 = -V_2$ ................(i)

Applying KVL in loop 3,

$R_{31}I_1 + R_{32}I_2 + R_{33}I_3 = V_3$

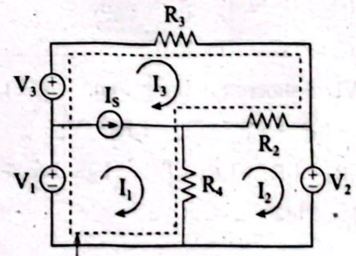
or, $-R_1I_1 - R_2I_2 + (R_1 + R_2 + R_3)I_3 = V_3$ ................(ii)

In loop 1,

$I_1 = -I_s$ ..........(iii)

Solving equations (i), (ii), and (iii), we get the value of $I_1$, $I_2$, and $I_3$.

**b. Current source present in the common branch between any two loops**




Supermesh

In this case, we have to construct a supermesh including two loops which contain the current source in their common branch.

Applying KVL in loop 2,

$R_{21}I_1 + R_{22}I_2 + R_{23}I_3 = -V_2$

or, $-R_4I_1 + (R_2 + R_4)I_2 - R_2I_3 = -V_2$ ................(i)

Applying KVL in supermesh (loop1 and loop 3),

$-R_3I_3 - R_2I_3 + R_2I_2 - R_4I_1 + R_4I_2 + V_1 + V_3 = 0$ .......(ii)

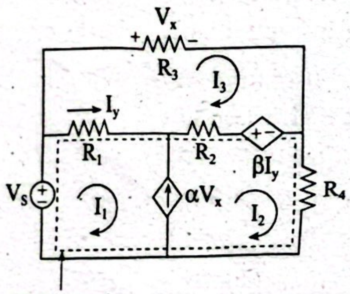
In common branch between loop 1 and loop 3, we have

$I_3 - I_1 = I_s$ ..........(iii)

On solving equations (i), (ii), and (iii), we get the value of $I_1$, $I_2$, and $I_3$.

### 3. Circuit Containing Dependent Source in Addition to Independent Source

For this, we have to apply the same method explained so far but some extra work has to be done. That is, the dependent variable of dependent source must be expressed in terms of loop current.




Supermesh

Applying KVL in loop 3,

$R_{31}I_1 + R_{32}I_2 + R_{33}I_3 = \beta I_y$

or, $-R_1I_1 - R_2I_2 + (R_1 + R_2 + R_3)I_3 = \beta I_y$

or, $-R_1I_1 - R_2I_2 + (R_1 + R_2 + R_3)I_3 = \beta[I_1 - I_3]$ ................(1)

Applying KVL in supermesh (loop 1 and loop 2),

$-I_1R_1 + I_3R_1 - I_2R_2 + I_3R_2 - \beta I_y - I_2R_4 + V_s = 0$

or, $-I_1R_1 + I_3R_1 - I_2R_2 + I_3R_2 - \beta(I_1 - I_3) - I_2R_4 + V_s = 0$ ......(2)

Also, in common branch, we have

$I_2 - I_1 = \alpha V_x$

or, $I_2 - I_1 = \alpha(I_3R_3)$ .................(3)

On solving these three equations, we get the required values of currents.

## 1.1.2 Nodal Analysis

In electric circuits analysis, *nodal analysis*, *node-voltage analysis*, or the *branch current method* is a method of determining the voltage (potential difference) between *nodes* (points where elements or branches connect) in an electrical circuit in terms of the branch currents.

Nodal analysis writes an equation at each electrical node, requiring that the branch currents incident at a node must sum to zero. The branch currents are written in terms of the circuit node voltages.

Nodal analysis produces a compact set of equations for the network, which can be solved by hand if small, or can be quickly solved using linear algebra by computer. Because of the compact system of equations, many circuit simulation programs (e.g., SPICE) use nodal analysis as a basis.

### General Steps in nodal analysis:

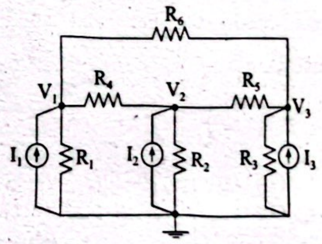
i. In a circuit of n nodes, select a node as a reference node, and assign voltages $V_1, V_2, \ldots\ldots, V_{n-1}$ to the remaining n–1 nodes (non-reference node). The reference node is assumed to have zero potential.

ii. Apply KCL to each of the (n – 1) non-reference node. Use Ohm's law to express the branch current in terms of the node voltages.

iii. Finally, solve the resulting (n – 1) equations to get the value of unknown node voltages.

In order to solve a circuit by nodal analysis, we have to classify the circuit on the basis of source as follows:

### 1. Circuit Containing only Independent Current Source




Applying KCL on node 1,

$$\frac{V_1 - 0}{R_1} + \frac{V_1 - V_2}{R_4} + \frac{V_1 - V_3}{R_6} = I_1$$

or, $\left(\frac{1}{R_1} + \frac{1}{R_4} + \frac{1}{R_6}\right)V_1 + \left(\frac{-1}{R_4}\right)V_2 + \left(\frac{-1}{R_6}\right)V_3 = I_1$ ...........(1)

or, $G_{11}V_1 + G_{12}V_2 + G_{13}V_3 = I_1$ ................(1a)

Applying KCL on node 2,

$$\frac{V_2 - 0}{R_2} + \frac{V_2 - V_1}{R_4} + \frac{V_2 - V_3}{R_5} = I_2$$

or, $\left(\frac{-1}{R_4}\right)V_1 + \left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5}\right)V_2 + \left(\frac{-1}{R_5}\right)V_3 = I_2$ .........(2)

or, $\frac{-1}{R_4}V_1 + \left(\frac{1}{R_2}+\frac{1}{R_4}+\frac{1}{R_5}\right)V_2 - \frac{1}{R_5}V_3 = I_2$ ....................(1)

$G_{21}V_1 + G_{22}V_2 + G_{23}V_3 = I_2$ .............(1a)

Applying KCL on node 3,

$$\frac{V_3-V_1}{R_6}+\frac{V_3-V_2}{R_5}+\frac{V_3-0}{R_3}=I_3$$

or, $\frac{-1}{R_6}V_1 - \frac{1}{R_5}V_2 + \left(\frac{1}{R_3}+\frac{1}{R_5}+\frac{1}{R_6}\right)V_3 = I_3$ ..........(2)

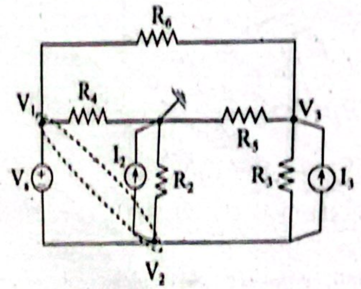
$G_{31}V_1 + G_{32}V_2 + G_{33}V_3 = I_3$ ..........(2a)

At node 1,

$V_1 - 0 = V_s$ ..................(3)

By solving equations (1), (2), and (3), we get the values of $V_1$, $V_2$, and $V_3$.

**b. Voltage source not involving reference node**




In this case, we have to construct a supernode including two non reference nodes to which the two terminals of voltage source is connected. And then, we have to apply KCL.

Applying KCL on node 3,

$$\frac{V_3-V_1}{R_6}+\frac{V_3-V_2}{R_3}+\frac{V_3-0}{R_5}=I_3$$

or, $\left(\frac{-1}{R_6}\right)V_1 + \left(\frac{-1}{R_3}\right)V_2 + \left(\frac{1}{R_3}+\frac{1}{R_5}+\frac{1}{R_6}\right)V_3 = I_3$ ....................(1)

or, $G_{31}V_1 + G_{32}V_2 + G_{33}V_3 = I_3$

Applying KCL on supernode ($V_1$ and $V_2$),



$$\frac{V_1-0}{R_4}+\frac{V_1-V_3}{R_6}+\frac{V_2-0}{R_2}+\frac{V_2-V_3}{R_3}=-I_2-I_3$$

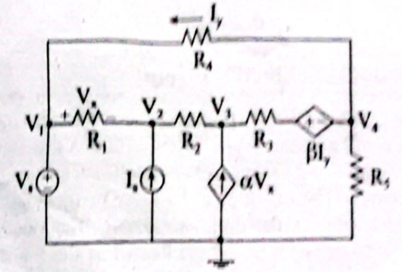
or, $\left(\frac{1}{R_4}+\frac{1}{R_6}\right)V_1 + \left(\frac{1}{R_2}+\frac{1}{R_3}\right)V_2 + \left(\frac{-1}{R_6}+\frac{-1}{R_3}\right)V_3 = -I_2 - I_3$ .....(2)

Also, at node 1 and 2,

$V_1 - V_2 = V_s$ ..................(3)

Solving equations (1), (2), and (3), we get the values of $V_1$, $V_2$, and $V_3$.

**3. Circuit Containing Dependent Source in Addition to Independent Source**




Applying KCL at node 2,

$G_{21}V_1 + G_{22}V_2 + G_{23}V_3 + G_{24}V_4 = I_s$

or, $\frac{-1}{R_1}V_1 + \left(\frac{1}{R_1}+\frac{1}{R_2}\right)V_2 + \frac{-1}{R_2}V_3 - 0V_4 = I_s$ ....................(1)

Applying KCL at node 3,

$$\frac{V_3-V_2}{R_2}+\frac{V_3-\beta I_y-V_4}{R_3}=\alpha V_x$$

or, $\frac{V_3-V_2}{R_2}+\frac{V_3-\beta\left(\frac{V_4-V_1}{R_4}\right)-V_4}{R_3}=\alpha(V_1-V_2)$

or, $\left(\frac{\beta}{R_3R_4}-\alpha\right)V_1 + \left(\frac{-1}{R_2}+\alpha\right)V_2 + \left(\frac{1}{R_2}+\frac{1}{R_3}\right)V_3 + \left(\frac{-\beta}{R_3R_4}-\frac{1}{R_3}\right)V_4 = 0$

........(2)

Applying KCL at node 4,

$$\frac{V_4-V_1}{R_4}+\frac{V_4-0}{R_5}+\frac{V_4+\beta I_y-V_3}{R_3}=0$$

$$\text{or, } \frac{V_4 - V_1}{R_4} + \frac{V_4 - 0}{R_5} + \frac{V_4 + \beta\left(\frac{V_4 - V_1}{R_4}\right) - V_3}{R_3} = 0$$

$$\text{or, } \left(\frac{-\beta}{R_3R_4} - \frac{1}{R_4}\right)V_1 + 0V_2 + \left(\frac{-1}{R_3}\right)V_3 + \left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} + \frac{\beta}{R_3R_4}\right)V_4 = 0 \qquad \text{.............(3)}$$

Also, at node 1,

$V_1 - 0 = V_s$ .......................(4)

Solving equations (1), (2), (3), and (4), we get the values of $V_1$, $V_2$, $V_3$, and $V_4$.

## 1.2 Transients in Electric Circuit

The methods of solving electric circuits by dc or ac sinusoidal circuit theory do not, in general, give the values of the voltages and currents in a circuit immediately following a change in the circuit, such as that occurs upon the closing of a switch. The values given by these methods are valid only after a lapse of time known as the *transient period*. The values of voltage and current during the transient period are known as the *transient responses*, or simply as *transients*. The word "transient" means transitory, short-lived.

When only resistors which are energy-dissipating elements, are present in an electric circuit, there is no. transient behavior. Consider the figure below containing only pure resistor where the current takes its final (maximum) value instantaneously without any delay.

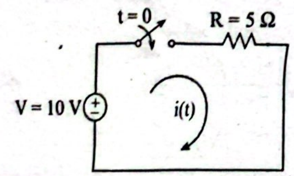



*Figure 1.1: A circuit containing resistor*

$$i = \frac{V_o}{R} = \frac{10}{5} = 2\text{ A}$$

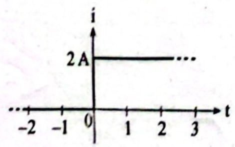



*Figure 1.2: Response curve*



But if the circuit contains energy storing elements (inductor or capacitor or both), the response does not take its maximum value instantaneously. There will be some delay in time.

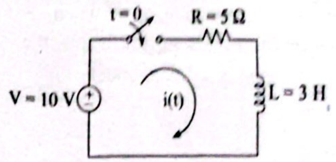



*Figure 1.3: A circuit containing resistor and inductor*

$$i = \frac{V_o}{R}(1 - e^{-t/\tau})$$

where $\tau = \frac{L}{R}$ = time constant

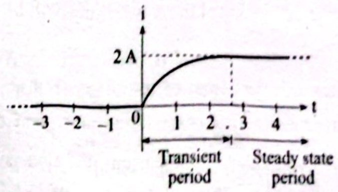



*Figure 1.4: Response curve*

Now, let's define transients in a very simple way. When an input is applied to a circuit containing energy storing elements, the response does not take its final value immediately. There will be some delay in the time known as *transient period* and the value of the response during this period is known as *transient response* or simply *transients*. The time, when the response takes its steady state value or constant value is known as *steady-state period* and the value at response during this period is known as *steady-state response*.

### Some Remarks on Notation

The use of lowercase letters such as v, e, i, q is to imply that the quantity involved is a variable with time, that is, a waveform. Occasionally, to emphasize this fact, these variables are written in the form

$i = i(t)$

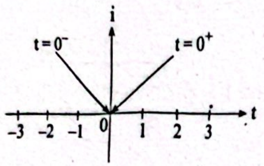
$v = v(t)$

The use of uppercase letters such as $V_o$, $I_o$, etc. is to imply that the quantity involved is a constant (or dc value).

$V_o$ = dc or constant voltage

$I_o$ = dc or constant current

In order to distinguish two states of the network, a notation has to be employed. At the reference time, $t = 0$, one or more switches operate. We assume that switches act in zero time. To differentiate between the time immediately before and immediately after the operation of a switch, – and + signs are used respectively. Thus conditions existing just before the switch is operated are designated as $i(0^-)$, $v(0^-)$, etc.; conditions after as $i(0^+)$, $v(0^+)$, etc.




$t = 0^+$ implies the time just after zero but very closed to zero.

$t = 0^-$ implies the time just before zero but very closed to zero.

As a limit, the difference of these two times will be zero i.e., $0^+ - 0^- \cong 0$.

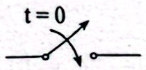
In general,

$t = 0^+$ denotes the time just after any change is made in the circuit.

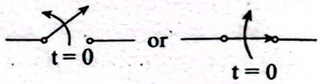
$t = 0^-$ denotes the time just before any change is made in the circuit.

Thus, $i(0^+)$ denotes the value of the current just after any change is made in the circuit and $i(0^-)$ denotes the value of current just before any change is made in the circuit.

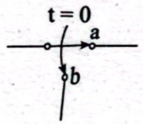
1. The figure illustrates that initially, the switch was open and at any time $t = 0$, it is closed.




2. Initially, the switch was closed and at any time $t = 0$, it is opened.



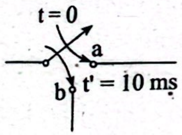

3. Initially, the switch was at position a and at any time $t = 0$, it is shifted to the position b.




4. i. $0 \le t \le t' \Rightarrow$ position a

   ii. $t > t' \Rightarrow$ position b

   Initially, the switch was opened and at any time $t = 0$, it is connected to position a and after 10 ms, it is shifted to position b.





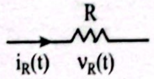

## 1.3 Characteristics of Various Network Elements

The voltage and current relations for different circuit elements are discussed below:

1. **Resistor (R)**

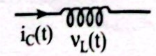
$$v_R = R\, i_R$$

$$\therefore i_R = \frac{v_R}{R}$$




2. **Inductor (L)**

$$v_L = L\frac{di_L}{dt}$$




$$\text{or, } di_L = \frac{1}{L} v_L\, dt$$

$$\text{or, } \int di_L = \frac{1}{L}\int v_L\, dt$$

$$\therefore i_L = \frac{1}{L}\int v_L\, dt$$

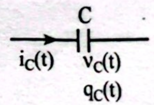
3. **Capacitor (C)**

$$i_C = \frac{dq}{dt} = C\frac{dv_C}{dt}$$

$$\text{or, } dq = i_C\, dt$$

$$\text{or, } \int dq = \int i_C\, dt$$

$$\therefore q = \int i_C\, dt$$




$$\text{Also, } i_C = C\frac{dv_C}{dt}$$

$$dv_C = \frac{1}{C} i_C\, dt$$

$$\text{or, } \int dv_C = \frac{1}{C}\int i_C\, dt$$

$$\therefore v_C = \frac{1}{C}\int i_C\, dt$$

In circuit analysis, generally either $i_L$ or $i_C$ or $v_C$ or $q$ is taken as dependent variable and time $t$ is taken as independent variable.

## 1.4 Initial Condition

The value of the dependent variable and its derivatives just after any change made in the circuit is known as *initial condition*. Generally, in circuit analysis, either $i_L$ or $v_C$ or q or $i_C$ is taken as dependent variable and time t is taken as independent variable. If the $n^{th}$ order circuit contains inductor, then the value of $i_L(0^+)$, $i'_L(0^+)$, $i''_L(0^+)$, ......., $i_L^{(n+1)}(0^+)$ are the initial conditions.

Initial (or final) conditions must be known to evaluate the arbitrary constants that comes in the general solution of differential equations. Initial (or final) conditions assist us to know the knowledge of the behavior of the elements at the instant of switching, indispensable in understanding nonlinear switching circuits; and knowledge of the initial value of one or more derivatives of a response, helpful in anticipating the form of that response, thus serving as a check on our solution.

The number of initial conditions required is equal to the order of the differential equation for a unique solution.

**1. Initial Condition for Inductor (L)**

By observation, $i_L(0^-) = 0$

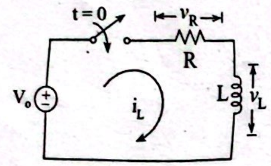



*Figure 1.5*

Suppose that the current through inductor can change instantaneously i.e., $i_L(0^+)$ = some value of current = constant.

$di_L = i_L(0^+) - i_L(0^-) = \text{constant}$

$dt = 0^+ - 0^- = 0$

$$\frac{di_L}{dt} = \frac{\text{constant}}{0} = \infty$$

$$\text{or, } L\frac{di_L}{dt} = L \times \infty = \infty$$

or, $v_L = \infty$ which is impossible. So, our conclusion is wrong. Hence, we conclude that current through inductor cannot change instantaneously i.e., $i_L(0^+) = i_L(0^-)$ .......... (1)

The equation (1) is known as *continuity relation for inductor.*

Though the current through inductor cannot change instantaneously but the voltage does i.e., $v_L(0^+) \neq v_L(0^-)$.

To verify that voltage across inductor can change instantaneously, consider figure 1.5.

By observation at $t = 0^-$,



$v_L(0^-) = 0$, $i_L(0^-) = 0$

From continuity relation for the inductor,

$i_L(0^+) = i_L(0^-) = 0$

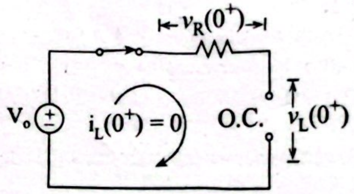



*Figure 1.6: Equivalent circuit at $t = 0^+$.*

From figure 1.6,

$v_L(0^+) = V_o$

and $v_L(0^+) \neq v_L(0^-)$

Thus, voltage across inductor can change instantaneously.

From figure (1.5) and (1.6), we conclude that an initially de-energized inductor can be replaced by an open circuit at $t = 0^+$.

**2. Initial Condition for Capacitor (C)**

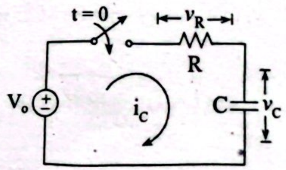



*Figure 1.7*

By observation at $t = 0^-$, $v_C(0^-) = 0$

Suppose that the voltage across capacitor can change instantaneously i.e., $v_C(0^+)$ = some value of voltage = constant.

$dv_C = v_C(0^+) - v_C(0^-) = \text{constant}$

$dt = 0^+ - 0^- = 0$

$$\frac{dv_C}{dt} = \frac{\text{constant}}{0} = \infty$$

$$\text{or, } C\frac{dv_C}{dt} = C \times \infty = \infty$$

$\therefore$ $i_C = \infty$ which is impossible. So, our conclusion is wrong.

Hence, we conclude that the voltage across capacitor cannot change instantaneously i.e., $v_C(0^+) = v_C(0^-)$. ......... (1)

Equation (1) is known as *continuity relation for capacitor in terms of voltage.*

Equation (1) can also be written as

$$\frac{q(0^+)}{C} = \frac{q(0^-)}{C}$$

or, $q(0^+) = q(0^-)$ ............. (1a)

Equation (1a) is also known as *continuity relation for capacitor in terms of charge.*

Though the voltage across capacitor cannot change instantaneously but the current does i.e., $i_C(0^+) \neq i_C(0^-)$.

To prove that current through capacitor can change instantaneously, consider figure 1.7.

By observation at $t = 0^-$,

$i_C(0^-) = 0$, $v_C(0^-) = 0$

From continuity relation for capacitor,

$v_C(0^+) = v_C(0^-) = 0$

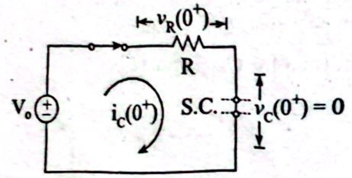



*Figure 1.8: Equivalent circuit at t = 0⁺.*

From figure 1.8,

$$i_C(0^+) = \frac{V_o}{R}$$

$\therefore i_C(0^+) \neq i_C(0^-)$

Thus, current through the capacitor can change instantaneously.

From figure (1.7) and (1.8), we conclude that an initially de-energized capacitor can be replaced by a short circuit at $t = 0^+$.

3. **Initial Condition for Resistor (R)**

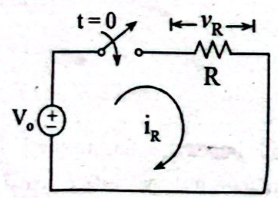



*Figure 1.9*



By observation at $t = 0^-$,

$i_R(0^-) = 0$ .........(1)

$v_R(0^-) = 0$ .........(2)

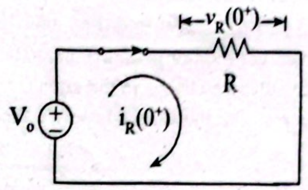



*Figure 1.10 Equivalent circuit at t = 0⁺.*

From figure 1.10,

$$i_R(0^+) = \frac{V_o}{R} \quad \text{.........(3)}$$

$$v_R(0^+) = i_R(0^+)\,R = \frac{V_o}{R} \times R = V_o \quad \text{.........(4)}$$

From equations (1) and (3),

$i_R(0^+) \neq i_R(0^-)$

From equations (2) and (4),

$v_R(0^+) \neq v_R(0^-)$

Thus, both voltage and current of resistor can change instantaneously.

## 1.5 Equivalent Circuit for Inductance and Capacitance

A *short circuit* (S.C.) is a two-terminal device for which the voltage must be zero when the current is not zero. This statement is written mathematically as ($v = 0$, $i \neq 0$). A short, heavy slab of good conducting material, such as copper, is usually a good approximation to a short circuit. An *open circuit* (O.C.) is a two-terminal device for which the current must be zero when the voltage is not zero: ($i = 0$, $v \neq 0$). The air surrounding two terminals is usually a good approximation to an open circuit.

The voltage across the inductor is given by

$$v_L = L\frac{di_L}{dt} \quad \text{...........(i)}$$

When an inductor has been in a circuit driven by dc sources for a long time, all the voltages and currents in the circuit settle down to steady, constant values or $i_L = I_L$ = a constant. Applying equation (i) to this situation,

$$v_L = L\frac{d}{dt}(I_L) = L\frac{d}{dt}(\text{a constant}) = L \times 0 = 0$$

Thus in dc steady state for a pure inductor, $i_L \neq 0$, $v_L = 0$. These are the conditions for a short circuit.

In dc steady state, a pure inductor can be represented by a short circuit. However, an actual, physical inductance coil still presents the resistance of its wire. Equation $i_L(0^+) = i_L(0^-)$, states that if an inductor is initially deenergized before a change in circuit configuration occurs, $i_L(0^-) = 0$ A, then the current is still zero instantly after the change in the circuit, $i_L(0^+) = 0$ A, regardless of the new voltage across the inductor. These are the conditions for an open circuit.

For initial conditions, a deenergized inductor can be represented by an open circuit. For an inductor with an initial current, $i_L(0^-) = I_{initial}$, so that $i_L(0^+) = I_{initial}$, the equivalent circuit for an inductor with initial current in it at the instant just after a change in circuit configuration is a current generator whose value and direction is that of $I_{initial}$.

The current through capacitor is given by

$$i_C = C\frac{dv_C}{dt} \qquad \text{.........(ii)}$$

When a capacitor has been in a circuit driven by dc sources for a long time, all the voltages and currents settle down to steady, constant values or $v_C = V_C$ = a constant. Applying equation (ii) to this situation,

$$i_C = C\frac{d}{dt}(V_C) = C\frac{d}{dt}(\text{a constant}) = C \times 0 = 0$$

Thus in dc steady state for a pure capacitor, $v_C \neq 0$, $i_C = 0$. These are the conditions for an open circuit.

In dc steady state, a capacitor is represented by an open circuit. Equation $v_C(0^+) = v_C(0^-)$, states that if a capacitor is initially deenergized before a change in circuit configuration occurs, $v_C(0^-) = 0$ V, then the voltage is still zero instantly after the change in the circuit, $v_C(0^+) = 0$ V, regardless of the new current in the capacitor. These are the conditions for a short circuit.

For initial conditions, a deenergized capacitor can be represented by a short circuit. For a capacitor with initial voltage, $v_C(0^-) = V_{initial}$, so that $v_C(0^+) = V_{initial}$, the equivalent circuit for a capacitor with initial voltage at the instant just after a change in circuit configuration is a battery whose voltage and polarity is that of $V_{initial}$.

1. **At $t = 0^+$ i.e., just after any change**

   i. **Inductor**

      a. Initially **de-energized inductor** [$i_L(0^-) = 0$] is replaced by an **open circuit.**



      b. Initially **energized inductor** [$i_L(0^-)$ = some value = $I_{initial}$] is replaced by a **current source** having same magnitude as that of $I_{initial}$.

   ii. **Capacitor**

      a. Initially **deenergized capacitor** [$v_C(0^-) = 0$ or $q(0^-) = 0$] is replaced by a **short circuit.**

      b. Initially **energized capacitor** [$v_C(0^-)$ = some value = $V_{initial}$] is replaced by a **voltage source** having same magnitude and polarity as that of $V_{initial}$.

2. **At $t = 0^-$ or $t = \infty$ i.e., in steady state**

   i. Inductor is replaced by a short circuit.

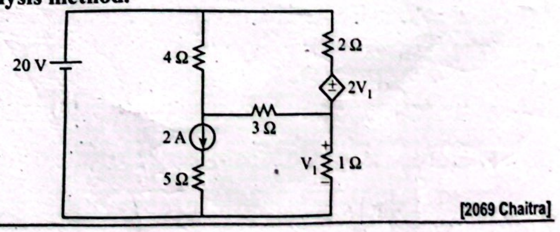
   ii. Capacitor is replaced by an open circuit.

*Table 1.1: Equivalent circuits for L and C*

| Conditions / Element | DC Steady-state, $t = 0^-$ or $t = \infty$ | Initial conditions starting from rest, $t = 0^+$ | General initial conditions, $t = 0^+$ |
|---|---|---|---|
| *L* | S.C. | O.C. | Current generator $i_g = i_L(0^+) = i_L(0^-) = I_{initial}$ |
| *C* | O.C. | S.C. | Battery $V_s = v_C(0^+) = v_C(0^-) = V_{initial}$ |

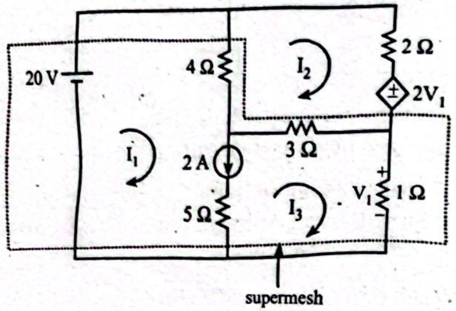
## SOLVED PROBLEMS

1. **In the given circuit, determine voltage across 1 Ω resistor using mesh analysis method.**




[2069 Chaitra]

***Solution:***




Applying KVL in supermesh,

$20 - 4I_1 + 4I_2 - 3I_3 + 3I_2 - I_3 = 0$

or, $20 = 4I_1 - 7I_2 + 4I_3$ ............(1)

Also, in the branch containing current source,

$I_1 - I_3 = 2$

or, $I_1 = 2 + I_3$ ..............(2)

From (1) and (2),

$20 = 8 + 4I_3 - 7I_2 + 4I_3$

or, $12 = 8I_3 - 7I_2$ .............(3)

Applying KVL in loop (2),

$-4I_1 + 9I_2 - 3I_3 = -2V_1$

or, $-4I_1 + 9I_2 - 3I_3 = -2I_3$

or, $-4I_1 + 9I_2 - I_3 = 0$ .............(4)

From (4) and (2),

$-4(2 + I_3) + 9I_2 - I_3 = 0$

or, $-8 - 4I_3 + 9I_2 - I_3 = 0$

or, $9I_2 - 5I_3 = 8$

or, $I_2 = \frac{8 + 5I_3}{9}$ ........................(5)

From (3) and (5),

$12 = \left(\frac{8}{9} + \frac{5}{9}I_3\right)(-7) + 8I_3$

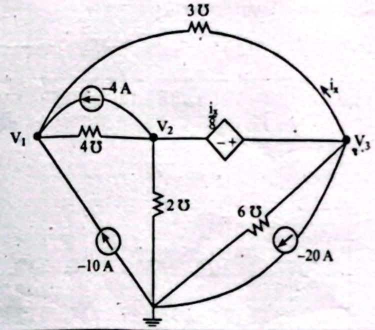


or, $12 = \frac{-56}{9} - \frac{35}{9}I_3 + 8I_3$

or, $I_3 = \frac{164}{37} = 4.434$

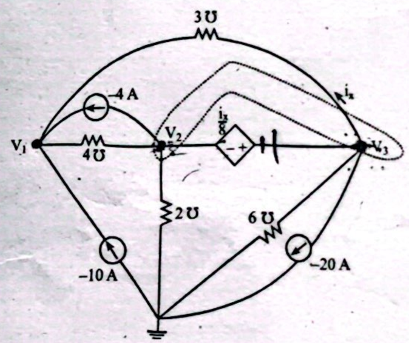
$\therefore V_1 = 1 \times I_3 = 4.434$ V

**2.** **Find the node voltages $V_1$, $V_2$, and $V_3$ for the circuit shown in figure.**




[2066 Bhadra]

***Solution:***




Applying KCL at node 1,

$3(V_1 - V_3) + 4(V_1 - V_2) = -10 + (-4)$

or, $7V_1 - 4V_2 - 3V_3 = -14$ ............(i)

Applying KCL at supernode (node 2 and node 3),

$4(V_2 - V_1) + 2V_2 + 6V_3 + 3(V_3 - V_1) = -(-4) - (-20)$

or, $-7V_1 + 6V_2 + 9V_3 = 24$ ...........(ii)

$$V_3 - V_2 = \frac{i_x}{8}$$

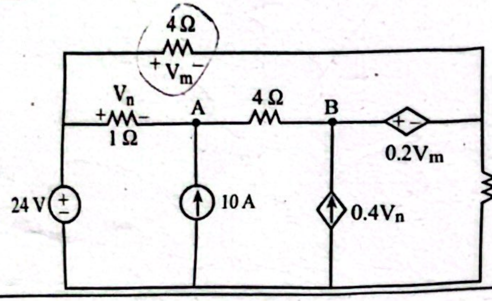
$$\text{or, } V_3 - V_2 = \frac{3(V_3 - V_1)}{8}$$

$$\text{or, } 0.375V_1 - V_2 + 0.625V_3 = 0 \qquad \ldots\ldots\ldots\text{(iii)}$$

Solving equations (i), (ii), and (iii), we get

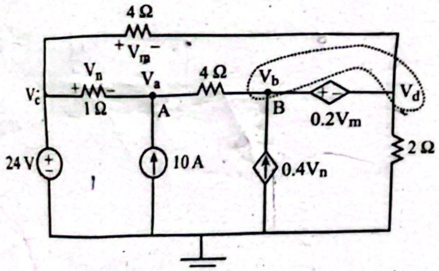
$V_1 = -1.0568$ V, $V_2 = 0.53409$ V, $V_3 = 1.4886$ V

**3. Using nodal analysis, determine the current through 4 Ω resistor connected between terminals A and B for the network of following figure.**



[2067 Mag]

***Solution:***



At node C and reference node,

$V_C - 0 = 24$

or, $V_C = 24$ V

Applying KCL at supernode (node $V_b$ and $V_d$),

$$\frac{V_b - V_a}{4} + \frac{V_d - 0}{2} + \frac{V_d - V_c}{4} = 0.4V_n$$



or, $V_b - V_a + 2V_d + V_d - 24 = 1.6V_n$

or, $V_b - V_a + 3V_d - 24 = 1.6(V_c - V_a)$

or, $0.6V_a + V_b + 3V_d = 62.4$ ……….. (i)

Also, at supernode,

$V_b - V_d = 0.2V_m$

or, $V_b - V_d = 0.2(V_c - V_d)$

or, $V_b - V_d + 0.2V_d = 4.8$

or, $V_b - 0.8V_d = 4.8$ ……….. (ii)

Applying KCL at node a,

$$\text{or, } \frac{V_a - V_c}{1} + \frac{V_a - V_b}{4} = 10$$

or, $4V_a - 4 \times 24 + V_a - V_b = 40$

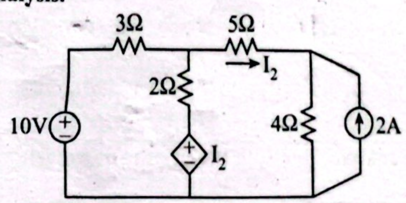
or, $5V_a - V_b = 136$ ……….. (iii)

Solving equations (i), (ii), and (iii), we get

$V_a = 29.831$ V, $V_b = 13.158$ V, $V_d = 10.447$ V

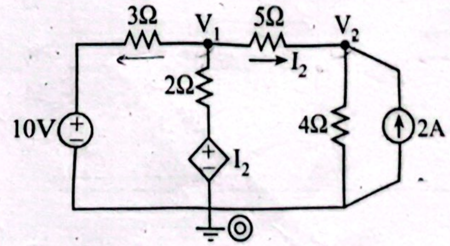
The current through 4 Ω resistor $= \dfrac{V_a - V_b}{4} = 4.168$ A

**4. Find the current through each branch of following figure using nodal analysis.**



[2070 Ashadh]

***Solution:***



Applying KCL at node 1, we get

$$\frac{V_1-10}{3}+\frac{V_1-I_2}{2}+\frac{V_1-V_2}{5}=0$$

$$\text{or, }\frac{V_1-10}{3}+\frac{V_1-\left(\frac{V_1-V_2}{5}\right)}{2}+\frac{V_1-V_2}{5}=0$$

$$\text{or, }\frac{V_1-10}{3}+\frac{5V_1-V_1+V_2}{10}+\frac{V_1-V_2}{5}=0$$

or, $28V_1 - 3V_2 = 100$ ....... (1)

Applying KCL at node 2, we get

$$\frac{V_2-V_1}{5}+\frac{V_2-0}{4}=2$$

$$\text{or, }\frac{4V_2-4V_1+5V_2}{20}=2$$

or, $4V_1 - 9V_2 = -40$ ....... (2)

Solving equations (1) & (2), we get

$V_1 = 4.25$ V, $V_2 = 6.33$ V

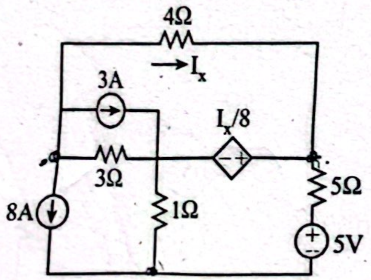
The current through 3Ω resistor $=\frac{V_1-10}{3}=-1.916$ A

The current through 5Ω resistor $(I_2)=\frac{V_1-V_2}{5}=-0.416$ A

The current through 2Ω resistor $=\frac{V_1-I_2}{2}=2.333$ A

The current through 4Ω resistor $=\frac{V_2-0}{4}=1.582$ A

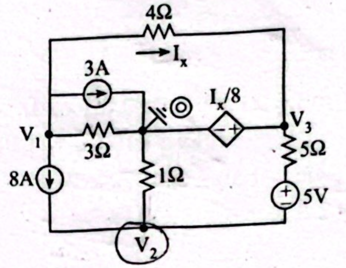
**5. Using nodal analysis, find $I_x$ in the circuit shown below.**




[2071 Shrawan]



***Solution:***




Applying KCL at node 1, we get

$$\frac{V_1-0}{3}+\frac{V_1-V_3}{4}=-8-3$$

or, $7V_1 - 3V_3 = -132$ ......... (1)

Applying KCL at node 2, we get

$$\frac{V_2-0}{1}+\frac{V_2+5-V_3}{5}=8$$

or, $6V_2 - V_3 = 35$ ............. (2)

At node 3,

$$V_3-0=\frac{I_x}{8}$$

$$\text{or, }V_3=\frac{\frac{V_1-V_3}{4}}{8}$$

$$\text{or, }V_3=\frac{V_1-V_3}{32}$$

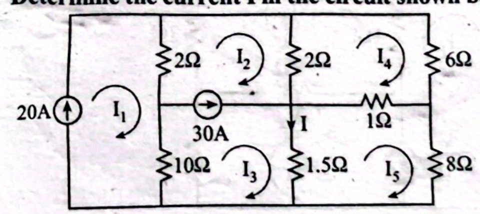
or, $-V_1 + 33V_3 = 0$ ........ (3)

Solving equations (1), (2), (3), we get

$V_1 = -19.105$ V, $V_2 = 5.736$ V, $V_3 = -0.578$ V

Hence, $I_x=\frac{V_1-V_3}{4}=-4.631$ A

**6. Determine the current I in the circuit shown below using loop analysis.**




[2064 Falgun]

**Solution:**

In loop 1,

$I_1 = 20$ ........ (1)

Applying KVL in super mesh (loop 2 & loop 3), we get

$2(I_2 - I_1) + 2(I_2 - I_4) + 1.5(I_3 - I_5) + 10(I_3 - I_1) = 0$

or, $4I_2 + 11.5I_3 - 2I_4 - 1.5I_5 = 240$ ......... (2)

Also from loop 2 & loop 3,

$-I_2 + I_3 = 30$ ............. (3)

Applying KVL in loop 4, we get

$2(I_4 - I_2) + 6I_4 + 1(I_4 - I_5) = 0$

or, $-2I_2 + 9I_4 - I_5 = 0$ .......... (4)

Applying KVL in loop 5, we get

$1(I_5 - I_4) + 8I_5 + 1.5(I_5 - I_3) = 0$

or, $-1.5I_3 - I_4 + 10.5I_5 = 0$ ....... (5)

From equation (4),

$I_5 = -2I_2 + 9I_4$ .......... (6)

Putting $I_5$ in equations (2) & (5),

$4I_2 + 11.5I_3 - 2I_4 - 1.5(-2I_2 + 9I_4) = 240$

or, $7I_2 + 11.5I_3 - 15.5I_4 = 240$ ........ (7)

$-1.5I_3 - I_4 + 10.5(-2I_2 + 9I_4) = 0$

or, $-21I_2 - 1.5I_3 + 93.5I_4 = 0$ ............ (8)

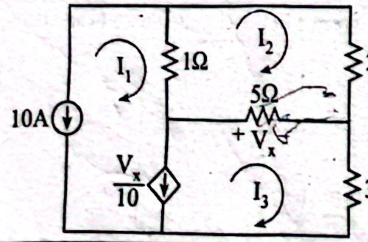
Solving equations (2), (3), (4), (5), we get

$I_1 = 20A, I_2 = -6.604\ A, I_3 = 23.396\ A, I_4 = -1.107\ A,$

$I_5 = -2I_2 + 9I_4 = -2\times(-6.604) + 9\times(-1.107) = 3.245\ A$

$I = I_3 - I_5 = 23.396 - 3.245 = 20.151\ A$

**7. Find the currents through each resistor using mesh analysis.**




**Solution:**

In loop 1,

$I_1 = -10$ ............ (1)

Applying KVL in loop 2, we get

$1(I_2 - I_1) + 2(I_2) + 5(I_2 - I_3) = 0$

or, $8I_2 - I_1 - 5I_3 = 0$

or, $8I_2 - (-10) - 5I_3 = 0$

or, $8I_2 - 5I_3 = -10$ ........ (2)

From supermesh (loop 1 & loop 3), we get

$I_1 - I_3 = \frac{V_x}{10}$

or, $I_1 - I_3 = \frac{1}{10}[5(I_3 - I_2)]$

or, $2I_1 + I_2 - 3I_3 = 0$

or, $2(-10) + I_2 - 3I_3 = 0$

or, $I_2 - 3I_3 = 20$ ....... (3)

Solving equations (2) & (3), we get

$I_2 = -6.842\ A, I_3 = -8.947\ A$

Hence,

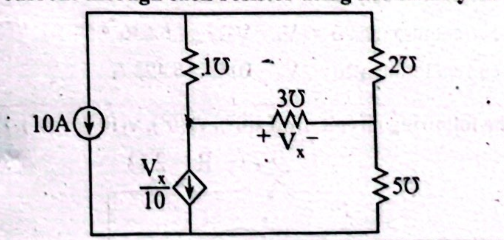
The current through 1Ω resistor = $I_1 - I_2 = -3.158\ A$

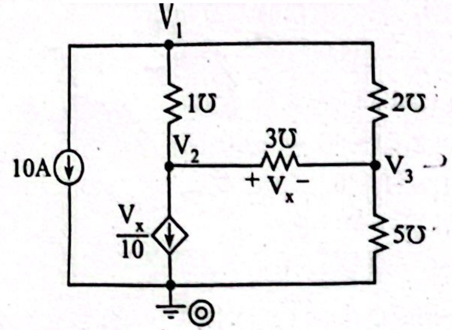
The current through 2Ω resistor = $I_2 = -6.842\ A$

The current through 5Ω resistor = $I_3 - I_2 = -2.105\ A$

The current through 3Ω resistor = $I_3 = -8.947\ A$

**8. Find the current through each resistor using nodal analysis.**






Applying KCL at node 1, we get

$(V_1 - V_2)1 + (V_1 - V_3)2 = -10$

or, $3V_1 - V_2 - 2V_3 = -10$ .......... (1)

Applying KCL at node 2, we get

$(V_2 - V_3)3 + (V_2 - V_1)1 = -\frac{V_x}{10}$

or, $4V_2 - 3V_3 - V_1 = -\frac{1}{10}(V_2 - V_3)$

or, $-10V_1 + 41V_2 - 4V_3 = 0$ .......... (2)

Applying KCL at node 3, we get

$(V_3 - V_1)2 + (V_3 - 0)5 + (V_3 - V_2)3 = 0$

or, $2V_1 + 3V_2 - 10V_3 = 0$ ....... (3)

Solving equations (1), (2), (3), we get

$V_1 = -4.617$ V, $V_2 = -1.252$ V, $V_3 = -1.299$ V

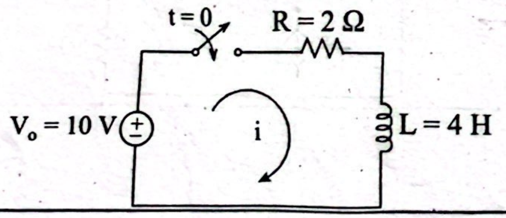
The current through 1℧ $= (V_1 - V_2)1 = -3.365$ A

The current through 3℧ $= (V_2 - V_3)3 = 0.141$ A

The current through 2℧ $= (V_1 - V_3)2 = -6.636$ A

The current through 5℧ $= (V_3 - 0)5 = -6.495$ A

**9. In the following circuit, find $i(0^+)$, $v_R(0^+)$, $v_L(0^+)$, $i'(0^+)$, $i''(0^+)$.**






***Solution:***

$$i'(0^+) = \frac{di(0^+)}{dt},\ i''(0^+) = \frac{d^2i(0^+)}{dt^2}$$

By observation, at $t = 0^-$, $i(0^-) = 0$.

From continuity relation,

$i(0^+) = i(0^-) = 0$

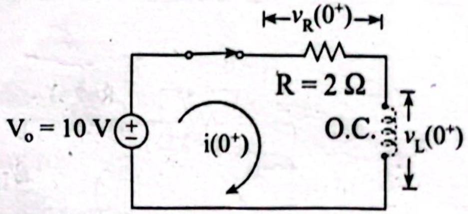



*Figure 1: Equivalent circuit at $t = 0^+$.*

From figure 1, $v_L(0^+) = 10$ V

$v_R(0^+) = i(0^+)\,R = 0$

Applying KVL for $t > 0$,

$v_R + v_L = V_o$

or, $2i + 4\frac{di}{dt} = 10$

or, $\frac{di}{dt} + \frac{1}{2}i = \frac{10}{4}$ .............. (1)

Put $t = 0^+$ in equation (1), we have

$$\frac{di(0^+)}{dt} + \frac{1}{2}i(0^+) = \frac{10}{4}$$

or, $\frac{di(0^+)}{dt} = \frac{10}{4}$ A/s .............. (2)

Differentiating (1) w.r.t. t, we get

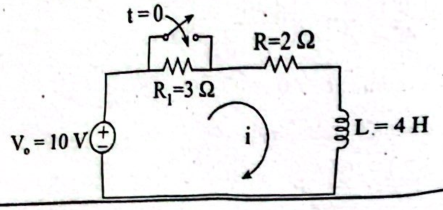
$\frac{d^2i}{dt^2} + \frac{1}{2}\frac{di}{dt} = 0$ ................ (3)

Put $t = 0^+$,

$$\frac{d^2i(0^+)}{dt^2} + \frac{1}{2}\frac{di(0^+)}{dt} = 0$$

or, $\frac{d^2i(0^+)}{dt^2} = -\frac{10}{8}$ A/s$^2$

**10. In the following circuit shown, find i, $v_{R1}$, $v_R$, $v_L$, i', i'' at $t = 0^+$.**




***Solution:***

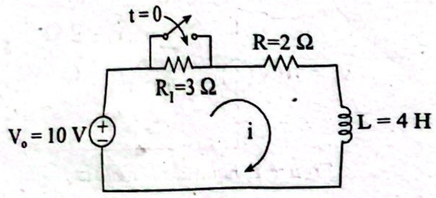



*Figure 1*

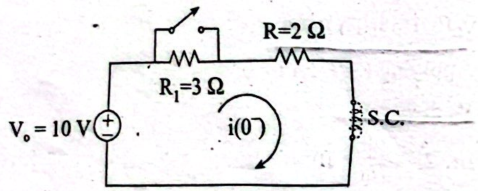



*Figure 2: Equivalent circuit at $t = 0^-$.*

From figure 2,

$$i(0^-) = \frac{10}{3+2} = 2\text{ A}$$

From continuity relation for inductor,

$$i(0^+) = i(0^-) = 2\text{ A}$$

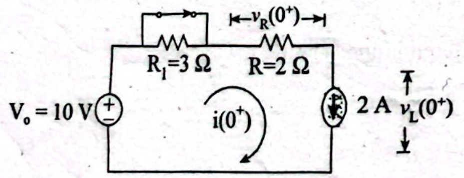



*Figure 3: Equivalent circuit at $t = 0^+$.*

$v_R(0^+) = i(0^+)\,R = 4\text{ V}$

Applying KVL in figure 3,

$10 - v_R(0^+) - v_L(0^+) = 0$

or, $v_L(0^+) = 10 - v_R(0^+)$

$= 10 - 4 = 6\text{ V}$

Applying KVL in figure 1,

$v_R + v_L = V_o$

or, $2i + 4\frac{di}{dt} = 10$

or, $\frac{di}{dt} + \frac{1}{2}i = \frac{10}{4}$ .............. (1)

Put $t = 0^+$ in above equation,

$$\frac{di(0^+)}{dt} + \frac{1}{2}i(0^+) = \frac{10}{4}$$

$$\therefore \frac{di(0^+)}{dt} = \frac{6}{4}\text{ A/s} \qquad \frac{\cancel{6}\,3}{\cancel{4}\,2}$$

Differentiating equation (1) w.r.t. t, we get

$$\frac{d^2i}{dt^2} + \frac{1}{2}\frac{di}{dt} = 0$$

Put $t = 0^+$ in above equation,

$$\frac{d^2i(0^+)}{dt^2} + \frac{1}{2}\frac{di(0^+)}{dt} = 0$$

$$\text{or, } \frac{d^2i(0^+)}{dt^2} + \frac{1}{2} \times \frac{6}{4} = 0$$

$$\text{or, } \frac{d^2i(0^+)}{dt^2} = -\frac{3}{4}\text{ A/s}^2$$

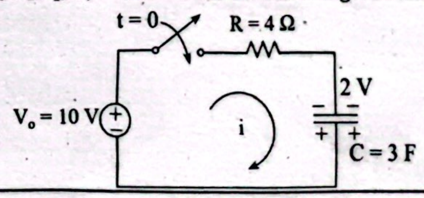
**Alternative method to find $\frac{di(0^+)}{dt}$:**

$$v_L = L\frac{di}{dt}$$

$$\text{or, } \frac{di}{dt} = \frac{v_L}{L}$$

Put $t = 0^+$, $\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{6}{4}\text{ A/s}$

**11. Find i, $v_R$, $v_C$, q, i', i'' at $t = 0^+$ in the following circuit.**



***Solution:***

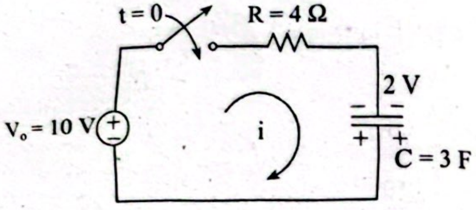



***Figure 1***

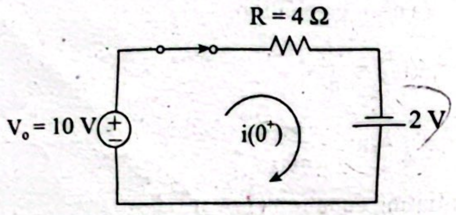



***Figure 2: Equivalent circuit at t = 0⁺.***

By observation at $t = 0^-$,

$v_C(0^-) = 2$ V

$q(0^-) = C\,v_C(0^-) = 3 \times 2 = 6$ C

From continuity relation for capacitor,

$v_C(0^+) = v_C(0^-) = 2$ V

$q(0^+) = q(0^-) = 6$ C

$$i(0^+) = \frac{10+2}{4} = 3 \text{ A}$$

$v_R(0^+) = i(0^+)\,R = 12$ V

Applying KVL for $t > 0$ in figure 1,

$v_R + v_C = 10$

$$\text{or, } 4i + \frac{1}{3}\int i\,dt = 10$$

$$\text{or, } 4\frac{di}{dt} + \frac{i}{3} = 0$$

$$\text{or, } \frac{di}{dt} + \frac{i}{12} = 0 \qquad \text{........(1)}$$

Put $t = 0^+$,

$$\frac{di(0^+)}{} \quad \frac{i(0^+)}{}$$

Differentiating equation (1) w.r.t. t, we get

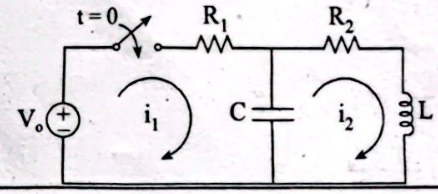
$$\frac{d^2i}{dt^2} + \frac{1}{12}\frac{di}{dt} = 0$$

Put $t = 0^+$,

$$\frac{d^2i(0^+)}{dt^2} + \frac{1}{12}\frac{di(0^+)}{dt} = 0$$

$$\text{or, } \frac{d^2i(0^+)}{dt^2} - \frac{1}{12} \times \frac{1}{4} = 0 \quad \therefore \frac{d^2i(0^+)}{dt^2} = \frac{1}{48} \text{ A/s}^2$$

**12. Find $i_1, i_2, i_1', i_2', i_1'', i_2''$ at $t = 0^+$.**




***Solution:***

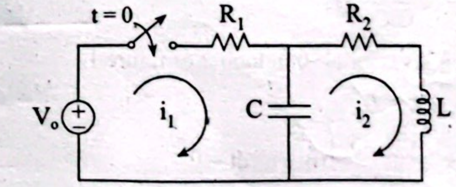



***Figure 1***

By observation at $t = 0^-$ in figure 1,

$i_L(0^-) = 0$, $v_C(0^-) = 0$

From continuity relation for inductor and capacitor,

$i_2(0^+) = i_L(0^+) = i_L(0^-) = 0$

$v_C(0^+) = v_C(0^-) = 0$

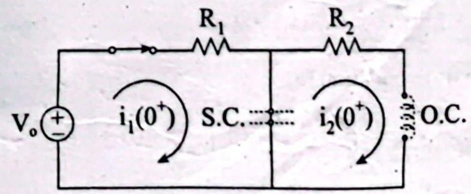



***Figure 2: Equivalent circuit at t = 0⁺.***

From figure 2

Applying KVL for $t > 0$ in figure 1 in first loop,

$V_o - v_{R1} - v_C = 0$

or, $v_{R1} + v_C = V_o$

or, $i_1R_1 + \frac{1}{C}\int(i_1 - i_2)\,dt = V_o$

Differentiating above equation w.r.t. t, we get

$R_1\frac{di_1}{dt} + \frac{1}{C}(i_1 - i_2) = 0$

or, $\frac{di_1}{dt} + \frac{1}{CR_1}i_1 - \frac{1}{CR_1}i_2 = 0$ ....... (1)

Put $t = 0^+$,

$\frac{di_1(0^+)}{dt} + \frac{1}{CR_1}i_1(0^+) - \frac{1}{CR_1}i_2(0^+) = 0$

or, $\frac{di_1(0^+)}{dt} + \frac{1}{CR_1}\frac{V_o}{R_1} - 0 = 0$

$\therefore \frac{di_1(0^+)}{dt} = \frac{-V_o}{CR_1^2}$ A/s .............. (2)

Applying KVL for $t > 0$ in loop 2 of figure 1,

$v_{R2} + v_L + v_C = 0$

or, $i_2R_2 + L\frac{di_2}{dt} + \frac{1}{C}\int(i_2 - i_1)\,dt = 0$ .......... (3)

Put $t = 0^+$ in equation (3),

$i_2(0^+)R_2 + L\frac{di_2(0^+)}{dt} + v_C(0^+) = 0$

$\therefore \frac{di_2(0^+)}{dt} = 0$ ............. (4)

Differentiating equation (3) w.r.t. t, we get

$R_2\frac{di_2}{dt} + L\frac{d^2i_2}{dt^2} + \frac{1}{C}(i_2 - i_1) = 0$ ............. (5)

Put $t = 0^+$ in equation (5),

$R_2\frac{di_2(0^+)}{dt} + L\frac{d^2i_2(0^+)}{dt^2} + \frac{1}{C}i_2(0^+) - \frac{1}{C}i_1(0^+) = 0$

or, $0 + L\frac{d^2i_2(0^+)}{dt^2} + 0 - \frac{1}{C}\frac{V_o}{R_1} = 0$

$\therefore \frac{d^2i_2(0^+)}{dt^2} = \frac{V_o}{LCR_1}$ A/s$^2$



Differentiating equation (1), w.r.t. t, we have

$\frac{d^2i_1}{dt^2} + \frac{1}{CR_1}\frac{di_1}{dt} - \frac{1}{CR_1}\frac{di_2}{dt} = 0$ ............. (6)

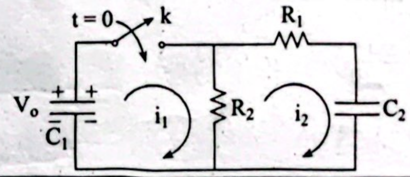
Put $t = 0^+$ in equation (6),

$\frac{d^2i_1(0^+)}{dt^2} + \frac{1}{CR_1}\frac{di_1(0^+)}{dt} - \frac{1}{CR_1}\frac{di_2(0^+)}{dt} = 0$

or, $\frac{d^2i_1(0^+)}{dt^2} + \frac{1}{CR_1}\left(\frac{-V_o}{CR_1^2}\right) - 0 = 0$

$\therefore \frac{d^2i_1(0^+)}{dt^2} = \frac{V_o}{C^2R_1^3}$ A/s$^2$

**13. In the given network, the capacitor $C_1$ is charged to voltage $V_o$ and switch k is closed at $t = 0$. When $R_1 = 2$ MΩ, $V_o = 1000$ V, $R_2 = 1$ MΩ, $C_1 = 10$ μF, $C_2 = 20$ μF, solve for $i_1$, $i_2$, $\frac{di_2}{dt}$, $\frac{d^2i_2}{dt^2}$ at $t = 0^+$.**




[2075 Ashwin]

***Solution:***

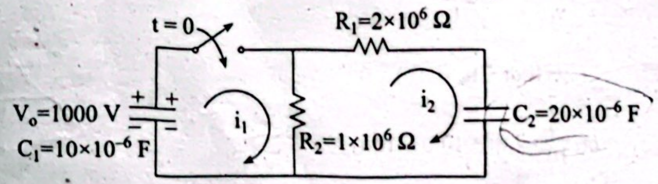



***Figure 1***

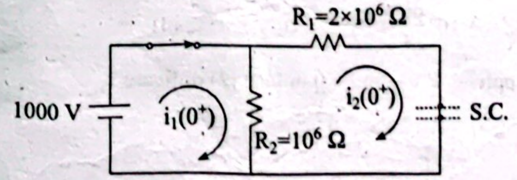



***Figure 2: Equivalent circuit at $t = 0^+$***

Applying KVL in loop 1 of figure 2

$1000 = 10^6 i_1(0^+) - 10^6 i_2(0^+)$

or, $10^3 i_1(0^+) - 10^3 i_2(0^+) = 1$ .................. (1)

Applying KVL in loop 2 of figure 2,

$-10^6 i_1(0^+) + 3 \times 10^6 i_2(0^+) = 0$

or, $-10^3 i_1(0^+) + 3 \times 10^3 i_2(0^+) = 0$ ............ (2)

Adding equations (1) and (2), we get

$2 \times 10^3 i_2(0^+) = 1$

$\therefore i_2(0^+) = \dfrac{1}{2 \times 10^3}$ ............. (3)

From equations (1) and (3),

$10^3 i_1(0^+) - 10^3 \times \dfrac{1}{2\times10^3} = 1$

or, $i_1(0^+) = \dfrac{3}{2\times10^3}$ A

Hence, $i_2(0^+) = \dfrac{1}{2\times10^3} = \dfrac{1}{2} \times 10^{-3}\text{A} = \dfrac{1}{2}$ mA

$i_1(0^+) = \dfrac{3}{2} \times 10^{-3}$ A $= \dfrac{3}{2}$ mA

Now, applying KVL for t > 0 in loop (1) of figure 1,

$v_{C_1} + v_{R_2} = 0$

or, $\dfrac{1}{C_1}\int i_1 dt + R_2(i_1 - i_2) = 0$

or, $\dfrac{1}{10^{-5}}\int i_1 dt + 10^6 i_1 - 10^6 i_2 = 0$

Differentiating w.r.t. t, we get

$10^5 i_1 + 10^6 \dfrac{di_1}{dt} - 10^6 \dfrac{di_2}{dt} = 0$ ........... (4)

Applying KVL for t > 0 in loop (2) of figure 1,

$v_{R_1} + v_{C_2} + v_{R_2} = 0$

or, $R_1 i_2 + \dfrac{1}{C_2}\int i_2\, dt + R_2(i_2 - i_1) = 0$

or, $2\times10^6 i_2 + \dfrac{1}{20\times10^{-6}}\int i_2\, dt + 10^6 i_2 - 10^6 i_1 = 0$

or, $3\times10^6 i_2 + 5\times10^4 \int i_2\, dt - 10^6 i_1 = 0$

Differentiating w.r.t. t, we get

$3\times10^6 \dfrac{di_2}{dt} + 5\times10^4 i_2 - 10^6 \dfrac{di_1}{dt} = 0$ ............. (5)

Adding equations (4) and (5), we get

$10^5 i_1 + 5\times10^4 i_2 + 2\times10^6 \dfrac{di_2}{dt} = 0$ .......... (6)

Put $t = 0^+$ in equation (6),

$10^5 i_1(0^+) + 5\times10^4 i_2(0^+) + 2\times10^6 \dfrac{di_2(0^+)}{dt} = 0$

or, $10^5 \times \dfrac{3}{2\times10^3} + 5\times10^4 \times \dfrac{1}{2\times10^3} + 2\times10^6 \dfrac{di_2(0^+)}{dt} = 0$

or, $\dfrac{di_2(0^+)}{dt} = -8.75\times10^{-5}$ A/s ............ (7)

Put $t = 0^+$ in equation (4),

$10^5 i_1(0^+) + 10^6 \dfrac{di_1(0^+)}{dt} - 10^6 \dfrac{di_2(0^+)}{dt} = 0$

or, $10^5 \times \dfrac{1}{2\times10^3} + 10^6 \dfrac{di_1(0^+)}{dt} - 10^6 \times -8.75\times10^{-5} = 0$

or, $50 + 10^6 \dfrac{di_1(0^+)}{dt} + 87.5 = 0$

or, $\dfrac{di_1(0^+)}{dt} = -1.375\times10^{-4}$ A/s ................ (8)

Differentiating equation (6) w.r.t. t, we get

$10^5 \dfrac{di_1}{dt} + 5\times10^4 \dfrac{di_2}{dt} + 2\times10^6 \dfrac{d^2 i_2}{dt^2} = 0$ ............ (9)

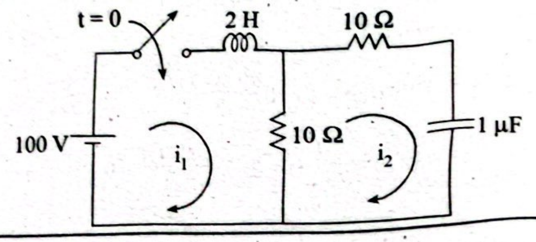
Put $t = 0^+$ in (9),

$10^5 \dfrac{di_1(0^+)}{dt} + 5\times10^4 \dfrac{di_2(0^+)}{dt} + 2\times10^6 \dfrac{d^2 i_2(0^+)}{dt} = 0$

or, $10^5\times(-1.375\times10^{-4}) + 5\times10^4\times(-8.75\times10^{-5}) + 2\times10^6 \dfrac{d^2 i_2(0^+)}{dt^2} = 0$

$\therefore \dfrac{d^2 i_2(0^+)}{dt^2} = -0.379315$ A/s$^2$

**14. In the given network, both the energy storing elements are initi relaxed i.e., no current is flowing through the inductor and charge is accumulated across the capacitor before application voltage. The switch k is closed at t = 0. Find the values of $i_1$, $i_2$, $\frac{di_2}{dt}$, $\frac{d^2i_1}{dt^2}$, $\frac{d^2i_2}{dt^2}$ at $t = 0^+$.**




[2074 Ash

*Solution:*

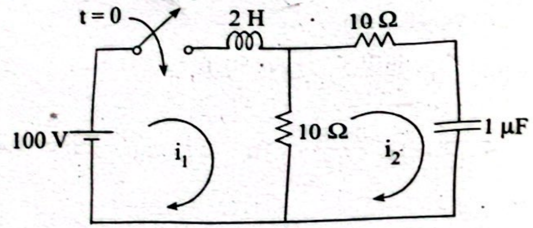



*Figure 1*

By inspection at $t = 0^-$,

$i_1(0^-) = 0$ and $v_c(0^-) = 0$

From continuity relation for inductor and capacitor,

$i_1(0^+) = i_1(0^-) = 0$

$v_c(0^+) = v_c(0^-) = 0$

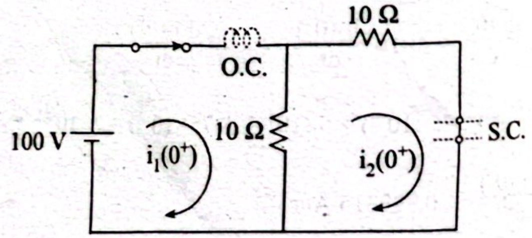



*Figure 2: Equivalent circuit at $t = 0^+$.*



From figure 2,

$i_2(0^+) = 0$

Applying KVL for $t > 0$ in loop 1 of figure 1,

$100 = v_1 + v_{10\Omega}$

or, $100 = L\frac{di_1}{dt} + 10(i_1 - i_2)$

or, $100 = 2\frac{di_1}{dt} + 10(i_1 - i_2)$ ................(1)

Put $t = 0^+$ in equation (1),

$$100 = \frac{di_1(0^+)}{dt} + 10\,[i_1(0^+) - i_2(0^+)]$$

or, $100 = 2\frac{di_1(0^+)}{dt} + 10\,(0 - 0)$

$\therefore \frac{di_1(0^+)}{dt} = \frac{100}{2} = 50$ A/s ................(2)

Applying KVL for $t > 0$ in loop 2 of figure 1,

$v_{10\Omega} + v_c + v_{10\Omega} = 0$

or, $10i_2 + \frac{1}{C}\int i_2\,dt + 10\,(i_2 - i_1) = 0$

or, $10i_2 + \frac{1}{10^{-6}}\int i_2\,dt + 10i_2 - 10i = 0$

or, $20\,i_2 + 10^6 \int i_2\,dt - 10\,i_1 = 0$

Differentiating w.r.t. t, we get

$20\frac{di_2}{dt} + 10^6\,i_2 - 10\frac{di_1}{dt} = 0$ ..................(3)

Put $t = 0^+$ in equation (3),

$$20\frac{di_2(0^+)}{dt} + 10^6\,i_2(0^+) - 10\frac{di_1(0^+)}{dt} = 0$$

or, $20\frac{di_2(0^+)}{dt} + 10^6 \times 0 - 10 \times 50 = 0$

or, $\frac{di_2(0^+)}{dt} = \frac{10 \times 50}{20} = 25$ A/s ................(4)

Differentiating equation (1) w.r.t. t, we get

$$0 = 2\frac{d^2i_1}{dt^2} + 10\frac{di_1}{dt} - 10\frac{di_2}{dt} \qquad \text{..................(5)}$$

Put $t = 0^+$ in equation (5),

$$0 = 2\frac{d^2i_1(0^+)}{dt^2} + 10\frac{di_1(0^+)}{dt} - 10\frac{di_2(0^+)}{dt}$$

$$\text{or, } 0 = 2\frac{d^2i_1(0^+)}{dt^2} + 10 \times 50 - 10 \times 25$$

$$\text{or, } \frac{d^2i_1(0^+)}{dt^2} = -125 \text{ A/s}^2$$

Differentiating equation (3) w.r.t. t, we get

$$20\frac{d^2i_2}{dt^2} + 10^6\frac{di_2}{dt} - 10\frac{d^2i_1}{dt^2} = 0 \qquad \text{...........(6)}$$

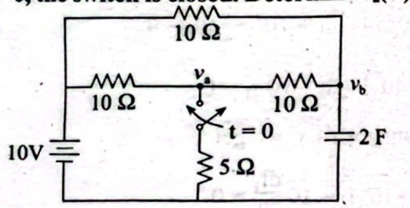
Put $t = 0^+$ in equation (6),

$$20\frac{d^2i_2(0^+)}{dt^2} + 10^6\frac{di_2(0^+)}{dt} - 10\frac{d^2i_1(0^+)}{dt^2} = 0$$

$$\text{or, } 20\frac{d^2i_2(0^+)}{dt^2} + 10^6 \times 25 - 10 \times (-125) = 0$$

$$\therefore \frac{d^2i_2(0^+)}{dt^2} = -1250062.5 \text{ A/s}^2$$

**15. In the network shown in figure below, a steady state is reached with switch open. At t=0, the switch is closed. Determine $v_a(0^-)$ and $v_a(0^+)$.**




[2073 Shr...]

