Derivatives

1

Exercise - 1

1. Find, from the first principles, the derivatives of the following functions

(i) $\sqrt{x}$

Sol^n. Let $y = \sqrt{x}$

Let Δx and Δy be the small increments in x and y respectively.

Then,

$$y + \Delta y = \sqrt{x + \Delta x}$$

$y + \Delta y = \sqrt{x + \Delta x}$

$$\text{or, } \Delta y = \sqrt{x + \Delta x} - y = \sqrt{x + \Delta x} - \sqrt{x}$$

$\Delta y = \sqrt{x + \Delta x} - y$

$\sqrt{x + \Delta x} - \sqrt{x}$

$$\text{or, } \frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$$

$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$

$$\text{Now, } \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \times \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x 0 \sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x + 0} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \text{ Ans.}$$

(ii) $\tan^{-1}x$

Sol^n. Let $f(x) = \tan^{-1}x$

$f(x) = \tan^{-1}(x)$

By the definition of derivatives, we have,

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h \to 0} \frac{F(x + h) - f(x)}{h}$$

$$\therefore \lim_{h\to 0} \frac{\tan^{-1}(x+h)-\tan^{-1}x}{h} \quad \ldots\ldots\ldots (i)$$

Let, $\tan^{-1}x = y$ and $\tan^{-1}(x+h) = y+k$, then

as $h \to 0$, $k \to 0$

Also, $x = \tan y$ and $x + h = \tan(y+k)$

Hence, from (i), we get,

$\tan^{-1}x = y$

$\tan^{-1}(x+h) = y+k$

$x = \tan y$ $\quad x+h = \tan(y+k)$

$$f'(x) = \lim_{k\to 0} \frac{y+k-y}{\tan(y+k)-\tan y}$$

$$= \lim_{k\to 0} \frac{k \times \cos(y+k).\cos y}{\sin(y+k).\cos y - \sin y.\cos(y+k)}$$

$$= \lim_{k\to 0} \frac{k\cos(y+k).\cos y}{\sin(y+k-y)}$$

$$= \lim_{k\to 0} \frac{k\cos(y+k).\cos y}{\sin k}$$

$$= \lim_{k\to 0} \left(\frac{k}{\sin k}\right).\cos(y+k).\cos y$$

$$= 1.\cos y.\cos y$$

$$= \cos^2 y$$

$$= \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\therefore f'(x) = \frac{1}{1+x^2} \text{ Ans.}$$

(iii) $\log\cos x$

Sol^n. Let $f(x) = \log\cos x$

By definition of derivative, we have,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\text{or, } \frac{d}{dx}(\log\cos x) = \lim_{h\to 0} \frac{\log\cos(x+h)-\log\cos x}{h} \quad \ldots\ldots (i)$$

Put $\cos x = y$ and $\cos(x+h) = y+k$

So that $h \to 0$, $k \to 0$ hence, the equation (i) becomes

$$\frac{d}{dx}(\log\cos x) = \lim_{h\to 0} \frac{\log(y+k)-\log y}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}} \cdot \frac{y}{y.h}$$

$$= 1.\lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h}.\frac{1}{y}$$

$$\left(\because \lim_{h\to 0} \frac{\log(1+x)}{x} = 1\right)$$

$$= 1.\frac{1}{y}\lim_{h\to 0} 2\sin\frac{2\sin\left(\frac{x+h+x}{2}\right).\sin\left(\frac{x-x-h}{2}\right)}{h}$$

$$= 1.\frac{1}{y}\lim_{h\to 0} \sin\left(x+\frac{h}{2}\right).\frac{\left(-\sin\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= \frac{1}{y}\lim_{h\to 0} -\sin\left(x+\frac{h}{2}\right)$$

$$= \frac{1}{y} - \sin x = -\frac{\sin x}{\cos x} = -\tan x$$

(iv) $e^{\sqrt{x}}$

Sol^n. Let $f(x) = e^{\sqrt{x}}$

By the definition of derivative, we have,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\therefore \frac{d}{dx}\left(e^{\sqrt{x}}\right) = \lim_{h\to 0} \frac{e^{\sqrt{x+h}}-e^{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} e^{\sqrt{x}}\left(\frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{h}\right)$$

Let $y = \sqrt{x+h}-\sqrt{x}$

Then when $h \to 0$, $y \to 0$

$$\therefore \frac{d}{dx}\left(e^{\sqrt{x}}\right) = \lim_{h\to 0} e^{\sqrt{x}}\left(\frac{e^y-1}{h}\right)$$

$$= e^{\sqrt{x}} \lim_{h\to 0} \frac{e^y-1}{y} \times \frac{y}{h}$$

$$= e^{\sqrt{x}} \lim_{h\to 0} \frac{y}{h}$$

$$= e^{\sqrt{x}} \cdot 1 \times \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{\left(\sqrt{x+h}-\sqrt{x}\right)}$$

$$= e^{\sqrt{x}} \cdot \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}-\sqrt{x}\right)}$$

$$= e^{\sqrt{x}} \lim_{h\to 0} \frac{1}{\sqrt{x+h}-\sqrt{x}}$$

$$= e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \text{ Ans.}$$

(v) $e^{\cos x}$

Sol^n. Now, by the definition, we have,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^{\cos(x+h)} - e^{\cos x}}{h}$$

Let $\cos x = y$ and $\cos(x+h) = y+k$

Then $k = \sin(x+h) - \sin x$ and as $h \to 0$, $k \to 0$

$$\therefore \frac{d}{dx}(e^{\cos x}) = \lim_{h\to 0} \frac{e^{y+k}-e^{y}}{h} = \lim_{h\to 0} \frac{e^{y}\left(e^{k}-1\right)}{h}$$

$$= e^{y} \lim_{h\to 0} \frac{e^{k}-1}{k} \cdot \lim_{h\to 0} \frac{k}{h}$$

$$= e^{y} \left[\lim_{h\to 0} \frac{e^{x}-1}{x} = 1\right]$$

$$= e^{y} \cdot \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h}$$

$$= e^{y} \cdot \lim_{h\to 0} \frac{2\sin\frac{(x+h+x)}{2} \cdot \frac{\sin(x-x-h)}{2}}{h}$$

$$= e^{y} \cdot \lim_{h\to 0} \frac{\sin\left(x+\frac{h}{2}\right)}{h} \cdot \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= e^{y} \cdot \lim_{h\to 0} -\sin\left(x+\frac{h}{2}\right) \cdot \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= e^{y} \cdot (-\sin x)$$

$$= -e^{\cos x} \sin x \text{ Ans.}$$

(vi) x^x

Sol^n. Let $f(x) = x^x = e^{\log x^x} = e^{x\log x}$

By the definition of derivatives, when,

$$f'(x) = \lim_{h\to 0} \frac{e^{(x+h)\log(x+h)} - e^{x\log x}}{h}$$

$$= \lim_{h\to 0} e^{x\log x} \left[\frac{e^{(x+h)\log(x+h)-x\log x}-1}{h}\right]$$

Put $(x+h)\log(x+h) - x\log x = y$, then as $h \to 0$, $y \to 0$

Hence equation (i) becomes,

$$f'(x) = \lim_{h\to 0} e^{x\log x} \left[\frac{e^{y}-1}{h}\right]$$

$$= e^{x\log x} \lim_{y\to 0} \frac{e^{y}-1}{y} \times \lim_{h\to 0} \frac{y}{h}$$

$$= e^{x\log x} \cdot 1 \cdot \lim_{h\to 0} \frac{y}{h}$$

$$= x^{x} \lim_{h\to 0} \left[\frac{(x+h)\log(x+h) - x\log x}{h}\right]$$

$$= x^{x} \lim_{h\to 0} \left[\frac{x\{\log(x+h) - \log x\}}{h} + \log(x+h)\right]$$

$$= x^{x} \cdot \lim_{h\to 0} \left[\frac{x\log\left(1+\frac{h}{x}\right)}{h} + \log(x+h)\right]$$

$$= x^{x} \lim_{h\to 0} \left[\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}} + \log(x+h)\right]$$

$$= x^{x}(1+\log x) \text{ Ans.} \quad \left(\because \lim_{h\to 0} \frac{\log(1+x)}{x} = 1\right)$$

Higher Order Derivatics

2

Differential calculus is the study of derivative, i.e., the study of change of functions with-respect to the change in input. It is the mathematical study of change motion, growth or decay etc.

Successive differentiation

If $y = f(x)$ be a differentiable functions of x, then its derivative $\frac{dy}{dx}$ is called the first order derivative of y. In general a function of x. If $\frac{dy}{dx}$ is differentiable then its derivative is called the second order derivative. Similarly, the derivative of $\frac{d^2y}{dx^2}$ is called the third derivative of y and is denoted by $\frac{d^2y}{dx^2}$ and so on.

$\therefore$ The successive differentiation of the function $y = f(x)$ are denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n} \ldots$$

y = UV, U & V are fn of x and whose n-th derivative are known then

Leibinitz theorem

If $y = UV$, where U and V are the functions of x and whose n^{th} derivative are known then

$$y_n = (UV)_n = {}^nC_0 U_n V + {}^nC_1 U_{n-1} V_1 + {}^nC_2 U_{n-2} V_2 + {}^nC_3 U_{n-3} V_3 \ldots {}^nC_n U V_n$$

$y_n = (UV)_n = {}^nC_0 U_n V + {}^nC_1 U_{n-1} V_1 + {}^nC_2 U_{n-2} V_2 + \ldots {}^nC_n U V_n$

Exercise - 2

1. Find the nth derivative of the following

(i) $y = (a-bx)^m$

Sol^n. Here,

$y = (a-bx)^m$ .....(i)

Diff^n (i) w.r.t x, successively, we get

$y_1 = m(a-bx)^{m-1} \cdot (-b)$

$y_2 = m(m-1)(a-bx)^{m-2}(-b)^2$

$y_3 = m(m-1)(m-2)(a-bx)^{m-3}(-b)^3$

.....

$y_n = m(m-1)(m-2) \ldots\ldots (m-n+1)(a-bx)^{m-n}(-b)^n$

$= (-1)^n b^n m(m-1)(m-2) \ldots\ldots (m-n+1)(a-bx)^{m-n}$

$= (-1)^n b^n m(m-1)(m-2) \ldots\ldots$

$$(m-n+1)\frac{(m-n)\ldots\ldots 3.2.1(a-bx)^{m-n}}{(m-n)\ldots\ldots 3.2.1}$$

$$y_n = \frac{(-1)^n b^n m!}{(m-n)!}(a-bx)^{m-n}$$

(ii) $y = (2-3x)^n$

Sol^n. Here, $y = (2-3x)^n$

Diff^n (i) w.r.t x, successively, we get

$y_1 = n(2-3x)^{n-1}(-3)$

$y_2 = n(n-1)(2-3x)^{n-2}(-3)^2$

$y_3 = n(n-1)(n-2)(2-3x)^{n-3}(-3)^3$

$y_n = n(n-1)(n-2) \ldots\ldots (n-(n-1)(2-3x)^{n-n}(-3)^n$

$y_n = n(n-1)(n-2) \ldots (n-n+1)(2-3x)^0(-3)^n$

$y_n = (-1)^n 3^n n!$

(iii) $y = \frac{1}{\sqrt{x}}$

Sol^n. $y = \frac{1}{\sqrt{x}}$ ...../...(1)

Diff^n (1) w.r.t x, successively, we get

$y = x^{\frac{1}{2}}$

$y_1 = \left(\frac{-1}{2}\right)x^{\frac{-1}{2}-1}$

$y_2 = \left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)x^{\frac{-1}{2}-2}$

$y_3 = \left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)x^{\frac{-1}{2}-3}$

.....

$y_n = \left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)\ldots\left(\frac{-1}{2}-n+1\right)x^{\frac{-1}{2}-n}$

$$= \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-3}{2}\right)\ldots\ldots\left\{-\left(\frac{2n-1}{2}\right)\right\}x^{\frac{-1}{2}-n}$$

$$= \frac{1}{2^n}(-1)^n\{1.3.5\ldots\ldots(2n-1)\}x^{\frac{-1}{2}-n}$$

$$y_n = \frac{1}{2^n}(-1)^n.\frac{1.3.5\ldots\ldots(2n-1)}{x^{n+\frac{1}{2}}} \text{ Ans.}$$

(iv) $y = x^{2n}$

Sol^n. Here, $y = x^{2n}$

Diff^n (1) successively w.r.t x, we get

$y_1 = 2nx^{2n-1}$

$y_2 = 2n(2n-1)x^{2n-2}$

$y_3 = 2n(2n-1)(2n-2)x^{2n-3}$

$y_3 = 2n(2n-1)(2n-2)(2n-3)x^{2n-4}$

.....

$y_n = 2n(2n-1)(2n-2)\ldots\ldots(2n-n+1)x^{2n-n}$

$y_n = 2n(2n+1)(2n-2)\ldots\ldots(n+1)x^n$

multiplying nominator and denominates by n (n – 1) (n – 2) .....3.2.1, we get

$$y_n = 2n(2n-1)(2n-2)\ldots\ldots(n+1)\times\frac{n(n-1)(n-2)\ldots\ldots3.2.1}{n(n-1)(n-2)\ldots\ldots3.2.1}x^n$$

Seperating even and odd brackets,

$$y_n = \frac{[2n(2n-2)(2n-4)\ldots\ldots6.4.2]\{(2n-1)(2n-3)(2n-5)\ldots\ldots5.3.1\}x^n}{n!}$$

$$= \frac{2^n\{n(n-1)(n-2)\ldots\ldots3.2.1\}\{(2n-1)(2n-3)(2n-5)\ldots\ldots5.3.1\}x^n}{n!}$$

$$= 2^n\frac{n!}{n!}(2n-1)(2n-3)(2n-5)\ldots\ldots5.3.1x^n$$

$$\therefore y_n = 2^n\{1, 3, 5\ldots\ldots(2n-3)(2n-1)\}x^n$$

(v) $y = \dfrac{1}{a-x}$

Sol^n. Here, $y = \dfrac{1}{a-x} = (a-x)^{-1}$

Diff^n, w.r.t x, successively, we get

$y_1 = (-1)(a-x)^{-2}(-1) = 1.(a-x)^{-2}$

$y_2 = (-2)(a-x)^{-3}(-1) = 1.2(a-x)^{-3}$

$y_3 = 1.2.3(a-x)^{-4} = 1.2.3(a-x)^{-4}$

.....

$y_n = 1.2.3\ldots\ldots n(a-x)^{-(n+1)}$

$= n!(a-x)^{-(n+1)}$

$$y_n = \frac{n!}{(a-x)^{n+1}} \text{ Ans.}$$

(vi) $y = \dfrac{x^n}{x-1}$

Sol^n. Here, $y = \dfrac{x^n}{x-1}$

By actual division, we get

$$y = (x^{n-1} + x^{n-2} + \ldots\ldots + 1) + \frac{1}{x-1}$$

$$\left[\because \frac{x^5}{x-1} = \left(x^4 + x^3 + x^2 + x + 1\right) + \frac{1}{x-1}\right]$$

Diff^n, w.r. t. x up to n times,

$$y_n = \frac{0 + (-1).n!}{(x-1)^{n+1}}$$

$$\therefore y_n = \frac{(-1)^n n!}{(x-1)^{n+1}}$$

(vii) $y = \dfrac{x}{a+bx}$

Sol^n. Here, $y = \dfrac{x}{a+bx}$

$$y = \frac{1}{b}\left[\frac{a+bx-a}{a+bx}\right]$$

$$= \frac{1}{b}\left[\frac{a+bx}{a+bx} - \frac{a}{a+bx}\right]$$

$$= \frac{1}{b}\left[1 - \frac{a}{a+bx}\right] = \frac{1}{b} - \frac{a}{b}\left[\frac{a}{a+bx}\right]$$

$$y = \frac{1}{b} - \frac{a}{b}(a+bx)^{-1}$$

Diff^n, w.r.t x, successively, we get

$$y_1 = 0 - \frac{a}{b}(-1)(a+bx)^{-2}.b$$

$$y_2 = -\frac{a}{b}[(-1)(-2)(a+bx)^{-3}.b^2]$$

$$y_3 = \frac{-a}{b}\left[(-1)(-2)(-3)(a+bx)^{-4}.b^3\right]$$

.....

.....

$$y_n = \frac{-a}{b}\left[(-1)(-2)(-3)\ldots\ldots(-n)(a+bx)^{-(n+1)}b^n\right]$$

$$y_n = \frac{-a}{b}\left[(-1)^n n!(a+bx)^{-(n+1)} b^n\right]$$

$$\therefore y_n = \frac{(-1)^{n+1} a b^{n-1} n!}{(a+bx)^{n+1}}$$

(viii) $y = \frac{x}{x^2+a^2}$

Sol^n. Here, $y = \frac{x}{x^2+a^2}$

$$= \frac{x}{(x-ai)(x+ai)} = \frac{1}{2}\left[\frac{1}{x+ai} + \frac{1}{x-ai}\right]$$

Diff^n, up to nsteps, we get

$$y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(x+ai)^{n+1}} + \frac{(-1)^n n!}{(x-ai)^{n+1}}\right]$$

$$= \frac{1}{2}(-1)^n n!\left[(x+ai)^{-(n+1)} + (x-ai)^{-(n+1)}\right]$$

Putting $x = r\cos\theta$ and $a = r\sin\theta$

So that

$\tan\theta = \frac{a}{x}$ and $r = \frac{a}{\sin\theta}$, then

$$\therefore y_n = \frac{(-1)n!}{2}\left[(r\cos\theta + ir\sin\theta)^{-(n+1)} + (r\cos\theta - ir\sin\theta)^{-(n+1)}\right]$$

$$= \frac{(-1)^n n!}{2r^{n+1}} r^{-(n+1)}\left[(\cos\theta + i\sin\theta)^{-(n+1)} + (\cos\theta - i\sin\theta)^{-(n+1)}\right]$$

$$= \frac{(-1)^n n!}{2r^{n+1}}\left[\cos\theta(n+1)\theta - i\sin(n+1)\theta + \cos(n+1)\theta\right]$$

$$= \frac{(-1)^n n!}{2r^{n+1}} 2\cos(n+1)\theta$$

But $r = \frac{a}{\sin\theta}$ then $\frac{1}{r^{n+1}} = \frac{\sin^{n+1}\theta}{a^{n+1}}$

$$\therefore y_n = (-1)^n n! \times \frac{\sin^{n+1}\theta}{a^{n+1}} \cos(n+1)\theta, \text{ where, } \theta = \tan^{-1}\frac{a}{x}$$

$$\text{or, } y_n = \frac{(-1)^n n!}{a^{n+1}} \sin^{n+1}\theta \cos(n+1)\theta$$

(ix) $y = = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

Sol^n. Here, $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

We know that $\tan^{-1}\left(\frac{A+B}{1-AB}\right) = \tan^{-1}A + \tan^{-1}B$

$\therefore y = \tan^{-1}\frac{1+x}{1-x} = \tan^{-1}1 + \tan^{-1}x$

Diff^n w.r.t x, we get

$$y_1 = \frac{1}{1+x^2} = \frac{1}{x^2+1^2}$$

Diff^n up to $(n-1)$ steps in (i), we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{1^{n+1}} \sin^n\theta . \sin\theta \text{ where, } \theta = \tan^{-1}x$$

We know that if $y = \frac{1}{x^2+a^2}$

$$y_n = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta . \sin(n+1)\theta, \theta = \tan^{-1}\frac{a}{x}$$

$$\therefore y_n = (-1)^{n-1}(n-1)! \sin^n\theta . \sin n\theta$$

Where, $\theta = \tan^{-1}\frac{1}{x}$

(x) $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Sol^n. Let $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Put $x = \tan\theta$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

But $\theta = \tan^{-1}x$ then $y = \frac{1}{2}\tan^{-1}x$

Diff^n w.r.t x, we get

$y_1 = \frac{1}{2}\left(\frac{1}{x^2+1}\right)$

Again, differentiating up to n – 1 steps

$y_n = \frac{1}{2}D^{n-1}\left(\frac{1}{x^2+1^2}\right)$ .....(i)

Again, Applying $D^n\left(\frac{1}{x^2+a^2}\right) = \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta.\sin(n+1)\theta$

$\therefore \theta = \tan^{-1}\frac{a}{x}$

putting 1 for a and n – 1 steps for n, then (i) becomes

$y_n = \frac{1}{2}\frac{(-1)^n(n-1)!}{1^{n+1}}\sin^n\theta.\sin\theta$, where $\theta = \tan^{-1}\frac{1}{x}$

$= \frac{1}{2}(-1)^{n-1}(n-1)!\sin^n\theta$ where $\theta = \tan^{-1}\frac{1}{x}$

2. **If $y = \frac{1}{x^x}$, then show that $y_1(1) = -1$**

Sol^n: Here, $y = \frac{1}{x^x} = x^{-x}$

Taking log on both sides,

$\log y = -x\log x$

Diff^n w.r.t x, we get

$\frac{1}{y}\frac{dy}{dx} = -\left(x.\frac{1}{x} + \log x.1\right)$ $\quad \therefore \frac{dy}{dx} = -y(1+\log x)$.....(i)

At x = 1, equation (i) gives

$y_1(1) = -y(1)(1+\log 1) = -1(1+0) = -1$

3. **If $y = \tan^{-1}x$, then prove that $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$**

Sol^n. Given $y = \tan^{-1}x$ .....(ii)

Diff^n (i) w.r.t. x, we get

$y_1 = \frac{1}{1+x^2}$

or, $y_1(1+x^2) = 1$.....(2)

Diff^n (2) w.r.t x, up to n times, using leibinitz these

$y_{n+1}(1+x^2) + ny_n.2x + \frac{n(n-1)}{2}2y_{n-1} = 0$

or, $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

5. **If $y = e^{a\tan^{-1}x}$, then prove that $(1+x^2)y_{n+1} + (2nx+2x-a)y_{n+1} + n(n+1)y_n = 0$**

Sol^n. Given, $y = e^{a\tan^{-1}x}$ ..... (i)

Diff^n. (i) w. r. t. x, we get

$y_1 = e^{a\tan^{-1}x}.\frac{a}{1+x^2}$

or, $y_1 = \frac{ay}{1+x^2}$ $\qquad$ or, $y_1(1+x^2) = ay$

Again, diff^n. we get,

$(1+x^2)y_2 + y_1.2x = ay_1$

$\therefore (1+x^2)y_2 + (2x-a)y_1 = 0$ ..... (i)

Again, diff^n (ii) w. r. t. x up to n times

Using leibnitz theorem, we get,

$[1+x^2)y_{n+1} + ny_{n+1}.2x + \frac{n(n-1)}{2}y_n.2]$

$+ [(2x-a)y_{n+1} + n.y_n^2] = 0$

or, $(1+x^2)y_{n+2} + ny_{n+1}.2x + n^2y_n - ny_n - (2x-n)y_{n+1}$

$+ 2ny_n = 0$

or, $(1+x^2)y_{n+2} + (2nx+2x-a)y_{n+1} + n(n+1)y_n = 0$

6. **If $y = ae^{mx} + be^{-mx}$ then, prove that $y_2 - m^2y = 0$**

Sol^n. Given, $y = ae^{mx} + be^{-mx}$ ..... (i)

Diff^n. (i) w. r. t. x, we get

$y_1 = a.me^{mx} - b.me^{-mx}$

$y_2 = am^2e^{mx} + bm^2e^{-mx}$

$= m^2(ae^{mx} + be^{-mx})$

or, $y_2 = m^2y$

$\therefore y_2 - m^2y = 0$

7. **If $y = e^{ax}\sin bx$ Prove that $y_{n+1} = 2ay_n - (a^2+b^2)y_{n-1}$**

Sol^n. Given $y = e^{ax}\sin bx$ ..... (i)

$y_1 = ae^{ax}\sin bx + e^{ax}.b\cos bx$

$= a(e^{ax}\sin bx) + be^{ax}\cos bx$

or, $y_1 = ay + be^{ax}\cos bx$

Again, differentiating w. r. t. x, we get,

$y_2 = ay_1 + b[ae^{ax}\cos bx - e^{ax}.b\sin bx]$

$= ay_1 + a(be^{ax}\cos bx) - b^2(e^{ax}\sin bx)]$

Since from (ii) $be^{ax}\cos bx = y_1 - ay$

$\therefore y_2 = ay_1 + a(y_1 - ay) - b^2y$

or, $y_2 = 2ay_1 - a^2y - b^2y$

or, $y_2 - 2ay_1 + (a^2+b^2)y = 0$ ..... (ii)

Again, differentiating (ii) up to n – 1 steps, we get,

$y_{2+n-1} - 2ay_{1+n-1} + (a^2+b^2)y_{n-1} = 0$

or, $y_{n+1} - 2ay_n + (a^2+b^2)y_{n-1} = 0$

8. **If $y = x^{n-1}\log x$, then prove that $xy_n = (n-1)!$.**

Sol^n. Given, $y = x^{n-1}\log x$ ..... (i)

Differentiating w. r. t. x, we get

$y_1 = x^{n-1}\frac{1}{x} + (n-1)x^{n-2}\log x$

or, $xy_1 = x^{n-1} + (n-1)x^{n-1}\log x$

or, $xy_1 = x^{n-1} + (n-1).y$

or, $xy_1 - (n-1)y = x^{n-1}$ ..... (ii)

Differentiating (ii) w. r. t. x using leibinitz theorem up to n – 1 steps

$y_{1+n-1}x + (n-1)y_{1+n-2} - (n-1)y_{n-1} = (n-1)!$

$\left[\because \frac{d^3(x^3)}{dx^3} = 3!\right]$

or, $y_n x + (n-1)y_{n-1} - (n-1)y_{n-1} = (n-1)!$

or, $xy_n = (n-1)!$

9. **If $y = a\cos(\log x) + b\sin(\log x)$, then prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$**

Sol^n. Given,

$y = a\cos(\log x) + b\sin(\log x)$ ..... (i)

Differentiating (i) w. r. t. x, we get,

$y_1 = -a\sin(\log x).\frac{1}{x} + b\cos(\log x).\frac{1}{x}$

or, $xy_1 = -a\sin(\log x) + b\cos(\log x)$

Again, Differentiating w. r. t. x, we get

$xy_2 + y_1 = -a\cos(\log x).\frac{1}{x} - b\sin(\log x).\frac{1}{x}$

$= -\frac{1}{x}[a\cos(\log x) + b\sin(\log x)]$

or, $x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$

or, $x^2y_2 + xy_1 = -y$

$\therefore x^2y_2 + xy_1 + y = 0$ ..... (ii)

Again, Differentiating (ii) w. r. t. x, up to n times,

Using leibinitz theorem,

$[x^2y_{n+2} + xy_{n+1}.2x + n(n-1)y_n.2] + [xy_{n+1} + xy_n] + y_n = 0$

or, $x^2y_{n+2} + 2nxy_{n+1} + x^2y_n - xy_n + xy_{n+1} + xy_n + y_n = 0$

or, $x^2y_{n+1} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

10. **If $y = \sin^{-1}x$, then prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$**

Sol^n. Given, $y = \sin^{-1}x$ ..... (i)

Differentiating (i) w. r. t. x, we get,

$y_1 = \frac{1}{\sqrt{1-x^2}}$

Squaring we get, $y_1^2(1-x^2) = 1$

Again, Differentiating, we get

$2y_1y_2(1-x^2) + y_1^2(-2x) = 0$

or, $2y_1[y_2(1-x^2) - xy_1] = 0$

or, $y_2(1-x^2) - xy_1 = 0$ ..... (ii)

again, Differentiating (ii) w. r. t. x, up to n times,

Using leibinitz theorem, we get

$[y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2)]$

$-[xy_{n+1} + ny_n] = 0$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n+1)}{2}2y_n - xy_{n+1} - xy_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

11. **If $y = e^{a\sin^{-1}x}$, then prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$**

Sol^n. Given, $y = e^{a\sin^{-1}x}$ ..... (i)

Differentiating (i) w. r. t. x, we get,

$y_1 = e^{a\sin^{-1}x}.\frac{1}{\sqrt{1-x^2}}a$ $\quad\therefore$ Squaring, we get,

$y_1^2(1-x^2) = a^2\left(e^{a\sin^{-1}x}\right)^2$

or, $y_1^2(1-x^2) = a^2y^2$

or, $y_1^2(1-x^2) - a^2y^2 = 0$

Differentiating (i) w. r. t. x, we get

$(1-x^2)2y_1y_2 + y_1^2(-2x) - 2a^2yy_1 = 0$

or, $2y_1[(1-x^2)y_2 - xy_1 - a^2y] = 0$

or, $(1-x^2)y_2 - xy_1 - a^2y = 0$ ..... (ii)

Differentiating (ii) up to n times, using leibinitz theorem w. r. t. x,

$\therefore [(1-x^2)y_{n+2} + ny_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2)]$

$-[xy_{n+1} + ny_n] - a^2y_n = 0$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - x^2y_n + ny_n - xy_{n+1} - xy_n - a^2y_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

12. **If $y = e^{x^2}$ then show that $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$**

Sol^n. Given, $y = e^{x^2}$ ..... (i)

Differentiating (i) w. r. t. x, we get

$y_1 = e^{x^2}.2x$

or, $y_1 = 2xy$ ..... (ii)

Differentiating (ii) up to n steps using leibnitz theorem, we get,

$y_{n+1} = 2(xy_n + ny_{n-1})$

or, $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$

13. **If $y = (\sin^{-1}x)^2$ then show that $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - n^2y_n = 0$**

Sol^n. Given, $y = (\sin^{-1}x)^2$ ..... (i)

Differentiating (i) w. r. t. x, we get,

$$y_1 = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$$

or, $\sqrt{1-x^2}\,.\,y_1 = 2\sin^{-1}x$

Squaring, $(1-x^2)\,y_1^2 = 4(\sin^{-1}x)^2$

or, $(1-x^2)\,y_1^2 = 4y$

or, $(1-x^2)\,y_1^2 - 4y = 0$

Again, Differentiating, we get

$(1-x^2)\,2y_1y_2 + y_1^2(-2x) - 4y_1 = 0$

or, $2y_1\,[(1-x^2)\,y_2 - xy_1] - 2y_1 = 0$

or, $(1-x^2)\,y_2 - xy_1 - 2 = 0$ ..... (ii)

Again, Differentiating (ii) w. r. t. x up to n times, using leibinitz theorem,

$$[(1-x^2)\,y_{n+2} + ny_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2)]$$
$$-[xy_{n+1} + xy_n] = 0$$

or, $(1-x^2)\,y_{n+2} - 2nxy_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n = 0$

or, $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - n^2y_n = 0$

14. **If $y = \sin(m\sin^{-1}x)$, then show that $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} + (m^2-n^2)\,y_n = 0$**

Sol^n. Given, $y = \sin(m\sin^{-1}x)$ ..... (i)

Differentiating (i) w. r. t. x, we get,

$$y_1 = \cos(m\sin^{-1}x)\,.\,m\,.\,\frac{1}{\sqrt{1-x^2}}$$

Squaring gives, $(1-x^2)\,y_1^2 = m^2\cos^2(m\sin^{-1}x)$

$= m^2[1-\sin^2(m\sin^{-1}x)]$

$= m^2[1-y^2]$

or, $(1-x^2)\,y_1^2 - m^2 + m^2y^2 = 0$

Again, Differentiating w. r. t. x, we get,

$(1-x^2)\,2y_1y_2 + (-2x)\,y_1^2 + 2m^2yy_1 = 0$

or, $2y_1\,[(1-x^2)\,y_2 - xy_1 + m^2y] = 0$

$\therefore (1-x^2)\,y_2 - xy_1 + m^2y = 0$ ..... (ii)

Differentiating (ii) w. r. t. x. up to n steps using leibnitz theorem,

$$[(1-x^2)\,y_{n+2} + ny_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2)]$$
$$-(xy_{n+1} + ny_n] + m^2y_n = 0$$

or, $(1-x^2)\,y_{n+2} - 2nxy_{n+1} - x^2y_n + xy_n - xy_{n+1} - ny_n + m^2y_n = 0$

or, $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} + (m^2-n^2)\,y_n = 0$

Mean-Value Theorem

3

Continous and differentiable functions

A function f(x) is said to be continuous at x = a if,

(i) f(a) is finite i.e., f(a) exists.

(ii) $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = f(a)$

A function f(x) is differentiable at x = a if Right hand derivative (RHD) and Left -hand derivative (LHD) exists and R. H. D. = L. H. D.

i.e., $\lim_{x\to a+} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a-} \frac{f(x)-f(a)}{x-a}$

or, $\lim_{x\to o} \frac{f(a+h)-f(a)}{h} = \lim_{x\to o} \frac{f(a-h)-f(a)}{h}$

Some important results

(i) IF function f(x) is finitely differentiation (derivative is finite) in the interval (a, b) then it is continuous in the interval [a, b], i.e., i.e. every differentiable function is continous in the given interval. But converse is not necessary true.

(ii) Algebraic, trigonometric, inverse trigonometric, logarithmic and exponential function are continuous and differentiable (with some exceptions)

(iii) Addition, substraction, product and quotient of two or more, continuous and differentiable functions are also continuous and differentiable.

(iv) f(x) and f'(x) are differentiable and continuous if f''(x) exists.

(v) A function is said to be differentiable if its derivative is neither indeterminant nor infinite.

Exercise - 3

1. **Verify Rolle's theorem for the function.**

(i) **$f(x) = x^2, x \in [-1, 1]$**

Sol^n. Given, $f(x) = x^2$

Since, x^2 is continuous in [– 1, 1]

So, f(x) is continuous in [– 1, 1]

$\therefore f(-1) = 1$

$f(1) = 1$

$\therefore f(-1) = f(1)$

$f'(x) = 2x$

$\therefore f'(c) = 2c$ and $f'(c) = 0 \Rightarrow 2c = 0 \quad \therefore c = 0$

$\therefore$ Clearly, $c = 0$ belongs to the open interval $(-1, 1)$

$\therefore f'(c) = 0$, Hence Rolle's theorem is verifies.

(ii) $f(x) = e^{-x} \sin x,\ x \in [0, \pi]$

Sol^n. Given, $f(x) = e^{-x} \sin x$

Since, $e^{-x} \sin x$ is continuous in $[0, \pi]$

So, $f(x)$ is contineous in $[0, \pi]$

$\therefore f(0) = 0 = f(\pi)$ and $f'(x) = \dfrac{e^{x}(\cos x - \sin x)}{e^{2x}} = \dfrac{\cos x - \sin x}{e^{x}}$

$\therefore f'(x) = 0 \Rightarrow \cos x - \sin x = 0$

$\cos x = \sin x \quad \therefore x = \dfrac{\pi}{4}$

Thus $f'\left(\dfrac{\pi}{4}\right) = 0$ and $\dfrac{\pi}{4} \in (0, \pi)$

Hence, Rolle's theorem is verified.

(iii) $f(x) = \left(\dfrac{\sin x - \cos x}{e^{-x}}\right);\ x \in \left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

Sol^n. Given, $f(x) = \left(\dfrac{\sin x - \cos x}{e^{-x}}\right)$

Since, $\left(\dfrac{\sin x - \cos x}{e^{-x}}\right)$ is continuous at $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

So, $f(x)$ is continuous at $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

or, $f\left(\dfrac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left[\sin\dfrac{\pi}{4} - \cos\dfrac{5\pi}{4}\right] = e^{\frac{\pi}{4}}\left[\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}\right] = 0$

and $f\left(\dfrac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left[\sin\dfrac{5\pi}{4} - \cos\dfrac{5\pi}{4}\right] = 0$

$\therefore f\left(\dfrac{\pi}{4}\right) = f\left(\dfrac{5\pi}{4}\right)$

$\therefore f'(x) = e^{x}(\sin x - \cos x) + e^{x}(\cos x + \sin x)$

$\therefore f'(x) = 0$ given,

$e^{x}(\sin x - \cos x) + e^{x}(\cos x + \sin x) = 0$

$2e^{x} \sin x - e^{-x} \cos x + e^{x} \cos x = 0$

$\therefore 2e^{x} \sin x = 0$

$2e^{c} \sin c = 0$

$\sin C = 0,\ e^{c} \neq 0$

$\therefore \sin C = \sin \pi$

$C = \pi$

$\therefore \pi$ belongs to $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

Hence, Rolle's theorem is verified.

(iv) $f(x) = \log\left\{\dfrac{x^2 + ab}{(a+b)x}\right\},\ x \in [a, b]$

Sol^n. Cleay, $\log\left\{\dfrac{x^2 + ab}{(a+b)\,x}\right\}$ is continuous in $[a, b]$

because it is logarithm function,

$\therefore f(c)$ is contineous in $[a, b]$

$\therefore f(a) = \log\left[\dfrac{a^2 + ab}{(a+b)\,a}\right] = \log 1 = 0$

and $f(b) = \log\left[\dfrac{b^2 + ab}{(a+b)\,b}\right] = \log\left[\dfrac{b(a+b)}{b(a+b)}\right] = \log 1 = 0$

$\therefore f'(x) = \dfrac{1}{\dfrac{x^2 + ab}{(a+b)\,x}} \times \left[\dfrac{(a+b)\,x\,.\,2x - (x^2 + ab)\,(a+b)^2}{(a+b)^2\,x^2}\right]$

$= \dfrac{(a+b)\,x}{x^2 + ab} \times \dfrac{(a+b)\,(x^2 - ab)}{(a+b)^2\,.\,x^2}$

$\therefore f'(x) = \dfrac{x^2 - ab}{x\,(x^2 + ab)}$

$\therefore f'(C) = 0 \Rightarrow \dfrac{C^2 - ab}{c\,(c^2 + ab)} = 0 \Rightarrow C^2 - ab = 0$

$\because C \pm \sqrt{ab} \in [a, b]$

Hence Rolle's theorem is verified.

(v) $f(x) = \dfrac{x(x+3)}{e^{\frac{x}{2}}},\ x \in [-3, 0)$

Sol^n. Since, $\left\{\dfrac{x(x+3)}{e^{\frac{x}{2}}}\right\}$ is continuous at $[-3, 0]$

So, $f(x)$ is continuous at $[-3, 0]$

$$\therefore f(x) = \left\{\frac{x(x+3)}{e^{\frac{x}{2}}}\right\}$$

$$f(-3) = \left\{\frac{-3(-3+3)}{e^{\frac{3}{2}}}\right\} = 0$$

$f(0) = 0 \therefore f(-3) = f(0)$

$$\therefore f'(x) = (2x+3)\, e^{-\frac{x}{2}} + (x^2+3x)\, e^{-\frac{x}{2}} \left(-\frac{1}{2}\right)$$

$$= e^{-\frac{x}{2}}\left[2x+3-\left(\frac{x^2+3x}{2}\right)\right]$$

$$= \left(-\frac{x^2+x+6}{2}\right) e^{-\frac{x}{2}}$$

$$\therefore f'(c) = 0 \Rightarrow \frac{-c^2+c+c}{2} = 0$$

$\Rightarrow c^2 - c - 6 = 0$

$\therefore c = -2$ and $c = 3$

$\therefore -2 \in (-3, 0)$

Hence Rolle's theorem is verified.

2. Verify Lagrenge's mean value theorem for the function.

(i) $f(x) = e^x \quad x \in [1, 2]$

Sol^n. Given that $f(x) = e^x$ ..... (i), $x \in [1, 2]$

Have, $a = 1, b = 2$ [exponential functions are continuous]

$\therefore f(a) = f(1) = e^1 = e$

$\therefore f(b) = f(2) = e^2 = e^2$

Diff^n (i) w. r. t. x we get,

$f'(x) = e^x$. $f'(c) = e^c$

$\therefore$ So that $f(2) - f(1) = (2-1) f'(c)$

or, $e^2 - 2 = 1\, e^c$

or, $e(e-1) = e^c$

or, $C = \log\{e(e-1)\} = \log e + \log(e-1)$

$= 1 + \log(e-1)$

[$\because e \in (2, 3) \therefore e - 1 < 2 \therefore \log(e-1) = 0 +$ decima

Hence verified.

(ii) $f(x) = x^2$, $x \in [1, 2]$

Sol^n. Given that $f(x) = x^2$, clearly, $f(x)$ is conditineous in $[1, 2]$ an differential in $(1, 2)$

Diff^n. (i) w. r. t. x we get

$\therefore f''(x) = 2x$ ..... (ii)

Here, $a = 1, b = 2$

$\therefore f(a) = f(1) = 1^2 = 1$

$f(b) = f(2) = 2^2 = 4$

$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{4-1}{2-1} = 3$$

Now, $f'(c) = 2c$

and $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ gives,

$2c = 3$

$\therefore C = \frac{3}{2}$, clearly, $\frac{3}{2} \in (1, 2)$

Hence verified.

(iii) $f(x) = \log x, x \in [1, e]$

Sol^n. Given that, $f(x) = \log x$ ..... (i), $x \in [1, e]$

Here, $a = 1, b = e$

$\therefore f(a) = \log 1 = 0$

$f(b) = \log e = 1$

$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{1-0}{e-1} = \frac{1}{e-1}$$

$f'(x) = \frac{1}{x}$

$\therefore f'(c) = \frac{1}{c}$

Also, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ gives,

$\frac{1}{c} = \frac{1}{e-1}$, $c = e - 1 \in (1, e)$

Hence verified.

(iv) $f(x) = Ax^2 + Bx + C, \ x \in [a, b]$

Sol^n. Given that $f(x) = Ax^2 + Bx + C$ ..... (i) $x \in [a, b]$

$\therefore f(a) = Aa^2 + Ba + c$

$f(b) = Ab^2 + Bb + c$

$f(b) - f(a) = Ab^2 + Bb + c - Aa^2 - Ba - c$

$= A(b^2 - a^2) + B(b - a)$

$= (b - a)\{A(b + a) + B\}$

so that $\dfrac{f(b)-f(a)}{b-a} = \dfrac{(b-a)\{A(b+a)+B\}}{(b-a)} = A(b+a) + B$

clearly, $f(x)$ is differentiable in (a, b) and contineous in $[a, b]$

Now, Differentiating (i) w.r.t x, we get

$f'(x) = 2Ax + B$

$\therefore f'(c) = 2Ac + B$

Here. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ gives,

$2Ac + B = A(b + a) + B$

or, $2Ac = A(b + a)$

or, $c = \frac{a+b}{2}$ $\therefore \frac{a+b}{2} \in (a, b)$ hence, verifies

v) $f(x) = \frac{x^2+1}{x}, x \in \left[\frac{1}{2}, 2\right]$

Sol^n. Given that $f(x) = \frac{x^2+1}{x}$ .....(i), $x \in \left[\frac{1}{2}, 2\right]$

Clearly, f(x) is differentiable in $\left(\frac{1}{2}, 2\right)$ and contineous in $\left[\frac{1}{2}, 2\right]$

Here, $f\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^2+1}{\frac{1}{2}} = \frac{\frac{1}{4}+1}{\frac{1}{2}} = \frac{5}{4} \times \frac{2}{1} = \frac{5}{2}$

$f(2) = \frac{2^2+1}{2} = \frac{5}{2}$

$\therefore \frac{f(2) - f\left(\frac{1}{2}\right)}{2 - \frac{1}{2}} = \frac{\frac{5}{2} - \frac{5}{2}}{\frac{3}{2}} = 0$

from (i) $f(x) = \frac{x^2+1}{x}$

$f(x) = \frac{x^2}{x} + \frac{1}{x} = x + x^{-1}$

$\therefore f^1(x) = 1 - \frac{1}{x^2}$

So that $f^1(c) = 1 - \frac{1}{c^2}$

$\therefore f^1(c) = \frac{f(2) - f\left(\frac{1}{2}\right)}{2 - \frac{1}{2}}$ gives

$1 - \frac{1}{c^2} = 0$

or, $\frac{c^2-1}{c^2} = 0$

or, $c^2 - 1 = 0, c^2 = 1, c = \pm 1$

So that there exists $1 \in \left(\frac{1}{2}, 2\right)$ such that

$f^1(1) = \frac{f(2) - f\left(\frac{1}{2}\right)}{2 - \frac{1}{2}}$

= Hence verified

(vi) $f(x) = x^2 + 3x + 2, x \in [0, 2]$

Sol^n. Given that $f(x) = x^2 + 3x + 2$ .....(i), $x \in [0, 2]$

clearly, f(x) is differentiable, in (0, 2)

and contineous in [0, 2]

Here, $f(a) = a^2 + 3 \times 0 + 2 = 2$

and $f(2) = 2^2 + 3 \times 2 + 2 = 12$

$\therefore \frac{f(2) - f(0)}{2 - 0} = \frac{12 - 2}{2} = \frac{10}{2} = 5$

Diff^n (i) w.r.t x, we get

$f^1(x) = 2x + 3$

$\therefore f^1(c) = 2c + 3$

$\therefore f^1(c) = \frac{f(2) - f(0)}{2 - 0}$ gives

$\therefore 2c + 3 = 5$

or, $2c = 2$

$\therefore c = 1 \in (0, 2)$

3. Verify Cauchy's mean value theorem.

(i) $f(x) = x^4, g(x) = 3x^2, x \in [1, 2]$

$\therefore f(1) = 1^4 = 1, g(1) = 3 \times 1^2 = 3$

$f(2) = 2^4 = 16, g(2) = 3 \times 2^2 = 12$

Now,

$f^1(x) = 4x^3, g^1(x) = 6x$

Since, f(x) and g(x) are Diff^n in (1, 2) and contineous in [1, 2]

So that by cauchy's mean value theorem there exist a point $c \in (1, 2)$ such that

$\frac{f^1(c)}{g^1(c)} = \frac{f(2) - f(1)}{g(2) - g(1)}$

or, $\frac{4c^3}{2c} = \frac{16 - 1}{12 - 3}$

or, $\frac{2c^2}{3} = \frac{15}{9}$

or, $c^2 = \frac{15}{9} \times \frac{3}{2}$

$\therefore c^2 = \frac{5}{2}$

$\therefore c = \pm\sqrt{\frac{5}{2}}$ Hence, verified

(ii) $f(x) = x^2, g(x) = x^3; x \in [0, 1]$

Sol^n. Given that $f(x) = x^2$ .....(i)

and $g(x) = x^3$ .....(ii), $x \in [0, 1]$

$\therefore f(0) = 0^2 = 0, g(0) = 0^3 = 0$

$f(1) = 1^2 = 1, g(1) = 1^3 = 1$

Now, $f^1(x) = 2x$ and $g^1(x) = 3x^2$

Since, f(x) and g(x) are diff^n in (0, 1) and continuous in [0, 1], then by cauchy's mean value theorem, these exist a point $c \in (0, 1)$ s.t.

$$\frac{f^1(c)}{g^1(c)} = \frac{f(1) - f(0)}{g(1) - g(0)}$$

or, $\frac{2c}{3c^2} = \frac{1}{1}$ $\therefore c = \frac{2}{3}$

clearly, $\frac{2}{3} \in (0, 1)$ Hence, verified

(4) In the mean value theorem $f(x+h) = f(x) + hf^1(x + \theta h)$, find θ if

(i) $f(x) = \frac{1}{x}$

Sol^n. (i) Given that

$f(x) = \frac{1}{x}$ .....(i)

$\therefore f^1(x) = -\frac{1}{x^2}$

Now, $f(x + h) = \frac{1}{x+h}$

$f^1(x + \theta h) = -\frac{1}{(x+\theta h)^2}$

So that

$f(x+h) = f(x) + hf^1(x + \theta h)$ gives,

$$\frac{1}{x+h} = \frac{1}{x} + h\left\{-\frac{1}{(x+\theta h)^2}\right\}$$

or, $x(x + \theta h)^2 = (x + h)[x^2 + 2x\theta h + \theta^2h^2 - hx]$

or, $x\{x^2 + 2x\theta h + \theta^2h^2\} = x^3 + 2x^2\theta h + x\theta^2h^2 - hx^2 + hx^2 + 2x\theta h^2 + \theta^2h^3 - h^2x$

or, $x^3 + 2x^2\theta h + x\theta^2h^2 = x^3 + 2x^2\theta h + x\theta^2h^2 + 2x\theta h^2 + \theta^2h^3 - h^2x$

or, $\theta(2xh^2 + \theta h^3) = h^2x$

or, $\theta h^2(2x + \theta h) = h^2x$

or, $\theta^2h + 2\theta x - x = 0$

$$\therefore \theta = \frac{-2x \pm \sqrt{(2x)^2 - 4n(x)}}{2h} = -\frac{2x \pm 2\sqrt{x^2 + xh}}{2h}$$

$$\therefore \theta = -\frac{x \pm \sqrt{x^2 + xh}}{h}$$

(ii) $f(x) = e^x$

Sol^n. Given that $f(x) = e^x$ .....(i)

$f(x+h) = e^{x+h}$

$f^1(x) = e^x$

$\therefore f^1(x + \theta h) = e^{x+\theta h}$

Now,

$f(x + h) = f(x) + hf^1(x + \theta h)$ gives,

$e^{x+h} = e^x$ the $x + \theta h$

or, $e^h = 1 + he^{\theta h}$

or, $he^{\theta h} = e^h - 1$

or, $e^{\theta h} = \frac{e^h - 1}{h}$ or,

or, $\theta h = \log\left(\frac{e^h - 1}{h}\right)$

$\theta = \frac{1}{h}\log\left(\frac{e^h - 1}{h}\right)$

(iii) $f(x) = \log x$

Sol^n. Given that

$f(x) = \log x$ .....(i)

$\therefore f^1(x) = \frac{1}{x}$

So that $f(x + h) = \log(x + h)$

and $f^1(x + \theta h) = \frac{1}{x + \theta h}$

Now, $f(x + h) = f(x) + hf^1(x + \theta h)$

or, $\log(x + h) = \log x + h.\frac{1}{x + \theta h}$

or, $\log\left(\frac{x+h}{x}\right) = \frac{h}{x + \theta h}$

or, $\frac{x + \theta h}{h} = \frac{1}{\log\left(\frac{x+h}{x}\right)}$

or, $\frac{x}{h}+\theta=\frac{1}{\log\left(1+\frac{h}{x}\right)}$

$\therefore \theta=\frac{1}{\log\left(1+\frac{h}{x}\right)}-\frac{x}{h}$

5. **In the mean value theorem, $f(a+h)=f(a)+hf^{1}(a+\theta h)$ find θ if $f(x)=\sqrt{x}$, $a=1, h=3$.**

Sol^n. We have.