***Solution:***

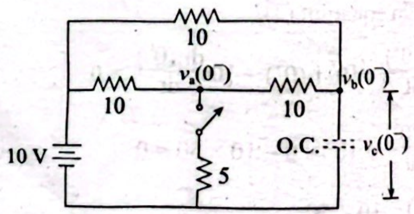



*Figure 1: Equivalent circuit at t = 0⁻*



From figure 1, $v_c(0^-) = 10$ V and $v_a(0^-) = 0$

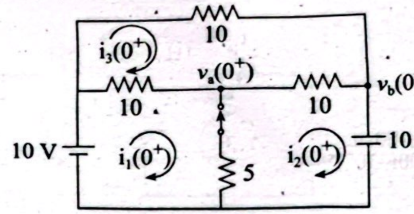



*Figure 2: Equivalent circuit at $t = 0^+$*

Applying KVL in loop 1,

$15i_1(0^+) - 5i_2(0^+) - 10i_3(0^+) = 10$ ........... (1)

Applying KVL in loop 2,

$-5i_1(0^+) + 15i_2(0^+) - 10i_3(0^+) = -10$ ........... (2)

Applying KVL in loop 3,

$-10i_1(0^+) - 10i_2(0^+) + 30i_3(0^+) = 0$ ............ (3)

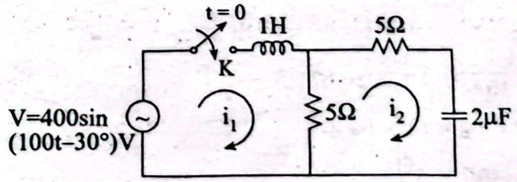
Solving (1), (2), (3), we get

$i_1(0^+) = 0.75$ A , $i_2(0^+) = -0.25$ A, $i_3(0^+) = 0.25$ A

$v_a(0^+) = v_{5\Omega}(0^+) = 5i_1(0^+) - 5i_2(0^+)$

$= 5 \times 0.75 - 5 \times (-0.25) = 5$ V

**16. In the given circuit, both the energy storing elements are initially relaxed i.e.; no current is flowing through the inductor & no charge is accumulated across capacitor before application of voltage. The switch K is closed at t = 0. Find the values of $i_1$, $i_2$, $\frac{di_1}{dt}$, $\frac{di_2}{dt}$, $\frac{d^2i_1}{dt^2}$, $\frac{d^2i_2}{dt^2}$ at $t = 0^+$.**




[2075 Chaitra]

**Solution:**

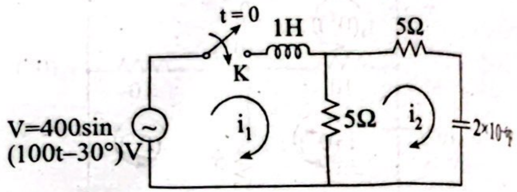



***Figure 1***

By observation at $t = 0^-$,

$i_1(0^-) = 0$, $v_C(0^-) = 0$

From continuity relation for inductor and capacitor, we get

$i_1(0^+) = i_1(0^-) = 0$ .......... (1)

$v_C(0^+) = v_C(0^-) = 0$ .......... (2)

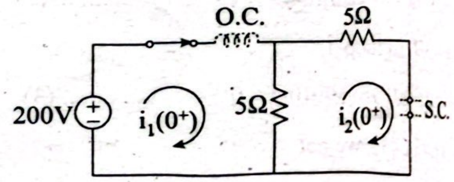



***Figure 2:** Equivalent circuit at $t = 0^+$*

From figure 2, $i_2(0^+) = 0$ ............ (3)

Applying KVL for $t > 0$ in loop (1) of figure 1, we get

$$1\frac{di_1}{dt} + 5i_1 - 5i_2 = 400\sin(100t - 30°) \text{ .............. (4)}$$

Putting $t = 0^+$ in equation (4), we get

$$\frac{di_1(0^+)}{dt} + 5i_1(0^+) - 5i_2(0^+) = 400\sin(-30°)$$

$$\therefore \frac{di_1(0^+)}{dt} = -200 \text{ A/s ......... (5)}$$

Applying KVL in loop (2) for $t > 0$ of figure 1, we get

$$5i_2 + \frac{1}{2\times10^{-6}}\int i_2\,dt + 5(i_2 - i_1) = 0$$

Differentiating w.r.t. t, we get

$$5\frac{di_2}{dt} + 5\times10^5 i_2 + 5\frac{di_2}{dt} - 5\frac{di_1}{dt} = 0$$

$$\text{or, } 10\frac{di_2}{dt} - 5\frac{di_1}{dt} + 5\times10^5 i_2 = 0 \text{ .......... (6)}$$



Put $t = 0^+$ in equation (6), we get

$$10\frac{di_2(0^+)}{dt} - 5\frac{di_1(0^+)}{dt} + 5\times10^5 i_2(0^+) = 0$$

$$\text{or, } 10\frac{di_2(0^+)}{dt} - 5(-200) = 0$$

$$\therefore \frac{di_2(0^+)}{dt} = -100 \text{ A/s ........ (7)}$$

Differentiating equation (4) w.r.t. t, we get

$$\frac{d^2i_1}{dt^2} + 5\frac{di_1}{dt} - 5\frac{di_2}{dt} = 400\times100\cos(100t - 30°) \text{ ........ (8)}$$

Put $t = 0^+$ in equation (8),

$$\frac{d^2i_1(0^+)}{dt^2} + 5\frac{di_1(0^+)}{dt} - 5\frac{di_2(0^+)}{dt} = 400\times100\cos(-30°)$$

$$\text{or, } \frac{d^2i_1(0^+)}{dt^2} - 1000 + 500 = 34641$$

$$\text{or, } \frac{d^2i_1(0^+)}{dt^2} = 35141 \text{ A/s}^2 \text{ ................ (9)}$$

Differentiating equation (6) w.r.t. t, we get

$$10\frac{d^2i_2}{dt^2} - 5\frac{d^2i_1}{dt^2} + 5\times10^5\frac{di_2}{dt} = 0 \text{ .................. (10)}$$

Put $t = 0^+$ in (10),

$$10\frac{d^2i_2(0^+)}{dt^2} - 5\frac{d^2i_1(0^+)}{dt^2} + 5\times10^5\frac{di_2(0^+)}{dt} = 0$$

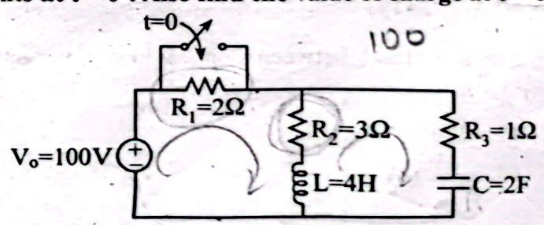
$$\text{or, } 10\frac{d^2i_2(0^+)}{dt^2} - 5(35141) + 5\times10^5(-100) = 0$$

$$\therefore \frac{d^2i_2(0^+)}{dt^2} = 5017570.5 \text{ A/s}^2$$

$V_C = 20\times3$

$i = \frac{100}{2+3} = \frac{100}{5} = 20$

**7. For the circuit shown below, find currents and voltages across each elements at $t = 0^+$. Also find the value of charge at $t = 0^+$.**




**[2078 Chaitra]**

**Solution:**

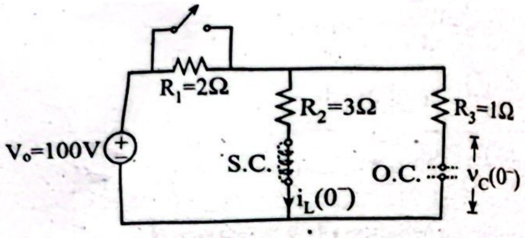



*Figure 1: Equivalent circuit at $t = 0^-$*

From figure 1,

$$i_L(0^-) = \frac{V_o}{R_1 + R_2} = \frac{100}{2+3} = 20\text{ A}$$

$$v_C(0^-) = v_{R_2}(0^-) = 20 \times 3 = 60\text{ V}$$

From continuity relation,

$i_L(0^+) = i_L(0^-) = 20$ A

$v_C(0^+) = v_C(0^-) = 60$ V

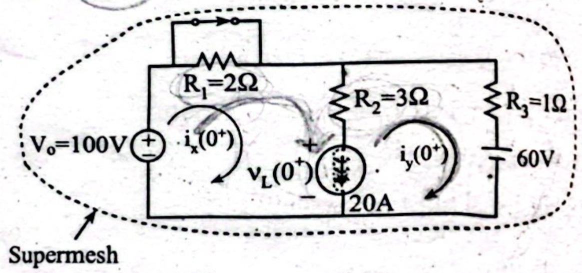



*Figure 2: Equivalent circuit at $t = 0^+$*

Applying KVL in supermesh of figure 2,

$V_o - 1i_y(0^+) - 60 = 0$

or, $100 - i_y(0^+) - 60 = 0$

$\therefore$ $i_y(0^+) = 40$ A ....... (1)

Also, from common branch between loop x & loop y, we get

$i_x(0^+) - i_y(0^+) = 20$

or, $i_x(0^+) - 40 = 20$

$\therefore$ $i_x(0^+) = 60$ .......... (2)



Now,

$i_{R_1}(0^+) = 0 \Rightarrow v_{R_1}(0^+) = 0$

$i_{R_2}(0^+) = i_L(0^+) = 20\text{ A} \Rightarrow v_{R_2}(0^+) = 3 \times 20 = 60\text{ V}$

$i_{R_3}(0^+) = i_y(0^+) = 40\text{ A} \Rightarrow v_{R_3}(0^+) = 1 \times 40 = 40\text{ V}$

$i_C(0^+) = i_y(0^+) = 40$ A

Also from figure 2,

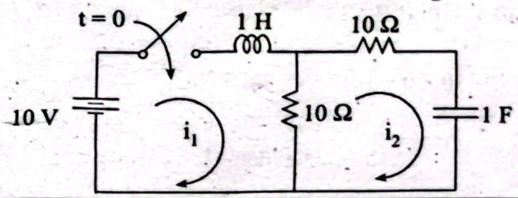
$+V_o - v_{R_2}(0^+) - v_L(0^+) = 0$

or, $v_L(0^+) = 100 - 60$

$\therefore$ $v_L(0^+) = 40$ V

And, $q(0^+) = Cv_C(0^+) = 2 \times 60 = 120$ C

**18. Obtain the value of $i_1$, $i_2$, $\frac{di_1}{dt}$, $\frac{di_2}{dt}$, $\frac{d^2i_1}{dt^2}$, $\frac{d^2i_2}{dt^2}$ at $t = 0^+$ if the switch is closed at $t = 0$ in the circuit shown in figure.**




[2074 Chaitra, 2079 Chaitra*]

**Solution:**

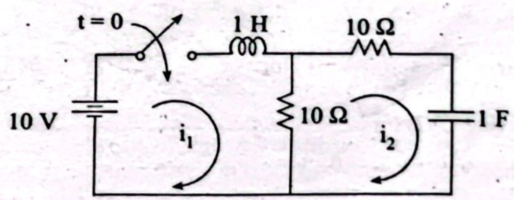



*Figure 1*

By observation at $t = 0^-$,

$i_1(0^-) = 0$ and $v_c(0^-) = 0$

From continuity relation for inductor and capacitor,

$i_1(0^+) = i_1(0^-) = 0$

$v_c(0^+) = v_c(0^-) = 0$

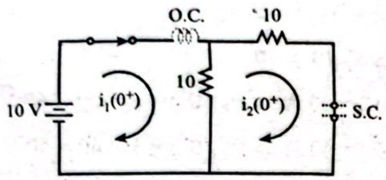



*Figure 2: Equivalent circuit at $t = 0^+$*

From figure 2,

$i_2(0^+) = 0$

Applying KVL for $t > 0$ in figure 1,

$10 = v_L + v_{10\Omega}$

or, $1\dfrac{di_1}{dt} + 10(i_1 - i_2) = 10$ ..............(1)

Put $t = 0^+$ in equation (1),

$$\frac{di_1(0^+)}{dt} + 10i_1(0^+) - 10i_2(0^+) = 10$$

or, $\dfrac{di_1(0^+)}{dt} + 10 \times 0 - 10 \times 0 = 10$

$\therefore \dfrac{di_1(0^+)}{dt} = 10$ A/s ..............(2)

Applying KVL for $t > 0$ in outer loop,

$10 = v_L + v_{10\Omega} + v_c$

or, $10 = 1\dfrac{di_1}{dt} + 10i_2 + \dfrac{1}{1}\int i_2\,dt$ ............(3)

Applying KVL for $t > 0$ in loop 2,

$v_{10\Omega} + v_c + v_{10\Omega} = 0$

or, $10i_2 + \int i_2 dt + 10(i_2 - i_1) = 0$

or, $20i_2 - 10i_1 + \int i_2\,dt = 0$

Differentiating above equation w.r.t. t, we get

$20\dfrac{di_2}{dt} - 10\dfrac{di_1}{dt} + i_2 = 0$ ..............(4)

Put $t = 0^+$ in equation (4),

$$20\frac{di_2(0^+)}{dt} - 10\frac{di_1(0^+)}{dt} + i_2(0^+) = 0$$



or, $20\dfrac{di_2(0^+)}{dt} - 10 \times 10 + 0 = 0$

$\therefore \dfrac{di_2(0^+)}{dt} = \dfrac{100}{20} = 5$ A/s ...............(5)

Differentiating equation (1) w.r.t. t, we get

$\dfrac{d^2i_1}{dt^2} + 10\dfrac{di_1}{dt} - 10\dfrac{di_2}{dt} = 0$ ...............(6)

Put $t = 0^+$ in equation (6),

$$\frac{d^2i_1(0^+)}{dt^2} + 10\frac{di_1(0^+)}{dt} - 10\frac{di_2(0^+)}{dt} = 0$$

or, $\dfrac{d^2i_1(0^+)}{dt^2} + 10 \times 10 - 10 \times 5 = 0$

or, $\dfrac{d^2i_1(0^+)}{dt^2} = -50$ A/s$^2$ ...................(7)

Differentiating equation (4) w.r.t. t, we get

$$20\frac{d^2i_2}{dt^2} - 10\frac{d^2i_1}{dt^2} + \frac{di_2}{dt} = 0$$

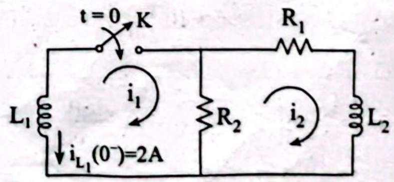
Put $t = 0^+$,

$$20\frac{d^2i_2(0^+)}{dt^2} - 10\frac{d^2i_1(0^+)}{dt^2} + \frac{di_1(0^+)}{dt^2} = 0$$

or, $20\dfrac{d^2i_2(0^+)}{dt^2} - 10 \times (-50) + 5 = 0$

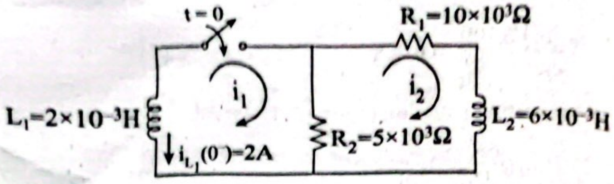
$\therefore \dfrac{d^2i_2(0^+)}{dt^2} = -\dfrac{101}{4}$ A/s$^2$

**19. In the given network of figure below, inductor $L_1$ is energized and the switch K is closed at $t = 0$. When each element has following values $R_1 = 10$ KΩ, $R_2 = 5$ KΩ, $L_1 = 2$ mH, $L_2 = 6$ mH, solve for $i_1$, $i_2$, $\dfrac{di_1}{dt}$, $\dfrac{di_2}{dt}$, $\dfrac{d^2i_1}{dt^2}$, $\dfrac{d^2i_2}{dt^2}$ at $t = 0^+$.**




**[2079 Ashwin]**

***Solution:***




$i_{L_1}(0^-) = 2A,\ i_{L_2}(0^-) = 0$

$\therefore\ i_1(0^+) = -i_{L_1}(0^-) = -2A$

$i_2(0^+) = i_{L_2}(0^-) = 0$

Applying KVL for $t > 0$ in loop (1),

$v_{L_1} + v_{R_2} = 0$

or, $L_1\dfrac{di_1}{dt} + R_2(i_1 - i_2) = 0$

or, $2 \times 10^{-3}\dfrac{di_1}{dt} + 5 \times 10^3(i_1 - i_2) = 0$ .........(1)

Put $t = 0^+$ in equation (1),

$2 \times 10^{-3}\dfrac{di_1(0^+)}{dt} + 5 \times 10^3 i_1(0^+) - 5 \times 10^3 i_2(0^+) = 0$

or, $2 \times 10^{-3}\dfrac{di_1(0^+)}{dt} = -5 \times 10^3(-2) + 0$

$\therefore\ \dfrac{di_1(0^+)}{dt} = \dfrac{10 \times 10^3}{2 \times 10^{-3}} = 5 \times 10^6$ A/s ....... (2)

Applying KVL for $t > 0$ in loop (2),

$v_{R_1} + v_{L_2} + v_{R_2} = 0$

or, $R_1 i_2 + L_2\dfrac{di_2}{dt} + R_2(i_2 - i_1) = 0$

or, $10 \times 10^3 i_2 + 6 \times 10^{-3}\dfrac{di_2}{dt} + 5 \times 10^3 i_2 - 5 \times 10^3 i_1 = 0$

or, $15 \times 10^3 i_2 - 5 \times 10^3 i_1 + 6 \times 10^{-3}\dfrac{di_2}{dt} = 0$ ......... (3)

Put $t = 0^+$ in equation (3),

$15 \times 10^3 i_2(0^+) - 5 \times 10^3 i_1(0^+) + 6 \times 10^{-3}\dfrac{di_2(0^+)}{dt} = 0$

$\therefore\ \dfrac{di_2(0^+)}{dt} = \dfrac{5 \times 10^3(-2)}{6 \times 10^{-3}} = -\dfrac{5}{3} \times 10^6$ A/s ......... (4)



Differentiating equation (1), we get

$2 \times 10^{-3}\dfrac{d^2i_1}{dt^2} + 5 \times 10^3\dfrac{di_1}{dt} - 5 \times 10^3\dfrac{di_2}{dt} = 0$ ........ (5)

Put $t = 0^+$ in equation (5),

$2 \times 10^{-3}\dfrac{d^2i_1(0^+)}{dt^2} + 5 \times 10^3\dfrac{di_1(0^+)}{dt} - 5 \times 10^3\dfrac{di_2(0^+)}{dt} = 0$

or, $2\times10^{-3}\dfrac{d^2i_1(0^+)}{dt^2} + 5\times10^3\times5\times10^6 - 5\times10^3\left(-\dfrac{5}{3}\times10^6\right) = 0$

or, $\dfrac{d^2i_1(0^+)}{dt^2} = \dfrac{-\dfrac{100}{3}\times10^9}{2\times10^{-3}} = -\dfrac{50}{3}\times10^{12}$ A/s² ......... (6)

Differentiating equation (3), we get

$15 \times 10^3\dfrac{di_2}{dt} - 5 \times 10^3\dfrac{di_1}{dt} + 6 \times 10^{-3}\dfrac{d^2i_2}{dt^2} = 0$ .......... (7)

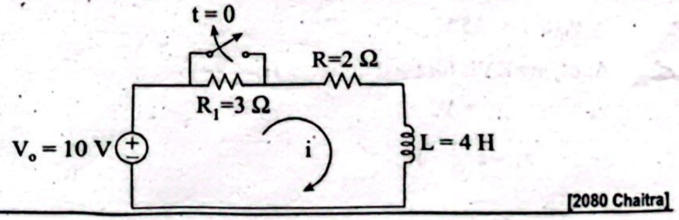
Put $t = 0^+$ in equation (7),

$15 \times 10^3\dfrac{di_2(0^+)}{dt} - 5 \times 10^3\dfrac{di_1(0^+)}{dt} + 6 \times 10^{-3}\dfrac{d^2i_2(0^+)}{dt^2} = 0$

or, $15\times10^3\times\left(-\dfrac{5}{3}\times10^6\right) - 5\times10^3\times(5\times10^6) + 6\times10^{-3}\dfrac{d^2i_2(0^+)}{dt^2} = 0$

$\therefore\ \dfrac{d^2i_2(0^+)}{dt^2} = \dfrac{50\times10^9}{6\times10^{-3}} = \dfrac{25}{3}\times10^{12}$ A/s²

**20. Find $i_L$, $v_{R1}$, $v_R$, $v_L$, $i'$, $i''$ at $t = 0^+$ in the following figure.**




[2080 Chaitra]

***Solution:***

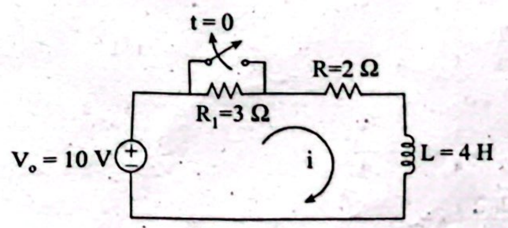



***Figure 1***

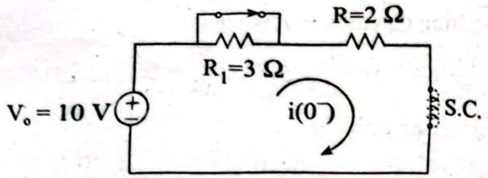



*Figure 2: Equivalent circuit at $t = 0^-$.*

From continuity relation for inductor,

$i(0^+) = i(0^-) = \frac{10}{2} = 5$ A

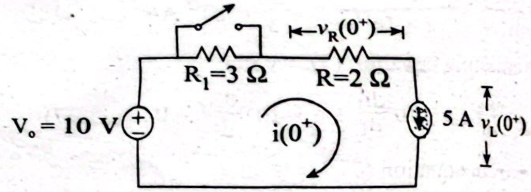



*Figure 3: Equivalent circuit at $t = 0^+$.*

$v_{R1}(0^+) = i(0^+)\,R_1 = 5 \times 3 = 15$ V

$v_R(0^+) = i(0^+)\,R = 5 \times 2 = 10$ V

Applying KVL in figure 3,

$10 - v_{R1}(0^+) - v_R(0^+) - v_L(0^+) = 0$

or, $10 - 15 - 10 - v_L(0^+) = 0$

$\therefore v_L(0^+) = -15$ V

Applying KVL for t > 0 in figure 1,

$v_R + v_{R1} + v_L = V_o$

or, $2i + 3i + 4\frac{di}{dt} = 10$

or, $4\frac{di}{dt} + 5i = 10$

or, $\frac{di}{dt} + \frac{5}{4}i = \frac{10}{4}$ ..................(1)

Put $t = 0^+$,

$\frac{di(0^+)}{dt} + \frac{5}{4}i(0^+) = \frac{10}{4}$

or, $\frac{di(0^+)}{dt} + \frac{5}{4} \times 5 = \frac{10}{4}$

$\therefore \frac{di(0^+)}{dt} = \frac{-15}{4}$ A/s

Differentiating equation (1) w.r.t. t, we have

$\frac{d^2i}{dt^2} + \frac{5}{4}\frac{di}{dt} = 0$

Put $t = 0^+$,

$\frac{d^2i(0^+)}{dt^2} + \frac{5}{4}\frac{di(0^+)}{dt} = 0$

or, $\frac{d^2i(0^+)}{dt^2} + \frac{5}{4} \times \frac{-15}{4} = 0$

$\therefore \frac{d^2i(0^+)}{dt^2} = \frac{75}{16}$ A/s²

Chapter – 2

# TRANSIENT ANALYSIS IN RLC CIRCUIT BY CLASSICAL METHOD

## 2.1 Introduction

*First-order circuits* are circuits that contain only one energy storage element (capacitor or inductor), and that can, therefore, be described using only a first order differential equation. The two possible types of first-order circuits are

i. RC (resistor and capacitor)

ii. RL (resistor and inductor)

These circuits are used in many applications: noise removers, delay lines, filters (high, low, band, notch, equalizers), signal coupling and decoupling in amplifiers, timers (the popular RC time constant…), sinusoidal and relaxation oscillators, etc.

*Second-order circuits* are circuits that contain both inductor and capacitor and that can, therefore, be described using a second order differential equation. Series RLC circuit and parallel RLC circuit are examples of second-order circuits. RLC circuits have many applications as oscillator circuits. Radio receivers and television sets use them for tuning to select a narrow frequency range from ambient radio waves. In this role, the circuit is often referred to as a tuned circuit. An RLC circuit can be used as a band-pass filter, band-stop filter, low-pass filter or high-pass filter. The tuning application, for instance, is an example of band-pass filtering.

## 2.2 Solution for General First Order Differential Equation

Consider a general first order nonhomogeneous differential equation given by the expression

$$\frac{di}{dt} + Pi = Q \qquad \text{................ (1)}$$

where i = dependent variable which may be either $i_L$ or $i_C$ or $v_C$ or q.

t = time which is independent variable.

P = constant which depends upon circuit component.

Q = source or input or excitation or driving force which may be either dc source or sinusoidal source or exponential source.

Multiplying equation (1) with I.F. $= e^{\int Pdt} = e^{P\int dt} = e^{Pt}$, we get

$$e^{Pt}\frac{di}{dt} + Pi\,e^{Pt} = Q\,e^{Pt}$$

$$\text{or, } \frac{d}{dt}(ie^{Pt}) = Q\,e^{Pt}$$

Integrating both sides w.r.t. t,

$$\int \frac{d}{dt}(ie^{Pt})\,dt = \int Q\,e^{Pt}\,dt$$

$$\text{or, } i\,e^{Pt} = \int Q\,e^{Pt}\,dt + k$$

$$\text{or, } i = e^{-Pt}\int Q\,e^{Pt}\,dt + ke^{-Pt} \qquad \text{........ (2)}$$

The first term of equation (2) is known as *particular integral (P.I.)* or *forced response*; the second is known as *complementary function (C.F.)* or *force free response* or *natural response.*

Equation (2) is the solution of equation (1).

i = P.I. + C.F.

i = force response + force free response (or natural response)

$$\text{or, } i = i_f + i_N \qquad \text{..........(3)}$$

For example, if we have, $\frac{dv_C}{dt} + 3v_C = 10$

$$i \to v_C,\ P \to 3,\ Q \to 10$$

$$v_C = e^{-3t}\int Q\,e^{3t}\,dt + ke^{-3t}$$

$$= \frac{10}{3} + ke^{-3t}$$

If $v_C(0^+) = 0$, then $k = \frac{-10}{3}$

$$v_C = \frac{10}{3} - \frac{10}{3}e^{-3t}$$

## 2.3 Step Response of First Order System

### Step Response

If the input of a system is dc, then the response in the circuit is known as *step response.*

$$f(t) = 0 \text{ if } t < 0$$
$$= k \text{ if } t > 0$$

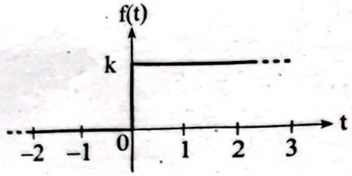



*Figure 2.1: Graphical representation of a step response*

**1) Step response of first order RL circuit without any initial value of current through inductor**

Consider the RL circuit shown in the following figure. The resistance can be the wire resistance of an actual inductor, added resistance, or combination of both. The switch has been open a very long time. The switch is closed at $t = 0$.

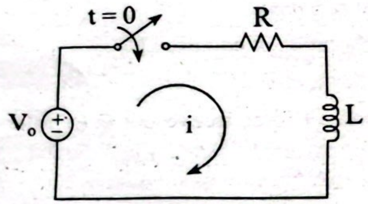



*Figure 2.2: Series RL circuit with dc source voltage*

Applying KVL for $t > 0$,

$$v_R + v_L = V_o$$

$$\text{or, } iR + L\frac{di}{dt} = V_o$$

$$\text{or, } \frac{di}{dt} + \frac{R}{L}i = \frac{V_o}{L} \qquad \text{..........(i)}$$

Comparing equation (i) with general first order differential equation, we have

$$i \rightarrow i,\ P \rightarrow \frac{R}{L},\ Q = \frac{V_o}{L}$$

The solution of equation (i) is given by

$$i = e^{-Pt}\int Q\, e^{Pt}\, dt + ke^{-Pt}$$

$$\text{or, } i = e^{\frac{-Rt}{L}}\int \frac{V_o}{L} e^{\frac{Rt}{L}}\, dt + ke^{\frac{-Rt}{L}}$$

$$\text{or, } i = e^{\frac{-Rt}{L}} \times e^{\frac{Rt}{L}} \times \frac{V_o}{\cancel{L}} \times \frac{\cancel{L}}{R} + ke^{\frac{-Rt}{L}}$$



$$\text{or, } i = \frac{V_o}{R} + ke^{\frac{-Rt}{L}} = I_o + ke^{\frac{-t}{\tau}} \qquad \text{..........(ii)}$$

$$\text{where } I_o = \frac{V_o}{R},\ \tau = \frac{L}{R} = \text{time constant}$$

Put $t = 0^+$ in equation (ii),

$$i(0^+) = I_o + ke^{\frac{0^+}{\tau}}$$

$$\text{or, } k = i(0^+) - I_o \qquad \text{..........(iii)}$$

By observation at $t = 0$, $i(0^-) = 0$

From continuity relation for inductor,

$$i(0^+) = i(0^-) = 0 \qquad \text{..........(iv)}$$

From equations (iii) and (iv), we get

$$k = -I_o \qquad \text{..........(v)}$$

Now, equation (ii) becomes

$$i = I_o - I_o e^{-t/\tau}$$

$$i = I_o(1 - e^{-t/\tau}) = \frac{V_o}{R}(1 - e^{-Rt/L})$$

The voltage across inductor is given by

$$v_L = L\frac{di}{dt}$$

$$\text{or, } v_L = L\frac{d}{dt}\left[\frac{V_o}{R}(1 - e^{-Rt/L})\right]$$

$$\text{or, } v_L = L\frac{V_o}{R}\frac{d}{dt}(1 - e^{-Rt/L})$$

$$\text{or, } v_L = L\frac{V_o}{R}\left(0 + \frac{R}{L}e^{-Rt/L}\right)$$

$$\therefore v_L = V_o e^{-Rt/L} = V_o e^{-t/\tau}$$

The voltage across resistor is

$$v_R = iR = \left[\frac{V_o}{R}(1 - e^{-Rt/L})\right]R = V_o(1 - e^{-Rt/L}) = V_o(1 - e^{-t/\tau})$$

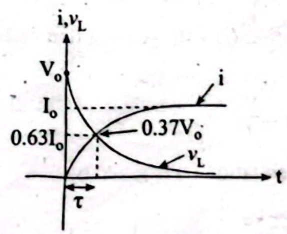



*Figure 2.3: Variation of current and voltage of inductor with time*

**Time constant ($\tau$)**

We have,

$i = I_o(1 - e^{-t/\tau})$

Put $t = \tau$ in above equation,

$i(\tau) = I_o(1 - e^{-1}) = 0.63I_o$

Hence, *time constant* can be defined as the time taken by the resp[onse to] reach 63% of its maximum value.

Also, $v_L = V_o e^{-t/\tau}$

Put $t = \tau$ in above equation,

$v_L(\tau) = V_o e^{-1} = 0.37V_o$

So, time constant may also be defined as the time taken by the res[ponse] to decrease upto 37%| of initial value.

2) **Step response of first order RL circuit with some initial val[ue of] current through inductor**

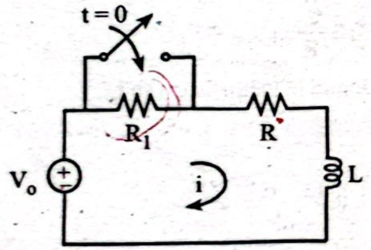



*Figure 2.4: RL circuit with arbitrary initial condition.*

Applying KVL for $t > 0$,

$v_R + v_L = V_o$

or, $iR + L\dfrac{di}{dt} = V_o$

or, $\dfrac{di}{dt} + \dfrac{R}{L}i = \dfrac{V_o}{L}$ ........... (i)

Comparing equation (i) with general first order differential equatio[n, we] have

$i \rightarrow i,\ P \rightarrow \dfrac{R}{L},\ Q = \dfrac{V_o}{L}$

The solution of equation (i) is given by

$i = e^{-Pt}\int Q\, e^{Pt}\, dt + ke^{-Pt}$

or, $i = e^{\frac{-Rt}{L}}\int \dfrac{V_o}{L} e^{\frac{Rt}{L}}\, dt + ke^{\frac{-Rt}{L}}$

or, $i = e^{\frac{-Rt}{L}} \times e^{\frac{Rt}{L}} \times \dfrac{V_o}{L} \times \dfrac{L}{R} + ke^{\frac{-Rt}{L}}$

or, $i = \dfrac{V_o}{R} + ke^{\frac{-Rt}{L}} = I_o + ke^{\frac{-t}{\tau}}$ .............. (ii)

where $I_o = \dfrac{V_o}{R}$, $\tau = \dfrac{L}{R}$ = time constant

Put $t = 0^+$ in equation (ii),

$i(0^+) = I_o + ke^{\frac{0^+}{\tau}}$

or, $k = i(0^+) - I_o$ ................ (iii)

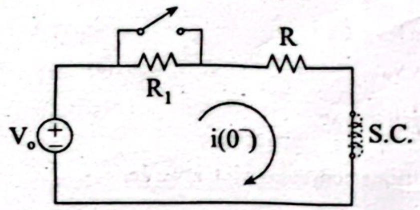



*Figure 2.5: Equivalent circuit at $t = 0^-$.*

From equivalent circuit at $t = 0^-$ as shown in figure 2.5,

$i(0^-) = \dfrac{V_o}{R_1 + R} = I_{initial}$

From continuity relation for inductor,

$i(0^+) = i(0^-) = I_{initial}$ .............(iv)

From equations (iii) and (iv), we get

$k = I_{initial} - I_o$

or, $k = -(I_o - I_{initial})$ ............. (v)

Now, equation (ii) becomes

$i = I_o - (I_o - I_{initial})\, e^{-t/\tau}$

The voltage across inductor is given by

$v_L = L\dfrac{di}{dt}$

or, $v_L = L\dfrac{d}{dt}[\,I_o - (I_o - I_{initial})\, e^{-t/\tau}]$

$\therefore v_L = (I_o - I_{initial}) R e^{-t/\tau}$

The voltage across resistor is

$v_R = iR = R[I_o - (I_o - I_{initial})] e^{-t/\tau}$

**3) Step response of first order RC circuit without any initial va voltage across capacitor**

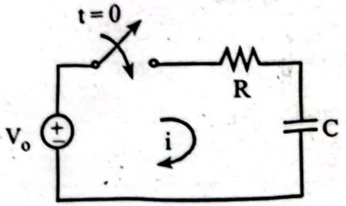



*Figure 2.6: RC circuit excited by dc voltage*

Applying KVL for t > 0,

$v_R + v_C = V_o$ ..............(a)

or, $iR + \frac{1}{C}\int i \, dt = V_o$

Differentiating both sides w.r.t. t, we get

$R\frac{di}{dt} + \frac{1}{C} i = 0$

or, $\frac{di}{dt} + \frac{1}{CR} i = 0$ ........... (i)

Comparing equation (i) with general first order differential equatio have

$i \rightarrow i, P \rightarrow \frac{1}{CR}, Q \rightarrow 0$

The solution of equation (i) is given by

$i = e^{-Pt} \int Q e^{Pt} dt + ke^{-Pt}$

or, $i = 0 + ke^{\frac{-t}{CR}}$

or, $i = ke^{\frac{-t}{\tau}}$ ..........(ii)

where $\tau = CR$ = time constant.

Put $t = 0^+$ in equation (ii),



$i(0^+) = ke^{\frac{0^+}{\tau}}$

or, $k = i(0^+)$ ........... (iii)

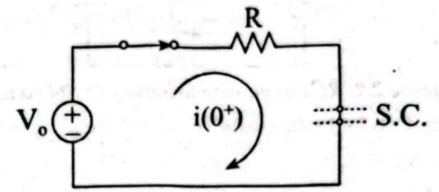



*Figure 2.7: Equivalent circuit at $t = 0^+$.*

From equivalent circuit at $t = 0^+$ (figure 2.7),

$i(0^+) = \frac{V_o}{R} = I_o$ (say) .............(iv)

From equations (iii) and (iv), we get

$k = I_o$

Now, equation (ii) becomes

$i = I_o e^{-t/\tau} = \frac{V_o}{R} e^{-t/\tau}$

The voltage across resistor is

$v_R = iR = R\frac{V_o}{R} e^{-t/\tau} = V_o e^{-t/\tau}$

The voltage across capacitor is obtained from equation (a),

$v_C = V_o - V_R$

or, $v_C = V_o - iR$

or, $v_C = V_o - R\frac{V_o}{R} e^{-t/\tau}$

$\therefore v_C = V_o (1 - e^{-t/\tau})$

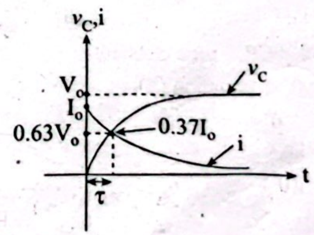



*Figure 2.8: Variation of current and voltage of capacitor with time*

**4) Step response of first order RC circuit with some initial value of voltage across capacitor**

The figure below is a general RC series circuit being supplied by a dc source. The capacitor is assumed to have an arbitrary initial condition, that is, initial charge, and therefore voltage, on it.

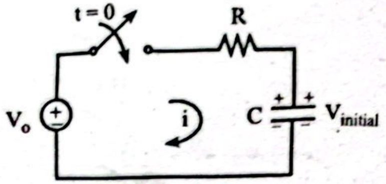



*Figure 2.9: RC circuit with arbitrary initial condition.*

Applying KVL for $t > 0$,

$$v_R + v_C = V_o \quad \text{.............(a)}$$

or, $iR + \frac{1}{C}\int i\, dt = V_o$

Differentiating both sides w.r.t. t, we get

$$R\frac{di}{dt} + \frac{1}{C} i = 0$$

$$\text{or, } \frac{di}{dt} + \frac{1}{CR} i = 0 \quad \text{............... (i)}$$

Comparing equation (i) with general first order differential equation, we have

$$i \to i,\ P \to \frac{1}{CR},\ Q \to 0$$

The solution of equation (i) is given by

$$i = e^{-Pt}\int Q\, e^{Pt}\, dt + ke^{-Pt}$$

$$\text{or, } i = 0 + ke^{\frac{-t}{CR}}$$

$$\text{or, } i = ke^{\frac{-t}{\tau}} \quad \text{...........(ii)}$$

where $\tau = CR$ = time constant.

Put $t = 0^+$ in equation (ii),

$$i(0^+) = ke^{\frac{0^+}{\tau}}$$

$$\text{or, } k = i(0^+) \quad \text{................ (iii)}$$

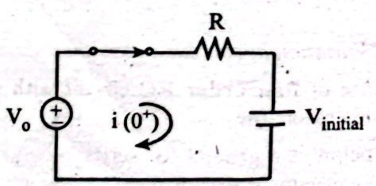



*Figure 2.10: Equivalent circuit at $t = 0^+$.*



From equivalent circuit at $t = 0^+$ (figure 2.10),

$$i(0^+) = \frac{V_o - V_{initial}}{R} \quad \text{.............(iv)}$$

From equations (iii) and (iv), we get

$$k = \frac{V_o - V_{initial}}{R}$$

Now, equation (ii) becomes

$$i = \left(\frac{V_o - V_{initial}}{R}\right) e^{-t/\tau}$$

The voltage across resistor is

$$v_R = iR = R\left(\frac{V_o - V_{initial}}{R}\right) e^{-t/\tau} = (V_o - V_{initial})\, e^{-t/\tau} \quad \text{........(v)}$$

The voltage across capacitor is obtained from equations (a) and (v),

$$v_C = V_o - V_R$$

$$\therefore v_C = V_o - (V_o - V_{initial})\, e^{-t/\tau}$$

## 2.4 Solution of Second Order Homogenous Differential Equation

A second order homogeneous differential equation is of the form

$$\frac{d^2i}{dt^2} + a_1\frac{di}{dt} + a_2 i = 0 \quad \text{........ (1)}$$

The auxiliary (or characteristic) equation of (1) is given by

$$s^2 + a_1 s + a_2 = 0$$

The two possible roots of above equation are given by

$$s_1, s_2 = \frac{a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$\text{or, } s_1, s_2 = \frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_2}}{2} \quad \text{............(2)}$$

**Case I:** If $a_1^2 < 4a_2$, then the roots will be complex, and is given by

$$s_1, s_2 = \frac{-a_1}{2} \pm \frac{\sqrt{-(4a_2 - a_1^2)}}{2}$$

$$= -\alpha \pm j\beta$$

where $\alpha = \frac{a_1}{2}$, $\beta = \frac{\sqrt{(4a_2 - a_1^2)}}{2}$.

Hence, the solution of equation (1) is given by

$i = e^{-\alpha t}(k_1 \cos\beta t + k_2 \sin\beta t)$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

**Case II:** If $a_1^2 = 4a_2$, then the roots will be real and equal, and is given by

$$s_1, s_2 = \frac{-a_1}{2} = -\alpha$$

Hence, the solution of equation (1) is given by

$i = (k_1 + k_2 t)\, e^{-\alpha t}$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

**Case III:** If $a_1^2 > 4a_2$, then the roots will be real and unequal, and is given

$$s_1 = \frac{a_1}{2} - \frac{\sqrt{a_1^2 - 4a_2}}{2},\ s_2 = \frac{a_1}{2} + \frac{\sqrt{a_1^2 - 4a_2}}{2}$$

Hence, the solution of equation (1) is given by

$i = k_1 e^{s_1 t} + k_2 e^{s_2 t}$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

## 2.5 Step Response for Second Order System

### 1. Steps response for RLC series circuit

A series RLC circuit with a dc source is shown in Figure 2.11.

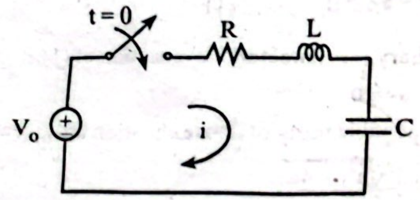



*Figure 2.11: A series RLC circuit with a dc source*

Applying KVL for $t > 0$,

$v_R + v_L + v_C = V_o$

$$\text{or, } iR + L\frac{di}{dt} + \frac{1}{C}\int i\, dt = V_o \qquad \text{......... (1)}$$

Differentiating equation (1) w.r.t. t, we get

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

$$\text{or, } \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0 \qquad \text{.........(2)}$$



This is a second order homogenous differential equation.

The auxiliary equation of (2) is given by

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

The two possible roots of above equation are given by

$$s_1, s_2 = \frac{\frac{-R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4\times\frac{1}{LC}}}{2}$$

$$= \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$$

$$\text{or, } s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} \qquad \text{............(3)}$$

where $\alpha = \frac{R}{2L}$ = damping coefficient,

$\omega_n = \frac{1}{\sqrt{LC}}$ = natural frequency.

**Case I:** If $\alpha < \omega_n$, then the system is said to be *underdamped* and roots will be complex which are given by

$$s_1, s_2 = -\alpha \pm \sqrt{-(\omega_n^2 - \alpha^2)} = -\alpha \pm j\omega_d$$

where $\omega_d = \sqrt{\omega_n^2 - \alpha^2}$ = damped natural frequency

Hence, the solution of equation (2) is given by

$$i = e^{-\alpha t}(k_1 \cos\omega_d t + k_2 \sin\omega_d t) \qquad \text{........ (4)}$$

Put $t = 0^+$,

$i(0^+) = k_1$ ........... (5)

By observation at $t = 0^-$,

$i(0^-) = 0,\ v_C(0^-) = 0$

From continuity relation for inductor and capacitor,

$i(0^+) = i(0^-) = 0$ ........ (6)

$v_C(0^+) = v_C(0^-) = 0$ ............(7)

From equations (5) and (6),

$k_1 = 0$

Now, equation (4) is reduced to

$i = k_2 e^{-\alpha t} \sin\omega_d t$ ........... (8)

Differentiating equation (8) w.r.t. t, we have

$$\frac{di}{dt} = k_2 (\omega_d e^{-\alpha t} \cos\omega_d t - e^{-\alpha t} \sin\omega_d t)$$

Put $t = 0^+$,

$$\frac{di(0^+)}{dt} = k_2 \omega_d$$

$$\text{or, } k_2 = \frac{1}{\omega_d} \frac{di(0^+)}{dt} \quad \text{.................(9)}$$

Put $t = 0^+$ in equation (1),

$$i(0^+) R + L\frac{di(0^+)}{dt} + v_C(0^+) = V_o \quad \text{.......... (10)}$$

From equations (6), (7) and (10), we get

$$R\times 0 + L\frac{di(0^+)}{dt} + 0 = V_o$$

$$\frac{di(0^+)}{dt} = \frac{V_o}{L} \quad \text{.......... (11)}$$

From equations (9) and (11),

$$k_2 = \frac{1}{\omega_d} \times \frac{V_o}{L} \quad \text{......... (12)}$$

Now, equation (8) becomes

$$i = \frac{V_o}{\omega_d L} e^{-\alpha t} \sin\omega_d t \quad \text{......... (13)}$$

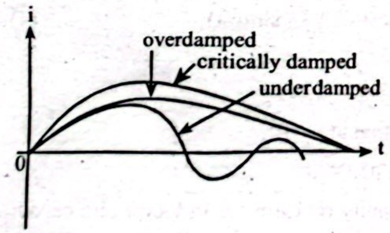



*Figure 2.12: Response*

**Case II:** If $\alpha = \omega_n$, then the system is said to be *critically damped*, and roots will be real and equal which are given by

$$s_1, s_2 = -\alpha$$

Hence, the solution of equation (2) is given by

$$i = (k_1 + k_2 t) e^{-\alpha t} \quad \text{......... (14)}$$

Put $t = 0^+$,

$$i(0^+) = k_1$$



From equations (6) and (7),

$$i(0^+) = 0$$

$$\therefore k_1 = 0$$

Now, equation (14) becomes

$$i = k_2 t e^{-\alpha t} \quad \text{.........(15)}$$

Differentiating equation (15) w.r.t. t, we get

$$\frac{di}{dt} = k_2 [e^{-\alpha t} - \alpha t e^{-\alpha t}] \quad \text{......... (16)}$$

Put $t = 0^+$,

$$\frac{di(0^+)}{dt} = k_2 \quad \text{........ (17)}$$

From equation (11),

$$\frac{di(0^+)}{dt} = \frac{V_o}{L}$$

$$\therefore k_2 = \frac{V_o}{L} \quad \text{...........(18)}$$

Now, from equations (18) and (15), we have

$$i = \frac{V_o}{L} t e^{-\alpha t} \quad \text{............(19)}$$

When the system is critically damped,

$$\alpha = \omega_n$$

$$\text{or, } \frac{R_{cr}}{2L} = \frac{1}{\sqrt{LC}}$$

$$\text{or, } R_{cr} = 2\sqrt{\frac{L}{C}}$$

Thus, critical resistance ($R_{cr}$) depends upon the value of inductance and capacitance.

**Case III:** If $\alpha > \omega_n$, then the system is said to be *overdamped*, and the roots will be real and unequal which are given by

$$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_n^2}, \quad s_2 = -\alpha + \sqrt{\alpha^2 - \omega_n^2}$$

Hence, the solution of equation (2) is given by

$$i = k_1 e^{s_1 t} + k_2 e^{s_2 t} \quad \text{.......... (20)}$$

Put $t = 0^+$,

$$i(0^+) = k_1 + k_2 \quad \text{......... (21)}$$

From equations (21) and (6),

$k_1 + k_2 = 0$

or, $k_2 = -k_1$ ........ (22)

Differentiating equation (20) w.r.t. t, we get

$$\frac{di}{dt} = k_1 s_1 e^{s_1 t} + k_2 s_2 e^{s_2 t} \quad \text{.........} (23)$$

Put $t = 0^+$ in (23),

$$\frac{di(0^+)}{dt} = k_1 s_1 + k_2 s_2 \quad \text{..............} (24)$$

Using equation (11),

$$\frac{V_o}{L} = k_1 s_1 + k_2 s_2 \quad \text{............}(25)$$

From equations (22) and (25),

$$k_1 s_1 - k_1 s_2 = \frac{V_o}{L}$$

$$\text{or, } k_1 (s_1 - s_2) = \frac{V_o}{L}$$

$$\text{or, } k_1 = \frac{V_o}{L\,(s_1 - s_2)}$$

$$\therefore k_2 = \frac{-V_o}{L\,(s_1 - s_2)}$$

Putting values of $k_1$ and $k_2$ in equation (20), we get

$$i = \frac{V_o}{L\,(s_1 - s_2)} e^{s_1 t} + \frac{-V_o}{L\,(s_1 - s_2)} e^{s_2 t}$$

$$= \frac{V_o}{L\,(s_1 - s_2)} \left(e^{s_1 t} - e^{s_2 t}\right)$$

**2) Step response of parallel RLC circuit**

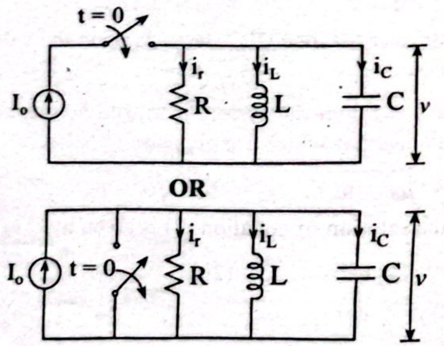



*Figure 2.13: A parallel RLC circuit*



Applying KCL for $t > 0$,

$i_C + i_L + i_r = I_o$

$$\text{or, } C\frac{dv}{dt} + \frac{1}{L}\int v\,dt + \frac{v}{R} = I_o$$

$$\text{or, } C\frac{dv}{dt} + \frac{1}{L}\int v\,dt + Gv = I_o \quad \text{..........} (1)$$

Differentiating above equation w.r.t. t, we get

$$C\frac{d^2v}{dt^2} + \frac{1}{L}v + G\frac{dv}{dt} = 0$$

$$\text{or, } \frac{d^2v}{dt^2} + \frac{v}{LC} + \frac{G}{C}\frac{dv}{dt} = 0 \quad \text{.........}(2)$$

where $G = \frac{1}{R}$ = conductance.

The auxiliary equation of (2) is

$$s^2 + \frac{G}{C}s + \frac{1}{LC} = 0$$

$$s_1, s_2 = \frac{-\frac{G}{C} \pm \sqrt{\left(\frac{G}{C}\right)^2 - 4\times\frac{1}{LC}}}{2} = \frac{-G}{2C} \pm \sqrt{\left(\frac{G}{2C}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$$

$$\therefore s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} \quad \text{.........} (3)$$

where $\alpha = \frac{G}{2C}$ = damping coefficient, $\omega_n = \frac{1}{\sqrt{LC}}$ = natural frequency.

**Case I:** If $\alpha < \omega_n$, then the system is said to be *underdamped* and roots will be complex which are given by

$$s_1, s_2 = -\alpha \pm \sqrt{-(\omega_n^2 - \alpha^2)} = -\alpha \pm j\omega_d$$

where $\omega_d = \sqrt{\omega_n^2 - \alpha^2}$ = damped natural frequency

Hence, the solution of equation (2) is given by

$$v = e^{-\alpha t}(k_1 \cos\omega_d t + k_2 \sin\omega_d t) \quad \text{........} (4)$$

Put $t = 0^+$,

$v(0^+) = k_1$ ........... (5)

By observation at $t = 0^-$,

$i_L(0^-) = 0$, $v(0^-) = 0$

From continuity relation for inductor and capacitor,

$i_L(0^+) = i_L(0^-) = 0$ ........ (6)

$$v(0^+) = v(0^-) = 0 \quad \text{............(7)}$$

From equations (5) and (7),

$$k_1 = 0$$

Now, equation (4) is reduced to

$$v = k_2 e^{-\alpha t} \sin\omega_d t \quad \text{........... (8)}$$

Differentiating equation (8) w.r.t. t, we have

$$\frac{dv}{dt} = k_2 (\omega_d e^{-\alpha t} \cos\omega_d t - e^{-\alpha t} \sin\omega_d t)$$

Put $t = 0^+$,

$$\frac{dv(0^+)}{dt} = k_2 \omega_d$$

$$\text{or, } k_2 = \frac{1}{\omega_d} \frac{dv(0^+)}{dt} \quad \text{................(9)}$$

Put $t = 0^+$ in equation (1),

$$C\frac{dv(0^+)}{dt} + i_L(0^+) + Gv(0^+) = I_o$$

$$\text{or, } C\frac{dv(0^+)}{dt} + 0 + 0 = I_o$$

$$\text{or, } \frac{dv(0^+)}{dt} = \frac{I_o}{C} \quad \text{............(10)}$$

From equations (9) and (10), we get

$$k_2 = \frac{I_o}{\omega_d C}$$

Hence, equation (8) becomes

$$v = \frac{I_o}{\omega_d C} e^{-\alpha t} \sin\omega_d t \quad \text{........... (11)}$$

**Case II:** If $\alpha = \omega_n$, then the system is said to be *critically damped*, and roots will be real and equal which are given by

$$s_1, s_2 = -\alpha$$

Hence, the solution of equation (2) is given by

$$v = (k_1 + k_2 t) e^{-\alpha t} \quad \text{.......... (14)}$$

Put $t = 0^+$,

$$v(0^+) = k_1 \quad \text{.............(15)}$$

From equations (15) and (7),

$$k_1 = 0$$



Now, equation (14) becomes

$$v = k_2 t e^{-\alpha t} \quad \text{.........(15)}$$

Differentiating equation (15) w.r.t. t, we get

$$\frac{dv}{dt} = k_2 (e^{-\alpha t} - \alpha t e^{-\alpha t}) \quad \text{......... (16)}$$

Put $t = 0^+$,

$$\frac{dv(0^+)}{dt} = k_2 \quad \text{........ (17)}$$

From equations (10) and (17),

$$k_2 = \frac{I_o}{C} \quad \text{...........(18)}$$

Hence, equation (15) becomes

$$v = \frac{I_o}{C} t e^{-\alpha t} \quad \text{...........(19)}$$

When the system is critically damped,

$$\alpha = \omega_n$$

$$\text{or, } \frac{G_{cr}}{2C} = \frac{1}{\sqrt{LC}}$$

$$\text{or, } G_{cr} = 2\sqrt{\frac{C}{L}}$$

Thus, critical conductance ($G_{cr}$) depends upon the value of inductance and capacitance.

**Case III:** If $\alpha > \omega_n$, then the system is said to be *overdamped*, and the roots will be real and unequal which are given by

$$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_n^2}, \quad s_2 = -\alpha + \sqrt{\alpha^2 - \omega_n^2}$$

Hence, the solution of equation (2) is given by

$$v = k_1 e^{s_1 t} + k_2 e^{s_2 t} \quad \text{......... (20)}$$

Put $t = 0^+$,

$$v(0^+) = k_1 + k_2 \quad \text{......... (21)}$$

From equations (21) and (7),

$$k_1 + k_2 = 0$$

$$\text{or, } k_2 = -k_1 \quad \text{........ (22)}$$

Differentiating equation (20) w.r.t. t, we get

$$\frac{dv}{dt} = k_1 s_1 e^{s_1 t} + k_2 s_2 e^{s_2 t} \quad \text{......... (23)}$$

Put $t = 0^+$ in (23),

$$\frac{dv(0^+)}{dt} = k_1 s_1 + k_2 s_2 \quad \text{.............} (24)$$

Using equation (10),

$$\frac{I_o}{C} = k_1 s_1 + k_2 s_2 \quad \text{.................}(25)$$

From equation (22) and (25),

$$k_1 s_1 - k_1 s_2 = \frac{I_o}{C}$$

$$\text{or, } k_1 (s_1 - s_2) = \frac{I_o}{C}$$

$$\text{or, } k_1 = \frac{I_o}{C(s_1 - s_2)}$$

$$\therefore k_2 = -k_1 = \frac{-I_o}{C(s_1 - s_2)}$$

Putting values of $k_1$ and $k_2$ in equation (20), we get

$$v = \frac{I_o}{C(s_1 - s_2)} e^{s_1 t} + \frac{-I_o}{C(s_1 - s_2)} e^{s_2 t}$$

$$\therefore v = \frac{I_o}{C(s_1 - s_2)} (e^{s_1 t} - e^{s_2 t})$$

## 2.6 Solution for Nonhomogeneous Second Or Differential Equation

A second order nonhomogeneous differential equation is of the form

$$\frac{d^2 i}{dt^2} + a_1 \frac{di}{dt} + a_2 i = Q \quad \text{............} (1)$$

The solution of above differential equation is given as

$$i = \text{C.F.} + \text{P.I.}$$

$$\text{or, } i = i_N + i_f \quad \text{...........}(2)$$

In the above equation, i represents the total solution, $i_f$ is called the f *component* of the solution, and $i_n$ is called the *natural component* o solution.

The *total solution, i* is the actual waveform that could be observed by m of an oscilloscope. The *forced response, $i_f$* is the response due to the pres of the driving force. After the lapse of the transient adjustment period total response settles down to the forced response. For this reason, the f response is often called the *steady state response*. The *natural response* the response that would take place if the source were reduced to zero and the system responded to its initial stored energy. It is often called the *source-free* or *transient response*. Its purpose in the total solution when a source is present is to act as a buffer between the initial value of the circuit variable as determined by the initial conditions and the value supplied at the initial instant by the source. After making up the initial discrepancy, the transient component of the response must decay to zero to ultimately allow the source to establish the steady-state conditions.

To find the particular integral, we have to apply method of undetermined coefficient. According to this method, we have to select a trial solution for particular integral on the basis of source as given in the following table.

| Source (Q) | Trial solution for P.I. |
|---|---|
| 1. Q = dc source ($V_o$ or $I_o$) | P.I. = constant = A |
| 2. Q = exponential source ($V_o e^{-\alpha t}$ or $I_o e^{-\alpha t}$) | P.I. = $A e^{-\alpha t}$ if denominator ≠ 0.<br>= $At e^{-\alpha t}$ if denominator = 0. |
| 3. Q = sinusoidal source ($V_m \sin\omega t$ or $V_m \cos\omega t$) | P.I. = $A \cos\omega t + B \sin\omega t$<br>= $k \sin(\omega t + \phi)$<br>= $k \cos(\omega t - \psi)$<br>where $k = \sqrt{A^2 + B^2}$<br>$\phi = \tan^{-1}\left(\frac{A}{B}\right)$, $\psi = \tan^{-1}\left(\frac{B}{A}\right)$ |

**Procedure to find undetermined coefficient of particular integral:**

- Select a trial solution for particular integral on the basis of source as given in the above table.
- Replace the dependent variable of differential equation by the value of trial solution.
- Finally, perform algebraic manipulation to get the value of undetermined coefficient.

## 2.7 Response of Circuit Excited by Exponential Source

**1. Response of RL Circuit Excited by Exponential Source**

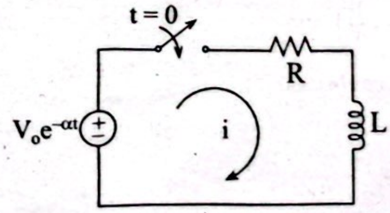



*Figure 2.14: An RL circuit excited by exponential source*

Applying KVL for $t > 0$,

$$v_R + v_L = V_o e^{-\alpha t}$$

$$\text{or, } iR + L\frac{di}{dt} = V_o e^{-\alpha t}$$

$$\text{or, } \frac{di}{dt} + \frac{R}{L}i = \frac{V_o}{L}e^{-\alpha t} \qquad \text{........... (1)}$$

The solution of above differential equation is given by

$i = \text{C.F.} + \text{P.I.}$

$$i = i_N + i_f \qquad \text{..................... (2)}$$

To find $i_N$, the auxiliary equation (or characteristic equation) is written as

$$s + \frac{R}{L} = 0$$

$$\text{or, } s = -\frac{R}{L}$$

$$i_N = ke^{-\frac{R}{L}t} \qquad \text{..........(3)}$$

To find $i_f$,

**Case I:** If $\alpha \neq \frac{R}{L}$, then let the trial solution for P.I. is

$$i_f = Ae^{-\alpha t} \qquad \text{............. (4)}$$

From equations (1) and (4),

$$\frac{d(Ae^{-\alpha t})}{dt} + \frac{R}{L}(Ae^{-\alpha t}) = \frac{V_o}{L}e^{-\alpha t}$$

$$\text{or, } A(-\alpha)\,e^{-\alpha t} + \frac{R}{L}(Ae^{-\alpha t}) = \frac{V_o}{L}e^{-\alpha t}$$

$$\text{or, } \left(-\alpha + \frac{R}{L}\right)Ae^{-\alpha t} = \frac{V_o}{L}e^{-\alpha t}$$

$$\text{or, } A = \frac{\frac{V_o}{L}}{\left(-\alpha + \frac{R}{L}\right)} \qquad \text{........... (5)}$$

From equations (4) and (5),

$$i_f = \frac{\frac{V_o}{L}}{-\alpha + \frac{R}{L}}e^{-\alpha t} \qquad \text{......... (6)}$$

From (2), (3), and (6), we have

$$i = ke^{-\frac{R}{L}t} + \frac{\frac{V_o}{L}}{-\alpha + \frac{R}{L}}e^{-\alpha t} \qquad \text{........... (7)}$$

Put $t = 0^+$ in (7),

$$i(0^+) = k + \frac{\frac{V_o}{L}}{-\alpha + \frac{R}{L}}$$

$$\text{or, } k = i(0^+) - \frac{\frac{V_o}{L}}{-\alpha + \frac{R}{L}} \qquad \text{............... (8)}$$

By observation at $t = 0^-$,

$i(0^-) = 0$

From continuity relation for inductor,

$$i(0^+) = i(0^-) = 0 \qquad \text{......... (9)}$$

Now, equation (8) reduces to

$$k = i(0^+) - \frac{\frac{V_o}{L}}{-\alpha + \frac{R}{L}}$$

$$\text{or, } k = -\frac{\frac{V_o}{L}}{-\alpha + \frac{R}{L}}$$

From (7) and (10),

$$i = -\frac{\frac{V_o}{L}}{-\alpha + \frac{R}{L}}e^{-\frac{R}{L}t} + \frac{\frac{V_o}{L}}{-\alpha + \frac{R}{L}}e^{-\alpha t}$$

**Case II:** If $\alpha = \frac{R}{L}$, then let the trial solution for P.I. is

$$i_f = At\,e^{-\alpha t} \qquad \text{............ (10)}$$

From (1) and (10),

$$\frac{d(Ate^{-\alpha t})}{dt} + \frac{R}{L}(Ate^{-\alpha t}) = \frac{V_o}{L}e^{-\alpha t}$$

$$\text{or, } A[e^{-\alpha t} - \alpha t\, e^{-\alpha t}] + \frac{R}{L}(Ate^{-\alpha t}) = \frac{V_o}{L} e^{-\alpha t}$$

$$\text{or, } A\left(1 - \alpha t + \frac{R}{L}t\right) = \frac{V_o}{L}$$

Since $\alpha = \frac{R}{L}$, we have

$$A = \frac{V_o}{L} \quad \text{.......... (11)}$$

From (10) and (11),

$$i_f = \frac{V_o}{L} t\, e^{-\alpha t} \quad \text{.......... (12)}$$

From equations (2), (3), and (12),

$$i = ke^{-\frac{R}{L}t} + \frac{V_o}{L} te^{-\alpha t} \quad \text{........... (13)}$$

Put $t = 0^+$ in above equation,

$$i(0^+) = k \quad \text{.......... (14)}$$

From (14) and (9),

$$k = 0 \quad \text{....... (15)}$$

From (15) and (13),

$$i = \frac{V_o}{L} t\, e^{-\alpha t}$$

**2) Response of RC Circuit Excited by Exponential Source**

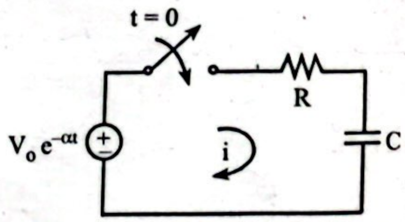



*Figure 2.15: An RC circuit excited by exponential source*

Applying KVL for $t > 0$,

$$v_R + v_C = V_o e^{-\alpha t}$$

$$\text{or, } iR + \frac{1}{C}\int i\, dt = V_o\, e^{-\alpha t}$$

Differentiating above equation w.r.t. t, we get

$$R\frac{di}{dt} + \frac{i}{C} = -\alpha V_o\, e^{-\alpha t}$$

$$\text{or, } \frac{di}{dt} + \frac{i}{CR} = \frac{-\alpha V_o}{R} e^{-\alpha t} \quad \text{........... (1)}$$

The solution of above differential equation is given by

$i = \text{C.F.} + \text{P.I.}$

$$\text{or, } i = i_N + i_f \quad \text{........... (2)}$$

To find $i_N$, the A.E. of (1) is

$$s + \frac{1}{CR} = 0$$

$$\text{or, } s = \frac{-1}{CR}$$

$$\therefore i_N = ke^{-(1/CR)\,t} \quad \text{........... (3)}$$

To find $i_f$,

**Case I:** If $\alpha \neq \frac{1}{CR}$, then let the trial solution for P.I. is

$$i_f = Ae^{-\alpha t} \quad \text{.......... (4)}$$

From equations (1) and (4),

$$\frac{d(Ae^{-\alpha t})}{dt} + \frac{1}{CR}(Ae^{-\alpha t}) = \frac{-\alpha V_o}{R} e^{-\alpha t}$$

$$\text{or, } -\alpha Ae^{-\alpha t} + \frac{1}{CR} Ae^{-\alpha t} = -\alpha\frac{V_o}{R} e^{-\alpha t}$$

$$\text{or, } Ae^{-\alpha t}\left(-\alpha + \frac{1}{CR}\right) = -\alpha\frac{V_o}{R} e^{-\alpha t}$$

$$\text{or, } A = \frac{-\frac{\alpha V_o}{R}}{-\alpha + \frac{1}{CR}} \quad \text{.......... (5)}$$

From (4) and (5),

$$i_f = \frac{-\frac{\alpha V_o}{R}}{-\alpha + \frac{1}{CR}} e^{-\alpha t} \quad \text{......... (6)}$$

From (2), (3), and (6),

$$i = ke^{-(1/CR)\,t} + \frac{-\frac{\alpha V_o}{R}}{-\alpha + \frac{1}{CR}} e^{-\alpha t} \quad \text{.................(7)}$$

Put $t = 0^+$ in equation (7),

$$i(0^+) = k + \frac{-\frac{\alpha V_o}{R}}{-\alpha + \frac{1}{CR}}$$

$$\text{or, } k = i(0^+) - \left(\frac{-\frac{\alpha V_o}{R}}{-\alpha + \frac{1}{CR}}\right) \quad \text{..................(8)}$$

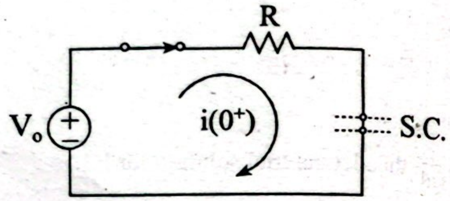



*Figure 2.16: Equivalent circuit at $t = 0^+$.*

From equivalent circuit at $t = 0^+$ (figure 2.16),

$$i(0^+) = \frac{V_o}{R} \quad \text{..................(9)}$$

From equations (8) and (9),

$$k = \frac{V_o}{R} - \left(\frac{-\frac{\alpha V_o}{R}}{-\alpha + \frac{1}{CR}}\right) \quad \text{..............(10)}$$

From (7) and (10),

$$i = \left[\frac{V_o}{R} - \left(\frac{-\frac{\alpha V_o}{R}}{-\alpha + \frac{1}{CR}}\right)\right] e^{-\frac{1}{CR}t} + \frac{-\frac{\alpha V_o}{R}}{-\alpha + \frac{1}{CR}} e^{-\alpha t} \quad \text{.........(11)}$$

**Case II:** If $\alpha = \frac{1}{CR}$, then let the trial solution for P.I. is

$$i_f = Ate^{-\alpha t} \quad \text{.......... (12)}$$

From equations (1) and (12),

$$\frac{d(Ate^{-\alpha t})}{dt} + \frac{1}{CR}(Ate^{-\alpha t}) = \frac{-\alpha V_o}{R} e^{-\alpha t}$$

$$\text{or, } A[e^{-\alpha t} - \alpha t\, e^{-\alpha t}] + \frac{1}{CR}(Ate^{-\alpha t}) = \frac{-\alpha V_o}{R} e^{-\alpha t}$$



$$\text{or, } A\left(1 - \alpha t + \frac{1}{CR}t\right) = \frac{-\alpha V_o}{R}$$

Since $\alpha = \frac{1}{CR}$, we have

$$A = \frac{-\alpha V_o}{R} \quad \text{..................(13)}$$

From equations (12) and (13),

$$i_f = \left(\frac{-\alpha V_o}{R}\right) te^{-\alpha t} \quad \text{...............(14)}$$

From equations (2), (3), and (14),

$$i = ke^{-\frac{1}{CR}t} + \left(\frac{-\alpha V_o}{R}\right) te^{-\alpha t} \quad \text{...............(15)}$$

Put $t = 0^+$ in equation (15),

$$i(0^+) = k \quad \text{.............(16)}$$

From (9) and (16),

$$k = \frac{V_o}{R} \quad \text{..........................(17)}$$

From (15) and (17),

$$i = \frac{V_o}{R} e^{-\frac{1}{CR}t} + \left(\frac{-\alpha V_o}{R}\right) te^{-\alpha t}$$

### 3) Response of Series RLC Circuit Excited by Exponential Source

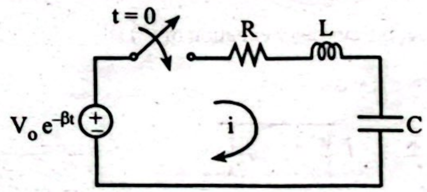



*Figure 2.17: A series RLC circuit excited by exponential source*

Applying KVL for $t > 0$,

$$v_R + v_L + v_C = V_o e^{-\beta t}$$

$$\text{or, } iR + L\frac{di}{dt} + \frac{1}{C}\int i\, dt = V_o e^{-\beta t} \quad \text{........... (1)}$$

Differentiating equation (1) w.r.t. t, we get

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{i}{C} = -\beta V_o e^{-\beta t}$$

or, $\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i=\frac{-\beta V_o}{L}e^{-\beta t}$ ............. (2)

Equation (2) is a second order nonhomogeneous differential equation. Its solution is given by

$i = C.F. + P.I.$

or, $i = i_N + i_f$ .......... (3)

To find $i_f$, let the trial solution for particular integral be

$i_f = Ae^{-\beta t}$ .......... (4)

From equations (2) and (4),

$$\frac{d^2(Ae^{-\beta t})}{dt^2}+\frac{R}{L}\frac{d(Ae^{-\beta t})}{dt}+\frac{1}{LC}(Ae^{-\beta t})=\frac{-\beta V_o}{L}e^{-\beta t}$$

or, $A(-\beta)^2 e^{-\beta t}+\frac{R}{L}A(-\beta)e^{-\beta t}+\frac{1}{LC}(Ae^{-\beta t})=\frac{-\beta V_o}{L}e^{-\beta t}$

or, $A=\dfrac{-\beta\frac{V_o}{L}}{\beta^2-\frac{\beta R}{L}+\frac{1}{LC}}$ ............... (5)

Now, equation (4) reduces to

$i_f=\dfrac{-\beta\frac{V_o}{L}}{\beta^2-\frac{\beta R}{L}+\frac{1}{LC}}e^{-\beta t}$ .......... (6)

To find $i_N$, the auxiliary equation of (2) is

$s^2+\frac{R}{L}s+\frac{1}{LC}=0$

$$s_1, s_2=\frac{\frac{-R}{L}\pm\sqrt{\left(\frac{R}{L}\right)^2-4\times\frac{1}{LC}}}{2}=\frac{-R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\left(\frac{1}{\sqrt{LC}}\right)^2}$$

or, $s_1, s_2=-\alpha\pm\sqrt{\alpha^2-\omega_n^2}$ .......... (7)

**Case I:** If $\alpha<\omega_n$, then the roots will be complex given from equation (7).

$s_1, s_2=-\alpha\pm\sqrt{-(\omega_n^2-\alpha^2)}=-\alpha\pm j\omega_d$

Hence, the complementary function is given by

$i_N=e^{-\alpha t}(k_1\cos\omega_d t+k_2\sin\omega_d t)$ ............(8)

From equations (3), (6), and (8),



$$i=e^{-\alpha t}(k_1\cos\omega_d t+k_2\sin\omega_d t)+\frac{-\beta\frac{V_o}{L}}{\beta^2-\frac{\beta R}{L}+\frac{1}{LC}}e^{-\beta t}$$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

**Case II:** If $\alpha=\omega_n$, then the roots will be real and equal as given from equation (7).

$s_1, s_2=-\alpha$

Hence, the complementary function is given by

$i_N=(k_1+k_2t)e^{-\alpha t}$ ........... (9)

From equations (3), (6), and (9),

$$i=(k_1+k_2t)e^{-\alpha t}+\frac{-\beta\frac{V_o}{L}}{\beta^2-\frac{\beta R}{L}+\frac{1}{LC}}e^{-\beta t}$$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

**Case III:** If $\alpha>\omega_n$, then the roots will be real and unequal as given by equation (7).

$s_1=-\alpha+\sqrt{\alpha^2-\omega_n^2}$, $s_2=-\alpha-\sqrt{\alpha^2-\omega_n^2}$

Hence, the complementary function is given by

$i_N=k_1e^{s_1t}+k_2e^{s_2t}$ ......... (10)

From equations (3), (6), and (10),

$$i=(k_1e^{s_1t}+k_2e^{s_2t})+\frac{-\beta\frac{V_o}{L}}{\beta^2-\frac{\beta R}{L}+\frac{1}{LC}}e^{-\beta t}$$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

**4) Response of Parallel RLC Circuit Excited by Exponential Source**

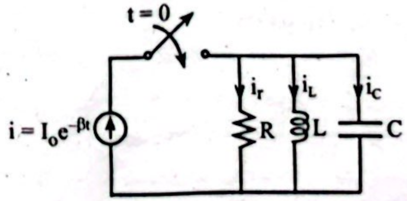



*Figure 2.18: A parallel RLC circuit excited by exponential source*

Applying KCL for t > 0,

$i_C + i_L + i_r = I_o e^{-\beta t}$

or, $C\frac{dv}{dt} + \frac{1}{L}\int v\,dt + \frac{v}{R} = I_o e^{-\beta t}$

or, $C\frac{dv}{dt} + \frac{1}{L}\int v\,dt + Gv = I_o e^{-\beta t}$ .......... (1)

where $G = \frac{1}{R}$ = conductance.