$f(x)=\sqrt{x}$

$\therefore f^{1}(x)=\frac{1}{2}x^{\frac{-1}{2}}=\frac{1}{2\sqrt{x}}$

Also, $f(a+h)=\sqrt{a+h}$

and $f^{1}(a+\theta h)=\frac{1}{2\sqrt{a+\theta h}}$

Now,

$f(a+h)=\sqrt{a}+hf^{1}(a+\theta h)$gives,

$\sqrt{a+h}=\sqrt{a}+h.\frac{1}{2\sqrt{1+\theta.3}}$

or, $2=1+\frac{3}{2\sqrt{3\theta+1}}$

or, $\frac{1}{\sqrt{3\theta+1}}=\frac{2}{3}$ or, $\sqrt{3\theta+1}=\frac{3}{2}$

$\therefore \sqrt{3\theta+1}=\frac{3}{2}$

$\therefore 3\theta+1=\frac{9}{4}$

or, $3\theta=\frac{5}{4}$

$\therefore \theta=\frac{5}{12}$

6. **Assuming the validity of expansion find the expansion of the following function vary maclaurin's theorem.**

(i) **$\log(1+x)$**

Sol^n. Let $f(x)=\log(1+x)$ .....(i)

$\therefore f(0)=\log(1+0)=\log 1=0$

$f^{1}(x)=\frac{1}{1+x}$

$\therefore f^{1}(0)=\frac{1}{1+0}=\frac{1}{1}=1$

$f^{ii}(x)=-\frac{1}{(x+1)^2}=\frac{-1}{(0+1)^2}=-1$

$f^{iii}(x)=\frac{2}{(x+1)^3}\Rightarrow f^{iii}(0)=\frac{2}{(0+1)^3}=2$

$f^{iv}(x)=\frac{-6}{(x+1)^4}\Rightarrow f^{iv}(0)=\frac{-6}{(0+1)^4}-6$

We have Maclaurin's series of f(x) is given by

$f(x)=f(a)+xf^{1}(0)+\frac{x^2}{2!}f^{ii}(0)+\frac{x^3}{3!}f^{iii}(0)+\frac{x^4}{4!}f^{iv}(0)+$.....

$\log(1+x)=0+\frac{x}{1!}.1+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2)+\frac{x^4}{4!}(-0)+$.....

$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+$.....

(ii) **sinhx**

Sol^n. Let

$f(x)=\sinh x$

$f(0)=\sinh 0=0$

$f^{1}(x)=\cosh x$

$f^{1}(0)=\cosh.0=1$

$f^{ii}(x)=\sinh x$

$f^{ii}(0)=0$

$f^{iii}(x)=\cosh x\Rightarrow f^{iii}(0)=1$

$f^{iv}(x)=\sinh x\ \therefore f^{v}(0)=0$

$f^{v}(x)=\cosh x\ \therefore f^{v}(0)=1$

By maclaurin's Theorem, we have

$f(x)=f(0)+$

$\frac{x}{1!}f^{1}(0)+\frac{x^2}{2!}f^{ii}(0)+\frac{x^3}{3!}f^{iii}(0)+\frac{x^4}{4!}f^{iv}\times 0+\frac{x^5}{5!}x^{v}(0)+$.....

$\therefore \sinh x=\frac{x}{1!}+\frac{x^2}{2!}+0+\frac{x^5}{3!}\times 1+\frac{x^4}{4!}\times 0+\frac{x^5}{5!}\times 1+$.....

Hence, $\sinh x=\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}1$.....

(iii) **coshx**

Sol^n. Let $f(x)=\cos hx$

$\therefore f(0)=1$

$f^{1}(x)=\sinh x$

$\therefore f^{1}90)=0$

$f^{ii}(x)=\cosh x$

$\therefore f^{ii}(x)=\cosh x$

$\therefore f^{iv}(0) = 1$

$f^{v}(x) = \sinh x$

$\therefore f^{v}(0) = 0$

$f^{vi}(x) = \cosh x$

$\therefore f^{vi}(x) = 1$

By maclaurin's theorem we have

$$f(x) = f(0) + \frac{x}{1!}f^{i}(0) + \frac{x^2}{2!}f^{ii}(0) + \frac{x^3}{3!}f^{iii}(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots$$

$$\therefore \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots$$

(iv) tanhx

Sol^n. Let $f(x) = \tanh x$

$\therefore f(0) = 0$

$f^{i}(x) = \operatorname{sech}^2 x$

$f^{i}(0) = 1$

$f^{ii}(x) = -2\operatorname{sech}x.\ \tanh x$

$\therefore f^{ii}(0) = 0$

$f^{iii}(x) = -2\operatorname{sech}^4 x + 4\operatorname{sech}^2 x.\ \tanh^2 x$

$\therefore f^{iii}() = -2$

$f^{iv}(x) = -8\operatorname{sech}^2 x\ .\ \operatorname{sech}x.\ \tan dx + 8\operatorname{sech}^2 x.\ \tanh x.\ \operatorname{sech}^2 x + 8\operatorname{sech}x\ .\ \operatorname{sech}x.\ \tanh x.\ \tanh x$

$+ 16\operatorname{sech}^4 x\ \tanh x + 8\operatorname{sech}^2 x\ \tanh^3 x$

$\therefore f^{iv}(0) = 0$

$\therefore f^{v}(0) = 16$

We know, the Maclaurin's saries in infinite form is

$f(x) = f(0)$

$$+ \frac{x}{1!}f^{i}(0) + \frac{x^2}{2!}f^{ii}(0) + \frac{x^3}{3!}f^{iii}(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots\ldots$$

$$\therefore \tanh x = 0 + \frac{x.1}{1!} + \frac{x^2}{2!}\times 0 + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}\times 0 + \frac{x^5}{5!}.16\ldots\ldots$$

so that

$$\tanh x = x - \frac{2x^3}{3!} + \frac{6x^5}{5!} + \ldots$$

(v) tanx

Sol^n. Let $f(x) = \tan x$

$\therefore f(0) = 0$

$f^{i}(x) = \sec^2 x$

$\therefore f^{i}(0) = 1$

$f^{ii}(x) = 2\sec x.\ \sec x.\ \operatorname{tg}anx = 2\tan x\,(1 + \tan^2 x)$

$= 2\tan x + 2\tan^3 x$

$\therefore f^{ii}(0) = 0$

Again, $f^{iii}(x) = 2\sec^2 x + 6\tan^2 x\ \sec^2 x$

$= 2(\tan^2 x) + 6\tan^2 x\,(1 + \tan^2 x)$

$= 2 + 8\tan^2 x + 6\tan^4 x$

$\therefore f^{iii}0) = 2$

$f^{iv}(x) = 0 + 16\tan x.\ \sec^2 x + 24\tan^3 x.\ \sec^2 x$

$= 16\tan^3(1 + \tan^2 x) + 24\tan^3 x\ (\text{if } \tan^2 x)$

$= 16\tan x + 40\tan^3 x + 24\tan^5 x$

$\therefore f^{iv}(0) = 0$

Again,

$f^{v}(x) = 16\sec^2 x + 120\tan^2 x\ \sec^2 x + 120\tan^4 x\ \sec^2 x$

$\therefore f^{v}(0) = 16$

By maclaurin's series, we have,

$f(x) = f(0) +$

$$\frac{x}{1!}f^{i}(0) + \frac{x^2}{2!}f^{ii}(0) + \frac{x^3}{3!}f^{iii}(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots\ldots$$

$$\therefore \tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots$$

(vi) e^xsinx

sol^n. Let $f(x) = e^x \sin x$

$f^{i}(x) = e^x \cos x + e^x \sin x$

$\therefore f^{i}(x) = e^x \cos x + e^x \sin x$

$f^{ii}(x) = -e^x \sin x + e^x \cos x + e^x \cos x + e^x \sin x$

$f^{ii}(x) = 2e^x \cos x$

$\therefore f^{ii}(0) = 2$

Again,

$f^{iii}(x) = 2\,[e^x \cos x - e^x \sin x]$

$\therefore f^{iii}(0) = 2\,[1 - 0] = 2$

By maclauin's series, we have,

$$f(x) = f(0) + \frac{x}{1!}f^{i}(0) + \frac{x^2}{2!}f^{ii}(0) + \frac{x^3}{3!}f^{iii}(0) + \ldots\ldots$$

$$\therefore e^x \sin x = x + x^2 + \frac{x^3}{3} + \ldots\ldots$$

(vii) log (2 + sinx)

Sol^n. Let $f(x) = \log(2 + \sin x)$

$\therefore f(0) = \log(1 + \sin 0) = \log 1 = 0$

$$f^{i}(x) = \frac{\cos x}{1 + \sin x}$$

$$\therefore f^{i}(x) = \frac{\cos o}{1 + \sin 0} = 1$$

$$f^{ii}(x) = \frac{-(1 + \sin x)\sin x - \cos x.\cos x}{(1 + \sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$\therefore f^{ii}(x) = \frac{-1}{(1+\sin x)^2} \therefore f^{ii}(0) = \frac{-1}{1+\sin 0} = -1$$

Again, Diff^n, w.r.t, we get

$$f^{iii}(x) = (-1)(-1)\frac{\cos x}{(1+\sin x)^2} = \frac{\cos x}{(1+\sin x)^2}$$

$\therefore f^{iii}(0) = 1$

Again, diff^n w.r.t x, we get

$$f^{iv}(x) = \frac{(1+\sin x) - \cos x 2.(1+\sin x).\cos x}{(1+\sin x)^4}$$

$$= \frac{-\sin x(1+\sin x)^2 - 2(1-\sin^2 x)(1+\sin x)}{(1+\sin x)^4}$$

$$= (1+\sin x) + 2\left[\frac{-\sin x - 2(1-\sin x}{(1+\sin x)^4}\right]$$

$$= \frac{-\sin x - 2 + 2\sin x}{(1+\sin x)^2}$$

$\therefore f^{iv}(0) = -2$

By Maclavrin's expansion of f(x) is

$$f(x) = f(0) + xf^{i}(0) + \frac{x^2}{2!}f^{i}(0) + \frac{x3}{3!}f^{ii}(0) + \frac{x4}{4!}f^{iv}(0) + \ldots\ldots$$

$$\therefore \log(1+\sin x) = x - \frac{x^2}{2!} + \frac{x3}{3!} - \frac{x4}{4!} + \ldots\ldots$$

Here, $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{12} + \ldots\ldots$

(viii) **Log (secx)**

Sol^n. Let $f(x) = \log \sec x$

$f(0) = \log 1 = 0$

$f^{i}(x) = \frac{1}{\sec x}.\sec x.\tan x \Rightarrow f^{i}(0) = 0$

$f^{ii}(x) \sec^2 x.\tan x \Rightarrow f^{ii}(0) = 0$

$f^{iii}(x) = 2\sec^2 x.\tan x \Rightarrow f^{iii}(0) = 0$

$f^{iv}(x) = 2[2\sec^2 x \tan^2 x + \sec^2 x]$

or, $f^{iv}(x) = 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x$

$= 6\sec^4 x - 4\sec^2 x$

$\therefore f^{iv}(0) = 6 - 4 = 2$

$f^{v}(x) = 24\sec^4 x \tan x - 8\sec^2 x \tan x$

$\therefore f^{v}(0) \Rightarrow 0$

$f^{vi}(x) = 24[\sec^4 x + \sec^2 x + 4\sec^4 x + \tan^2 x]$

$- 8[2\sec^2 x + an^2 x + \sec^4 x]$

$\therefore f^{vi}(0) = 24 - 8 = 16$

By maclaurin's series, we have,

$$f(x) = f(0) + xf^{i}(0) + \frac{x^2}{2!}f^{ii}(0) + \frac{x^3}{3!}f^{iii}(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots$$

$$\therefore \log(\sec x) = \frac{x^2}{2!} + \frac{2x^4}{4!} + \frac{16x^6}{6!} + \ldots\ldots$$

(ix) $\frac{e^x}{1+e^x}$

Sol^n. Let $f(x) = \frac{e^x}{1+e^x}$ $\therefore f(0) = \frac{1}{1+1} = \frac{1}{2}$

$$f^{i}(x) = \frac{e^x}{(1+e^x)^2}, f^{i}(0) \Rightarrow \frac{1}{(1+1)^2} = \frac{1}{4}$$

Again,

$$f^{ii}(x) = \frac{(1+e^x)^2 e^x - 2(1+e^x)e^x e^x}{(1+e^x)^4}$$

$$= \frac{(1+e^x)(1+e^x).e^x - 2e^{2x})}{(1+e^x)^4}$$

$$f^{ii}(x) = \frac{e^x - e^{2x}}{\left(1+e^x\right)^3}$$

$$\therefore f^{ii}(0) = \frac{1-1}{2^3} = 0$$

$$\text{Also, } f^{iii}(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x})3.(1+e^x).e^x}{(1+e^x)^6}$$

$$= (1+e^x)\left(\frac{(1+e^x)^2(e^x - 2e^{2x}) - 3e^x(e^x e^{2x}}{(1+e^x)^6}\right)$$

$$= \frac{(1+e^x)(e^x - 2e^{2x}) - 3e^x(e^x - e^{2x})}{(1+e^x)^5}$$

$$\therefore f^{iii}(0) = \frac{(1+1)^2(1-2) - 3.1(1-1)}{(1+1)^5} = -\frac{1}{2^3} = \frac{-1}{8}$$

Hence, By maclaurin's expansion series

$$f(x) = f(0) + \frac{x}{1!}f^{i}(0) + \frac{x^2}{2!}f^{ii}(0) + \frac{x^3}{3!}f^{iii}(0) + \ldots\ldots$$

$$\therefore \frac{e^x}{1+e^x} = \frac{1}{2} + x.\frac{1}{4} + \frac{x^2}{2}\times 0 + \frac{x^3}{6}\left(\frac{-1}{8}\right) + \ldots\ldots$$

Hence, $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \ldots\ldots$

(x) $\log(1+e^x)$

Sol^n. Let, $f(x) = \log(1+e^x)$

$\therefore f(0) = \log(1+1) = \log 2$

$$f^{i}(x) = \frac{e^x}{1+e^x} \Rightarrow f^{i}(0) = \frac{1}{1+1} \Rightarrow \frac{1}{2}$$

$$f^{ii}(x) = \frac{(1+e^x).e^x - e^x.e^x}{(1+e^x)^2} \Rightarrow f^{ii}(0) = \frac{(1+1)-1}{(1+1)^2} = \frac{1}{4}$$

Hence By maclaurin's expansion, we have

$$f(x) = f(0) + \frac{x}{1!}f^{i}(0) + \frac{x^2}{2!}f^{ii}(0) + \ldots\ldots$$

or, $\log(1+e^x) = \log 2 + x.\frac{1}{2} + \frac{x^2}{2}.\frac{1}{4} + \ldots\ldots$

$$\therefore \log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} + \ldots\ldots$$

(xi) $e^{\sin x}$

Sol^n. Let $f(x) = e^{\sin x} \Rightarrow f(0) = e^0 = 1$

$f^{i}(x) = e^{\sin x}.\cos x \Rightarrow f(x)\cos x \therefore f^{i}(0) = 1$

$f^{ii}(x) = f^{i}(x)\cos x - f(x)\sin x$

$\therefore f^{ii}(0) = 1\times 1 - 0 = 1$

$\therefore f^{iii}(x) = f^{ii}(x)\cos x - 2f^{i}(x)\sin x - f(x)\cos x$

$\therefore f^{iii}(0) = 1 + 1 - 0 - 0 - 1 = 0$

By Maclaurin's expansion, we have,

$$f(x) = f(0) + \frac{x}{1!}f^{i}(0) + \frac{x^2}{2!}f^{ii}(0) + \frac{x^3}{3!}f^{iii}(0) + \ldots\ldots$$

$$\therefore e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{3x^4}{4!}$$

(xii) $x\cot x$

Sol^n. Let $f(x) = x\cot x = x\frac{\cos x}{\sin x}$

$$= \frac{x\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\right]}{x - \frac{x^3}{3!} + \frac{x^5}{5!}}$$

$$= \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots}{1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\ldots\right)}$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\right)\left[\left(1 - \frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\ldots\right)\right]^{-1}$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\right)\left(1 + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\right)\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\ldots\right)^2\right) + \ldots$$

$\ldots\ldots\ldots\ldots$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\right)\left[1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\ldots + \frac{x^4}{(3!)^2} + \ldots\right]$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\right)\left(1 + \frac{x^2}{360}x^4 + \ldots\ldots\right)$$

$$= 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + \ldots\ldots \frac{x^2}{2!} - \frac{x^4}{6+2!} \ldots\ldots\ldots\ldots \frac{x^4}{4!+\ldots\ldots}$$

$$= 1 + \left(\frac{1}{6} - \frac{1}{2}\right)x^2 + \left(\frac{7}{3600} - \frac{1}{12} + \frac{1}{24}\right)x^4 + \ldots\ldots$$

$$\therefore f(x) = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 + \ldots\ldots$$

7. Assuuming the validity of expansion claurin's series.

(i) $\sin^{-1}x = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \ldots\ldots$

and hence show that

$$\cos x = \frac{\pi}{2} - x - \frac{x^3}{3!} \ldots\ldots\ldots\ldots$$

Sol^n. Let $f(x) = \sin^{-1}x$

$\therefore f(0) = 0$

$$\therefore f^{i}(x) = \frac{1}{\sqrt{1-x^2}}$$

$\therefore f^{i}(0) = 1$

Squaring, we get $(f^{i}(x))^2 = \frac{1}{1-x^2}$

or, $(1-x^2)(f^{i}(x))^2 = 1$

Differentiating w.r.t x, we get

$(1-x^2)\,2f^{i}(x)\,f^{ii}(x) + \{f^{i}(x)\}^2(-2x) = 0$

or, $(1-x^2)\,2f'(x)\,f''(x) + \{f'(x)\}^2(-2x) = 0$

or, $(1-x^2)\,f''(x) - xf'(x) = 0$ .....(i)

When $x = 0$

$f''(0) - 0f'(0) = 0 \Rightarrow f''(0) = 0$

Differentiating, n times by using Leibnitz's rule on (i) we get

$(1-x^2)\,f^{n+2}(x) + nc_1\,f^{n+1}(x)(-2x) + nc_2\,f^{n}(x)(-2) - [xf^{n+1}(x) + nc_1 f^{n}(x)] = 0$

or, $(1-x^2)\,f^{n+2}\,9x) - nf^{n+1}(x).\,2x - n(n-1)\,f^{n}(x) - x\,f^{n+1}(x) - nf^{n}(x) = 0$

or, $(1-x^2)\,f^{n+2}(x) - (2n+1)\,f^{n+1}(x) - n^2\,f^{n}(x) = 0$ .....(ii)

Putting $x = 0$ in (ii)

$\therefore f^{n+2}(0) = n^2 f^{n}(0)$ .....(iii)

Putting $n = 0, 1, 2, 3,$ .....in (iii), we get

$f^{iv}(0) = 1^2.\,f''(0) = 1 \times 1 = 1$

$f^{v}(0) = 2^2 f'''(0) = 4 \times 0 = 0$

$f^{vi}(0) = 3^2 f^{iv}(0) = 9 \times 1 = 9$

By the maclaurin's expansion, we have,

$$f(x) = f(0) + \frac{x}{1!}f^{i}(0) + \frac{x^2}{2!}f^{ii}(0) + \frac{x^3}{3!}f^{iii}(0) + \ldots\ldots$$

$$\therefore \sin^{-1}x = 0 + x.1 + \frac{x^2}{2}\times 0 + \frac{x^3}{3!}\times 1 + \frac{x^4}{4!}\times 0 + \frac{x^5}{5!}\times 9 + \ldots\ldots$$

$$\therefore \sin^{-1}x = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \ldots\ldots\ldots\ldots$$

Again, we know that

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\therefore \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$$

$$\frac{\pi}{2} - x - \frac{x^3}{3!} - \frac{9x^5}{5!} + \ldots\ldots\ldots \text{ proved}$$

(ii) $\sec x = 1 + \frac{x^2}{2} + \frac{5}{24} + \ldots\ldots$

Sol^n. Let $f(x) = \sec x$

$\therefore f(0) = 1$

$f'(x) = \sec x.\,\tan x$

$f'(0) = 0$

Again, $f''(x) = \sec x \tan^2 x + \sec^2 x$

$f''(x) = \sec x \tan^2 x + \sec^3 x$

$f''(x) = 2\sec^3 x - \sec x$

$\therefore f''(0) = 0$

Diff^n w.r.tx, we get

$\therefore f'''(x) = 6\sec^2 x.\,\sec x.\,\tan x - \sec x.\,\tan x$

$\therefore f'''(0) = 0$

again diff^n, w.r.t x, we get

$f^{iv}(x) = 5\left[\sec x.\sec^2 x + \sec x \tan^2 x\right] + 6[\sec^2 x \tan^4 + 3\sec x.\,\tan^2 x\,\sec^2 x]$

$\therefore f^{iv}(0) = 5[1+0] + 6[\,) + 0] = 5$

Hence, by maclaurin's expansion of f(x), we have,

$$f(x) = f(0) + \frac{x}{1!}f^{i}(0) + \frac{x^2}{2}f^{ii}(0) + \frac{x^3}{3!}f^{iii}(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots$$

$$\text{or, } \sec x = 1 + x \times 0 + \frac{x^2}{2}\times 1 + \frac{x^3}{6}\times 0 + \frac{x^4}{24}\times 5 + \ldots\ldots$$

$$\therefore \sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots\ldots \text{ proved.}$$

(iii) $\text{Log}\,(1 + x + x^3) = x + \frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4} + \ldots\ldots$

Sol^n. Let $f(x) = \log(1 + x + x^2)$

$$= \log\left\{\frac{(1-x)(1+x+x^2)}{(1-x)}\right\}$$

$$= \log\left\{\frac{(1-x)^3}{(1-x)}\right\}$$

$\therefore f(x) = \log(1-x^3) - \log(1-x)$ ..... (i)

But, we have,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots\ldots$$

So that,

$$\log(1-x) = -x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots$$

$$\text{and } \log(1-x^3) = -x^3 - \frac{1}{2}\left(x^3\right)^2 - \frac{1}{3}\left(x^3\right)^3 - \frac{1}{4}\left(x^3\right)^4 - \frac{1}{5}\left(x^3\right)^5 \ldots\ldots$$

$$= -x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \frac{x^{12}}{4} \ldots\ldots\ldots\ldots$$

Them from (i),

$$f(x) = \left(-x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \frac{x^{12}}{4} \ldots\ldots\right) - \left\{-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots\right\}$$

$$= x + \frac{x^2}{2} + \left(\frac{x^3}{3} - x^3\right) + \frac{x^4}{4} + \ldots\ldots\ldots\ldots$$

$$\therefore f(x) = x + \frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4} + \ldots\ldots\ldots\ldots$$

(iv) $x\operatorname{cosec}x = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \ldots\ldots\ldots$

Solⁿ. Let $f(x) = x\cos\operatorname{sec}x$

Suppose,

$x\operatorname{cosec}x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots\ldots$ (i)

Where $a_0, a_1, a_2 \ldots\ldots\ldots\ldots$ are constant

or, $\frac{x}{\sin x} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots$

or, $x = \sin x\,(a_0 + a_1x = a_2x^2 = a_3x^3 + a_4x^4 + \ldots\ldots\ldots)$

or, $x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\right)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \ldots\ldots)$

or, $x = a_0x + a_1x^2 + \left(a_2 - \frac{a^2}{3!} + \frac{a_0}{5!}\right)x^5 + \ldots\ldots\ldots\ldots$

Equating the coefficient of like terms from both sides, we get,

$a_0 = 1,\ a_1 = 0,\ a_2 - \frac{a_0}{3!} = 0$

or, $a_2 = \frac{a_0}{3!} = \frac{1}{6},\ a_3 = \frac{a_1}{3!} = 0$

or, $a_4 = -\frac{a_2}{3!} + \frac{a_0}{5!} = 0$

or, $a_4 = \frac{a_2}{3!} - \frac{a_0}{5!} = \frac{1}{6\times 3!} - \frac{1}{5!}$

$= \frac{1}{3!}\left(\frac{1}{6} - \frac{1}{20}\right) = \frac{1}{3!}\left(\frac{20-6}{120}\right) = \frac{1}{3\times 2}\times\frac{14}{120} = \frac{7}{360}$

Substituting the values of a_0, a_1, a_2, a_3 and a_4 in (i)

$\therefore x\operatorname{cosec}x = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \ldots\ldots\ldots\ldots\ldots$

(v) $\frac{1}{\sqrt{x^2+1}} = 1 - \frac{x^2}{2!} + \frac{9}{4!}x^4 \ldots\ldots\ldots$

Solⁿ. Let $f(x) = \frac{1}{\sqrt{x^2+1}}$

$\therefore f(0) = 1$

Squaring on both sides, we get,

$\therefore \{f(x)\}^2 = \frac{1}{x^2+1}$

or, $(x^2+1)\{f(x)\}^2 = 1 \ldots\ldots\ldots$ (i)

Diffⁿ. w. r. t. x, we get,

$(1+x^2).\,2f(x).\,f'(x) + 2x.\,\{f(x)\}^2 = 0 \ldots\ldots\ldots$ (ii)

Putting $x = 0$

$\therefore 2f(0)\times f'(0) + 2\times 0\,\{f(0)\}^2 = 0$

or, $2\times 1\,f'(0) = 0$

$\therefore f'(0) = 0$

Again, from (ii)

$(1+x^2)\,f'(x) + x\,f(x) = 0$

Again, diffⁿ w. r. t. x n times by using leibnited rule, we get,

$(1+x^2)\,f^{n+1}(x) = {}^nC_1\,f^n(x).\,2x + {}^nC_2\,f^{n-1}(x).\,2 + xf^n(x) + {}^nC_1(x) = 0$

or, $(1+x^2)\,f^{n+1}(x) + n\,f^n(x).\,2x + n(n-1)\,f^{n-1}(x) + xf^n(x) + nf^{n-1}(x) = 0$

or, $(1+x^2)\,f^{n+1}(x) + (2n+1)\,xf^n(x) + (x^2 - n + n)\,f^{n-1}(x) = 0$

or, $(1+x^2)\,f^{n+1}(x) + (2n+1)\,xf^n(x) + n^2f^{n-1}(x) = 0$

Put $n = 0$

$\therefore f^{n+1}(0) + 0 + n^2\,f^{n-1}(0) = 0$

or, $f^{n+1}(0) = -n^2f^{n-2}(0)$

Putting $n = 1, 2, 3, \ldots\ldots\ldots\ldots$

$\therefore f''(0) = -1^2.\,f(0) = -1\times 1 = -1$

$f'''(0) = -2^2.\,f'(0) = -4\times 0 = 0$

$f^{iv}(0) = -3^2.\,f''(0) = -9\times(-1) = 9$

$\therefore$ By maclaurin's series, we have,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}(a) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots$

$\therefore \frac{1}{\sqrt{1+x^2}} = 1 + 0 - \frac{x^2}{3!} + 0 + \frac{x^4}{4!}\times 9 \ldots\ldots\ldots\ldots$

Hence, $\frac{1}{\sqrt{1+x^2}} = 1 - \frac{x^2}{2!} + \frac{9}{4!}x^4 + \ldots\ldots\ldots$

(vi) $\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \ldots\ldots\ldots\ldots$

Solⁿ. Let $f(x) = \frac{x}{e^x - 1}$

When $x = 0 \Rightarrow f(0) = \frac{0}{0}$

$\therefore$ Suppose, $\frac{x}{e^x - 1} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots\ldots$ (i)

or, $x = (e^x - 1)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots)\ldots\ldots\ldots$ (ii)

But $e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) - 1$

$e^x - 1 + x$

$\therefore e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\ldots\ldots\ldots$

Use it in equation (ii), we get,

$$\therefore x = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$$

$$\text{or, } x = a_0x + \left(a_1 + \frac{a_0}{2!}\right)x^2 + \left(a_2 + \frac{a_1}{2!} + \frac{a_0}{3!}\right)x^3$$

$$\left(a_3 + \frac{a_2}{2!} + \frac{a_1}{3!} + \frac{a_4}{4!} + \ldots\right)x^4 + \ldots\ldots\ldots$$

Equating the coefficient of like terms,
We get,

$$a_0 = 1, a_1 = \frac{a_0}{2} = 0,\ a_2 + \frac{a_1}{2!} + \frac{a_0}{3!} = 0 \text{ and } a_3 + \frac{a_2}{2!} + \frac{a_1}{3!} + \frac{a_4}{4!} = 0$$

Solving these equations, we get,

$$a_0 = 1, a_1 = -\frac{1}{2}, a_2 = \frac{1}{12}, a_3 = 0, a_4 = \frac{1}{720}$$

$\therefore$ From (i)

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{2} + \frac{x^2}{12} - \frac{1}{720}x^4 + \ldots\ldots\ldots\ldots$$

8. If $y = e^{a\sin^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$ then show that

(i) $(1-x^2)y_2 - xy_1 - a^2y = 0$

(ii) $(n+1)(n+2)a_{n+2} - (n^2 + a^2)a_n = 0$

and hence obtain the expansion of $e^{a\sin^{-1}x}$.

Sol^n. Given, that $y = e^{a\sin^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$
Put $x = 0$, then $e^0 = a_0$
an $a_0 = 1$

$$\therefore y_1 = \frac{ae^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

By squaring both sides,

$(1-x^2)y_1^2 = a^2(e^{a\sin^{-1}x})^2$
or, $(1-x^2)y_1^2 = a^2y^2$
Diff. we get,
$(1-x^2)2y_1y_2 + (-2x)y_1^2 = a^2 2yy_1$
or, $(1-x^2)y_2 - xy_1 = a^2y$
or, $(1-x^2)y_2 - xy_1 - a^2y = 0$
New, differentiating n times using leibinitz theorem, we get,

$$(1-x^2)y_{n+2} + ny_{n+1}(-2x) + \frac{n(n-1)}{2}(-2)y_n - xy_{n-1} - ny_n - a^2y_n = 0$$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - xy_n - a^2y_n = 0$
or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - n + n + a^2)y_n = 0$
or, $(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 + a^2)y_n$

Also, $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$ and $a_0 = 1$
Now, $y_1 = a_1 + 2a_2x + 3a_3x^2 + \ldots\ldots$
Put $x = 0$
$\therefore (y_1)_0 = a_1$

$$\text{But, } y_1 = \frac{ae^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

$\therefore$ At $x = 0$, $y_1 = a$
So that $(y_1)_0 = a_1 = a$
$y_2 = 2a_2 + 2.3.a_3.x + \ldots\ldots$
$\therefore (y_2)_0 = 2a_2 = 2!\,a_2$
$y_3 = 1.2.3\,a_3 + \ldots\ldots$
$\therefore (y_3)_0 = 3!\,a_3$
$y_n = n!\,a_n \qquad \therefore (y_n)_0 = n!\,a_n$
Again, putting $x = 0$ in (i), we get

$$(y_{n+2})_0 = (n^2 + a^2)(y_n)_0$$

$$\text{or, } (n+2)!\,a_{n+2} = (n^2 + a^2)\frac{n!}{n!}a_n$$

$$\text{or, } (n+2)(n+1)a_{n+2} = (n^2 + a^2)a_n$$

$$\text{or, } (n+2)(n+1)a_{n+2} - (n^2 + a^2)a_n = 0$$

Indeterminant Form

4

We have studied certain rules to evaluated the limits. But some limits can not be evaluated by using these rules. These limits are known as indetermination forms. The are seven type of indeterminate forms given as,

(i) $\frac{0}{0}$ (ii) $\frac{\infty}{\infty}$ (iii) $0 \times \infty$ (iv) $\infty - \infty$

(v) 1^{∞} (vi) 0^{0} (vii) ∞^{0}

These limits can be evaluated by using L'Hospital Rule.

L'Hospital rule

If f(x) and f(x) are two continuous functions such that their derivative f'(x) and g'(x) are continuous at x = a and f(a) = g(a) = 0 then

$$\lim_{x\to a}\frac{f(x)}{f(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} \text{ where } g'(a) \neq 0.$$

(i) $\lim_{x\to 0}\frac{\sin x}{x} = 1$ (ii) $\lim_{x\to 0}\frac{\tan x}{x} = 1$

(iii) $\lim_{x\to 0}\frac{a^x - 1}{x} = \log a$ (iv) $\lim_{x\to 0}\frac{e^x - 1}{x} = 1$

(v) $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$ (vi) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

(vii) $\lim_{x\to 0}\frac{\sin^{-1}x}{x} = 1$ (viii) $\lim_{x\to 0}\frac{\sinh x}{x} = 1$

(ix) $\lim_{x\to 0}\frac{\tan^{-1}x}{x} = 1$

Exercise - 4

1. **Evaluate the following limits.**

(i) $\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x - \sin x}$

Sol^n. Here, $\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x - \sin x}\left[\frac{0}{0}\right]$ from

$$\therefore \lim_{x\to 0}\frac{e^x + e^{-x} - 2}{1 - \cos x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{e^x - e^{-x}}{\sin x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{e^x + e^{-x}}{\cos x}$$

$$\therefore \frac{1+1}{1} = 2$$

(ii) $\lim_{x\to 2}\frac{x^3 - 2x^2 + 2x - 4}{x^2 - 5x + 2}$

Sol^n. $\lim_{x\to 2}\frac{x^3 - 2x^2 + 2x - 4}{x^2 - 5x + 2}\left[\frac{0}{0}\right]$

Here, $\lim_{x\to 2}\frac{3x^2 - 4x + 2}{2x - 5}$

$$= \frac{3\times 2^2 - 4\times 2 + 2}{2\times 2 - 5} = \frac{6}{-1} = -6$$

(iii) $\lim_{x\to 0}\frac{\tan x - x}{x - \sin x}$

Sol^n. Here, $\lim_{x\to 0}\frac{\tan x - x}{x - \sin x}\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{\sec^2 x - 1}{1 - \cos x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{2\sec x \cdot \sec x \cdot \tan x}{\sin x}$$

$$= \lim_{x\to 0}\frac{2\sec^2 x \cdot \sin x}{\sin x \cos x}$$

$$= 2$$

(iv) $\lim_{x\to 0}\frac{x\cos x - \log(1+x)}{x^2}$

Sol^n. Here, $\lim_{x\to 0}\frac{x\cos x - \log(1+x)}{x^2}\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{-x\sin x + \cos x - \frac{1}{1+x}}{2x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\sin x - x\cos x - \sin x + \frac{1}{(1+x)^2}}{2}$$

$$= \frac{1}{2}$$

(v) $\lim_{x\to 0} \frac{x-\sin^{-1}x}{\sin^3 x}$

Sol^n. $\lim_{x\to 0} \frac{x-\sin^{-1}x}{\left(\frac{\sin x}{x}\right)^3 x^3}$

$$= \lim_{x\to 0} \frac{x-\sin^{-1}x}{x^3}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{1-\frac{1}{\sqrt{1-x^2}}}{3x^2}$$

$$= \lim_{x\to 0} \frac{1-\left(1-x^2\right)^{-\frac{3}{2}}}{3x^2}$$

$$= \lim_{x\to 0} 0-\frac{\frac{1}{2}(1-x^2)^{\frac{-3}{2}}(-2x)}{6x}$$

$$= \lim_{x\to 0} \frac{-(1-x^2)^{-\frac{3}{2}}}{6}$$

$$= -\frac{1}{6}$$

(vi) $\lim_{x\to 0} \frac{2\sin x-\sin 2x}{\tan^3 x}$

Sol^n. Here, $\lim_{x\to 0} \frac{2\sin x-\sin 2x}{\left(\frac{\tan x}{x}\right)^3 . x^3}$

$$= \lim_{x\to 0} \frac{2\sin x-\sin 2x}{x^3}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{2\cos x-2\cos 2x}{3x^2}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-2\sin x+4\sin 2x}{6x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-2\cos x+8\cos 2x}{6}$$

$$= \frac{-2+8}{6} = \frac{6}{6} = 1$$

ii) $\lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$

ol^n. $\lim_{x\to 0} \frac{\tan x - x}{x^3 . \left(\frac{\tan x}{x}\right)}$

$$= \lim_{x\to 0} \frac{\tan x - x}{x^3}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{1}{3}\sec^2 x \frac{\tan x}{x}$$

$$= \lim_{x\to 0} \frac{\sec^2 x}{3}$$

$$= \frac{1}{3} \qquad \left[\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$$

$$= 1$$

viii) $\lim_{x\to 1} \frac{1+\log x - x}{1-2x+x^2}$

ol^n. Here, $\lim_{x\to 1} \frac{1+\log x - x}{1-2x+x^2}\left[\frac{0}{0}\right]$

$$= \lim_{x\to 1} \frac{0+\frac{1}{x}-1}{0-2+2x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 1} \frac{-\frac{1}{x^2}}{2} = -\frac{1}{2}$$

ix) $\lim_{x\to 0} \frac{\cosh x - \cos x}{x\sin x}$

Sol^n. $\lim_{x\to 0} \frac{\cosh x - \cos x}{x^2\left(\frac{\sin x}{x}\right)}$

$$= \lim_{x\to 0} \frac{\cosh x - \cos x}{x^2}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\sinh x + \sin x}{2x} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\cosh x + \cos x}{2}$$

$$= \frac{2}{2} = 1$$

(x) $\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

Sol^n. Here, $\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} \left[\frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{e^x - e^{\sin x}.\cos x}{1 - \cos x} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{e^x - e^{\sin x}\cos^2 x + e^{\sin x}\sin x}{\sin x} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{e^x - e^{\sin x}\cos^3 x + e^{\sin x}\sin 2x + e^{\sin x}\cos x \sin x + e^{\sin x}\cos}{\cos x}$$

$$= \frac{1-1+0+0+1}{1} = 1$$

(xi) $\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3}$

Sol^n. Here, $\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3} \left[\frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{3x^2} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\sin x - \sin x - x\cos x + \sin x}{6x} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\cos x - \cos x - \cos x + x\sin x + \cos x}{6}$$

$$= \frac{-1-1-1+0+1}{6} = \frac{-2}{6} = -\frac{1}{3}$$

(xii) $\lim_{x\to 0} \frac{x\sin x - \sin^2 x}{x^4}$

Sol^n. Here, $\lim_{x\to 0} \frac{x\sin x - \sin^2 x}{x^4} \left[\frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\sin x + x\cos x - \sin 2x}{4x^3} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\cos x + \cos x - x\sin x - 2\cos 2x}{12x^2} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\sin x - \sin x - \sin x - x\cos x + 4\sin 2x}{24x} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\cos x - \cos x - \cos x - \cos x + x\sin x + 8\cos 2x}{24}$$

$$= \frac{-1-1-1-1+8}{24} = \frac{4}{24} = \frac{1}{6}$$

(xiii) $\lim_{x\to 0} \frac{xe^x - (1+x)\log(1+x)}{x^2}$

Sol^n. Here, $\lim_{x\to 0} \frac{xe^x - (1+x)\log(1+x)}{x^2}$

$$= \lim_{x\to 0} \frac{xe^x + e^x - (1+x)\log(1+x)}{x^2}$$

$$= \lim_{x\to 0} \frac{xe^x + e^x - (1+x).\frac{1}{1+x} - \log(1+x)}{2x}$$

$$= \lim_{x\to 0} \frac{xe^2 + e^x - 1 - \log(1+x)}{2x} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{xe^x + e^x + e^x - 0 - \frac{1}{1+x}}{2}$$

$$= \frac{0+1+1-1}{2} = \frac{1}{2}$$

2. Evaluate the following limits.

(i) $\lim_{x\to 0} \frac{\log \tan x}{\cot x}$

Sol^n. Here, $\lim_{x\to 0} \frac{\log \tan x}{\cot x} \left[\frac{\infty}{\infty}\right]$

$$= \lim_{x\to 0} \frac{\frac{1}{\tan x}.\sec^2 x}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to 0} \frac{\frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x}}{-\frac{1}{\sin^2 x}}$$

$$= \lim_{x\to 0} -\frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x} \times \frac{\sin^2 x}{1}$$

$$= \lim_{x\to 0} \tan x$$

$$= 0$$

(ii) $\lim\limits_{x\to\frac{\pi}{2}} \frac{\tan 5x}{\tan x}$

Sol^n. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\tan 5x}{\tan x}\left[\frac{0}{0}\right]$

$$= \lim_{x\to\frac{\pi}{2}} \frac{5\sec^2 5x}{\sec^2 x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{5\cos^2 x}{\cos^2 5x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-10\cos x\sin x}{-2\cos 5x\sin 5x\,.5}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\sin 2x}{\sin 10x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{2\cos 2x}{10\cos 10x} = \frac{1}{5}$$

(iii) $\lim\limits_{x\to 0} x\log(\tan x)$

Sol^n. $\lim\limits_{x\to 0} \frac{\log(\tan x)}{\frac{1}{x}}\left[\frac{\infty}{\infty}\right]$

$$= \lim_{x\to 0} \frac{\frac{1}{\tan x}\sec^2 x}{-\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{\sec^2 x\, x}{\tan x\; 1}$$

$$= \lim_{x\to 0} \sec^2 x \left[\lim_{x\to 0}\frac{x}{\tan x} = 1\right]$$

$= 1$

(iv) $\lim\limits_{x\to 1} (a-x)\tan\left(\frac{\pi x}{2a}\right)$

Sol^n. $\lim\limits_{x\to 1} \frac{a-x}{\cot\frac{\pi x}{2a}}\left[\frac{0}{0}\right]$

$$= \lim_{x\to 1} \frac{-1}{-\operatorname{cosec}^2\frac{\pi x}{2a}\cdot\frac{\pi}{2a}} = \frac{2a}{\pi}$$

(v) $\lim\limits_{x\to 0} x\log x$

Sol^n. $\lim\limits_{x\to 0} x\log x$

$$= \lim_{x\to 0}\frac{\log x}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0}\frac{1}{-x}\times\frac{x^2}{1} = \lim_{x\to 0} -x = 0$$

(vi) $\lim\limits_{x\to 0} x\log\sin^2 x$

Sol^n. $\lim\limits_{x\to 0} x\log\sin^2 x$

$$= \lim_{x\to 0}\frac{\log\sin^2 x}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{1}{\sin^2 x}2\sin x\cos x}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0} -\frac{x^2\sin 2x}{\sin^2 x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{-2x\sin 2x - 2x^2\cos 2x}{\sin 2x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{-2\sin 2x - 4x\cos 2x - 4x\cos 2x + 4x^2\sin 2x}{2\cos 2x}$$

$$= \frac{0}{1} = 0$$

(vii) $\lim\limits_{x\to 0} \log_{\tan x}\tan 2x$

Sol^n. $\lim\limits_{x\to 0}\frac{\log\tan 2x}{\log\tan x}\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{\frac{1}{\tan 2x}2\sec^2 2x}{\frac{1}{\tan x}\sec^2 x} = \lim_{x\to 0}\frac{\frac{\cos 2x}{\sin 2x}\times\frac{2}{\cos^2 2x}}{\frac{1}{\cos x}\times\frac{\cos x}{\sin x}}$$

$$= \lim_{x\to 0}\frac{2\cos x\sin x}{\sin 2x\cos 2x} = \lim_{x\to 0}\frac{\sin 2x}{\sin 2x\cos 2x} = 1$$

3. Evaluate the following limits

(i) $\lim\limits_{x\to\frac{\pi}{2}} (\sec x - \tan x)$

Sol^n. $\lim\limits_{x\to\frac{\pi}{2}}\left[\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right] = \lim\limits_{x\to\frac{\pi}{2}}\left[\frac{1-\sin x}{\cos x}\right] = \frac{0}{0}$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{\sin x} = \frac{\cos 90}{\sin 90} = \frac{0}{1} = -1$$

(ii) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

Sol^n. $\lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)} = \lim_{x\to 0}\frac{e^x-1}{e^x-1+xe^x}\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{e^x}{e^x+e^x+xe^x} = \frac{1}{1+1} = \frac{1}{2}$$

(iii) $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$

Sol^n. $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right) = \lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cos^2 x}{\sin^2 x}\right)$

$$= \lim_{x\to 0}\frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x}$$

$$= \lim_{x\to 0}\frac{\sin^2 x - x^2\cos^2 x}{x^4}\times \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2$$

$$= \lim_{x\to 0}\frac{\sin^2 x - x^2\cos^2 x}{x^4}\times 1^2$$

$$= \lim_{x\to 0}\frac{\sin 2x - 2x\cos^2 x + x^2\sin 2x}{4x^3}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{2\cos 2x - 2\cos^2 x + 2x\sin 2x + 2x\sin 2x + 2x^2\cos 2x}{12x^2}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{-2\cos 2x + 4\sin 2x + 8x\cos 2x + 4x\cos 2x - 4x^2\sin 2x}{24x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{2\sin 2x + 12x\cos 2x - 4x^2\sin 2x}{24x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{4\cos 2x + 12\cos ex - 24x\sin 2x - 8x\sin 2x - 8x^2\cos 3x}{24}$$

$$= \lim_{x\to 0}\frac{4+12-0-0-0}{24} = \frac{16}{24} = \frac{2}{3}$$

(iv) $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]$

Sol^n. $\lim_{x\to 0}\frac{x-\log(1+x)}{x^2}\left[\frac{0}{0}\right] = \lim_{x\to 0}\frac{1-\frac{1}{1+x}}{2x}\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{\frac{1}{(1+x)^2}}{2} = \lim_{x\to 0}\frac{1}{2(1+x)^2} = \frac{1}{2}$$

(v) $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{\log x}\right)$

Sol^n. $\lim_{x\to 1}\frac{\log x - x + 1}{\log x(x-1)}\left[\frac{0}{0}\right]$

$$= \lim_{x\to 1}\frac{\frac{1}{x}-1}{\frac{1}{x}(x-1)+\log x} = \lim_{x\to 1}\frac{\frac{1-x}{x}}{\frac{(x-1)+x\log x}{1}}$$

$$= \lim_{x\to 1}\frac{1-x}{(x-1)+x\log x}\left[\frac{0}{0}\right] = \lim_{x\to 1}\frac{-1}{1+\log x + x\cdot\frac{1}{x}}$$

$$= \lim_{x\to 1}\frac{-1}{2+\log x} = -\frac{1}{2}$$

4. Evaluate

(i) $\lim_{x\to\pi}(\sin x)^{\tan x}$

Sol^n. Let $y = (\sin x)^{\tan x}$

Taking log on both sides, we get,

$\log y = \tan x \log \sin x$

Taking limit as $x \to \pi$ on both sides,

$$\lim_{x\to\pi}\log y = \lim_{x\to\pi}[\tan x \,.\, \log \sin x]$$

$$= \lim_{x\to\pi}\left[\frac{\log\sin x}{\cot x}\right]\left(\frac{\infty}{\infty}\text{ from}\right)$$

$$\text{or, } \log\left(\lim_{x\to\pi} y\right) = \lim_{x\to\pi}\frac{\frac{1}{\sin x}.\cos x}{-\operatorname{cosec}^2 x} = \lim_{x\to\pi}\left[\frac{-\cos x}{\operatorname{cosec} x}\right]$$

$$= \lim_{x\to\pi}[-\sin x\cos x] = \lim_{x\to\pi}\left[-\frac{\sin 2x}{2}\right] = \frac{0}{2} = 0$$

$$\therefore \log\left(\lim_{x\to\pi} y\right) = 0$$

$$\lim_{x\to\pi} y = e^0$$

$$\therefore \lim_{x\to\pi}(\sin x)^{\tan x} = 1$$

(ii) $\lim_{x\to 0} x^{2\sin x}$

Sol^n. Here, $\lim_{x\to 0} x^{2\sin x}$

Let, $y = x^{2\sin x}$

Taking log on both sides,

$\log y = 2\sin x \log x$

Taking limit as $x \to 0$ on bothsides

$$\lim_{x\to 0} (\log y) = \lim_{x\to 0} [2\sin x \log x]$$

$$= \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0}\left[\frac{2\log x}{\operatorname{cosec} x}\right]\left(\frac{\infty}{\infty}\text{form}\right)$$

$$= \lim_{x\to 0}\left[\frac{2\cdot\frac{1}{x}}{-\operatorname{cosec} x\cot x}\right] = \lim_{x\to 0}\frac{-2\sin^2 x}{x\cos x}$$

$$= \lim_{x\to 0}\left[\frac{\cos 2x - 1}{x\cos x}\right] = \lim_{x\to 0}\left[\frac{-2\sin 2x}{\cos x - x\sin x}\right] = \frac{2x}{1}$$

$$\log\left(\lim_{x\to 0} y\right) = 0$$

$$\therefore \lim_{x\to 0} y = e^0 = 1$$

$$\therefore \lim_{x\to 0} x^{2\sin x} = 1$$

(iii) $\lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x}$

Sol^n. Let $y = \left(\frac{1}{x^2}\right)^{\tan x}$

Taking log on both sides, we get,

$$\log y = \tan x \log\left(\frac{1}{x^2}\right)$$

Taking limit as $x \to 0$

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \tan x\,[\log 1 - \log x^2]$$

$$= \lim_{x\to 0} \tan x\,[-2\log x]$$

$$\text{or, } \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0}\frac{-2\log x}{\cot x}\left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0}\frac{-\frac{2}{x}}{-\operatorname{cosec}^2 x} = \lim_{x\to 0}\frac{2}{x}\sin^2 x$$

$$= \lim_{x\to 0} 2\sin x\cdot\left(\frac{\sin x}{x}\right) = \lim_{x\to 0} 2\sin x = 0$$

$$\text{or, } \log\left(\lim_{x\to 0} y\right) = 0$$

$$\therefore \lim_{x\to 0} y = e^0$$

$$\text{or, } \lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x} = 1$$

(iv) $\lim_{x\to 0} (\cot x)^{\frac{1}{\log x}}$

Sol^n. Let $y = (\cot x)^{\frac{1}{\log x}}$

Taking log on both sides we get,

$$\log y = \frac{1}{\log x}\log\cot x$$

Taking limit as $x \to 0$

$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0}\frac{\log\cot x}{\log x}\left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0}\frac{\frac{1}{\cot x}\cdot(-\operatorname{cosec}^2 x)}{\frac{1}{x}}$$