Differentiating equation (1) w.r.t. t, we get

$$C\frac{d^2v}{dt^2} + \frac{1}{L}v + G\frac{dv}{dt} = -\beta I_o e^{-\beta t}$$

or, $\frac{d^2v}{dt^2} + \frac{G}{C}\frac{dv}{dt} + \frac{1}{LC}v = \frac{-\beta I_o}{C}e^{-\beta t}$ .........(2)

Equation (2) is a second order nonhomogeneous differential equ Its solution is given by

v = C.F. + P.I.

or, $v = v_N + v_f$ ....................(3)

To find particular integral ($v_f$), let the trial solution for P.I. be

$v_f = Ae^{-\beta t}$ ................(4)

From equations (2) and (4),

$$\frac{d^2(Ae^{-\beta t})}{dt^2} + \frac{G}{C}\frac{d(Ae^{-\beta t})}{dt} + \frac{1}{LC}(Ae^{-\beta t}) = \frac{-\beta I_o}{C}e^{-\beta t}$$

or, $A(-\beta)^2 e^{-\beta t} + \frac{G}{C}A(-\beta)e^{-\beta t} + \frac{1}{LC}(Ae^{-\beta t}) = \frac{-\beta I_o}{C}e^{-\beta t}$

or, $A = \dfrac{\frac{-\beta I_o}{C}}{\left(\beta^2 - \frac{\beta G}{C} + \frac{1}{LC}\right)}$ ................(5)

From equations (4) and (5),

$v_f = \dfrac{\frac{-\beta I_o}{C}}{\left(\beta^2 - \frac{\beta G}{C} + \frac{1}{LC}\right)}e^{-\beta t}$ ................(6)

To find $v_N$, the auxiliary equation of (2) is

$$s^2 + \frac{G}{C}s + \frac{1}{LC} = 0$$



$$s_1, s_2 = \frac{\frac{-G}{C} \pm \sqrt{\left(\frac{G}{C}\right)^2 - 4\times\frac{1}{LC}}}{2}$$

$$= \frac{-G}{2C} \pm \sqrt{\left(\frac{G}{2C}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_n^2}$$ ........... (7)

where $\alpha = \frac{G}{2C}$ = damping coefficient, $\omega_n = \frac{1}{\sqrt{LC}}$ = natural frequency.

**Case I:** If $\alpha < \omega_n$, then the system is said to be *underdamped* and roots will be complex which are given by

$s_1, s_2 = -\alpha \pm \sqrt{-(\omega_n^2 - \alpha^2)} = -\alpha \pm j\omega_d$

where $\sqrt{\omega_n^2 - \alpha^2} = \omega_d$ is damped natural frequency.

Hence, the complementary function is given by

$v_N = e^{-\alpha t}(k_1 \cos\omega_d t + k_2 \sin\omega_d t)$ ............ (8)

From equations (3), (6), and (8),

$$v = e^{-\alpha t}(k_1 \cos\omega_d t + k_2 \sin\omega_d t) + \frac{\frac{-\beta I_o}{C}}{\left(\beta^2 - \frac{\beta G}{C} + \frac{1}{LC}\right)}e^{-\beta t}$$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

**Case II:** If $\alpha = \omega_n$, then the system is said to be *critically damped*, and the roots will be real and equal which are given by

$s_1, s_2 = -\alpha$

Hence, the complementary function of equation (2) is given by

$v_N = (k_1 + k_2 t)e^{-\alpha t}$ ......... (9)

From equations (3), (6), and (9),

$$v = (k_1 + k_2 t)e^{-\alpha t} + \frac{\frac{-\beta I_o}{C}}{\left(\beta^2 - \frac{\beta G}{C} + \frac{1}{LC}\right)}e^{-\beta t}$$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

**Case III:** If $\alpha > \omega_n$, then the system is said to be *overdamped*, and the roots will be real and unequal which are given by

$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_n^2}$, Applying KCL for $t > 0$,

$$i_C + i_L + i_r = I_o e^{-\beta t}$$

$$\text{or, } C\frac{dv}{dt} + \frac{1}{L}\int v\,dt + \frac{v}{R} = I_o e^{-\beta t}$$

$$\text{or, } C\frac{dv}{dt} + \frac{1}{L}\int v\,dt + Gv = I_o e^{-\beta t} \qquad \text{.......... (1)}$$

where $G = \frac{1}{R}$ = conductance.

Differentiating equation (1) w.r.t. t, we get

$$C\frac{d^2v}{dt^2} + \frac{1}{L}v + G\frac{dv}{dt} = -\beta I_o e^{-\beta t}$$

$$\text{or, } \frac{d^2v}{dt^2} + \frac{G}{C}\frac{dv}{dt} + \frac{1}{LC}v = \frac{-\beta I_o}{C} e^{-\beta t} \qquad \text{.........(2)}$$

Equation (2) is a second order nonhomogeneous differential eq

Its solution is given by

$v$ = C.F. + P.I.

$$\text{or, } v = v_N + v_f \qquad \text{.....................(3)}$$

To find particular integral ($v_f$), let the trial solution for P.I. be

$$v_f = Ae^{-\beta t} \qquad \text{................(4)}$$

From equations (2) and (4),

$$\frac{d^2(Ae^{-\beta t})}{dt^2} + \frac{G}{C}\frac{d(Ae^{-\beta t})}{dt} + \frac{1}{LC}(Ae^{-\beta t}) = \frac{-\beta I_o}{C}e^{-\beta t}$$

$$\text{or, } A(-\beta)^2 e^{-\beta t} + \frac{G}{C}A(-\beta)e^{-\beta t} + \frac{1}{LC}(Ae^{-\beta t}) = \frac{-\beta I_o}{C}e^{-\beta t}$$

$$\text{or, } A = \frac{\frac{-\beta I_o}{C}}{\left(\beta^2 - \frac{\beta G}{C} + \frac{1}{LC}\right)} \qquad \text{................(5)}$$

From equations (4) and (5),

$$v_f = \frac{\frac{-\beta I_o}{C}}{\left(\beta^2 - \frac{\beta G}{C} + \frac{1}{LC}\right)}e^{-\beta t} \qquad \text{................(6)}$$

To find $v_N$, the auxiliary equation of (2) is

$$s^2 + \frac{G}{C}s + \frac{1}{LC} = 0$$



$$s_1, s_2 = \frac{\frac{-G}{C} \pm \sqrt{\left(\frac{G}{C}\right)^2 - 4\times\frac{1}{LC}}}{2}$$

$$= \frac{-G}{2C} \pm \sqrt{\left(\frac{G}{2C}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} \qquad \text{........... (7)}$$

where $\alpha = \frac{G}{2C}$ = damping coefficient, $\omega_n = \frac{1}{\sqrt{LC}}$ = natural frequency.

**Case I:** If $\alpha < \omega_n$, then the system is said to be *underdamped* and roots will be complex which are given by

$$s_1, s_2 = -\alpha \pm \sqrt{-(\omega_n^2 - \alpha^2)} = -\alpha \pm j\omega_d$$

where $\sqrt{\omega_n^2 - \alpha^2} = \omega_d$ is damped natural frequency.

Hence, the complementary function is given by

$$v_N = e^{-\alpha t}(k_1 \cos\omega_d t + k_2 \sin\omega_d t) \qquad \text{............ (8)}$$

From equations (3), (6), and (8),

$$v = e^{-\alpha t}(k_1 \cos\omega_d t + k_2 \sin\omega_d t) + \frac{\frac{-\beta I_o}{C}}{\left(\beta^2 - \frac{\beta G}{C} + \frac{1}{LC}\right)}e^{-\beta t}$$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

**Case II:** If $\alpha = \omega_n$, then the system is said to be *critically damped*, and the roots will be real and equal which are given by

$$s_1, s_2 = -\alpha$$

Hence, the complementary function of equation (2) is given by

$$v_N = (k_1 + k_2 t)e^{-\alpha t} \qquad \text{.......... (9)}$$

From equations (3), (6), and (9),

$$v = (k_1 + k_2 t)e^{-\alpha t} + \frac{\frac{-\beta I_o}{C}}{\left(\beta^2 - \frac{\beta G}{C} + \frac{1}{LC}\right)}e^{-\beta t}$$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

**Case III:** If $\alpha > \omega_n$, then the system is said to be *overdamped*, and the roots will be real and unequal which are given by

$$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_n^2},$$

$s_2 = -\alpha + \sqrt{\alpha^2 - \omega_n^2}$

Hence, the complementary function of equation (2) is given by

$v = k_1 e^{s_1 t} + k_2 e^{s_2 t}$ .......... (10)

From equations (3), (6), and (10),

$$v = k_1 e^{s_1 t} + k_2 e^{s_2 t} + \frac{\frac{-\beta I_0}{C}}{\left(\beta^2 - \frac{\beta G}{C} + \frac{1}{LC}\right)} e^{-\beta t}$$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

## 2.8 Response of Circuit Excited by Sinusoidal Source

**1) Response of RL Circuit Excited by Sinusoidal Source**

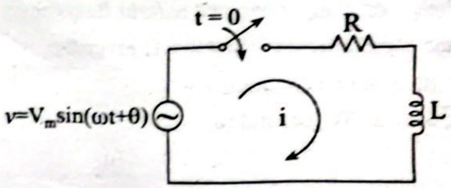



*Figure 2.19: A series RL circuit driven by a sinusoidal ac source*

Applying KVL for $t > 0$,

$v_R + v_L = V_m \sin(\omega t + \theta)$

or, $iR + L\frac{di}{dt} = V_m \sin(\omega t + \theta)$

or, $\frac{di}{dt} + \frac{R}{L} i = \frac{V_m}{L} \sin(\omega t + \theta)$ ................ (1)

The solution of equation (1) is given by

$i = C.F. + P.I.$

or, $i = i_N + i_f$ ................ (2)

To find $i_N$, the auxiliary equation of (1) is

$s + \frac{R}{L} = 0$

or, $s = -\frac{R}{L}$

$\therefore i_N = k\, e^{-\frac{R}{L}t}$ .............. (3)

To find $i_f$, let the trial solution for P.I. be



$i_f = A \cos(\omega t + \theta) + B \sin(\omega t + \theta)$ ............. (4)

From equations (1) and (4),

$$\frac{d[A \cos(\omega t + \theta) + B \sin(\omega t + \theta)]}{dt} + \frac{R}{L}[A\cos(\omega t + \theta) + B \sin(\omega t + \theta)] = \frac{V_m}{L} \sin(\omega t + \theta)$$

or, $-A\omega \sin(\omega t+\theta) + B\omega \cos(\omega t+\theta) + \frac{R}{L}[A \cos(\omega t + \theta) + B \sin(\omega t + \theta)] = \frac{V_m}{L} \sin(\omega t + \theta)$

Equating the coefficients of sine and cosine terms, we get

$-A\omega + \frac{R}{L} B = \frac{V_m}{L}$ ................ (5)

$B\omega + \frac{R}{L} A = 0$ ................ (6)

From equation (6),

$A = \left(\frac{-\omega L}{R}\right) B$ .............. (7)

From equations (5) and (7),

$-\left(\frac{-\omega L}{R}\right) B\omega + \frac{R}{L} B = \frac{V_m}{L}$

or, $\frac{\omega^2 L}{R} B + \frac{R}{L} B = \frac{V_m}{L}$

or, $B = \frac{V_m R}{(\omega L)^2 + R^2}$ .............. (8)

From equations (8) and (7),

$A = \left(\frac{-\omega L}{R}\right) \times \frac{V_m R}{R^2 + (\omega L)^2} = \frac{-V_m(\omega L)}{R^2 + (\omega L)^2}$ ............. (9)

Equation (4) can be written as

$i_f = \sqrt{A^2 + B^2} \sin\left(\omega t + \theta + \tan^{-1}\frac{A}{B}\right)$ ............. (10)

Now,

$$\sqrt{A^2 + B^2} = \frac{V_m}{R^2 + (\omega L)^2} \sqrt{R^2 + (\omega L)^2} = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}}$$

$$= \frac{V_m}{\sqrt{R^2 + X_L^2}} = \frac{V_m}{|Z_L|}$$

$$\tan^{-1}\frac{A}{B} = \tan^{-1}\left(\frac{-\omega L}{R}\right) = -\tan^{-1}\left(\frac{X_L}{R}\right)$$

Hence, equation (10) becomes

$$i_f = \frac{V_m}{|Z_L|}\sin\left(\omega t + \theta - \tan^{-1}\frac{X_L}{R}\right) \quad \text{............ (11)}$$

From (2), (3), and (11),

$$i = ke^{-\frac{R}{L}t} + \frac{V_m}{|Z_L|}\sin\left(\omega t + \theta - \tan^{-1}\frac{X_L}{R}\right) \quad \text{............ (12)}$$

Put $t = 0^+$ in equation (12),

$$i(0^+) = k + \frac{V_m}{|Z_L|}\sin\left(\theta - \tan^{-1}\frac{X_L}{R}\right)$$

$$\text{or, } k = i(0^+) - \frac{V_m}{|Z_L|}\sin\left(\theta - \tan^{-1}\frac{X_L}{R}\right) \quad \text{............ (13)}$$

By observation at $t = 0^-$,

$i(0^-) = 0$

From continuity relation for inductor,

$i(0^+) = i(0^-) = 0$ ......... (14)

From (13) and (14),

$$k = -\frac{V_m}{|Z_L|}\sin\left(\theta - \tan^{-1}\frac{X_L}{R}\right) \quad \text{............ (15)}$$

From equations (12) and (15),

$$i = -\frac{V_m}{|Z_L|}\sin\left(\theta - \tan^{-1}\frac{X_L}{R}\right)e^{-\frac{R}{L}t} + \frac{V_m}{|Z_L|}\sin\left(\omega t + \theta - \tan^{-1}\frac{X_L}{R}\right)$$

## 2) Response of RC Circuit Excited by Sinusoidal Source

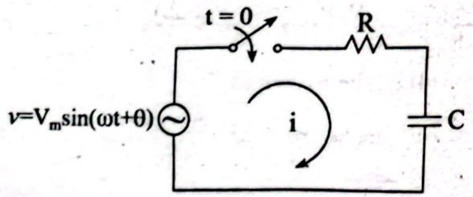



*Figure 2.20: An RC circuit driven by a sinusoidal source*

Applying KVL for $t > 0$,

$v_R + v_C = V_m \sin(\omega t + \theta)$



$$\text{or, } iR + \frac{1}{C}\int i\,dt = V_m \sin(\omega t + \theta)$$

Differentiating above equation w.r.t. t, we get

$$R\frac{di}{dt} + \frac{i}{C} = V_m\omega\cos(\omega t + \theta)$$

$$\text{or, } \frac{di}{dt} + \frac{i}{CR} = \frac{V_m\omega}{R}\cos(\omega t + \theta) \quad \text{........... (1)}$$

The solution of above differential equation is given by

$i = \text{C.F.} + \text{P.I.}$

or, $i = i_N + i_f$ ........... (2)

To find $i_N$, the A.E. of (1) is

$$s + \frac{1}{CR} = 0$$

$$\text{or, } s = \frac{-1}{CR}$$

$\therefore i_N = ke^{-(1/CR)t}$ ........... (3)

To find $i_f$, let the trial solution for P.I. be

$i_f = A\cos(\omega t + \theta) + B\sin(\omega t + \theta)$ ............. (4)

From equations (1) and (4),

$$\frac{d[A\cos(\omega t + \theta) + B\sin(\omega t + \theta)]}{dt} + \frac{1}{CR}[A\cos(\omega t + \theta) + B\sin(\omega t + \theta)] = \frac{V_m\omega}{R}\cos(\varphi t + \theta)$$

$$\text{or, } -A\omega\sin(\omega t+\theta) + B\omega\cos(\omega t+\theta) + \frac{1}{CR}[A\cos(\omega t + \theta) + B\sin(\omega t + \theta)] = \frac{V_m\omega}{R}\cos(\omega t + \theta)$$

Equating the coefficients of sine and cosine terms, we get

$$-A\omega + \frac{1}{CR}B = 0 \quad \text{............... (5)}$$

$$B\omega + \frac{1}{CR}A = \frac{V_m\omega}{R} \quad \text{................ (6)}$$

From equation (5),

$$A = \left(\frac{1}{\omega CR}\right)B \quad \text{.............. (7)}$$

From equations (6) and (7),

$$\text{or, } B = \frac{V_m\omega^2C^2R}{1+(\omega CR)^2} \quad \text{.................. (8)}$$

From equations (8) and (7),

$$A = \left(\frac{1}{\omega CR}\right) \times \frac{V_m\omega^2C^2R}{1+(\omega CR)^2} = \frac{V_m\omega C}{1+(\omega CR)^2} \quad \text{.................. (9)}$$

Equation (4) can be written as

$$i_f = \sqrt{A^2+B^2}\,\sin\left(\omega t+\theta+\tan^{-1}\frac{A}{B}\right) \quad \text{.................. (10)}$$

Now,

$$\sqrt{A^2+B^2} = \frac{V_m\omega C}{\sqrt{1+\omega^2C^2R^2}} = \frac{V_m}{\sqrt{\frac{1}{(\omega C)^2}+(R)^2}}$$

$$= \frac{V_m}{\sqrt{R^2+X_C^2}} = \frac{V_m}{|Z_C|}$$

$$\tan^{-1}\frac{A}{B} = \tan^{-1}\left(\frac{\frac{1}{\omega C}}{R}\right) = \tan^{-1}\left(\frac{X_C}{R}\right)$$

Hence, equation (10) becomes

$$i_f = \frac{V_m}{|Z_C|}\sin\left(\omega t+\theta+\tan^{-1}\frac{X_C}{R}\right) \quad \text{.............. (11)}$$

From (2), (3), and (11),

$$i = ke^{-(1/CR)\,t} + \frac{V_m}{|Z_C|}\sin\left(\omega t+\theta+\tan^{-1}\frac{X_C}{R}\right) \quad \text{.................. (12)}$$

Put $t = 0^+$ in equation (12),

$$i(0^+) = k + \frac{V_m}{|Z_C|}\sin\left(\theta+\tan^{-1}\frac{X_C}{R}\right)$$

$$\text{or, } k = i(0^+) - \frac{V_m}{|Z_C|}\sin\left(\theta+\tan^{-1}\frac{X_C}{R}\right) \quad \text{.............. (13)}$$

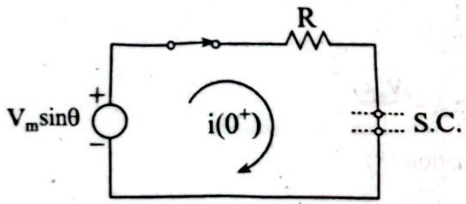



*Figure 2.21: Equivalent circuit at $t = 0^+$.*



From equivalent circuit at $t = 0^+$,

$$i(0^+) = \frac{V_m\sin\theta}{R}$$

Then, equation (13) becomes

$$k = \frac{V_m\sin\theta}{R} - \frac{V_m}{|Z_C|}\sin\left(\theta+\tan^{-1}\frac{X_C}{R}\right) \quad \text{........... (14)}$$

From equations (12) and (14),

$$i = \left[\frac{V_m\sin\theta}{R} - \frac{V_m}{|Z_C|}\sin\left(\theta+\tan^{-1}\frac{X_C}{R}\right)\right]e^{-(1/CR)t} + \frac{V_m}{|Z_C|}\sin\left(\omega t+\theta+\tan^{-1}\frac{X_C}{R}\right)$$

### 3) Response of RLC Series Circuit Excited by Sinusoidal Source

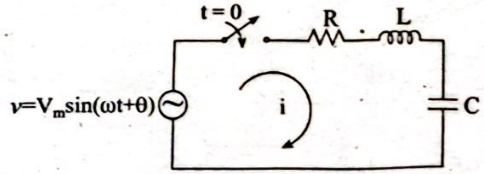



*Figure 2.22: A series RLC circuit excited by sinusoidal source.*

Applying KVL for $t > 0$,

$$v_R + v_L + v_C = V_m\sin(\omega t+\theta)$$

$$\text{or, } iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = V_m\sin(\omega t+\theta) \quad \text{............ (1)}$$

Differentiating equation (1) w.r.t. t, we get

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = V_m\omega\cos(\omega t+\theta)$$

$$\text{or, } \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{V_m\omega}{L}\cos(\omega t+\theta) \quad \text{............ (2)}$$

Equation (2) is a second order nonhomogeneous differential equation. Its solution is given by

$i = \text{C.F.} + \text{P.I.}$

or, $i = i_N + i_f$ ......... (3)

To find $i_f$, let the trial solution for P.I. be

$i_f = A\cos(\omega t+\theta) + B\sin(\omega t+\theta)$ ............. (4)

From equations (2) and (4),

$$\frac{d^2[A\cos(\omega t+\theta)+B\sin(\omega t+\theta)]}{dt^2}+\frac{R}{L}\frac{d[A\cos(\omega t+\theta)+B\sin(\omega t+\theta)]}{dt}$$

$$+\frac{1}{LC}[A\cos(\omega t+\theta)+B\sin(\omega t+\theta)]=\frac{V_m\omega}{L}\cos(\omega t+\theta)$$

Solving and equating the coefficients of sine and cosine terms, we get two equations which can be solved for A and B. Then, substituting the value of A and B in equation (4) gives $i_f$ as

$$i_f=\frac{V_m}{|Z|}\sin\left(\omega t+\theta-\tan^{-1}\frac{X_L-X_C}{R}\right) \quad \text{.............(5) for } X_L>X_C$$

where $|Z|=\sqrt{R^2+(X_L-X_C)^2}$

To find $i_N$, the auxiliary equation of (2) is

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0$$

$$s_1, s_2=\frac{\frac{-R}{L}\pm\sqrt{\left(\frac{R}{L}\right)^2-4\times\frac{1}{LC}}}{2}$$

$$=\frac{-R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\left(\frac{1}{\sqrt{LC}}\right)^2}$$

$$\text{or, } s_1, s_2=-\alpha\pm\sqrt{\alpha^2-\omega_n^2} \quad \text{.......... (6)}$$

**Case I:** If $\alpha<\omega_n$, then the roots will be complex as given from eq. (6).

$$s_1, s_2=-\alpha\pm\sqrt{-(\omega_n^2-\alpha^2)}=-\alpha\pm j\omega_d$$

Hence, the complementary function is given by

$$i_N=e^{-\alpha t}(k_1\cos\omega_d t+k_2\sin\omega_d t) \quad \text{............(7)}$$

From equations (3), (5), and (7),

$$i=e^{-\alpha t}(k_1\cos\omega_d t+k_2\sin\omega_d t)+\frac{V_m}{|Z|}\sin\left(\omega t+\theta-\tan^{-1}\frac{X_L-X_C}{R}\right)$$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

**Case II:** If $\alpha=\omega_n$, then the roots will be real and equal as given from equation (6).

$$s_1, s_2=-\alpha$$

Hence, the complementary function is given by

$$i_N=(k_1+k_2t)e^{-\alpha t} \quad \text{.......... (8)}$$



From equations (3), (5), and (8),

$$i=(k_1+k_2t)e^{-\alpha t}+\frac{V_m}{|Z|}\sin\left(\omega t+\theta-\tan^{-1}\frac{X_L-X_C}{R}\right)$$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

**Case III:** If $\alpha>\omega_n$, then the roots will be real and unequal as given by equation (6).

$$s_1=-\alpha+\sqrt{\alpha^2-\omega_n^2}, \quad s_2=-\alpha-\sqrt{\alpha^2-\omega_n^2}$$

Hence, the complementary function is given by

$$i_N=k_1e^{s_1t}+k_2e^{s_2t} \quad \text{......... (9)}$$

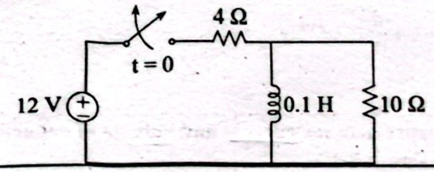
From equations (3), (5), and (9),

$$i=(k_1e^{s_1t}+k_2e^{s_2t})+\frac{V_m}{|Z|}\sin\left(\omega t+\theta-\tan^{-1}\frac{X_L-X_C}{R}\right)$$

The values of $k_1$ and $k_2$ can be obtained by using initial conditions.

## SOLVED PROBLEMS

**1. Find the expression of voltage and current of inductor as a function of time for t > 0.**




***Solution:***

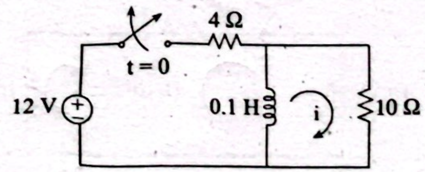



***Figure 1***

Applying KVL for t > 0,

$$v_{10\Omega}+v_L=0$$

$$\text{or, } 10i+0.1\frac{di}{dt}=0$$

$$\text{or, } \frac{di}{dt}+100i=0 \quad \text{............ (1)}$$

$i \rightarrow i,\ P \rightarrow 100,\ Q \rightarrow 0$

$i = e^{-Pt}\int Q\, e^{Pt}\, dt + ke^{-Pt}$

or, $i = ke^{-100t}$ ................ (2)

Put $t = 0^+$ in equation (2),

$i(0^+) = k$ ............... (3)

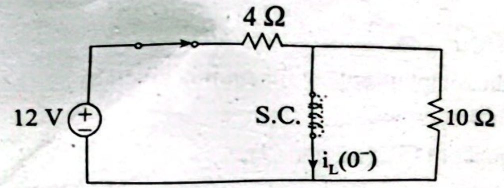



*Figure 2: Equivalent circuit at $t = 0^-$.*

From figure 2, $i_L(0^-) = \frac{12}{4} = 3$ A

From continuity relation for inductor, $i(0^+) = -i_L(0^-) = -3$

From equation (3), $k = -3$

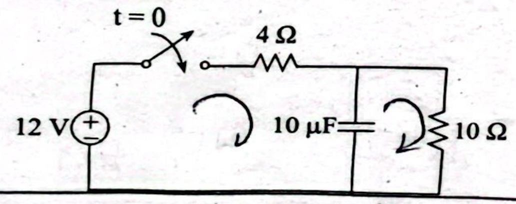
From equation (2), $i = -3e^{-100t}$ ............ (4)

The voltage across inductor is given by

$v_L = L\frac{di}{dt} = 0.1\frac{d}{dt}(-3e^{-100t})$

$\therefore v_L = 30\, e^{-100t}$

**2. Find the expression for current and voltage of capacitor for t>0 the figure shown below.**




***Solution:***

??

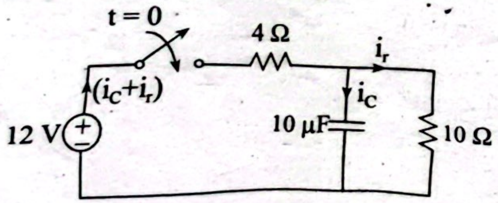



*Figure 1*



Applying KVL (in left loop) for $t > 0$,

$12 - v_{4\Omega} - v_C = 0$ ............(1)

or, $v_{4\Omega} + v_C = 12$

or, $4(i_r + i_c) + \frac{1}{10\times10^{-6}}\int i_c\, dt = 12$

Differentiating above equation w.r.t. t, we have

$4\frac{di_r}{dt} + 4\frac{di_C}{dt} + 10^5\, i_C = 0$ ............ (2)

Applying KVL (in outer loop) for $t > 0$,

$12 - v_{4\Omega} - v_{10\Omega} = 0$

or, $v_{4\Omega} + v_{10\Omega} = 12$

or, $4(i_r + i_c) + 10i_r = 12$

or, $14i_r + 4i_c = 12$

$\therefore i_r = \frac{12 - 4i_c}{14} = \frac{6 - 2i_c}{7}$ ............. (3)

From equations (2) and (3),

$4\frac{d}{dt}\left(\frac{6-2\,i_c}{7}\right) + 4\frac{di_c}{dt} + 10^5\, i_c = 0$

or, $\left(0 - \frac{8}{7}\frac{di_c}{dt}\right) + 4\frac{di_c}{dt} + 10^5\, i_c = 0$

or, $\frac{20}{7}\frac{di_c}{dt} + 10^5\, i_c = 0$

or, $\frac{di_c}{dt} + \frac{7\times10^5}{20}\, i_c = 0$ ................ (4)

The solution of above differential equation is

$i_c = ke^{\frac{-7\times10^{-5}}{20}t}$ ................ (5)

Put $t = 0^+$ in (4),

$i_c(0^+) = k$

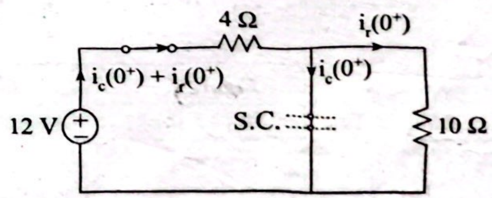



*Figure 2: Equivalent circuit at $t = 0^+$.*

From figure 2,

$$i_c(0^+) = \frac{12}{4} = 3\text{ A}$$

$$\therefore k = 3$$

Now, equation (5) becomes

$$\therefore i_c = 3e^{\frac{-7\times10^{-5}}{20}t} \quad \text{............. (7)}$$

From equations (3) and (7), we get

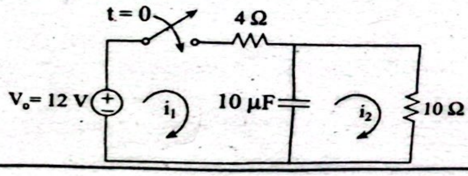
$$i_r = \frac{6}{7} - \frac{2}{7} \times 3e^{\frac{-7\times10^{-5}}{20}t}$$

$$i_r = \frac{6}{7} - \frac{6}{7}e^{\frac{-7\times10^{-5}}{20}t} \quad \text{............(8)}$$

From equation (1),

$$v_C = 12 - v_{4\Omega}$$
$$= 12 - 4(i_c + i_r)$$
$$= 12 - 4\left[3e^{\frac{-7\times10^{-5}}{20}t} + \frac{6}{7} - \frac{6}{7}e^{\frac{-7\times10^{-5}}{20}t}\right] = \frac{60}{7} - \frac{60}{7}e^{\frac{-7\times10^{-5}}{20}t}$$

**3. Find the expression for $i_1$ and $i_2$ and voltage across capacito function of time for t > 0.**




[2080 A

***Solution:***

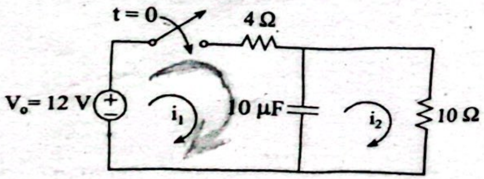



***Figure 1***

Applying KVL (in first loop) for t > 0,

$$v_{4\Omega} + v_c = V_o$$

$$\text{or, } 4i_1 + \frac{1}{10\times10^{-6}}\int(i_1 - i_2)\,dt = 12$$



$$\text{or, } 4i_1 + \frac{1}{10^{-5}}\int(i_1 - i_2)\,dt = 12 \quad \text{..........(1)}$$

Applying KVL (in second loop) for t > 0,

$$v_{10\Omega} + v_C = 0 \quad \text{..................(A)}$$

$$\text{or, } 10i_2 + \frac{1}{10\times10^{-6}}\int(i_2 - i_1)\,dt = 0$$

$$\text{or, } 10i_2 - \frac{1}{10^{-5}}\int(i_1 - i_2)\,dt = 0 \quad \text{............... (2)}$$

Adding equations (1) and (2), we have

$$4i_1 + 10i_2 = 12$$

$$\text{or, } i_1 = \frac{6 - 5i_2}{2} \quad \text{........(3)}$$

Substituting the value of $i_1$ in equation (2), we have

$$10i_2 - \frac{1}{10^{-5}}\int\left[\frac{6-5i_2}{2} - i_2\right]dt = 0$$

Differentiating w.r.t. t, we get

$$10\frac{di_2}{dt} - 10^5\left[\frac{6-5i_2}{2} - i_2\right] = 0$$

$$\text{or, } 10\frac{di_2}{dt} + 3.5\times10^5 i_2 = 3\times10^5$$

$$\text{or, } \frac{di_2}{dt} + 3.5\times10^4 i_2 = 3\times10^4 \quad \text{...........(4)}$$

Solving equation (4), we get

$$\boxed{i_2 = \frac{3\times10^4}{3.5\times10^4} + ke^{-3.5\times10^4 t}}$$

$$\text{or, } i_2 = 0.857 + ke^{-3.5\times10^4 t} \quad \text{............(5)}$$

Put $t = 0^+$ in equation (5), we get

$$i_2(0^+) = 0.857 + k$$

$$\text{or, } k = i_2(0^+) - 0.857 \quad \text{...............(6)}$$

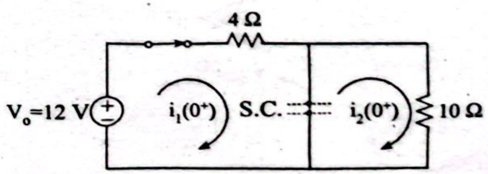



***Figure 2: Equivalent circuit at t = 0⁺***

From figure 2,

$i_2(0^+) = 0$ .............(7)

From equations (6) and (7),

$k = -0.857$

Thus, equation (5) becomes

$i_2 = 0.857 - 0.857e^{-3.5\times10^4 t}$ ...............(8)

From equations (3) and (8),

$$i_1 = \frac{6 - 5i_2}{2} = 3 - 2.5i_2 = 3 - 2.5(0.857 - 0.857e^{-3.5\times10^4 t})$$

$\therefore i_1 = 0.8575 + 2.1425e^{-3.5\times10^4 t}$

Now voltage across capacitor is calculated as

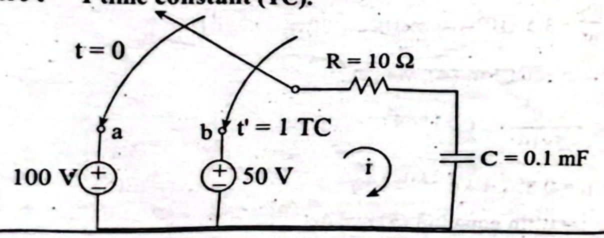
$v_{10\Omega} + v_C = 0$

or, $v_C = -v_{10\Omega}$

or, $v_C = -10i_2$

$\therefore v_C = -10(0.857 - 0.857e^{-3.5\times10^4 t}) = -8.57 + 8.57e^{-3.5\times10^4 t}$

**4. For the given circuit, find the expression of current through capacitor in the intervals**
**i. $0 \le t < t'$**
**ii. $t \ge t'$**
**where t' = 1 time constant (TC).**




***Solution:***

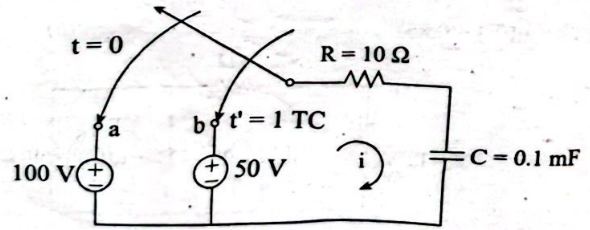



***Figure 1***



Applying KVL for interval $0 \le t < t'$,

$v_R + v_C = 100$ .......... (A)

or, $10i + \frac{1}{0.1\times10^{-3}}\int i\,dt = 100$

or, $10i + 10^4\int i\,dt = 100$

Differentiating above equation w.r.t. t, we get

$10\frac{di}{dt} + 10^4 i = 0$

or, $\frac{di}{dt} + 10^3 i = 0$ ......... (1)

The solution of above differential equation is

$i = ke^{-10^3 t}$ .......... (2)

Put $t = 0^+$,

$i(0^+) = k$ ......... (3)

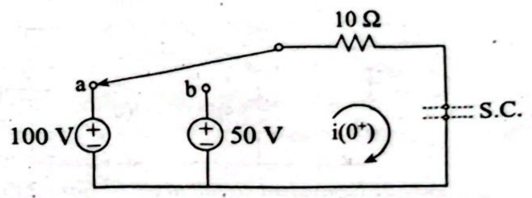



***Figure 2: Equivalent circuit at $t = 0^+$ (for $0 \le t < t'$).***

From figure 2,

$i(0^+) = \frac{100}{10} = 10$ A

$\therefore k = 10$

Now, equation (2) becomes

$i = 10e^{-10^3 t}$ ............. (5)

From equation (A),

$v_C = 100 - v_R$

or, $v_C = 100 - 10i = 100 - 100e^{-10^3 t}$ ........ (6)

The initial voltage for the second interval can be obtained by substituting $t = t' = 1\text{ms} = 1\times10^{-3}$ s in equation (6).

$v_C(0^+) = v_C(t') = 100 - 100e^{-10^3\times10^{-3}}$

$= 63$ V

Applying KVL for the interval $t \ge t'$,

$v_R + v_C = 50$

or, $10i + \frac{1}{10^{-3}} \int i \, dt = 50$

Differentiating above equation w.r.t. t, we get

$10\frac{di}{dt} + 10^4 i = 0$

or, $\frac{di}{dt} + 10^3 i = 0$ ............ (7)

The solution of above differential equation is

$i = ke^{-10^3 t}$ ......... (8)

Put $t = 0^+$,

$i(0^+) = ke^{-10^3 t'}$

or, $k = i(0^+)\, e^{10^3 t'}$ ......... (9)

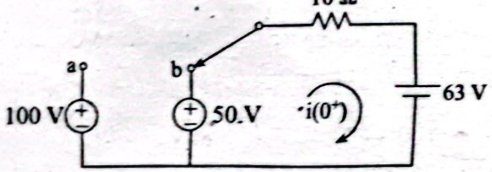



*Figure 3: Equivalent circuit at $t = 0^+$ (for $t \geq t'$).*

From figure 3,

$i(0^+) = \frac{50-63}{10} = -1.3$

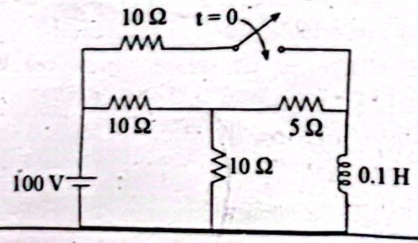
$\therefore k = -1.3e^{10^3 t'}$

Now, equation (8) becomes

$i = -1.3\, e^{10^3 t'}\, e^{-10^3 t}$

$= -1.3\, e^{-10^3 (t - t')}$

**5. Find the expression for current and voltage of inductor for t > 0.**






***Solution:***

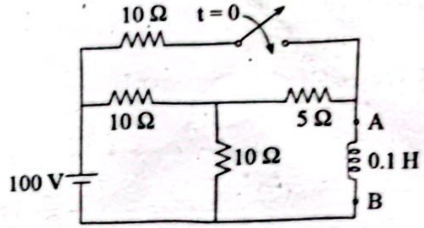



*Figure 1*

To find $R_{Th}$,

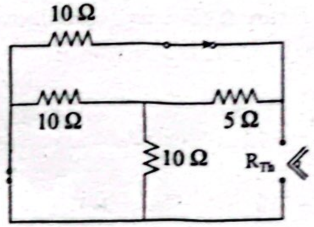



*Figure 2*

$R_{Th} = [(10//10) + 5] // 10 = 5$

To find $V_{Th}$,

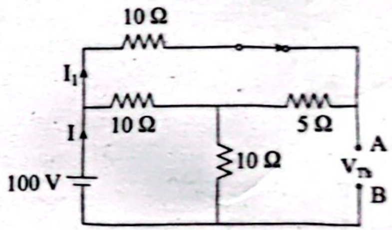



*Figure 3*

$R_{eq} = [(10 + 5) // 10] + 10 = \frac{15 \times 10}{15 + 10} + 10 = 16\ \Omega$

From figure 3,

$I = \frac{100}{16} = 6.25$ A

$I_1 = \frac{10}{10 + 15} \times 6.25 = 2.5$ A

Applying KVL in figure 3,

$V_{Th} - 5I_1 - 10I = 0$

$V_{Th} = 5 \times 2.5 + 10 \times 6.25 = 75$ V

Now, the Thevenin's equivalent circuit is

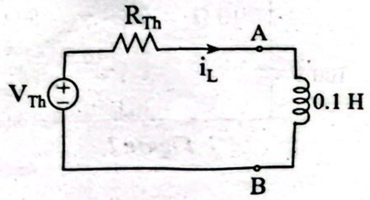



*Figure 4: Thevenin's equivalent circuit*

Applying KVL for $t > 0$ in figure 4,

$V_{Rth} + v_L = V_{Th}$

or, $5i_L + 0.1\dfrac{di_L}{dt} = 75$

or, $\dfrac{di_L}{dt} + 50i_L = 750$ ...........(1)

The solution of above differential equation is calculated as

$i = e^{-Pt}\int Q\, e^{Pt}\, dt + ke^{-Pt}$

$i \rightarrow i_L,\ P \rightarrow 50,\ Q \rightarrow 750$

$i_L = e^{-50t}\int 750\, e^{50t}\, dt + ke^{-50t}$

or, $i_L = \dfrac{750}{50} + ke^{-50t}$

or, $i_L = 15 + ke^{-50t}$ ........(2)

Put $t = 0^+$,

$i_L(0^+) = 15 + k$

$k = i_L(0^+) - 15$ ......... (3)

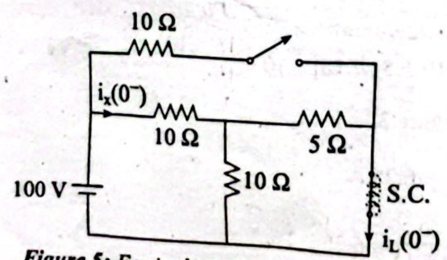



*Figure 5: Equivalent circuit at $t = 0^-$.*



From figure 5, total current supplied by source

$$i_x = \frac{100}{\left(\frac{10}{3} + 10\right)} = 7.5 \text{ A}$$

From figure 5, current through inductor is given by

$$i_L(0^-) = \frac{10}{10 + 5} \times 7.5 = 5 \text{ A}$$

From continuity relation for inductor,

$i_L(0^+) = i_L(0^-) = 5$ A

From equation (3),

$k = 5 - 15 = -10$

Hence, equation (2) becomes

$i_L = 15 - 10e^{-50t}$ ............(4)

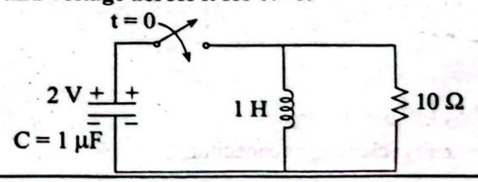
The voltage across inductor is given by

$v_L = L\dfrac{di_L}{dt}$

or, $v_L = 0.1\dfrac{d}{dt}(15 - 10e^{-50t})$

$\therefore v_L = 50\, e^{-50t}$

**6. For the given circuit, find the expression for current through inductor and voltage across it for $t > 0$.**




***Solution:***

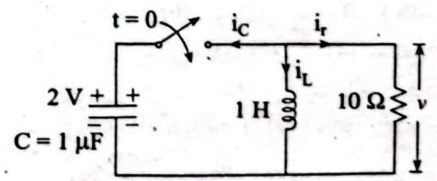



*Figure 1*

Applying KCL for $t > 0$,

$i_C + i_L + i_r = 0$ ............ (A)

$$\text{or, } C\frac{dv}{dt}+\frac{v}{R}+\frac{1}{L}\int v\,dt=0$$

$$\text{or, } 1\times10^{-6}\frac{dv}{dt}+\frac{v}{10}+\frac{1}{1}\int v\,dt \qquad \text{......... (1)}$$

Differentiating equation w.r.t. t, we get

$$10^{-6}\frac{d^2v}{dt^2}+\frac{1}{10}\frac{dv}{dt}+v=0$$

$$\text{or, } \frac{d^2v}{dt^2}+\frac{1}{10^{-5}}\frac{dv}{dt}+\frac{1}{10^{-6}}v=0$$

$$\text{or, } \frac{d^2v}{dt^2}+10^5\frac{dv}{dt}+10^6v=0 \qquad \text{....... (2)}$$

The auxiliary equation of (2) is

$s^2+10^5s+10^6=0$

$$s_1, s_2=\frac{-10^5\pm\sqrt{(10^5)^2-4\times10^6}}{2}$$

$$=\frac{-10^5}{2}\pm\frac{99979.998}{2}$$

$\therefore\ s_1, s_2=-10, -99990$

The solution of equation (2) is

$$v=k_1e^{-10t}+k_2e^{-99990t} \qquad \text{........ (3)}$$

Put $t=0^+$ in (3),

$$v(0^+)=k_1+k_2 \qquad \text{......... (4)}$$

By observation at $0^-$,

$v(0^-)=2\ V,\ i_L(0^-)=0$

From continuity relation for capacitor,

$$v(0^+)=v(0^-)=2\ V \qquad \text{.............. (5)}$$

$$i_L(0^+)=i_L(0^-)=0 \qquad \text{............... (6)}$$

From equations (4) and (5),

$$k_1+k_2=2 \qquad \text{...........(7)}$$

Differentiating equation (3) w.r.t. t, we get

$$\frac{dv}{dt}=-10k_1e^{-10t}-99990\,k_2e^{-99990t} \qquad \text{...........(8)}$$

Put $t=0^+$ in equation (8),

$$\frac{dv(0^+)}{dt}=-10k_1-99990k_2 \qquad \text{........... (9)}$$

Put $t=0^+$ in equation (1),

$$10^{-6}\frac{dv(0^+)}{dt}+\frac{v(0^+)}{10}+i_L(0^+)=0 \qquad \text{.........(10)}$$

Using equations (5) and (6), equation (10) reduces to

$$10^{-6}\frac{dv(0^+)}{dt}+\frac{2}{10}+0=0$$

$$\text{or, } \frac{dv(0^+)}{dt}=-2\times10^5 \qquad \text{.............. (11)}$$

From equations (11) and (9),

$-2\times10^5=-10k_1-99990k_2$

$$\text{or, } 10k_1+99990k_2=2\times10^5 \qquad \text{........... (12)}$$

Solving (12) and (7), we get

or, $10(2-k_2)+99990k_2=2\times10^5$

or, $20-10k_2+99990k_2=2\times10^5$

or, $k_2=2.0002$

From equation (7),

$k_1=2-2.0002=-0.0002=-2\times10^{-4}$

From equation (3),

$$v=-2\times10^{-4}e^{-10t}+2.0002e^{-99990t} \qquad \text{............ (13)}$$

From equation (A),

$i_L=-i_C-i_r$

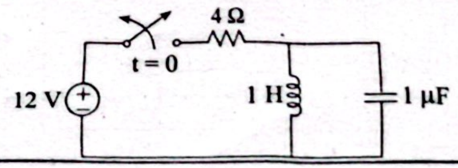
$$=-C\frac{dv}{dt}-\frac{v}{R}$$

$$=-10^{-6}\frac{d}{dt}(-2\times10^{-4}e^{-10t}+2.0002e^{-99990t})-\frac{1}{10}(-2\times10^{-4}e^{-10t}+2.0002e^{-99990t})$$

$$\therefore\ i_L=1.9998\times10^{-5}e^{-10t}-2.0002\times10^{-5}e^{-99990t}$$

---

**7. Find the expression for current and voltage of capacitor for t>0.**



***Solution:***

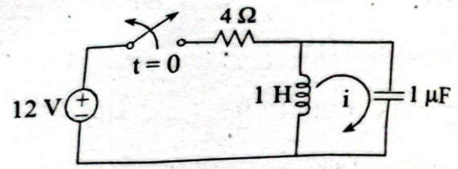



***Figure 1***

Applying KVL for $t > 0$,

$v_L + v_C = 0$ .............. (A)

$$\text{or, } L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0$$

$$\text{or, } 1\frac{di}{dt} + \frac{1}{10^{-6}}\int i\,dt = 0$$

$$\text{or, } \frac{di}{dt} + 10^6\int i\,dt = 0 \quad \text{......... (1)}$$

Differentiating equation (1) w.r.t. t, we get

$$\frac{d^2i}{dt^2} + 10^6 i = 0 \quad \text{....... (2)}$$

The auxiliary equation of (2) is

$$s^2 + 10^6 = 0$$

$$s_1, s_2 = \frac{0 \pm \sqrt{0^2 - 4\times10^6}}{2}$$

$$= \pm j\,10^3$$

Hence, the solution of (2) is given by

$$i = e^{-\alpha t}(k_1 \cos\omega_d t + k_2 \sin\omega_d t)$$

$$i = e^{0t}(k_1 \cos10^3t + k_2 \sin10^3t)$$

$$\text{or, } i = k_1 \cos10^3t + k_2 \sin10^3t \quad \text{.............. (3)}$$

Put $t = 0^+$ in equation (3),

$$i(0^+) = k_1 \quad \text{...............(4)}$$

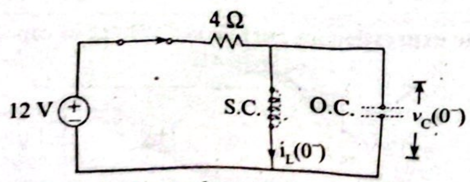



*Figure 2: Equivalent circuit at $t = 0^-$.*



From figure 2,

$$i_L(0^-) = \frac{12}{4} = 3,\ v_C(0^-) = 0$$

From continuity relation for inductor and capacitor,

$$i(0^+) = -i_L(0^-) = -3$$

$$v_C(0^+) = v_C(0^-) = 0$$

From equation (4),

$$k_1 = -3$$

Substituting the value of $k_1$ in equation (3), we get

$$i = -3\cos10^3t + k_2\sin10^3t \quad \text{...............(5)}$$

Differentiating equation (5) w.r.t. t, we get

$$\frac{di}{dt} = 3\times10^3\sin10^3t + k_2\times10^3\cos10^3t \quad \text{..........(6)}$$

Put $t = 0^+$ in equation (6),

$$\frac{di(0^+)}{dt} = k_2\times10^3$$

$$\text{or, } k_2 = 10^{-3}\frac{di(0^+)}{dt} \quad \text{................(7)}$$

Put $t = 0^+$ in equation (1),

$$\frac{di(0^+)}{dt} + v_C(0^+) = 0$$

$$\text{or, } \frac{di(0^+)}{dt} + 0 = 0$$

$$\text{or, } \frac{di(0^+)}{dt} = 0 \quad \text{................(8)}$$

From equations (7) and (8),

$$k_2 = 0 \quad \text{...................(9)}$$

From equations (9) and (5), we get

$$i = -3\cos10^3t + 0$$

$$\therefore i = -3\cos10^3t \quad \text{.......................(10)}$$

From equation (A),

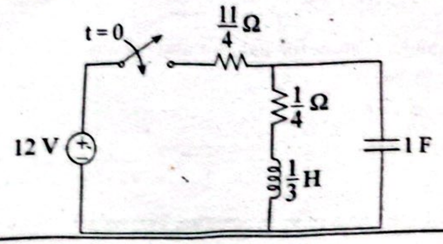
$$v_C = -v_L$$

$$\text{or, } v_C = -L\frac{di}{dt}$$

$$\text{or, } v_C = -1\frac{d}{dt}(-3\cos10^3t)$$

$$\therefore v_C = -3\times10^3\sin10^3t$$

**8. Find the expression for current and voltage of capacitor in the circuit for t > 0.**




***Solution:***

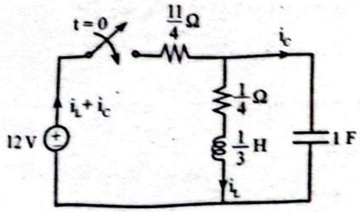



***Figure 1***

Applying KVL for t > 0,

$$\frac{11}{4}(i_L + i_C) + \frac{1}{4}i_L + \frac{1}{3}\frac{di_L}{dt} = 12$$

$$\text{or, } 3i_L + \frac{11}{4}i_C + \frac{1}{3}\frac{di_L}{dt} = 12 \quad \text{..........(1)}$$

Applying KVL in outer loop for t > 0,

$$\frac{11}{4}(i_L + i_C) + \frac{1}{1}\int i_C\, dt = 12$$

$$\text{or, } i_L = \frac{48}{11} - \frac{11}{4}i_C - \frac{4}{11}\int i_C\, dt \quad \text{..............(2)}$$

From equations (1) and (2), we get

$$\frac{11}{12}\frac{di_C}{dt} + \frac{722}{132}i_C + \frac{12}{11}\int i_C\, dt = 1 \quad \text{..........(3)}$$

Differentiating equation (3) w.r.t. t, we get

$$\frac{11}{12}\frac{d^2i_C}{dt^2} + \frac{722}{132}\frac{di_C}{dt} + \frac{12}{11}i_C = 0$$

$$\text{or, } \frac{d^2i_C}{dt^2} + 5.966\frac{di_C}{dt} + 1.19i_C = 0 \quad \text{..........(4)}$$



The characteristic equation of (4) is

$$s^2 + 5.966s + 1.19 = 0$$

$$s_1, s_2 = \frac{-5.966 \pm \sqrt{(5.966)^2 - 4\times1.19}}{2}$$

$$= -5.742, -0.19$$

The solution of equation (4) is given by

$$i_C = k_1 e^{-5.742t} + k_2 e^{-0.19t} \quad \text{..............(5)}$$

Put $t = 0^+$ in equation (5),

$$i_C(0^+) = k_1 + k_2 \quad \text{....................(6)}$$

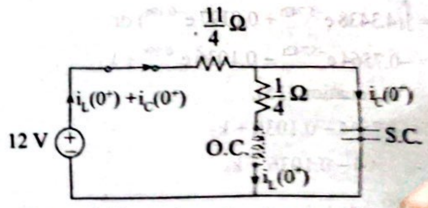



***Figure 2: Equivalent circuit at t = 0⁺.***

From figure 2,

$$i_C(0^+) = \frac{12}{\frac{11}{4}} = \frac{48}{11} \quad \text{.............(7)}$$

Also, $i_L(0^+) = 0$ ....................(8)

and $v_C(0^+) = 0$ ....................(9)

From equations (6) and (7),

$$k_1 + k_2 = \frac{48}{11} \quad \text{............(10)}$$

Differentiating equation (5) w.r.t. t, we get

$$\frac{di_C}{dt} = -5.742k_1 e^{-5.742t} - 0.19k_2 e^{-0.19t} \quad \text{.........(11)}$$

Put $t = 0^+$ in (11),

$$\frac{di_C(0^+)}{dt} = -5.742k_1 - 0.19k_2 \quad \text{..........(12)}$$

Put $t = 0^+$ in (3),

$$\frac{di_C(0^+)}{dt} = -24.946 \quad \text{..............(13)}$$

From equations (12) and (13),

$5.742k_1 + 0.19k_2 = 24.946$ ..........(14)

Solving equations (10) and (14), we get

$k_1 = 4.3438,\ k_2 = 0.0197$

Hence, equation (5) becomes

$i_C = 4.3438\, e^{-5.742t} + 0.0197\, e^{-0.19t}$

The voltage across capacitor is given by

$v_C = \frac{1}{1}\int i_C\, dt$

or, $v_C = \int (4.3438\, e^{-5.742t} + 0.0197\, e^{-0.19t})\, dt$

or, $v_C = -0.7564\, e^{-5.742t} - 0.1036\, e^{-0.19t} + k_3$ .............(15)

Put $t = 0^+$ in equation (15),

$v_C(0^+) = -0.7564 - 0.1036 + k_3$

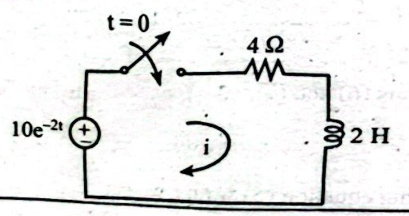
or, $0 = -0.7564 - 0.1036 + k_3$

or, $k_3 = 0.86$

Hence, equation (15) becomes

$v_C = -0.7564\, e^{-5.742t} - 0.1036\, e^{-0.19t} + 0.86$

**9. Find the expression for current and voltage of inductor for t>0 in the following circuit.**




***Solution:***

Applying KVL for $t > 0$,

$v_L + v_R = 10\, e^{-2t}$

or, $L\frac{di}{dt} + iR = 10\, e^{-2t}$

or, $2\frac{di}{dt} + 4i = 10\, e^{-2t}$

or, $\frac{di}{dt} + 2i = 5\, e^{-2t}$ .............(1)



The solution of equation (1) is given by

$i = \text{C.F.} + \text{P.I.}$

or, $i = i_N + i_f$ ............. (2)

To find $i_N$, the auxiliary equation of (1) is

$s + 2 = 0$

or, $s = -2$

$\therefore\ i_N = k\, e^{-2t}$ ............ (3)

To find $i_f$, let the trial solution for P.I. is

$i_f = At\, e^{-2t}$ ......... (4) [since $\alpha = \frac{R}{L}$ ]

From (1) and (4),

$\frac{d(At e^{-2t})}{dt} + 2(At\, e^{-2t}) = 5\, e^{-2t}$

or, $A[e^{-2t} - 2t\, e^{-2t}] + 2At\, e^{-2t}) = 5\, e^{-2t}$

or, $A = 5$

Thus, equation (4) becomes

$i_f = 5t\, e^{-2t}$ ............ (5)

From equations (2), (3), and (5),

$i = k\, e^{-2t} + 5t\, e^{-2t}$ .............(6)

Put $t = 0^+$ in equation (6),

$i(0^+) = k$ ................(7)

By observation at $t = 0^-$,

$i(0^-) = 0$

From continuity relation for inductor,

$i(0^+) = i(0^-) = 0$ ..............(8)

From equations (7) and (8),

$k = 0$

Equation (6) is then reduced to

$i = 5t\, e^{-2t}$

The voltage across inductor is given by

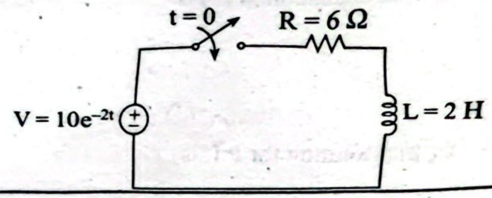
$v_L = L\frac{di}{dt}$

or, $v_L = 2\frac{d(5t\, e^{-2t})}{dt}$

or, $v_L = 2\times5(e^{-2t} - 2t\,e^{-2t})$

$\therefore v_L = 10e^{-2t} - 20t\,e^{-2t}$

**10. Find the expression for current and voltage of inductor for t>**




***Solution:***

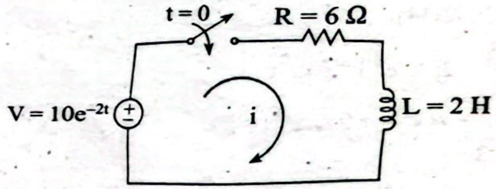



*Figure 1*

Applying KVL for $t > 0$,

$v_L + v_R = 10\,e^{-2t}$

or, $L\dfrac{di}{dt} + iR = 10\,e^{-2t}$

or, $2\dfrac{di}{dt} + i\times6 = 10\,e^{-2t}$

or, $\dfrac{di}{dt} + 3i = 5\,e^{-2t}$ ............ (1)

The solution of equation (1) is given by

$i = \text{C.F.} + \text{P.I.}$

or, $i = i_N + i_f$ ........... (2)

To find $i_N$, the A.E. of (1) is

$s + 3 = 0$

or, $s = -3$

$\therefore i_N = k\,e^{-3t}$ ............ (3)

To find $i_f$, let the trial solution be

$i_f = A\,e^{-2t}$ ......... (4) [ since $\alpha \neq \frac{R}{L}$ ]



From equations (1) and (4),

$\dfrac{d(A\,e^{-2t})}{dt} + 3(A\,e^{-2t}) = 5\,e^{-2t}$

or, $-2\,Ae^{-2t} + 3A\,e^{-2t} = 5\,e^{-2t}$

or, $A = 5$

Now, equation (4) becomes

$i_f = 5\,e^{-2t}$ ............(5)

From equations (2), (3), and (5),

$i = k\,e^{-3t} + 5\,e^{-2t}$ ................(6)

Put $t = 0^+$ in equation (6),

$i(0^+) = k + 5$

or, $k = i(0^+) - 5$ ................(7)

By observation at $t = 0^-$,

$i(0^-) = 0$

From continuity relation for inductor,

$i(0^+) = i(0^-) = 0$ ..............(8)

From equations (7) and (8),

$k = -5$

Equation (6) is then reduced to

$i = -5e^{-3t} + 5\,e^{-2t}$

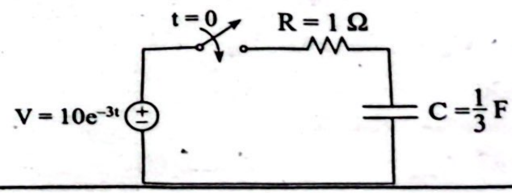
The voltage across inductor is given by

$v_L = L\dfrac{di}{dt}$

or, $v_L = 2\dfrac{d(-5e^{-3t} + 5\,e^{-2t})}{dt}$

$\therefore v_L = 30e^{-3t} - 20e^{-2t}$

**11. In the given figure, find the expression for voltage and current of capacitor for t > 0.**



***Solution:***

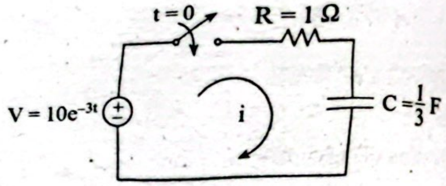



***Figure 1***

Applying KVL for $t > 0$,

$v_R + v_C = 10\,e^{-3t}$ ..................(a)

or, $1 \times i + \frac{1}{\left(\frac{1}{3}\right)}\int i\,dt = 10\,e^{-3t}$

or, $i + 3\int i\,dt = 10\,e^{-3t}$

Differentiating above equation w.r.t. t, we get

$\frac{di}{dt} + 3i = -30\,e^{-3t}$ ............... (1)

The solution of equation (1) is given by

$i = \text{C.F.} + \text{P.I.}$

or, $i = i_N + i_f$ ................. (2)

To find $i_N$, the A.E. of (1) is

$s + 3 = 0$

or, $s = -3$

$\therefore\ i_N = k\,e^{-3t}$ ............... (3)

To find $i_f$, let the trial solution of particular integral be

$i_f = At\,e^{-3t}$ ............ (4) [since $\alpha = \frac{1}{CR}$]

From equations (1) and (4),

$\frac{d(Ate^{-3t})}{dt} + 3(At\,e^{-3t}) = -30e^{-3t}$

or, $A[e^{-3t} - 3t\,e^{-3t}] + 3At\,e^{-3t} = -30e^{-3t}$

or, $A = -30$

Thus, equation (4) becomes

$i_f = -30t\,e^{-3t}$ ................ (5)

From equations (2), (3), and (5),



$i = k\,e^{-3t} - 30t\,e^{-3t}$ ..............(6)

Put $t = 0^+$ in equation (6),

$i(0^+) = k$ ................(7)

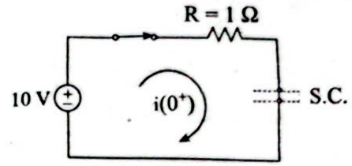



***Figure 2:** Equivalent circuit at $t = 0^+$.*

From equivalent circuit at $t = 0^+$,

$i(0^+) = \frac{10}{1} = 10$ ................(8)

From equations (7) and (8),

$k = 10$

Equation (6) is then reduced to

$i = 10\,e^{-3t} - 30t\,e^{-3t}$

The voltage across capacitor is given from equation (a),

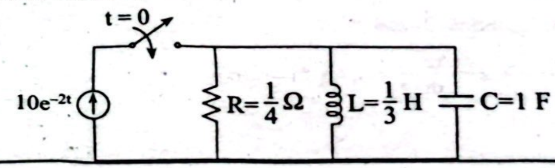
$v_C = 10\,e^{-3t} - v_R$

or, $v_C = 10\,e^{-3t} - iR$

or, $v_C = 10\,e^{-3t} - (10\,e^{-3t} - 30t\,e^{-3t})$

$\therefore\ v_C = 30t\,e^{-3t}$

**12. In the given figure, find the expression for voltage and current of capacitor for $t > 0$.**




***Solution:***

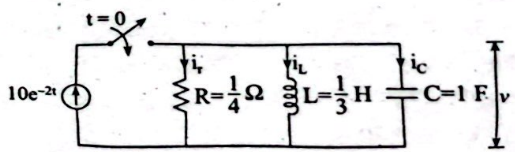



***Figure 1***

Applying KCL for $t > 0$,

$$i_C + i_L + i_r = 10\, e^{-2t}$$

$$\text{or, } C\frac{dv}{dt} + \frac{1}{L}\int v\, dt + \frac{v}{R} = 10\, e^{-2t}$$

$$\text{or, } \frac{dv}{dt} + 3\int v\, dt + 4v = 10\, e^{-2t} \qquad \text{.......... (1)}$$

Differentiating equation (1) w.r.t. t, we get

$$\frac{d^2v}{dt^2} + 3v + 4\frac{dv}{dt} = -20\, e^{-2t}$$

$$\text{or, } \frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 3v = -20\, e^{-2t} \qquad \text{............(2)}$$

Equation (2) is a second order nonhomogeneous differential equ
Its solution is given by

$$v = \text{C.F.} + \text{P.I.}$$

$$\text{or, } v = v_N + v_f \qquad \text{....................(3)}$$

To find $v_N$, the auxiliary equation of (2) is

$$s^2 + 4s + 3 = 0$$

$$s_1, s_2 = \frac{-4 \pm \sqrt{(4)^2 - 4\times3\times1}}{2}$$

$$= -3, -1$$

$$\text{Hence, } v_N = k_1e^{-3t} + k_2e^{-t} \qquad \text{...................(4)}$$

To find particular integral ($v_f$), let the trial solution for P.I. be

$$v_f = Ae^{-2t} \qquad \text{........................(5)}$$

From equations (2) and (5),

$$\frac{d^2(Ae^{-2t})}{dt^2} + 4\frac{d(Ae^{-2t})}{dt} + 3(Ae^{-2t}) = -20\, e^{-2t}$$

or, $A = 20$

Thus, equation (5) becomes

$$v_f = 20e^{-2t} \qquad \text{........................(6)}$$

From (3), (4), and (6),

$$v = k_1e^{-3t} + k_2e^{-t} + 20e^{-2t} \qquad \text{...................(7)}$$

Put $t = 0^+$ in (7),

$$v(0^+) = k_1 + k_2 + 20$$

$$\text{or, } k_1 + k_2 = v(0^+) - 20 \qquad \text{.....................(8)}$$



Differentiating equation (7) w.r.t. t, we get

$$\frac{dv}{dt} = -3k_1e^{-3t} - k_2e^{-t} - 40e^{-2t} \qquad \text{...............(9)}$$

Put $t = 0^+$ in equation (9),

$$\frac{dv(0^+)}{dt} = -3k_1 - k_2 - 40 \qquad \text{.................(10)}$$

Put $t = 0^+$ in equation (1),

$$\frac{dv(0^+)}{dt} + i_L(0^+) + 4v(0^+) = 10 \qquad \text{.................(11)}$$

By observation at $t = 0^-$,

$$i_L(0^-) = 0$$

$$v(0^-) = 0$$

From continuity relation for inductor and capacitor,

$$i_L(0^+) = i_L(0^-) = 0 \qquad \text{...............(12)}$$

$$v(0^+) = v(0^-) = 0 \qquad \text{...........(13)}$$

From equations (11), (12), and (13),

$$\frac{dv(0^+)}{dt} + 0 + 0 = 10$$

$$\text{or, } \frac{dv(0^+)}{dt} = 10 \qquad \text{................(14)}$$

From (10) and (14),

$$10 = -3k_1 - k_2 - 40$$

$$\text{or, } 3k_1 + k_2 = -50 \qquad \text{....................(15)}$$

From (13) and (8),

$$k_1 + k_2 = 0 - 20$$

$$\text{or, } k_1 + k_2 = -20 \qquad \text{.................(16)}$$

Solving (15) and (16), we get

$$k_1 = -15,\ k_2 = -5$$

Hence, equation (7) becomes

$$v = -15e^{-3t} - 5e^{-t} + 20e^{-2t} \qquad \text{...................(7)}$$

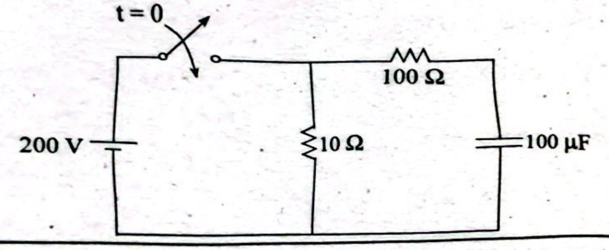
The current through capacitor is given by

$$i_c = C\frac{dv}{dt}$$

$$\text{or, } i_c = \frac{d(-15e^{-3t} - 5e^{-t} + 20e^{-2t})}{dt}$$

$$\therefore i_c = 45e^{-3t} + 5e^{-t} - 40e^{-2t}$$

**13. In the circuit shown below, if the switch is closed at $t = 0$, find expression for voltage across capacitor for $t > 0$ using classical method.**




[2074 A...

***Solution:***

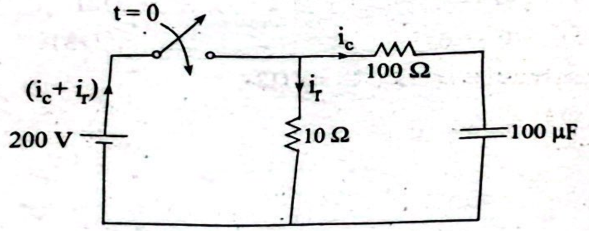



***Figure 1***

Applying KVL for $t > 0$ in left loop,

$200 = 10\, i_r$

or, $i_r = 20$ ..................(1)

Applying KVL for $t > 0$ in right loop,

$v_{100\Omega} + v_c - v_{10\Omega} = 0$ ................(A)

$$\text{or, } 100\, i_c + \frac{1}{100 \times 10^{-6}}\int i_c\, dt - 10\, i_r = 0$$

$$\text{or, } 100\, i_c + 10^4 \int i_c\, dt - 10 \times 20 = 0$$

Differentiating above equation w.r.t. t, we get

$$100\frac{di_c}{dt} + 10^4 i_c = 0$$

$$\text{or, } \frac{di_c}{dt} + 100\, i_c = 0 \quad \text{............(2)}$$



The solution of above differential equation is given by

$i_c = k\, e^{-100t}$ ..................(3)

Put $t = 0^+$ in equation (3),

$i_c(0^+) = k$ ............(4)

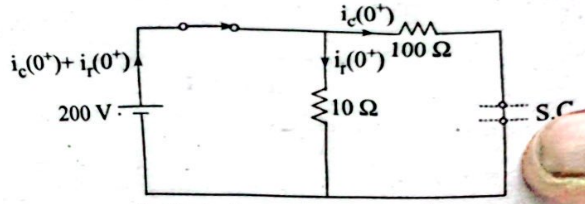



***Figure 2: Equivalent circuit at $t = 0^+$***

From figure 2,

$$i_c(0^+) = \frac{200}{100} = 2$$

From equations (4) and (5),

$k = 2$ ..................(6)

From equations (3) and (6),

$i_c = 2\, e^{-100t}$ ....................(7)

From equation (A),

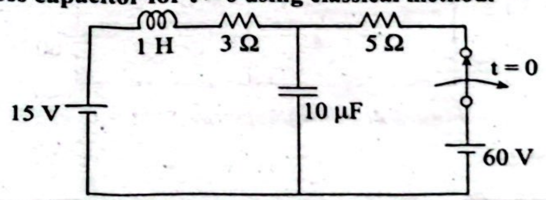
$v_c = v_{10\Omega} - v_{100\Omega}$

or, $v_c = 10 i_r - 100 i_c$

or, $v_c = 10 \times 20 - 100 \times 2\, e^{-100t}$

$\therefore\ v_c = 200 - 200\, e^{-100t}$

**14. Keeping the switch closed for a long time, the switch is opened at $t = 0$ in the circuit shown in figure. Find the expression for voltage across capacitor for $t > 0$ using classical method.**




[2074 Ashwin]

**Solution:**

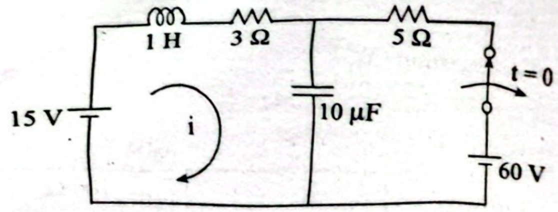



*Figure 1*

Applying KVL for $t > 0$ in figure 1,

$$v_L + v_{3\Omega} + v_c = 15$$

$$\text{or, } 1\frac{di}{dt} + 3i + \frac{1}{10 \times 10^{-6}}\int i\,dt = 15 \quad \text{.......................(1)}$$

Differentiating equation (1) w.r.t. t, we get

$$\frac{d^2i}{dt^2} + 3\frac{di}{dt} + 10^5\, i = 0 \quad \text{.......................(2)}$$

The characteristic equation of (2) is

$$s^2 + 3s + 10^5 = 0$$

$$\text{or, } s_1, s_2 = \frac{-3 \pm \sqrt{3^2 - 4 \times 10^5}}{2} = -1.5 \pm j316.22$$

The solution of equation (2) is

$$i = e^{-1.5t}[k_1 \cos 316.22t + k_2 \sin 316.22t]$$

Put $t = 0^+$ in equation (3),

$$i(0^+) = k_1 \quad \text{.............(4)}$$

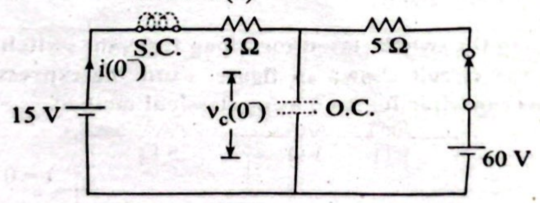



*Figure 2: Equivalent circuit at $t = 0^-$*

From figure 2,

$$i(0^-) = \frac{15 - 60}{3 + 5} = \frac{-45}{8}$$



Also, $v_c(0^-) - 5i(0^-) - 60 = 0$

$$\text{or, } v_c(0^-) - 5 \times \left(\frac{-45}{8}\right) - 60 = 0$$

$$\text{or, } v_c(0^-) = 31.875 \text{ V}$$

From continuity relation for inductor and capacitor,

$$i(0^+) = i(0^-) = \frac{-45}{8} \text{ A} \quad \text{............(5)}$$

$$v_c(0^+) = v_c(0^-) = 31.875 \text{ V} \quad \text{............(6)}$$

From equations (4) and (5),

$$k_1 = -\frac{45}{8} \quad \text{............(7)}$$

From equations (3) and (7),

$$i = e^{-1.5t}\left[-\frac{45}{8}\cos 316.22t + k_2 \sin 316.22t\right] \quad \text{..........(8)}$$

Differentiating equation (8) w.r.t. t, we get

$$\frac{di}{dt} = -\frac{45}{8}[-1.5e^{-1.5t}\cos 316.22t - 316.22 \sin 316.22t\, e^{-1.5t}]$$
$$+k_2[-1.5e^{-1.5t}\sin 316.22t + 316.22\, e^{-1.5t}\cos 316.22t] \quad \text{........(9)}$$

Put $t = 0^+$ in equation (9),

$$\frac{di(0^+)}{dt} = \frac{45 \times 1.5}{8} + k_2 \times 316.22$$

$$\text{or, } k_2 = -0.0266823 + \frac{1}{316.22}\frac{di(0^+)}{dt} \quad \text{.............(10)}$$

Put $t = 0^+$ in equation (1),

$$\frac{di(0^+)}{dt} + 3i(0^+) + v_c(0^+) = 15 \quad \text{.................(11)}$$

From equations (5), (6), and (11),

$$\frac{di(0^+)}{dt} + 3 \times \left(-\frac{45}{8}\right) + 31.875 = 15$$

$$\frac{di(0^+)}{dt} = 0 \quad \text{...........(12)}$$

From equations (10) and (12),

$$k_2 = -0.0266823 \quad \text{....................(13)}$$

From equations (8) and (13),

$$i = e^{-1.5t}\left[-\frac{45}{8}\cos 316.22t - 0.0266823 \sin 316.22t\right]$$

$$v_c(t) = \frac{1}{C}\int i(t)\,dt$$

$$v_c(t) = \frac{1}{10\times 10^{-6}}\int\left[\frac{-45}{8}e^{-1.5t}\cos 316.22t - 0.0266823e^{-1.5t}\sin 316.22t\right]$$

$$v_c(t) = -562500\int e^{-1.5t}\cos 316.22t - 2668.23\int e^{-1.5t}\sin 316.22t$$

$$v_c(t) = -562500\frac{e^{-1.5t}}{(-1.5)^2+(316.22)^2}(-1.5\ \cos 316.22t + 316.22\ \sin 316.22t) - 2668.23\frac{e^{-1.5t}}{(-1.5)^2+(316.22)^2}(-1.5\ \sin 316.22t - 316.22\cos 316.22t) + C$$

$$v_c(t) = 5.62514e^{-1.5t}(-1.5\cos 316.22t + 316.22\sin 316.22t) - 0.0266e^{-1.5t}(-1.5\sin 316.22t - 316.22\cos 316.22t) + C \quad \text{.........(14)}$$

Put $t = 0^+$ in equation (14),

$$v_c(0^+) = -5.62514(-1.5) - 0.0266(-316.22) + C$$

From equation (6),

$$31.875 = 8.43771 + 8.411452 + C$$

or, $C = 15.025$

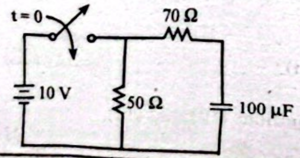
Then, equation (14) becomes

$$v_c(t) = [-5.62514e^{-1.5t}(-1.5\cos 316.22t + 316.22\sin 316.22t) - 0.0266e^{-1.5t}(-1.5\sin 316.22t - 316.22\cos 316.22t) + 15.025]$$

$$\text{or, } v_c(t) = 8.437e^{-1.5t}\cos 316.22t - 1778.78e^{-1.5t}\sin 316.2t + 0.0399e^{-1.5t}\sin 316.22t + 8.411452\, e^{-1.5t}\cos 316.22t + 15.025$$

$$\therefore v_c(t) = 16.8484e^{-1.5t}\cos 316.22t - 1778.7401e^{-1.5t}\sin 316.22t + 15.025$$

**15. In figure below, if the switch is closed at t = 0, find the time when the current from the battery reaches to 500 mA. Use classical method.**




[207...]



***Solution:***

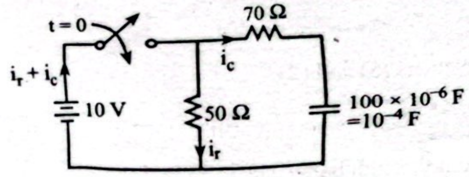



***Figure 1***

Applying KVL for $t > 0$ in left loop,

$$10 = v_{50\Omega}$$

or, $50 i_r = 10$

or, $i_r = \frac{1}{5}$ .............(1)

Applying KVL for $t > 0$ in outer loop,

$$10 = v_{70\Omega} + v_c$$

$$\text{or, } 10 = 70 i_c + \frac{1}{10^{-4}}\int i_c\,dt$$

Differentiating above equation w.r.t. t, we get

$$70\frac{di_c}{dt} + 10^4 i_c = 0$$

$$\text{or, } \frac{di_c}{dt} + 142.85 i_c = 0$$

The solution of above differential equation is given by

$$i_c = k\, e^{-142.85t} \quad \text{...............(2)}$$

Put $t = 0^+$ in equation (2),

$$i_c(0^+) = k \quad \text{.............(3)}$$

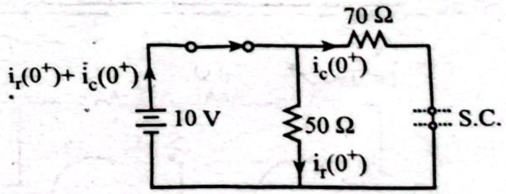



***Figure 2: Equivalent circuit at $t = 0^+$***

From figure 2,

$$i_c(0^+) = \frac{10}{70} = \frac{1}{7} \quad \text{...............(4)}$$

From equations (3) and (4),

$$k = \frac{1}{7} \quad \text{..............(5)}$$

From equations (5) and (2),

$$i_c = \frac{1}{7}\, e^{-142.85t} \quad \text{..............(6)}$$

The current through battery is

$$i_b = i_c + i_r$$

$$\text{or, } i_b = \frac{1}{7} e^{-142.85t} + \frac{1}{5} \quad \text{..............(7)}$$

To find the time to reach the battery current $500 \times 10^{-3}$A, we write

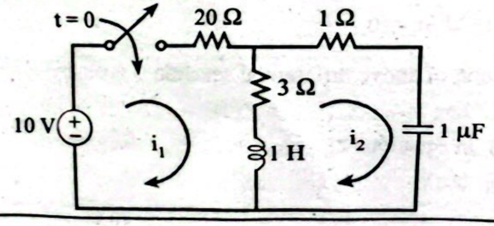
$$500 \times 10^{-3} = \frac{1}{7} e^{-142.85t} + \frac{1}{5}$$

$$\text{or, } \frac{1}{7} e^{-142.85t} = 0.5 - \frac{1}{5} = \frac{2.5 - 1}{5}$$

$$\text{or, } -142.85t = \ln 2.1$$

$$\therefore t = -\frac{\ln 2.1}{142.85} = -5.19 \times 10^{-3} \text{ sec}$$

**16. Find the time expression for loop currents i1 and i2 for t > 0 in the circuit using classical method.**




[2074 Chait]

***Solution:***

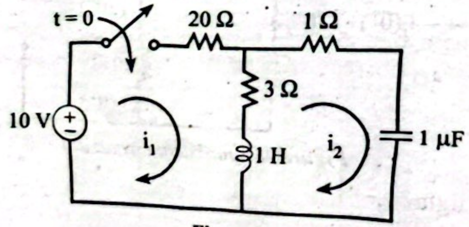



*Figure 1*



Applying KVL for t > 0 in loop 1,

$$10 = v_{20\Omega} + v_{3\Omega} + v_L$$

$$\text{or, } 20i_1 + 3(i_1 - i_2) + 1\frac{d}{dt}(i_1 - i_2) = 10$$

$$\text{or, } 23i_1 - 3i_2 + \frac{d}{dt}(i_1 - i_2) = 10 \quad \text{..............(1)}$$

Applying KVL for t > 0 in loop 2,

$$v_{1\Omega} + v_c + v_L + v_{3\Omega} = 0$$

$$\text{or, } i_2 + \frac{1}{10^{-6}}\int i_2\, dt + \frac{d}{dt}(i_2 - i_1) + 3(i_2 - i_1) = 0 \quad \text{..............(2)}$$

Adding equations (2) and (1),

$$20i_1 + i_2 + 10^6 \int i_2\, dt = 10$$

$$\text{or, } i_1 = \frac{1}{2} - \frac{1}{20} i_2 - 5 \times 10^4 \int i_2\, dt \quad \text{..............(3)}$$

From equations (1) and (3),

$$23\left(\frac{1}{2} - \frac{1}{20} i_2 - 5 \times 10^4 \int i_2 dt\right) - 3i_2 + \left(0 - \frac{1}{20}\frac{di_2}{dt} - 5 \times 10^4 i_2\right) - \frac{di_2}{dt} = 0$$

$$\text{..................... (A)}$$

Differentiating above equation w.r.t. t, we get

$$0 - \frac{23}{20}\frac{di_2}{dt} - 5 \times 10^4\, i_2 - 3\frac{di_2}{dt} - \frac{1}{20}\frac{d^2 i_2}{dt^2} - 5 \times 10^4 \frac{di_2}{dt} - \frac{d^2 i_2}{dt^2} = 0$$

$$\text{or, } -\frac{21}{20}\frac{d^2 i_2}{dt^2} - 50004.15\frac{di_2}{dt} - 5 \times 10^4\, i_2 = 0$$

$$\text{or, } \frac{d^2 i_2}{dt^2} + 476194.42\frac{di_2}{dt} + 47619.04 i_2 = 0 \quad \text{..............(4)}$$

The auxiliary equation of (4) is

$$s^2 + 476194.42s + 47619.04 = 0$$

$$\therefore s_1, s_2 = \frac{-476194.42 \pm \sqrt{(476194.42)^2 - 4 \times 47619.04}}{2}$$

$$= \frac{-476194.42 \pm 476194.22}{2} = -476194.22, -0.11$$

The solution of equation (4) is given by

$$i_2 = k_1 e^{-476194.22t} + k_2 e^{-0.11t} \quad \text{.............(5)}$$

Put $t = 0^+$ in (5),

$i_2(0^+) = k_1 + k_2$ .......... (6)

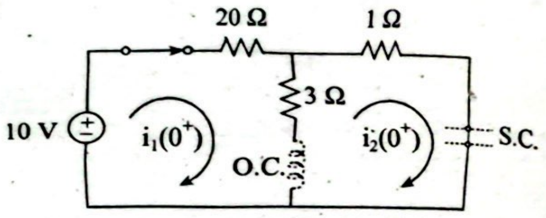



*Figure 2: Equivalent circuit at $t = 0^+$*

From figure 2,

$$i_1(0^+) = i_2(0^+) = \frac{10}{20+1} = \frac{10}{21} \quad \text{............. (7)}$$

From (6) and (7),

$$k_1 + k_2 = \frac{10}{21} \quad \text{............. (8)}$$

Differentiating equation (5) w.r.t. t, we get

$$\frac{di_2}{dt} = -476194.22k_1e^{-476194.22t} - 0.11k_2e^{-0.11t} \quad \text{......... (9)}$$

Put $t = 0^+$ in (9),

$$\frac{di_2(0^+)}{dt} = -476194.22k_1 - 0.11k_2 \quad \text{.......... (10)}$$

Put $t = 0^+$ in equation (A),

$$23\left[\frac{1}{2} - \frac{1}{20}i_2(0^+) - \frac{1}{20}v_c(0^+)\right] - 3i_2(0^+)\left[-\frac{1}{20}\frac{di_2(0^+)}{dt} - 5\times10^4 i_2(0\right.$$

$$\frac{di_2(0^+)}{dt} = 0$$

We have, $v_c(0^+) = v_c(0^-) = 0$ and $i_2(0^+) = \frac{10}{21}$

So, $\frac{di_2(0^+)}{dt} = 39695.31$

Putting the value of $\frac{di_2(0^+)}{dt}$ in equation (10), we get

$$39695.31 = -476194.22k_1 - 0.11k_2 \quad \text{..........(B)}$$



From equations (B) and (8), we have

$k_1 = -0.083,\ k_2 = 0.559$

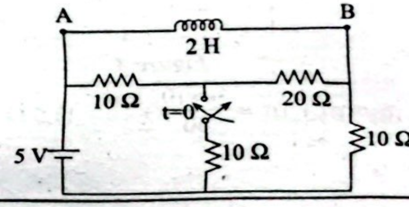
From equation (5),

$$i_2 = -0.083e^{-476194.22t} + 0.559\,e^{-0.11t}$$

Putting $i_2$ in equation (3), we get

$$i_2 = 0.5 - 4.56\times10^{-3}e^{-476194.22t} + 254090.88\,e^{-0.11t}$$

**17. In the circuit, find the current in the inductor for t > 0. Use classical method.**




[2072 Chaitra]

***Solution:***

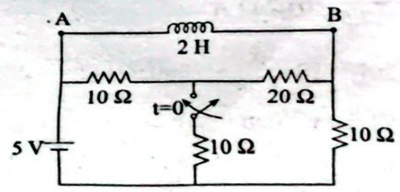



*Figure 1*

The Thevenin's equivalent circuit for figure 1 is given in figure 2.

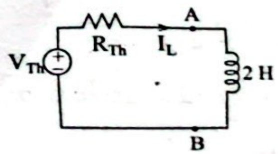
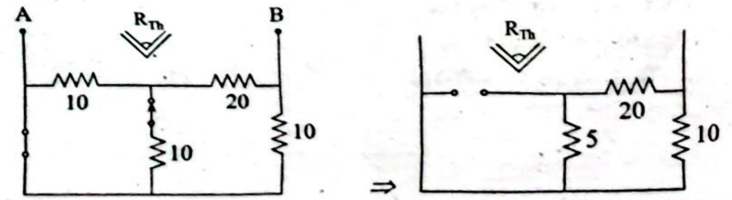



*Figure 2*

For finding $R_{Th}$,



$$R_{Th} = 25//10 = \frac{25\times10}{25+10} = \frac{250}{35} = 7.14\ \Omega$$

**For finding $V_{Th}$,**

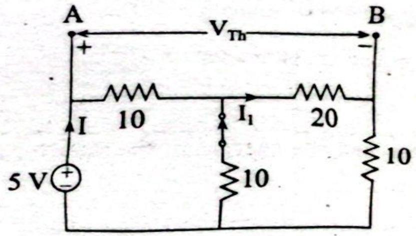



*Figure 3*

$$R_T = [(20+10)//10] + 10 = \frac{30\times10}{40} + 10 = 17.5\ \Omega$$

$$I = \frac{5}{17.5}\ \text{A}$$

$$I_1 = \frac{10}{10+30} \times \frac{5}{17.5} = 0.0742\ \text{A}$$

$$V_{Th} = 10I + 20I_1 = 10\times\frac{5}{17.5} + 20\times0.07142 = 4.285\ \text{V}$$

Applying KVL for $t > 0$ in figure 2,

$$V_{Th} = i_L R_{Th} + 2\frac{di_L}{dt}$$

$$\text{or, } 2\frac{di_L}{dt} + 7.14 i_L = 4.285$$

$$\text{or, } \frac{di_L}{dt} + 3.57 i_L = 2.1425 \quad \text{............ (1)}$$

The solution of above differential equation is

$$I_L = \frac{2.1425}{3.57} + ke^{-3.57t}$$

$$\text{or, } i_L = 0.60014 + ke^{-3.57t} \quad \text{............ (2)}$$

Put $t = 0^+$ in equation (2),

$$i_L(0^+) = 0.60014 + k$$

$$\text{or, } k = i_L(0^+) - 0.60014 \quad \text{........... (3)}$$



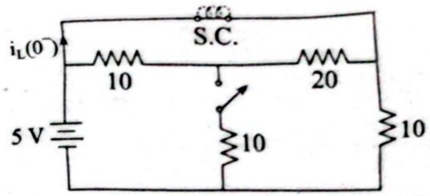



*Figure 4: Equivalent circuit at $t = 0^-$*

From figure 4,

$$i_L(0^-) = \frac{5}{10} = 0.5\ \text{A}$$

$$\therefore i_L(0^+) = i_L(0^-) = 0.5 \quad \text{............. (4)}$$

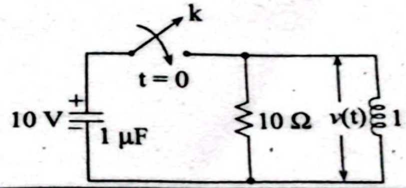
From equations (3) and (4),

$$k = 0.5 - 0.60014 = -0.10014$$

From equation (2),

$$i_L = 0.60014 - 0.10014e^{-3.57t}$$

**18. In the circuit shown in figure below, capacitor C has an initial voltage $V_c$ = 10 volts and at the same instant, current through the inductor L is zero. The switch k is closed at t = 0. Find out the expression for the voltage v(t) across the inductor L using classical method.**




[2072 Chaitra]

***Solution:***

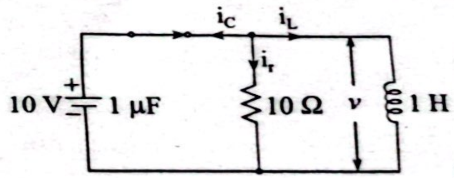



*Figure 1*

Applying KCL for $t > 0$,

$$i_c + i_r + i_L = 0$$

$$\text{or, } 1\times10^{-6}\frac{dv}{dt} + \frac{v}{10} + \frac{1}{1}\int v\,dt = 0 \quad \text{................. (1)}$$

Differentiating w.r.t. time, we get

$$10^{-6}\frac{d^2v}{dt^2} + \frac{1}{10}\frac{dv}{dt} + v = 0$$

$$\text{or, } \frac{d^2v}{dt^2} + 10^5\frac{dv}{dt} + 10^6 v = 0 \quad \text{.................. (2)}$$

Auxiliary equation of (2) is given by

$$s^2 + 10^5 s + 10 = 0$$

$$s_1, s_2 = \frac{-10^5 \pm \sqrt{(10^5)^2 - 4\times 10^6}}{2} = -10.0001, -99989.99$$

$$\therefore v = k_1 e^{-10.001t} + k_2 e^{-99989.99t} \quad \text{.............. (8)}$$

Put $t = 0^+$ in equation (3),

$$v(0^+) = k_1 + k_2 \quad \text{................ (4)}$$

$$v(0^-) = 10$$

$$v(0^+) = v(0^-) = 10 \quad \text{................ (5)}$$

Equation (4) now becomes

$$k_1 + k_2 = 10 \quad \text{.................. (6)}$$

Differentiating equation (3) w.r.t. time,

$$\frac{dv}{dt} = -10.001\, k_1 e^{-10.001t} - 99989.99\, k_2 e^{-99989.99t} \quad \text{...............(7)}$$

Put $t = 0^+$ in equation (7),

$$\frac{dv(0^+)}{dt} = -10.001k_1 - 99989.99k_2 \quad \text{................ (8)}$$

Put $t = 0^+$ in equation (1),

$$10^{-6}\frac{dv(0^+)}{dt} + \frac{v(0^+)}{10} + i_L(0^+) = 0$$

$$\text{or, } 10^{-6}\frac{dv(0^+)}{dt} + \frac{10}{10} + 0 = 0$$

$$\text{or, } \frac{dv(0^+)}{dt} = \frac{-1}{10^{-6}} = -10^6 \quad \text{.............. (9)}$$

From equations (8) and (9),

$$-10^6 = -10.001k_1 - 99989.99k_2$$

$$\text{or, } +10.001k_1 + 99989.99k_2 = 10^6$$

$$\text{or, } 10.001(10 - k_2) + 99989.99k_2 = 10^6$$



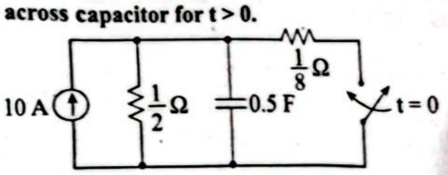
$$\text{or, } 99979.989k_2 = 999900$$

$$\text{or, } k_2 = 10.001$$

$$\therefore k_1 = 10 - 10.001 = -0.001$$

$$\text{Hence, } v = 0.001e^{-10.001t} + 10.001e^{-99989.99t}$$

**19. Using classical method in the circuit shown in figure below, find the voltage across capacitor for t > 0.**




***Solution:***

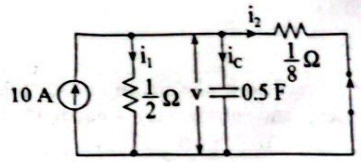



***Figure 1***

Applying KCL for $t > 0$,

$$i_1 + i_2 + i_c = 10$$

$$\text{or, } \frac{v}{\frac{1}{2}} + \frac{v}{\frac{1}{8}} + 0.5\frac{dv}{dt} = 10$$

$$\text{or, } 10v + \frac{1}{2}\frac{dv}{dt} = 20$$

$$\text{or, } \frac{dv}{dt} + 20v = 20 \quad \text{..............(1)}$$

The solution of above differential equation is

$$v = \frac{20}{20} + ke^{-20t}$$

$$\text{or, } v = 1 + ke^{-20t} \quad \text{.......... (2)}$$

To find k, we put $t = 0^+$ in equation (2),

$$v(0^+) = 1 + k$$

$$\text{or, } k = v(0^+) - 1 \quad \text{.............. (3)}$$

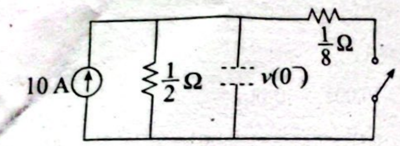



*Figure 2: Equivalent circuit at $t = 0^-$.*

$$v(0^-) = v_{1/2\Omega}(0^-) = \frac{1}{2} \times 10 = 5\text{ V}$$

From continuity relation for capacitor,

$$v(0^+) = v(0^-) = 5 \qquad \text{............ (4)}$$

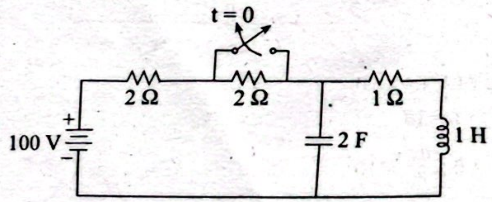
From (3) and (4),

$$k = 5 - 1 = 4$$

From equation (2), we get

$$v = 1 + 4e^{-20t}$$

**20. Using classical method in the circuit shown below, find the current through inductor and voltage across capacitor for t > 0.**




[2073 Shr...]

***Solution:***

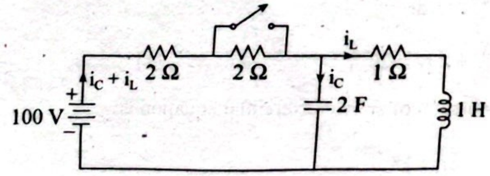



*Figure 1*

KVL in left loop for t > 0 gives

$$(2 + 2)(i_C + i_L) + \frac{1}{2}\int i_C\,dt = 100$$

$$\text{or, } 4(i_C + i_L) + \frac{1}{2}\int i_C\,dt = 100 \qquad \text{............ (1)}$$



KVL for t > 0 in outer loop gives

$$(2+2)(i_C + i_L) + i_L + 1\frac{di_L}{dt} = 100$$

$$\text{or, } 4(i_C + i_L) + i_L + \frac{di_L}{dt} = 100$$

$$\text{or, } 4i_C + 5i_L + \frac{di_L}{dt} = 100 \qquad \text{.............. (2)}$$

From equation (1),

$$i_L = \frac{100}{4} - i_C - \frac{1}{8}\int i_C\,dt \qquad \text{................ (3)}$$

From (2) and (3),

$$4i_C + 5\left[\frac{100}{4} - i_C - \frac{1}{8}\int i_C\,dt\right] + 0 - \frac{di_C}{dt} - \frac{1}{8}i_C = 100$$

$$\text{or, } -i_C - \frac{5}{8}\int i_C\,dt - \frac{di_C}{dt} - \frac{1}{8}i_C = 100 - 125$$

$$\text{or, } -\frac{9}{8}i_C - \frac{5}{8}\int i_C\,dt - \frac{di_C}{dt} = -25$$

Differentiating above equation w.r.t. t, we get

$$\frac{d^2i_C}{dt^2} + \frac{9}{8}\frac{di_C}{dt} + \frac{5}{8}i_C = 0 \qquad \text{............ (4)}$$

A.E. of (4) is

$$s^2 + \frac{9}{8}s + \frac{5}{8} = 0$$

$$\text{or, } s^2 + 1.125s + 0.625 = 0$$

$$s_1, s_2 = \frac{-1.125 \pm \sqrt{(1.125)^2 - 4 \times 0.625}}{2}$$

$$= \frac{-1.125 \pm j\sqrt{1.234375}}{2}$$

$$= \frac{-1.125 \pm j1.11024}{2} = -0.5625 \pm j0.5555$$

$$\therefore i_C = e^{-0.5625t}(k_1 \cos 0.5555t + k_2 \sin 0.5555t) \qquad \text{............(5)}$$

Put $t = 0^+$ in (5),

$$i_C(0^+) = k_1 \qquad \text{............ (6)}$$

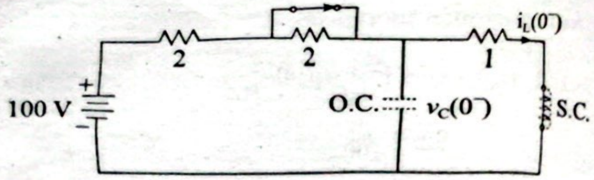



*Figure 2: Equivalent circuit at $t = 0^-$*

From figure 2,

$$i_L(0^-) = \frac{100}{2+1} = \frac{100}{3}\,A$$

$$v_C(0^-) = v_{1\Omega}(0^-) = \frac{100}{3} \times 1 = \frac{100}{3}\,V$$

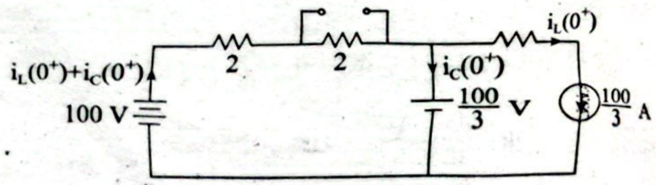



*Figure 3: Equivalent circuit at $t = 0^+$*

From figure (3),

$$100 - 4\,[i_C(0^+) + i_L(0^+)] - \frac{100}{3} = 0$$

or, $-4i_C(0^+) - 4\times\frac{100}{3} = -\frac{200}{3}$

or, $-4i_C(0^+) = \frac{200}{3}$ ............... (7)

From (6) and (7),

$k_1 = -\frac{50}{3}$ ........................ (8)

Differentiating equatioh (5) w.r.t. t and then putting $t = 0^+$, we get

$\frac{di_C}{dt} = k_1 \cos(0.5555t)\ e^{-0.5625t}\ (-0.5625) + k_2\ e^{-0.5625t}\ \cos\ (0.555$
$(0.5555)$

or, $\frac{di_C(0^+)}{dt} = k_1(-0.5625) + k_2(0.5555)$

or, $\frac{di_C(0^+)}{dt} = \frac{-50}{3}\times(-0.5625) + k_2\times0.5555$

or, $\frac{di_C(0^+)}{dt} = \frac{75}{8} + k_2\times0.5555$ ............. (9)



Put $t = 0^+$ in equation (A),

$$\frac{9}{8}i_C(0^+) + \frac{5}{4}v_C(0^+) + \frac{di_C(0^+)}{dt} = 25$$

or, $\frac{9}{8}\times\left(\frac{-50}{3}\right) + \frac{5}{4}\times\frac{100}{3} + \frac{di_C(0^+)}{dt} = 25$

or, $\frac{di_C(0^+)}{dt} = \frac{25}{12} = 2.083$ ............. (10)

From (9) and (10),

$$2.083 = \frac{75}{8} + k_2\times0.5555$$

or, $k_2 = -13.12$ ................ (11)

From (5), (8), and (11),

$$i_C = e^{-0.5625t}\left[\frac{-50}{3}\cos 0.5555t - 13.12\sin 0.5555t\right] \text{......}$$

$$v_C(t) = \frac{1}{C}\int i_C(t)\,dt$$

$$= \frac{1}{2}\int\left(\frac{-50}{3}\right)e^{-0.5625t}\cos 0.5555t\,dt - \frac{1}{2}\int(13.12)\,e^{-0.5625t}\sin 0.5555t\,dt$$

$$= \frac{-25}{3}\left[\frac{e^{-0.5625t}}{0.3164 + 0.3085}(-0.5625\cos0.5555t + 0.5555\sin 0.5555t)\right]$$

$$-\frac{13.12}{2}\left[\frac{e^{-0.5625t}}{0.3164 + 0.3085}(-0.5625\sin 0.5555t - 0.5555\cos 0.5555t)\right] + k$$

$$= \frac{e^{-0.5625t}}{0.6249}\,[8.3315\cos0.5555t - 0.939\sin0.5555t] + k \quad \text{.........(13)}$$

Put $t = 0^+$ in equation (13),

$$\frac{100}{3} = 8.3315 \times \frac{1}{0.6249} + k$$

or, $k = 20$

From equation (13),

$$v_C(t) = \frac{e^{-0.5625t}}{0.6249}\,[8.3315\cos0.5555t - 0.939\sin0.5555t] + 20$$

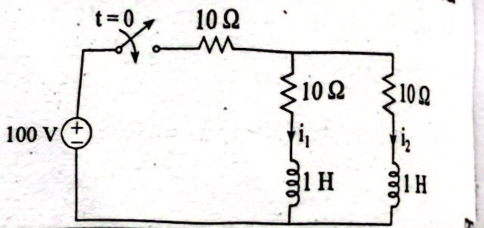
Similarly, from (1)

$i_L(t) = 25 - i_C(t) - 0.25v_C(t)$

or, $i_L(t) = 25 - e^{-0.5625t}[-50/3 \cos 0.5555t - 13.12 \sin 0.5555t] - \dfrac{e^{-0.5625t}}{0.6249}[8.3315 \cos 0.5555t - 0.939 \sin 0.5555t] + 20$

$i_L(t) = 45 - e^{-0.5625t}(-13.33 \cos 0.5555t - 13.49 \sin 0.5555t)$

**21. In the circuit shown, the switch is closed at $t = 0$ with the previously unenergized. For the element values shown in diagram, find $i_1(t)$ and $i_2(t)$ for $t > 0$ by classical method.**




***Solution:***

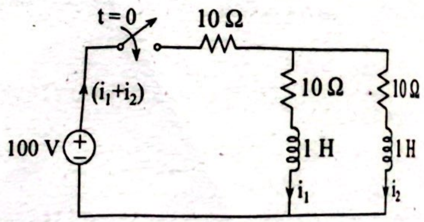



***Figure 1***

Applying KVL for $t > 0$ in left loop,

$$100 - 10(i_1+i_2) - 10i_1 - 1\frac{di_1}{dt} = 0$$

$$\text{or, } \frac{di_1}{dt} + 20i_1 + 10i_2 = 100 \qquad \text{.......... (1)}$$

Applying KVL for $t > 0$ in outer loop,

$$100 - 10(i_1+i_2) - 10i_2 - 1\frac{di_2}{dt} = 0$$

$$\text{or, } \frac{di_2}{dt} + 20i_2 + 10i_1 = 100$$



$$\text{or, } i_1 = 10 - 2i_2 - \frac{1}{10}\frac{di_2}{dt} \qquad \text{.......... (2)}$$

From equations (1) and (2), we get

$$\left[0 - 2\frac{di_2}{dt} - \frac{1}{10}\frac{d^2i_2}{dt^2}\right] + 20\left[10 - 2i_2 - \frac{1}{10}\frac{di_2}{dt}\right] + 10i_2 = 100$$

$$\text{or, } -\frac{1}{10}\frac{d^2i_2}{dt^2} - 4\frac{di_2}{dt} - 30i_2 = -100$$

$$\text{or, } \frac{d^2i_2}{dt^2} + 40\frac{di_2}{dt} + 300i_2 = 1000 \qquad \text{................ (3)}$$

The solution of equation (3) is

$$i_2 = \text{P.I.} + \text{C.F.}$$

$$\text{or, } i_2 = i_{2f} + i_{2N} \qquad \text{.............. (4)}$$

$$i_{2f} = \frac{1000}{300} = \frac{10}{3} \qquad \text{............. (5)}$$

To find C.F., A.E. of (3) is

$$s^2 + 40s + 300 = 0$$

$$\text{or, } s_1, s_2 = -10, -30$$

$$\therefore\ i_{2N} = k_1e^{-10t} + k_2e^{-30t} \qquad \text{.............. (6)}$$

From equations (4), (5), (6),

$$i_2 = \frac{10}{3} + k_1e^{-10t} + k_2e^{-30t} \qquad \text{.............. (7)}$$

Put $t = 0^+$ in (7),

$$i_2(0^+) = \frac{10}{3} + k_1 + k_2$$

$$\text{or, } k_1 + k_2 = i_2(0^+) - \frac{10}{3} \qquad \text{.............. (8)}$$

From observation at $t = 0^-$,

$$i_2(0^-) = 0,\ i_1(0^-) = 0$$

From continuity relation for inductor,

$$i_2(0^+) = i_2(0^-) = 0 \qquad \text{............. (9)}$$

Also, $i_1(0^+) = i_1(0^-) = 0$

From equations (8) and (9),

$$k_1 + k_2 = -\frac{10}{3} \quad \text{................ (10)}$$

Differentiating equation (7) w.r.t. t, we get

$$\frac{di_2}{dt} = 0 - 10k_1e^{-10t} - 30k_2e^{-30t} \quad \text{................ (11)}$$

Put $t = 0^+$ in equation (11),

$$\frac{di_2(0^+)}{dt} = -10k_1 - 30k_2 \quad \text{........... (12)}$$

Put $t = 0^+$ in equation (2),

$$i_1(0^+) = 10 - 2\, i_2(0^+) - \frac{1}{10}\frac{di_2(0^+)}{dt}$$

$$\text{or, } 0 = 10 - 2 \times 0 - \frac{1}{10}\frac{di_2(0^+)}{dt}$$

$$\therefore \frac{di_2(0^+)}{dt} = 100 \quad \text{................ (13)}$$

From equations (12) and (13),

$10k_1 + 30k_2 = -100$ ................ (14)

Solving equations (10), (14), we get

$k_1 = 0,\ k_2 = -3.333$

Therefore, equation (7) becomes

$$i_2 = \frac{10}{3} - 3.333\, e^{-30t} \quad \text{............. (15)}$$

From equations (2) and (15),

$$i_1 = 10 - 2\left(\frac{10}{3} - 3.333e^{-30t}\right) - \frac{1}{10}(0 + 99.9e^{-30t})$$

$$\therefore i_1 = \frac{10}{3} - 3.333e^{-30t}$$

**Alternative method:**

Since the impedances in two branches are same, equal current flows through them.

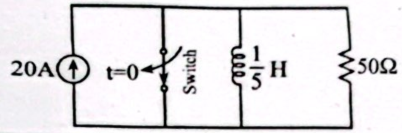
i.e., $i_1 = i_2$

Total current (i) = $i_1 + i_2 = 2i_1$

Applying KVL in left loop will give the answer.



**22. Find the expression of inductor current for t > 0 using classical method.**




[2078 Chaitra]

***Solution:***

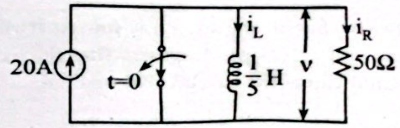



***Figure 1***

Applying KCL for $t > 0$,

$i_L + i_R = 20$ .......... (A)

$$\text{or, } \frac{1}{L}\int v\, dt + \frac{v}{R} = 20$$

$$\text{or, } \frac{1}{1/5}\int v\, dt + \frac{v}{50} = 20$$

$$\text{or, } 5\int v\, dt + \frac{v}{50} = 20$$

Differentiating, we get

$$5v + \frac{1}{50}\frac{dv}{dt} = 0$$

$$\text{or, } \frac{dv}{dt} + 250v = 0 \quad \text{...... (1)}$$

Solving equation (1), we get

$\therefore\ v = ke^{-250t}$ ........ (2)

Put $t = 0^+$ in (2),

$v(0^+) = k$ ........ (3)

As $i_L(0^-) = 20$ A,

$v(0^+) = 20 \times 50 = 1000$ V

$\therefore\ k = 1000$ ........... (4)

From equations (2) & (4),

$v = 1000\, e^{-250t}$ ..... (5)

From equation (A),

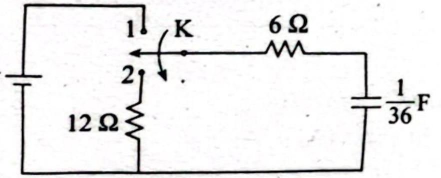
$i_L = 20 - i_R$

or, $i_L = 20 - \frac{v}{R}$

or, $i_L = 20 - \frac{1000e^{-250t}}{50}$

$\therefore\ i_L = 20 - 20e^{-250t}$

**23. In the network shown, the switch is moved from 1 to 2 at t = 0, f... the element value given on diagram, find the expression for volta... and current of capacitor by classical method.**




[2079 Ashw...]

***Solution:***

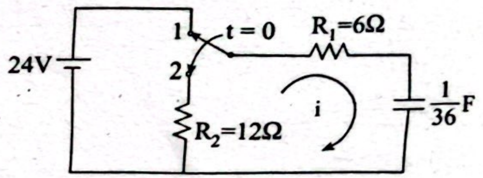



***Figure 1***

Apply KVL for $t > 0$,

$v_{R_1} + v_C + v_{R_2} = 0$ ....... (A)

or, $iR_1 + \frac{1}{C}\int i\,dt + iR_2 = 0$

or, $6i + \frac{1}{1/36}\int i\,dt + 12i = 0$

or, $18i + 36\int i\,dt = 0$

Differentiating, we get

$18\frac{di}{dt} + 36i = 0$

or, $\frac{di}{dt} + 2i = 0$ ...... (1)



Solving equation (1), we get

$i = ke^{-2t}$ ...... (2)

Put $t = 0^+$ in equation (2),

$i(0^+) = k$ ....... (3)

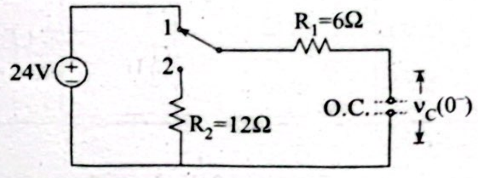



***Figure 2:** Equivalent circuit at t = 0⁻*

From figure 2, $v_C(0^-) = 24$ V

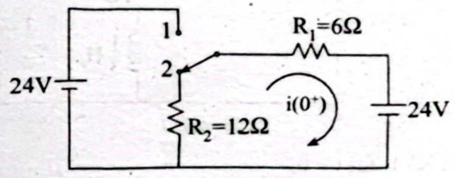



***Figure 3:** Equivalent circuit at t = 0⁺*

From figure 3,

$-12i(0^+) - 6i(0^+) - 24 = 0$

or, $i(0^+) = -\frac{24}{18} = -\frac{4}{3}$ A ....... (4)

From equations (4) & (3),

$k = -\frac{4}{3}$ ......... (5)

From equations (5) & (2),

$i = -\frac{4}{3}e^{-2t}$ ......... (6)

From equation (A),

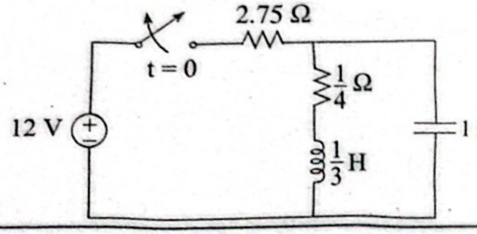
$v_C = -v_{R_1} - v_{R_2}$

or, $v_C = -6i - 12i = -18i$

or, $v_C = -18\left[-\frac{4}{3}e^{-2t}\right]$

$\therefore\ v_C = 24e^{-2t}$

**24. For the given circuit, find the expression for current and voltag[e] inductor for t > 0.**




[2079 Ch

***Solution:***

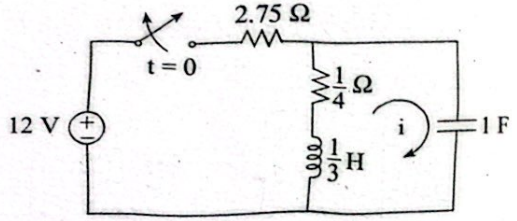



***Figure 1***

Applying KVL for t > 0,

$$v_R + v_L + v_C = 0$$

$$\text{or, } iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0$$

$$\text{or, } i\times\frac{1}{4} + \frac{1}{3}\frac{di}{dt} + \frac{1}{1}\int i\,dt = 0$$

$$\text{or, } \frac{1}{4}i + \frac{1}{3}\frac{di}{dt} + \int i\,dt = 0 \qquad \text{........ (1)}$$

Differentiating equation (1) w.r.t. t, we get

$$\frac{1}{3}\frac{d^2i}{dt^2} + \frac{1}{4}\frac{di}{dt} + i = 0$$

$$\text{or, } \frac{d^2i}{dt^2} + \frac{3}{4}\frac{di}{dt} + 3i = 0 \qquad \text{........ (2)}$$

The auxiliary equation of (2) is

$$s^2 + \frac{3}{4}s + 3 = 0$$

$$s_1, s_2 = \frac{-\frac{3}{4} \pm \sqrt{\left(\frac{3}{4}\right)^2 - 4\times1\times3}}{2} = -0.375 \pm j\,1.69$$

Hence, the solution of equation (2) is given by

$$i = e^{-0.375t}(k_1\cos 1.69t + k_2\sin 1.69t) \qquad \text{.......... (3)}$$

Put $t = 0^+$ in equation (3),

$$i(0^+) = k_1 \qquad \text{...........(4)}$$

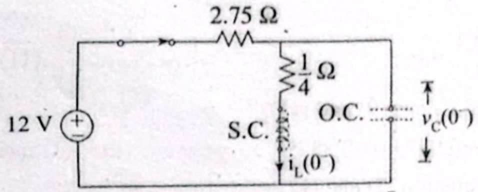



***Figure 2: Equivalent circuit at $t = 0^-$.***

From figure 2,

$$i_L(0^-) = \frac{12}{2.75 + \frac{1}{4}} = 4$$

$$v_C(0^-) = v_{\frac{1}{4}\Omega} = \frac{1}{4}\times 4 = 1$$

From continuity relation for inductor and capacitor,

$$i(0^+) = -i_L(0^-) = -4 \qquad \text{.............(5)}$$

$$v_C(0^+) = +v_C(0^-) = 1 \qquad \text{.............(6)}$$

From equations (4) and (5),

$$k_1 = -4$$

Now, equation (3) becomes

$$i = e^{-0.375t}(-4\cos 1.69t + k_2\sin 1.69t) \qquad \text{.......... (7)}$$

Differentiating equation (7) w.r.t. t, we get

$$\frac{di}{dt} = (1.5 + 1.69k_2)e^{-0.375t}\cos 1.69t + (6.76 - 0.375k_2)e^{-0.375t}\sin 1.69t \qquad \text{................(8)}$$

Put $t = 0^+$ in equation (8),

$$\frac{di(0^+)}{dt} = 1.5 + 1.69k_2$$

$$\text{or, } k_2 = 0.5917\frac{di(0^+)}{dt} - 0.8875 \qquad \text{.............(9)}$$

Put $t = 0^+$ in equation (1),

$$\frac{1}{4}i(0^+) + \frac{1}{3}\frac{di(0^+)}{dt} + v_C(0^+) = 0 \quad \text{.................(10)}$$

From equations (10), (5), and (6), we get

$$\frac{1}{4}\times -4 + \frac{1}{3}\frac{di(0^+)}{dt} + 1 = 0$$

$$\text{or, } \frac{di(0^+)}{dt} = 0 \quad \text{.....................(11)}$$

From equations (9) and (11),

$$k_2 = 0 - 0.8875 = -0.8875 \quad \text{................(12)}$$

From equation (7) and (12),

$$i = e^{-0.375t}(-4\cos 1.69t - 0.8875\sin 1.69t)$$

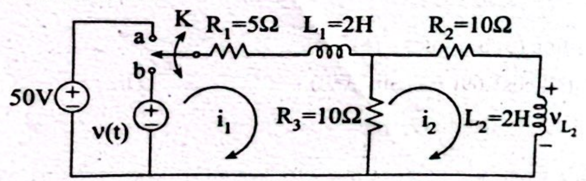
The voltage across inductor is given by

$$v_L = L\frac{di}{dt}$$

$$\text{or, } v_L = \frac{1}{3}\frac{d}{dt}[e^{-0.375t}(-4\cos 1.69t - 0.8875\sin 1.69t)]$$

$$\therefore v_L = e^{-0.375t}(2.36\sin 1.69t + 0.001\cos 1.69t)$$

**25. In circuit shown in figure below, switch is changed from positio to 'b' at time t = 0. Find expression for loop currents $i_1$ and $i_2$ for using classical approach, where v(t) = 30sin(50t + 30°) volts, and instantaneous polarity is as indicated in figure. Evaluate also $v_L$ 3 milliseconds.**




[2080 Ch

***Solution:***

Applying KVL for t > 0 in loop (1),

$$v_{R_1} + v_{L_1} + v_{R_3} = v(t)$$

$$\text{or, } 5i_1 + 2\frac{di_1}{dt} + 10(i_1 - i_2) = 30\sin(50t + 30°)$$

$$\text{or, } 15i_1 + 2\frac{di_1}{dt} - 10i_2 = 30\sin(50t + 30°) \text{ ........ (1)}$$



Applying KVL for t > 0 in loop (2),

$$v_{R_2} + v_{L_2} + v_{R_3} = 0$$

$$\text{or, } 10i_2 + 2\frac{di_2}{dt} + 10(i_2 - i_1) = 0$$

$$\text{or, } 20i_2 + 2\frac{di_2}{dt} - 10i_1 = 0$$

$$\text{or, } i_1 = \frac{1}{5}\frac{di_2}{dt} + 2i_2 \text{ ........ (2)}$$

From equations (1) & (2), we have

$$15\left[\frac{1}{5}\frac{di_2}{dt} + 2i_2\right] + 2\left[\frac{1}{5}\frac{d^2i_2}{dt^2} + 2\frac{di_2}{dt}\right] - 10i_2 = 30\sin(50t+30°)$$

$$\text{or, } \frac{2}{5}\frac{d^2i_2}{dt^2} + 5\frac{di_2}{dt} + 20i_2 = 30\sin(50t + 30°)$$

$$\text{or, } \frac{d^2i_2}{dt^2} + \frac{25}{2}\frac{di_2}{dt} + 50i_2 = 75\sin(50t + 30°) \text{ ....... (3)}$$

The solution of equation (3) is given by

$$i_2 = \text{C.F.} + \text{P.I.} = i_{2N} + i_{2f} \text{ ........ (4)}$$

To find C.F. i.e. $i_{2N}$, the auxilary equation of (3) is

$$s^2 + \frac{25}{2}s + 50 = 0$$

$$\therefore s_1, s_2 = \frac{-\frac{25}{2} \pm \sqrt{\left(\frac{25}{2}\right)^2 - 4\times 50}}{2} = \frac{-\frac{25}{2} \pm j6.614}{2} = -6.25 \pm j3.307$$

$$\therefore i_{2N} = e^{-6.25t}[k_1\cos 3.307t + k_2\sin 3.307t] \text{ ...... (5)}$$

To find P.I. i.e. $i_{2f}$, let

$$i_{2f} = A\cos(50t + 30°) + B\sin(50t + 30°) \text{ ......... (6)}$$

From equations (6) & (3),

$$(-50A)(50)\cos(50t+30°) - 2500B\sin(50t+30°) + \frac{25}{2}[-50A\sin(50t+30°) + 50B\cos(50t + 30°)] + 50[A\cos(50t + 30°) + B\sin(50t + 30°)] = 75\sin(50t + 30°)$$

Equating coefficients of sin & cosine terms on both sides, we get

$$-2500B - 625A + 50B = 75$$

$$\text{or, } 2450B + 625A = -75 \text{ ....... (7)}$$

$$\text{Also, } -2500A + 625B + 50A = 0$$

or, $-2450A + 625B = 0$ ....... (8)

Solving equations (7) & (8), we get

$A = -7.496\times10^{-3}$, $B = -28.7\times10^{-3}$ .

Now, equation (6) can be written as

$$i_{2f} = \sqrt{A^2+B^2}\,\sin\left(50t + 30° + \tan^{-1}\frac{A}{B}\right) \text{ ..... (9)}$$

Now,

$$\sqrt{A^2+B^2} = \sqrt{(-7.496\times10^{-3})^2 + (-28.7\times10^{-3})^2} = 29.66\times10^{-3} \text{ ...... (10)}$$

$$\tan^{-1}\frac{A}{B} = \tan^{-1}\left[\frac{7.496}{28.7}\right] = 14.64°$$

Hence, equation (9) becomes

$$i_{2f} = 29.66\times10^{-3}\sin(50t + 30° + 14.64°)$$
$$= 29.66\times10^{-3}\sin(50t + 44.64°) \text{ ........ (11)}$$

Now, from equations (4), (5) & (11),

$$i_2 = e^{-6.25t}[k_1\cos3.307t + k_2\sin3.307t] + 29.66\times10^{-3}\sin(50t+44.64°) \text{ ............... (12)}$$

Put $t = 0^+$ in equation (12),

$i_2(0^+) = k_1 + 29.66\times10^{-3}\sin(44.64°)$

or, $k_1 = i_2(0^+) - 20.84\times10^{-3}$ ....... (13)

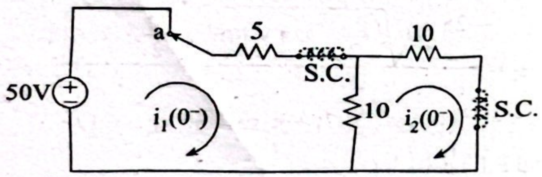



*Figure 2: Equivalent circuit at $t = 0^-$*

From figure (2),

$15i_1(0^-) - 10i_2(0^-) = 50$ ....... (14)

$-10i_1(0^-) + 20i_2(0^-) = 0$ ........ (15)

Solving (14) & (15), we get

$i_2(0^-) = 2.5$ A, $i_1(0^-) = 5$ A

From continuity relation for inductor,

$i_2(0^+) = i_2(0^-) = 2.5$A ........ (16)

$i_1(0^+) = i_1(0^-) = 5$ A ................(16a)

From equations (13) & (16), we get



$k_1 = 2.479$ ........ (17)

Differentiating equation (12), we get

$$\frac{di_2}{dt} = k_1[-3.307\sin(3.307t)e^{-6.25t} + (-6.25)e^{-6.25t}\cos3.307t] + k_2[3.307\cos3.307te^{-6.25t} - 6.25e^{-6.25t}\sin(3.307t)] + 29.66\times50\times10^{-3}\cos(50t + 44.64°) \text{ ......... (18)}$$

Put $t = 0^+$ in equation (18),

$$\frac{di_2(0^+)}{dt} = -6.25k_1 + 3.307k_2 + 29.66\times50\times10^{-3}\times711.5\times10^{-3} \text{ .....(19)}$$

Put $t = 0^+$ in equation (2),

$$i_1(0^+) = \frac{1}{5}\frac{di_2(0^+)}{dt} + 2i_2(0^+)$$

$$\therefore \frac{di_2(0^+)}{dt} = 0 \text{ ........ (20)}$$

From (19), (20), (17),

$k_2 = 4.366$ ....... (21)

From (17), (21), (12), we get

$$i_2 = e^{-625t}(2.479\cos3.307t + 4.366\sin3.307t) + 29.66\times10^{-3}\sin(50t+44.64°) \text{ ........ (22)}$$

From (22) & (2),

$$i_1 = e^{-6.25t}(1.623\sin3.307t + 4.7476\cos3.3707t) + 0.2966\cos(50t + 44.64°) + 0.05932\sin(50t + 44.64°)$$

Now, $v_{L_2} = L_2\frac{di_2}{dt} = 2\frac{di_2}{dt}$

$$= (-70.971\sin3.307t - 2.104\cos3.307t)e^{-6.25t} + 2.966\cos(50t + 44.64°)$$

For $t = 3 \times 10^{-3}$ s,

$v_{L_2} = 0.0279$ V

CHAPTER - 3

# TRANSIENT ANALYSIS IN RLC CIRCUIT BY LAPLACE TRANSFORM

## 3.1 The Laplace Transform

In mathematics, the *Laplace transform* is an integral transform named after its discoverer Pierre-Simon Laplace. It takes a function of a real variable t (often time) to a function of a complex variable *s* (complex frequency).

For a given function f(t), the Laplace transform is given by the expression

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

where s is a complex number, $s = \sigma + j\omega$.

All f(t) should be thought of as being multiplied by u(t) i.e., $f(t) = 0$ for $t<0$.

The Laplace transform method of solving differential equations offers a number of advantages over the classical methods that were discussed in the previous chapter. For example:

i. The solution of differential equations is routine and progresses systematically.

ii. The method gives the total solution–the particular integral and the complementary function–in one operation.

iii. Initial conditions are automatically specified in the transformed equations. Further, the initial conditions are incorporated into the problem as one of the first steps rather than as the last step.

iv. There is no need to solve for arbitrary constants separately.

A short table of Laplace transforms is given in Table 3.1.

*Table 3.1: Table of transforms*

| f(t)* | F(s) |
|---|---|
| 1. u(t) | $\frac{1}{s}$ |
| 2. t | $\frac{1}{s^2}$ |
| 3. $\frac{t^{n-1}}{(n-1)!}$, n = integer | $\frac{1}{s^n}$ |
| 4. $e^{at}$ | $\frac{1}{s-a}$ |
| 5. $t\,e^{at}$ | $\frac{1}{(s-a)^2}$ |
| 6. $\frac{1}{(n-1)!}\, t^{n-1}\, e^{at}$ | $\frac{1}{(s-a)^n}$ |
| 7. $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ |
| 8. $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ |
| 9. $\sin(\omega t+\theta)$ | $\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$ |
| 10. $\cos(\omega t+\theta)$ | $\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$ |
| 11. $e^{-at}\sin \omega t$ | $\frac{\omega}{(s+\alpha)^2+\omega^2}$ |
| 12. $e^{-at}\cos \omega t$ | $\frac{(s+\alpha)}{(s+\alpha)^2+\omega^2}$ |

* All f(t) should be thought of as being multiplied by u(t) i.e., $f(t) = 0$ for $t < 0$.

**Some important transforms:**

$L\{i(t)\} = I(s) = I,\qquad L\{v(t)\} = V(s) = V,$

$L\{V_o\} = \frac{V_o}{s},\qquad L\{I_o\} = \frac{I_o}{s},$

$L\{i(0^+)\} = \frac{i(0^+)}{s},\qquad L\{v(0^+)\} = \frac{v(0^+)}{s}$

### 3.1.1 Some Important Properties of Laplace Transform

**i) Linearity**

$$L\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$$

**ii) Time differentiation**

$$L\{f'(t)\} = L\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0^+)$$

$$L\{f''(t)\} = L\left\{\frac{d^2}{dt^2} f(t)\right\} = s^2F(s) - sf(0^+) - f'(0^+)$$

$$L\{f^n(t)\} = L\left\{\frac{d^2}{dt^2} f(t)\right\} = s^nF(s) - s^{(n-1)}f(0^+) - s^{(n-2)}f'(0^+) - \ldots - f^{(n-1)}$$

$$\text{Thus, } L\left\{\frac{di}{dt}\right\} = sI - i(0^+)$$

**iii) Time integration**

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$$

$$\text{Thus, } L\left\{\int_0^t i\,dt\right\} = \frac{I}{s}$$

**iv) Frequency differentiation**

$$L\{tf(t)\} = \frac{-dF(s)}{ds}$$

**v) Frequency integration**

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

**iv) Shifting Theorem**

If $L\{f(t)\} = F(s)$, then

$$L\{e^{-\alpha t} f(t)\} = F(s+\alpha)$$

$$L\{e^{\alpha t} f(t)\} = F(s-\alpha)$$

### 3.1.2 Inverse Laplace transform

The operation of obtaining f(t) from its Laplace transform F(s) is know[n as] the *inverse Laplace transform.*

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\,e^{st}\,ds$$

Though one could evaluate the *inverse Laplace transform of F(s)* by u[sing the] above equation, normally *the transform table* is used to obtain the inv[erse] Laplace transform.

## 3.2 Use of Partial Fraction Expansion in Analysis usi[ng] Laplace Transform

A differential equation of the general form

$$a_0\frac{d^ni}{dt^n} + a_1\frac{d^{n-1}i}{dt^{n-1}} + \ldots + a_{n-1}\frac{di}{dt} + a_ni = v(t)$$

becomes, as a result of the *Laplace transform,* an algebraic equation wh[ich] may be solved for the unknown as



$$I(s) = \frac{L[v(t)] + \text{initial condition terms}}{a_0s^n + a_1s^{n-1} + \ldots + a_{n-1}s + a_n}$$

The general form of this equation is a quotient of polynomial in s. Let the numerator and denominator polynomials be designated P(s) and Q(s) respectively as

$$I(s) = \frac{P(s)}{Q(s)}$$

In general, the transform expression for I(s) must be broken into simpler terms before any practical transform table can be used.

As the first step in the expansion of the quotient P(s)/Q(s), we check to see that the order of the polynomial P is less than that of Q. If this condition is not fulfilled, we divide the numerator by the denominator to obtain an expansion in the form

$$\frac{P(s)}{Q(s)} = B_0 + B_1s + B_2s^2 + \ldots + B_{m-n}s^{m-n} + \frac{P_1(s)}{Q(s)}$$

where m and n are the orders of the numerator and denominator respectively.

Next, we factor the denominator polynomial Q(s),

$$Q(s) = a_0s^n + a_1s^{n-1} + \ldots + a_n = a_0(s-s_1)\ldots(s-s_n)$$

or, very compactly, $Q(s) = a_0\prod_{j=1}^{n}(s-s_j)$

where $\Pi$ indicates a product of factors, and $s_1, s_2, \ldots, s_n$ are the n roots of the equation $Q(s) = 0$. Now, the possible form of these roots are:

**Case-I:** If all roots of $Q(s) = 0$ are simple, then the partial fraction expansion is

$$\frac{P_1(s)}{(s-s_1)(s-s_2)\ldots(s-s_n)} = \frac{k_1}{s-s_1} + \frac{k_2}{s-s_2} + \ldots + \frac{k_n}{s-s_n}$$

where k's are real constants called *residues.*

**Case-II:** If a root of $Q(s) = 0$ is of multiplicity of r, then the partial fraction expansion for the repeated root is

$$\frac{P_1(s)}{(s-s_1)^r} = \frac{k_{11}}{s-s_1} + \frac{k_{12}}{(s-s_1)^2} + \ldots + \frac{k_{1r}}{(s-s_1)^r}$$

and there will be similar terms for every other repeated root.

**Case-III:** If two roots form a complex conjugate pair, then the p[artial] fraction expansion is

$$\frac{P_1(s)}{Q_1(s)(s+\alpha+j\omega)(s+\alpha-j\omega)} = \frac{k_1}{(s+\alpha+j\omega)} + \frac{k_1^*}{(s+\alpha-j\omega)} + \ldots$$

where $k_1^*$ is the complex conjugate of $k_1$. In other words, whe[n] are conjugates, so are the partial fraction expansion coeffic[ients] expansion of the type shown above is necessary for each complex conjugate roots.

In an expansion of a quotient of polynomials by partial fractions, i[t] necessary to use a combination of the three rules given above.