$$= \lim_{x\to 0} -\frac{\sin x}{\cos x}\cdot\frac{x}{\sin^2 x} = \lim_{x\to 0}\frac{2x}{\sin 2x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{-2}{2\cos 2x} = \frac{-2}{2} = -1$$

$$\therefore \log\left(\lim_{x\to 0} y\right) = -1$$

$$\lim_{x\to 0} y = e^{-1}$$

$$\therefore \lim_{x\to 0} (\cot x)^{\frac{1}{\log x}} = e^{-1}$$

(v) $\lim_{x\to\infty} x^{\frac{1}{x}}$

Sol^n. Let $y = x^{\frac{1}{x}}$

Taking log on both sides, we get

$$\log y = \frac{1}{x}\log x$$

Taking limit as $x \to 0$

$$\therefore \lim_{x\to\infty} \log y = \lim_{x\to\infty}\frac{\log x}{x}$$

$$\text{or, } \log\left(\lim_{x\to\infty} y\right) = \lim_{x\to\infty}\frac{\log x}{x}\left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{\frac{1}{x}}{1}$$

$$= \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

$$\therefore \log\left(\lim_{x\to\infty} y\right) = 0$$

$$\lim_{x\to\infty} y = e^0$$

$$\therefore \lim_{x\to\infty} x^{\frac{1}{x}} = 1$$

(vi) $\lim_{x\to 0} (\cot^2 x)^{\sin x}$

Sol^n. Let $y = (\cot^2 x)^{\sin x}$

Taking log on both sides, we get,

$\log y = \sin x \log(\cot^2 x)$

Taking limit as $x \to 0$

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log \cot^2 x}{\operatorname{cosec} x}\left[\frac{\infty}{\infty}\right]$$

$$\text{or, } \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{\frac{1}{\cot^2 x}.2\cot x.(-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x \cot x}$$

$$= \lim_{x\to 0} \frac{2\cot x.\operatorname{cosec}^2 x}{\cot^2 x \operatorname{cosec} x \cot x} = \lim_{x\to 0} \frac{2}{\sin x} \times \frac{\sin^2 x}{\cos^2 x}$$

$$= \lim_{x\to 0} \frac{2\sin x}{\cos^2 x} = \frac{0}{1} = 1$$

$$\therefore \lim_{x\to 0} y = 0^0$$

$$\lim_{x\to 0} (\cot^2 x)^{\sin x} = 1$$

5. Evaluate

(i) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

Sol^n. Let $y = (\cos x)^{\frac{1}{x^2}}$

Taking log on both sides, we get,

$\log y = \frac{1}{x^2} \log \cos x$

Taking limit as $x \to 0$

$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log \cos x}{x^2}$$

$$\text{or, } \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{\log \cos x}{x^2}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x} = \lim_{x\to 0} \frac{-\sin x}{2x \cos x}$$

$$= \lim_{x\to 0} -\frac{1}{2\cos x}$$

$$\log\left(\lim_{x\to 0} y\right) = -\frac{1}{2}$$

$$\therefore \lim_{x\to 0} y = e^{-\frac{1}{2}}$$

$$\lim_{x\to 0} (\cos x)^{\frac{1}{x}} = e^{-\frac{1}{2}}$$

(ii) $\lim_{x\to \frac{\pi}{2}} (\sin x)^{\tan x}$

Sol^n. Let $y = (\sin x)^{\tan x}$

Taking $\log y = \tan x \log \sin x$

Taking limit as $x = \frac{\pi}{2}$

$$\lim_{x\to \frac{\pi}{2}} \log y = \lim_{x\to \frac{\pi}{2}} \tan x \log \sin x$$

$$\text{or, } \log\left(\lim_{x\to \frac{\pi}{2}} y\right) = \lim_{x\to \frac{\pi}{2}} \frac{\log \sin x}{\cot x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\frac{1}{\sin x}\cos x}{-\operatorname{cosec}^2 x} = \lim_{x\to \frac{\pi}{2}} \frac{-\cos x}{\sin x} \times \frac{\sin^2 x}{1}$$

$$= \lim_{x\to \frac{\pi}{2}} -\cos x \sin x = -\cos\frac{\pi}{2}$$

$$\therefore \log\left(\lim_{x\to \frac{\pi}{2}} y\right) = 0$$

$$\lim_{x\to \frac{\pi}{2}} y = e^0$$

$$\therefore \lim_{x\to \frac{\pi}{2}} (\sin x)^{\tan x} = 1$$

(iii) $\lim_{x\to 0} (\cos x)^{\cot^2 x}$

Sol^n. Let $y = (\cos x)^{\cot^2 x}$

Taking log on both sides, we get,

$\log y = \cot^2 x \log \cos x$

Taking limit as $x \to 0$

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \cot^2 x \log \cos x$$

$$\text{or, } \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{\log \cos x}{\tan^2 x} = \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2\tan x \sec^2 x}$$

$$= \lim_{x\to 0} \frac{-\tan x}{2\tan x \sec^2 x} = \lim_{x\to 0} \frac{-1}{2\sec^2 x}$$

$$\log\left(\lim_{x\to 0} y\right) = -\frac{1}{2}$$

$$\therefore \lim_{x\to 0} y = e^{-\frac{1}{2}}$$

$$\lim_{x\to 0} (\cos x)^{\cot^2 x} = e^{-\frac{1}{2}}$$

(iv) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Sol^n. Let $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Taking log on both sides, we get,

$$\log y = \frac{1}{x^2} \log\left(\frac{\tan x}{x}\right)$$

$$\text{or, } \log y = \frac{\log\left(\frac{\tan x}{x}\right)}{x^2}$$

Taking limit as $x \to 0$

$$\lim_{x\to 0} (\log y) = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x^2}$$

$$\text{or, } \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{\frac{1}{\frac{\tan x}{x}}\left[\frac{x\sec^2 - \tan x}{x^2}\right]}{2x}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{2x^3} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\sec^2 x + 2x\sec^2 x \tan x - \sec^2 x}{6x^2}$$

$$= \lim_{x\to 0} \frac{\sec^2 x \tan x}{3x}$$

$$= \lim_{x\to 0} \frac{\sec^2 x}{3}\left(\frac{\tan x}{x}\right)$$

$$\log\left(\lim_{x\to 0} y\right) = \frac{1}{3} \quad \left[\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$$

$$\therefore \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$$

(v) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Sol^n. Let $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Taking log on both sides, we get,

$$\log y = \frac{1}{x}\log\left(\frac{\tan x}{x}\right)$$

Taking limit as $x \to 0$

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x}$$

$$\text{or, } \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{\frac{1}{\frac{\tan x}{x}}\left[\frac{x\sec^2 x - \tan x}{x^2}\right]}{1}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\sec^2 x + 2x\sec^2 x \tan x - \sec^2 x}{2x}$$

$$= \lim_{x\to 0} \frac{2x\sec^2 x \tan x}{2x}$$

$$= \lim_{x\to 0} \sec^2 x \tan x$$

$$\log\left(\lim_{x\to 0} y\right) = 0$$

$$\therefore \lim_{x\to 0} y = e^0.$$

$$\therefore \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}} = 1$$

(vi) $\lim_{x\to a} \left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$

Sol^n. Let $y = \left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$

Taking log on both sides, we get,

$$\log y = \tan\frac{\pi x}{2a} \log\left(2-\frac{x}{a}\right)$$

taking limit as $x \to a$

$$\therefore \lim_{x\to a} \log y = \lim_{x\to a} \frac{\log\left(2-\frac{x}{a}\right)}{\cot\frac{\pi x}{2a}}$$

$$= \lim_{x\to a} \frac{\frac{1}{2-\frac{x}{a}}\left[-\frac{1}{a}\right]}{-\operatorname{cosec}^2\frac{\pi x}{2a}\cdot\frac{\pi}{2a}} = \frac{1}{a}\cdot\frac{2a}{\pi} = \frac{2}{\pi}$$

$$\therefore \lim_{x\to a} \log y = \frac{2}{\pi}$$

$$\log\left(\lim_{x\to a} y\right) = \frac{2}{\pi}$$

$$\therefore \lim_{x\to a} y = e^{\frac{2}{\pi}}$$

$$\therefore \lim_{x\to a} \left(2-\frac{x}{a}\right)^{\tan\frac{\pi a}{2a}} = e^{\frac{2}{\pi}}$$

(vii) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

Sol^n. Let $y = \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

Taking log on both sides, we get,

$$\log y = \frac{1}{x}\log\left(\frac{\sin x}{x}\right)$$

Taking limit as $x \to 0$

$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x}$$

$$= \lim_{x\to 0} \frac{x\cos x - \sin x}{x^2}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{2x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\sin x - x\cos x}{2}$$

$$= \frac{0}{2} = 0$$

$$\therefore \log\left(\lim_{x\to 0} y\right) = 0$$

$$\lim_{x\to 0} y = e^0$$

$$\therefore \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = 1$$

6.

(i) If $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$ be finite, find the value of a and the limit.

Sol^n. Here, $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2}$$

When $x = 0$ it takes $\frac{2+a}{0} = \frac{0}{0}$ form

$\therefore 2 + a = 0 \Rightarrow a = -2$

Then the limit is $\lim_{x\to 0} \frac{2\cos x2x + a\cos x}{3x^2}$

$$= \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x}$$

$$= \lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6}$$

$$= \frac{-8+2}{6} = \frac{-6}{6} = 1$$

(ii) **If $\lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x}$ be finite, then the value of 'a' and the limit.**

Sol^n. $\lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x}$

$$= \lim_{x\to 0} \frac{a\sin x - \sin 2x}{x^3\left(\frac{\tan x}{x}\right)^3}$$

$$= \lim_{x\to 0} \frac{a\sin x - \sin 2x}{x^3}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{a\cos x - 2\cos 2x}{3x^2} \quad \text{..... (i)}$$

Here, when $x \to 0$ the numinator is not zero but denominater is zero.

Let finite limit, numinator should be are

$\therefore \lim_{x\to 0} a\cos x - 2\cos 2x = 0 \quad [\because a - 2 = 0 \Rightarrow a = 2]$

Then from (i), we get,

$$\lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{3x^2}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{6x}\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{6}$$

$$= \frac{-2+8}{6} = \frac{6}{6} = 1$$

(iii) **If $\lim_{x\to 0} \frac{(a+b\cos x)x - c\sin x}{x^5} = 1$ then find the value of a, b and c**

Sol^n. Given, $\lim_{x\to 0} \frac{(a+b\cos x)x - c\sin x}{x^5} = 1\left[\frac{0}{0}\right]$

or, $\lim_{x\to 0} \frac{(a+b\cos x) - bx\sin x - c\cos x}{5x^4} = 1 \quad \text{..... (i)}$

or, $\frac{a+b-c}{0} = 1$

or, $a + b - c = 0 \quad \text{..... (ii)}$

Then (i) gives,

$$\lim_{x\to 0} \frac{a + b\cos x - bx\sin x - a\cos x - b\cos x}{5x^4} = 1\left(\frac{0}{0}\text{ from}\right)$$

or, $\lim_{x\to 0} \frac{-bx\cos x - b\sin x + a\sin x}{20x^3} = 1\left(\frac{0}{0}\text{ from}\right)$

or, $\lim_{x\to 0} \frac{-b\sin x - b\cos x + bx\sin x - b\cos x + a\cos x}{60x^2} = 1\left(\frac{0}{0}\text{ from}\right)$

or, $\frac{-2b+a}{0} = 1$

or, $a = 2b$

Then equation (iii) gives,

$$\lim_{x\to 0} \frac{-2b\cos x + bx\sin x + 2b\cos x}{60x^2} = 1$$

or, $\lim_{x\to 0} \frac{bx\cos x + b\sin x}{120x} = 1\left(\frac{0}{0}\text{ from}\right)$

or, $\frac{2b}{120} = 1$

or, $2b = 120$

$b = 60$

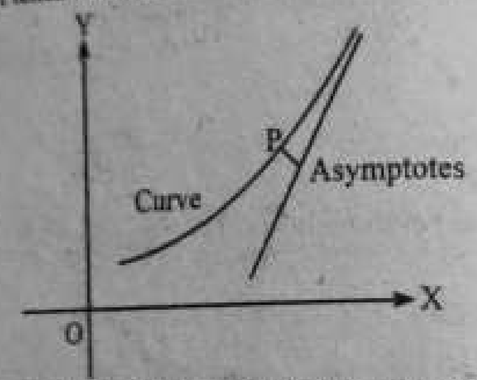
The from equation (iv) $a = 120$ from (ii) $c = 180$

$$\left.\begin{aligned} a &= 120 \\ \therefore b &= 60 \\ c &= 180 \end{aligned}\right\}$$

Asymptotes

5

A straight line is called an asymptotes of an infinite curve, t perpendicular distance of a point P on the curve form the straigh tends to zero as P tends to infinity along the curve.



Note : A curve which is finitely bounded has no asymptotes at all.

The asymptotes are classified into two groups

(i) Parallel asymptoes and

(ii) Oblique asymptots

Determination of Parallel Asymptotes

1. Horizontal asymptotes

Rule - I : If a line $y = b$ is an asymptotes to the curve $y - f(x)$ parallel axis, then $y \to b$ as $x \to \infty$

i.e. $\lim_{x \to \infty} y = b$ or, $\lim_{x \to \infty} f(x) = b$

Rule - II : In an algebrais curve of degree n, if x^n is absent the asymptotes parallel to X-axis can be obtained by equating the coeff of highest degree therm in X to zero.

2. Vertical asymptotes

Rule - I : If a line $x = a$ is an asymptotes to the curve $y = f(x)$, parall y-axis, then $y \to \infty$ as $x \to a$

i.e. $\lim_{x \to a} y = \infty$ or $\lim_{x \to a} f(x) = \infty$

Rule - II : In an algebraic curve of degree n if y^n is absent parallel to y-axis, then the asymptotes parallel to y-axis (Vertical asymptotes) can be obtained by equating the coefficient of highest degree term in y to zero.

Oblique asymptoes

In an algebraic curve of degree n if x^n, y^n and both present there is oblique asymptoes which is of the form $y = mx + c$. Where we have to find the value of m and c. For,

(i) Find $\phi_n(m)$ by putting $x = 1$ and $y = m$ in the highest degree terms of the equation.

(ii) Solve $\phi_n(m) = 0$ and let its roots $m_1, m_2, m_3 \ldots$

(iii) Find $\phi_{n-1}(m)$ by putting $x = 1$ and $y = m$ in the next-lower degree term. Similarly $\phi_{n-2}(m)$

(iv) For the value of $C_1, C_2 \ldots$ use $C_1 = \dfrac{-\phi_{n-1}(m)}{\phi_n'(m)}$

Then required asymtotesare $y = m_1x + C_1$, $y = m_2x + C_2$

(v) If the roots are repeated, then, we use the formula to find the roots,

$$\frac{C^2}{2}\phi_n''(m) + C\,\phi_{n-1}(m) + \phi_{n-2}(m) = 0$$

(vi) If these roots are equal, we use the formula to find the C.

$$\frac{C^3}{3!}\phi_n''(m) + \frac{C^2}{2!}\phi_{n-1}''(m) + \frac{C}{1!}\phi_{n-2}'(m) + \phi_{n-3}(m) = 0$$

If more then 3 –values of m are repeated, the procesure as above applied.

Exercise - 5

1. Find the asymptotes of the followings.

(i) $y = \dfrac{3x}{x-2}$

Solⁿ. Given, $y = \dfrac{3x}{x-2}$

or, $xy - 2y = 3x$

or, $3x - xy + 2y = 0$

Thus, equation of second degree and x^2 and y^2 are absent.

So these the asyptotes parallel to x and y-axies.

To find the asymptotes, parallel to x-axis

$\therefore y - 3 = 0 \Rightarrow y = 3$

Again, to find the asymptotes parallel to x-axis

$x - 2 = 0$

$x = 2$

$\left.\begin{array}{l} x = 2 \\ \therefore y = 3 \end{array}\right\}$ are required asymptoes.

(ii) $y = \dfrac{4x^2 + 4x - 3}{x^2 - 4x + 3}$

Sol^n. $x^3y - 4xy + 3y - 4x^2 - 4x + 3 = 0$

This equation is of degree third also,

x^3 and y^3 are both absent,

therefore these are asymptotes parallel to both axes.

To find the asymptotes parallel tox-axis

$y - 4 = 0 \Rightarrow y = 4$

Again, The asymptotes parallel to y-axis is,

$x^2 - 4x + 3 = 0$

or, $x^2 - 3x - x + 3 = 0$

or, $x(x-3) - 1(x-3) = 0$

or, $(x-1)(x-3) = 0$

$\therefore x = 1, 3$

Thus, required asymptoes are $x = 1$ and $x = 3$, $y = 4$

(iii) $y = \dfrac{x^2 + 2x - 1}{x}$

Sol^n. Here, $y = \dfrac{x^2 + 2x - 1}{x}$

or, $xy - x^2 - 2x + 1 = 0$

This equation is of degree 2nd.

Since, x^2 is present there is oblique asympotes

$\therefore$ Put $x = 1$ and $y = m$ in 2nd degree and first degree term, we

$\phi_2(m)\ 1 - m$

or, Now, $\phi_2(m) = 0$ gives $m = 1$

and $\phi_1(m) = 2 \times 1 = 2$

Now, $C = \dfrac{-\phi_1(m)}{\phi_2(m)} = \dfrac{-2}{-1} = 2$

The oblique asymptotes is

$y = mx + C$

$y = 1 . x + 2$

$\therefore y = x + 2$

Thus, the required asymptotes are $y = x + 2$

and since y^2 is absent

These is asymptotes parallel to y-axis,

$-x = 0 \Rightarrow x = 0$

$\therefore$ Required asymptotes is $x = 0$

(iv) $y = \dfrac{x^2}{x^2 + 1}$

Sol^n. $x^2y + y = x^2$

or, $x^2y - x^2 + y = 0$

Thus equation is of degree third,

Since, x^3 and y^3 is absent, there is asymptotes parallel to x-axis and y-axis.

The asymptoes parallel to x-axis are obtained by equating the coefficient of highest degree term of x to zero.

i.e., $y - 1 = 0 \Rightarrow y = 1$

Again, equating the coefficient of highest degree term of y to zero,

i.e. $x^2 + 1 = 0$ $\quad x = \pm i$

Which doesn't gives the real asymptotes.

$\therefore$ The equation of asymptoes is $y = 1$

(v) $y = \dfrac{(x+2)^2 (x-3)}{x-1}$

Sol^n. $y = \dfrac{(x^2 + 4x + 4)(x-3)}{x-1}$

or, $x^3 + x^2 - 8x - 12 - xy + y = 0$

or, $x^3 + x^2 - xy - 8x + y - 12 = 0$

This equation is of degree third, so it has 3 asymptotes. Since, x^3 is present and y^3 is absent these is adymptoes parallel to y-axis.

i.e. $-x = 1$ $\quad \therefore x = 1$

Again, due to the present of x^3 in (i) there is asymptoes of the form $y = mx + C$

Put $x = 1$ and $y = m$ in third and second degree term.

$\therefore \phi_3(m) = 1$

$\therefore \phi_2(m) = 0 \Rightarrow 1 = 0$ which is impossible.

$\therefore$ Only the asymptoes is $x = 1$

Also, $\phi_3^1(m) = -3m^2$

$\therefore C = \dfrac{-\phi_2(m)}{\phi_3^1(m)} = \dfrac{-(-3a)}{-3m^2} = \dfrac{-a}{m^2}$

or, $C = \dfrac{-a}{1^2} = -a$

$\therefore$ The asymptotes is $y = mx + C$

i.e. $y = x - a$

Hence, $y = x - a$ is only an oblique asymptoes.

(vi) $x^3 - y^3 = 3ax^2$

Sol^n. Given, $x^3 - y^3 = 3ax^2$

or, $x^3 - y^3 - 3ax^2 = 0$ ..... (i)

This equation is of degree third and x^3 and y^3 are both present. So, these are no asymptotes parallel to x and y axes. To find the oblique asymptotes of the form $y = mx + C$,

Put $x = 1$ and $y = m$ in the third and second degree term, we get,

$\phi_3(m) = 1 - m^3$

and $\phi_2(m) = -3a$

Now, $\phi_3(m) = 0$ gives,

$1 - m^3 = 0$

or, $(m-1)(m^2 = m + 1) = 0$

$\therefore$ Only real value is $m = 1$

Also, $\phi_3^1(m) = -3m^2$

$$\therefore C = \frac{-\phi_2(m)}{\phi_3^1(m)} = \frac{-(-3a)}{-3m^2} = \frac{-a}{m^2}$$

or, $C = \frac{-a}{1^2} = -a$

$\therefore$ The asymptote is $y = mx + C$

i.e., $y = x - a$

Hence, $y = x - a$ is only an asymptotes.

(vii) $y(y-1)^2 - x^2y = 0$

Solⁿ. Here, given equation is,

$y(y-1)^2 - x^2y = 0 \Rightarrow y(y^2 - 2y + 1) - x^2y = 0$

or, $y^3 - 2y^2 + y - x^2y = 0$

or, $y^3 - x^2y - 2y^2 + y = 0$

This equation is of degree third,

So, it has must there asymptotes,

Since, y^2 is present, there is no asymptoes parallel to y-axis.

since, x^3 is absent, these is asymptoes parallel to x-axis.

So, equating the coefficient of highest degree term of x to zero.

$\therefore -y = 0 \Rightarrow y = 0$

Since, y^2 is present so,

There is of the form $y = mx + C$ is an oblique asymptoes.

To find the value of m and C put $x = 1$ and $y = m$ in the third degree and second degree term.

$\therefore \phi_3(m) = m^3 - 1^2 . m \Rightarrow \phi_3^1(m) = 3m^2 - 1$

$\therefore \phi_3(m) = 0 \Rightarrow m^3 - m = 0$

$m(m^2 - 1) = 0$

$m = 0, m^2 = 1$

$\therefore m = \pm 1$

$\therefore m = 0, m = -1$

Again, $\phi_2(m) = -2m^2$

$$\text{Now, } C = \frac{-\phi_2(m)}{\phi_3(m)} = \frac{-(-2m^2)}{3m^2 - 1} = \frac{2m^2}{3m^2 - 1}$$

$$\text{When } m = 0 \text{ then } C = \frac{2 \times 0}{3 \times 0 - 1} = 0$$

$$\text{When } m = 1, \text{ then } C = \frac{2 \times 1^2}{3 \times 1^2 - 1} = \frac{2}{2} = 1$$

$$\text{When } m = -1 \text{ then, } C = \frac{2 \times (-1)^2}{3 \times (-1)^2 - 1} = \frac{2}{2} = 1$$

$\therefore y = mx + C$ gives,

$y = 0 . x + 0$

i.e. $y = 0$

$y = 1 . x + 1$

$y = -1 . x + 1$

$\therefore y + x - 1 = 0$

Thus, the required asymptotes are

$y = 0, y - x - 1 = 0, y + x - 1 = 0$

(viii) $xy^2 - a^2(x - a) = 0$

Solⁿ. $xy^2 - a^2x + a^3 = 0$

Thus, equation is of degree third

So, it has at most 3rd asymptotes,

Since, x^3 and y^3 is absent, then the asymptotes parallel to x and y-axis.

$\therefore$ To find the asymptotes parallel to x-axis, the equate the coefficient of highest degree term of x to zero.

$\therefore y^2 - a^2 = 0$

$\therefore y^2 = a^2$

$y = \pm a$

Again, To find the asymptotes parallel to y-axis

i.e. coefficient of $y^2 = 0$

$x = 0$

Thus, the required asymptoes are $x = 0, y = \pm a$

(ix) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$

Solⁿ. Here, $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$

or, $a^2y^2 + b^2x^2 = x^2y^2$

or, $a^2y^2 + b^2x^2 - x^2y^2 = 0$

This equation is of degree 4, so, it has at most 4 asymptotes. Since x^4 and y^4 is absent, there is asymptotes parallel to X and Y-axis.

To find the asymptotes parallel to x-axis equation the coefficient of highest degree term to y to zero.

i.e. $x^2 - a^2 = 0 \Rightarrow x^2 = a^2 \quad \therefore x = \pm a$

Similarly, Asymptotes parallel to y-axis is

$y^2 - b^2 = 0 \qquad \Rightarrow y = \pm b$

$\therefore$ The required asymptotes are $x = \pm a$, $y = \pm b$

(x) $(y-a)^2(x^2-a^2)=x^4+a^4$

Sol^n. Given equation is,

$(y^2-2ay+a^2)(x^2-a^2)=x^4+a^4$

or, $x^2y^2-a^2y^2-2ax^2y+2a^3y+a^2x^2-a^4=x^4+a^4$

or, $x^2y^2-x^4-2ax^2y+a^2x^2-a^2y^2+2a^3y-2a^4=0$

or, $x^4-x^2y^2+2ax^2y-a^2x^2+a^2y^2-2a^3y+2a^4=0$

This equation is degree four, so,

There are four asymptotes.

Since, x^4 is present and y^4 is absent, there is asymptotes oblique and parallel to y-axis.

Since, y^4 is absent, there is asymptotes parallel to y-axis.

i.e. $-x^2+a^2=0$

$\therefore x=\pm a$

Again, to find the oblique asymptotes of the form $y=mx+C$

Put $x=1$ and $y=$ in fourth and third degree terms.

$\phi_4(m)=1-m^2$, $\phi_3(m)=2am$

Now, $\phi_4'(m)=-2m$

and $\phi_4(m)=0$ gives,

$1-m^2=0$ or, $m^2=1$ $\therefore m=\pm 1$

Now, $C=-\dfrac{\phi_3(m)}{\phi_4(m)}=\dfrac{-2am}{-2m}=a$

$\therefore$ Equation of asymptotes are;

$y=\pm 1x+a$

or, $y\pm x=a$

Thus, the equation of asymptotes are $x=\pm a$, $y\pm a=a$

(xi) $x^3+xy^2-ay^2=0$

Sol^n. Given equation is $x^3+xy^2-ay^2=0$

This equation is degree third, since x^3 is present there is oblique asymptotes and since y^3 is absent, there is asymptotes parallel to y-axis.

So, the asymptotes parallel to y-axis is $x-a=0$

$\therefore x=a$

To find the oblique asymptotes of the form $y=mx+C$

Put $x=1$ and $y=m$ in the 3rd and 2nd degree term, we get,

$\phi_3(m)=1+m^2=m^2+1$

$\phi_2(m)=-m^2a$

Now, $\phi_3(m)=0$ gives,

$m^2+1=0$

From which is not possible to find the real values of x.

Thus, the real asymptotes is $x=a$

(xii) $x^2y^2=a^2(x^2+y^2)$

Sol^n. Given equation is $x^2y^2=a^2x^2+a^2y^2$

$x^2y^2-a^2x^2-a^2y^2=0$

(iii) $x^2-y^2+2x+2y+3=0$ / $y=x+2$, $y=-x$

This equation is of degree four,

So, since x^4 and y^4 is absent there is asymptotes parallel to x-axis and y-axis

To find the asymptotes parallel to x-axis equating the coefficient of highest degree term of x to zero.

i.e. $y^2-a^2=0$ $\therefore y^2=a^2$, $y=\pm a$

Again, To find the asymptotes parallel to parallel to y-axis, equating the coefficient of highest degree terms y to zero.

i.e. $x^2-a^2=0$

$x^2=a^2$ i.e., $x=\pm a$

$\therefore$ The required asymptotes are $x=\pm a$, $y=\pm a$

2. Find the asymptotes of the following curves.

(i) $y^3-x^2y+2y^2+4y+x=0$

Sol^n. Given equation is $y^3-x^2y+2y^2+4y+x=0$

This equation is of degree third, here, y^3 is present and x^3 is absent.

Since x^3 is absent, there is asymptotes parallel to x-axis.

i.e. $-y=0$ $\Rightarrow y=0$

Again, since y^3 is present, so there is oblique asymptotes of the form $y=mx+C$

Put $x=1$ and $y=m$ is the third and 2nd degree term, we get,

$\phi_3(m)=m^3-m$ and $\phi_2(m)=2m^2$

$\phi'_3(m)=3m^2-1$

Now, $\phi_3(m)=0$ gives,

$m^3-m=0$

$m(m^2-1)=0$

or, $m=0$, $m=\pm 1$

We have, $C=\dfrac{\phi_2(m)}{\phi_3(m)}=\dfrac{-2m^2}{3m^2-1}$

When $m=0$ then $C=0$

When $m=1$, then $C=\dfrac{-2}{3-1}=\dfrac{-2}{2}=-1$

The asymptotes be $y=-x-1$

$\therefore$ The asymptotes be $y=-x-1$

i.e. $x+y+1=0$

Then required asymptotes is $y=0$ and $x+y+1=0$

(ii) $y^3+x^2y+2xy^2-y+1=0$

Sol^n. Given equation is $y^3+x^2y+2xy^2-y+1=0$

This equation is of degree third

So, there are 3 asymptotes parallel or imaginary

Here, x^3 is absent. So, these is asymptotes parallel to x-axis i.e., $y=0$

As, y^3 is present, these is oblique asymptotes of the form $y=mx+C$

Put $x = 1$ and $y = m$ in the third and 2nd degree term, we get,

$\phi_3(m) = m^3 + m + 2m^2$ and $\phi_2(m) = 0$

Now, $\phi_3(m) = 0$ gives,

$\therefore m^3 + 2m^2 + m = 0$

or, $m(m^2 + 2m + 1) = 0$

or, $m(m+1)^2 = 0$

$\therefore m = 0, m = -1, -1$

Since, $m = -1$ is repreated twice so that to find C we shall use the formula,

$$\frac{C^2}{2!}\phi''_3(m) + C\phi'_2(m) + \phi_1(m) = 0$$

But $\phi'_3 = 0$ and $\phi_1(m) = -m$

$\therefore$ From (i) $\frac{C^2}{2!}(6m+4) + C.0 - m = 0$

$$\frac{C^2}{2!}(-6+4) + 1 = 0$$

or, $-C^2 + 1 = 0 \quad \Rightarrow x + y = \pm 1$

$\therefore$ Oblique asymptotes are $y = -1 . x \pm 1$

or, $x + y = \pm 1$

$\therefore$ Required asymptotes are $y = 0, x + y = \pm 1$

(iii) $\mathbf{x^2y^2 - 4(x-y)^2 + 2y - 3 = 0}$

Sol^n. Given equation is, $x^2y^2 - 4(x^2 - 2xy + y^2) + 2y - 3 = 0$

or, $x^2y^2 - 4x^2 + 8xy - 4y^2 + 2y - 3 = 0$

This equation is of degree four

Both x^4 and y^4 are absent

There is asymptotes parallel to x and y-axis

To find the asymptotes parallel to x-axis, equation the coefficient of highest degree, term of x to zero.

i.e. $y^2 - 4 = 0$ or, $y^2 = 4 \therefore y = \pm 2$

Again, to find the asymptotes parallel to y-axis equating the coefficient of highest degree term of y to zero.

i.e. $x^2 - 4 = 0$

or, $x^2 = 4 \quad \Rightarrow x = \pm 2$

Hence, required asymptotes are, $x = \pm 2$ and $y = \pm 2$

(iv) $\mathbf{x^2(x-y)^2 - a^2(x^2+y^2) = 0}$

Sol^n. Given, equation is,

$x^2(x-y)^2 - a^2(x^2+y^2) = 0$

or, $x^2(x^2 - 2xy + y^2) - a^2x^2 - a^2y^2 = 0$

or, $x^4 - 2x^3y + x^2y^2 - a^2x^2 - a^2y^2 = 0$

This equation is of degree four, so there are four asymptotes, since y^4 is absent, there is asymptotes parallel to y-axis,

i.e., $x^2 - a^2 = 0$

$\therefore x = \pm a$

But x^4 is present so, there is no asymptotes parallel to x-axis. So, there is asymptotes of the form $y = mx + C$

So, put $x = 1$ and $y = m$ in the 4th and 3rd degree terms,

We get, $\phi_4(m) = 1 + m^2 - 2m$

and $\phi_3(m) = 0$

$\therefore \phi'_4(m) = 2m - 2$

or, $\phi''_4(m) = 2$

Now, $\phi_4(m) = 0$ gives,

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \qquad$ or, $m = 1, 1$

Also, $\phi_2(m) = a^2 - a^2m^2$

Here, 1 is the reapeted value of m.

So, to find the value of C, we get.

$C^2\phi''_4(m) + C\phi'_3(m) + \phi_2(m) = 0$

or, $\frac{C^2}{2!}2 + C \times 0 - a^2 - a^2m^2 = 0$

or, $C^2 - a^2 - a^2 = 0$

or, $C^2 - 2a^2 = 0$

$C^2 = 2a^2$

$C = \pm a\sqrt{2}$

i.e. $x - y = \pm a\sqrt{2}$

Thus, the required asymptotes be

$x = \pm a, x - y = \pm a\sqrt{2}$

(v) $\mathbf{x^3 + y^3 - xy^2 - x^2y + x^2 - y^2 = 1}$

Sol^n. Given equation is,

$x^3 + y^3 - xy^2 - x^2y + x^2 - y^2 = 1$

This equation of degree third, so, there are at most 3 asymptotes, since, x^3 and y^3 is present, there is no asymptotes parallel to x and y-axis.

To find the oblique asymptotes of the form $y = mx + C$

Put $x = 1$ and $y = m$ in third and 2nd degree term, we get,

$\phi_3(m) = 1 + m^3 - m^2 - m$

$\phi_2(m) = 1 - m^2$

Now, $\phi_3(m) = 0$ gives,

$m^3 - m^2 - m + 1 = 0$

$m^2(m-1) + (c-1) = 0$

$(m-1)(m^2-1) = 0$

$\therefore m = 1, 1, -1$

Also, $\phi'_3(m) = 3m^2 - 2m - 1$

For $m = -1$

$$C = \frac{-\phi_2(m)}{\phi_3(m)} = \frac{-(1-m^2)}{3m^2 - 2m - 1} = \frac{-(1-1)}{3+2-1} = \frac{-0}{4} = 0$$

$\therefore y = -x + 0$

$y + x = 0$

Again, $m = 1$ is repeated root. So, to find the value of C, we use,

$\frac{C^2}{2!}\phi''_3(m) + C\phi'_2(m) + \phi_1(m) = 0$

But $\phi''_3(m) = 6m - 2$

$\phi'_2(m) = -2m, \phi_1(m) = 0$

Then, $\frac{C^2}{2}(6m-2) + C(-2m) + 0 = 0$

$\frac{C^2}{2!}(6m-2) - 2m = 0$

or, $C^2(3-2) - C = 0$

$C^2 - C = 0$

$C(C-1) = 0, C = 0, C = 1$

$\therefore y = 1.x + 0$

$y = x$

and $y = 1.x + 1$

$y = x + 1$

$\therefore$ The required asymptotes are

$x + y = 0, x - y = 0, x - y + 1 = 0$

(vi) $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$

Sol^n. Given equation is of degree third

Hence, x^3 and y^3 are both present

So, there are no asymptotes parallel to the axes.

To find the oblique asymptotes of the form

$y = mx + C$

Put $x = 1$ and $y = m$ in the third and 2nd degree term. So there are no asymptotes parallel to axes.

So, $\phi_3(m) = m^3 - 6m^2 + 11m - 6$

$\phi_2(m) = 0$

Now, $\phi_3(m) = 0$ gives,

$m^3 - 6m^2 + 11m - 6 = 0$

$m^3 - m^2 - 5m^2 + 5m + 6m - 6 = 0$

or, $m^2(m+1) - 5m(m-1) + 6(m-1) = 0$

or, $(m^2 - 5m + 6)(m-1) = 0$

or, $(m^2 - 3m - 2m + 6)(m-1) = 0$

or, $(m-1)(m-3)(m-2) = 0$

$\therefore m = 1, 2, 3$

Also, $\phi'_3(m) = 3m^2 - 12m + 11$

Now, $C = \frac{-\phi_2(m)}{\phi_3(m)} = \frac{0}{3m^2 + 2m + 11} = 0$

Which shows that $C = 0$ for $m = 1, 2, 3$

$\therefore y = 1.x, y = 2x, y = 3x$

i.e., $x - y = 0, 2x - y = 0, 3x - y = 0$ are the required asymptotes.

(vii) $x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0$

Sol^n. Given equation is

$x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0$

This equation is of degree third. Here, y^3 is absent so, there are asymptotes parallel to y-axis

i.e, $x = 0$

Again, To find the asymptotes of oblique which of the form

$y = mx + c$

put $x = 1$ and $y = m$ in the 3rd and 2nd degree term

$\phi_3(m) = 1 - 2m + m^2$

$\phi_2(m) = 1 - m$

Now, $\phi_3((m) = 0$ give $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

Since, the roots of m is repeated, so, to find c, we use the formula,

$\frac{c^2}{2}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$

But $\phi_3'(m) = -2 + 2m$

$\therefore \phi_3''(m) = 2$ and $\phi_2'(m) = -1$

$\phi_1(m) = 0$

$\therefore \frac{c^2}{2!}2 + c(-1) + 0 = 0$

$\therefore c^2 - c = 0$

or, $c(c-1) = 0$

$c = 0$ and $c = 1$

Then $y = mx + c$ gives

$y = x + 0$

i.e, $y = x$

and $y = x + 1$

i.e, $x - y + 1 = 0$

Thus, required asymptotes are $x = 0, x - y = 0; x - y + 1 = 0$

(viii) $x^3 - 4y^3 + 3x^2y + y - x + 3 = 0$

Sol^n. Given equation is

$x^3 - 4y^3 + 3x^2y + y - x + 3 = 0$

This equation is of degree third here, x^3 and y^3 are both present hence, there are 20 asymptotes parallel to y and x-axis so, here is oblique asymptotes put $x = 1$ and $y = m$ in the third and 2nd degree term.

$\phi_3(m) = 1 + 3m - 4m^3$

$\phi_2(m) = 0$

Now, $\phi_3(m) = 0$ gives

$-4m^3 + 3m + 1 = 0$

or, $4m^3 - 3m - 1 = 0$
or, $4m^3 - 4m^2 + 4m^2 - 4m + m - 1 = 0$
or, $4m^2(m-1) + 4m(m-1) + 1(m-1) = 0$
or, $(4m^2 + 4m + 1)(m-1) = 0$
$(2m+1)^2(m-1) = 0$
or, $m = 1, \frac{-1}{2}, \frac{-1}{2}$

When $m = 1$, then

$$c = \frac{-\phi_2(m)}{\phi_3^1(m)} = \frac{-0}{\phi_3^1(m)} = 0$$

and the asymptotes is $y = 1.x + 0 \Rightarrow y = x$
i.e. $x - y = 0$

Again,

$m = \frac{-1}{2}$ is repeated roots, so to find c, we have,

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

But $\phi_2(m) = 0$
$\phi_1(m) = -1 + m$
and $\phi_3''(m) = -24m$
Then, we have,

$$\frac{c^2}{2!}(-24m) + c.0 + (m-1) = 0$$

or, $-12mc^2 + m - 1 = 0$

When $m = \frac{-1}{2}$

$$\therefore -12\left(\frac{-1}{2}\right)c^2 - \frac{1}{2} - 1 = 0$$

or, $6c^2 - \frac{3}{2} = 0$ The required asymptotes are

$c^2 = \frac{1}{4}$, $y = -\frac{1}{2}x \pm 1$

(ix) $(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0$

Sol^n. Given,

$(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0$
or, $x^3 + 2x^2y + x^2 - xy^2 - 2y^3 - y^2 + x + y + 1 = 0$
or, $x^3 - 2y^3 + 2x^2y - xy^2 + x^2 - y^2 + x + y + 1 = 0$

This equation is of degree third and both x^3 and y^3 is present, so there is oblique asymptotes of the form $y = mx + C$ put $x = 1$ and $y = m$ in the 3rd and 2nd degree term the get

$\phi_3(m) = 1 + 2m - m^2 - 2m^3$

ix) $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$

$\phi_2(m) = 1 - m^2$

Now, $\phi_3(m) = 0$ gives
$1 + 2m - m^2 - 2m^3 = 0$
or, $2m^3 + m^2 - 2m - 1 = 0$
or, $m^2(2m+1) - 1(2m+1) = 0$
or, $(m^2 - 1)(2m + 1) = 0$
or, $(m-1)(m+1)(2m+1) = 0$
$\therefore m = 1, -1, \frac{-1}{2}$

Also, $c = \frac{-\phi_2(m)}{\phi_3^1(m)} = \frac{m^2 - 1}{2 - 2m - 6m^2} = \frac{1-1}{2-2-6} = 0$

When $m = 1$, then $c = 0$

when $m = \frac{-1}{2}$, then

$$c = \frac{\frac{1}{4} - 1}{2 + 1 - \frac{6}{4}} = \frac{-1}{2}$$

The required asymptotes are

$y = 1.x + 0$; $y = -1.x + 0$, $y = \frac{1}{2}x - \frac{1}{2}$

i.e. $y = x$, $y + x = 0$; $x + 2y + 1 = 0$.

(x) $y^3 + 2xy^2 + x^2y - y + 1 = 0$

Sol^n. Given equation is

$y^3 + 2xy^2 + x^2y - y + 1 = 0$

This equation is of degree 3rd, here,
y^3 is present and x^3 is asymptotes parallel to x – axis,
i.e, $y = 0$ [coefficient of x^2]
since, y^3 is present, there asymptotes of the form $y = mx + c$
For,
Put $x = 1$ and $y = m$ in 3rd and 2nd degree term,
$\phi_3(m) = m^3 + 2m^2 + m$
$\phi_2(m) = 0$
Now, $\phi_3(m) = 0$ gives
$m^3 + 2m^2 + m = 0$; i.e. $m(m^2 + 2m + 1) = 0$
i.2. $m(m+1)^2 = 0$
$\Rightarrow m = 0, m = -1, -1$

when $m = 0$, $c = \frac{-\phi_2(m)}{\phi_3^1(m)} = \frac{-0}{\phi_3^1(m)} = 0$

$\therefore y = 0$ is the asymptotes.

Again,

since $m = -1$ is repeated root.

To find the value of c, we use

$$\frac{c^2}{2}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

But

$\phi_3'(m) = 3m^2 + 4m + 1$

$\phi_3''(m) = 6m + 4$

and $\phi_2'(m) = 0$, $\phi_1(m) = -m$

Then we have

$$\frac{c^2}{2}(6m+4) + c.0 + (-m) = 0$$

$= c^2(3m+2) - m = 0$

When $m = -1$

$[3(-1) + 0^2]c^2 - (-1) = 0$

or, $-c^2 + 1 = 0$

or, $c^2 = 1$

$c = \pm 1$

Then, $y = -1x \pm 1$ i.e, $x + y = \pm 1$

Here, the required asymptotes be $y = 0$; $x + y = \pm 1$

3. Find the asymptotes of the following curve

(i) $r\cos\theta = a\sin\theta$

Sol^n. Given curve is

$r\cos\theta = a\sin\theta$

or, $\frac{1}{r} = \frac{1}{a}\frac{\cos\theta}{\sin\theta}$

or, $\frac{1}{r} = \frac{1}{a}\cot\theta$

or, $\frac{1}{r} = f(\theta)$, where $f(\theta) = \frac{1}{a}\cot\theta$

Now,

$f(\theta) = 0$ implies

$\frac{1}{a}\cot\theta = 0$ or, $\cot\theta = 0$

or, $\theta = \frac{\pi}{2} = \alpha$ (say)

Also, $f'(\theta) = \frac{-\operatorname{cosec}^2\theta}{a}$

$f'(\alpha) = \frac{-\operatorname{cosec}^2\alpha}{as} = \frac{-\operatorname{cosec}^2\frac{\pi}{2}}{a} = \frac{-1}{a}$

Therefore, the required asymptotes is

$r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$

or, $r\sin\left(0 - \frac{\pi}{2}\right) = \frac{1}{\frac{-1}{a}}$

or, $r\sin\left(\frac{\pi}{2} - \theta\right) = -a$

or, $-r\cos\theta = -a$

or, $r\cos\theta = a$

(ii) $r\theta = a$

Sol^n. Given curve is $r\theta = a$

or, $\frac{1}{r} = \frac{\theta}{a}$

or, $\frac{1}{r} = \frac{\theta}{a} = f(\theta)$

where,

$f(\theta) = \frac{\theta}{a}$

New, $f(\theta) = 0$ implies

$\frac{\theta}{a} = 0 \therefore \theta = 0$

$\therefore \alpha = 0$ (say)

Also, $f'(\theta) = \frac{1}{a}$

$\therefore f'(\alpha) = \frac{1}{a}$

So that the equation of asymptotes is

$r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$

or, $r\sin(\theta - 0) = \frac{1}{\frac{1}{a}}$ or, $r\sin\theta = a$

(iii) $2r^2 = \tan 2\theta$

Sol^n. Given curve is $2r^2 = \tan 2\theta$

or, $\frac{1}{r^2} = \frac{2}{\tan 2\theta}$

or, $\frac{1}{r^2} = 2\cot 2\theta$

or, $\frac{1}{r^2} = f(\theta)$

Where, $f(\theta) = 0$ gives

$2\cot 2\theta = 0$

or, $\frac{2\cos 2\theta}{\sin 2\theta} = 0$

$\therefore \cos 2\theta = 0$

or, $2\theta = \frac{\pi}{2}$

or, $\theta = \frac{\pi}{4}$

$\therefore \alpha = \frac{\pi}{4}$

Also,

$f'(\theta) = -4\operatorname{cosec}^2 2\theta$

As $f'(\alpha) = -4\operatorname{cosec}^2 2\alpha$

the required asymptotes of the curve is $r\sin(\theta - \alpha) = \frac{1}{f^1(\alpha)}$

or, $r\sin\left(\theta - \frac{\pi}{4}\right) = \frac{-1}{4\operatorname{cosec}^2 2.\frac{\pi}{4}}$

or, $r\sin\left(\theta - \frac{\pi}{4}\right) = \frac{-1}{4}\sin^2\frac{\pi}{2}$

or, $r\sin\left(\theta - \frac{\pi}{4}\right) = \frac{-1}{4r}$ .....(*)

But

$\frac{-1}{4r} = \frac{-1}{4}\sqrt{\frac{2}{\tan 2\theta}}$

$\therefore$ At $\theta = \frac{\pi}{4}$

$\frac{-1}{4r} = \frac{-1}{4} \times 0 = 0$

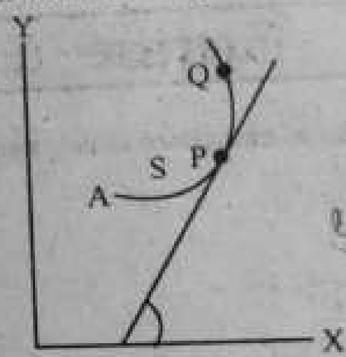
$\therefore \sin\left(\theta - \frac{\pi}{4}\right) = 0$

$\theta - \frac{\pi}{4} = 0 \therefore \theta = \frac{\pi}{4}$

Curvature

6

Curvature: Let P and Q be two neighboring points on the curve. Let Arc Ap = s the curve. Let Are Ap = s, where A is fixed point on the curve. Let ψ be the angle made by the tangent at p with x – axis. Then $\frac{d\psi}{ds}$ is called curvature of the curve at the point p. Thus, the curvature is defined as the rate of turning of the tangent w.r.t arc– length.



$\rho = \frac{(r_1^2 + r^2)^{3/2}}{r^2 + 2r_1^2 - r\,r_2}$

Radius of curvature:- Radins of curvature of the curve at any point is defined as the reciprocal of the curvature at that point and is denoted by ρ. $\therefore \rho = \frac{ds}{d\psi}$ some formula for radins- of curvature.

Cartesian Curve:

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \text{ or, } \frac{\left(1 + y_1^2\right)^{\frac{3}{2}}}{y_2}$$

$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$

Note: The radius of curvature ρ is positive or negative according as $\frac{d^2y}{dx^2}$ is positive or negative.

Thus, indicates the curve is either concave or convex.

If y_1 becomes infinite at any point, we use the formula

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} \text{ or, } \rho = \frac{\left(1+x_1^2\right)^{\frac{3}{2}}}{x_2}$$

i.e. the tangent is perpendicular to x – axis.