## 3.3 Heaviside's Partial Fraction Expansion Theore[m]

Consider the case in which Q(s) has only distinct roots. Let

$$\frac{P_1(s)}{Q(s)} = \frac{k_1}{s-s_1} + \frac{k_2}{s-s_2} + \frac{k_3}{s-s_3} + .... + \frac{k_n}{s-s_n}$$

Then, any of the coefficients $k_1, k_2, k_3, ...., k_n$ can be evaluated by m[ultiplying] by the denominator of that coefficient and setting s to the value of th[e] the denominator. In other words, to find the coefficient of $k_j$,

$$k_j = \left[(s-s_j)\frac{P_1(s)}{Q(s)}\right]_{s=s_j}$$

To consider a general case of r-repeated roots, let

$$\frac{P(s)}{Q(s)} = \frac{R(s)}{(s-s_j)^r} = \frac{k_{j1}}{s-s_j} + \frac{k_{j2}}{(s-s_j)^2} + ....$$

$$+ \frac{k_{jn}}{(s-s_j)^n} + .... + \frac{k_{jr}}{(s-s_j)^r} \qquad ......(i)$$

where n is any term in the partial fraction expansion and R(s) is define[d]

$$R(s) = \frac{P(s)}{Q(s)}(s-s_j)^r$$

Multiplying equation (i) by $(s-s_j)^r$ gives

$$R(s) = k_{j1}(s-s_j)^{r-1} + k_{j2}(s-s_j)^{r-2} + .... + k_{jr} \qquad ......(ii)$$

From equation (ii), we can visualize the method to be used to evaluat[e] coefficient. If we let $s = s_j$, all terms in the equation disappear ex[cept] which can be evaluated. Next, differentiate the equation once with res[pect to] s. The term $k_{jr}$ will vanish, but $k_{j,r-1}$ will remain without a multi[plying] function of s. Again, $k_{j,r-1}$ can be evaluated by letting $s = s_j$. To fi[nd the] general term $k_{jn}$, differentiate equation (ii) $(r-n)$ times and let $s = s_j$; th[en]

$$k_{jn} = \frac{1}{(r-n)!}\frac{d^{r-n}R(s)}{ds^{r-n}}\bigg|_{s=s_j}$$

$$\text{or, } k_{jn} = \frac{1}{(r-n)!}\frac{d^{r-n}}{ds^{r-n}}\left[\frac{P(s)}{Q(s)}(s-s_j)^r\right]\bigg|_{s=s_j}$$

$$\mathcal{L}\left\{\frac{di}{dt}+\frac{R}{L}i\right\}=\mathcal{L}\left\{\frac{V_o}{L}\right\}$$

$$\text{or, } sI - i(0^+) + \frac{R}{L}I = \frac{\frac{V_o}{L}}{s} \quad \text{............ (1)}$$

By observation at $t = 0^-$,

$i(0^-) = 0$

From continuity relation for inductor,

$i(0^+) = i(0^-) = 0$ ........... (2)

From equations (1) and (2),

$$sI - 0 + \frac{R}{L}I = \frac{\frac{V_o}{L}}{s}$$

$$\text{or, } I\left(s+\frac{R}{L}\right) = \frac{\frac{V_o}{L}}{s}$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{s\left(s+\frac{R}{L}\right)}$$

$$\text{or, } I = \frac{V_o}{L}\left[\frac{1}{s\left(0+\frac{R}{L}\right)}+\frac{1}{\left(s+\frac{R}{L}\right)\left(\frac{-R}{L}\right)}\right]$$

$$\text{or, } I = \frac{V_o}{R}\left(\frac{1}{s}-\frac{1}{s+\frac{R}{L}}\right)$$

Taking inverse Laplace transform on both sides, we get

$$i = \frac{V_o}{R}\left(1-e^{-\frac{R}{L}t}\right)$$

$$\therefore i = I_o(1-e^{-t/\tau}) \text{ where } I_o = \frac{V_o}{R}$$

**2) Step Response of RL Circuit with some Initial Value of Cur- through Inductor**

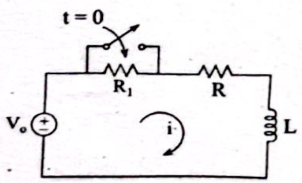



*Figure 3.2: RL circuit with some initial value of current through induc-*



Applying KVL for $t > 0$,

$v_R + v_L = V_o$

$$\text{or, } iR + L\frac{di}{dt} = V_o$$

$$\text{or, } \frac{di}{dt}+\frac{R}{L}i = \frac{V_o}{L}$$

Taking Laplace transform on both sides,

$$\mathcal{L}\left\{\frac{di}{dt}+\frac{R}{L}i\right\}=\mathcal{L}\left\{\frac{V_o}{L}\right\}$$

$$\text{or, } sI - i(0^+) + \frac{R}{L}I = \frac{\frac{V_o}{L}}{s} \quad \text{............ (1)}$$

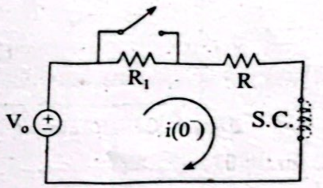



*Figure 3.3: Equivalent circuit at $t = 0^-$.*

From figure 3.3,

$$i(0^-) = \frac{V_o}{R_1+R} = I_{initial} \text{ (say)}$$

From continuity relation for inductor,

$i(0^+) = i(0^-) = I_{initial}$ ........... (2)

From equations (1) and (2),

$$sI - I_{initial} + \frac{R}{L}I = \frac{\frac{V_o}{L}}{s}$$

$$\text{or, } I\left(s+\frac{R}{L}\right) = \frac{\frac{V_o}{L}}{s} + I_{initial}$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{s\left(s+\frac{R}{L}\right)} + \frac{I_{initial}}{s+\frac{R}{L}}$$

$$\text{or, } I = \frac{V_o}{L}\left[\frac{1}{s\left(0+\frac{R}{L}\right)}+\frac{1}{\left(s+\frac{R}{L}\right)\left(\frac{-R}{L}\right)}\right] + \frac{I_{initial}}{s+\frac{R}{L}}$$

$$\text{or, } I = \frac{V_o}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right) + \frac{I_{initial}}{s+\frac{R}{L}}$$

Taking inverse Laplace transform on both sides, we get

$$i = \frac{V_o}{R}[1 - e^{-\frac{R}{L}t}] + I_{initial}\, e^{-\frac{R}{L}t}$$

$$\text{or, } i = I_o[1 - e^{-t/\tau}] + I_{initial}\, e^{-t/\tau}$$

$$\therefore i = I_o - (I_o - I_{initial})\, e^{-t/\tau}$$

**3) Step Response of RC Circuit without any Initial Value of V across Capacitor**

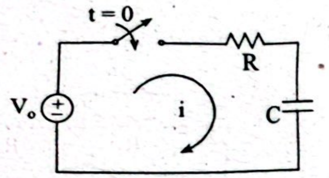



*Figure 3.4: RC circuit*

Applying KVL for $t > 0$,

$$v_R + v_C = V_o$$

$$\text{or, } iR + \frac{1}{C}\int_{-\infty}^{t} i\, dt = V_o$$

$$\text{or, } iR + \frac{1}{C}\int_{-\infty}^{0} i dt + \frac{1}{C}\int_{0}^{t} i\, dt = V_o$$

$$\text{or, } iR + v_C(0^+) + \frac{1}{C}\int_{0}^{t} i\, dt = V_o$$

Taking Laplace transform on both sides, we get

$$RI + \frac{v_C(0^+)}{s} + \frac{1}{C}\frac{I}{s} = \frac{V_o}{s} \qquad \text{........... (1)}$$

By observation at $t = 0^-$,

$$v_C(0^-) = 0$$

From continuity relation for capacitor,

$$v_C(0^+) = v_C(0^-) = 0 \qquad \text{........... (2)}$$

From (1) and (2),

$$RI + 0 + \frac{1}{C}\frac{I}{s} = \frac{V_o}{s}$$



$$\text{or, } I\left(R + \frac{1}{sC}\right) = \frac{V_o}{s}$$

$$\text{or, } I = \frac{V_o}{s\left(R + \frac{1}{sC}\right)}$$

$$\text{or, } I = \frac{V_o}{s \times \frac{R}{s}\left(s + \frac{1}{CR}\right)}$$

$$\text{or, } I = \frac{\frac{V_o}{R}}{s + \frac{1}{CR}}$$

Taking inverse Laplace transform on both sides, we get

$$i = \frac{V_o}{R} e^{-\frac{t}{CR}}$$

**4) Step Response of RC Circuit with Some Initial Value of Voltage across Capacitor**

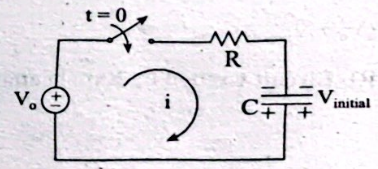



*Figure 3.5: RC circuit with initial value of voltage across capacitor*

Applying KVL for $t > 0$,

$$v_R + v_C = V_o$$

$$\text{or, } iR + \frac{1}{C}\int_{-\infty}^{t} i\, dt = V_o$$

$$\text{or, } iR + \frac{1}{C}\int_{-\infty}^{0} i\, dt + \frac{1}{C}\int_{0}^{t} i\, dt = V_o$$

$$\text{or, } iR + v_C(0^+) + \frac{1}{C}\int_{0}^{t} i\, dt = V_o$$

Taking Laplace transform on both sides, we get

$$RI + \frac{v_C(0^+)}{s} + \frac{1}{C}\frac{I}{s} = \frac{V_o}{s} \qquad \text{........... (1)}$$

By observation at $t = 0^-$,

$$v_C(0^-) = V_{initial}$$

From continuity relation for capacitor,

$v_C(0^+) = -v_C(0^-) = -V_{initial}$ ............ (2)

From (1) and (2),

$$RI - \frac{V_{initial}}{s} + \frac{1}{C}\frac{I}{s} = \frac{V_o}{s}$$

$$\text{or, } I\left(R + \frac{1}{sC}\right) = \frac{V_o}{s} + \frac{V_{initial}}{s}$$

$$\text{or, } I = \frac{V_o}{s\left(R + \frac{1}{sC}\right)} + \frac{V_{initial}}{s\left(R + \frac{1}{sC}\right)}$$

$$\text{or, } I = \frac{V_o}{s \times \frac{R}{s}\left(s + \frac{1}{CR}\right)} + \frac{V_{initial}}{s \times \frac{R}{s}\left(s + \frac{1}{CR}\right)} = \frac{\frac{V_o}{R}}{s + \frac{1}{CR}} + \frac{\frac{V_{initial}}{R}}{s + \frac{1}{CR}}$$

Taking inverse Laplace transform on both sides, we get

$$i = \frac{V_o}{R} e^{-\frac{t}{CR}} + \frac{V_{initial}}{R} e^{-\frac{t}{CR}}$$

$$\therefore i = \frac{1}{R} e^{-\frac{t}{CR}} (V_o + V_{initial})$$

**5) Response of RL Circuit Excited by Exponential Source**

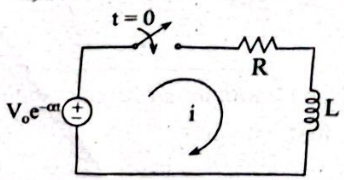



*Figure 3.6: RL circuit excited by exponential source*

Applying KVL for $t > 0$,

$v_R + v_L = V_o e^{-\alpha t}$

$$\text{or, } iR + L\frac{di}{dt} = V_o e^{-\alpha t}$$

$$\text{or, } \frac{di}{dt} + \frac{R}{L} i = \frac{V_o}{L} e^{-\alpha t}$$

Taking Laplace transform on both sides, we get

$$sI - i(0^+) + \frac{R}{L} I = \frac{V_o}{L}\frac{1}{s+\alpha} \quad \text{............ (1)}$$

$i(0^-) = 0$

$\therefore i(0^+) = i(0^-) = 0$ ......... (2)



From (1) and (2),

$$sI - 0 + \frac{R}{L} I = \frac{V_o}{L}\frac{1}{s+\alpha}$$

$$\text{or, } \left(s + \frac{R}{L}\right) I = \frac{V_o}{L}\frac{1}{s+\alpha}$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{(s+\alpha)\left(s + \frac{R}{L}\right)} \quad \text{............... (3)}$$

**Case I:** If $\alpha = \frac{R}{L}$, then equation (3) becomes

$$I = \frac{\frac{V_o}{L}}{(s+\alpha)^2}$$

Taking inverse Laplace transform on both sides, we get

$$i = \frac{V_o}{L} e^{-\alpha t} \quad \text{............ (4)}$$

**Case II:** If $\alpha \neq \frac{R}{L}$, then equation (3) becomes

$$I = \frac{\frac{V_o}{L}}{(s+\alpha)\left(s + \frac{R}{L}\right)} = \frac{V_o}{L}\left[\frac{1}{(s+\alpha)\left(-\alpha + \frac{R}{L}\right)} + \frac{1}{\left(s + \frac{R}{L}\right)\left(-\frac{R}{L} + \alpha\right)}\right]$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{\left(-\alpha + \frac{R}{L}\right)}\left[\frac{1}{s+\alpha} - \frac{1}{s + \frac{R}{L}}\right]$$

Taking inverse Laplace transform on both sides, we get

$$i = \frac{\frac{V_o}{L}}{\left(\alpha + \frac{R}{L}\right)}\left(e^{-\alpha t} - e^{-\frac{R}{L}t}\right)$$

**6) Response of RC Circuit Excited by Exponential Source**

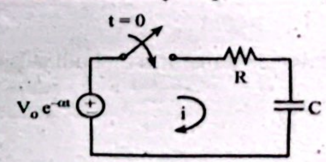



*Figure 3.7: RC circuit excited by exponential source*

Applying KVL for $t > 0$,

$$v_R + v_L = V_o e^{-\alpha t}$$

$$\text{or, } iR + \frac{1}{C}\int_{-\infty}^{t} i\,dt = V_o e^{-\alpha t}$$

$$\text{or, } iR + v_C(0^+) + \frac{1}{C}\int_{0}^{t} i\,dt = V_o e^{-\alpha t}$$

Taking Laplace transform on both sides, we get

$$IR + \frac{v_C(0^+)}{s} + \frac{1}{C}\frac{I}{S} = V_o \frac{1}{s+\alpha} \quad \text{............ (1)}$$

$$v_C(0^-) = 0$$

$$\therefore v_C(0^+) = v_C(0^-) = 0 \quad \text{.................... (2)}$$

From equations (1) and (2),

$$I\left(R + \frac{1}{sC}\right) = \frac{V_o}{s+\alpha}$$

$$\text{or, } I = \frac{V_o}{(s+\alpha)\left(R + \frac{1}{sC}\right)} = \frac{V_o}{(s+\alpha)\frac{R}{s}\left(s + \frac{1}{CR}\right)}$$

$$\text{or, } I = \frac{\left(\frac{V_o}{R}\right)s}{(s+\alpha)\left(s + \frac{1}{CR}\right)} \quad \text{.................. (3)}$$

**Case I:** If $\alpha = \frac{1}{CR}$, then equation (3) becomes

$$I = \frac{\left(\frac{V_o}{R}\right)s}{(s+\alpha)^2}$$

$$\text{or, } I = \frac{V_o}{R}\left[\frac{(s+\alpha) - \alpha}{(s+\alpha)^2}\right]$$

$$\text{or, } I = \frac{V_o}{R}\left[\frac{1}{s+\alpha} - \frac{\alpha}{(s+\alpha)^2}\right]$$

Taking inverse Laplace transform on both sides, we get

$$i = \frac{V_o}{R}\left(e^{-\alpha t} - \alpha t\, e^{-\alpha t}\right) \quad \text{......................... (4)}$$



**Case II:** If $\alpha \neq \frac{1}{CR}$, then equation (3) becomes

$$I = \frac{\left(\frac{V_o}{R}\right)s}{(s+\alpha)\left(s + \frac{1}{CR}\right)}$$

$$\text{or, } I = \frac{V_o}{R}\left[\frac{(-\alpha)}{(s+\alpha)\left(-\alpha + \frac{1}{RC}\right)} + \frac{\frac{-1}{RC}}{\left(s + \frac{1}{RC}\right)\left(-\frac{1}{RC} + \alpha\right)}\right]$$

$$\text{or, } I = \frac{\frac{V_o}{R}}{\left(-\alpha + \frac{1}{CR}\right)}\left[\frac{-\alpha}{s+\alpha} + \frac{\frac{1}{CR}}{s + \frac{1}{CR}}\right]$$

Taking inverse Laplace transform on both sides, we get

$$i = \frac{\frac{V_o}{R}}{\left(-\alpha + \frac{1}{CR}\right)}\left[-\alpha\, e^{-\alpha t} + \frac{1}{CR} e^{-\frac{t}{CR}}\right]$$

**7) Response of RL Circuit Excited by Sinusoidal Source**

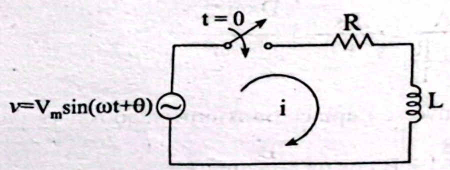



*Figure 3.8: Series RL circuit driven by a sinusoidal ac source*

Applying KVL for $t > 0$,

$$v_R + v_L = V_m \sin(\omega t + \theta)$$

$$\text{or, } iR + L\frac{di}{dt} = V_m \sin(\omega t + \theta)$$

$$\text{or, } \frac{di}{dt} + \frac{R}{L}i = \frac{V_m}{L}\sin(\omega t + \theta)$$

$$\text{or, } \frac{di}{dt} + \frac{R}{L}i = \frac{V_m}{L}[\sin \omega t \cos \theta + \cos \omega t \sin \theta]$$

Taking Laplace transform on both sides, we get

$$sI - i(0^+) + \frac{R}{L}I = \frac{V_m}{L}\left[\frac{\omega \cos\theta}{s^2+\omega^2} + \frac{s \sin\theta}{s^2+\omega^2}\right]$$

or, $sI - i(0^+) + \frac{R}{L}I = \frac{V_m}{L}\left[\frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}\right]$

$i(0^-) = 0$

$\therefore i(0^+) = i(0^-) = 0$ ................ (2)

From (1) and (2),

$$I\left(s+\frac{R}{L}\right) = \frac{V_m}{L}\left[\frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}\right]$$

$$\text{or, } I = \frac{V_m}{L}\left[\frac{\omega\cos\theta + s\sin\theta}{(s^2+\omega^2)\left(s+\frac{R}{L}\right)}\right] \quad \text{.............. (3)}$$

$$\text{Let } \frac{V_m}{L}\left[\frac{\omega\cos\theta + s\sin\theta}{(s^2+\omega^2)\left(s+\frac{R}{L}\right)}\right] = \frac{A}{s+\frac{R}{L}} + \frac{Bs+D}{s^2+\omega^2}$$

By partial fraction, we will get the values of A, B, and D.

Hence, equation (3) becomes

$$I = \frac{A}{s+\frac{R}{L}} + \frac{Bs+D}{(s^2+\omega^2)}$$

$$\text{or, } I = \frac{A}{s+\frac{R}{L}} + \frac{Bs}{s^2+\omega^2} + \frac{D\times\omega\frac{1}{\omega}}{s^2+\omega^2}$$

Taking inverse Laplace transform on both sides, we get

$$i = A\,e^{-\frac{R}{L}t} + B\cos\omega t + \frac{D}{\omega}\sin\omega t$$

$$\text{or, } i = A\,e^{-\frac{R}{L}t} + \sqrt{B^2 + \left(\frac{D}{\omega}\right)^2}\,\sin\left[\omega t + \text{Tan}^{-1}\frac{B}{\left(\frac{D}{\omega}\right)}\right]$$

**8) Response of RC Circuit Excited by Sinusoidal Source**

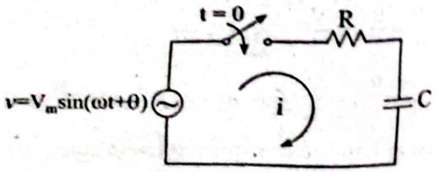



*Figure 3.9: RC circuit excited by sinusoidal source*



Applying KVL for $t > 0$,

$v_R + v_C = V_m\sin(\omega t + \theta)$

$$\text{or, } iR + \frac{1}{C}\int_{-\infty}^{t} i\,dt = V_m\sin(\omega t+\theta)$$

$$\text{or, } iR + \frac{1}{C}\int_{-\infty}^{0} i\,dt + \frac{1}{C}\int_{0}^{t} i\,dt = V_m\sin(\omega t+\theta)$$

$$\text{or, } iR + v_C(0^+) + \frac{1}{C}\int_{0}^{t} i\,dt = V_m[\sin\omega t\cos\theta + \cos\omega t\sin\theta]$$

Taking Laplace transform on both sides, we get

$$RI + \frac{v_C(0^+)}{s} + \frac{1}{C}\frac{I}{s} = V_m\left[\frac{\omega\cos\theta}{s^2+\omega^2} + \frac{s\sin\theta}{s^2+\omega^2}\right]$$

$$\text{or, } RI + \frac{v_C(0^+)}{s} + \frac{1}{C}\frac{I}{s} = V_m\left[\frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}\right] \quad \text{........(1)}$$

By observation at $t = 0^-$,

$v_c(0^-) = 0$

From continuity relation for capacitor,

$v_c(0^+) = v_c(0^-) = 0$ ............(2)

From equations (1) and (2),

$$RI + 0 + \frac{1}{C}\frac{I}{s} = V_m\left[\frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}\right]$$

$$\text{or, } I\left(R + \frac{1}{Cs}\right) = V_m\left[\frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}\right]$$

$$\text{or, } I = \frac{V_m s}{R}\left[\frac{\omega\cos\theta + s\sin\theta}{(s^2+\omega^2)\left(s+\frac{1}{RC}\right)}\right] \quad \text{.............. (3)}$$

$$\text{Let } \frac{V_m s}{R}\left[\frac{\omega\cos\theta + s\sin\theta}{(s^2+\omega^2)\left(s+\frac{1}{RC}\right)}\right] = \frac{A}{s+\frac{1}{RC}} + \frac{Bs+D}{s^2+\omega^2}$$

By partial fraction, we will get the values of A, B, and D.

Hence, equation (3) becomes

$$I = \frac{A}{s+\frac{1}{RC}} + \frac{Bs+D}{s^2+\omega^2}$$

$$\text{or, } I = \frac{A}{s + \frac{1}{RC}} + \frac{Bs}{s^2 + \omega^2} + \frac{D \times \omega \times \frac{1}{\omega}}{s^2 + \omega^2}$$

Taking inverse Laplace transform on both sides, we get

$$i = A\, e^{-\frac{1}{RC}t} + B \cos \omega t + \frac{D}{\omega} \sin \omega t$$

$$\text{or, } i = A\, e^{-\frac{1}{RC}t} + \sqrt{B^2 + \left(\frac{D}{\omega}\right)^2} \sin\left[\omega t + \tan^{-1} \frac{B}{\left(\frac{D}{\omega}\right)}\right]$$

### 3.4.2 Application of Laplace Transform in the Second Order System

**1) Step Response of RLC Series Circuit**

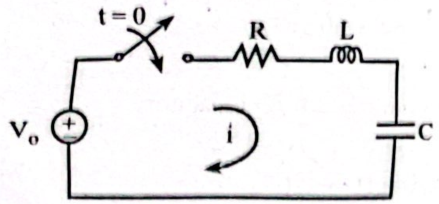



*Figure 3.10: Series RLC circuit excited by dc voltage source*

Applying KVL for $t > 0$,

$$v_R + v_L + v_C = V_o$$

$$\text{or, } iR + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\, dt = V_o$$

$$\text{or, } L\frac{di}{dt} + iR + \frac{1}{C}\int_{-\infty}^{0} i\, dt + \frac{1}{C}\int_{0}^{t} i\, dt = V_o$$

$$\text{or, } L\frac{di}{dt} + iR + v_C(0^+) + \frac{1}{C}\int_{0}^{t} i\, dt = V_o$$

Taking Laplace transform on both sides, we get

$$L[sI - i(0^+)] + RI + \frac{v_C(0^+)}{s} + \frac{1}{C}\frac{I}{s} = \frac{V_o}{s} \qquad \text{.......... (1)}$$

By observation at $t = 0^-$,

$v_C(0^-) = 0$ and $i(0^-) = 0$

From continuity relation for capacitor and inductor,

$$v_C(0^+) = v_C(0^-) = 0 \qquad \text{.......... (2)}$$

$$i(0^+) = i(0^-) = 0 \qquad \text{.......... (3)}$$

From (1), (2), (3),

$$L[sI - 0] + RI + 0 + \frac{1}{C}\frac{I}{s} = \frac{V_o}{s}$$

$$\text{or, } I\left(sL + R + \frac{1}{sC}\right) = \frac{V_o}{L}$$

$$\text{or, } I\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = \frac{V_o}{L}$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \qquad \text{.................. (4)}$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{(s - s_1)(s - s_2)} \qquad \text{.................. (5)}$$

$$\text{where } s_1, s_2 = \frac{\frac{-R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4 \times \frac{1}{LC}}}{2} = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$$

$$\text{or, } s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} \qquad \text{.......... (6)}$$

where $\alpha = \frac{R}{2L}$ and $\omega_n = \frac{1}{\sqrt{LC}}$

**Case I:** If $\alpha < \omega_n$, then the roots will be complex and equation (4) can be written as

$$I = \frac{\frac{V_o}{L}}{s^2 + 2 \times s \times \frac{R}{2L} + \left(\frac{R}{2L}\right)^2 + \left(\frac{1}{\sqrt{LC}}\right)^2 - \left(\frac{R}{2L}\right)^2}$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{\left(s + \frac{R}{2L}\right)^2 + \left[\sqrt{\left(\frac{1}{\sqrt{LC}}\right)^2 - \left(\frac{R}{2L}\right)^2}\,\right]^2}$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{(s + \alpha)^2 + \left(\sqrt{\omega_n^2 - \alpha^2}\right)^2}$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{(s+\alpha)^2 + \omega_d^2}\ ;\ \omega_d = \sqrt{\omega_n^2 - \alpha^2}$$

$$\text{or, } I = \frac{\left(\frac{V_o}{L}\right)\omega_d \frac{1}{\omega_d}}{(s+\alpha)^2 + (\omega_d)^2}$$

Taking inverse Laplace transform, we get $i = \frac{V_o}{\omega_d L} e^{-\alpha t} \sin \omega_d t$

**Case II:** If $\alpha = \omega_n$, then roots will be real and equal, and given from (6),

$s_1, s_2 = -\alpha$

Hence, equation (5) can be written as

$$\text{or, } I = \frac{\frac{V_o}{L}}{(s+\alpha)(s+\alpha)}$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{(s+\alpha)^2}$$

Taking inverse Laplace transform on both sides, we get

$$i = \frac{V_o}{L} t e^{-\alpha t}$$

**Case III:** If $\alpha > \omega_n$, then roots will be real and unequal, and given from equation (6),

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_n^2},\quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_n^2}$$

Hence, equation (5) can be written as

$$I = \frac{\frac{V_o}{L}}{(s-s_1)(s-s_2)}$$

$$\text{or, } I = \frac{V_o}{L}\left[\frac{1}{(s-s_1)(s_1-s_2)} + \frac{1}{(s-s_2)(s_2-s_1)}\right]$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{(s_1-s_2)}\left[\frac{1}{(s-s_1)} - \frac{1}{(s-s_2)}\right]$$

Taking inverse Laplace transform on both sides, we get

$$i = \frac{\frac{V_o}{L}}{(s_1-s_2)}\left(e^{s_1 t} \ldots\right.$$



**2) Step Response of RLC Parallel Circuit**

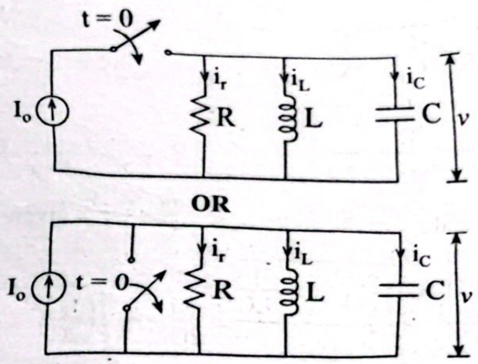



*Figure 3.11: A parallel RLC circuit*

Applying KCL for $t > 0$,

$$i_C + i_L + i_r = I_o$$

$$\text{or, } C\frac{dv}{dt} + \frac{1}{L}\int_{-\infty}^{t} v\, dt + \frac{v}{R} = I_o$$

$$\text{or, } C\frac{dv}{dt} + \frac{1}{L}\int_{-\infty}^{0} v\, dt + \frac{1}{L}\int_{0}^{t} v\, dt + Gv = I_o$$

$$\text{or, } C\frac{dv}{dt} + \frac{\psi(0^+)}{L} + \frac{1}{L}\int_{0}^{t} v\, dt + Gv = I_o$$

Taking Laplace transform on both sides, we get

$$C\,[sV - v_C(0^+)] + \frac{\psi(0^+)}{sL} + \frac{1}{L}\times\frac{V}{s} + GV = \frac{I_o}{s} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

By observation at $t = 0^-$,

$i_L(0^-) = 0$ and $v_C(0^-) = 0$

From continuity relation for inductor and capacitor,

$$i_L(0^+) = i_L(0^-) = 0$$

$$\therefore \psi(0^+) = Li_L(0^+) = 0 \qquad \ldots\ldots\ldots(2)$$

$$v_C(0^+) = v_C(0^-) = 0 \qquad \ldots\ldots\ldots(3)$$

From equations (1), (2), and (3),

$$C\,[sV - 0] + 0 + \frac{1}{L}\times\frac{V}{s} + GV = \frac{I_o}{s}$$

$$\text{or, } V\left(Cs + \frac{1}{sL} + G\right) = \frac{I_o}{s}$$

$$\text{or, } V = \frac{\frac{I_o}{C}}{s^2 + \frac{G}{C}s + \frac{1}{LC}} \quad \ldots\ldots\ldots\ldots(4)$$

$$\text{or, } V = \frac{\frac{I_o}{C}}{(s - s_1)(s - s_2)} \quad \ldots\ldots\ldots\ldots(5)$$

where $s_1$ and $s_2$ are the roots of $s^2 + \frac{G}{C}s + \frac{1}{LC}$ given by

$$s_1, s_2 = \frac{\frac{-G}{C} \pm \sqrt{\left(\frac{G}{C}\right)^2 - 4\times\frac{1}{LC}}}{2} = \frac{-G}{2C} \pm \sqrt{\left(\frac{G}{2C}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$$

$$\therefore s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} \quad \ldots\ldots\ldots\ldots(6)$$

where $\alpha = \frac{G}{2C}$ = damping coefficient, $\omega_n = \frac{1}{\sqrt{LC}}$ = natural frequen[cy]

**Case I:** If $\alpha < \omega_n$, then the system is said to be *underdamped* and roo[ts will] be complex, and equation (4) can be written as

$$V = \frac{\frac{I_o}{C}}{s^2 + \frac{G}{C}s + \frac{1}{LC}}$$

$$\text{or, } V = \frac{\frac{I_o}{C}}{s^2 + 2\times s\times\frac{G}{2C} + \left(\frac{G}{2C}\right)^2 + \left(\frac{1}{\sqrt{LC}}\right)^2 - \left(\frac{G}{2C}\right)^2}$$

$$\text{or, } V = \frac{\frac{I_o}{C}}{\left(s + \frac{G}{2C}\right)^2 + \left[\sqrt{\left(\frac{1}{\sqrt{LC}}\right)^2 - \left(\frac{G}{2C}\right)^2}\right]^2}$$

$$\text{or, } V = \frac{\frac{I_o}{C}}{(s+\alpha)^2 + \left(\sqrt{\omega_n^2 - \alpha^2}\right)^2}$$

$$\text{or, } V = \frac{\frac{I_o}{C}}{(s+\alpha)^2 + \omega_d^2} \text{ where } \omega_d = \sqrt{\omega_n^2 - \alpha^2}$$



$$\text{or, } V = \frac{\left(\frac{I_o}{C}\right)\times\omega_d\times\frac{1}{\omega_d}}{(s+\alpha)^2 + (\omega_d)^2}$$

Taking inverse Laplace transform, we get

$$v = \frac{I_o}{\omega_d C} e^{-\alpha t} \sin\omega_d t$$

**Case II:** If $\alpha = \omega_n$, then roots will be real and equal, and given from equation (6),

$$s_1, s_2 = -\alpha$$

Hence, equation (5) can be written as

$$\text{or, } V = \frac{\frac{I_o}{C}}{(s+\alpha)(s+\alpha)}$$

$$\text{or, } V = \frac{\frac{I_o}{C}}{(s+\alpha)^2}$$

Taking inverse Laplace transform on both sides, we get

$$v = \frac{I_o}{C} t\, e^{-\alpha t}$$

**Case III:** If $\alpha > \omega_n$, then roots will be real and unequal, and given from equation (6),

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_n^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_n^2}$$

Hence, equation (5) can be written as

$$V = \frac{\frac{I_o}{C}}{(s - s_1)(s - s_2)}$$

$$\text{or, } V = \frac{I_o}{C}\left[\frac{1}{(s - s_1)(s_1 - s_2)} + \frac{1}{(s - s_2)(s_2 - s_1)}\right]$$

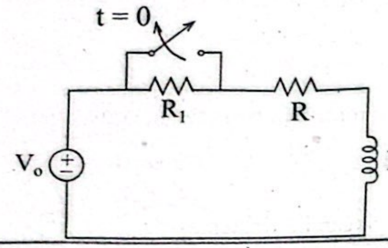
$$\text{or, } V = \frac{\frac{I_o}{C}}{(s_1 - s_2)}\left[\frac{1}{(s - s_1)} - \frac{1}{(s - s_2)}\right]$$

Taking inverse Laplace transform on both sides, we get

$$v = \frac{\frac{I_o}{C}}{(s_1 - s_2)}\left(e^{s_1 t} - e^{s_2 t}\right)$$

## SOLVED PROBLEMS

**1. Find the expression for current through inductor for $t > 0$ using Laplace transform method.**




***Solution:***

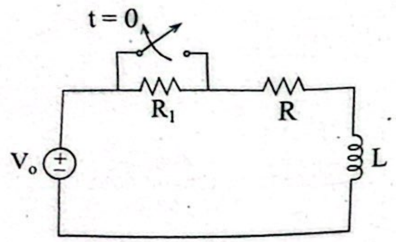



***Figure 1***

Applying KVL for $t > 0$,

$$v_{R1} + v_R + v_L = V_o$$

$$\text{or, } iR_1 + iR + L\frac{di}{dt} = V_o$$

$$\text{or, } i(R_1 + R) + L\frac{di}{dt} = V_o$$

$$\text{or, } iR_x + L\frac{di}{dt} = V_o \text{ where } R_x = R_1 + R$$

$$\text{or, } \frac{di}{dt} + \frac{R_x}{L}i = \frac{V_o}{L}$$

Taking Laplace transform on both sides,

$$\mathcal{L}\left\{\frac{di}{dt} + \frac{R_x}{L}i\right\} = \mathcal{L}\left\{\frac{V_o}{L}\right\}$$

$$\text{or, } sI - i(0^+) + \frac{R_x}{L}I = \frac{\frac{V_o}{L}}{s} \qquad \text{............ (1)}$$



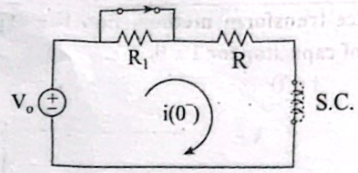



***Figure 2:*** *Equivalent circuit at $t = 0^-$*

From figure (2),

$$i(0^-) = \frac{V_o}{R} = I_{initial} \text{ (say)}$$

From continuity relation for inductor,

$$i(0^+) = i(0^-) = I_{initial} \qquad \text{.......... (2)}$$

From equations (1) and (2),

$$sI - I_{initial} + \frac{R_x}{L}I = \frac{\frac{V_o}{L}}{s}$$

$$\text{or, } I\left(s + \frac{R_x}{L}\right) = \frac{\frac{V_o}{L}}{s} + I_{initial}$$

$$\text{or, } I = \frac{\frac{V_o}{L}}{s\left(s + \frac{R_x}{L}\right)} + \frac{I_{initial}}{s + \frac{R_x}{L}}$$

$$\text{or, } I = \frac{V_o}{L}\left[\frac{1}{s\left(0 + \frac{R_x}{L}\right)} + \frac{1}{\left(s + \frac{R_x}{L}\right)\left(\frac{-R_x}{L}\right)}\right] + \frac{I_{initial}}{s + \frac{R_x}{L}}$$

$$\text{or, } I = \frac{V_o}{R_x}\left(\frac{1}{s} - \frac{1}{s + \frac{R_x}{L}}\right) + \frac{I_{initial}}{s + \frac{R_x}{L}}$$

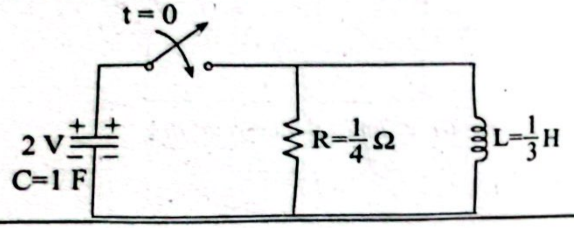
Taking inverse Laplace transform on both sides, we get

$$i = \frac{V_o}{R_x}\left(1 - e^{-\frac{R_x}{L}t}\right) + I_{initial}\, e^{-\frac{R_x}{L}t}$$

$$\text{or, } i = I_o(1 - e^{-t/\tau}) + I_{initial}\, e^{-t/\tau};\ \tau = \frac{L}{R_x}$$

$$\therefore\ i = I_o - (I_o - I_{initial})\, e^{-t/\tau}$$

2. **Using Laplace transform method, find the expression for cur**
**and voltage of capacitor for $t > 0$.**




***Solution:***

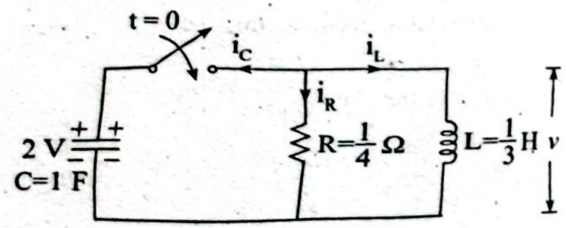



***Figure 1***

Applying KCL for $t > 0$,

$i_C + i_R + i_L = 0$

$$\text{or, } C\frac{dv}{dt} + \frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\,dt = 0$$

$$\text{or, } 1\frac{dv}{dt} + \frac{v}{\left(\frac{1}{4}\right)} + \frac{1}{\left(\frac{1}{3}\right)}\int_{-\infty}^{t} v\,dt = 0$$

$$\text{or, } \frac{dv}{dt} + 4v + 3\int_{-\infty}^{0} v\,dt + 3\int_{0}^{t} v\,dt = 0$$

$$\text{or, } \frac{dv}{dt} + 4v + 3\psi(0^+) + 3\int_{0}^{t} v\,dt = 0$$

Taking Laplace transform on both sides,

$$[sV - v(0^+)] + 4V + \frac{3\psi(0^+)}{s} + 3\frac{V}{s} = 0 \quad \ldots\ldots\ldots\ldots(1)$$

By observation at $t = 0^-$,

$v(0^-) = 2$ V and $i_L(0^-) = 0$

From continuity relation of capacitor and inductor,

$v(0^+) = v(0^-) = 2$ V $\quad \ldots\ldots\ldots(2)$



$i_L(0^+) = i_L(0^-) = 0$

$\therefore \psi(0^+) = Li_L(0^+) = 0 \quad \ldots\ldots\ldots(3)$

From equations (1), (2), and (3), we get

$$[sV - 2] + 4V + 0 + 3\frac{V}{s} = 0$$

$$\text{or, } V\left(s + 4 + \frac{3}{s}\right) = 2$$

$$\text{or, } V(s^2 + 4s + 3) = 2s$$

$$\text{or, } V = \frac{2s}{s^2 + 4s + 3} = \frac{2s}{(s+1)(s+3)}$$

$$\text{or, } V = \frac{2(-1)}{(s+1)(-1+3)} + \frac{2(-3)}{(s+3)(-3+1)}$$

$$\text{or, } V = \frac{-1}{s+1} + \frac{3}{s+3}$$

Taking inverse Laplace transform, we have

$v = -e^{-t} + 3e^{-3t}$

$$\text{Now, } i_C = C\frac{dv}{dt} = 1\frac{d(-e^{-t} + 3e^{-3t})}{dt} = e^{-t} - 9e^{-3t}$$

**Alternative method:**

Applying KCL for $t > 0$,

$i_C + i_R + i_L = 0$

$$\text{or, } i_C + \frac{v}{\left(\frac{1}{4}\right)} + \frac{1}{\frac{1}{3}}\int_{-\infty}^{t} v\,dt = 0$$

$$\text{or, } i_C + 4v + 3\int_{-\infty}^{0} v\,dt + 3\int_{0}^{t} v\,dt = 0$$

$$\text{or, } i_C + 4v + 3\psi(0^+) + 3\int_{0}^{t} v\,dt = 0$$

Taking Laplace transform on both sides,

$$I_C + 4V + 3\frac{\psi(0^+)}{s} + 3\frac{V}{s} = 0 \quad \ldots\ldots\ldots\ldots(1)$$

By observation at $t = 0^-$,

$i_L(0^-) = 0$

From continuity relation for inductor,

$i_L(0^+) = i_L(0^-) = 0$

$\therefore\ \psi(0^+) = L\, i_L(0^+) = 0$ ................ (2)

From equations (1) and (2),

$$I_C + 4V + 0 + 3\frac{V}{s} = 0$$

$$\text{or, } I_C + 4V + \frac{3V}{s} = 0$$

$$\text{or, } (4s + 3)V + sI_C = 0 \quad \text{................ (3)}$$

$$\text{Also, } i_C = C\frac{dv}{dt}$$

$$\text{or, } i_C = 1\frac{dv}{dt}$$

Taking Laplace transform on both sides, we get

$$I_C = sV - v(0^+) \quad \text{................(4)}$$

By observation at $t = 0^-$,

$$v(0^-) = 2\text{ V}$$

From continuity relation for capacitor,

$$v(0^+) = v(0^-) = 2\text{ V} \quad \text{................(5)}$$

From (4) and (5),

$$I_C = sV - 2$$

$$\text{or, } sV - I_C = 2 \quad \text{........... (6)}$$

Putting equation (3) and (6) in matrix form,

$$\begin{bmatrix} 4s+3 & s \\ s & -1 \end{bmatrix}\begin{bmatrix} V \\ I_C \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$V = \frac{\Delta_1}{\Delta} = \frac{-2s}{-(4s+3) - s^2}$$

$$\text{or, } V = \frac{2s}{(s^2 + 4s + 3)}$$

$$\text{or, } V = \frac{2s}{(s+1)(s+3)}$$

$$\text{or, } V = \frac{2(-1)}{(s+1)(-1+3)} + \frac{2(-3)}{(s+3)(-3+1)}$$

$$\text{or, } V = \frac{-1}{s+1} + \frac{3}{s+3}$$

Taking inverse Laplace transform on both sides, we get

$$v = -e^{-t} + 3e^{-3t}$$



$$\text{Also, } I_C = \frac{\Delta_2}{\Delta} = \frac{8s + 6}{-(4s+3) - s^2}$$

$$\text{or, } I_C = \frac{-(8s+6)}{(s^2 + 4s + 3)}$$

$$\text{or, } I_C = \frac{-(8s+6)}{(s+1)(s+3)}$$

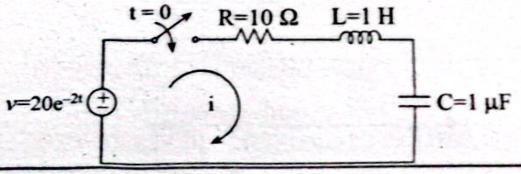
$$\text{or, } I_C = -\left[\frac{8(-1)+6}{(s+1)(-1+3)} + \frac{8(-3)+6}{(s+3)(-3+1)}\right]$$

$$\text{or, } I_C = \frac{1}{s+1} - \frac{9}{s+3}$$

Taking inverse Laplace transform on both sides, we get

$$i_C = e^{-t} - 9e^{-3t}$$

**3. Using Laplace transform method, find the expression for current i in the circuit for t > 0.**




***Solution:***

Applying KVL for $t > 0$,

$$v_R + v_L + v_C = 20e^{-2t}$$

$$\text{or, } i_R + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 20e^{-2t}$$

$$\text{or, } 10i + 1\frac{di}{dt} + \frac{1}{10^{-6}}\int_{-\infty}^{t} i\,dt = 20e^{-2t}$$

$$\text{or, } 10i + \frac{di}{dt} + 10^6\int_{-\infty}^{0} i\,dt + 10^6\int_{0}^{t} i\,dt = 20e^{-2t}$$

$$\text{or, } 10i + \frac{di}{dt} + v_C(0^+) + 10^6\int_{0}^{t} i\,dt = 20e^{-2t}$$

Taking Laplace transform on both sides,

$$10I + sI - i(0^+) + \frac{v_C(0^+)}{s} + 10^6\frac{I}{s} = 20\frac{1}{s+2} \quad \text{........(1)}$$

$$i(0^-) = 0,\ v_C(0^-) = 0$$

$\therefore i(0^+) = i(0^-) = 0$ ............ (2)

$v_C(0^+) = v_C(0^-) = 0$ ............ (3)

From (1), (2), and (3),

$$I\left(10 + s + \frac{10^6}{s}\right) = \frac{20}{s+2}$$

$$\text{or, } I\,(s^2 + 10s + 10^6) = \frac{20s}{s+2}$$

$$\text{or, } I = \frac{20s}{(s+2)\,(s^2 + 10s + 10^6)} \quad \text{.................. (4)}$$

$$\text{Let } \frac{20s}{(s+2)\,(s^2 + 10s + 10^6)} = \frac{A}{s+2} + \frac{Bs + D}{s^2 + 10s + 10^6}$$

Solving, we get

$A = -4\times10^{-5}$, $B = 1.467\times10^{-4}$, $D = 20$

Then, equation (4) becomes

$$I = \frac{-4\times10^{-5}}{s+2} + \frac{1.467\times10^{-4}s + 20}{s^2 + 10s + 10^6}$$

$$\text{or, } I = \frac{-4\times10^{-5}}{s+2} + \frac{1.467\times10^{-4}s + 20}{s^2 + 2\times s\times5 + (5)^2 + 10^6 - 25}$$

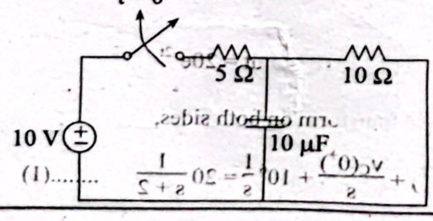
$$\text{or, } I = \frac{-4\times10^{-5}}{s+2} + \frac{1.467\times10^{-4}(s+5) + 20 - 5\times1.467\times10^{-4}}{(s+5)^2 + (999.98)^2}$$

$$\text{or, } I = \frac{-4\times10^{-5}}{s+2} + \frac{1.467\times10^{-4}(s+5)}{(s+5)^2 + (999.98)^2} + \frac{19.99\times\frac{999.98}{999.98}}{(s+5)^2 + (999.98)^2}$$

Taking inverse Laplace transform,

$$i = -4\times10^{-5}e^{-2t} + 1.467\times10^{-4}e^{-5t}\cos 999.98t + 0.0199\,e^{-5t}\sin 999.98t$$

**4. For the circuit shown, find the current and voltage of capacitor for t > 0 using Laplace transform method.**




[2074 Ashwin]

**Solution:**



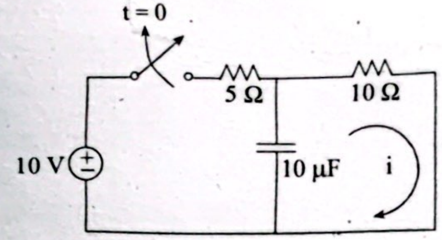



***Figure 1***

Applying KVL for t > 0,

$v_{10\Omega} + v_c = 0$ .................(A)

$$\text{or, } 10\,i + \frac{1}{10\times10^{-6}}\int_{-\infty}^{t} i\,dt = 0$$

$$\text{or, } 10i + 10^5\int_{-\infty}^{0} i\,dt + 10^5\int_{0}^{t} i\,dt = 0$$

$$\text{or, } 10\,i + v_c(0^+) + 10^5\int_{0}^{t} i\,dt = 0$$

Taking Laplace transform, we get

$$10I + \frac{v_c(0^+)}{s} + 10^5\frac{I}{s} = 0 \quad \text{..................(1)}$$

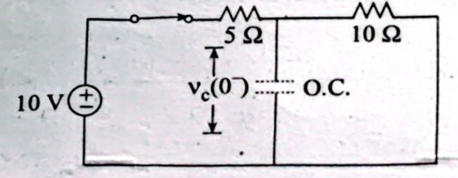



***Figure 2: Equivalent circuit at t = 0⁻***

From figure 2,

$$v_c(0^-) = v_{10\Omega}(0^-) = \frac{10}{10+5}\times10 = \frac{20}{3}\text{ V}$$

From continuity relation for capacitor,

$$v_c(0^+) = -v_c(0^-) = -\frac{20}{3} \quad \text{...............(2)}$$

From equations (1) and (2),

$$10I - \frac{\frac{20}{3}}{s} + 10^5\frac{I}{s} = 0$$

$$\text{or, } I\left(10 + \frac{10^5}{s}\right) = \frac{\frac{20}{3}}{s}$$

$$\text{or, } I = \frac{20/3}{10s + 10^5} = \frac{20/3}{10(s + 10^4)}$$

$$\text{or, } I = \frac{2/3}{s + 10^4}$$

Taking inverse Laplace transform, we get

$$i = \frac{2}{3}\, e^{-10^4 t} \qquad \text{.......................(3)}$$

From equation (A),

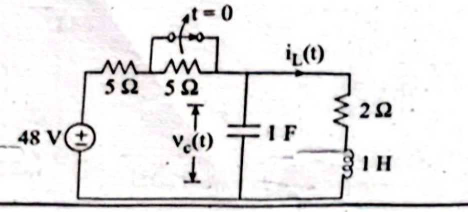
$v_c = -v_{10\Omega}$

or, $v_c = -10i$

or, $v_c = -10 \times \frac{2}{3} e^{-10^4 t}$

$\therefore\ v_c = -\frac{20}{3} e^{-10^4 t}$

**5. After being closed for a long time, if the switch in the circuit sh... is opened at t = 0, obtain the expression for $i_L(t)$ and $v_c(t)$ for t... using Laplace transform method.**




[2074 ...]

***Solution:***

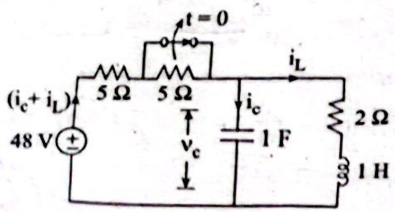



***Figure 1***



Applying KVL t > 0 in left loop,

$48 = 5\,(i_c + i_L) + 5(i_c + i_L) + v_c$

or, $48 = 10i_L + 10i_c + v_c$

or, $48 = 10i_L + 10\times C\frac{dv_c}{dt} + v_c$

or, $48 = 10i_L + 10\frac{dv_c}{dt} + v_c$

Taking Laplace transform on both sides, we get

$$\frac{48}{s} = 10I_L + 10\,[sV_c - v_c(0^+)] + V_c \qquad \text{...........(1)}$$

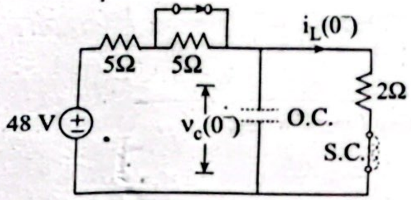



***Figure 2: Equivalent circuit at t = 0⁻***

From figure 2,

$$i_L(0^-) = \frac{48}{5+2} = \frac{48}{7}$$

$$v_c(0^-) = v_{2\Omega}(0^-) = 2 \times \frac{48}{7} = \frac{96}{7}$$

From continuity relation for inductor and capacitor,

$$i_L(0^+) = i_L(0^-) = \frac{48}{7} \qquad \text{................(2)}$$

$$v_c(0^+) = v_c(0^-) = \frac{96}{7} \qquad \text{................(3)}$$

From equations (1) and (3),

$$10I_L + (10s + 1)V_c - 10 \times \frac{96}{7} = \frac{48}{s}$$

$$\text{or, } 10I_L + (10s + 1)V_c = \frac{48}{s} + \frac{960}{7} \qquad \text{.............(4)}$$

Applying KVL for t >0 in right loop,

$v_c - v_{2\Omega} - v_L = 0$

or, $v_c - 2i_L - \frac{di_L}{dt} = 0$.

or, $\frac{di_L}{dt} + 2i_L - v_c = 0$

Taking Laplace transform on both sides, we get

$sI_L - i_L(0^+) + 2I_L - V_c = 0$

or, $sI_L - \frac{48}{7} + 2I_L - V_c = 0$

or, $(s+2)I_L - V_c = \frac{48}{7}$ ......................(5)

Arranging equations (4) and (5) in matrix form, we get

$$\begin{bmatrix} 10 & (10s+1) \\ (s+2) & -1 \end{bmatrix}\begin{bmatrix} I_L \\ V_c \end{bmatrix} = \begin{bmatrix} \frac{48}{s}+\frac{960}{7} \\ \frac{48}{7} \end{bmatrix}$$

By applying Cramer's rule,

$$I_L = \frac{\Delta 1}{\Delta} = \frac{\begin{vmatrix} \left(\frac{48}{s}+\frac{960}{7}\right) & (10s+1) \\ \frac{48}{7} & -1 \end{vmatrix}}{\begin{vmatrix} 10 & (10s+1) \\ (s+2) & -1 \end{vmatrix}}$$

or, $I_L = \dfrac{-\frac{48}{s} - \frac{960}{7} - \frac{48}{7}(10s+1)}{-10-(10s+1)(s+2)}$

or, $I_L = \dfrac{-(4320s+672)}{-7s(10s^2+21s+12)}$

or, $I_L = \dfrac{4320s+672}{70s\left(s^2+\frac{21}{10}s+\frac{12}{10}\right)}$

or, $I_L = \dfrac{4320s+672}{70s(s^2+2.1s+1.2)}$

or, $I_L = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+2.1s+1.2}$

$A = \left|\dfrac{4320s+672}{70s(s^2+2.1s+1.2)}\times s\right|_{s=0} = \dfrac{672}{70\times 1.2} = 8$

So, $I_L = \dfrac{8}{s} + \dfrac{Bs+C}{s^2+2.1s+1.2}$ .............(6)



$\dfrac{4320s+672}{70s(s^2+2.1s+1.2)} = \dfrac{8s^2+16.8s+9.6+Bs^2+Cs}{s(s^2+2.1s+1.2)}$

or, $4320s + 672 = 560s^2 + 1176s + 672 + 70Bs^2 + 70Cs$

Equating the coefficient of s,

$4320 = 1176 + 70C$

or, $C = 44.91$

Equating the coefficient of $s^2$,

$0 = 560 + 70B$

or, $B = -8$

Now, equation (6) becomes

$I_L = \dfrac{8}{s} + \dfrac{(-8s+44.91)}{s^2+2.1s+1.2}$

or, $I_L = \dfrac{8}{s} - \dfrac{8s}{s^2+2\times s\times 1.05+(1.05)^2+0.0975} + \dfrac{44.91}{s^2+2\times s\times 1.05+(1.05)^2+0.0975}$

or, $I_L = \dfrac{8}{s} - \dfrac{8(s+1.05-1.05)}{(s+1.05)^2+(\sqrt{0.0975})^2} + \dfrac{44.91}{(s+1.05)^2+(\sqrt{0.0975})^2}$

or, $I_L = \dfrac{8}{s} - \dfrac{8(s+1.05)}{(s+1.05)^2+(0.3122)^2} + \dfrac{8.4}{0.3122}\times\dfrac{0.3122}{(s+1.05)^2+(0.3122)^2} + \dfrac{44.01}{0.3122}\times\dfrac{0.3122}{(s+1.05)^2+(0.3122)^2}$

Taking inverse Laplace transform on both sides,

$i_L = 8 - 8\,e^{-1.05t}\cos(0.3122t) + 26.9058\,e^{-1.05t}\sin(0.3122t) + 143.85\,e^{-1.05t}\sin(0.3122t)$

$\therefore i_L = 8 - 8\,e^{-1.05t}\cos(0.3122t) + 170.755\,e^{-1.05t}\sin(0.3122t)$ A

Again by Cramer's rule,

$$V_c = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 10 & (48/5+960/7) \\ s+2 & 48/5 \end{vmatrix}}{\Delta}$$

$$= \frac{\frac{480}{5} - (s+2)\left(\frac{48}{5}+\frac{960}{7}\right)}{-10-(10s^2+s+20s+2)}$$

$$\text{or, } V_c = \frac{960s^2 + 2256s - 2688}{70s(s^2 + 2.1s + 1.2)}$$

$$\text{or, } V_c = \frac{A}{s} + \frac{Bs + C}{s^2 + 2.1s + 1.2} \qquad \text{.................(7)}$$

$$A = \left| \frac{960s^2 + 2256s - 2688}{70s(s^2 + 2.1s + 1.2)} \times 5 \right|_{s=0} = \frac{2688}{70 \times 1.2} = -32$$

Equation (7) now becomes

$$V_c = \frac{-32}{s} + \frac{Bs + C}{s^2 + 2.1s + 1.2} \qquad \text{.............(8)}$$

$$\text{or, } \frac{960s^2 + 2256s - 2688}{70} = -32s^2 - 67.25 - 38.4 + Bs^2 + Cs$$

$$\text{or, } 960s^2 + 2256s - 2688 = -2240s^2 - 4704s - 2688 + 70Bs^2 + 70Cs$$

Equating the coefficient of $s^2$,

$960 = -2240 + 70B$

or, $B = 45.71$

Equating the coefficient of s,

$2256 = -4704 + 70C$

or, $C = 99.42$

Hence, equation (8) becomes

$$\text{or, } V_c = \frac{-32}{s} + \frac{45.71s + 99.42}{(s + 1.05)^2 + (0.3122)^2}$$

$$\text{or, } V_c = \frac{-32}{s} + 45.71\left[\frac{s + 1.05}{(s + 1.05)^2 + (0.3122)^2}\right] - \frac{45.71 \times 1.05}{0.3122}$$

$$\frac{0.3122}{(s + 1.05)^2 + (0.3122)^2} + \frac{99.42}{0.3122} \times \frac{0.3122}{(s + 1.05)^2 + (0.3122)^2}$$

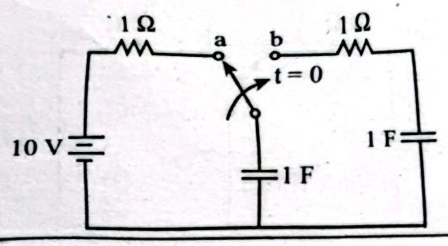
Taking inverse Laplace transform on both sides, we get

$$v_c = -32 + 45.71e^{-1.05t}\cos 0.3122t - 153.733e^{-1.5t}\sin 0.3122t + 318.4\ e^{-1.05t}\sin 0.3122t$$

$$v_c = -32 + 45.71e^{-1.05t}\cos 0.3122t + 1.64.71e^{-1.05t}\sin 0.3122t \text{ volts}$$

**6. Keeping the switch at position 'a' for a long time, if the switch is moved to position 'b' at t = 0 in the circuit shown below, find the expression for current through and voltage across capacitor using Laplace transform method.**






[2074 Chaitra]

***Solution:***

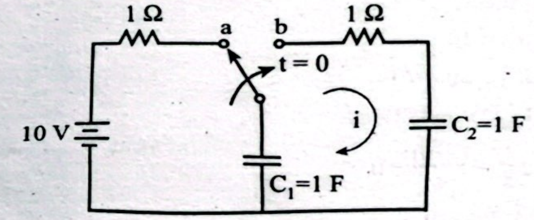



***Figure 1***

Applying KVL for t > 0,

$$v_{1\Omega} + v_{C_1} + v_{C_2} = 0$$

$$\text{or, } i + \frac{1}{1}\int_{-\infty}^{t} i\,dt + \frac{1}{1}\int_{-\infty}^{t} i\,dt = 0$$

$$\text{or, } 0 + \int_{-\infty}^{0} i\,dt + \int_{0}^{t} i\,dt + \int_{-\infty}^{0} i\,dt + \int_{0}^{t} i\,dt = 0$$

$$\text{or, } i + v_{c1}(0^+) + \int_{0}^{t} i\,dt + v_{c2}(0^+) + \int_{0}^{t} i\,dt = 0$$

$$\text{or, } i + 2\int_{0}^{t} i\,dt + v_{c1}(0^+) + v_{c2}(0^+) = 0$$

Taking Laplace transform on both sides, we get

$$I + 2\frac{I}{s} + \frac{v_{c1}(0^+)}{s} + \frac{v_{c2}(0^+)}{s} = 0 \qquad \text{...........(1)}$$

By inspection at $t = 0^-$,

$$v_{c2}(0^-) = 0$$

$$\therefore\ v_{c2}(0^+) = v_{c2}(0^+) = 0 \qquad \text{..............(2)}$$

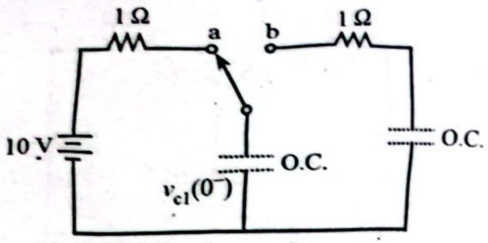



*Figure 2: Equivalent circuit at $t = 0^-$*

From figure 2,

$$v_{c1}(0^-) = 10$$

$$\therefore\ v_{c1}(0^+) = -v_{c1}(0^-) = -10 \qquad ..........(3)$$

From (1), (2), and (3),

$$I + 2\frac{I}{s} + \frac{(-10)}{s} = 0$$

$$\text{or, } I\left(\frac{s+2}{s}\right) = \frac{10}{s}$$

$$\text{or, } I = \frac{10}{s+2}$$

Taking inverse Laplace transform on both sides, we get

$$i = 10\,e^{-2t}$$

$$v_{c2} = \frac{1}{1}\int_{-\infty}^{t} i\,dt$$

$$\text{or, } v_{c2} = \int_{-\infty}^{0} i\,dt + \int_{0}^{t} i\,dt$$

$$\text{or, } v_{c2} = v_{c2}(0^+) + \int_{0}^{t} i\,dt$$

Taking Laplace transform on both sides,

$$V_{c2} = \frac{v_{c2}(0^+)}{s} + \frac{I}{s}$$

$$\text{or, } V_{c2} = 0 + \frac{\frac{10}{(s+2)}}{s}$$

$$\text{or, } V_{c2} = \frac{10}{s(s+2)}$$



$$\text{or, } V_{c2} = 10\left[\frac{1}{s(0+2)} + \frac{1}{(s+2)(-2)}\right]$$

$$\text{or, } V_{c2} = 5\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

Taking inverse Laplace transform on both sides,

$$v_{c2} = 5[1 - e^{-2t}]$$

Again,

$$v_{c1} = \frac{1}{1}\int_{-\infty}^{t} i\,dt$$

$$\text{or, } v_{c1} = v_{c1}(0^+) + \int_{0}^{t} i\,dt$$

Taking Laplace transform on both sides,

$$V_{c1} = \frac{v_{c1}(0^+)}{s} + \frac{I}{s}$$

$$\text{or, } V_{c1} = \frac{-10}{s} + \frac{10}{s(s+2)}$$

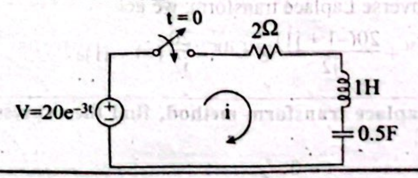
$$\text{or, } V_{c1} = \frac{-10}{s} + 5\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

$$\text{or, } V_{c1} = \frac{-5}{s} - \frac{5}{s+2}$$

Taking inverse Laplace transform on both sides,

$$v_{c1} = -5 - 5e^{-2t}$$

**7. In the series R–L–C circuit shown in figure, there is no initial charge on capacitor. If the switch S is closed at t = 0, determine the expression for current and voltage for all elements for t > 0, using Laplace transform method.**




[2075 Chaitra]

***Solution:***

Applying KVL for t > 0,

$$v_r + v_L + v_C = 20e^{-3t}$$

$$\text{or, } 2i + 1\frac{di}{dt} + \frac{1}{0.5}\int_{-\infty}^{t} i\,dt = 20e^{-3t}$$

or, $2i + \frac{di}{dt} + 2\int_{-\infty}^{0} i\,dt + 2\int_{0}^{t} i\,dt = 20e^{-3t}$

or, $2i + \frac{di}{dt} + v_C(0^+) + 2\int_{0}^{t} i\,dt = 20e^{-3t}$

Taking Laplace transform on both sides , we get

$2I + sI - i(0^+) + \frac{v_C(0^+)}{s} + 2\frac{I}{s} = 20\frac{1}{s+3}$ .......... (1)

Now, $i(0^-) = 0$, $v_C(0^-) = 0$

Hence, $i(0^+) = i(0^-) = 0$ .......... (2)

$v_C(0^+) = v_C(0^-) = 0$ ............. (3)

From equations (1), (2), (3), we get

$I\left(2 + s + \frac{2}{s}\right) = \frac{20}{s+3}$

or, $I\left(\frac{2s + s^2 + 2}{s}\right) = \frac{20}{s+3}$

or, $I = \frac{20s}{(s+3)(s^2+2s+2)}$

or, $I = \frac{20s}{(s+3)(s+1-j1)(s+1+j1)}$

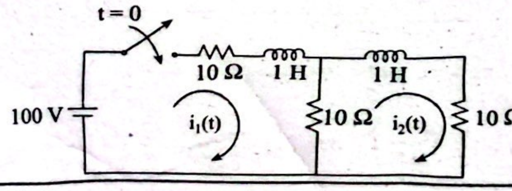
or, $I = \frac{20(-3)}{(s+3)(9-6+2)} + \frac{20(-1+j1)}{(s+1-j1)-(-1+j1+1+j1)} + \frac{20(-1-j1)}{(s+1+j1)(-1-j1+1-j1)}$

or, $I = \frac{-60}{5(s+3)} + \frac{20(-1+j1)}{j2(s+1-j1)} + \frac{20(-1-j1)}{-j2(s+1+j1)}$

Taking inverse Laplace transform, we get

$i = -12e^{-3t} + \frac{20(-1+j1)}{j2}e^{-(1-j1)t} - \frac{20}{j2}(-1-j1)e^{-(1+j1)t}$

**8. Using Laplace transform method, find the expression for $i_1$ and i for $t > 0$.**




[2072 Chaitra]



***Solution:***

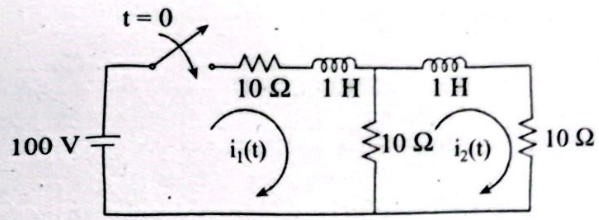



***Figure 1***

Applying KVL for $t > 0$ in loop 1,

$10i_1 + 1\frac{di_1}{dt} + 10i_1 - 10i_2 = 100$

or, $20i_1 + \frac{di_1}{dt} - 10i_2 = 100$

Taking Laplace transform on both sides, we get

$20I_1 + [sI_1 - i_1(0^+)] - 10I_2 = \frac{100}{s}$

By observation at $t = 0^-$,

$i_1(0^-) = 0$

$\therefore i_1(0^+) = i_1(0^-) = 0$

or, $20I_1 + sI_1 - 10I_2 = \frac{100}{s}$

or, $(s+20)I_1 - 10I_2 = \frac{100}{s}$

Applying KVL for $t > 0$ in loop 2,

$1\frac{di_2}{dt} + 10i_2 + 10(i_2 - i_1) = 0$

or, $\frac{di_2}{dt} + 20i_2 - 10i_1 = 0$

Taking Laplace transform on both sides, we get

$[sI_2 - i_2(0^+)] + 20I_2 - 10I_1 = 0$

By observation at $t = 0^-$,

$i_2(0^-) = 0$

$\therefore i_2(0^+) = i_2(0^-) = 0$

or, $(s + 20)I_2 - 10I_1 = 0$

or, $-10I_1 + (s + 20)I_2 = 0$ ................(2)

$$\begin{bmatrix} (s+20) & -10 \\ -10 & (s+20) \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{100}{s} \\ 0 \end{bmatrix}$$

$$I_1(s) = \frac{\Delta_1}{\Delta} = \frac{\frac{100}{s}(s+20)}{(s+20)^2 - 100}$$

$$= \frac{100(s+20)}{s(s+30)(s+10)}$$

$$= \frac{100(20)}{s(30)(10)} + \frac{100(-30+20)}{(-30)(s+30)(-30+10)} + \frac{100(-10+20)}{(-10)(-10+30)(s+10)}$$

$$= \frac{20}{3s} - \frac{1.667}{s+30} - \frac{5}{s+10}$$

Taking inverse Laplace transform on both sides, we get

$$i_1(t) = \frac{20}{3} - 1.667e^{-30t} - 5e^{-10t}$$

$$I_2(s) = \frac{\Delta_2}{\Delta} = \frac{10\times100}{s(s+30)(s+10)} = \frac{A}{s} + \frac{B}{s+30} + \frac{C}{s+10}$$

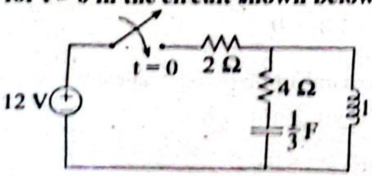
$$A = \frac{1000}{30\times10} = 3.33,\ B = \frac{1000}{-30\times(-2)} = 1.667,\ C = \frac{1000}{-10\times20} = -5$$

or, $I_2(s) = \frac{3.33}{s} + \frac{1.667}{s+30} + \frac{-5}{s+10}$

Taking inverse Laplace transform on both sides, we get

$i_2(t) = 3.33 + 1.667e^{-30t} - 5e^{-10t}$

**9. Using Laplace transform method, find the current through Inductor and capacitor for $t > 0$ in the circuit shown below.**




[2072 Chait...]



*Solution:*

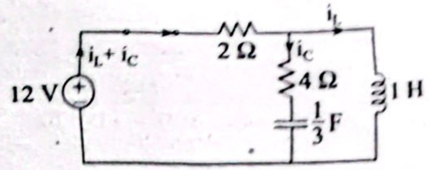



*Figure 1*

Applying KVL in left loop for $t > 0$,

$$2(i_L + i_C) + 4i_C + \frac{1}{\frac{1}{3}}\int_{-\infty}^{t} i_C\,dt = 12$$

or, $2i_L + 6i_C + 3\int_0^t i_C\,dt = 12$

Taking Laplace transform on both sides, we get

$$2I_L + 6I_C + 3\frac{I_C}{s} = \frac{12}{s}$$

or, $2I_L + \frac{(6s+3)}{s}I_C = \frac{12}{s}$

or, $2sI_L + (6s + 3)I_C = 12$ .......... (1)

Applying KVL for $t > 0$ in outer loop,

$$2(i_L + i_C) + 1\frac{di_L}{dt} = 12 .$$

Taking Laplace transform on both sides, we get

$$2I_L + 2I_C + sI_L - i_L(0^+) = \frac{12}{s}$$

By observation at $t = 0^-$,

$i_L(0^-) = 0$

$\therefore i_L(0^+) = i_L(0^-) = 0$

or, $(s + 2)I_L + 2I_C = \frac{12}{s}$ ............. (2)

Writing equations (1) and (2) in matrix form,

$$\begin{bmatrix} 2s & (6s+3) \\ (s+2) & 2 \end{bmatrix}\begin{bmatrix} I_L \\ I_C \end{bmatrix} = \begin{bmatrix} 12 \\ \frac{12}{s} \end{bmatrix}$$

$$I_L = \frac{24 - \frac{12}{s}(6s+3)}{4s - (s+2)(6s+3)}$$

$$= \frac{24s - 72s - 36}{s(4s - 6s^2 - 3s - 12s - 6)} = \frac{48s + 36}{s(6s^2 + 11s + 6)}$$

Let $\frac{48s+36}{s(6s^2+11s+6)} = \frac{A}{s} + \frac{Bs + C}{6s^2 + 11s + 6}$.

Put $s = 0$,

$$A = \frac{36}{6} = 6$$

Put $s = 1$,

$$\frac{48+36}{6+11+6} = 6 + \frac{B+C}{6+11+6}$$

or, $B + C = -54$ .............(x)

Put $s = 2$,

$$\frac{48\times2+36}{2(6\times2^2+11\times2+6)} = \frac{6}{2} + \frac{2B+C}{6\times2^2+11\times2+6}$$

or, $2B + C = -90$ .............(y)

Solving (x) and (y), we get

$B = -36$, $C = -18$

So, $\frac{48s+36}{s(6s^2+11s+6)} = \frac{6}{s} - \frac{36s+18}{6s^2+11s+6} = \frac{6}{s} - \frac{36s+18}{6\left(s^2+\frac{11}{6}s+1\right)}$

$$= \frac{6}{s} - \frac{6\left(s+\frac{11}{12}\right) + 3 - 6\times\frac{11}{12}}{\left(s+\frac{11}{12}\right)^2 + \left(\frac{\sqrt{23}}{12}\right)^2}$$

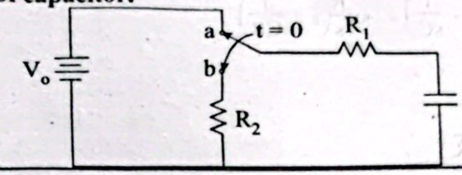
$$\text{or, } I_L = \frac{6}{s} - 6\frac{s+\frac{11}{12}}{\left(s+\frac{11}{12}\right)^2 + \left(\frac{\sqrt{23}}{12}\right)^2} - 6.255\frac{\frac{\sqrt{23}}{12}}{\left(s+\frac{11}{12}\right)^2 + \left(\frac{\sqrt{23}}{12}\right)^2}$$

Taking inverse Laplace transform, we get

$$= 6 - 6\,e^{-\frac{11}{12}t}\cos\frac{\sqrt{23}}{12}t - 6.255\,e^{-\frac{11}{12}t}\sin\frac{\sqrt{23}}{12}t$$



**10. Use Laplace transform method in the circuit to find the voltage and current of capacitor.**




***Solution:***

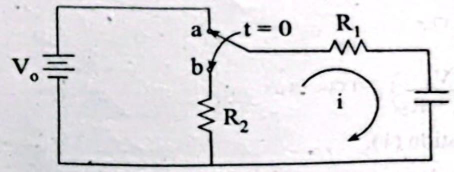



***Figure 1***

Applying KVL for $t > 0$,

$v_{R1} + v_{R2} + v_C = 0$ ............... (1)

or, $i(R_1 + R_2) + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0$

or, $i(R_1 + R_2) + v_C(0^+) + \frac{1}{C}\int_{0}^{t} i\,dt = 0$

Taking Laplace transform on both sides, we get

$$I(R_1 + R_2) + \frac{v_C(0^+)}{s} + \frac{1}{C}\frac{I}{s} = 0$$

or, $I\left[(R_1 + R_2) + \frac{1}{sC}\right] = -\frac{v_C(0^+)}{s}$ ........... (2)

By observation at $t = 0^-$,

$v_C(0^-) = V_o$

From continuity relation for capacitor,

$v_C(0^+) = v_C(0^-) = V_o$ .................(3)

From equations (2) and (3), we get

$$I\left[(R_1 + R_2) + \frac{1}{sC}\right] = -\frac{V_o}{s}$$

or, $I\left(R + \frac{1}{sC}\right) = -\frac{V_o}{s}$ where $R = R_1 + R_2$

$$\text{or, } I = \frac{V_o}{s\left(R + \frac{1}{sC}\right)} = \frac{V_o}{s \times \frac{R}{s}\left(s + \frac{1}{CR}\right)}$$

$$\text{or, } I = \frac{\frac{V_o}{R}}{s + \frac{1}{CR}}$$

Taking inverse Laplace transform, we get

$$i = \frac{V_o}{R} e^{-(1/CR)t}$$

$$\text{or, } i = \left(\frac{V_o}{R_1 + R_2}\right) e^{-[1/C(R_1+R_2)]t} \quad \text{............. (3)}$$

From equation (1),

$$v_C = -v_{R1} - v_{R2}$$

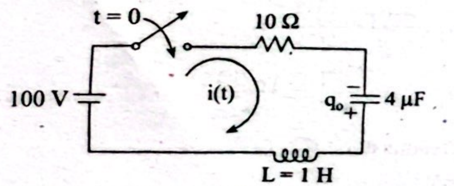
$$\text{or, } v_C = -i(R_1 + R_2)$$

$$\text{or, } v_C = -i\,R$$

$$\text{or, } v_C = -R\left[\frac{V_o}{R} e^{-(1/CR)t}\right]$$

$$\therefore v_C = -V_o e^{-\frac{1}{C(R_1+R_2)}t}$$

**11. In the circuit shown in figure below, the capacitor has an ini[tial] charge of $q_o$ = 800 μC with polarity shown in the figure. Fi[nd] current i(t) if the switch is closed at t = 0 using Laplace transfor[m] method.**




[2072 Ka...]

***Solution:***

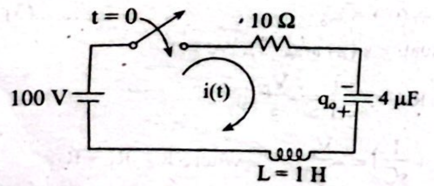



***Figure 1***



Applying KVL for t > 0,

$$100 = v_L + v_{10\Omega} + v_C$$

$$100 = 1\frac{di(t)}{dt} + 10\, i(t) + \frac{1}{4\times10^{-6}}\int_{-\infty}^{0} i(t)\,dt + \frac{1}{4\times10^{-6}}\int_{0}^{t} i(t)\,dt$$

$$100 = 1\frac{di(t)}{dt} + 10\, i(t) + v_c(0^+) + 25000\int_{0}^{t} i(t)\,dt$$

Taking Laplace transform on both sides,

$$\frac{100}{s} = sI(s) - i(0^+) + 10I(s) + \frac{v_c(0^+)}{s} + 250000\frac{I(s)}{s} \quad \text{......... (i)}$$

By observation at $t = 0^-$,

$$q(0^-) = 800\ \mu C = 800\times10^{-6}\ C$$

$$i(0^-) = 0$$

$$\therefore v_c(0^-) = \frac{800\times10^{-6}}{4\times10^{-6}} = 200\ V$$

From continuity equation for capacitor and inductor,

$$i(0^+) = i(0^-) = 0$$

$$v_c(0^+) = -v_c(0^-) = -200\ V$$

So, equation (i) becomes

$$\frac{100}{s} = sI(s) - 0 + 10I(s) - \frac{200}{s} + 250000\frac{I(s)}{s}$$

$$\text{or, } I(s)\left[s + 10 + \frac{250000}{s}\right] = \frac{100}{s} + \frac{200}{s}$$

$$\text{or, } [s^2 + 10s + 250000]\, I(s) = 100 + 200$$

$$\text{or, } I(s) = \frac{300}{s^2 + 10s + 250000}$$

$$\text{or, } I(s) = \frac{300}{s^2 + 2\times s\times 5 + 25 + 250000 - 25}$$

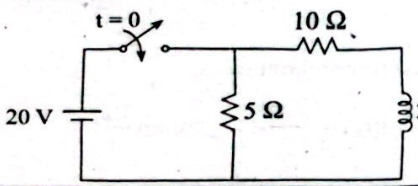
$$\text{or, } I(s) = \frac{300}{(s+5)^2 + (\sqrt{249975})^2} \times \frac{\sqrt{249975}}{\sqrt{249975}}$$

$$\text{or, } I(s) = 0.6\frac{\sqrt{249975}}{(s+5)^2 + (\sqrt{249975})^2}$$

Taking inverse Laplace transform, we get

$$i(t) = 0.6\, e^{-5t} \sin(499.97t)$$

**12. In the circuit shown in figure, obtain the expression for vo... across the inductor if the switch is closed at t = 0 using La... transform method.**




[2075 Asw...]

***Solution:***

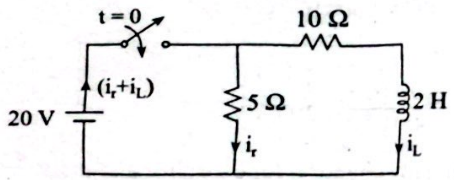



***Figure 1***

Applying KVL for $t > 0$ in left loop of figure 1,

$$20 = 5i_r$$

$$\text{or, } i_r = 4$$

Taking Laplace transform, we get

$$I_r = \frac{4}{5} \quad \text{.............. (1)}$$

Applying KVL for $t > 0$ in outer loop of figure 1,

$$20 = 10i_L + 2\frac{di_L}{dt}$$

$$\text{or, } \frac{di_L}{dt} + 5i_L = 10$$

Taking Laplace transform, we get

$$sI_L - i_L(0^+) + 5I_L = \frac{10}{s}$$

$$(s+5)\, I_L - i_L(0^+) = \frac{10}{s} \quad \text{........... (2)}$$



By observation at $t = 0^-$,

$$i_L(0^-) = 0$$

From continuity relation for inductor,

$$i_L(0^+) = i_L(0^-) = 0 \quad \text{............ (3)}$$

From equations (2) and (3),

$$(s+5)\, I_L = \frac{10}{s}$$

$$\text{or, } I_L = \frac{10}{s(s+5)}$$

$$\text{or, } I_L = 10\left[\frac{1}{s(0+5)} + \frac{1}{(s+5)(-5)}\right]$$

$$\text{or, } I_L = 2\left[\frac{1}{s} - \frac{1}{5(s+5)}\right]$$

Taking inverse Laplace transform,

$$i_L = 2\left(1 - \frac{1}{5}e^{-5t}\right)$$

Hence, voltage across inductor is given by

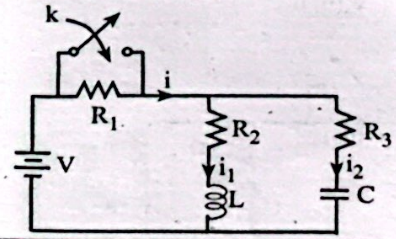
$$v_L = L\frac{di_L}{dt}$$

$$\text{or, } v_L = 2\frac{d}{dt}\left[2 - \frac{2}{5}e^{-5t}\right]$$

$$\text{or, } v_L = 2\left[0 + \frac{10}{5}e^{-5t}\right]$$

$$\therefore\ v_L = 4e^{-5t}$$

**13. In the network shown below, a steady state is reached with the switch k open with V = 100 V, $R_1$ = 10 Ω, $R_2$ = 20 Ω, $R_3$ = 20 Ω, L = 1H and C = 1 μF. At time t = 0, the switch is closed. Evaluate the current $i_1$ and $i_2$ using Laplace transform method for t > 0.**




[2074 Chaitra, 2079 Chaitra]

***Solution:***

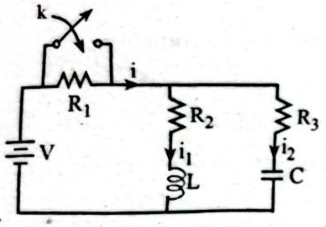



***Figure 1***

Applying KVL for $t > 0$ in left loop,

$$V = v_{R2} + v_L$$

$$\text{or, } 100 = 20i_1 + 1\frac{di_1}{dt}$$

Taking Laplace transform on both sides, we get

$$\frac{100}{s} = 20I_1 + sI_1 - i_1(0^+) \qquad \text{..........(1)}$$

Applying KVL for $t > 0$ in outer loop,

$$V = v_{R3} + v_c$$

$$\text{or, } 100 = 20i_2 + \frac{1}{10^{-6}}\int_{-\infty}^{t} i_2\,dt$$

$$\text{or, } 100 = 20i_2 + 10^6\int_{-\infty}^{0} i_1\,dt + 10^6\int_{0}^{t} i_2\,dt$$

$$\text{or, } 100 = 20i_2 + v_c(0^+) + 10^6\int_{0}^{t} i_2\,dt$$

Taking Laplace transform on both sides, we get

$$\frac{100}{s} = 20I_2 + \frac{v_c(0^+)}{s} + 10^6\frac{I_2}{s}$$

$$\text{or, } I_2\left(20 + \frac{10^6}{s}\right) = \frac{100}{s} - \frac{v_c(0^+)}{s} \qquad \text{...........(2)}$$

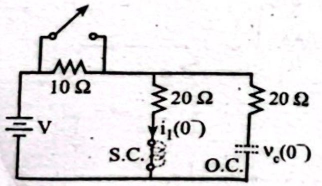



*Figure 2: Equivalent circuit at $t = 0^-$*



From figure 2,

$$I_1(0^-) = \frac{100}{10 + 20} = \frac{10}{3}\text{ A}$$

$$v_c(0^-) = v_{20\Omega}(0^-) = \frac{20}{20 + 10} \times 100 = \frac{200}{3}\text{ V}$$

From continuity relation for inductor and capacitor,

$$i_1(0^+) = i_1(0^-) = \frac{10}{3} \qquad \text{................(3)}$$

$$v_c(0^+) = v_c(0^-) = \frac{200}{3} \qquad \text{................(4)}$$

From equations (1) and (3), we get

$$(s + 20)\,I_1 = \frac{100}{s} + \frac{10}{3}$$

$$\text{or, } I_1 = \frac{100}{s(s + 20)} + \frac{10}{3(s + 20)}$$

$$\text{or, } I_1 = 100\left[\frac{1}{s(0 + 20)} + \frac{1}{(s + 20)(-20)}\right] + \frac{10}{3(s + 20)}$$

$$\text{or, } I_1 = 5\left[\frac{1}{s} - \frac{1}{s + 20}\right] + \frac{10}{3(s + 20)}$$

Taking inverse Laplace transform on both sides, we get

$$i_1 = 5(1 - e^{-20t}) + \frac{10}{3}e^{-20t}$$

$$\text{or, } i_1 = 5 - \frac{5}{3}e^{-20t}$$

From equations (2) and (4), we get

$$I_2\left(\frac{20s + 10^6}{s}\right) = \frac{100}{s} - \frac{200}{3s}$$

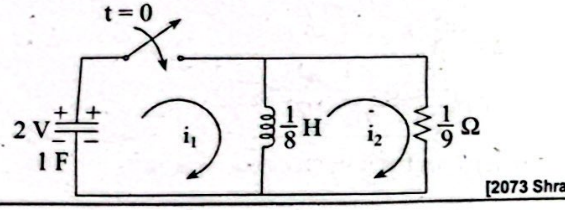
$$\text{or, } I_2\left(\frac{20s + 10^6}{s}\right) = \frac{300 - 200}{3s}$$

$$\text{or, } I_2 = \frac{100}{3(20s + 10^6)} = \frac{100}{20\times3(s + 5\times10^4)} = \frac{5}{3(s + 5\times10^4)}$$

Taking inverse Laplace transform on both sides, we get

$$i_2 = \frac{5}{3}\,e^{-5\times10^4 t}$$

**14. Using Laplace transform, find the currents $i_1$ and $i_2$ for $t > 0$.**




[2073 Shrawan, 2079 Ashwin]

***Solution:***

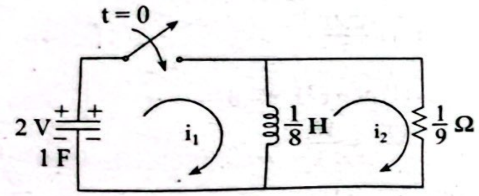



***Figure 1***

Applying KVL for t > 0 in loop 1,

$$\frac{1}{C}\int_{-\infty}^{t} i_1\, dt + \frac{1}{8}\frac{d}{dt}(i_1 - i_2) = 0$$

$$\text{or, } v_C(0^+) + \frac{1}{1}\int_{0}^{t} i_1\, dt + \frac{1}{8}\frac{d}{dt}(i_1 - i_2) = 0$$

Taking Laplace transform, we get

$$\frac{v_C(0^+)}{s} + \frac{I_1}{s} + \frac{1}{8}[s(I_1 - I_2) - \{i_1(0^+) - i_2(0^+)\}] = 0$$

By observation at $t = 0^-$,

$$i_L(0^-) = 0$$

$$i_1(0^+) - i_2(0^+) = i_L(0^+) = i_L(0^-) = 0$$

$$\text{or, } -\frac{2}{s} + \frac{I_1}{s} + \frac{1}{8}s(I_1 - I_2) = 0$$

$$\text{or, } I_1\left(\frac{1}{s} + \frac{s}{8}\right) - \frac{s}{8}I_2 = \frac{2}{s}$$

$$\text{or, } \frac{I_1(8 + s^2) - s^2 I_2}{8s} = \frac{2}{s}$$

$$\text{or, } (s^2 + 8)I_1 - s^2 I_2 = 16 \qquad \text{............ (1)}$$



Applying KVL for t > 0 in loop 2,

$$\frac{1}{9}i_2 + \frac{1}{8}\frac{d}{dt}(i_2 - i_1) = 0$$

Taking Laplace transform on both sides, we get

$$\frac{1}{9}I_2 + \frac{1}{8}[s(I_2 - I_1) - \{i_2(0^+) - i_1(0^+)\}] = 0$$

$$\text{or, } \frac{1}{9}I_2 + \frac{s}{8}(I_2 - I_1) = 0$$

$$\text{or, } -\frac{s}{8}I_1 + \left(\frac{1}{9} + \frac{s}{8}\right)I_2 = 0$$

$$\text{or, } \frac{-9sI_1 + (8 + 9s)I_2}{72} = 0$$

$$\text{or, } -9sI_1 + (9s + 8)I_2 = 0 \qquad \text{............... (2)}$$

From (1) and (2),

$$\begin{bmatrix} s^2 + 8 & -s^2 \\ -9s & 9s + 8 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 16 \\ 0 \end{bmatrix}$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{16(9s + 8)}{(s^2 + 8)(9s + 8) - 9s^3} = \frac{2(9s + 8)}{s^2 + 9s + 8} = \frac{2(9s + 8)}{(s + 1)(s + 8)}$$

$$\text{or, } I_1 = \frac{2(-9+8)}{(s + 1)(-1+8)} + \frac{2(-72 + 8)}{(s + 8)(-8 + 1)}$$

$$\text{or, } I_1 = \frac{-2}{7(s + 1)} + \frac{128}{7(s + 8)}$$

Taking inverse Laplace transform on both sides, we get

$$i_1 = -\frac{2}{7}e^{-t} + \frac{128}{7}e^{-8t}$$

$$\text{Also, } I_2 = \frac{\Delta_2}{\Delta} = \frac{16\times 9s}{8s^2 + 72s + 64} = \frac{18s}{s^2 + 9s + 8} = \frac{18s}{(s+1)(s+8)}$$

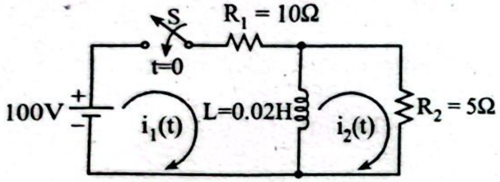
$$\text{or, } I_2 = \frac{-18}{(s + 1)(-1 + 8)} + \frac{18(-8)}{(s + 8)(-8 + 1)}$$

$$\text{or, } I_2 = \frac{-18}{7(s + 1)} + \frac{20.57}{s + 8}$$

Taking inverse Laplace transform on both sides, we get

$$i_2 = -\frac{18}{7}e^{-t} + 20.57\, e^{-8t}$$

**15. In the circuit below, find $i_1(t)$ and $i_2(t)$ and the output voltage across $5\ \Omega$ when the switch is closed at $t > 0$ using Laplace method.**




[2080 Chaitra]

***Solution:***

Using KVL for $t > 0$ in loop 1, we get

$$100 - v_{R_1} - v_L = 0$$

or, $v_L + v_{R_1} = 100$

or, $L\frac{d}{dt}(i_1 - i_2) + R_1 i_1 = 100$

or, $0.02\frac{d}{dt}(i_1 - i_2) + 10i_1 = 100$

Taking Laplace transform, we get

$$0.02[s(I_1 - I_2) - \{i_1(0^+) - i_2(0^+)\}] + 10I_1 = \frac{100}{s} \quad \text{............. (1)}$$

By observation at $t = 0^-$,

$i_L(0^-) = 0 \Rightarrow i_1(0^-) - i_2(0^-) = 0$

From continuity relation for inductor,

$i_1(0^+) - i_2(0^+) = i_1(0^-) - i_2(0^-) = 0$ .............. (2)

From (1) & (2),

$$0.02s(I_1 - I_2) + 10I_1 = \frac{100}{s}$$

or, $\frac{2}{100}s(I_1 - I_2) + 10I_1 = \frac{100}{s}$

or, $(2s + 1000)I_1 - 2sI_2 = \frac{10000}{s}$ ......... (3)

Applying KVL for $t > 0$ in loop 2,

$v_L + v_{R_2} = 0$

or, $L\frac{d}{dt}(i_2 - i_1) + i_2R_2 = 0$

or, $0.02\frac{d}{dt}(i_2 - i_1) + 5i_2 = 0$



Taking Laplace transform, we get

$$0.02[s(I_2 - I_1) - \{i_2(0^+) - i_1(0^+)\}] + 5I_2 = 0$$

or, $0.02s(I_2 - I_1) + 5I_2 = 0$

or, $\frac{2}{100}s(I_2 - I_1) + 5I_2 = 0$

or, $(2s + 500)I_2 - 2sI_1 = 0$

or, $2sI_1 - (2s + 500)I_2 = 0$ ......... (4)

Putting equations (3) & (4) in matrix form,

$$\begin{bmatrix} (2s+1000) & -2s \\ 2s & -(2s+500) \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{10000}{s} \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\therefore I_1 = \frac{\Delta_1}{\Delta} = \frac{-\frac{10000}{s}(2s + 500)}{-(2s + 1000)(2s + 500) + 4s^2}$$

$$= \frac{-10000(2s + 500)}{s[-(4s^2 + 1000s + 2000s + 500000) + 4s^2]}$$

$$= \frac{-10000(2s + 500)}{-s(3000s + 500000)}$$

$$= \frac{10000(2s + 500)}{3000s\left(s + \frac{500}{3}\right)}$$

$$= \frac{10(2s + 500)}{3s\left(s + \frac{500}{3}\right)}$$

$$= \frac{10}{3}\left[\frac{(2 \times 0 + 500)}{s\left(0 + \frac{500}{3}\right)} + \frac{2\left(-\frac{500}{3}\right) + 500}{\left(s + \frac{500}{3}\right)\left(-\frac{500}{3}\right)}\right]$$

$$= \frac{10}{3} \times \frac{500}{\frac{500}{3}(s)} + \frac{10}{3} \times \frac{500}{3} \times \left(-\frac{3}{500}\right)\frac{1}{\left(s + \frac{500}{3}\right)}$$

$$= \frac{10}{s} - \frac{10}{3}\frac{1}{\left(s + \frac{500}{3}\right)}$$

Taking inverse Laplace transform, we get

$$i_1 = 10 - \frac{10}{3} e^{-(500/3)t}$$

Also, by Cramer's rule,

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{-20000}{-(3000s + 500000)}$$

$$\text{or, } I_2 = \frac{20000}{3000\left(s + \frac{500}{3}\right)} = \frac{20}{3\left(s + \frac{500}{3}\right)}$$

Taking inverse Laplace transform, we get

$$i_2 = \frac{20}{3} e^{-(500/3)t}$$

Now, voltage across 5Ω resistor is

$$v_{R_2} = i_2 R_2 = \left[\frac{20}{3} e^{-(500/3)t}\right] 5 = \frac{100}{3} e^{-(500/3)t}$$

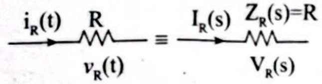


Chapter - 4

# NETWORK TRANSFER FUNCTION AND FREQUENCY RESPONSE

## 4.1 Transform Impedance and Admittance of Different Circuit Elements

**1) Resistor (R)**




$v_R = Ri_R$

Taking Laplace transform on both sides,

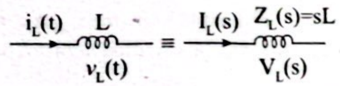
$V_R(s) = R\, I_R(s)$

$$\text{or, } R = \frac{V_R(s)}{I_R(s)} = Z_R(s) = \text{Transform impedance of resistor}$$

$$\text{or, } \frac{1}{R} = \frac{I_R(s)}{V_R(s)} = Y_R(s) = \text{Transform admittance of resistor}$$

$$\therefore Z_R(s) = \frac{1}{Y_R(s)} = R$$

**2) Inductor (L)**




$$v_L = L\frac{di}{dt}$$

Taking Laplace transform on both sides,

$V_L(s) = L[sI_L(s) - i_L(0^+)]$

Let $i_L(0^+) = 0$

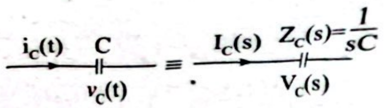
So, $V_L(s) = sLI_L(s)$

$$\text{or, } sL = \frac{V_L(s)}{I_L(s)} = Z_L(s) = \text{Transform impedance of inductor}$$

$$\text{or, } \frac{1}{sL} = \frac{I_L(s)}{V_L(s)} = Y_L(s) = \text{Transform admittance of inductor}$$

$$\therefore Z_L(s) = \frac{1}{Y_L(s)} = sL$$

3) **Capacitor (C)**




$$i_C = C\frac{dv_C}{dt}$$

Taking Laplace transform on both sides,

$I_C(s) = C[sV_C(s) - v_C(0^+)]$

Let $v_C(0^+) = 0$

So, $I_C(s) = Cs\,V_C(s)$

or, $\frac{1}{sC} = \frac{V_C(s)}{I_C(s)} = Z_C(s)$ = Transform impedance of capacitor

or, $sC = \frac{I_C(s)}{V_C(s)} = Y_C(s)$ = Transform admittance of capacitor

$\therefore Z_C(s) = \frac{1}{Y_C(s)} = \frac{1}{sC}$

## 4.2 Transfer Function or Network Function [H(s), T(s), or N(s)]

The ratio of a voltage or current response to the source voltage or current in the transformed circuit is called a *transfer function*. The transfer function indicates how the network transfers an input from the source to an output response. Transfer functions are also called *network functions*.

### 4.2.1 Network Function of Two-Port Network (TPN)

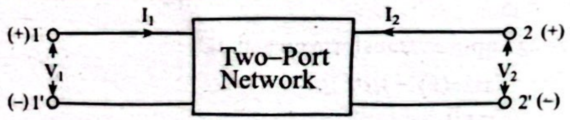



*Figure 4.1: Two-port network with standard reference directions for voltages and currents.*

Consider an arbitrary network made up entirely of passive elements. To indicate the general nature of the network, let it be represented by the symbol of a rectangle (or a box). If a conductor is fastened to any node in the network and brought out of the box for access, the end of this conductor is designated as *terminal*. Terminals are required for connecting driving forces to the network, for connecting some other network (say, a load), or for



making measurements. The minimum number of terminals that are useful is two. Also, the terminals are associated in pairs, one pair for a driving force, another pair for the load, etc. Two associated terminals are given the name *terminal pair* or *port*, suggesting a port of entry into the network.

A *two-port network* (TPN) is one of the examples of multiport network. It consists of four terminals. (1, 1', 2, and 2'). One pair of terminal (1, 1') makes input port or port 1, and another pair of terminal (2, 2') makes output port or port 2.

There are four variables ($V_1$, $I_1$, $V_2$, $I_2$) of a TPN. $V_1$ and $I_2$ are the voltage and current at the output port with the reference direction and polarities as shown in figure.

Network function is a ratio of any two variable. The transfer function may have the dimensions of ohms, mhos, or be a dimensionless ratio depending on the quantities taken as input and output.

1) **Driving Point Network Function**

It is the ratio of two quantities of same port i.e., $\frac{V_1}{I_1}, \frac{I_1}{V_1}, \frac{V_2}{I_2}, \frac{I_2}{V_2}$.

$Z_{11}(s) = \frac{V_1(s)}{I_1(s)}$ is called *driving point impedance of port 1* or *input driving point impedance.*

$Y_{11}(s) = \frac{I_1(s)}{V_1(s)}$ is called *driving point admittance of port 1* or *input driving point admittance.*

$Z_{22}(s) = \frac{V_2(s)}{I_2(s)}$ is called *driving point impedance of port 2* or *output driving point impedance.*

$Y_{22}(s) = \frac{I_2(s)}{V_2(s)}$ is called *driving point admittance of port 2* or *output driving point admittance.*

2) **Transform Network Function**

It is the ratio of two quantities of different port

i.e., $\frac{V_2}{I_1}, \frac{I_1}{V_2}, \frac{V_1}{I_2}, \frac{I_2}{V_1}, \frac{V_2}{V_1}, \frac{V_1}{V_2}, \frac{I_2}{I_1}, \frac{I_1}{I_2}$.

a. **Forward transform network function**

It is the ratio of quantity of port 2 to that of port 1.

$G_{21}(s) = \frac{V_2(s)}{V_1(s)}$ is called *forward voltage ratio transfer function* or *forward voltage gain* or simply, *voltage gain.*

$Y_{21}(s) = \frac{I_2(s)}{V_1(s)}$ is called *forward admittance.*

$Z_{21}(s) = \frac{V_2(s)}{I_1(s)}$ is called *forward impedance.*

$\alpha_{21}(s) = \frac{I_2(s)}{I_1(s)}$ is called *forward current ratio transfer function* or *forward current gain* or simply, *current gain.*

**b. Reverse transform network function**

It is the ratio of quantity of port 1 to that of port 2.

$G_{12}(s) = \frac{V_1(s)}{V_2(s)}$ is called *reverse voltage ratio transfer function* or *reverse voltage gain.*

$Y_{12}(s) = \frac{I_1(s)}{V_2(s)}$ is called *reverse admittance.*

$Z_{12}(s) = \frac{V_1(s)}{I_2(s)}$ is called *reverse impedance.*

$\alpha_{12}(s) = \frac{I_1(s)}{I_2(s)}$ is called *reverse current ratio transfer function* or *reverse current gain.*

All the above transfer functions can be summarized by the expression

$$Z_{xy} = \frac{V_x}{I_y}$$

1. If $x = y$, then it is called *driving point network function.*
2. If $x \neq y \Rightarrow$ Transform network function.
   a. If $x > y$, then it is called *forward transform network function.*
   b. If $x < y$, then it is called *reverse transform network function.*

**Procedure to find network function of two-port network (TPN):**

1. Transform the circuit in the s domain.
2. Remember that $V_2$ is not a source but it is a voltage drop.
3. Check either TPN is terminated or unterminated at port 2. If it is unterminated (open circuit), then $I_2 = 0$.

## 3 Poles and Zeros of Network Function

ie network function is always the ratio of two polynomial in s. Hence, it n be written as



$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0 s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1}s + a_n}{b_0 s^m + b_1 s^{m-1} + \ldots\ldots + b_{m-1}s + b_m}$$

where a and b coefficients are real and positive for networks of passive elements and no controlled sources. Now the equation $P(s) = 0$ has n roots, and $Q(s) = 0$ similarly has m roots. Both P(s) and Q(s) may be written as a product of linear factors involving these roots

$$N(S) = K\frac{(s - z_1)(s - z_2)\ldots\ldots(s - z_n)}{(s - p_1)(s - p_2)\ldots\ldots(s - p_m)}$$

where $K = \frac{a_0}{b_0}$ is a constant known as the *scale factor*, and $z_1, z_2, \ldots\ldots, z_n, p_1, p_2, \ldots\ldots, p_m$ are complex frequencies. When the variable s has the values $z_1, z_2, \ldots\ldots, z_n$, the network function vanishes; such complex frequencies are known as the *zeros* of the network function. When the variable s has the values $p_1, p_2, \ldots\ldots, p_m$, the network function becomes infinite; such complex frequencies are known as the *poles* of the network function. The factors $(s - z_1)(s - z_2)\ldots\ldots(s - z_n)$ are known as *zero factors*; the factors $(s - p_1)(s - p_2)\ldots\ldots(s - p_m)$ are known as *pole factors*. Thus, poles and zeros are known as critical frequencies. A network function is completely specified by its poles, zeros, and the scale factor.

## 4.4 Normal form of Second Order System

The auxiliary equation of second order RLC series circuit is given by

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$\text{or, } s^2 + \frac{R}{R_{cr}}s + \frac{1}{LC} = 0$$

$$\text{where } R_{cr} = 2\sqrt{\frac{L}{C}}$$

$$\text{or, } s^2 + \zeta\frac{2\sqrt{\frac{L}{C}}}{L}s + \frac{1}{LC} = 0$$

$$\text{where } \zeta = \frac{R}{R_{cr}} = \text{damping ratio}$$

$$\text{or, } s^2 + 2\zeta\frac{1}{\sqrt{LC}}s + \frac{1}{\sqrt{(LC)}^2} = 0$$

$$\text{or, } s^2 + 2\zeta\,\omega_n s + \omega_n^2 = 0 \qquad \ldots\ldots\ldots (1)$$

Equation (1) is known as normal form of second order system.

If $R < R_{cr}$, $\zeta < 1$, then root will be complex, and system is said to be *underdamped*.

If $R = R_{cr}$, $\zeta = 1$, then root will be real and equal, and system is said to be *critically damped*.

If $R > R_{cr}$, $\zeta > 1$, then root will be real and equal, and system is said to be *overdamped*.

In steady state, s may be replaced by $j\omega$.

Hence, equation (1) becomes

$$(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2 = 0$$

$$\text{or, } -\left(\frac{\omega}{\omega_n}\right)^2 + 2j\zeta\frac{\omega}{\omega_n} + 1 = 0$$

$$\text{or, } 1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2j\zeta\frac{\omega}{\omega_n} = 0 \qquad \text{........ (2)}$$

Equation (2) is known as normal form of second order system in steady state.

$$1 - \mu^2 + 2j\zeta\mu = 0 \qquad \text{.......... (2a)}$$

where $\mu = \frac{\omega}{\omega_n}$.

## 4.5 Frequency Response of Network

The response given by the system when input frequency $\omega$ is changed over a certain range is called *frequency response* of network or the system.

Let N(s) be any network function either driving-point or transfer. A network function is always a complex number and can be written in rectangular form as

$$N(s) = R(s) + jX(s)$$

where R(s) = real part of N(s), X(s) = imaginary part of N(s).

If the network function is written in polar form, it is expressed as

$$N(s) = |N(s)| \angle\phi(s)$$

$$\text{where } |N(s)| = \sqrt{[R(s)]^2 + [X(s)]^2} \qquad \text{..........(1)}$$

$$\phi(s) = \tan^{-1}\left[\frac{X(s)}{R(s)}\right] \qquad \text{.......... (2)}$$

For networks in the sinusoidal steady state, $s = j\omega$.



$$\therefore |N(j\omega)| = \sqrt{\{R(j\omega)\}^2 + \{X(j\omega)\}^2} \qquad \text{.........(1a)}$$

$$\phi(j\omega) = \tan^{-1}\left[\frac{X(j\omega)}{R(j\omega)}\right] \qquad \text{.......... (2a)}$$

From equations (1a) and (2a), we see that both magnitude and phase angle depend upon frequency. There is one graph known as Bode diagram which shows the variation of magnitude and phase angle with respect to frequency. There are two plots in Bode diagram:

1. *Magnitude* of network function in dB on vertical axis against frequency in log scale on horizontal axis.
2. *Phase angle* of network function in degree on vertical axis against frequency in log scale on horizontal axis.

$$(X)_{dB} = 20 \log |X| \text{ dB}$$

$$\text{So, } (3 + j4)_{dB} = 20 \log (\sqrt{3^2 + 4^2}) = 20 \log 5 \text{ dB}$$

$$(j\omega)_{dB} = 20 \log \omega \text{ dB}$$

$$\left(\frac{2}{j\omega}\right)_{dB} = 20 \log \frac{2}{\omega} \text{ dB}$$

$$\left(\frac{4}{(j\omega)^2}\right)_{dB} = 20 \log \frac{4}{(\omega)^2} \text{ dB}$$

## 4.6 Bode-Plot

A network function containing constant term, simple zeros, multiple poles at origin, simple poles, and complex poles are given by

$$N(s) = \frac{k'(s + z_1)(s + z_2)\text{.......}(s + z_n)}{(s)^r (s + p_1)(s + p_2)\text{........}(s + p_m)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$\text{or, } N(s) = \frac{k' z_1 z_2 \text{.....} z_n \left(1 + \frac{s}{z_1}\right)\left(1 + \frac{s}{z_2}\right)\text{.......}\left(1 + \frac{s}{z_n}\right)}{p_1 p_2 \text{.......} p_m \omega_n^2 (s)^r \left(1 + \frac{s}{p_1}\right)\left(1 + \frac{s}{p_2}\right)\text{.....}\left(1 + \frac{s}{p_m}\right)\left[1 + \left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n}\right]}$$

$$\text{or, } N(s) = \frac{k(1 + sT_{z1})(1 + sT_{z2})\text{......}(1 + sT_{zn})}{(s)^r (1 + sT_{p1})(1 + sT_{p2})\text{....}(1 + sT_{pm})\left[1 + \left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n}\right]}$$

where $T_{z1} = \frac{1}{z_1}$, $T_{p1} = \frac{1}{p_1}$

In steady state, $s \rightarrow j\omega$

$$\therefore N(j\omega) = \frac{k\,(1 + j\omega T_{z1})\,(1 + j\omega T_{z2}) \,......\, (1 + j\omega T_{zn})}{(j\omega)^r\,(1 + j\omega T_{p1})\,(1 + j\omega T_{p2}) \,......\, (1 + j\omega T_{pm})\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2j\zeta\frac{\omega}{\omega_n}\right]} \quad ...(1)$$

Equation (1) is known as time constant form of network function in steady state. From equation (1), we also conclude that a network function may either consist of any of the following factors or product of any two or all.

i) **Constant gain:** [k]

ii) **One zero at origin:** $[s]^{+1} \Rightarrow [j\omega]^{+1}$

iii) **Multiple zeros at origin:** $[s]^{+r} \Rightarrow [j\omega]^{+r}$

iv) **One pole at origin:** $[s]^{-1} \Rightarrow [j\omega]^{-1}$

v) **Multiple poles at origin:** $[s]^{-r} \Rightarrow [j\omega]^{-r}$

vi) **Simple zero:** $[s + z]^{+1} \Rightarrow [1 + j\omega T]^{+1}$

vii) **Simple pole:** $[s + p]^{-1} \Rightarrow [1 + j\omega T]^{-1}$

viii) **Complex zero:** $[s^2 + 2\zeta\omega_n s + \omega^2{}_n]^{+1} \Rightarrow \left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}\right]^{+1}$

ix) **Complex pole:** $[s^2 + 2\zeta\omega_n s + \omega^2{}_n]^{-1} \Rightarrow \left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}\right]^{-1}$

**Summary for magnitude plot:**

| Network function | Factors | Corner frequency | Slope |
|---|---|---|---|
| Constant gain | [k] | Low | 0 dB/decade |
| One zero at origin | $[j\omega]^{+1}$ | Low (+) | +20 dB/decade |
| Two zero at origin | $[j\omega]^{+2}$ | Low (+2) | +40 dB/decade |
| One pole at origin | $[j\omega]^{-1}$ | Low (–) | –20 dB/decade |
| Two pole at origin | $[j\omega]^{-2}$ | Low (–2) | –40 dB/decade |
| Simple zero | $[1 + j\omega T]^{+1}$ | $\frac{1}{T}(+)$ | +20 dB/decade |
| Simple pole | $[1 + j\omega T]^{-1}$ | $\frac{1}{T}(-)$ | –20 dB/decade |



| Network function | Factors | Corner frequency | Slope |
|---|---|---|---|
| Complex zero | $\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}\right]^{+1}$ | $\omega_n$ (+ +) | +40 dB/decade |
| Complex pole | $\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}\right]^{-1}$ | $\omega_n$ (– –) | –40 dB/decade |

**Summary for phase plot:**

$$\text{Effective frequency} = \frac{\text{Corner frequency}}{10}$$

| Network function | Factors | Corner frequency | Effective frequency | Slope |
|---|---|---|---|---|
| Constant gain | [k] | Low | Low | 0°/decade |
| One zero at origin | $[j\omega]^{+1}$ | Low (+) | Low (+) | 0°/decade |
| Two zero at origin | $[j\omega]^{+2}$ | Low (+2) | Low (+2) | 0°/decade |
| One pole at origin | $[j\omega]^{-1}$ | Low (–) | Low (–) | 0°/decade |
| Two pole at origin | $[j\omega]^{-2}$ | Low (–2) | Low (–2) | 0°/decade |
| Simple zero | $[1 + j\omega T]^{+1}$ | $\frac{1}{T}(+)$ | $\frac{0.1}{T}(+)$ | +45°/decade |
| Simple pole | $[1 + j\omega T]^{-1}$ | $\frac{1}{T}(-)$ | $\frac{0.1}{T}(-)$ | –45°/d |
| Complex zero | $\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}\right]^{+1}$ | $\omega_n$ (+ +) | $0.1\omega_n$ (+ +) | +90°/dec |
| Complex pole | $\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}\right]^{-1}$ | $\omega_n$ (– –) | $0.1\omega_n$ (– –) | –90°/decade |

## 4.7 Filters

A *filter* is an electronic network which passes or allows without loss (or gain), transmission of electric signal within certain frequency range but disallows transmission of electric signal outside this frequency range. Therefore, filter is usually a *frequency-selective network*.

Filters have numerous applications, some of which are as follows:

1. Filters are used in communication systems for suppressing noise.
2. Filters are employed in high-performance stereo systems, where certain ranges of audio frequencies need to be amplified or suppressed for best sound quality and power efficiency.
3. Filters are used in digital image processing to enhance image.
4. Filters are used in biomedical instruments to interface sensors with diagnostic equipments and data logging.

Filters can be classified as *passive filters* and *active filters*. A *passive filter* is built with passive components such as resistors, capacitors, and inductors. An *active filter* is built with transistors or op-amps (providing voltage amplification, and isolation or buffering) in addition to resistors, and capacitors.

Filter may also be classified as:

(i) **low-pass** (ii) **high-pass**

(iii) **band-pass** (iv) **band-stop**

(v) **all-pass**

A *low-pass filter* is one in which the passband extends from $f = 0$ to $f = f_c$, where $f_c$ is known as the cutoff frequency. A *high-pass filter* is the complement of the low-pass filter in that the frequency range from 0 to $f_c$ is the stopband and from $f_c$ to infinity is the pass band. A *band-pass filter* is one in which the frequencies extending from $f_{c1}$ to $f_{c2}$ are passed, while signals at all other frequencies are stopped. A *band-stop filter* is the complement of the band-pass filter where signal components at the frequencies from $f_{c1}$ to $f_{c2}$ are stopped and all others are passed. These filters are also sometimes referred to as *notch filters* because of the "notch" in their transmission characteristic.

The circuit diagrams and transfer characteristics for different types of filters are illustrated by the diagrams that follow.



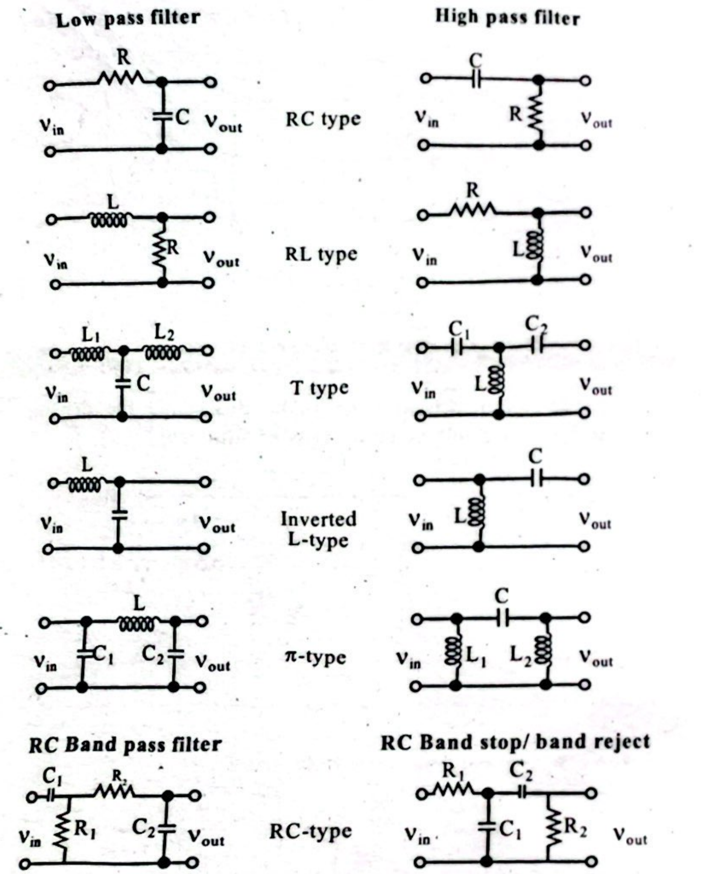
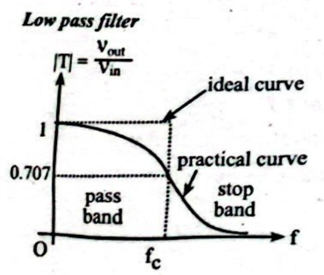
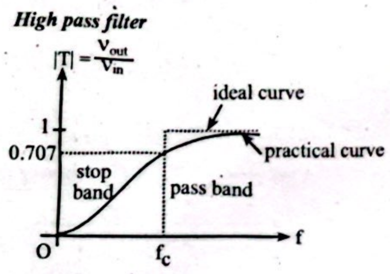



***Figure 4.2:*** *Different types of filters*



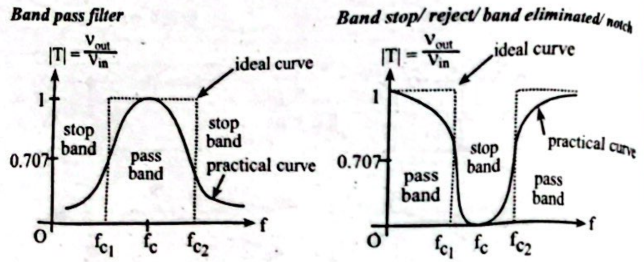
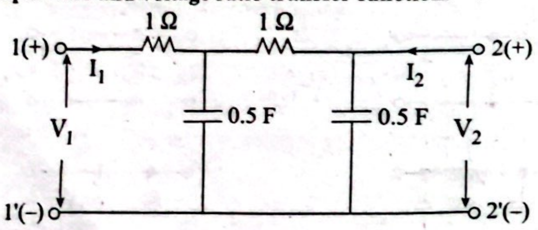



*Figure 4.3: Transfer characteristics of different types of filters*

## SOLVED PROBLEMS

**1. For the given two-port network, determine the driving point impedance and voltage ratio transfer function.**




[2074 Ashwin]

***Solution:***

The transform network is shown in figure 1.

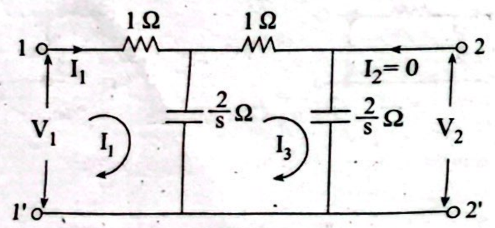



***Figure 1***

Since port 2 is unterminated, $I_2 = 0$ as shown in figure 1.

Applying KVL and arranging in matrix form,

$$\begin{bmatrix} \left(1+\frac{2}{s}\right) & -\frac{2}{s} \\ -\frac{2}{s} & \left(1+\frac{4}{s}\right) \end{bmatrix} \begin{bmatrix} I_1 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \end{bmatrix}$$



By Cramer's rule,

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} V_1 & -2/s \\ 0 & (1+4/s) \end{vmatrix}}{(1+2/s)(1+4/s) - \frac{4}{s^2}} = \frac{V_1\left(\frac{s+4}{s}\right)}{1+\frac{6}{s}+\frac{4}{s^2}}$$

$$\text{or, } I_1 = \frac{V_1 s(s+4)}{s^2+6s+4}$$

$$I_3 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} \left(1+\frac{2}{s}\right) & V_1 \\ -2/s & 0 \end{vmatrix}}{1+\frac{6}{s}+\frac{4}{s^2}} = \frac{\frac{2}{s}V_1}{(s^2+6s+4)/s^2} = \frac{2V_1 s}{s^2+6s+4}$$

Driving-point impedance of port 1 is

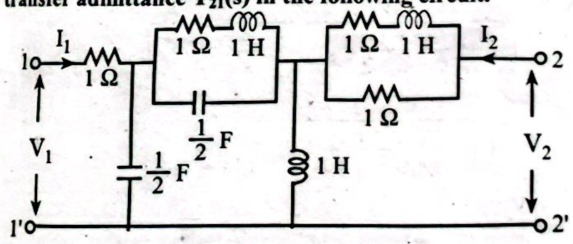
$$Z_{11} = \frac{V_1}{I_1} = \frac{V_1}{\frac{V_1 s(s+4)}{s^2+6s+4}} = \frac{s^2+6s+4}{s(s+4)}$$

Voltage ratio transfer function is

$$G_{21} = \frac{V_2}{V_1} = \frac{I_3 \times \frac{2}{s}}{V_1} = \frac{\frac{2}{s}\left[\frac{2V_1 s}{(s^2+6s+4)}\right]}{V_1} = 4(s^2+6s+4)$$

$$\text{and } G_{12} = \frac{1}{4(s^2+6s+4)}$$

**2. Find the forward voltage ratio transfer function $G_{21}(s)$ and forward transfer admittance $Y_{21}(s)$ in the following circuit.**

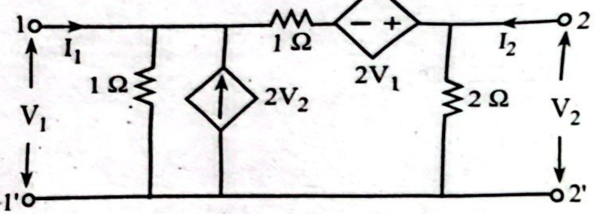



[2074 Chaitra]

***Solution:***

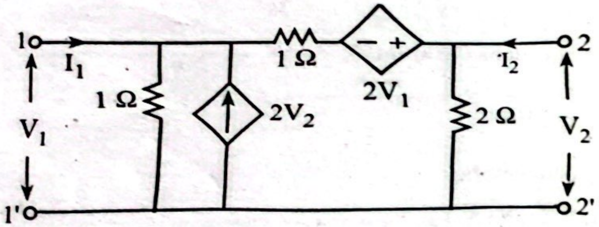
Since port 2 is unterminated, $I_2 = 0$ and the transform network is given by

**2. Find transmission and admittance parameters for the given two-port network and check its reciprocity and symmetry.**



[2074 Chaitra]

***Solution:***



For Y parameters,

$I_1 = Y_{11}V_1 + Y_{12}V_2$ .......... (1)

$I_2 = Y_{21}V_1 + Y_{22}V_2$ ........ (2)

Applying $V_1$ at port 1 and make port 2 short circuited i.e., $V_2 = 0$ as shown in figure 1. Then, from equations (1) and (2),

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \qquad ............(3)$$

and $$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} \qquad ................(4)$$

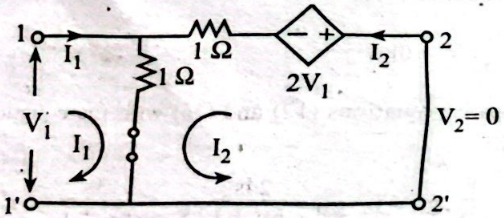


*Figure 1*

Applying KVL in loop 1,

$V_1 = I_1 + I_2$ ..............(5)

*Applying KVL in loop 2,*

$-2V_1 = 2I_2 + I_1$

or, $I_1 = -2V_1 - 2I_2$ .............(6)

From (5) and (6),

$V_1 = (-2V_1 - 2I_2) + I_2$

or, $3V_1 = -I_2$ ..............(7)

or, $\frac{I_2}{V_1} = -3 \Rightarrow Y_{21} = \frac{I_2}{V_1} = -3\ \Omega^{-1}$

From (7) and (5),

$V_1 = I_1 - 3V_1$

or, $I_1 = 4V_1$

or, $\frac{I_1}{V_1} = 4$

$\therefore\ Y_{11} = \frac{I_1}{V_1} = 4\ \Omega^{-1}$

Applying $V_2$ at port 2 and making port 1 short circuited i.e., $V_1 = 0$ as shown in figure 2. Then, from equations (1) and (2),

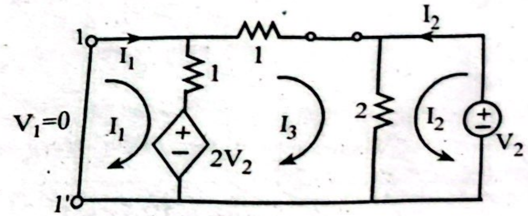


*Figure 2*

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \qquad ..............(8)$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} \qquad ................(9)$$

Applying KVL in loop 1,

$-2V_2 = I_1 - I_3$ ................(10)

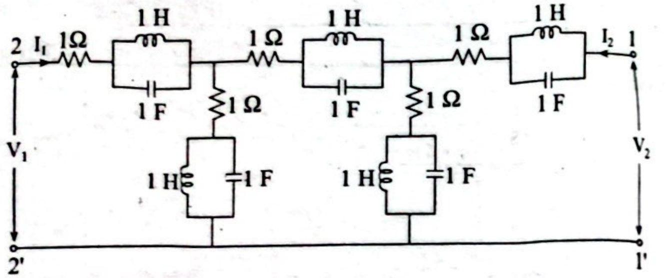
Applying KVL in loop 3,

$2V_2 = 4I_3 - I_1 + 2I_2$ .............(11)

Similarly,

$$Z_{12} = \frac{V_1}{I_2} = 0 \text{ V/A}$$

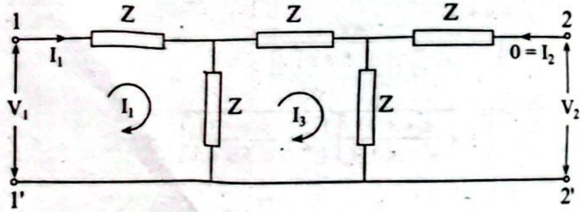
**4. For the two-port network shown in figure below, find the voltage ratio transfer function.**




[2073 Shrawan]

***Solution:***

$$\text{Let } Z = 1 + s \,//\, \frac{1}{s} = 1 + \frac{s \times \frac{1}{s}}{s + \frac{1}{s}} = 1 + \frac{s}{s^2+1} = \frac{s^2+s+1}{s^2+1}$$



$$\begin{bmatrix} 2s & -Z \\ -Z & 3Z \end{bmatrix} \begin{bmatrix} I_1 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \end{bmatrix}$$

$$I_3 = \frac{\Delta_2}{\Delta} = \frac{V_1 Z}{6Z^2 - Z^2} = \frac{V_1 Z}{5Z^2} = \frac{V_1}{5Z}$$

Now,

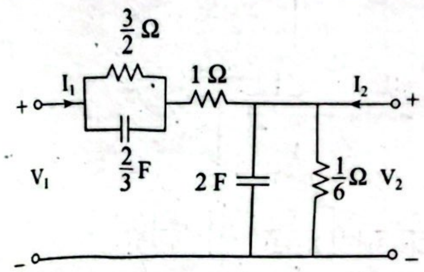
$$V_2 = I_3 Z$$

$$\text{or, } V_2 = \frac{V_1}{5Z} \times Z = \frac{V_1}{5}$$

$$G_{21} = \frac{V_2}{V_1} = \frac{\frac{V_1}{5}}{V_1} = \frac{1}{5}$$



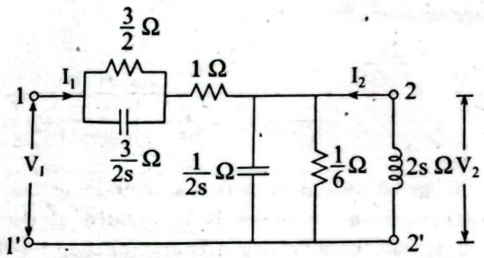
**5. Find the voltage ratio transfer function of the two-port network shown in figure below, if the port 2 is terminated with 2 H inductor.**

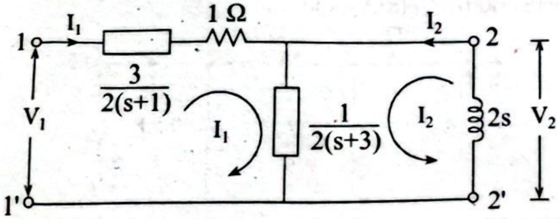



[2075 Ashwin]

***Solution:***

The transform network with port 2 terminated is shown below.




Applying KVL & arranging in matrix form,

$$\begin{bmatrix} \frac{3}{2(s+1)} + 1 + \frac{1}{2(s+3)} & \frac{1}{2(s+3)} \\ \frac{1}{2(s+3)} & \frac{1}{2(s+3)} + 2s \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \end{bmatrix}$$

$$I_2 = \frac{\Delta_2}{\Delta}$$

$$\Delta_2 = \begin{vmatrix} \frac{3}{2(s+1)} + 1 + \frac{1}{2(s+3)} & V_1 \\ \frac{1}{2(s+3)} & 0 \end{vmatrix} = \frac{-V_1}{2(s+3)}$$

$$\Delta = \begin{vmatrix} \frac{3}{2(s+1)} + 1 + \frac{1}{2(s+3)} & \frac{1}{2(s+3)} \\ \frac{1}{2(s+3)} & \frac{1}{2(s+3)} + 2s \end{vmatrix}$$

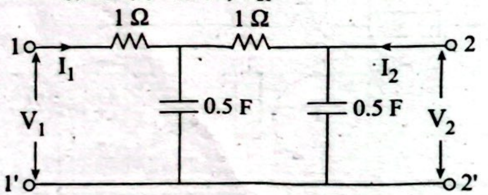
$$= \left[\frac{3}{2(s+1)} + 1 + \frac{1}{2(s+3)}\right] \times \left[\frac{1}{2(s+3)} + 2s\right] - \frac{1}{4(s+3)^2}$$

Now, $I_2 = \dfrac{\frac{-V_1}{2(s+3)}}{\Delta}$ and $V_2 = -I_2 \times 2s = \dfrac{\frac{2sV_1}{2(s+3)}}{\Delta}$

Voltage ratio transfer function is:

$$G_{21} = \frac{V_2}{V_1} = \frac{\frac{2s}{2(s+3)}}{\Delta} = \frac{\frac{2s}{2(s+3)}}{\left[\frac{3}{2(s+1)} + 1 + \frac{1}{2(s+3)}\right]\left[\frac{1}{2(s+3)} + 2s\right] - \frac{1}{4(s+3)^2}}$$

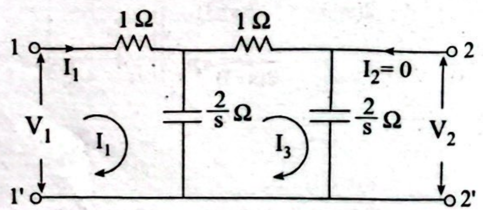
**6. For the given two port network, determine the driving point impedance. If this network is terminated at port 2 with 1 F capacitor, find the following network function for this terminated network: (i) $Z_{21}$ (ii) $Y_{21}$ (iii) $\alpha_{21}$**




[2075 Chaitra]

***Solution:***

The transform network is shown in figure.