Polar curve: Let $r = f(\theta)$ be the curve then

$$\rho = \frac{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r.\frac{d^2r}{d\theta^2}} \text{ or, } \frac{\left[\left(r^2+r_1^2\right)^{\frac{3}{2}}\right]}{r^2+2r_1^2-rr_2}$$

Exercise - 6

1. **Find the radius of curvature at any point (xk y) of the follow curves**

(i) $ay^2 = x^3$

Solⁿ. $ay^2 = x^3$ .....(1)

Diffⁿ(1) w.r.t x, we get

$$2ay\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\text{or } y_1 = \frac{3x^2}{2ay}$$

$$\text{or, } y_1^2 = \frac{9x^4}{4a^2y^2}$$

$$\text{or, } y_1^2 = \frac{9x^4a}{4a^2x^3}$$

$$\text{or, } y_1^2 = \frac{9x}{4a} \text{ .....(i)}$$

Again, Diffⁿ, we get

$$2y_1y_2 = \frac{9}{4a}$$

$$\text{or, } y_2 = \frac{9}{4a\times 2y_1} = \frac{9\times 2ay}{8a\times 3x^2} = \frac{y}{4x^2} = \frac{3}{4}\frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}x^2} = \frac{3}{4}\frac{x^{\frac{-1}{2}}}{a^{\frac{1}{2}}}$$

$$\therefore \text{Radius or curvature } (\rho) = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}$$

$$= \frac{\left(1+\frac{9x}{4a}\right)^{\frac{3}{2}}}{\frac{3}{4}\frac{x^{\frac{-1}{2}}}{a^{\frac{1}{2}}}}$$

$$= \frac{(4a+9x)^{\frac{3}{2}}}{(4a)^{\frac{3}{2}}} \times \frac{4a^{\frac{1}{2}}}{3x^{\frac{-1}{2}}}$$

$$= \frac{(4a+9x)^{\frac{3}{2}}}{a^{\frac{3}{2}}\times 8} \times \frac{4a^{\frac{1}{2}}}{3x^{\frac{-1}{2}}} = \frac{(4a+9x)^{\frac{3}{2}}x^{\frac{1}{2}}}{6a}$$

(ii) $xy = c^2$

Here $y = \frac{c^2}{x}$ .....(i)

Diffⁿ (i) w.r.t x; we get

$$y_1 = \frac{-c^2}{x^2}; y_2 = -c^2(-2x^{-3}) = \frac{2c^2}{x^3}$$

we know, by the formula,

$$\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{\left(1+\frac{c^4}{x^4}\right)^{\frac{3}{2}}}{\frac{2c^2}{x^3}} = \frac{\left(x^4+c^4\right)^{\frac{3}{2}}}{2c^2x^3}$$

But $c^2 = xy$

$\therefore c^4 = x^2y^2$

$$\therefore \rho = \frac{(x^4+x^2y^2)^{\frac{3}{2}}}{2c^2x^3} = \frac{x^3(x^2+y^2)^{\frac{3}{2}}}{2c^2x^3} = \frac{\left(x^2+y^2\right)^{\frac{3}{2}}}{2c^2}$$

(iii) $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

Sol^n. Here, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ .....(i)

Diff^n eq^n (i) w.r.t x, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=y_1=\frac{-2x}{a^2}\times\frac{b^2}{2y}=\frac{-b^2x}{a^2y}$$

$$\text{or, } \frac{d^2y}{dx^2}=y_2=\frac{-b^2}{a^2}\left[\frac{y-xy_1}{y^2}\right]$$

$$=\frac{-b^2}{a^2}\left[\frac{y-x\left(\frac{-b^2}{a^2}\right)\frac{x}{y}}{y^2}\right]=\frac{-b^2}{a^2}\left[\frac{a^2y^2+b^2x^2}{a^2y^3}\right]=\frac{-b^2}{a^2}\left[\frac{a^2b^2}{a^2y^3}\right]$$

$$\therefore \text{Radius of curvature } \rho=\frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}$$

$$=\frac{\left(1+\frac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}}}{\frac{b^2}{a^2}\left[\frac{a^2b^2}{a^2y^3}\right]}=\frac{-(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^6y^3}\times\frac{a^2y^3}{b^4}=\frac{-\left(a^4y^2+b^4x^2\right)^{\frac{3}{2}}}{a^4b^4}$$

$$=\frac{\left(a^4y^2+b^4x^2\right)^{\frac{3}{2}}}{a^4b^4}$$

(iv) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

Sol^n. Here,

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}} \quad .....(i)$$

Diff^n (i) w.r.t x, we get

$$\frac{2}{3}x^{\frac{-1}{3}}+\frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=\frac{-x^{\frac{-1}{3}}}{y^{\frac{-1}{3}}}=\frac{-y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\text{Now, } 1+y_1^2=1+\left(\frac{-y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)^2=\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}=\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}$$

$$\text{or, } 1+y_1^2=a^{\frac{2}{3}}x^{\frac{-2}{3}} \quad .....(ii)$$

Again, Diff^n w.r.t x we get

$$0+2y_1=a^{\frac{2}{3}}\left(\frac{-2}{3}x^{\frac{-5}{3}}\right)=\frac{-2}{3}a^{\frac{2}{3}}x^{\frac{-5}{3}}$$

$$\therefore y_2=\frac{-2}{3}\frac{a^{\frac{2}{3}}x^{\frac{-5}{3}}}{y_1}$$

$$\text{But } y_1=\frac{-y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\therefore y_2=\frac{1}{3}a^{\frac{2}{3}}.\frac{x^{\frac{-5}{3}}}{y^{\frac{1}{3}}}.x^{\frac{1}{3}}=\frac{1}{3}a^{\frac{2}{3}}x^{\frac{-4}{3}}y^{\frac{-1}{3}}$$

we know, the formula for radius of carucuture ρ in Cartesian form

$$\rho=\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}=\frac{\left(a^{\frac{2}{3}}x^{\frac{2}{3}}\right)^{\frac{3}{2}}}{\frac{1}{3}a^{\frac{2}{3}}x^{\frac{-4}{3}}.y^{\frac{-1}{3}}}=\frac{3ax^{-1}}{a^{\frac{2}{3}}.x^{-1}y^{\frac{-1}{3}}.x^{\frac{1}{3}}}$$

$$=3a^{\frac{1}{3}}x^{\frac{1}{3}}.y^{\frac{1}{3}}=3(axy)^{\frac{1}{3}}$$

(v) $y^2=4x$ at vertex (0, 0)

Sol^n. we have,

$y^2=4x$ .....(i)

Diff^n (i) w.r.t x, we get

$$2y\frac{dy}{dx}=4 \therefore \frac{dy}{dx}=\frac{4}{2y}=\frac{2}{y}$$

$$\therefore \frac{dy}{dx}\text{at}(0,0)=\infty$$

Thus, $\frac{dy}{dx}$ at the vertex is infinite,

Hence the formula,

$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$ is not applicable

in this case, here, we apply

$$\rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2}$$

Now, form (i)

$$x = \frac{y^2}{4}$$

$$\frac{dx}{dy} = \frac{1}{2}y$$

and $\frac{d^2x}{dy^2} = \frac{1}{2}$

At the vertex (0, 0), $x_1 = 0$ and $x_2 = \frac{1}{2}$

Hence,

$$\rho = \frac{\left(1+x_1^2\right)^{\frac{3}{2}}}{x_2} = \frac{(1+0)^{\frac{3}{2}}}{\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

(vi) $y = x^3 - 2x^2 + 7x$ **at (0, 0)**

Sol^n. Let $y = x^3 - 2x^2 + 7x$ .....(i)

Diff^n (i) w.r.t x, we get

$y_1 = 3x^2 - 4x + 7$

y_1 at (0, 0) is 7

and $y_2 = 6x - 4$

y_2 at (0, 0) = –4

$\therefore$ Radius of curvature $(\rho) = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{(1+49)^{\frac{3}{2}}}{-4} = \frac{(50)^{\frac{3}{2}}}{-4}$

$= \frac{-1}{2}125\sqrt{2}$ (ignore negative)

$= \frac{125\sqrt{2}}{2}$.

2. Find the radius of curvature at any point (r, θ) for the curve

(i) $r^3 = a^3 \cos 3\theta$

Sol^n. $r^3 = a^3 \cos 3\theta$ .....(i)

Diff^n eq^n (i) w.r.t θ, we get

$$3r^2 \frac{dr}{d\theta} = -3a^3 \sin 3\theta$$

or, $\frac{dr}{d\theta} = \frac{-3a^3 \sin 3\theta}{3r^2} = \frac{-ra^3 \sin 3\theta}{r^3} = \frac{-ra^3 \sin 3\theta}{a^3 \cos 3\theta}$

$r_1 = \frac{dr}{d\theta} = -r \tan 3\theta$

Again, Diff^n eq^n (ii), we get

$r_2 = -[r_1 \tan 3\theta + 3r\sec^3 3\theta]$

$= -[r\tan^2 3\theta + 3r\sec^3 3\theta]$

$= r\tan^2 3\theta = 3r\sec^3 3\theta$

Radius of curvature $(\rho) = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1 - rr_2}$

or, $\rho = \frac{\left(r^2 + r^2\tan^2 3\theta\right)^{\frac{3}{2}}}{r^2 + 2r^2\tan^2 3\theta - r^2\tan^2 3\theta + 3r^2\sec^2 3\theta}$

$= \frac{r^3 \sec^3 3a}{4r^2 \sec^2 3\theta} = \frac{1}{4} r \sec 3\theta$

$= \frac{r}{4\cos 3\theta}$

and from (i), $= \frac{r}{4.\frac{r^3}{a^3}} = \frac{a^3}{4r^2}$

$\therefore \rho = \frac{a^3}{4r^2}$

(ii) $r = a\cos\theta$

Sol^n. Let $r = a\cos\theta$ .....(i)

Diff^n (i) w.r.t θ, we get

$r_1 = -a\sin\theta$

and $r_2 = -a\cos\theta$

we know, the radius of the curvature

$$\rho = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} = \frac{\left(a^2\cos^2\theta + a^2\sin^2\theta\right)^{\frac{3}{2}}}{a^2\cos^2\theta + a^2\sin^2\theta + a^2\cos^2\theta}$$

$$= \frac{\left(a^2\right)^{\frac{3}{2}}}{2a^2} = \frac{a^3}{2a^2} = \frac{1}{2}a$$

(iii) $r^2\cos\theta = a^2$

Sol^n. Given, $r^2\cos\theta = a^2$

$\therefore r = a(1-\cos\theta)$

$$\therefore \rho = \frac{[2a.a(1-\cos\theta)]^{\frac{3}{2}}}{3a.a(1-\cos\theta)} = \frac{(2ar)^{\frac{3}{2}}}{3ar} = \frac{2^{\frac{3}{2}}}{3}(ar)^{\frac{3}{2}-1} = \frac{2\sqrt{2}\sqrt{ar}}{3} = \frac{2\sqrt{2a}}{3}$$

4. Show that the radius of curvature at any point (r, θ) of the curve $r^m = a^m \cos m\theta$ is $\dfrac{a^m}{(m+1)\,r^{m-1}}$.

Sol^n. We have $r^m = a^m \cos m\theta$ .....(i)

Taking log on both sides, we get

$m \log r = m \log a + \log \cos m\theta$

Diff^n w. r. t. θ, we get

$$m \cdot \frac{1}{r} \cdot r_1 = 0 - m\frac{\sin m\theta}{\cos m\theta}$$

$$\therefore \frac{mr_1}{r} = -m \tan m\theta$$

$r_1 = -r \tan m\theta$

Again, diff^n w. r. t. θ, we get

$r_2 = -[r_1 \tan m\theta + mr\sec^2 m\theta]$

$= -[-r\tan^2 m\theta + mr\sec^2 m\theta]$

$= r\tan^2 m\theta - mr\sec^2 m\theta$

We know the formula,

$$\rho = \frac{\left(r^2+r_1^2\right)^{\frac{3}{2}}}{r^2+2r_1^2-r_2} = \frac{\left(r^2+r^2\tan^2 m\theta\right)^{\frac{3}{2}}}{r^2+2r^2\tan^2 m\theta - r\left[r\tan^2 m\theta - mr\sec^2 m\theta\right]}$$

$$= \frac{\left(r^2\sec^2 m\theta\right)^{\frac{3}{2}}}{r^2\left(1+\tan^2 m\theta + m\sec^2 m\theta\right)} = \frac{r^3\sec^3 m\theta}{r^2\left(\sec^2 m\theta + m\sec^2 m\theta\right)}$$

$$= \frac{r^3\sec^2 m\theta}{r^2\sec^2 m\theta(m+1)} = \frac{r\sec m\theta}{m+1} = \frac{r}{(1+m)\cos m\theta}$$

But $\cos m\theta = \dfrac{r^m}{a^m}$ (from i)

$$\rho = \frac{r}{1+m}\cdot\frac{a^m}{r^m} = \frac{a^3}{(m+1)\,r^{m-1}}$$

5. Find the radius of curvature at any point for the curves.

(i) $x = at^2, y = 2at$

Sol^n. We have, $x = at^2$, $y = 2at$

$x' = 2at$, $y' = 2a$

$x'' = 2a$, $y'' = 0$

We know,

$$\text{Radius of curvature } (\rho) = \left[\frac{(x')^2+(y')^2}{x'y''-y'x''}\right]$$

$$= \left[\frac{4a^2t^2+4a^2}{-2a\,.\,2a}\right]^{\frac{3}{2}} = -\frac{\left(4a^2\right)^{\frac{3}{2}}\left(1+t^2\right)^{\frac{3}{2}}}{4a^2}$$

$$= -2a\left(1+t^2\right)^{\frac{3}{2}} = 2a\left(1+t^2\right)^{\frac{3}{2}}$$

[Ignore negative]

(ii) $x = a\cos\theta, y = a\sin\theta$

Sol^n. Diff^n both sides, w. r. t. θ, we get

$x' = -a\sin\theta$, $y' = a\cos\theta$

$x'' = -a\cos\theta$, $y'' = -a\sin\theta$

We know, the radius of curvature for the parametric curves

$$\rho = \left[\frac{(x')^2+(y')^2}{x'y''-y'x''}\right]^{\frac{3}{2}}$$

$$\therefore \rho = \left[\frac{a^2\sin^2\theta + a^2\cos^2\theta}{(-a\sin\theta)(-a\sin\theta)-(a\cos\theta)(-a\cos\theta)}\right]$$

$$\rho = \frac{\left(a^2\right)^{\frac{3}{2}}}{a^2\left(\sin^2\theta+\cos^2\theta\right)}$$

$$\therefore \rho = \frac{a^3}{a^2} = a$$

(iii) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = 0^\circ$

Sol^n. We have,

$x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$

Diff^n, w. r. t. θ, we get

$x' = a(1 + \cos\theta)$; $y' = a\sin\theta$

Again, diff^n, we get

$x'' = -a\sin\theta$, $y'' = a\cos\theta$

At $\theta = 0$

$x' = a(1 + 1)$, $y' = 0$

$x'' = 0$, $y'' = a$

We know the formula;

$$\rho = \left[\frac{(x')^2+(y')^2}{x'y''-y'x''}\right]^{\frac{3}{2}} = \left[\frac{(2a)^2+0^2}{2a\,.\,a}\right]^{\frac{3}{2}} = \left[\frac{4a^2}{2a^2}\right]^{\frac{3}{2}} = \frac{(2a)^3}{2a^2} = 4a$$

(iv) $x = a\cos\theta,\ y = b\sin\theta$

Sol^n. Diff^n both sides w. r. t. θ, we get

$x' = a\sin\theta,\ y' = b\cos\theta$

Again, differentiating, we get

$x'' = -a\cos\theta,\ y'' = -b\sin\theta$

We know the radius for radius of curvature.

$$\rho = \left[\frac{(x')^2+(y')^2}{x'y''-y'x''}\right]^{\frac{3}{2}}$$

$$= \frac{\left[a^2\sin^2\theta+b^2\cos^2\theta\right]^{\frac{3}{2}}}{(-a\sin\theta)(-b\sin\theta)-(-a\cos\theta)\,b\cos\theta}$$

$$= \frac{\left[a^2\sin^2\theta+b^2\cos^2\theta\right]^{\frac{3}{2}}}{ab}$$

6. Find the radius of curvature at any point (p, r) on the following curves whose pedal equations are

(i) $r^2 = 2ap$

Sol^n. Diff^n w. r. t. p, we get

$$2r\frac{dr}{dp} = 2a$$

$$\text{or, } \frac{dr}{dp} = \frac{2a}{2r} = \frac{a}{r}$$

$$\therefore \text{radius of curvature } \rho = r\frac{dr}{dp} = r\cdot\frac{a}{r} = a$$

(ii) $r^3 = a^2P$

Sol^n. Diff^n, w. r. t. P, we get

$$3r^2\frac{dr}{dP} = a^2$$

$$\therefore \frac{dr}{dP} = \frac{a^2}{3r^2}$$

$$\rho = r\frac{dr}{dP} = \frac{a^2r}{3r^2}$$

$$\therefore \text{Radius of curvature } (\rho) = \frac{a^2}{3r}$$

(iii) $P^2 = ar$

Sol^n. Diff^n w. r. t. to P, we get

$$2P = a\cdot\frac{dr}{dP}$$

$$\text{or, } \frac{dr}{dP} = \frac{2P}{a}$$

$$\therefore \rho = r\frac{dr}{dP} = r\cdot\frac{2P}{a} = \frac{2r}{a}\sqrt{ar}$$

$$\therefore \text{Radius of curvature } (\rho) = 2\sqrt{\frac{r^3}{a}}$$

7. Find the radius of curvature at the origin for the following curves:

(i) $4x^2 - 3xy + y^2 - 3y = 0$

Sol^n. The curve passess through the origin.

Here, we use substitution method to find the value of ρ.

$8x - 3xy_1 - 3y + 2yy_1 - 3y_1 = 0$

$$\therefore y_1 = \frac{8x-3y}{3-2y+3x}$$

y_1 at (0, 0) = 0

Again,

$8 - 3xy_2 - 6y_1 + 2yy_2 - 2y_1^2 - 3y_2 = 0$

$$\therefore y_2 = \frac{8-6y_1+2y_1^2}{3x-2y+3}$$

y_2 at (0, 0) $= \frac{8}{3}$ $\qquad \therefore \rho = \frac{(1+0)^{\frac{3}{2}}}{\frac{8}{3}} = \frac{3}{8}$

(ii) $x^3 + y^3 - 2x^2 + 6y = 0$

Sol^n. The curve passess through the origin

Here, we use the substitution method to find the value of ρ.

$$3x^2 + 3y^2\frac{dy}{dx} - 4x + 6y_1 = 0$$

$$\therefore y_1 = \frac{yx-3x^2}{3y^2+6}$$

y_1 at (0, 0) is 0

Again, $6x + 3y^2y_2 + 6yy_1^2 - 4 + 6y_2 = 0$

$$\therefore y_2 = \frac{4-6x-6yy_1^2}{3y^2+6}$$

$$\therefore \rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{(1+0)^{\frac{3}{2}}}{\frac{2}{3}} = \frac{3}{2}$$

8. Find the radius of curvature of the curve $y = x^2(x-3)$ at the points where, the tangent is parallel to x-axis.

Sol^n. We have, $y = (x-3)$ ..... (i)

Diff^n, we get,

$$\frac{dy}{dx} = 2x(x-3) + x^2 \qquad \left[\because \frac{d^2y}{dx^2} = 6x - 6\right]$$

$$= 3x^2 - 6x$$

If the tangent is parallel to x-axis,

Then, $\frac{dy}{dx} = 0$

or, $3x^2 - 6x = 0$

$\therefore x = 0, x = 2$

Then, if $x = 0, y = 0$ and if $x = 2, y = -4$

$\therefore$ Points are (0, 0) and (2, – 4)

y_1 at $(2, -4) = 3.4 - 12 = 12 - 12 = 0$

y_2 at $(2, -4) = 6.2 - 6 = 12 - 6 = 6$

$$\therefore \rho = \frac{\left(1+y^2\right)^{\frac{3}{2}}}{y_2} = \frac{(1+0)^{\frac{3}{2}}}{6} = \frac{1}{6}$$

9. Show that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the radius of curvatur the extremity of major axis is equal to the half of the la rectum.

Sol^n. We have, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ..... (i)

Diff^n (i) w. r. t. x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{b^2x}{a^2x} \quad \text{..... (ii)}$$

Now, the end of major axis of the ellipse is (a, 0)

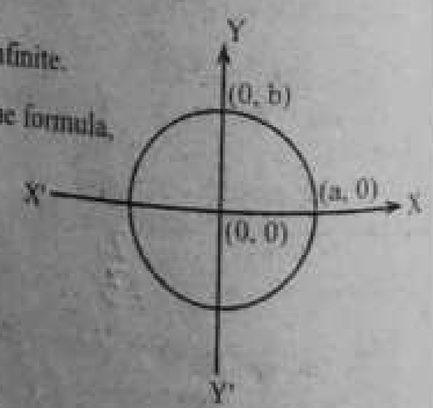
$\therefore x = a, y = 0$

$\therefore \frac{dy}{dx}$ at (a, 0) is ∞, i.e. infinite.

So, we can't find ρ form the formula,

$$\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}$$

Hence, we apply



$$\rho = \frac{\left(1+x_1^2\right)^{\frac{3}{2}}}{x_2}$$

Hence from equation (ii), we have,

$$\frac{dx}{dy} = -\frac{a^2y}{b^2x}$$

$\therefore \frac{dx}{dy}$ at $(a, 0) = 0$

Again, diff^n w. r. t. y, we get

$$\frac{d^2x}{dy^2} = -\frac{a^2}{b^2}\left[\frac{x.1 - y.\frac{dx}{dy}}{x^2}\right]$$

$$= -\frac{a^2}{b^2}\left[\frac{x - y\left(-\frac{a^2y}{b^2x}\right)}{x^2}\right] = -\frac{a^2}{b^2}\left[\frac{b^2x^2 + a^2y^2}{b^2x^3}\right]$$

$$\therefore \frac{d^2x}{dy^2} \text{ at } (a, 0) = -\frac{a^2}{b^2}\left[\frac{b^2a^2 + a^2.0}{b^2a^3}\right]$$

$$= \frac{-a^2b^2a^2}{b^2b^2a^3} = \frac{-a}{b^2}$$

We know, the formula,

$$\rho = \frac{\left(1+x_1^2\right)^{\frac{3}{2}}}{x_2} = \frac{(1+0)^{\frac{3}{2}}}{\frac{-a}{b^2}} = \frac{-b^2}{a} = \frac{b^2}{a}$$

$$\therefore \rho = \frac{b^2}{a} = \frac{1}{2}\left[\frac{2b^2}{a}\right] = \frac{1}{2} \text{ Latusrectum}$$

= Semi-latus-rectum

10. Find the chord of the curvature through the pole for the following curves.

(i) $r = a(1 + \cos\theta)$ ..... (i)

Sol^n. Here, $r = a(1 + \cos\theta)$ ..... (i)

The chord of curvature through the pole is $2P\frac{dr}{dP}$.

First, we change equation (i) in to pedal form

$r = a(1 + \cos\theta)$

$\frac{dr}{d\theta} =$

$\therefore \frac{dr}{d\theta} = -a\sin\theta$

Now, $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{-a\sin\theta} = \frac{a(1+\cos\theta)}{-a\sin\theta}$

$\therefore \tan\phi = -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2} = \tan\left(90^0 + \frac{\theta}{2}\right)$

$\therefore \phi = 90^0 + \frac{\theta}{2}$

Hence, $p = r\sin\phi$

$p = r\sin\left(90^0 + \frac{\theta}{2}\right) = r\cos\frac{\theta}{2}$

$p^2 = r^2\cos^2\frac{\theta}{2}$

$r^2\frac{(1+\cos\theta)}{2} = \frac{r^2}{2}\cdot\frac{r}{a} = \frac{r^3}{2a}$

$\therefore r^3 = 2ap^2$ ..... (ii)

Which is the pedal equation of the curves.

Again, diff^n (ii) w. r. t. P, we get

$3r^2\frac{dr}{dP} = 4aP$

$\therefore \frac{dr}{dP} = \frac{4aP}{3r^2}$

Hence, Chord of curvature through the pole

$= 2P\frac{dr}{dP} = 2P\frac{4aP}{3r^2} = \frac{8aP^2}{3r^2} = \frac{4.2aP^2}{3r^2} = \frac{4r^3}{3r^2} = \frac{4r}{3}$

(ii) $r^2\cos2\theta = a^2$

Sol^n. Given curve is, $r^2\cos2\theta = a^2$

or, $r^2 = a^2\sec2\theta$

or, $r\frac{dr}{d\theta} = a^2 . x\sec2\theta\tan2\theta$

or, $r\frac{dr}{d\theta} = a^2\sec2\theta\tan2\theta$

$\frac{d\theta}{r\,dr} = \frac{\cos2\theta . \cot2\theta}{a^2}$

$\therefore \frac{r^2}{r}\frac{d\theta}{dr} = \frac{r^2\cos2\theta . \cot2\theta}{a^2}$

or, $r\frac{d\theta}{dr} = \frac{a^2}{\cos2\theta}\cdot\frac{\cos2\theta . \cot2\theta}{a^2} = \cot2\theta$

$\frac{dr}{d\theta} = 0$

or, $\tan\phi = \tan\left(\frac{\pi}{2} - 2\theta\right)$

or, $\phi = \frac{\pi}{2} - 2\theta$

or, $P = r\sin\left(\frac{\pi}{2} - 2\theta\right) = r\cos2\theta$

or, $P = r\cdot\frac{a^2}{r^2}$

or, $P = \frac{a^2}{r}$

or, $Pr = a^2$

or, $P\frac{dr}{dP} + r = 0$

$\therefore$ Chord of curvature $P\frac{dr}{dP} = 2r$

(iii) $r = ae^{\theta\cot\alpha}$

Sol^n. Here, $r = ae^{\theta\cot\alpha}$ ..... (i)

Diff^n (i) w. r. t. θ, we get

$\therefore \frac{dr}{d\theta} = ae^{\theta\cot\alpha}\cot\alpha = r\cot\alpha$

Now, $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{r\cot\alpha} = \tan\alpha$ $[\because \phi = \alpha]$

Hence, $P = r\sin\phi$ gives $P = r\sin\alpha$ ..... (ii)

Which is pedal equation.