Applying KVL in loops (1) & (3), and arranging them in matrix from, we get

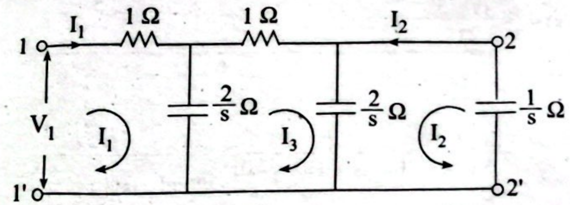
$$\begin{bmatrix} \left(1+\frac{2}{s}\right) & -\frac{2}{s} \\ -\frac{2}{s} & 1+\frac{4}{s} \end{bmatrix} \begin{bmatrix} I_1 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \end{bmatrix}$$

$$I_3 = \frac{\Delta_2}{\Delta} = \frac{+\frac{2}{s}V_1}{\left(1+\frac{2}{s}\right)\left(1+\frac{4}{s}\right) - \frac{4}{s^2}}, \quad I_1 = \frac{\Delta_1}{\Delta} = \frac{V_1\left(1+\frac{4}{s}\right)}{\left(1+\frac{2}{s}\right)\left(1+\frac{4}{s}\right) - \frac{4}{s^2}}$$

Driving point impedance is given by

$$Z_{11} = \frac{V_1}{I_1} = \frac{V_1}{\frac{V_1\left(1+\frac{4}{s}\right)}{\left(1+\frac{2}{s}\right)\left(1+\frac{4}{s}\right) - \frac{4}{s^2}}} = \frac{\left(1+\frac{2}{s}\right)\left(1+\frac{4}{s}\right) - \frac{4}{s^2}}{\left(1+\frac{4}{s}\right)}$$

If this TPN is terminated with 1 F capacitor, then the network becomes




Applying KVL in loops (1), (2), (3), and arranging them in matrix form, we get

$$\begin{bmatrix} \left(1+\frac{2}{s}\right) & 0 & -\frac{2}{s} \\ 0 & \frac{3}{s} & +\frac{2}{s} \\ -\frac{2}{s} & +\frac{2}{s} & 1+\frac{4}{s} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix}$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{V_1\left[\frac{3}{s}\left(1+\frac{4}{s}\right) - \frac{4}{s^2}\right]}{\left(1+\frac{2}{s}\right)\left[\frac{3}{s}\left(1+\frac{4}{s}\right) - \frac{4}{s^2}\right] - \frac{2}{s}\left[0+\frac{6}{s^2}\right]}$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{V_1\left[0+\frac{4}{s}\right]}{\left(1+\frac{2}{s}\right)\left[\frac{3}{s}\left(1+\frac{4}{s}\right)-\frac{4}{s^2}\right]-\frac{2}{s}\left[0+\frac{6}{s^2}\right]}$$

$$Z_{21} = \frac{V_2}{I_1} = \frac{-I_2 \times \frac{1}{s}}{I_1} = \frac{-\left(\frac{4}{s^2}\right)\left(\frac{1}{s}\right)}{\frac{3}{s}\left(1+\frac{4}{s}\right)-\frac{4}{s^2}}$$

$$Y_{21} = \frac{I_2}{V_1} = \frac{\frac{4}{s^2}}{\left(1+\frac{2}{s}\right)\left[\frac{3}{s}\left(1+\frac{4}{s}\right)-\frac{4}{s^2}\right]-\frac{12}{s^3}}$$

$$\alpha_{21} = \frac{I_2}{I_1} = \frac{\frac{4}{s^2}}{\frac{3}{s}\left(1+\frac{4}{s}\right)-\frac{4}{s^2}}$$

**7. For the given network function, plot the approximate Bode diagram.**

$$N(s) = \frac{(s+5)}{s^2+21s+20}$$

***Solution:***

$$N(s) = \frac{(s+5)}{s^2+21s+20} = \frac{(s+5)}{(s+1)(s+20)} = \frac{5\left(1+\frac{s}{5}\right)}{20\,(1+s)\left(1+\frac{s}{20}\right)}$$

Put $s \to j\omega$,

$$N(j\omega) = \frac{5\left(1+\frac{j\omega}{5}\right)}{20\,(1+j\omega)\left(1+\frac{j\omega}{20}\right)} = \frac{\left(\frac{1}{4}\right)\left(1+\frac{j\omega}{5}\right)}{(1+j\omega)\left(1+\frac{j\omega}{20}\right)}$$

The corner frequencies are:

Low, 1(–), 5(+), 20(–)

**Magnitude plot**

If a network function neither contains pole nor zero at origin, then for magnitude plot, starting point can be obtained as follows:

$$\text{Starting point} = (k)_{dB} = \left(\frac{1}{4}\right)_{dB} = 20\log\frac{1}{4} = -12.05\text{ dB}$$

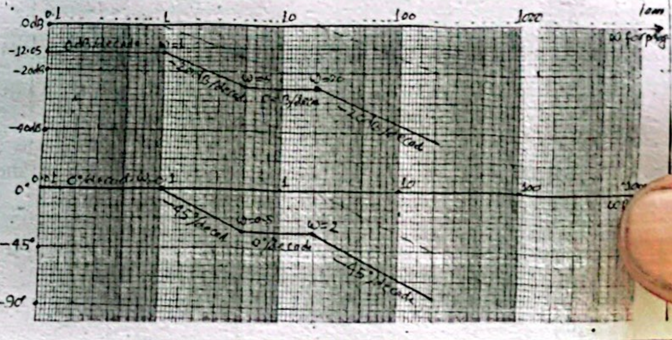


| Factors | Corner frequency | Initial slope | Final slope |
|---|---|---|---|
| $\frac{1}{4}$ | Low | ....... | 0 dB/decade |
| $[1+j\omega]^{-1}$ | 1 (–) | –20 dB/decade | –20 dB/decade |
| $\left[1+\frac{j\omega}{5}\right]^{+1}$ | 5 (+) | +20 dB/decade | 0 dB/decade |
| $\left[1+\frac{j\omega}{20}\right]^{-1}$ | 20 (–) | –20 dB/decade | –20 dB/decade |

**Phase plot**

If a network function neither contains pole nor zero at origin, then for phase plot, starting point will be 0°.

| Factors | Corner frequency | Effective frequency | Initial slope | Final slope |
|---|---|---|---|---|
| $\left(\frac{1}{4}\right)$ | Low | Low | ....... | 0°/decade |
| $[1+j\omega]^{-1}$ | 1 (–) | 0.1(–) | –45°/decade | – 45°/decade |
| $\left[1+\frac{j\omega}{5}\right]^{+1}$ | 5 (+) | 0.5 (+) | +45°/decade | 0°/decade |
| $\left[1+\frac{j\omega}{20}\right]^{-1}$ | 20 (–) | 2 (–) | –45°/decade | – 45°/decade |




**8. For the given network function, plot the approximate Bode diagram:** $N(s) = \frac{(s+5)}{s(s^2+21s+20)}$

***Solution:***

$$N(s) = \frac{(s+5)}{s(s^2+21s+20)} = \frac{(s+5)}{s(s+1)(s+20)} = \frac{5\left(1+\frac{s}{5}\right)}{s\,(20)\,(1+s)\left(1+\frac{s}{20}\right)}$$

Put $s = j\omega$,

$$N(j\omega) = \frac{\left(\frac{1}{4}\right)\left(1+\frac{j\omega}{5}\right)}{(j\omega)\,(1+j\omega)\left(1+\frac{j\omega}{20}\right)}$$

The corner frequencies are: Low(–), 1(–), 5(+), 20(–)

**Magnitude plot**

If a network function either contains pole or zero at origin, then for magnitude plot, starting point can be obtained as follows:

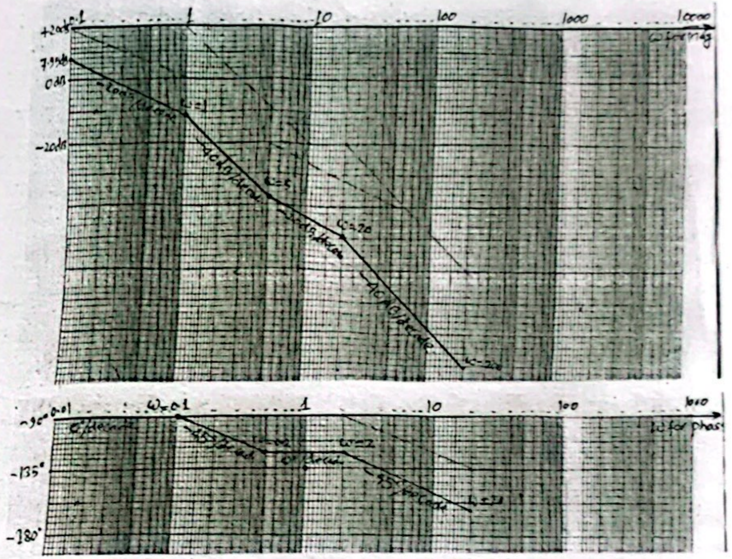
$$\text{Starting point} = \left|\frac{\left(\frac{1}{4}\right)}{(j\omega)}\right|_{dB} = 20\log\frac{\frac{1}{4}}{\omega} = 20\log\frac{\frac{1}{4}}{0.1} = 7.95\text{ dB}$$

| Factors | Corner frequency | Initial slope | Final slope |
|---|---|---|---|
| $\left(\frac{1}{4}\right)[j\omega]^{-1}$ | Low (–) | .......... | –20 dB/decade |
| $[1+j\omega]^{-1}$ | 1 (–) | –20 dB/decade | –40 dB/decade |
| $\left[1+\frac{j\omega}{5}\right]^{+1}$ | 5 (+) | +20 dB/decade | –20 dB/decade |
| $\left[1+\frac{j\omega}{20}\right]^{-1}$ | 20 (–) | –20 dB/decade | –40 dB/decade |

**Phase plot**

If a network function either contains pole or zero at origin, then for phase plot, starting point will be –90° or +90° respectively.

| Factors | Corner frequency | Effective frequency | Initial slope | Final slope |
|---|---|---|---|---|
| $\left(\frac{1}{4}\right)[j\omega]^{-1}$ | Low (–) | Low (–) | .......... | 0°/decade |
| $[1+j\omega]^{-1}$ | 1 (–) | 0.1 (–) | –45°/decade | –45°/decade |
| $\left[1+\frac{j\omega}{5}\right]^{+1}$ | 5 (+) | 0.5 (+) | +45°/decade | 0°/decade |
| $\left[1+\frac{j\omega}{20}\right]^{-1}$ | 20 (–) | 2 (–) | –45°/decade | –45°/decade |



**9. For the given network function, plót the approximate Bode diagram:** $N(s) = \dfrac{(s+5)}{s(s^2+21s+20)(s^2+2s+10)}$

***Solution:***

$$N(s) = \frac{(s+5)}{s(s^2+21s+20)(s^2+2s+10)}$$

$$= \frac{(s+5)}{s(s+1)(s+20)(s^2+2s+10)}$$

$$= \frac{5\left(1+\frac{s}{5}\right)}{20\times 10\,(s)\,(1+s)\left(1+\frac{s}{20}\right)\left[1+\frac{s^2}{10}+\frac{2}{10}s\right]}$$

For $s \to j\omega$,

$$N(j\omega) = \frac{\frac{1}{40}\left(1+\frac{j\omega}{5}\right)}{(j\omega)\,(1+j\omega)\left(1+\frac{j\omega}{20}\right)\left[1-\left(\frac{\omega}{\sqrt{10}}\right)^2+j\frac{2}{10}\omega\right]}$$

The corner frequencies are: Low(–), 1(–), $\sqrt{10}$(– –), 5(+), 20(–)

**Magnitude plot**

$$\text{Starting point} = \left|\frac{\left(\frac{1}{40}\right)}{(j\omega)}\right|_{dB} = 20\log\frac{\frac{1}{40}}{\omega} = 20\log\frac{\frac{1}{40}}{0.1} = 12.041 \text{ at } \omega = 0.1$$

| Factors | Corner frequency | Initial slope | Final slope |
|---|---|---|---|
| $\left(\frac{1}{40}\right)[j\omega]^{-1}$ | Low (–) | ……. | –20 dB/decade |
| $[1+j\omega]^{-1}$ | 1 (–) | –20 dB/decade | –40 dB/decade |
| $\left[1-\left(\frac{\omega}{\sqrt{10}}\right)^2+j\frac{2}{10}\omega\right]^{-1}$ | 3.16 (– –) | –40 dB/decade | –80 dB/decade |
| $\left[1+\frac{j\omega}{5}\right]^{+1}$ | 5 (+) | +20 dB/decade | –60 dB/decade |
| $\left[1+\frac{j\omega}{20}\right]^{-1}$ | 20 (–) | –20 dB/decade | –80 dB/decade |

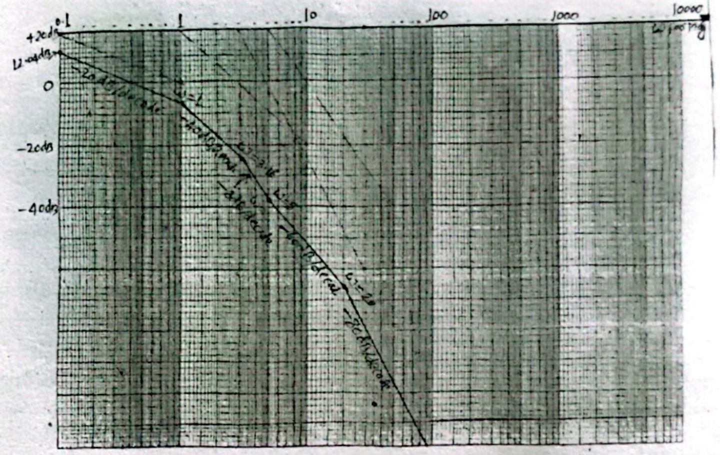
**Phase plot**

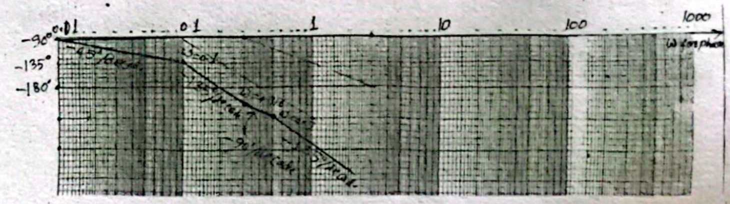
Starting point = –90°

| Factors | Corner frequency | Effective frequency | Initial slope | Final slope |
|---|---|---|---|---|
| $\left(\frac{1}{40}\right)[j\omega]^{-1}$ | Low (–) | Low (–) | ……. | 0°/ decade |
| $[1+j\omega]^{-1}$ | 1 (–) | 0.1 (–) | –45°/decade | –45°/decade |
| $\left[1-\left(\frac{\omega}{\sqrt{10}}\right)^2+j\frac{2}{10}\omega\right]^{-1}$ | 3.16 (– –) | 0.316 (– –) | –90°/decade | –135°/decade |



| Factors | Corner frequency | Effective frequency | Initial slope | Final slope |
|---|---|---|---|---|
| $\left[1+\frac{j\omega}{5}\right]^{+1}$ | 5 (+) | 0.5 (+) | +45°/decade | –90°/decade |
| $\left[1+\frac{j\omega}{20}\right]^{-1}$ | 20 (–) | 2 (–) | –45°/decade | –135°/decade |




**10. Sketch the Bode plot for the transfer function given by**

$$H(s) = \frac{200(s+1)}{s(s^2+2s+100)(s+5)}$$

***Solution:***

$$H(s) = \frac{200(s+1)}{s(s+5)(s^2+2s+100)}$$

$$\text{or, } H(s) = \frac{200(s+1)}{5\times 100\,(s)\left(1+\frac{s}{5}\right)\left[1+\frac{s^2}{100}+\frac{2}{100}s\right]}$$

Put $s = j\omega$,

$$H(j\omega) = \frac{\frac{2}{5}(1+j\omega)}{(j\omega)\left(1+\frac{j\omega}{5}\right)\left[1-\left(\frac{\omega}{10}\right)^2 + j\frac{2}{100}\omega\right]}$$

The corner frequencies are:

Low(−), 1(+), 5(−), 10(− −)

**Magnitude plot**

Starting point = $\left|\frac{2/5}{j\omega}\right|_{dB} = 20\log\frac{2/5}{\omega} = 20\log\frac{2/5}{0.1} = 12.04$ dB

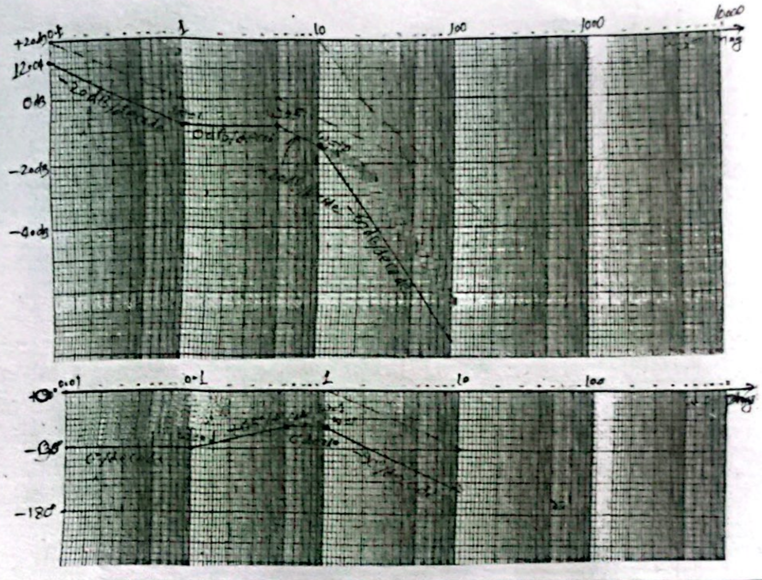
| Factors | Corner frequency | Initial slope | Final slope |
|---|---|---|---|
| $\left[\frac{2/5}{j\omega}\right]$ | Low (−) | ........ | −20 dB/decade |
| $[1+j\omega]^{+1}$ | 1 (+) | +20 dB/decade | 0 dB/decade |
| $\left[1+\frac{j\omega}{5}\right]^{-1}$ | 5 (−) | −20 dB/decade | −20 dB/decade |
| $\left[1-\left(\frac{\omega}{10}\right)^2 + j\frac{2}{100}\omega\right]^{-1}$ | 10 (− −) | −40 dB/decade | −60 dB/decade |

**Phase plot**

Starting point = −90°

| Factors | Corner frequency | Effective frequency | Initial slope | Final slope |
|---|---|---|---|---|
| $\left[\frac{2/5}{j\omega}\right]$ | Low (−) | Low (−) | ......... | 0°/ decade |
| $[1+j\omega]^{+1}$ | 1 (+) | 0.1 (+) | +45°/ decade | +45°/ decade |
| $\left[1+\frac{j\omega}{5}\right]^{-1}$ | 5 (−) | 0.5 (−) | −45°/ decade | 0°/ decade |
| $\left[1-\left(\frac{\omega}{10}\right)^2 + j\frac{2}{100}\omega\right]^{-1}$ | 10 (− −) | 1 (− −) | −90°/ decade | −90°/ decade |






**11. Draw the asymptotic Bode plot for the transfer function**

$$H(s) = \frac{(s+5)}{s(s^2+21s+20)(s^2+2s+100)}$$ [2076 Ashwin]

***Solution:***

$$H(s) = \frac{(s+5)}{s(s^2+21s+20)(s^2+2s+100)}$$

$$\text{or, } H(s) = \frac{5\left(1+\frac{s}{5}\right)}{s(s+20)(s+1)(s^2+2s+100)}$$

$$\text{or, } H(s) = \frac{5\left(1+\frac{s}{5}\right)}{20\times 100\,(s)(1+s)\left(1+\frac{s}{20}\right)\left[1+\frac{s^2}{100}+\frac{2}{100}s\right]}$$

$$\text{or, } H(s) = \frac{\left(\frac{1}{400}\right)\left(1+\frac{s}{5}\right)}{(s)(1+s)\left(1+\frac{s}{20}\right)\left[1+\left(\frac{s}{10}\right)^2+\frac{2}{100}s\right]}$$

Put $s = j\omega$,

$$H(j\omega) = \frac{\left(\frac{1}{400}\right)\left(1+\frac{j\omega}{5}\right)}{(j\omega)(1+j\omega)\left(1+\frac{j\omega}{20}\right)\left[1+\left(\frac{\omega}{10}\right)^2+j\frac{2}{100}\omega\right]}$$

The corner frequencies are:

Low(–), 1(–), 5(+), 10(– –), 20 (–)

**Magnitude plot**

Starting point = $\left|\frac{1/400}{j\omega}\right|_{dB} = 20\log\frac{1/400}{\omega} = 20\log\frac{1/400}{0.1} = 32.04$ dB

at $\omega = 0.1$.

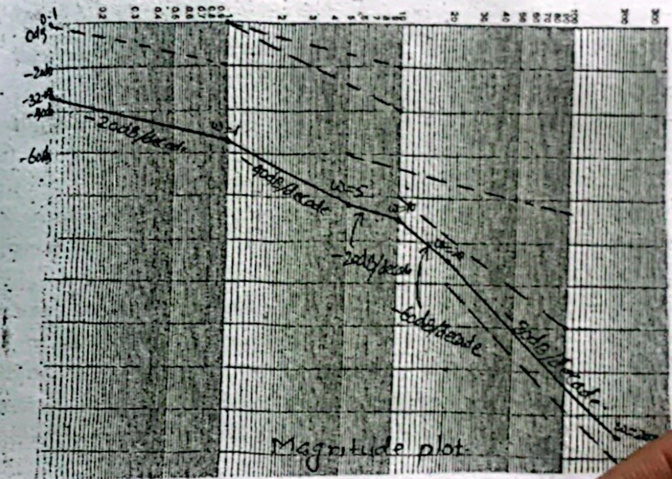
| Factors | Corner frequency | Initial slope | Final slope |
|---|---|---|---|
| $\left[\frac{1/400}{j\omega}\right]$ | Low (–) | ........ | –20 dB/decade |
| $[1+j\omega]^{-1}$ | 1 (–) | –20 dB/decade | –40 dB/decade |
| $\left[1+\frac{j\omega}{5}\right]^{+1}$ | 5 (+) | +20 dB/decade | –20 dB/decade |
| $\left[1-\left(\frac{\omega}{10}\right)^2+j\frac{2}{100}\omega\right]^{-1}$ | 10 (– –) | –40 dB/decade | –60 dB/decade |
| $\left[1+\frac{j\omega}{20}\right]^{-1}$ | 20 (–) | –20 dB/decade | –80 dB/decade |

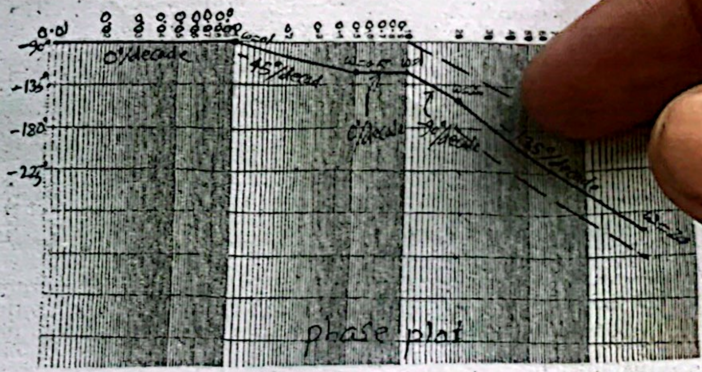
**Phase plot**

Starting point = –90°

| Factors | Corner frequency | Effective frequency | Initial slope | Final slope |
|---|---|---|---|---|
| $\left[\frac{1/400}{j\omega}\right]$ | Low (–) | Low (–) | ......... | 0°/ decade |
| $[1+j\omega]^{-1}$ | 1 (–) | 0.1 (–) | –45°/ decade | –45°/ decade |
| $\left[1+\frac{j\omega}{5}\right]^{+1}$ | 5 (+) | 0.5 (+) | +45°/ decade | 0°/ decade |
| $\left[1-\left(\frac{\omega}{10}\right)^2+j\frac{2}{100}\omega\right]^{-1}$ | 10 (– –) | 1 (– –) | –90°/ decade | –90°/ decade |
| $\left[1+\frac{j\omega}{20}\right]^{-1}$ | 20 (–) | 2 (–) | –45°/ decade | –135°/ decade |





Chapter - 5

# PARAMETERS OF TWO-PORT NETWORK (TPN)

## 5.1 Introduction

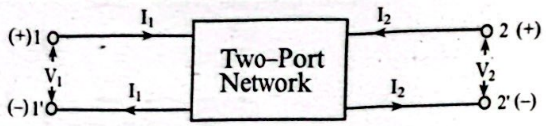



*Figure 5.1: A two-port network*

In the two-port network of figure above, we see four variables identified - two voltages and two currents. We assume that the variables are transform quantities and use $V_1$ and $I_1$ as variables at the input, port 1; and $V_2$ and $I_2$ as the variables at the output, port 2. Now only two of the four variables are independent, and the specification of any two of them determines the remaining two. For example, if $V_1$ and $V_2$ are specified, then $I_1$ and $I_2$ are determined. The dependence of two of the four variables on the other two is described in a number of ways, depending on which of the variables are chosen to be the independent variables. In this chapter, we study the six combinations which are listed in the table below. The names of the parameters are chosen to indicate dimensions (impedance, admittance), lack of consistent dimensions (hybrid), or the principal application of the parameter (transmission).

*Table 5.1: Different parameters of a two-port network*

| S. No. | Name | Function: Express | Function: In terms of | Matrix equation |
|---|---|---|---|---|
| 1. | Z-parameters (open-circuit impedance parameters) | $V_1, V_2$ | $I_1, I_2$ | $\begin{bmatrix}V_1\\V_2\end{bmatrix}=\begin{bmatrix}Z_{11}&Z_{12}\\Z_{21}&Z_{22}\end{bmatrix}\begin{bmatrix}I_1\\I_2\end{bmatrix}$ |
| 2. | Y-parameters (short-circuit admittance parameters) | $I_1, I_2$ | $V_1, V_2$ | $\begin{bmatrix}I_1\\I_2\end{bmatrix}=\begin{bmatrix}Y_{11}&Y_{12}\\Y_{21}&Y_{22}\end{bmatrix}\begin{bmatrix}V_1\\V_2\end{bmatrix}$ |
| 3. | T-parameters (transmission parameters) | $V_1, I_1$ | $V_2, -I_2$ | $\begin{bmatrix}V_1\\I_1\end{bmatrix}=\begin{bmatrix}A&B\\C&D\end{bmatrix}\begin{bmatrix}V_2\\-I_2\end{bmatrix}$ |
| 4. | T'-parameters (inverse transmission parameters) | $V_2, I_2$ | $V_1, -I_1$ | $\begin{bmatrix}V_2\\I_2\end{bmatrix}=\begin{bmatrix}A'&B'\\C'&D'\end{bmatrix}\begin{bmatrix}V_1\\-I_1\end{bmatrix}$ |
| 5. | h-parameters (hybrid parameters) | $V_1, I_2$ | $I_1, V_2$ | $\begin{bmatrix}V_1\\I_2\end{bmatrix}=\begin{bmatrix}h_{11}&h_{12}\\h_{21}&h_{22}\end{bmatrix}\begin{bmatrix}I_1\\V_2\end{bmatrix}$ |
| 6. | g-parameters (inverse hybrid parameters) | $I_1, V_2$ | $V_1, I_2$ | $\begin{bmatrix}I_1\\V_2\end{bmatrix}=\begin{bmatrix}g_{11}&g_{12}\\g_{21}&g_{22}\end{bmatrix}\begin{bmatrix}V_1\\I_2\end{bmatrix}$ |



### 1) Z-Parameters (Open-Circuit Impedance Parameters)

$(V_1, V_2) = f(I_1, I_2)$

$[V] = [Z]\,[I]$

$$\text{or, } \begin{bmatrix}V_1\\V_2\end{bmatrix}=\begin{bmatrix}Z_{11}&Z_{12}\\Z_{21}&Z_{22}\end{bmatrix}\begin{bmatrix}I_1\\I_2\end{bmatrix}$$

where $Z_{ij}$ are the open-circuit impedance (Z) parameters.

$$\therefore\ V_1 = Z_{11} I_1 + Z_{12} I_2 \qquad \ldots\ldots\ldots\ldots (1)$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2 \qquad \ldots\ldots\ldots\ldots (2)$$

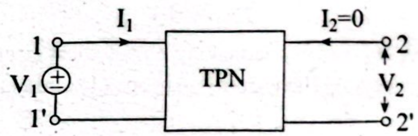



*Figure 5.2*

Applying $V_1$ at port 1 while making port 2 open circuited i.e., $I_2 = 0$ as shown in figure 5.2. Then, from equations (1) and (2),

$$Z_{11} = \frac{V_1}{I_1}\bigg|_{I_2=0} \text{ and } Z_{21} = \frac{V_2}{I_1}\bigg|_{I_2=0}$$

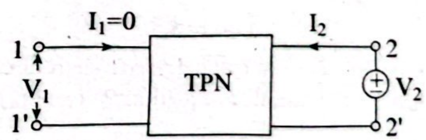



*Figure 5.3*

Applying $V_2$ at port 2 while making port 1 open circuited i.e., $I_1 = 0$ as shown in figure 5.3. Then, from equations (1) and (2),

$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$ and $Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$

The condition $I_1 = 0$ or $I_2 = 0$ implies an open circuit at port 1 or port 2. Since an open-circuit condition is specified for each of the functions in last four equations, the parameters are known as *open-circuit impedance* parameters.

**2) Y-Parameters (Short-Circuit Admittance Parameters)**

$(I_1, I_2) = f(V_1, V_2)$

or, $[I] = [Y][V]$

or, $\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

where $Y_{ij}$ are the short-circuit admittance (Y) parameters.

$\therefore I_1 = Y_{11} V_1 + Y_{12} V_2$ ..............(1)

$I_2 = Y_{21} V_1 + Y_{22} V_2$ ..............(2)

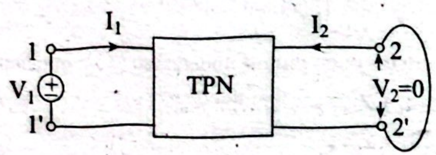



*Figure 5.4*

Applying $V_1$ at port 1 while making port 2 short circuited i.e., $V_2 = 0$ as shown in figure 5.4. Then, from equations (1) and (2),

$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$ and $Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$

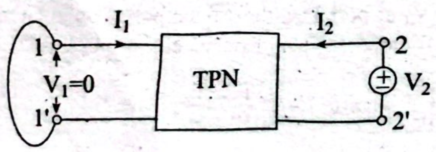



*Figure 5.5*

Applying $V_2$ at port 2 while making port 1 short circuited i.e., $V_1 = 0$ as shown in figure 5.5. Then, from equations (1) and (2),

$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$ and $Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$

The condition $V_1 = 0$ or $V_2 = 0$ is accomplished by shorting port 1 or port 2. Since a short-circuit condition is specified for each of the functions in last four equations, the parameters are known as *short-circuit admittance* parameters.



**3) T-Parameters (Transmission Parameters or Chain Parameters or ABCD Parameters)**

T-parameters are used in the analysis of power transmission line. The input and output ports are called sending and receiving ends respectively. T-parameters are useful in describing two-port networks which are connected in cascade (or in a chain arrangement).

The transmission parameters serve to relate the voltage and current at one port to voltage and current at the other port.

$(V_1, I_1) = f(V_2, -I_2)$

or, $\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$

$\therefore V_1 = AV_2 + B(-I_2)$ ..............(1)

$I_1 = CV_2 + D(-I_2)$ ..............(2)

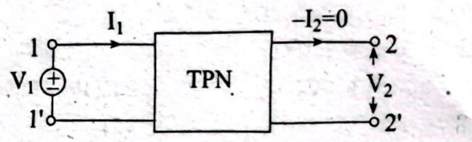



*Figure 5.6*

Applying $V_1$ at port 1 while making port 2 open circuited i.e., $I_2 = 0$ as shown in figure 5.6. Then, from equations (1) and (2),

$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$ and $C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$

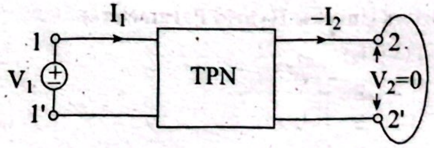



*Figure 5.7*

Applying $V_1$ at port 2 while making port 2 short circuited i.e., $V_2 = 0$ as shown in figure 5.7. Then, from equations (1) and (2),

$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0}$ and $D = \left.\frac{I_1}{-I_2}\right|_{V_2=0}$

**4) T'-Parameters (Inverse Transmission Parameters)**

$(V_2, I_2) = f(V_1, -I_1)$

$$\text{or,} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$$

$\therefore V_2 = A'V_1 + B'(-I_1)$ ...............(1)

$I_2 = C'V_1 + D'(-I_1)$ ...............(2)

From equations (1) and (2),

$$A' = \left.\frac{V_2}{V_1}\right|_{I_1=0}, \quad B' = \left.\frac{V_2}{-I_1}\right|_{V_1=0}$$

$$C' = \left.\frac{I_2}{V_1}\right|_{I_1=0}, \quad D' = \left.\frac{I_2}{-I_1}\right|_{V_1=0}$$

5) **h-Parameters (Hybrid Parameters)**

The hybrid parameters find wide usage in electronic circuits, especially in constructing models for transistors.

$(V_1, I_2) = f(I_1, V_2)$.

$$\text{or,} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$\therefore V_1 = h_{11} I_1 + h_{12} V_2$ ...............(1)

$I_2 = h_{21} I_1 + h_{22} V_2$ ...............(2)

From equations (1) and (2),

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}, \quad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}, \quad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

6) **g-Parameters (Inverse Hybrid Parameters)**

$(I_1, V_2) = f(V_1, I_2)$

$$\text{or,} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

$\therefore I_1 = g_{11} V_1 + g_{12} I_2$ ...............(1)

$V_2 = g_{21} V_1 + g_{22} I_2$ ...............(2)

From equations (1) and (2),

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0}, \quad g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0}$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0}, \quad g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0}$$



## 5.2 Interrelation between Parameters of Two-Port Networks

1) **Z-Parameter in terms of other Parameters**

For this, we have to write the corresponding parameter equation and then by algebraic manipulation, we have to express them in terms of the equations of Z-parameters. Finally, on comparison, we get the value of Z-parameters.

a) **Z-parameters in terms of Y-parameters**

Since $[Z] = [Y]^{-1}$

$$\text{or,} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}^{-1}$$

$$\text{or,} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{Y_{22}}{\Delta Y} & \frac{-Y_{12}}{\Delta Y} \\ \frac{-Y_{21}}{\Delta Y} & \frac{Y_{11}}{\Delta Y} \end{bmatrix}$$

where $\Delta Y = Y_{11} Y_{22} - Y_{12} Y_{21}$

On comparison, we get

$$Z_{11} = \frac{Y_{22}}{\Delta Y}, \; Z_{12} = -\frac{Y_{12}}{\Delta Y}, \; Z_{21} = -\frac{Y_{21}}{\Delta Y}, \; Z_{22} = \frac{Y_{11}}{\Delta Y}$$

b) **Z-parameters in terms of transmission parameters**

The transmission parameters equations are

$V_1 = AV_2 + B(-I_2)$ ............... (1)

$I_1 = CV_2 + D(-I_2)$ ............... (2)

Rewriting equation (2),

$V_2 = \frac{1}{C} I_1 + \frac{D}{C} I_2$ ...............(3)

From (3) and (1),

$$V_1 = A\left[\frac{1}{C} I_1 + \frac{D}{C} I_2\right] - BI_2$$

$\text{or, } V_1 = \frac{A}{C} I_1 + \left(\frac{AD - BC}{C}\right) I_2$ ...............(4)

The equations of Z-parameters are

$V_1 = Z_{11} I_1 + Z_{12} I_2$ ...............(5)

$V_2 = Z_{21} I_1 + Z_{22} I_2$ ...............(6)

Following are the conditions for reciprocity in terms of different parame...

i) $Z_{12} = Z_{21}$

ii) $Y_{12} = Y_{21}$

iii) $AD - BC = 1$ i.e., $\Delta T = 1$

iv) $A'D' - B'C' = 1$ i.e., $\Delta T' = 1$

v) $h_{12} = -h_{21}$

vi) $g_{12} = -g_{21}$

**Condition for Reciprocity in terms of g-Parameters**

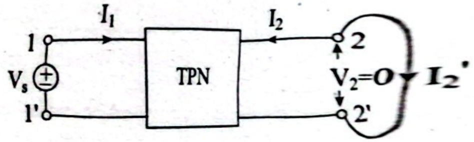



*Figure 5.8*

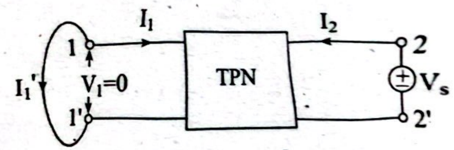



*Figure 5.9*

$(I_1, V_2) = f(V_1, I_2)$

$I_1 = g_{11} V_1 + g_{12} I_2$ ................ (1)

$V_2 = g_{21} V_1 + g_{22} I_2$ ................ (2)

Considering figure 5.8,

$V_1 = V_s,\ I_1 = I_1,\ V_2 = 0,\ I_2 = -I_2'$

From equation (2) and figure 5.8,

$0 = g_{21} V_s + g_{22}(-I_2')$

or, $\frac{V_s}{I_2'} = \frac{g_{22}}{g_{21}}$ ........... (3)

Considering figure 5.9,

$V_1 = 0,\ I_1 = -I_1',\ V_2 = V_s,\ I_2 = I_2$

From equation (1) and figure 5.9,

$I_1' = 0 + g_{12} I_2$

$I_2 = -\frac{I_1'}{g_{12}}$ ................ (4)



From equations (2) and (4), and figure 5.9,

$V_s = g_{22}\left(-\frac{1}{g_{12}} I_1'\right)$

or, $\frac{V_s}{I_1'} = -\frac{g_{22}}{g_{12}}$ .............. (5)

If the network is reciprocal, then

$\frac{V_s}{I_2'} = \frac{V_s}{I_1'}$

or, $\frac{g_{22}}{g_{21}} = \frac{-g_{22}}{g_{12}}$

$\boxed{\therefore g_{12} = -g_{21}}$

*In the similar manner, we can find out the conditions for reciprocity in terms of other parameters.*

## 5.4 Condition for Symmetry of Two-Port Network

A TPN is said to be *symmetrical*, if the port can be interchanged without changing port voltage and current. That is,

$$\left.\frac{V_s}{I_1}\right|_{I_2=0} = \left.\frac{V_s}{I_2}\right|_{I_1=0}$$

Following are the conditions for symmetry in terms of different parameters:

i) $Z_{11} = Z_{22}$

ii) $Y_{11} = Y_{22}$

iii) $A = D$

iv) $A' = D'$

v) $h_{11} \times h_{22} - h_{12} \times h_{21} = 1$ i.e., $\Delta h = 1$

vi) $g_{11} \times g_{22} - g_{12} \times g_{21} = 1$ i.e., $\Delta g = 1$

**Condition for Symmetry in terms of h-Parameters**

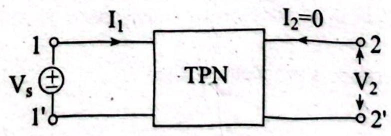



*Figure 5.10*

Comparing equations (3) and (4) with (6) and (5) respectively, we get

$$Z_{11} = \frac{A}{C}, Z_{12} = \frac{AD - BC}{C} = \frac{\Delta T}{C}, Z_{21} = \frac{1}{C}, Z_{22} = \frac{D}{C}$$

c) **Z-parameters in terms of inverse transmission parameters**

The inverse transmission parameters equations are

$V_2 = A'V_1 + B'(-I_1)$ .............. (1)

$I_2 = C'V_1 + D'(-I_1)$ ................ (2)

Rewriting equation (2),

$V_1 = \left(\frac{D'}{C'}\right) I_1 + \left(\frac{1}{C'}\right) I_2$ ................(3)

From equations (3) and (1),

$$V_2 = A'\left[\left(\frac{D'}{C'}\right) I_1 + \left(\frac{1}{C'}\right) I_2\right] - B'I_1$$

or, $V_2 = \left(\frac{A'D' - B'C'}{C'}\right) I_1 + \left(\frac{A'}{C'}\right) I_2$ .................(4)

The equations of Z-parameters are

$V_1 = Z_{11} I_1 + Z_{12} I_2$ ............(5)

$V_2 = Z_{21} I_1 + Z_{22} I_2$ ............(6)

Comparing equations (3) and (4) with (5) and (6) respectively, we get

$$Z_{11} = \frac{D'}{C'}, Z_{12} = \frac{1}{C'}, Z_{21} = \left(\frac{A'D' - B'C'}{C'}\right) = \frac{\Delta T'}{C'}, Z_{22} = \frac{A'}{C'}$$

d) **Z-parameters in terms of hybrid parameters**

The hybrid parameters equations are

$V_1 = h_{11}I_1 + h_{12}V_2$ .............. (1)

$I_2 = h_{21}I_1 + h_{22}V_2$ ................ (2)

Rewriting equation (2),

$V_2 = \left(-\frac{h_{21}}{h_{22}}\right) I_1 + \left(\frac{1}{h_{22}}\right) I_2$ ................(3)

From equations (3) and (1),

$$V_1 = h_{11}I_1 + h_{12}\left[\left(-\frac{h_{21}}{h_{22}}\right) I_1 + \left(\frac{1}{h_{22}}\right) I_2\right]$$

or, $V_1 = \left(\frac{\Delta h}{h_{22}}\right) I_1 + \left(\frac{h_{12}}{h_{22}}\right) I_2$ .................(4)

where $\Delta h = h_{11}h_{22} - h_{12}h_{21}$



The equations of Z-parameters are

$V_1 = Z_{11} I_1 + Z_{12} I_2$ ............(5)

$V_2 = Z_{21} I_1 + Z_{22} I_2$ ............(6)

Comparing equations (3) and (4) with (6) and (5) respectively, we get

$$Z_{11} = \frac{\Delta h}{h_{22}}, Z_{12} = \frac{h_{12}}{h_{22}}, Z_{21} = -\frac{h_{21}}{h_{22}}, Z_{22} = \frac{1}{h_{22}}$$

e) **Z-parameters in terms of inverse hybrid parameters**

The inverse hybrid parameters equations are

$I_1 = g_{11}V_1 + g_{12}I_2$ .............. (1)

$V_2 = g_{21}V_1 + g_{22}I_2$ ................ (2)

Rewriting equation (1),

$V_1 = \left(\frac{1}{g_{11}}\right) I_1 + \left(-\frac{g_{12}}{g_{11}}\right) I_2$ ................(3)

From equations (3) and (2),

$$V_2 = g_{21}V_1 + g_{22}\left(\frac{1}{g_{11}}\right) I_1 + \left(-\frac{g_{12}}{g_{11}}\right) I_2$$

or, $V_2 = \left(\frac{g_{21}}{g_{11}}\right) I_1 + \left(\frac{\Delta g}{g_{11}}\right) I_2$ .................(4)

where $\Delta g = g_{11}g_{22} - g_{12}g_{21}$

The equations of Z-parameters are

$V_1 = Z_{11} I_1 + Z_{12} I_2$ ............(5)

$V_2 = Z_{21} I_1 + Z_{22} I_2$ ............(6)

Comparing equations (3) and (4) with (5) and (6) respectively, we get

$$Z_{11} = \frac{1}{g_{11}}, Z_{12} = -\frac{g_{12}}{g_{11}}, Z_{21} = \frac{g_{21}}{g_{11}}, Z_{22} = \frac{\Delta g}{g_{11}}$$

***In the similar manner, we can express any of the parameters into the other parameters.***

## 5.3 Condition for Reciprocity of TPN

A TPN is said to be *reciprocal*, if the ratio of excitation to the response is unchanged with the interchange of position of excitation and response. That is,

$$\left.\frac{V_s}{I_2'}\right|_{V_2=0} = \left.\frac{V_s}{I_1'}\right|_{V_1=0}$$

Following are the conditions for reciprocity in terms of different parameters:

i) $Z_{12} = Z_{21}$

ii) $Y_{12} = Y_{21}$

iii) $AD - BC = 1$ i.e., $\Delta T = 1$

iv) $A'D' - B'C' = 1$ i.e., $\Delta T' = 1$

v) $h_{12} = -h_{21}$

vi) $g_{12} = -g_{21}$

**Condition for Reciprocity in terms of g-Parameters**

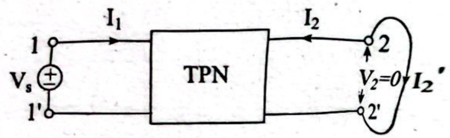



*Figure 5.8*

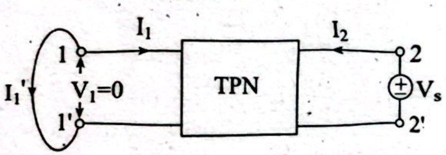



*Figure 5.9*

$(I_1, V_2) = f(V_1, I_2)$

$I_1 = g_{11} V_1 + g_{12} I_2$ ............... (1)

$V_2 = g_{21} V_1 + g_{22} I_2$ ............... (2)

Considering figure 5.8,

$V_1 = V_s, I_1 = I_1, V_2 = 0, I_2 = -I_2'$

From equation (2) and figure 5.8,

$0 = g_{21} V_s + g_{22} (-I_2')$

or, $\dfrac{V_s}{I_2'} = \dfrac{g_{22}}{g_{21}}$ ............ (3)

Considering figure 5.9,

$V_1 = 0, I_1 = -I_1', V_2 = V_s, I_2 = I_2$

From equation (1) and figure 5.9,

$-I_1' = 0 + g_{12} I_2$

or, $I_2 = -\dfrac{I_1'}{g_{12}}$ ............... (4)



From equations (2) and (4), and figure 5.9,

$V_s = g_{22}\left(-\dfrac{1}{g_{12}} I_1'\right)$

or, $\dfrac{V_s}{I_1'} = -\dfrac{g_{22}}{g_{12}}$ ............... (5)

If the network is reciprocal, then

$\dfrac{V_s}{I_2'} = \dfrac{V_s}{I_1'}$

or, $\dfrac{g_{22}}{g_{21}} = \dfrac{-g_{22}}{g_{12}}$

$\boxed{\therefore g_{12} = -g_{21}}$

***In the similar manner, we can find out the conditions for reciprocity in terms of other parameters.***

## 5.4 Condition for Symmetry of Two-Port Network

A TPN is said to be *symmetrical*, if the port can be interchanged without changing port voltage and current. That is,

$$\left.\frac{V_s}{I_1}\right|_{I_2=0} = \left.\frac{V_s}{I_2}\right|_{I_1=0}$$

Following are the conditions for symmetry in terms of different parameters:

i) $Z_{11} = Z_{22}$

ii) $Y_{11} = Y_{22}$

iii) $A = D$

iv) $A' = D'$

v) $h_{11} \times h_{22} - h_{12} \times h_{21} = 1$ i.e., $\Delta h = 1$

vi) $g_{11} \times g_{22} - g_{12} \times g_{21} = 1$ i.e., $\Delta g = 1$

**Condition for Symmetry in terms of h-Parameters**

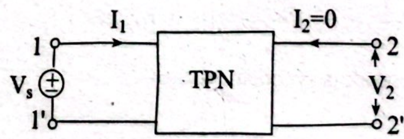



*Figure 5.10*

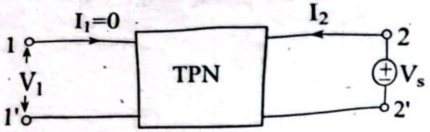



*Figure 5.11*

$(V_1, I_2) = f(I_1, V_2)$

$V_1 = h_{11} I_1 + h_{12} V_2$ ................ (1)

$I_2 = h_{21} I_1 + h_{22} V_2$ ................ (2)

Considering figure 5.10,

$V_1 = V_s, I_1 = I_1, V_2 = V_2, I_2 = 0$

From equation (2) and figure 5.10,

$0 = h_{21} I_1 + h_{22} V_2$

or, $V_2 = \frac{-h_{21}}{h_{22}} I_1$ ................ (3)

From equations (3) and (1), and figure 5.10,

$V_s = h_{11} I_1 + h_{12} \left(\frac{-h_{21}}{h_{22}} I_1\right)$

or, $\frac{V_s}{I_1} = \frac{h_{11} h_{22} - h_{12} h_{21}}{h_{22}} = \frac{\Delta h}{h_{22}}$ ................ (4)

Considering figure 5.11,

$V_1 = V_1, I_1 = 0, V_2 = V_s, I_2 = I_2$

From equation (1) and figure 5.11,

$V_1 = h_{12} V_s$

or, $\frac{V_1}{V_s} = h_{12}$ ................ (5)

From equations (2) and (3), and figure 5.11,

$I_2 = h_{22} V_s$

or, $\frac{V_s}{I_2} = \frac{1}{h_{22}}$ ................ (6)

If the network is symmetrical, then

$\frac{V_s}{I_1} = \frac{V_s}{I_2}$

or, $\frac{\Delta h}{h_{22}} = \frac{1}{h_{22}}$



$\boxed{\therefore \Delta h = 1}$

***In the similar manner, we can find out the conditions for symmetry in terms of other parameters.***

## 5.5 Interconnection of Two-Port Networks

A given two-port network, with some degree of complexity, can be built up from simpler two-port networks whose ports are interconnected in certain ways. Conversely, a two-port network can be designed by combining simple two-port structures as building blocks. From the designer's point of view, it is much easier to design simple blocks and to interconnect them to design a complex network in one piece. A further practical point is that it is much easier to shield smaller units and thus reduce parasitic capacitances to ground.

**1) Series Connection**

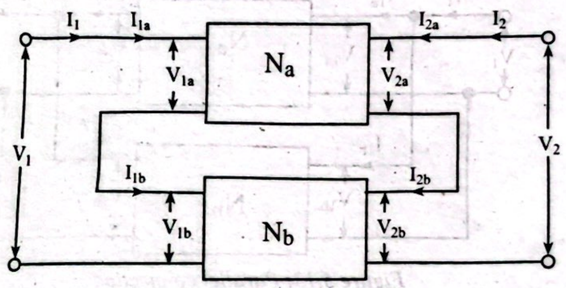



*Figure 5.12: Series connection*

The Z-parameter equations of two two-port networks $N_a$ and $N_b$ are respectively given as

$\begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix} = \begin{bmatrix} Z_{11a} & Z_{12a} \\ Z_{21a} & Z_{22a} \end{bmatrix} \begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix}$ ................ (1)

$\begin{bmatrix} V_{1b} \\ V_{2b} \end{bmatrix} = \begin{bmatrix} Z_{11b} & Z_{12b} \\ Z_{21b} & Z_{22b} \end{bmatrix} \begin{bmatrix} I_{1b} \\ I_{2b} \end{bmatrix}$ ................ (2)

If these two TPNs are connected in series as shown in figure, then

$I_1 = I_{1a} = I_{1b}$ and $I_2 = I_{2a} = I_{2b}$

Also, $V_1 = V_{1a} + V_{1b}$

or, $V_1 = (Z_{11a} I_{1a} + Z_{12a} I_{2a}) + (Z_{11b} I_{1b} + Z_{12b} I_{2b})$

or, $V_1 = (Z_{11a} + Z_{11b}) I_1 + (Z_{12a} + Z_{12b}) I_2$

or, $V_1 = Z_{11} I_1 + Z_{12} I_2$ ................ (3)

where $Z_{11} = Z_{11a} + Z_{11b}$, $Z_{12} = Z_{12a} + Z_{12b}$

Again, $V_2 = V_{2a} + V_{2b}$

Similar substitution will result

$V_2 = Z_{21} I_1 + Z_{22} I_2$ ................(4)

where $Z_{21} = Z_{21a} + Z_{21b}$, $Z_{22} = Z_{22a} + Z_{22b}$

**If T-parameters of two TPNs are given and they are connected in series, how to find equivalent T-parameters of the combination?**

**Ans:**

- Transform individual T-parameter into Z-parameter
- Find equivalent Z-parameter of the combination
- Transform equivalent Z-parameter into equivalent T-parameter

**2) Parallel Connection**

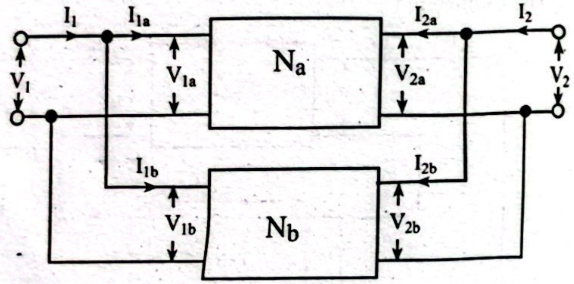



*Figure 5.13: Parallel connection*

For network $N_a$, the Y-parameter equations are

$$\begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix} = \begin{bmatrix} Y_{11a} & Y_{12a} \\ Y_{21a} & Y_{22a} \end{bmatrix} \begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix} \quad \text{.......................... (1)}$$

For network $N_b$, the Y-parameter equations are

$$\begin{bmatrix} I_{1b} \\ I_{2b} \end{bmatrix} = \begin{bmatrix} Y_{11b} & Y_{12b} \\ Y_{21b} & Y_{22b} \end{bmatrix} \begin{bmatrix} V_{1b} \\ V_{2b} \end{bmatrix} \quad \text{.......................... (2)}$$

If these two TPNs are connected in parallel as shown in figure, then

$V_1 = V_{1a} = V_{1b}$ and $V_2 = V_{2a} = V_{2b}$

Also, $I_1 = I_{1a} + I_{1b}$

or, $I_1 = (Y_{11a} V_{1a} + Y_{12a} V_{2a}) + (Y_{11b} V_{1b} + Y_{12b} V_{2b})$

or, $I_1 = V_1 (Y_{11a} + Y_{11b}) + V_2 (Y_{12a} + Y_{12b})$

or, $I_1 = V_1 Y_{11} + V_2 Y_{12}$ ................ (3)

where $Y_{11} = Y_{11a} + Y_{11b}$, $Y_{12} = Y_{12a} + Y_{12b}$



Again, $I_2 = I_{2a} + I_{2b}$

or, $I_2 = (Y_{21a} V_{1a} + Y_{22a} V_{2a}) + (Y_{21b} V_{1b} + Y_{22b} V_{2b})$

or, $I_2 = V_1 (Y_{21a} + Y_{21b}) + V_2 (Y_{22a} + Y_{22b})$

or, $I_2 = Y_{21} V_1 + Y_{22} V_2$ ..................(4)

where $Y_{21} = Y_{21a} + Y_{21b}$, $Y_{22} = Y_{22a} + Y_{22b}$

**If Z-parameters of two TPNs are given, they are connected in parallel, how to find equivalent Z-parameters of the combination?**

**Ans:**

- Transform individual Z-parameter into Y-parameter
- Find equivalent Y-parameter of the combination
- Transform equivalent Y-parameter into equivalent Z-parameter.

**3) Cascade Connection**

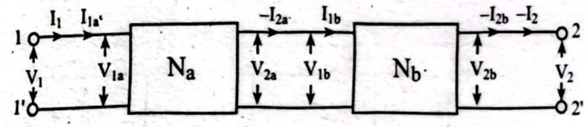



*Figure 5.14: Cascade connection of two-port networks*

For the network $N_a$, the transmission parameter equations are

$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix} \quad \text{................. (1)}$$

For the network $N_b$, the transmission parameter equations are

$$\begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix} = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix} \quad \text{........... (2)}$$

If these two TPNs are cascaded as shown in figure, then

$V_1 = V_{1a}$, $I_1 = I_{1a}$, $V_{2a} = V_{1b}$, $-I_{2a} = I_{1b}$, $V_2 = V_{2b}$, $I_2 = I_{2b}$

From equation (1),

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix} \quad \text{............... (3)}$$

From equations (2) and (3),

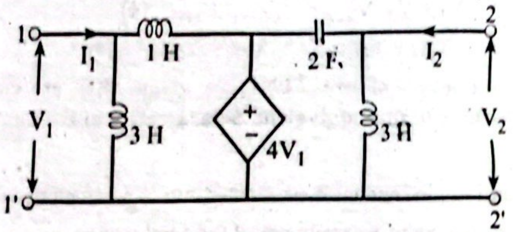
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \quad \text{...................... (4)}$$

where $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}$

## SOLVED PROBLEMS

1. **Find the Y and g parameters of the TPN and also find whether the network is reciprocal or not.**



***Solution:***

The transform network is shown in figure 1.

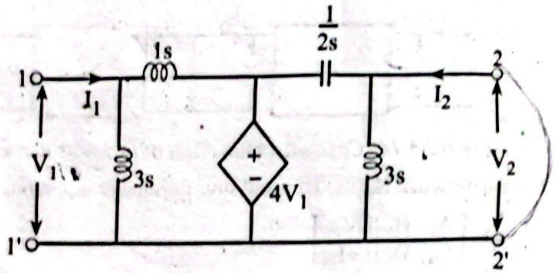


***Figure 1***

$(I_1, I_2) = f(V_1, V_2)$

$I_1 = Y_{11}V_1 + Y_{12}V_2$ ............(1)

$I_2 = Y_{21}V_1 + Y_{22}V_2$ ............(2)

Applying $V_1$ at port 1 and make port 2 short circuited as shown in figure 2. Then, from equations (1) and (2),

$Y_{11} = \frac{I_1}{V_1}\Big|_{V_2=0}$ ............(A)

and $Y_{21} = \frac{I_2}{V_1}\Big|_{V_2=0}$ ............(B)



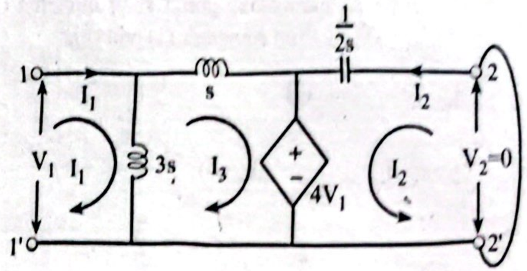


***Figure 2***

KVL in loop 1 gives

$V_1 = 3sI_1 - 3sI_3$ ............(3)

KVL in loop 2 gives

$-4V_1 - \frac{1}{2s}I_2 = 0$

or, $I_2 = \frac{-4V_1}{\frac{1}{2s}} = -8sV_1$ ............(4)

KVL in loop 3 gives

$-4V_1 = -3sI_1 + 4sI_3$

or, $4sI_3 = 3sI_1 - 4V_1$

or, $I_3 = \frac{3}{4}I_1 - \frac{1}{s}V_1$ ..................(5)

From equations (5) and (3),

$V_1 = 3sI_1 - 3s\left[\frac{3}{4}I_1 - \frac{1}{5}V_1\right]$

or, $V_1 = 3sI_1 - \frac{9s}{4}I_1 + 3V_1$

or, $I_1 = \frac{-8}{3s}V_1$ ..................(6)

From equations (A) and (6),

$Y_{11} = \frac{I_1}{V_1} = \frac{(-8/3s)V_1}{V_1} = \frac{-8}{3s}\ \Omega^{-1}$

From equations (B) and (4),

$Y_{21} = \frac{I_2}{V_1} = \frac{-8sV_1}{V_1} = -8s$

$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

$I_1 = y_{11}V_1 + y_{12}V_2$

$I_2 = y_{21}V_1 + y_{22}V_2$

Applying $V_2$ at port 2 and making port 1 short circuited i.e., $V_1 = 0$ as shown in figure 3. Then, from equations (1) and (2),

$$Y_{12} = \frac{I_1}{V_2}\Big|_{V_1=0} \qquad \text{.........(7)}$$

and $$Y_{22} = \frac{I_2}{V_2}\Big|_{V_1=0} \qquad \text{.........(8)}$$

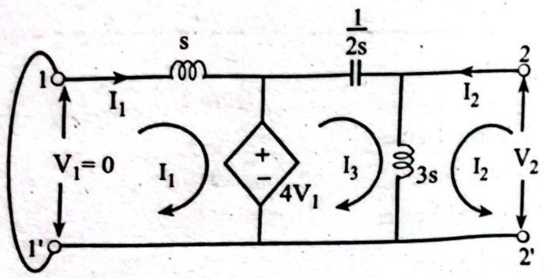



*Figure 3*

Applying KVL in loop 1,

$$-4V_1 = sI_1$$

$$\text{or, } I_1 = -\frac{4}{s}V_1$$

$$\text{or, } I_1 = 0 \qquad \text{....................(9)}$$

Applying KVL in loop 2,

$$V_2 = 3sI_2 + 3sI_3 \qquad \text{....................(10)}$$

Applying KVL in loop 3,

$$4V_1 = \left(3s + \frac{1}{2s}\right) I_3 + 3sI_2$$

$$\text{or, } I_3 = \frac{4}{\left(3s + \frac{1}{2s}\right)}V_1 - \frac{3s}{\left(3s + \frac{1}{2s}\right)}I_2$$

$$\text{or, } I_3 = 0 - \frac{3s}{3s + \frac{1}{2s}}I_2$$

$$\text{or, } I_3 = \frac{-6s^2}{6s^2 + 1}I_2 \qquad \text{................(11)}$$



From equations (11) and (10),

$$V_2 = 3sI_2 + 3s\left[\frac{-6s^2}{6s^2+1}\right]I_2$$

$$\text{or, } V_2 = \frac{18s^3 + 3s - 18s^3}{6s^2 + 1}I_2$$

$$\text{or, } I_2 = \frac{(6s^2+1)}{3s}V_2 \qquad \text{..................(12)}$$

From equations (8) and (12),

$$Y_{22} = \frac{I_2}{V_2} = \frac{[(6s^2+1)/3s]V_2}{V_2} = \frac{6s^2+1}{3s}$$

From equations (7) and (9),

$$Y_{12} = \frac{I_1}{I_2} = 0$$

For g parameters,

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$\text{or, } I_1 = \frac{-8}{3s}V_1 + 0V_2 \qquad \text{.................(14)}$$

$$\text{and, } I_2 = Y_{21}V_1 + Y_{22}V_2$$

$$\text{or, } I_2 = (-8s)V_1 + \left(\frac{6s^2+1}{3s}\right)V_2 \qquad \text{................(15)}$$

From equation (15),

$$V_2 = \left(\frac{8s}{6s^2+1}\right)3sV_1 + \frac{3s}{6s^2+1}I_2 \qquad \text{.................(16)}$$

From equation (14),

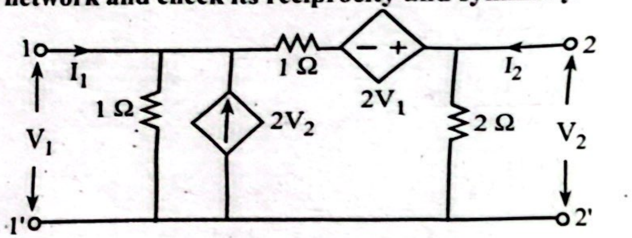
$$I_1 = -\frac{8}{3s}V_1 + 0I_2 \qquad \text{.................(17)}$$

Comparing equations (17) and (16) with the equations of g parameters, we get

$$g_{11} = -\frac{8}{3s},\ g_{12} = 0,\ g_{21} = \frac{24s^2}{6s^2+1},\ g_{22} = \frac{3s}{6s^2+1}$$

Since $g_{12} \neq g_{21}$, the network is not reciprocal.

**2. Find transmission and admittance parameters for the given two-port network and check its reciprocity and symmetry.**

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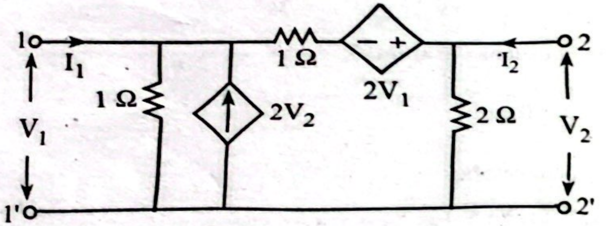
***Solution:***




For Y parameters,

$I_1 = Y_{11}V_1 + Y_{12}V_2$ .......... (1)

$I_2 = Y_{21}V_1 + Y_{22}V_2$ ........ (2)

Applying $V_1$ at port 1 and make port 2 short circuited i.e., $V_2 = 0$ as shown in figure 1. Then, from equations (1) and (2),

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \quad \text{...........(3)}$$

and $$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} \quad \text{...............(4)}$$

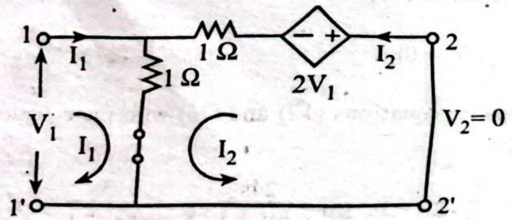



*Figure 1*

Applying KVL in loop 1,

$V_1 = I_1 + I_2$ ..............(5)



*Applying KVL in loop 2,*

$-2V_1 = 2I_2 + I_1$

or, $I_1 = -2V_1 - 2I_2$ .............(6)

From (5) and (6),

$V_1 = (-2V_1 - 2I_2) + I_2$

or, $3V_1 = -I_2$ ..............(7)

or, $\frac{I_2}{V_1} = -3 \Rightarrow Y_{21} = \frac{I_2}{V_1} = -3\ \Omega^{-1}$

From (7) and (5),

$V_1 = I_1 - 3V_1$

or, $I_1 = 4V_1$

or, $\frac{I_1}{V_1} = 4$

$\therefore\ Y_{11} = \frac{I_1}{V_1} = 4\ \Omega^{-1}$

Applying $V_2$ at port 2 and making port 1 short circuited i.e., $V_1 = 0$ as shown in figure 2. Then, from equations (1) and (2),

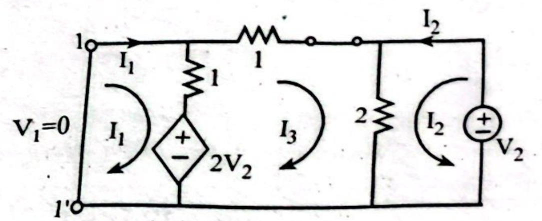



*Figure 2*

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \quad \text{..............(8)}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} \quad \text{................(9)}$$

Applying KVL in loop 1,

$-2V_2 = I_1 - I_3$ ...............(10)

Applying KVL in loop 3,

$2V_2 = 4I_3 - I_1 + 2I_2$ ............(11)

Applying KVL in loop 2,

$V_2 = 2I_2 + 2I_3$ ...............(12)

From equations (10), (11), and (12),

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 4 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -2V_2 \\ 2V_2 \\ V_2 \end{bmatrix}$$

$$I_1 = \frac{\Delta_1}{\Delta}$$

$$\Delta_1 = \begin{vmatrix} -2V_2 & 0 & -1 \\ 2V_2 & 2 & 4 \\ V_2 & 2 & 2 \end{vmatrix} = 2V_2(4-8) + (-1)[4V_2 - 2V_2]$$

$$= 8V_2 - 2V_2 = 6V_2$$

$$\Delta = \begin{vmatrix} 1 & 0 & -1 \\ -1 & 2 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1\,(4-8) + (-1)\,(-2-0) = -2$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{6V_2}{-2} = -3V_2$$

$$\therefore\ Y_{12} = \frac{I_1}{V_2} = -3$$

$$I_2 = \frac{\Delta_2}{\Delta}$$

$$\Delta_2 = \begin{vmatrix} 1 & -2V_2 & -1 \\ -1 & 2V_2 & 4 \\ 0 & V_2 & 2 \end{vmatrix} = 1(4V_2 - 4V_2) - 1(-1)(-4V_2 + V_2)$$

$$= 0 + (-3V_2) = -3V_2$$

$$I_2 = \frac{-3V_2}{-2}$$

$$\therefore\ Y_{22} = \frac{I_2}{V_2} = \frac{3}{2}$$

$$[Y] = \begin{bmatrix} 4 & -3 \\ -3 & \frac{3}{2} \end{bmatrix}$$

Now, $I_1 = 4V_1 + (-3)V_2$ ...............(13)

$I_2 = -3V_1 + \frac{3}{2}V_2$ .................(14)



From equation (13),

$V_1 = \frac{1}{4}I_1 + \frac{3}{4}V_2$ ...................(15)

From (14) and (15),

$$I_2 = -3\left(\frac{1}{4}I_1 + \frac{3}{4}V_2\right) + \frac{3}{2}V_2$$

or, $I_2 = -\frac{3}{4}I_1 - \frac{3}{4}V_2$

or, $\frac{3}{4}I_1 = -I_2 - \frac{3}{4}V_2$

or, $I_1 = -\frac{3}{4} \times \frac{4}{3}V_2 - \frac{4}{3}I_2$

or, $I_1 = -V_2 + \frac{4}{3}(-I_2)$ .................(16)

From (16) and (15),

$$V_1 = \frac{1}{4}\left(-V_2 - \frac{4}{3}I_2\right) + \frac{3}{4}V_2$$

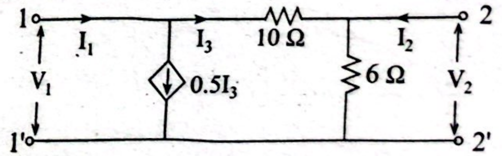
or, $V_1 = \frac{1}{2}V_2 + \frac{1}{3}(-I_2)$ ...............(17)

$\therefore\ A = \frac{1}{2}, B = \frac{1}{3}, C = -1, D = \frac{4}{3}$

Since $Y_{12} = Y_{21}$, the network is reciprocal.

$Y_{11} \neq Y_{22}$, the network is not symmetrical

**3. *For the TPN shown below, find transmission parameters and Y-parameters.***




[2073 Shrawan]

***Solution:***

$(V_1, I_1) = f(V_2, -I_2)$

$V_1 = AV_2 + B(-I_2)$ ............. (1)

$I_1 = CV_2 + D(-I_2)$ .......... (2)

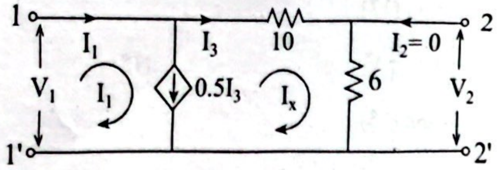



*Figure 1*

Applying $V_1$ at port 1 and making port 2 open circuited i.e., $I_2 = 0$ as shown in figure 1. Then, from equations (1) and (2),

$$A = \frac{V_1}{V_2}\bigg|_{I_2=0} \quad \text{............ (3)}$$

$$C = \frac{I_1}{V_2}\bigg|_{I_2=0} \quad \text{............ (4)}$$

KVL in outer loop gives

$V_1 - 10I_x - 6I_x = 0$

or, $I_x = \dfrac{V_1}{16}$ ........... (5)

Also, $I_1 - I_x = 0.5I_3$

or, $I_1 - I_x = 0.5I_x$

or, $I_1 = 1.5I_x = \dfrac{1.5V_1}{16}$ .......... (6)

Also, $V_2 = 6I_x$

or, $V_2 = 6 \times \dfrac{V_1}{16} = \dfrac{3V_1}{8}$ ......... (7)

$$\therefore A = \frac{V_1}{V_2} = \frac{V_1}{\frac{3V_1}{8}} = \frac{8}{3} = 2.6667$$

$$C = \frac{I_1}{V_2} = \frac{\frac{1.5V_1}{16}}{\frac{3V_1}{8}} = \frac{1.5\times 8}{3\times 16} = 0.25\ \Omega^{-1}$$

Applying $V_1$ at port 1 and making port 2 short circuited i.e., $V_2 = 0$ as shown in figure 2. Then, from equations (1) and (2),

$$B = \frac{V_1}{-I_2}\bigg|_{V_2=0} \quad \text{............ (a)}$$

$$D = \frac{I_1}{-I_2}\bigg|_{V_2=0} \quad \text{............ (b)}$$



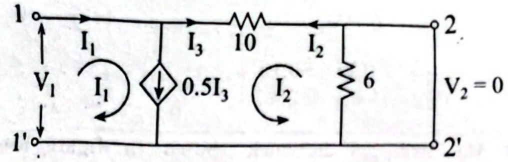



*Figure 2*

KVL in outer loop gives

$V_1 + 10I_2 = 0$

or, $V_1 = -10I_2$ .............. (8)

Also, $I_1 + I_2 = 0.5I_3$

or, $I_1 + I_2 = 0.5(-I_2)$

or, $I_1 = -1.5I_2$ .......... (9)

From equations (a) and (8),

$$B = \frac{V_1}{-I_2} = \frac{-10I_2}{-I_2} = 10\ \Omega$$

From (b) and (9),

$$D = \frac{I_1}{-I_2} = \frac{-1.5I_2}{-I_2} = 1.5$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 2.667 & 10 \\ 0.25 & 1.5 \end{bmatrix}$$

$V_1 = AV_2 + B(-I_2)$

or, $V_1 = 2.667V_2 + 10(-I_2)$ .......... (10)

$I_1 = CV_2 + D(-I_2)$

or, $I_1 = 0.25V_2 + 1.5(-I_2)$ .......... (11)

From equation (10),

$$I_2 = \frac{2.667}{10}V_2 - \frac{1}{10}V_1$$

or, $I_2 = -0.1V_1 + 0.2667V_2$ .......... (12)

From (11) and (12),

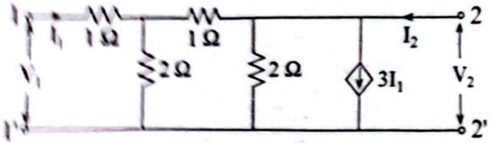
$I_1 = 0.25V_2 - 1.5(-0.1V_1 + 0.2667V_2)$

or, $I_1 = 0.25V_2 + 0.15V_1 - 0.4V_2$

or, $I_1 = 0.15V_1 - 0.15V_2$ ........... (13)

$\therefore \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 0.15 & -0.15 \\ -0.1 & 0.2667 \end{bmatrix}$

4. **For the two-port network shown in figure below, find Z parameters.**




[2072 Kartik]

*Solution:*

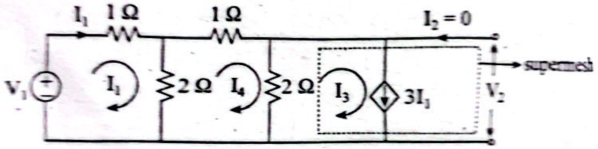
$V_1 = Z_{11}I_1 + Z_{12}I_2$ .......... (i)

$V_2 = Z_{21}I_1 + Z_{22}I_2$ ............ (ii)

Supply $V_1$ voltage at port 1 and port 2 is open circuited i.e., $I_2 = 0$.

$$Z_{11} = \frac{V_1}{I_1}\bigg|_{I_2=0}$$

$$Z_{21} = \frac{V_2}{I_1}\bigg|_{I_2=0}$$




Applying KVL on supermesh,

$V_2 = 2(-I_3 + I_4)$

or, $V_2 = 2(-3I_1 + I_4)$

or, $V_2 = -6I_1 + 2I_4$ ........... (i)

Also, $I_3 = 3I_1$

Applying KVL on loop 1,

$V_1 = I_1 + 2(I_1 - I_4)$

or, $2I_4 = 3I_1 - V_1$ ——— (ii)

From equations (i) and (ii),

$V_2 = -6I_1 + 3I_1 - V_1$



or, $V_2 = -3I_1 - V_1$ ............. (iii)

Applying KVL on loop 4,

$-2(I_4 - I_1) - I_4 - 2(I_4 - I_3) = 0$

or, $-2I_4 + 2I_1 - I_4 - 2I_4 + 2\times3I_1 = 0$

or, $5I_4 = 8I_1$

or, $5\left(\frac{3I_1 - V_1}{2}\right) = 8I_1$

or, $15I_1 - 5V_1 = 16I_1$

or, $V_1 = \frac{I_1}{-5}$

In (iii),

$V_2 = -3I_1 - \frac{I_1}{-5}$

or, $-5V_2 = 15I_1 - I_1$

or, $-5V_2 = 14I_1$ ........ (iv)

or, $Z_{21} = \frac{V_2}{I_1} = -\frac{14}{5}$

Solving (iii) and (iv),

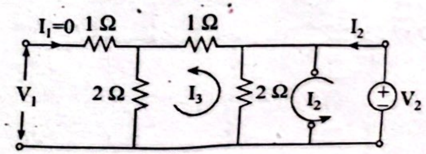
$\frac{14}{-5}I_1 = -3I_1 - V_1$

or, $-14I_1 = -15I_1 - 5V_1$

or, $-5V_1 = I_1$

or, $Z_{11} = \frac{V_1}{I_1} = \frac{-1}{5}$

Supply $V_2$ in port 2 and port 1 is open circuited i.e., $I_1 = 0$.




Applying KVL in loop 2,

$V_2 = 2(I_2 - I_3)$ ......... (i)

Also, $V_1 = 2I_3$ .......... (ii)

Applying KVL in loop 3,

$-5I_3 + 2I_2 = 0$

or, $5I_3 = 2I_2$ .......... (iii)

Solving equations (iii) and (i), we get

$$V_2 = 2I_2 - 2\left(\frac{2I_2}{5}\right)$$

$$\text{or, } V_2 = 2I_2 - \frac{4I_2}{5}$$

or, $5V_2 = 10I_2 - 4I_2$

or, $5V_2 = 6I_2$

$$\text{or, } Z_{22} = \frac{V_2}{I_2} = \frac{6}{5}$$

Solving (ii) and (iii),

$$V_1 = 2\left(\frac{2I_2}{5}\right)$$

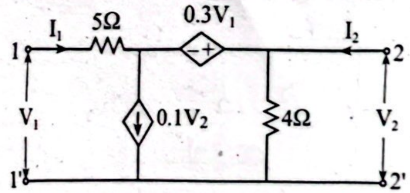
$$\text{or, } V_1 = \frac{4I_2}{5} = \frac{V_1}{I_2}$$

$$\text{or, } Z_{12} = \frac{4}{5}$$

Hence, the Z-parameters are

$$Z_{11} = \frac{-1}{5},\ Z_{12} = \frac{4}{5},\ Z_{21} = -\frac{14}{5},\ Z_{22} = \frac{6}{5}$$

5. **For the two-port network shown below, find h-parameters and T-parameters. Also check for reciprocity of network.**




[2075 Chaitra, 2078 Chaitra]

***Solution:***

The h-parameters equations of TPN is



$V_1 = h_{11}I_1 + h_{12}V_2$ ............ (1)

$I_2 = h_{21}I_1 + h_{22}V_2$ ............ (2)

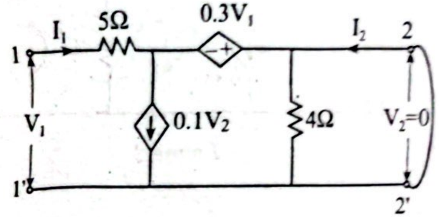



***Figure 1***

Applying $V_1$ at port 1and making port 2 short circuited as shown in figure 1, the resultant circuit is shown in figure 2.

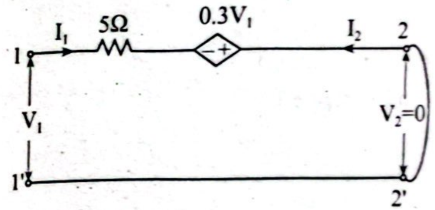



***Figure 2***

From figure 2,

$I_1 = -I_2$ ......... (3)

Applying KVL in figure 2, we get

$V_1 - 5I_1 + 0.3V_1 = 0$

or, $1.3V_1 = 5I_1$

$$\text{or, } V_1 = \frac{5}{1.3}I_1 \text{ .......... (4)}$$

When $V_2 = 0$, then from equation (1),

$$h_{11} = \frac{V_1}{I_1} = \frac{\frac{5}{1.3}I_1}{I_1} = \frac{5}{1.3}\,\Omega$$

When $V_2 = 0$, then from equation (2),

$$h_{21} = \frac{I_2}{I_1} = \frac{-I_1}{I_1} = -1$$

Applying $V_2$ at port 2, and making port 1 open circuited as shown in figure 3, then from equations (1) and (2), we get

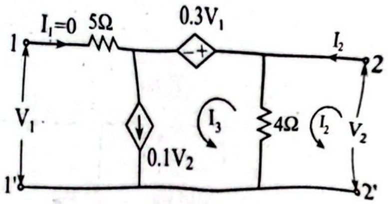



*Figure 3*

$$h_{12} = \frac{V_1}{V_2} \text{ ............ (5)}$$

$$h_{22} = \frac{I_2}{V_2} \text{ ............ (6)}$$

Applying KVL in loop 2 of figure 3,

$V_2 - 4I_2 + 4I_3 = 0$

or, $V_2 = 4I_2 - 4I_3$ .......... (7)

Also in loop 3 of figure 3,

$I_3 = 0.1V_2$

or, $I_3 = 0.1[4I_2 - 4I_3]$

or, $I_3 = \frac{0.4}{1.4} I_2$ ........ (8)

From (7) and (8),

$$V_2 = 4I_2 - 4\left(\frac{0.4}{1.4}\right) I_2$$

$$\text{or, } V_2 = 4I_2 - \frac{1.6}{1.4} I_2$$

$$\text{or, } V_2 = \frac{5.6I_2 - 1.6I_2}{1.4}$$

$$\text{or, } V_2 = \frac{4I_2}{1.4} \text{ ......... (9)}$$

From (6),

$$h_{22} = \frac{I_2}{V_2} = \frac{I_2}{\frac{4I_2}{1.4}} = \frac{1.4}{4} \mho$$

From figure 3,

$V_1 + 0.3V_1 - 4I_2 + 4I_3 = 0$

$$\text{or, } 1.3V_1 - 4I_2 + 4\left(\frac{0.4}{1.4} I_2\right) = 0$$



$$\text{or, } 1.3V_1 - 4I_2 + \frac{1.6}{1.4} I_2 = 0$$

$$\text{or, } V_1 = \frac{4}{1.82} I_2 \text{ .......... (10)}$$

From equation (5),

$$h_{12} = \frac{V_1}{V_2} = \frac{\frac{4}{1.82} I_2}{\frac{4I_2}{1.4}} = \frac{1.4}{1.82}$$

Thus h-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{5}{1.3} & \frac{1.4}{1.82} \\ -1 & \frac{1.4}{4} \end{bmatrix}$$

*Hence*, the h-parameters equations are

$$V_1 = \frac{5}{1.3} I_1 + \frac{1.4}{1.82} V_2 \text{ ........ (11)}$$

$$I_2 = -I_1 + \frac{1.4}{4} V_2 \text{ ........ (12)}$$

From equation (12),

$$I_1 = \frac{1.4}{4} V_2 - I_2 \text{ ......... (13)}$$

From (11) and (13),

$$V_1 = \frac{5}{1.3}\left[\frac{1.4}{4} V_2 - I_2\right] + \frac{1.4}{1.82} V_2$$

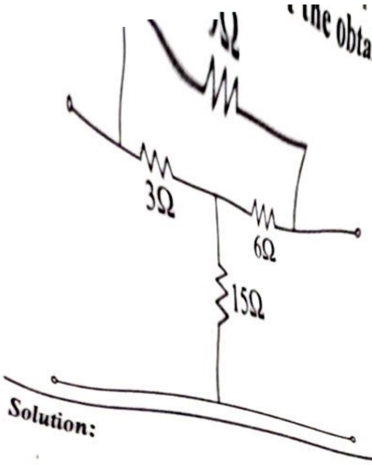
$$\text{or, } V_1 = \frac{7}{5.2} V_2 - \frac{5}{1.3} I_2 + \frac{1.4}{1.82} V_2$$

or, $V_1 = 2.1152V_2 - 3.846I_2$ ....... (14)

Hence, comparing equations (13) and (14) with the equation of T-parameters, we get

$$A = 2.1152,\ C = \frac{1.4}{4},\ B = 3.846,\ D = 1$$

$$\therefore \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 2.115 & 3.846 \\ \frac{1.4}{4} & 1 \end{bmatrix}$$




**Solution:**

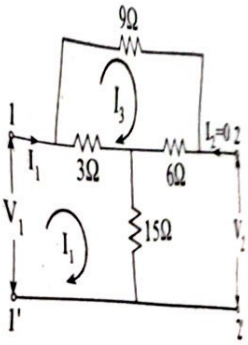



*Figure 1*

The open circuit impedance parameter equations are

$V_1 = Z_{11}I_1 + Z_{12}I_2$ .......... (1)

$V_2 = Z_{21}I_1 + Z_{22}I_2$ .......... (2)

Applying $V_1$ at port-1 while making port-2 open circuited as shown in figure 1, then from equations (1) & (2), we get

$$Z_{11} = \frac{V_1}{I_1} \text{ ........ (3) \& } Z_{21} = \frac{V_2}{I_1} \text{ ........ (4)}$$

Applying KVL in loop 1 & loop 3, and putting them in matrix form, we have

$$\begin{bmatrix} 18 & -3 \\ -3 & 18 \end{bmatrix}\begin{bmatrix} I_1 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \end{bmatrix}$$

$$\text{Now, } I_1 = \frac{\Delta_1}{\Delta} = \frac{18V_1}{324-9} = \frac{18V_1}{315} = \frac{2V_1}{35} \text{ .............. (5)}$$

$$I_3 = \frac{\Delta_2}{\Delta} = \frac{3V_1}{315} = \frac{V_1}{105} \text{ .............. (6)}$$

Also, from figure 1,

$$V_2 - 6I_3 - 15I_1 = 0$$

$$\text{or, } V_2 = 6 \times \frac{V_1}{105} + 15 \times \frac{2V_1}{35}$$

$$\therefore V_2 = \frac{96V_1}{105} = \frac{32V_1}{35} \text{ .............. (7)}$$

From equations (3), (5),

$$Z_{11} = \frac{V_1}{I_1} = \frac{V_1}{\frac{2V_1}{35}} = \frac{35}{2}\,\Omega$$

From equations (4), (5), (7),

$$Z_{21} = \frac{V_2}{I_1} = \frac{32V_1/35}{2V_1/35} = \frac{32}{2} = 16\,\Omega$$

Now, applying $V_2$ at port-2 while making port-1 open circuited as shown in figure 2, then from equations (1) & (2), we get

$$Z_{12} = \frac{V_1}{I_2} \text{ ........ (8)}$$

$$Z_{22} = \frac{V_2}{I_2} \text{ ........ (9)}$$

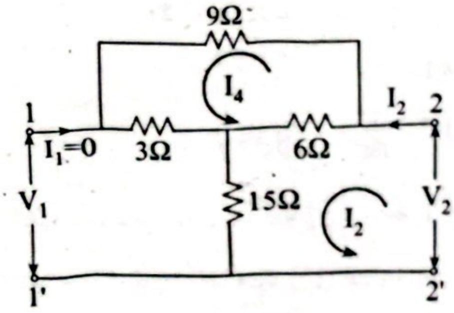



*Figure 2*

Now, applying KVL in loop 2 & loop 4, and putting them in matrix form, we get

$$\begin{bmatrix} 21 & -6 \\ -6 & 18 \end{bmatrix}\begin{bmatrix} I_2 \\ I_4 \end{bmatrix} = \begin{bmatrix} V_2 \\ 0 \end{bmatrix}$$

$$\text{Now, } I_2 = \frac{\Delta_1}{\Delta} = \frac{18V_2}{378-36} = \frac{18V_2}{342} = \frac{V_2}{19} \text{ ........................ (10)}$$

$$I_4 = \frac{\Delta_2}{\Delta} = \frac{6V_2}{342} = \frac{V_2}{57} \text{ ............................ (11)}$$

Also from figure (2),

$V_1 - 3I_4 - 15I_2 = 0$

or, $V_1 = 3 \times \frac{6V_2}{342} - 15 \times \frac{18V_2}{342} = \frac{18V_2 - 270V_2}{342}$

or, $V_1 = \frac{-252V_2}{342}$ ...... (12)

Now, from equations (8), (10), (12),

$Z_{12} = \frac{V_1}{I_2} = \frac{-252V_2/342}{18V_2/342} = -14\Omega$

From equations (9), (10),

$Z_{22} = \frac{V_2}{I_2} = \frac{V_2}{V_2/19} = 19\ \Omega$

Now, from equations (1) & (2),

$V_1 = \frac{35}{2} I_1 + (-14)I_2$ ........................ (13)

$V_2 = 16\ I_1 + 19I_2$ ............................ (14)

From equation (14),

$I_2 = -\frac{16}{19} I_1 + \frac{1}{19} V_2$ .................... (15)

From (15) & (13),

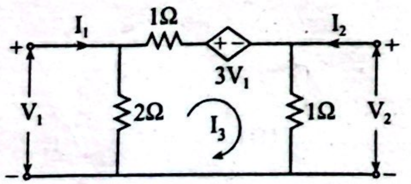
$V_1 = \frac{35}{2} I_1 - 14\left(-\frac{16}{19} I_1 + \frac{1}{19} V_2\right)$

or, $V_1 = \frac{1113}{38} I_1 + \left(-\frac{14}{19}\right) V_2$ ........................ (16)

Comparing equations (15) & (16) with the equations of h-parameters, we get

$h_{11} = \frac{1113}{38}, h_{12} = \frac{-14}{19}, h_{21} = \frac{-16}{19}, h_{22} = \frac{1}{19}$

**7. Find A, B, C, D parameters of the two-port network shown and check its reciprocity.**




[2079 Ashwin]



***Solution:***

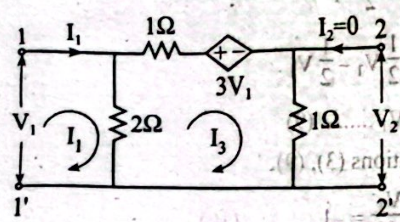



*Figure 1*

The ABCD parameters equations are

$V_1 = AV_2 + B(-I_2)$ ............. (1)

$I_1 = CV_2 + D(-I_2)$ ............... (2)

Applying $V_1$ at port-1, while making port-2 open circuited as shown in figure 1 i.e. $I_2 = 0$, then from equations (1) & (2), we get

$A = \frac{V_1}{V_2}$ ....... (3) & $C = \frac{I_1}{V_2}$ ........ (4)

Applying KVL in loop 1 & loop 3 of figure 1, we respectively get

$V_1 - 2I_1 + 2I_3 = 0$

or, $V_1 = 2I_1 - 2I_3$ ......... (5)

$-2I_3 + 2I_1 - I_3 - 3V_1 - I_3 = 0$

or, $-4I_3 + 2I_1 = 3V_1$ ......... (6)

From equation (5),

$I_3 = \frac{2}{2} I_1 - \frac{1}{2} V_1$

or, $I_3 = I_1 - \frac{1}{2} V_1$ ......... (7)

From equations (7) & (6),

$-4(I_1 - \frac{1}{2} V_1) + 2I_1 = 3V_1$

or, $-2I_1 + 2V_1 = 3V_1$

or, $I_1 = -\frac{1}{2} V_1$ ........ (8)

Also, from figure (1),

$V_2 - I_3 = 0$

or, $V_2 = I_3$

Applying $V_1$ at port 1 and making port 2 *open circuited* as shown in figure 1. Then, from equations (1) and (2),

$$Z_{11} = \frac{V_1}{I_1}\bigg|_{I_2=0},\ Z_{21} = \frac{V_2}{I_1}\bigg|_{I_2=0}$$

Applying KVL in loop 1,

$3I_1 - 2I_3 = V_1$ ............ *(3)*

Applying KVL in supermesh (loop 3 and loop 4),

$-2I_3 + 2I_1 - I_3 - 2I_4 - I_4 = 0$

or, $2I_1 - 3I_3 - 3I_4 = 0$ .......... *(4)*

Also, in the branch containing current source,

$I_3 - I_4 = 4V_1$

or, $I_3 - I_4 = 4\,(3I_1 - 2I_3)$

or, $12I_1 - 9I_3 + I_4 = 0$ ............ (5)

From (5),

$I_4 = 9I_3 - 12I_1$ ............... (6)

From (4) and (6),

$2I_1 - 3I_3 - 3(9I_3 - 12I_1) = 0$

or, $2I_1 - 3I_3 - 27I_3 + 36I_1 = 0$

or, $I_3 = \frac{38}{30}I_1 = \frac{19}{15}I_1$ .............. (7)

From (3) and (7),

$$3I_1 - 2\left(\frac{19}{15}\right)I_1 = V_1$$

or, $\frac{45I_1 - 38I_1}{15} = V_1$

or, $\frac{7}{15}I_1 = V_1$

or, $I_1 = \frac{15V_1}{7}$ ........... (8)

$$\therefore\ Z_{11} = \frac{V_1}{I_1} = \frac{V_1}{\frac{15V_1}{7}} = \frac{7}{15}\ \Omega$$

From (5), (7), (8),

$12I_1 - 9\left(\frac{19}{15}I_1\right) + I_4 = 0$

or, $I_4 = -\frac{9}{15}I_1$ .............. (9)

Now, $V_2 = 1 \times I_4 = 1\left(\frac{-9}{15}I_1\right) = \frac{-9}{15}I_1$ ........... (10)

$$Z_{21} = \frac{V_2}{I_1} = \frac{(-9/15)I_1}{I_1} = \frac{-9}{15}\ \Omega$$

Applying $V_2$ at port 2 and making port 1 open circuited i.e., $I_1 = 0$ as shown in figure 2. Then, from equations (1) and (2),

$$Z_{12} = \frac{V_1}{I_2}\bigg|_{I_1=0},\ Z_{22} = \frac{V_2}{I_2}\bigg|_{I_1=0}$$

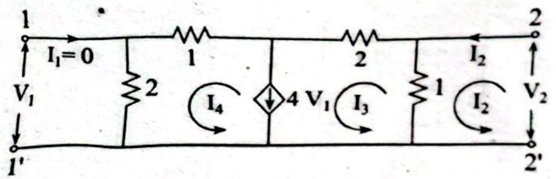



*Figure 2*

Applying KVL in loop 2,

$V_2 = I_1 - I_3$ ............ (11)

Applying KVL in supermesh (loop 3 and loop 4)

$-1I_3 + I_2 - 2I_3 - I_4 - 2I_4 = 0$

or, $I_2 - 3I_3 - 3I_4 = 0$ ............. (12)

Also, in the branch containing current source,

$I_3 - I_4 = 4V_1$

or, $I_3 - I_4 = 4(2I_4)$

or, $I_4 = \frac{1}{9}I_3$ ............ (13)

From (12) and (13),

$$I_2 - 3I_3 - 3 \times \frac{1}{9} I_3 = 0$$

or, $I_2 - 3I_3 - \frac{1}{3} I_3 = 0$

or, $I_3 = \frac{3}{10} I_2$ ............ (14)

From (11) and (14),

$$V_2 = I_2 - \frac{3}{10} I_2$$

or, $V_2 = \frac{7}{10} I_2$ ............ (15)

$\therefore \; Z_{22} = \frac{V_2}{I_2} = \frac{7}{10} \; \Omega$

Also, $V_1 = 2I_4$

$$Z_{12} = \frac{V_1}{I_2} = \frac{2I_4}{I_2} = \frac{2\left(\frac{1}{9} I_3\right)}{I_2} = \frac{\frac{2}{9}\left(\frac{3}{10} I_2\right)}{I_2} = \frac{1}{15} \; \Omega$$

Now, $[Z] = \begin{bmatrix} \frac{7}{15} & \frac{1}{15} \\ \frac{-9}{15} & \frac{7}{10} \end{bmatrix}$

$\therefore \; V_1 = \frac{7}{15} I_1 + \frac{1}{15} I_2$ ............ (16)

$V_2 = \frac{-9}{15} I_1 + \frac{7}{10} I_2$ ............ (17)

From (17),

$$\frac{9}{15} I_1 = \frac{7}{10} I_2 - V_2$$

or, $I_1 = \frac{15}{9}\left(\frac{7}{10} I_2 - V_2\right)$

or, $I_1 = -\frac{15}{9} V_2 + \frac{21}{18} I_2$

or, $I_1 = \left(\frac{-15}{9}\right) V_2 + \left(\frac{-21}{18}\right)(-I_2)$ .............. (18)



From (16) and (18),

$$V_1 = \frac{7}{15}\left(\frac{-15}{9} V_2 + \frac{21}{18} I_2\right) + \frac{1}{15} I_2$$

or, $V_1 = \frac{-7}{9} V_2 + \frac{11}{18} I_2$

or, $V_1 = \left(\frac{-7}{9}\right) V_2 + \left(\frac{-11}{18}\right)(-I_2)$

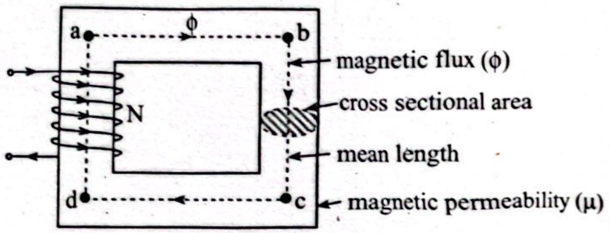
$\therefore \; [T] = \begin{bmatrix} \frac{-7}{9} & \frac{-11}{18} \\ \frac{-15}{9} & \frac{-21}{18} \end{bmatrix}$

Chapter - 6

# MAGNETIC CIRCUITS AND INDUCTION

## 6.1 Magnetic Circuit

A *magnetic circuit* is the path followed by a magnetic flux (analogous to the electric circuit as the path followed by electric current). It is analogous to electrical current (I).




In an electric circuit, the current flows due to an emf source. Similarly, in magnetic circuits the magnetic flux is produced by a quantity known as *MMF (magneto motive force)*.

MMF = NI

Where, N = number of turns in the winding

I = current flow through the winding

The current flow in any electric circuit is opposed by the resistance of the path. Similarly, the magnetic flux flow in a magnetic circuit is opposed by the reluctance (R) nature of the path.

$$R = \frac{l}{\mu A} = \frac{l}{\mu_o \mu_r A}$$

where, $l$ = mean length of the magnetic path

$\mu_o$ = permeability of the core

A = cross sectional area of core

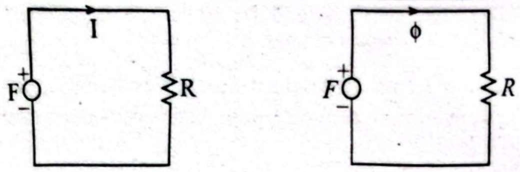
$\mu_r$ = relative permeability of the core

Reciprocal of reluctance is called permeance.

i.e., Permeance (P) $= \frac{1}{R}$



## 6.2 Ohm's Law of Magnetic Flux




*Electric circuit* *Magnetic circuit*

***Figure 6.1:** Analogy between electric and magnetic circuit*

Ohm's law: $\frac{V}{I} = R$ $\qquad \frac{F}{\phi} = R$

$V = IR \qquad F = \phi R$

$$\phi = B \times A$$

$$\Rightarrow \phi = \mu H \times A$$

$$\Rightarrow \phi = \frac{\mu NI}{l} \times A$$

$$\phi = \frac{NI}{(l/\mu A)}$$

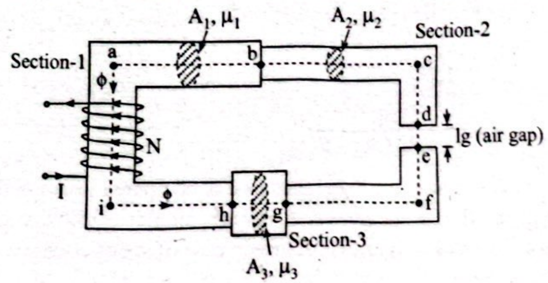
Hence, Ohm's law in magnetic circuit is, MMF (F) = $\phi$R

Then, the analogy is as:

| | Electrical | | Magnetic circuit |
|---|---|---|---|
| 1 | Current ( I ) | 1 | Magnetic flux ($\phi$) |
| 2 | EMF (E) | 2 | MMF (F) |
| 3 | Resistance (R) | 3 | Reluctance (R) |
| 4 | $I = \frac{E}{R}$ | 4 | $\phi = \frac{MMF}{R}$ |

## 6.3 Series and Parallel Magnetic Circuits

### 6.3.1 Series Magnetic Circuit



flux does not divide and passes through the different sections of the core, then those sections are said to be in series forming a series magnetic circuit as shown in figure.

Here, MMF = NI and the resultant flux 'ϕ' flow through each section of the core. Now, the reluctance of each part/ section can be calculated as:

**Section-1:**

Length $(L_1) = ba + ai + ih$

Area = $A_1$ and Permeability = $\mu_1$

$$\therefore R_1 = \frac{L_1}{\mu_1 A_1}$$

**Section-2:**

Length $(L_2) = bc + cd + ef + fg$

Area = $A_2$ and permeability = $\mu_2$

$$\therefore R_2 = \frac{L_2}{\mu_2 A_2}$$

**Section-3:**

Length $(L_3) = hg$; Area = $A_3$ and permeability = $\mu_3$

$$\therefore R_3 = \frac{L_3}{\mu_3 A_3}$$

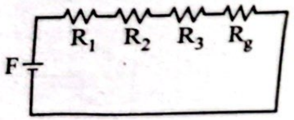
**Air gap:**

Length = $l_g$; Area = $A_g$ and permeability = $\mu_o$

$$\therefore R_g = \frac{l_g}{\mu_o A_g}$$

Then, total reluctance in series is given by,

$$R = R_1 + R_2 + R_3 + R_g$$




$$\therefore \phi = \frac{mmf}{R}$$

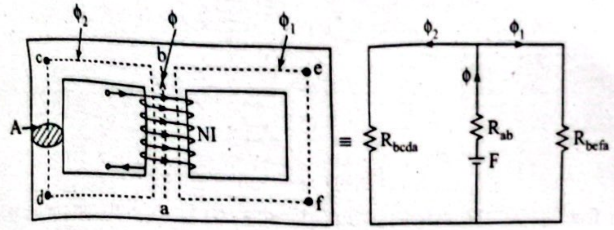
$$\therefore \phi = \frac{NI}{R_1 + R_2 + R_3 + R_g}$$

Air gap has very high reluctance in compared to iron core, it reduces the magnetic flux in the circuit. It is quite similar to addition of very high resistance in series with low resistance in case of series electric circuit.



### 6.3.2 Parallel Magnetic Circuit

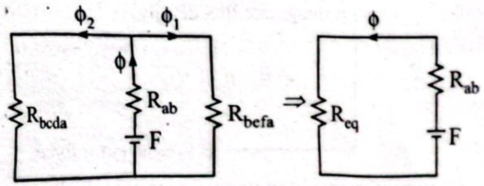
If the magnetic flux produced by MMF divides into two or more parallel paths in some sections of the magnetic circuit in a core, then those section are said to be in parallel as shown in figure.




Thus, in this case

$$R_{ab} = \frac{l_{ab}}{\mu A}; R_{bcda} = \frac{l_{bcda}}{\mu A}; R_{befa} = \frac{l_{befa}}{\mu A}$$

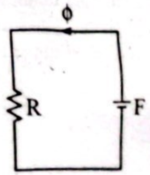
And, thus as an electrical circuit, the magnetic equivalent reluctance can be calculated.




$$R_{eq} = \frac{R_{bcda} \times R_{befa}}{R_{bcda} + R_{befa}}$$

$$R = R_{eq} + R_{ab}$$

$$\Rightarrow \phi_{net} = \phi = \frac{F}{R}$$




$$\therefore \phi = \frac{MMF}{R_{ab} + \frac{R_{bcda} \times R_{befa}}{R_{bcda} + R_{befa}}} = \frac{NI}{R_{ab} + \frac{R_{bcda} \times R_{befa}}{R_{bcda} + R_{befa}}}$$

## 6.4 B-H Relationship (Magnetization Characteristics)

As we know, $H = \frac{NI}{l} \Rightarrow H \propto I$

$$B = \frac{\phi}{A} \Rightarrow B \propto \phi$$

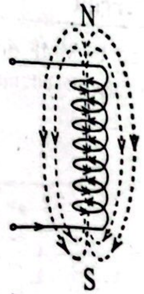



*Figure 6.2: Coil without core*

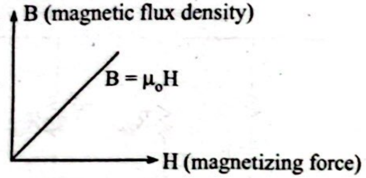
In free space, the magnetic flux density (B) is directly proportional to the magnetizing force (H)

i.e. $B \propto H$

or, $B = \mu_o H$; where $\mu_o$ = permeability of free space

$= 4\pi \times 10^{-7}$ H/m

Here, the relationship between B and H is linear one.




However, this is not the case in magnetic material which is used as core in all electrical machines and transformers. The relationship is strictly non-linear. A typical B-H curve for a magnetic material is shown below.

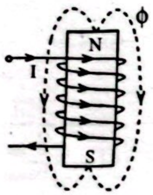
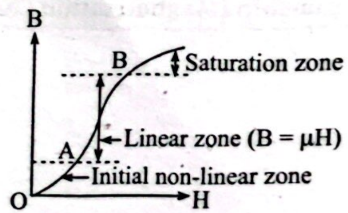



*Figure 6.3: Coil with iron core (ferromagnetic material)*

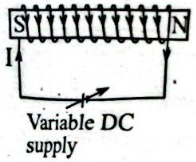





## 6.5 Hysteresis Curve/ B-H Curve / Magnetizing Curve

### 6.5.1 Hysteresis Loss in DC Excitation

Consider an electromagnet supplied by a variable DC supply. The magnetizing force inside the core is given by, $H = \frac{NI}{l}$, Hence varying I, H in the material can be varied and accordingly "B" will also vary.



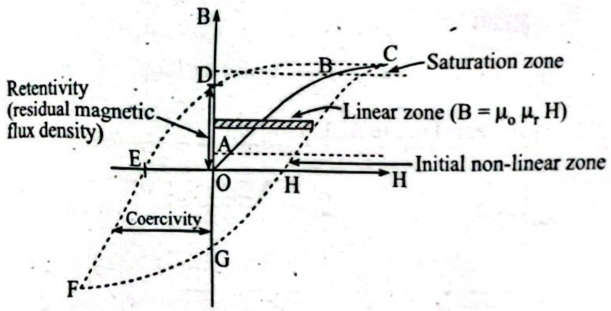




*Figure 6.4: Hysteresis loop*

There is a loss in the process of magnetization and demagnetization in the form of heat and is called hysteresis loss. While demagnetizing, there is a residual magnetic flux density even when magnetic field intensity (H) is zero and this residual magnetic flux density is called *retentivity*.

To make magnetic flux density zero, we give magnetic field intensity in opposite direction. The value of opposite magnetic field strength required to make 'B' zero is called *coercivity*.

The magnetic flux at any instant is given by,

$$\phi(t) = B(t)\,A$$

Then emf induced in the coil according to Faraday's law of electromagnetic induction,

$$e = N\frac{d\phi}{dt} = N\frac{d(B.A)}{dt} = NA\frac{dB}{dt} \quad \text{.............. (i)}$$

and also, magnetizing force is,

$$\therefore P = e \times I = NA\frac{dB}{dt}\frac{Hl}{N} = A\,L\,H\frac{dB}{dt}$$

Energy spent in small time interval, $(dw) = P\,dt$

or, $dw = A\,l\,H\,dB$

Energy spent in one cycle of magnetization (i.e. the complete hysteresis loop) $= \oint dw$

$W = Al\oint H.dB$

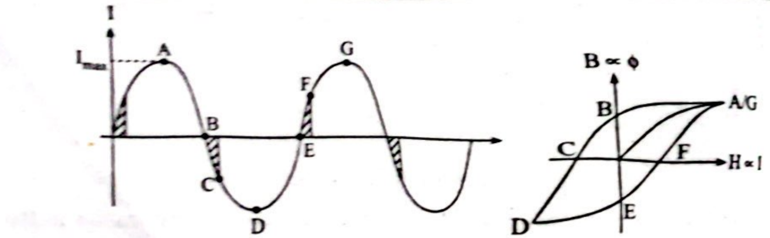
or, $H.dB$ = Shaded area

or, $\oint H.dB$ = Complete area of the loop

$\frac{W}{Al} = \oint H.dB$

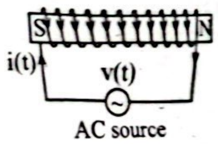
$\therefore$ Energy loss per unit volume = Area of the loop.

### 6.5.2 Hysteresis Loss in AC Excitation




- With the varying voltage source, the core inside the coil gets magnetized and demagnetized in each cycle causing hysteresis loss. The power loss due to hysteresis is

$P_h = \eta B_m^{1.6} f\, v$ (watts)




where, $\eta$ = Steinmetz constant

$= 502\ J/m^3$ (sheet steel)

$= 181\ J/m^3$ (silicon steel)

$B_m$ = maximum flux density in the core

$f$ = frequency of exciting current

$v$ = volume of iron core

**Note:** Addition of silicon reduces '$\eta$', hence reduces hysteresis loss.



## 6.6 Eddy Current Loss

The time varying flux in the core induces emf in the coils according to Faraday's law of electromagnetic induction. But since the core itself is a conductor (all magnetic materials are), emf will also be induced in the core resulting circulating currents in the core. These currents are known as eddy current and have a Power loss ($I^2R$) associated with it, the loss is known as eddy current loss.

This loss depends upon the

i. Resistivity of the material

ii. Mean length of the path of the circulating current for a given cross-sectional area.

The eddy current loss in the core is given by,

$P_e = KB_m^2 f^2 t^2 v$ (watts)

Where,

$B_m$ = maximum value of the flux density in the core

$f$ = frequency of exciting current

$v$ = volume of iron core

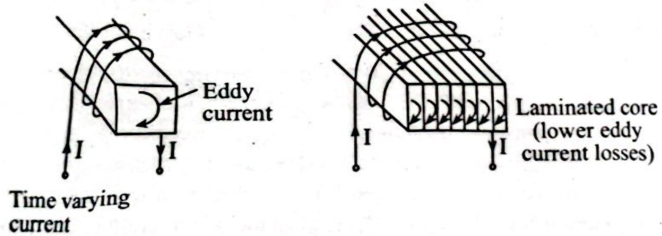
$t$ = thickness of each lamination

$K$ = constant, depending upon the nature of the core

In practical applications, the eddy current losses can be reduced by

i. Adding silicon to steel which will give a high resistivity of the material

ii. Dividing up the solid core into thin laminated core while making sure that each laminated core is insulated from each other.




## 6.7 Faraday's Law of Electromagnetic Induction, Statically and Dynamically Induced EMF

A gentleman named Michael Faraday paved a history in 1891 by discovering the relationship between electricity and magnetism. He observed the momentary with the circuit, when magnetic flux linking with the circuit changes momentary with respect to time. After his detail study of the phenomenon, he formulated some laws, which are well known as "Faraday's law of electromagnetic induction".

i. **First law:** "Whenever the magnetic flux linked with a conductor changes with respect to time, an emf will be induced in it." (magnetic flux linkage = cuts magnetic flux)

ii. **Second law:** "The magnitude of emf induced is equal to the time rate of change of magnetic flux linkage"

The magnetic flux-linkage could be changed in two different ways. Therefore, emf could be produced in two different ways.

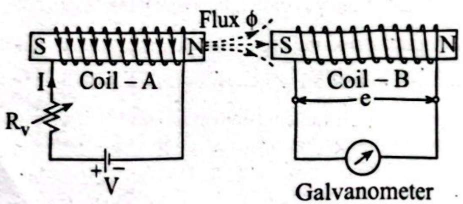
a. Statically induced emf

b. Dynamically induced emf

$$\left[e = N\frac{d\phi}{dt} = N\frac{d(BA)}{dt} = \underbrace{\left(NB\frac{dA}{dt}\right)}_{dynamically} + \underbrace{\left(NA\frac{dB}{dt}\right)}_{statically}\right]$$

### 6.7.1 Statically Induced EMF: (Coil Stationary, Field Changes)

In this method, there is no physical movement of conductor or coil, only the magnitude of magnetic flux is changed. Hence changing the flux-linkage.




When the value of DC current is varied by varying the resistance $R_v$ in coil-A, the produced magnetic flux also varies. (When current is not varied, the magnetic field remains constant.) The produced magnetic flux links the coil-B. When I is increased, change in flux-linkage i.e. increase in it occurs, and the galvanometer shows a deflection in one direction indicating induced emf causing current flow. When I is decreased, the observation is just opposite.

If the magnetic flux in the coil-B changes from $\phi_1$ to $\phi_2$ in a small time interval from $t_1$ to $t_2$, then according to second statement given by Faraday's law of electromagnetic induction, emf induced in a single turn of coil-B is given by,



$$E\,(\text{per turn}) = \frac{\phi_2 - \phi_1}{t_2 - t_1} = \frac{d\phi}{dt} = \text{Rate of change of flux}$$

For 'N' number of turns in the coil -B, then total emf induced across the coil is given by:

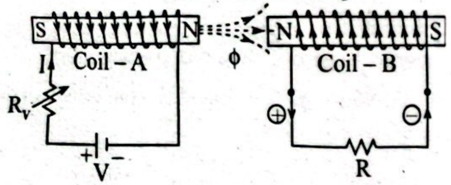
$$E = N\frac{d\phi}{dt} \text{ volts}$$

Taking the Lenz's law statements, "Direction of induced current/emf in the conductor will be such that the magnetic field set up by the induced current/emf opposes the cause by which the current/emf was induced."

$$E = -N\frac{d\phi}{dt}$$

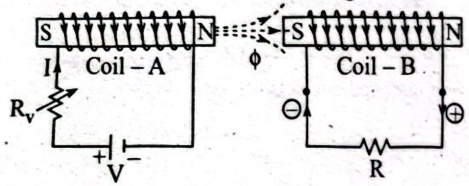
e.g.,

If flux is increasing




Emf is induced to oppose the cause i.e. to decrease the flux from coil A.

If flux is decreasing




Emf is induced to support the cause i.e. to increase the flux from coil A.

*Figure 6.5: Lenz law*

### 6.7.2 Dynamically Induced emf

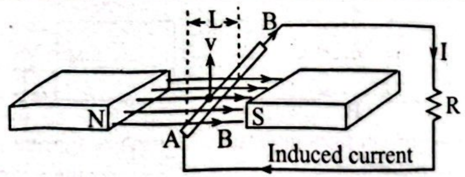



*Figure 6.6: Dynamically induced emf*

In this method, field is stationary and conductor moves to cut across it which is responsible for the change in flux linkage.

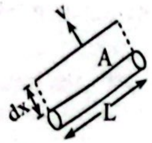



*Figure 6.7: Area swept by the conductor in dt time*

As shown in the figure 6.7, the conductor is moving upward in the magnetic field with velocity v. Here, in small time 'dt' the conductor sweeps a distance dx with velocity 'v'. Let, L be the length of conductor inside the magnetic field.

When the conductor moves in the magnetic field, there is a change in flux-linkage.

Now, the change in flux-linkage will be equal to the change in flux when conductor moves a distance dx, which is given by,

$d\phi = B \times A$ ; where A = area swept by conductor in dt time

or, $d\phi = B \times dx \times L$

or, $d\phi = B \times L \times v \times dt$ $\qquad (\because v = \frac{dx}{dt})$

or, $\frac{d\phi}{dt} = BLv$............. (i)

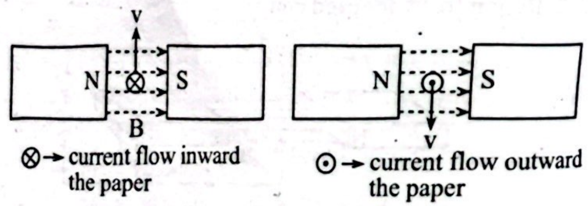
(flux change = flux cut = $L \times dx \times B$)

We know that, induced emf is given by,

$e = \frac{d\phi}{dt}$ ........... (ii)

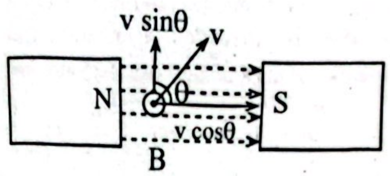
Thus, $E = BLv$

Here, L, B and v are vector quantity. The direction of induced emf (or current) can be find out by using Fleming's right hand rule.






If the direction of motion is inclined to the magnetic flux density as shown below,




Then, only the component of velocity perpendicular to field 'B' is taken. Since the component parallel to B has no change in flux linkage. Thus, in general, the induced emf for dynamic case is,

$E = BLv \sin\theta$

## 6.8 Force on Current Carrying Conductor in Magnetic Field

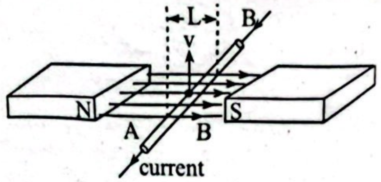



*Figure 6.8: Force developed on current carrying conductor in a magnetic field*

When a current carrying conductor is placed in a magnetic field, then a force will develop on the conductor, whose magnitude is given by,

$F = B.I.L$ (Newton)

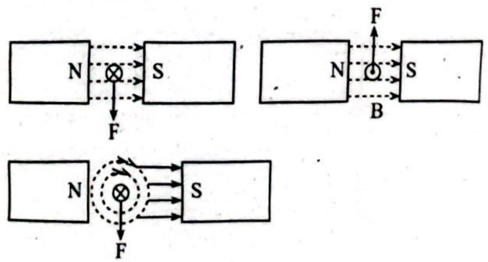
Where,

B = magnetic flux density (wb/m$^2$)

I = current passing through the conductor (A)

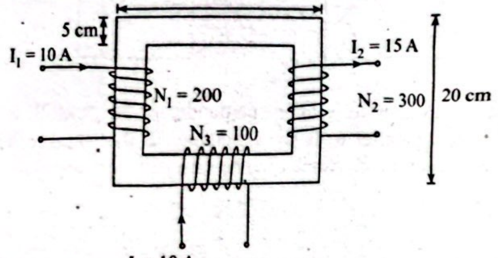
L = length of conductor lying within the magnetic field (m)

And the direction of force is given by the Fleming's left hand rule.



## SOLVED PROBLEMS

**1. Calculate the net magnetic flux in the core of the following magnetic circuit and show the direction of magnetic flux in the core. Given that cross sectional area of the core is 25 cm² and $\mu_r = 4000$.**




[2073, 2071 Bhadra]

***Solution:***

MMF due to current $I_1$ is: $F_1 = I_1N_1 = 10 \times 200 = 2000$ AT

MMF due to current $I_2$ is: $F_2 = I_2N_2 = 15 \times 300 = 4500$ AT

MMF due to current $I_3$ is: $F_3 = I_3N_3 = 10 \times 100 = 1000$AT

Cross-sectional area of iron core (A) = 25 cm² = $25\times10^{-4}$ m²

Total mean length of the iron core is:

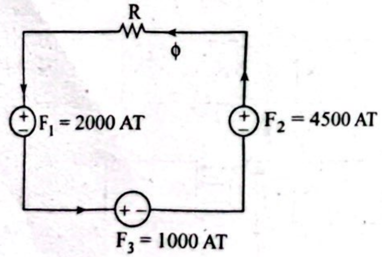
$l = 2 \times (20-5) + 2 \times (30-5) = 80$ cm = 0.8 m

Relative permeability of iron core ($\mu_r$) = 4000

$$\text{Reluctance (R)} = \frac{l}{\mu_o\mu_rA} = \frac{0.8}{4\pi \times 10^{-7} \times 4000 \times 25 \times 10^{-4}}$$

$$= 63661.97724 \text{ AT/wb}$$

The electrical analogy circuit of given magnetic circuit is as follows:






Net MMF (F) = $F_2 - (F_1 + F_2)$

$= 4500 - (1000 + 2000) = 1500$ AT

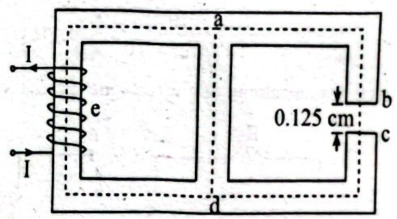
The magnetic flux in the core,

$$\phi = \frac{F}{R}$$

$$= \frac{1500}{63661.97724} = 0.02356 \text{ wb} = 23.56 \text{ mwb}$$

Direction: anti-clockwise

**2. A magnetic circuit shown below has cast iron core whose dimensions are given below:**




**Length (ab + cd) = 50 cm, cross-sectional area of path (ab + cd) = 25 sq. cm**
**Length (ad) = 20 cm, cross sectional area of path (ad) = 1.25 sq. cm**
**Length (dea) = 50 cm, cross-sectional area of path (dea) = 25 sq. cm**
**Determine the current 'I' required to produce a magnetic flux of 0.75 mwb in the central limb. Given that, number of turns in the coil is 500, $\mu_r = 2000$.** [2067 Poush]

***Solution:***

Number of turns of coil (N) = 500

Relative permeability ($\mu_r$) = 2000

Length (ab + cd); $l_1 = 50$ cm = 0.5 m

Length (ad), $l_2 = 20$ cm = 0.2 m

Length (dea), $l_3 = 50$ cm = 0.5 m

Length of air ($l_g$) = 0.25 cm = $0.25 \times 10^{-2}$ m

Cross-sectional area of path (ab + cd), $(A_1) = 2.5$ cm² = $25 \times 10^{-4}$ m²

Cross-sectional area of path (ad), $(A_2) = 12.5$ cm² = $12.5 \times 10^{-4}$ m²

Cross-sectional area of path (dea), $(A_3) = 25$ cm² = $25 \times 10^{-4}$ m²

Cross-sectional area of air gap, Ag = 25 cm² = $25 \times 10^{-4}$ m²

Now, the reluctance of,

Path (ab + cd), $R_1 = \frac{l_1}{\mu_0\mu_r A_1} = \frac{0.5}{4\pi \times 10^{-7} \times 2000 \times 25 \times 10^{-4}}$

$= 79577.47155$ AT/wb

Path (ad), $R_2 = \frac{l_2}{\mu_0\mu_r A_2} = \frac{0.2}{4\pi \times 10^{-7} \times 2000 \times 12.5 \times 10^{-4}}$

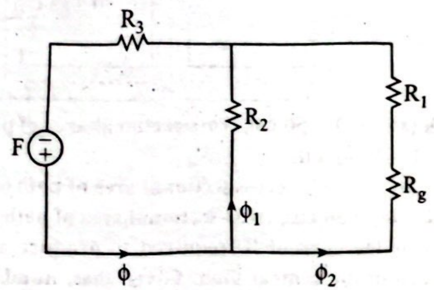
$= 63661.97724$ AT/wb

Path (dea), $R_3 = \frac{l_3}{\mu_0\mu_r A_3} = \frac{0.5}{4\pi \times 10^{-7} \times 2000 \times 25 \times 10^{-4}}$

$= 79577.47155$ AT/wb

Air gap, $R_g = \frac{l_g}{\mu_0 A_g} = \frac{0.25 \times 10^{-2}}{4\pi \times 10^{-7} \times 25 \times 10^{-4}} = 795774.7155$ AT/wb

The electrical analogy circuit of given magnetic circuit is:




Since in the parallel path mmf is same so,

$\phi_1 R_2 = \phi_2(R_1 + R_g)$

or, $\phi_2 = \frac{0.75 \times 63661.97724}{795774.47155 + 795774.7155} = 0.05454$ mwb

Total magnetic flux, $\phi = \phi_1 + \phi_2$

$= 0.75 + 0.05454 = 0.80454$ mwb

The equivalent reluctance of the magnetic circuit is:

$R = R_3 + R_2 // (R_1 + R_g)$

$= 79577.4715 + (63661.97724 // 795774.7155)$

$= 79577.4715 + 58946.27522 = 138523.7467$ AT/wb

As we know,

$F = NI = \phi R$

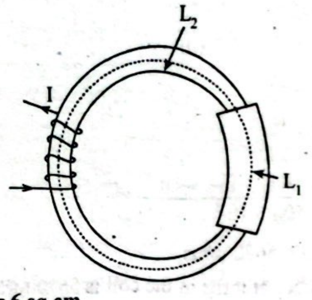


or, $I = \frac{\phi R}{N}$

or, $I = \frac{0.80454 \times 138523.7467}{500} = 0.222$ A

Hence, the current renatured to produce a magnetic flux of 0.75 mwb in the central limb is 0.222 A.

**3. An uneven ring shaped core (as shown below) has $\mu_r = 100$ and flux density in the larger section is 0.75 T. If the current through the coil is 500 mA. Determine the number of turns in the coil.**




**$L_1 = 10$ cm, $A_1 = 6$ sq.cm**
**$L_2 = 25$ cm, $A_2 = 4$ sq.cm** **[2070 Magh]**

***Solution:***

Relative permeability of iron core ($\mu_r$) = 100

Flux density in larger section ($B_2$) = 0.75 T

Current through the coil (I) = 500 mA $= 500 \times 10^{-3}$ A

$L_1 = 10$ cm $= 0.1$ m, $A_1 = 6$ cm$^2 = 6 \times 10^{-4}$ m$^2$, $L_2 = 25$ cm $= 0.25$ m,

$A_2 = 4$ cm$^2 = 4 \times 10^{-4}$ m$^2$

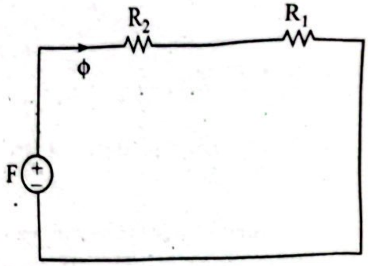
The reluctance of iron part $L_1$ is:

$R_1 = \frac{L_1}{\mu_0\mu_r A_1} = \frac{0.1}{4\pi \times 10^{-7} \times 100 \times 6 \times 10^{-4}} = 1326291.192$ AT/wb

The reluctance of iron part $L_2$ is:

$R_2 = \frac{L_2}{\mu_0\mu_r A_2} = \frac{0.25}{4\pi \times 10^{-7} \times 100 \times 4 \times 10^{-4}} = 4973591.972$ AT/wb

The electrical analogy circuit of given magnetic circuit is as below:




Since in series magnetic circuit same flux flows,

So, $\phi_1 = \phi_2 = 0.75 \times 6 \times 10^{-4}$

$= 4.5 \times 10^{-4}$ wb

$F = NI = \phi(R_2 + R_1)$

or, $N = \dfrac{\phi \times (R_2 + R_1)}{I}$

or, $N = \dfrac{4.5 \times 6299883.164 \times 10^{-4}}{500 \times 10^{-3}}$

$\therefore$ $N = 5669.89 \approx 5670$ turns.

Hence, the number of turns in the coil is 5670 turns.

**4. A circular iron core has a cross-sectional area of 5 sq. cm and mean length of 15 cm. It has two coils A and B with 100 turns and 500 turns respectively. The current in a coil A is changed from zero to 10 amp in 0.1 sec. Calculate the emf induced in the coil B. Given that the relative permeability of the core is 3000.** [2073 Ashwin]

***Solution:***

Cross-sectional area (A) = 5 cm$^2$ = $5 \times 10^{-4}$ m$^2$

Mean length of the iron core ($l$) = 15 cm = 0.15 m

Number of turns of coil A ($N_A$) = 100 turns

Number of turns of coil B ($N_B$) = 500 turns

Current change in coil A ($di_A$) = 10 A

Time taken to change current (dt) = 0.1 sec

Relative permeability of core ($\mu_r$) = 3000

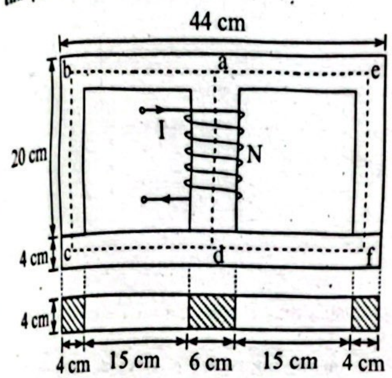
Now, the emf induced in the coil B is:

$$E_B = \frac{\mu_0\mu_r N_A N_B A}{l} \times \frac{di_A}{dt}$$

$$= \frac{4\pi\times10^{-7}\times3000\times100\times500\times5\times10^{-4}\times10}{0.15 \times 0.1} = 62.83 \text{ V}$$



**5. For the magnetic circuit shown, below calculate the Amp-turn (NI) required to establish a flux of 0.75 wb in the central limb. Given that $\mu_r$ = 4000 for iron core.**




[2074 Bhadra]

***Solution:***

Mean length of (ab + bc + cd), $L_1$ = 20+20+20 = 60 cm = 0.6 m

Area of (ab + bc + cd), $A_1 = 4 \times 4 = 16$ cm$^2$ = $16\times10^{-4}$ m$^2$

Mean length of (ae + ef + fd), $L_2$ = 20+20+20 = 60 cm = 0.6 m

Area of (ae + ef + fd), $A_2 = 4 \times 4 = 16$ cm$^2$ = $16\times10^{-4}$ m$^2$

Mean length of (ad), $L_3$ = 20 cm

Area of (ad), $A_3 = 4 \times 6 = 24$ cm$^2$ = $24\times10^{-4}$ m$^2$

The reluctance of the $L_1$ is:

$$R_1 = \frac{L_1}{\mu_0\mu_r A_1} = \frac{0.6}{4\pi \times 10^{-7} \times 4000 \times 16 \times 10^{-4}}$$

= 74603.87957 AT/wb

The reluctance of the $L_2$ is:

$$R_2 = \frac{L_2}{\mu_0\mu_r A_2} = \frac{0.6}{4\pi \times10^{-7} \times 4000 \times 16 \times 10^{-4}}$$

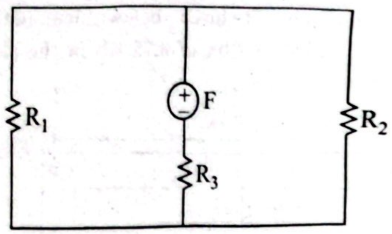
= 74603.87957 AT/wb

The reluctance of the $L_3$ is:

$$R_3 = \frac{L_3}{\mu_0\mu_r A_3} = \frac{0.20}{4\pi \times 10^{-7} \times 4000 \times 24 \times 10^{-4}}$$

= 16578.63991 AT/wb

The electrical analogy circuit of given magnetic circuit is:




The equivalent reluctance is:

$R = (R_1//R_2) + R_3$

$= 53880.57969$

$\therefore\ F = NI = \phi R$

$= 0.75 \times 53880.57969 = 40.4 \times 10^3 = 40.4 \times 10^3$ Amp-turn

**6. A steel ring of 12 cm mean radius and of circular cross-sectional 1 cm in radius has an air gap of 2 mm length. It is sound uniformly with 550 turns of wire carrying 3A of current. The air gap passes 60% of total magneto motive force. Find the total reluctance.** **[2074 Bhadra]**

***Solution:***

Mean length of steel ring $(l) = 2\pi r$

$= 2 \times \pi \times 12 = 75.398$ cm $= 0.75398$ m

Area of steel ring $(A) = \pi r^2 = \pi \times (1)^2 = 3.1415\ cm^2 = 3.1415 \times 10^{-4}\ m^2$

Length of the air gap $(l_g) = 2$ mm $= 2 \times 10^{-3}$ m

N = 550 turn

I = 3 A

Total magnetic flux, F = NI = 550 × 3 = 1650 AT

The reluctance of the air gap is:

$$R_g = \frac{l_g}{\mu_0 A} = \frac{2 \times 10^{-3}}{4\pi \times 10^{-7} \times 3.1415 \times 10^{-4}} = 5066208.598\ AT/wb$$

The mmf across the air gap is:

$$F_g = 60\%\ \text{of total mmf} = 1650 \times \frac{60}{100} = 990\ AT$$

$$\phi_g = \frac{F_g}{R_g} = 0.0001954\ wb$$

In series circuit, same flux flows through the circuit. So,

$\phi = \phi_g = 0.0001954$ wb



$$\text{Total reluctance } (R) = \frac{NI}{\phi}$$

$$= \frac{1650}{0.0001954} = 8443680.996\ AT/wb$$

**7. A magnetic circuit consists of a circular iron core having mean length of 10 cm and cross-sectional area of 100 square mm. The air gap is 2 mm and the core has 600 turns of winding. Calculate the magnitude of current to be passed through the winding to produce air gap flux of 1 tesla. (Permeability of iron = 4000)** **[2072 Magh]**

***Solution:***

Mean length of the circular iron core $(l_c) = 10$ cm $= 0.1$ m

Mean length of the air gap $(l_g) = 2$ mm $= 0.002$ m

Cross-sectional area of iron core $(A) = 100\ mm^2 = 10^{-2}\ m^2$

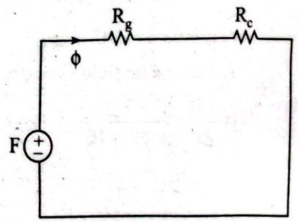
The reluctance of the iron core is:

$$R_c = \frac{l_c}{\mu_0 \mu_r A} = \frac{0.1}{4\pi \times 10^{-7} \times 4000 \times 10^{-4}} = 198943.6789\ AT/wb$$

The reluctance of the air gap is

$$R_g = \frac{l_g}{\mu_0 \mu_r A} = \frac{0.002}{4\pi \times 10^{-7} \times 1 \times 10^{-4}}$$

$= 15915494.31$ AT/wb




The electrical analogy circuit is

$\therefore\ \phi = BA = 1 \times 10^{-4} = 10^{-4}$ wb

$F = NI = \phi(R_c + R_g)$

$$\text{or, } I = \frac{\phi(R_c + R_g)}{N}$$

$$\therefore\ I = \frac{10^{-4} \times (198943.6789 + 15915494.31)}{600} = 2.685\ A$$

**8. An iron ring of mean length 1.2 m and cross-sectional area of 0.005 m² is wound with a coil of 900 turns. If a current of 2 A in the coil produces a flux density of density 1.2T in the iron ring. Calculate:**

**i. The mmf**

**ii. Total flux in the ring**

**iii. The magnetic field strength**

**iv. The relative permeability of iron at this flux density.** **[2077 Chaitra]**

***Solution:***

Mean length of iron ring ($l$) = 1.2 m

Cross-sectional area (A) = 0.005 $m^2$

Number of turns (N) = 900

Current in coil (I) = 2A

Flux density (B) = 1.2 T

We know,

i. mmf = NI

$= 900 \times 2 = 1800$ AT

ii. Total flux in the ring, $\phi = BA = 1.2 \times 0.005 = 0.006$ wb

iii. $\phi R = NI$

or, $R = \frac{1800}{0.006} = 3,00,000$ AT/wb

Now, $R = \frac{l}{\mu A}$

or, $\mu = \frac{1.2}{300000 \times 0.006}$

$\therefore \mu = 6.66 \times 10^{-4}$

The magnetic field strength,

$H = \frac{B}{\mu} = \frac{1.2}{6.66 \times 10^{-4}} = 1800$ A/m

iv. $\mu = \mu_o \mu_r$

$\therefore \mu_r = \frac{\mu}{\mu_o} = \frac{6.66 \times 10^{-4}}{4\pi \times 10^{-7}} = 530.52$

Hence, the relative permeability of the iron at given flux density is 530.52.

**9. An iron ring has mean length of 80 cm and cross-sectional area of 16 sq. cm has a radial air gap of 2 mm. The core is wound with a coil of 1000 turns. If a current of 6 amp is passed through the coil. Calculate the magnetic flux and magnetic flux density in the core. Given that the relative permeability of the core is 2000.** **[2077 Chaitra]**

***Solution:***

Given:

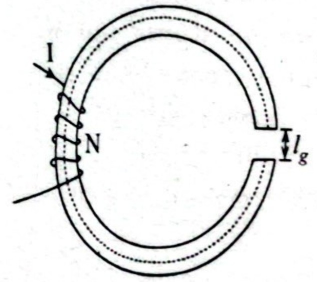
Mean length of iron ring ($l$) = 80 cm = 0.8 m

Cross-sectional area (A) = 16 $cm^2$ = $16 \times 10^{-4}$ $m^2$

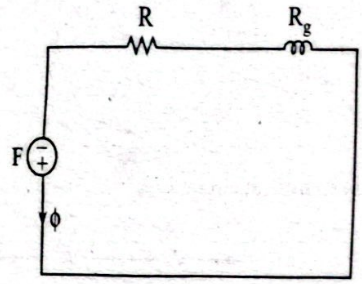
Radial length of air gap ($l_g$) = 2 mm = $2 \times 10^{-3}$ m

Number of turn (N) = 1000 turns

Current in coil (I) = 6 A




The electrical equivalent circuit is:




$R = \frac{l - l_g}{\mu_o \mu_r A} = \frac{0.8 - 2\times10^{-3}}{4\pi\times10^{-7}\times2000\times16\times10^{-4}} = 198446.3197$ AT/ωb

$R_g = \frac{l_g}{\mu_o A} = \frac{2\times10^{-3}}{4\pi\times10^{-7}\times16\times10^{-4}} = 994718.3943$ AT/ωb

$F = \phi R_{eq}$

or, $NI = \phi(R + R_g)$

$\therefore \phi = \frac{1000 \times 6}{198446.3197 + 994718.3943} = 5.028\times10^{-3}$ wb = 5.028 mwb

$\therefore B = \frac{\phi}{A} = \frac{5.028 \times 10^{-3}}{16 \times 10^{-4}} = 3.1429$ T

Hence, the magnetic flux and magnetic flux density are 5.028 mwb and 3.1429 T respectively.

**10. A magnetic circuit consists of a circular iron core having mean diameter of 10 cm, and cross-sectional area of 100mm² and air gap of 2 mm. The core has 600 turns of winding. Calculate the**

**magnitude of current to be passed through the winding to produce air gap flux density of 1 Tesla. Given that relative permeability of the core is 4000.** **[2078 Baishakh]**

***Solution:***

Mean diameter of circular iron core (d) = 10 cm = 0.1 m

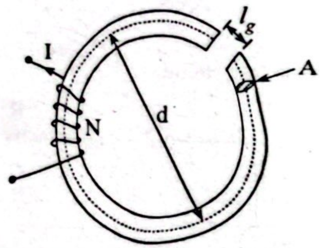
Cross-sectional area (A) = 100 mm$^2$ = $10^{-4}$ m$^2$

Length of air gap ($L_g$) = 2 mm = $2 \times 10^{-3}$ m

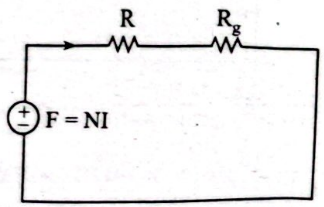
Number of turns of winding (N) = 600

Air gap flux density (B) = 1 Tesla

Relative permeability of core ($\mu_r$) = 4000




The electrical equivalent circuit is:




$\therefore$ $\phi = BA = 1 \times 10^{-4} = 10^{-4}$ wb

Mean length (L) = $\pi \times 0.1 = 0.3141$ m

The reluctance of the iron core,

$$R = \frac{L - L_g}{\mu_0 \mu_r A}$$

$$= \frac{0.3141 - 2\times10^{-3}}{4\pi\times10^{-7}\times4000\times10^{-4}} = 620903.2217 \text{ AT/wb}$$

The reluctance of the air gap,

$$R_g = \frac{L_g}{\mu_0 A}$$



$$= \frac{2\times10^{-3}}{4\pi\times10^{-7}\times10^{-4}} = 15915494.31 \text{ AT/wb}$$

The equivalent reluctance is

$R_{eq} = R + R_g$

$= 16536397.53$ AT/wb

We know,

$F = NI = \phi R_{eq}$

or, $600 \times I = 10^{-4} \times 16536397.53$

$\therefore$ $I = 2.756$ Amp

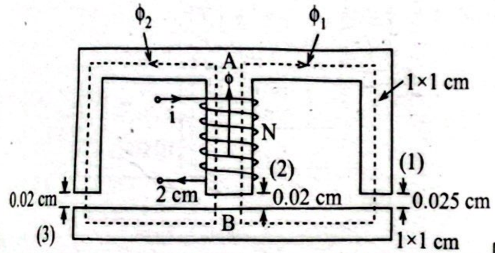
Hence, the required value of the current is 2.756 Amp.

**11. For the given magnetic circuit cost steel core with dimensions as shown:**

**Mean length from A to B through either outer limb = 0.5m**

**Mean length from A to B through central limb = 0.2m**

**In the magnetic circuit shown it is required to establish a flux of 0.75 mwb in the air-gap of the central limb. Determine the mmf of the exciting coil it for the core material (a) $\mu_r = \infty$ (b) $\mu_r = 5000$**




**[2071 Magh]**

***Solution:***

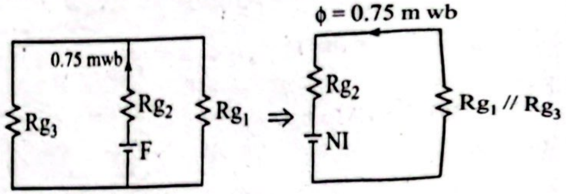
a. $\mu_r = \infty$, i.e. there are no mmf drops in the magnetic core. In this figure, two outer limbs are parallel magnetic circuit. Now, air gap reluctances are:

$$R_{g1} = \frac{0.025\times10^{-2}}{4\pi\times10^{-7}\times1\times10^{-4}} = 1.99\times10^{6} \text{ AT/W}$$

$$R_{g2} = \frac{0.02\times10^{-2}}{4\pi\times10^{-7}\times2\times10^{-4}} = 0.796\times10^{6}$$

$$R_{g3} = \frac{0.02\times10^{-2}}{4\pi\times10^{-7}\times1\times10^{-4}} = 1.592\times10^{6}$$

The electrical analog of the magnetic circuit is

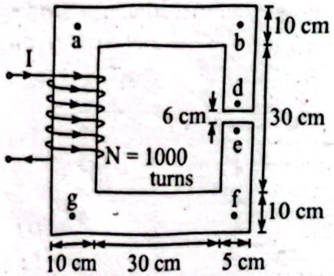



$\text{mmf} = \phi R = 0.75\times10^{-3}\,(R_{g1} // R_{g2} + R_{g3})$

$= 1230$ AT

b. For $\mu_r = 5000$, mmf = 1466 AT

**12. A two-legged magnetic core with an air gap is shown in figure below. The depth of the core is 5 cm, the length of the air gap in the core is 0.06 cm, and the number of turns on the coil is 1000. The relative permeability of iron core is 2000. Assume no fringing in air gap. How much current is required to produce an air-gap flux density of 0.5 T?**




Depth = 5 cm

**[2079 Ashwin]**

***Solution:***

$\mu_r = 2000$

$B = 0.5$ T (air gap)

$I = ?$

Magnetic flux ($\phi$)

= flux density in air gap × cross-sectional area of that part

$= 0.5 \times 5 \times 5 \times 10^{-4} = 12.5 \times 10^{-4}$ weber

Reluctance of section bagf $(R_{bagf}) = \dfrac{l_{ba} + l_{ag} + l_{gf}}{\mu_0\mu_r A}$

$= \dfrac{0.375 + 0.4 + 0.375}{4\pi \times 10^{-7} \times 2000 \times 50 \times 10^{-4}}$

$= 91514.09$

Reluctance of section bd + ef $(R_{bd+ef}) = \dfrac{(l_{bd} + l_{ef})}{\mu_0\mu_r A}$

$= \dfrac{(40 - 0.06) \times 10^{-2}}{4\pi \times 10^{-7} \times 2000 \times 25 \times 10^{-4}}$

$= 63566.48$

Reluctance of air gap $(R_g) = \dfrac{l_g}{\mu_0 A}$

$= \dfrac{0.06 \times 10^{-2}}{4\pi \times 10^{-7} \times 25 \times 10^{-4}} = 190985.93$

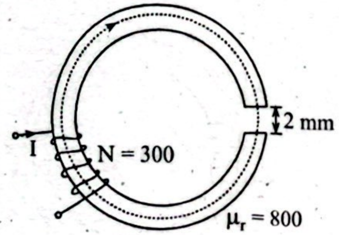
$\therefore$ Total reluctance $(R) = R_{bagf} + R_{(bd+ef)} + R_g = 346066.5$

Now, $\phi = \dfrac{NI}{R}$

or, $I = \dfrac{\phi R}{N} = \dfrac{12.5 \times 10^{-4} \times 346066.5}{1000} = 0.433$A

**13. An iron ring of 0.15 meter diameter and 0.001 m² in cross section with a saw cut 2 mm wide is wound with 300 turns of wire. The air gap flux density is 1 Tesla. The relative permeability of the iron is 800. Determine the exciting current and inductance.** **[2078 Poush]**

***Solution:***




Cross sectional area $(A) = 0.001\ m^2$

Air gap flux density $(B) = 1$ T

or, $B = \dfrac{\phi}{A}$

$\therefore \phi = BA = 1 \times 0.001 = 0.001$ weber

Mean length of iron ring ($l_C$) = $\pi d$ – length of air gap

$$= \pi \times 15 - 2 \times 10^{-3} = 0.469 \text{ m}$$

Reluctance of iron ring ($R_C$) $= \dfrac{l_C}{\mu_0 \mu_r A}$

$$= \frac{0.469}{4\pi \times 10^{-7} \times 800 \times 0.001} = 466522.9$$

Reluctance of air gap ($R_g$) $= \dfrac{l_g}{\mu_0 A}$

$$= \frac{2 \times 10^{-3}}{4\pi \times 10^{-7} \times 0.001} = 1591549.4$$

$\therefore$ Total reluctance (R) = $R_C + R_g$ = 2058072.3

Now, $\phi = \dfrac{NI}{R}$

$$\therefore\ I = \frac{\phi R}{N} = \frac{0.001 \times 2058072.3}{300} = 6.86 \text{ A}$$

$$\text{and, } L = \frac{N\phi}{I} = \frac{300 \times 0.001}{6.86} = 0.0437 \text{ H}$$

---

**14. An iron has a mean length of 1.5 m and cross-sectional area of 0.01 $m^2$. It has a radial air gap of 4 mm. The ring is wound with 250 turns. What dc current would be needed in the coil to produce a flux of 0.8 wb in the air gap?**
**Assume that $\mu_r$ = 400 and leakage factor is 1.25 (or 25%).**

**[2080 Chaitra, 2068 Bhadra]**

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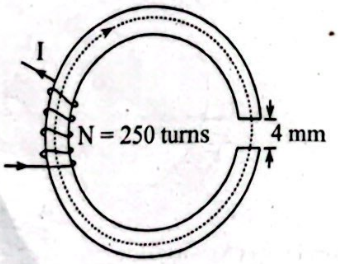
***Solution:***




The reluctance of the iron core is:

$$R_c = \frac{l_c}{\mu_0 \mu_r A}$$

$$= \frac{1.5 - (4 \times 10^{-3})}{4\pi \times 10^{-7} \times 400 \times 0.01} = 297619.7430 \text{ AT/wb}$$



The reluctance of the air gap is:

$$R_g = \frac{l_g}{\mu_0 A}$$

$$= \frac{4 \times 10^{-3}}{4\pi \times 10^{-7} \times 0.01} = 318309.8862 \text{ AT/wb}$$

The flux through the iron core is:

$$\phi_c = 1.25 \times 0.8 = 1 \text{ wb}$$

Now,

$$F = NI = \phi_c R_c + \phi_g R_g$$

$$\text{or, } I = \frac{\phi_c R_c + \phi_g R_g}{N}$$

$$\therefore\ I = \frac{0.8 \times 318309.8862 + 1 \times 297619.7436}{250}$$

$$= 2209.07061 \text{ A}$$