Differentiating (ii) w. r. t. P, we get

$\frac{dP}{dP} = \frac{dr}{dP}\sin\alpha$

$\therefore \frac{dr}{dP} = \frac{1}{\sin\alpha}$

Hence, the chord of curvature through the pole,

$= P\frac{dr}{dP} = 2P.\frac{1}{\sin\alpha} = \frac{2r\sin\alpha}{\sin\alpha} = 2r$

11. Show that the chord of curvature through the pole of the curve $r^m = a^m\cos m\theta$ is $\frac{2r}{m+1}$

Sol^n. Here, $r^m = a^m\cos m\theta$

$\therefore \frac{d\theta}{dr} = -\frac{r^{m-1}}{a^m\sin m\theta}$

$$\therefore r\frac{d\theta}{dr} = \frac{r^m}{a^m \sin m\theta} = \frac{a^m}{a^m}\frac{\cos m\theta}{\sin m\theta}$$

$$\therefore \tan\phi = r\frac{d\theta}{dr} = -\cot m\theta = \tan\left(\frac{\pi}{2} + m\theta\right)$$

$$\therefore \phi = \frac{\pi}{2} + m\theta$$

$$\therefore P = r\sin\phi = r\sin\left(\frac{\pi}{2} + m\theta\right) = r\cos m\theta$$

$$\text{or, } P = r \cdot \frac{r^m}{a^m} = \frac{r^{m+1}}{a^m}$$

$$\therefore \frac{dP}{dr} = \frac{(m+1)r^m}{a^m}$$

Now, the chord of curvature thought the pole

$$= 2P\frac{dr}{dP} = 2 \cdot \frac{r^{m+1}}{a^m} \cdot \frac{a^m}{(m+1)r^m} = \frac{2r}{m+1}$$

12. Show that the chord of curvature through the pole for the curve $p = f(r)$ is given by $\frac{2f(r)}{f'(r)}$.

Sol^n. Chord of curvature through the pole $= 2P\frac{dr}{dP}$

Since, $P = f(r)$

So, $1 = f'(r)\frac{dr}{dP}$

$$\therefore \frac{dr}{dP} = \frac{1}{f'(r)}$$

Now, the chord of curvature through the pole

$$= 2P\frac{dr}{dP} = 2f(r) \cdot \frac{1}{f'(r)} = \frac{2f(r)}{f'(r)}$$

13. Show that the chord of curvature parallel to y-axis for the curve $y = c\cosh\left(\frac{x}{c}\right)$ is double ordinate.

Sol^n. The required chord of curvature parallel to y-axis $= 2\rho\cos\psi$

First, we find ρ

We have, $y = C\cosh\left(\frac{x}{c}\right)$ ..... (i)

Diff^n. both sides w. r. t. 'x', we get,

$$\frac{dy}{dx} = C\sinh\frac{x}{C} \cdot \frac{1}{C} = \sinh\frac{x}{C}$$

$$\therefore \tan\psi = \sinh\frac{x}{C}$$

$$\text{Now, } \sec\psi = \sqrt{1+\tan^2\psi} = \sqrt{1+\sinh^2\left(\frac{x}{C}\right)} = \cosh\left(\frac{x}{C}\right) = \frac{y}{C}$$

$\therefore C\sec\psi$ ..... (ii)

Diff^n both sides w. r. t. s we get

$$\frac{dy}{ds} = C\sec\psi\tan\psi\frac{dx}{ds}$$

$$\text{or, } \sin\psi = C\sec\psi\tan\psi\frac{1}{\rho}$$

$$\therefore \rho = \frac{C\sec\psi + \tan\psi}{\sin\psi} = C\sec^2\psi$$

Hence, required chord of curvature parallel to y-axis,

$= 2\rho\cos\psi$

$= 2C\sec\psi$

$= 2y$ = double of the ordinate. Hence Proved.

14. Find the pedal equation of the following curves:

(i) $y^2 = 4a(x+a)$

Sol^n. We have, $y^2 = 4a(x+a)$ ..... (i)

Diff^n. (i) w. r. t. 'x', we get

$$2y\frac{dy}{dx} = 4a$$

$$\therefore \frac{dy}{dx} = \frac{2a}{y}$$

Take (X, Y) as the current coordinates, the equation of the tngent at any point (x, y) is given by,

$$Y - y = \frac{2a}{y}(X - x)$$

or, $2aX - Yy + y^2 - 2ax = 0$

Now, if the length of the perpendicular from the origin (0, 0) on the tangent is P,

$$\text{Then, } P = \frac{2a.0 - 0.y + y^2 - 2ax}{\sqrt{4a^2 + y^2}}$$

$$\therefore P^2 = \frac{(y^2 - 2ax)^2}{4a^2 + y^2}$$

But $y^2 = 4a(x+a)$

$$\therefore P^2 = \frac{[4a(x+a) - 2ax]^2}{4a^2 + 4a^2 + 4ax} = \frac{[2a(x+2a)^2]}{4a(2a+x)} = a(x+2a)$$

$$= a\sqrt{(x+2a)^2}$$

or, $P^2 = a\sqrt{x^2 + 4ax + 4a^2}$

or, $P^2 = a\sqrt{x^2 + y^2}$ $\quad [x^2 + y^2 = r^2]$

$= a\sqrt{r^2}$

$\therefore P^2 = ar$ is the required pedal equation.

(ii) $y^2 = 4ax$

Solⁿ. Here, $y^2 = 4ax$ ..... (i)

Diffⁿ. (i) w. r. t. 'x' we get,

$2y\,y_1 = 4a$

or, $y_1 = \frac{4a}{2y} = \frac{2a}{y}$

Therefore, the equation of tangent at (x, y)

$Y - y = y_1 (X - x)$

or, $Y - y = \frac{2a}{y}(X - x)$

or, $Yy - y^2 = 2aX - 2ax$

or, $2aX - Y_y = -2ax$

or, $2aX - Y_y + 2ax = 0$..... (i)

Length of perpendicular (0, 0) to the equation (i) is,

$$P = \frac{2ax}{\sqrt{(2a)^2 + y^2}}$$

or, $P = \frac{2ax}{\sqrt{4a^2 + y^2}}$ $\quad$ or, $P = \frac{4a^2x^2}{4a^2 + y^2} = \frac{4a^2x^2}{4a^2 + 4ax} = \frac{ax^2}{x + a}$

or, $ax^2 - p^2x - ap^2 = 0$ .....(ii)

We have, $r^2 = x^2 + y^2$

or, $r^2 = x^2 + 4ax$ $\quad [y^2 = 4ax]$

$\therefore x^2 + 4ax - r^2 = 0$ .....(iii)

Now, eliminating x from equation (ii) and (iii) by the rule of cross multiplication, we get,

$$\frac{x^2}{p^2r^2 + 4a^2r^2} = \frac{x}{ar^2 - ap^2} = \frac{1}{4a^2 + p^2}$$

or, $\frac{x^2}{p^2r^2 + 4a^2p^2} = \frac{1}{4a^2 + p^2}$ and $\frac{x}{ar^2 - ap^2} = \frac{1}{4a^2 + p^2}$

or, $x^2 = \frac{p^2r^2 + 4a^2p^2}{4a^2 + p^2}$ and $x = \frac{ar^2 - ap^2}{4a^2 + p^2}$

i.e. $= x^2 = \frac{(ar^2 - ap^2)^2}{(4a^2 + p^2)^2}$ $\quad \therefore \frac{p^2r^2 + 4a^2p^2}{(4a^2 + p^2)} = \frac{(ar^2 - ap^2)^2}{(4a^2 + p^2)^2}$

or, i.e., $(ar^2 - ap^2)^2 = (4a^2 + p^2)(p^2r^2 + 4a^2p^2)$

Which is the required equation.

(iii) $x^{2/3} + y^{2/3} = a^{2/3}$

Solⁿ. We have given curve is,

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ ..... (i)

Diffⁿ. (i) w. r. t. 'x', w get,

$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

Let us take (X, Y) as the current coordinates.

Then the equation of tangket at any point (x, y) is,

$Y - y = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(X - x)$

or, $Y y^{-\frac{1}{3}} - y^{\frac{2}{3}} = -X x^{-\frac{1}{3}} + x^{\frac{2}{3}}$

or, $X x^{-\frac{1}{3}} + Y y^{-\frac{1}{3}} = x^{\frac{2}{3}} + y^{\frac{2}{3}}$

or, $X x^{-\frac{1}{3}} + Y y^{-\frac{1}{3}} = a^{\frac{2}{3}}$

Now, if P is the length of the perpendicular from the origin (0, 0) to this tangent, then,

$$P = \frac{0.x^{-\frac{1}{3}} + y^{-\frac{1}{3}} - a^{\frac{2}{3}}}{\sqrt{x^{-\frac{1}{3}} + y^{-\frac{2}{3}}}} = \frac{-a^{\frac{2}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}$$

$= \frac{-a^{\frac{2}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}}{a^{\frac{1}{3}}} = a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}$ (Ignore negatively)

Now, $r^2 + 3p^2 = x^2 + y^2 + 3a^{\frac{2}{3}} \cdot x^{\frac{2}{3}} \cdot y^{\frac{2}{3}}$

$= \left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^3 + 3x^{\frac{2}{3}}y^{\frac{2}{3}}\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)$

But $a^{\frac{2}{3}} = x^{\frac{2}{3}} + y^{\frac{2}{3}}$

$$\text{or, } r^2 + 3p^2 = \left(a^{\frac{2}{3}}\right)^3 = a^2$$

$$\therefore r^2 + 3p^2 = a^2$$

Which is required pedal equations.

15. Find the pedal equation of the following curves

(i) $r = a(1 - \cos\theta)$

Solⁿ. We have,

$r = a(1 - \cos\theta)$ ..... (i)

Diffⁿ (i) w. r. t. θ, we get

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\text{Now, } \tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{a\sin\theta} = \frac{a(1-\cos\theta)}{a\sin\theta}$$

$$= \frac{2\sin^2\frac{\theta}{2}}{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\tan\phi = \tan\frac{\theta}{2}$$

$$\therefore \phi = \frac{\theta}{2}$$

Also, we know, $p = r\sin\phi$

$p = r\sin\phi$

$$\text{or, } p = r\sin\frac{\theta}{2}$$

$$\text{or, } p^2 = r^2\sin^2\frac{\theta}{2} \text{ ..... (ii)}$$

But $r = a(1 - \cos\theta)$

$$\text{or, } r = a \cdot 2\sin^2\frac{\theta}{2}$$

$$\therefore \sin^2\frac{\theta}{2} = \frac{r}{2a} \text{ ..... (iii)}$$

Then, from equation (ii), we have

$$p^2 = r^2 \cdot \frac{r}{2a}$$

$$\therefore r^3 = 2ap^2$$

(ii) $r = a(1 + \cos\theta)$

Solⁿ. We have, $r = a(1 + \cos\theta)$ ..... (i)

Diffⁿ (i) with respect to θ, we get

Gaurab Shrestha

$$\frac{dr}{d\theta} = a(0 - \sin\theta) = -a\sin\theta$$

$$\text{Now, } \tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{-a\sin\theta} = \frac{a(1+\cos\theta)}{-a\sin\theta}$$

$$= -\frac{a\,2\cos^2\frac{\theta}{2}}{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

$$\therefore \tan\phi = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\therefore \phi = \frac{\pi}{2} + \frac{\theta}{2}$$

Also, we know,

$p = r\sin\phi$

$$p = r\sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\text{or, } p = r\cos\frac{\theta}{2} \qquad \text{or, } p^2 = r^2\cos^2\frac{\theta}{2} \text{ ..... (ii)}$$

But, $r = a(1 + \cos\theta)$

$$\therefore r = a \cdot 2\cos^2\frac{\theta}{2}$$

$$\text{So, } \cos^2\frac{\theta}{2} = \frac{r}{2a} \text{ ..... (iii)}$$

Then form equation (ii) we have,

$$\therefore p^2 = r^2 \cdot \frac{r}{2a} \qquad \left[\because r^3 = 2ap^2\right]$$

(iii) $r^2 = a^2\cos 2\theta$

Solⁿ. We have, $r^2 = a^2\cos 2\theta$

$$\text{or, } 2\log r = \log a^2 + \log\cos 2\theta$$

$$\text{or, } 2\left(\frac{1}{r}\right)\frac{dr}{d\theta} = -\frac{2\sin 2\theta}{\cos 2\theta}$$

$$\text{or, } \tan\phi = -\cot 2\theta$$

$$\text{or, } \tan\phi = \tan\left(\frac{\pi}{2} + 2\theta\right)$$

$$\text{or, } \phi = \frac{\pi}{2} + 2\theta$$

Also, $p = r\sin\phi$

$$\text{or, } p = r\sin\left(\frac{\pi}{2} + 2\theta\right)$$

$$\text{or, } p = r\cos 2\theta$$

Gaurab Shrestha

or, $p = r \cdot \frac{r^2}{a^2}$

or, $r^3 = pa^2$ is the pedal equation.

(iv) $r^2 \cos 2\theta = a^2$

Sol^n. We have, $r^2\cos 2\theta = a^2$

or, $r^2 = a^2\sec 2\theta$ ..... (i)

Differentiating of (i) w. r. t. θ, we get

$2r\frac{dr}{d\theta} = a^2\sec 2\theta \cdot \tan 2\theta \cdot 2$

$\therefore \frac{dr}{d\theta} = \frac{a^2 \sec 2\theta \tan 2\theta}{2}$

But $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{\frac{a^2\cos 2\theta \cdot \tan 2\theta}{r}} = \frac{r^2}{a^2 \sec 2\theta \tan 2\theta}$

But, $r^2 = a^2\sec 2\theta$

$\therefore \tan\phi = \frac{r^2}{r^2 \tan 2\theta} = \cot 2\phi = \tan\left(\frac{\pi}{2} - \theta\right)$

$\therefore \phi = \frac{\pi}{2} - 2\theta$ ϕ gives,

$p = r\sin$

Again, $p = r\sin\left(\frac{\pi}{2} - \theta\right)$

or, $p = -r\cos 2\theta$

But, $r^2\cos 2\theta = a^2$

So, $\cos 2\theta = \frac{a^2}{r^2}$

$\therefore p = r - \frac{a^2}{r^2}$

$\therefore pr = a^2$ is pedal equation.

(v) $r^m = a^m \cos m\theta$

Sol^n. We have, $r^m = a^m \cos m\theta$ ..... (i)

Taking log on both sides of (i), we get,

$m\log r = m \log a + \log \cos m\theta$

Diff^n both sides with respect to θ, we get,

$m \cdot \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\cos m\theta}(-\sin m\theta)$

$m \cdot \frac{1}{r}\frac{dr}{d\theta} = -m + a\tan m\theta$

We know that, $r\frac{d\theta}{dr} = -\cot m\theta$

But we know, $\tan\phi = \tan\left(\frac{\pi}{2} + m\theta\right)$

$\therefore \phi = \frac{\pi}{2} + m\theta$

Again, from, $p = r\sin\phi$ gives,

$p = r\sin\left(\frac{\pi}{2} + m\theta\right)$

or, $p = r\cos m\theta$ ..... (ii)

But, $r^m = a^m \cos m\theta$

$\therefore \cos m\theta = \frac{r^m}{a^m}$

Then from (ii), we have,

$p = r \cdot \frac{r^m}{a^m}$ or, $r^{m+1} = a^m p$ which is required pedal equation.

(vi) $r = ae^{\theta\cot\alpha}$

Sol^n. We have, $r = ae^{\theta\cot\alpha}$

Diff^n both sides, w. r. t. θ, we get

$\frac{dr}{d\theta} = ae^{\theta\cot\alpha} \cdot \cot\alpha$

$\frac{dr}{d\theta} = r\cot\alpha$

Now, $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{r\cot\alpha} = \tan\alpha$

$\therefore \phi = \alpha$

Again, we know

$p = r\sin\phi$ which is required pedal equation.

$= r\sin\alpha$

Unit - II

Integration and its Application

Chapter - 7 Indefinite Integral

Chapter - 8 Definite Integral and Improper Integral

Chapter - 9 Reduction Formula and Beta Gamma Functions

Chapter - 10 Tracing of Curves

Chapter - 11 Quadrature-Area

Chapter - 12 Rectification (Arc-length)

Chapter - 13 Volume and Surface

Indefinite Imegral

7

Introduction

Integration- Integration as the inverse of differentiation and make this as our starting point; or else we may start with defining integration as a certain summation and that proceed to show that the result is identical with the reversal of a differentiation.

Let f(x) be differentiable function and its derivative with respect to x is F(x)

i.e. $\frac{d}{dx}[f(x)] = F(x)$ then,

the integration of F(x) with respect to x is,

i.e. $\int F(x)\,dx = f(x)$.

I. Fundamental properties

(i) $\int [f_1(x) \pm f_2(x) \pm f_3(x) \ldots\ldots\ldots \text{ to n terms}]\,dx$
$= \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \int f_3(x)\,dx + \ldots\ldots \text{ ton terms}$

(ii) $\int c\,f(x)\,dx = c\int f(x)\,dx$

II. Fundamental integrals

(i) $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

(ii) $\int \frac{1}{x^n}\,dx = -\frac{1}{(n-1)\,x^{n-1}} \quad (n \neq 1)$

(iii) $\int dx = x$

(iv) $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}$

(v) $\int \frac{1}{x}\,dx = \log(x)$

(vi) $\int e^{ax}\,dx = \frac{e^{ax}}{a}$

(vii) $\int e^x\,dx = e^x$

(viii) $\int a^x\,dx = \frac{a^x}{\log_e^a} \quad (a > 0)$

(ix) $\int \sin ax\, dx = -\frac{\cos ax}{a}$

(x) $\int \sin x\, dx = -\cos x$

(xi) $\int \cos ax\, dx = \frac{\sin ax}{a}$

(xii) $\int \cos x\, dx = \sin x$

(xiii) $\int \sec^2 x\, dx = \tan x$

(xiv) $\int \operatorname{cosec}^2 dx = -\cot x$

(xv) $\int \sec x \tan x\, dx = \sec x$

(xvi) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x$

(xvii) $\int \sin hx\, dx = \cosh x$

(xviii) $\int \cosh x\, dx = \sinh x$

(xix) $\int \tan hx\, dx = \log|(\cosh x)|$

(xx) $\int \coth x\, dx = \log|(\sinh x)|$

(xxi) $\int \operatorname{cosech} x\, dx = \log\left|\left(\tanh \frac{1}{2}x\right)\right|$

(xxii) $\int \operatorname{sech} x\, dx = 2\tan^{-1}(e^x)$

(xxiii) $\int \operatorname{sech}^2 x dx = \tanh x$

(xxiv) $\int \operatorname{cosech}^2 x dx = -\coth x$

Standard integral

(i) $\int \frac{f'(x)}{f(x)} dx = \log|f(x)|$

(ii) $\int \tan x\, dx = \log|\sec x|$

(iii) $\int \cot x\, dx = \log|\sin x|$

(iv) $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\ (a \neq 0)$

(v) $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|;\ |x| > |a|$

(vi) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|;\ |x| < |a|$

(vii) $\int \frac{1}{\sqrt{(x^2+a^2)}} dx = \log\left|x+\sqrt{x^2+a^2}\right| = \sinh^{-1}\frac{x}{a}$

(viii) $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x+\sqrt{x^2-a^2}\right| = \cosh^{-1}\frac{x}{a}$

(ix) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}\ (|x| < |a|)$

(x) $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$

(xi) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1}x$

(xii) $\int u\,v\, dx = u\int v\, dx - \int\left(\frac{dy}{dx}\cdot\int v\, dx\right)dx$

Integral of the product of two functions
= 1st function x integral of 2nd
Integral of (Differential coefficient of 1st × integral of 2nd)

(xiii) $\int e^{ax}\cos bx\, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx\,.\tan^{-1}\frac{b}{a}\right)$

(xiv) $\int e^{ax}\sin bx\, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx - \tan^{-1}\frac{b}{a}\right)$

(xv) $\int\sqrt{x^2+a^2}\, dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left|\left(x+\sqrt{x^2+a^2}\right)\right|$

$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

(xvi) $\int\sqrt{x^2-a^2}\, dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left|\left(x+\sqrt{x^2-a^2}\right)\right|$

$= \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

(xvii) $\int\sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

(xviii) $\int \operatorname{cosec} x\, dx = \log\left|\tan\frac{1}{2}x\right|$

$= \log|\operatorname{cosec} x - \cot x|$

(xix) $\int \sec x\, dx = \log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$

$= \log|\sec x + \tan x|$

Exercise - 7

Integrate the following.

1. $\int \frac{3x^2}{1+x^6}\,dx$

Solⁿ. Let $I = \int \frac{3x^2}{1+x^6}\,dx = \int \frac{3x^2}{1+\left(x^3\right)^2}\,dx$ ..... (i)

Put $x^3 = t$
Diff. w. r. t. x
$3x^2\,dx = dt$
Now (i) becomes,

$I = \int \frac{1}{1+t^2}\,dt = \tan^{-1}t + C$

$\therefore I = \tan^{-1}(x^3) + C$. Ans.

2. $\int \frac{1}{e^x+e^{-x}}\,dx$

Solⁿ. Let $I = \int \frac{1}{e^x+e^{-x}}\,dx = \int \frac{e^x}{e^{2x}+1}\,dx$

$I = \int \frac{e^x}{1+(e^x)^2}\,dx$ ..... (i)

Put $e^x = t$
Diff. w. r. t. x
$e^x\,dx = dt$
Now (i) becomes,

$I = \int \frac{1}{1+t^2}\,dt = \tan^{-1}t + C$

$\therefore I = \tan^{-1}(e^x) + C$ Ans.

3. $\int \frac{1}{e^x+1}\,dx$

Solⁿ. Let $I = \int \frac{1}{e^x+1}\,dx = \int \frac{e^{-x}}{1+e^{-x}}\,dx$ ..... (i)

Put $1+e^{-x} = t$
Diff. w. r. t. x
$-e^{-x}dx = dt$
or, $e^{-x}\,dx = -dt$

Now (i) become,

$I = \int \frac{-1}{t}\,dt = -\log t + C$

$\therefore I = -\log(1+e^{-x}) + C$ Ans.

4. $\int \sqrt{\frac{a+x}{a-x}}\,dx$

Solⁿ. Let $I = \int \sqrt{\frac{a+x}{a-x}}\,dx$

Rationalizing the numerator we have

$$= \int \sqrt{\frac{a+x}{a-x} \times \frac{a+x}{a+x}}\,dx$$

$$= \int \sqrt{\frac{(a+x)^2}{a^2-x^2}}\,dx$$

$$= \int \frac{a+x}{\sqrt{a^2-x^2}}\,dx$$

$$= a\int \frac{1}{\sqrt{a^2-x^2}}\,dx + \int \frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$= a\,.\sin^{-1}\frac{x}{a} - \frac{1}{2}\,.\,2\sqrt{a^2-x^2} + C$$

$\therefore I = a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + C$ Ans.

5. $\int \frac{4x+3}{3x^2+3x+1}\,dx$

Solⁿ. Let $I = \int \frac{4x+3}{3x^2+3x+1}\,dx$

$$= \int \frac{\frac{4}{6}(6x+3) - \frac{12}{6} + 3}{3x^2+3x+1}\,dx$$

$$= \frac{2}{3}\int \frac{6x+3}{3x^2+3x+1}\,dx + \frac{1}{3}\int \frac{1}{x^2+x+\frac{1}{3}}\,dx$$

$$= \frac{2}{3}\log(3x^2+3x+1) + \frac{1}{3}\int \frac{1}{x^2+2\,.\,x\,.\,\frac{1}{2}+\left(\frac{1}{2}\right)^2-\frac{1}{4}+\frac{1}{3}}\,dx$$

$$= \frac{2}{3}\log(3x^2+3x+1) + \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{12}}\right)^2}\,dx$$

$$= \frac{2}{3}\log(3x^2+3x+1) + \frac{\sqrt{12}}{3}\tan^{-1}\sqrt{12}\left(x+\frac{1}{2}\right) + C$$

$$\therefore I = \frac{2}{3}\log(3x^2+3x+1) + \frac{2}{\sqrt{3}}\tan^{-1}\sqrt{3}\,(2x+1) + C \text{ Ans.}$$

6. $\int \frac{1}{x\sqrt{x^4-1}}\,dx$

Sol^n. Let $I = \int \frac{1}{x\sqrt{x^4-1}}\,dx$

$$= \int \frac{x}{x^2\sqrt{(x^2)^2-1}}\,dx \quad \ldots\ldots (i)$$

Put $x^2 = t$

Diff. w.r.t. 'x'

$2x\,dx = dt$

$x\,dx = \frac{1}{2}dt$

Now, (i) becomes,

$$I = \frac{1}{2}\int \frac{1}{t\sqrt{t^2-1}}\,dt = \frac{1}{2}\sec^{-1}t + C$$

$$\therefore I = \frac{1}{2}\sec^{-1}x^2 + C \text{ Ans.}$$

7. $\int \frac{(x+1)}{\sqrt{4+8x-5x^2}}\,dx$

Sol^n. Let $I = \int \frac{x+1}{\sqrt{4+8x-5x^2}}\,dx$

$$= \int \frac{\frac{1}{10}(8-10x)+\frac{8}{10}+1}{\sqrt{4+8x-5x^2}}\,dx$$

$$= -\frac{1}{10}\int \frac{8-10x}{\sqrt{4+8x-5x^2}}\,dx + \frac{9}{5}\int \frac{1}{\sqrt{4+8x-5x^2}}\,dx$$

$$= \frac{-2}{10}\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\int \frac{1}{\sqrt{-\left(x^2-\frac{8}{5}x-\frac{4}{5}\right)}}\,dx$$

$$= -\frac{1}{5}\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\int \frac{1}{\sqrt{-\left\{x^2-2x\frac{4}{5}+\left(\frac{4}{5}\right)^2+\frac{16}{25}-\frac{4}{5}\right.}}}$$

$$= -\frac{1}{5}\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\int \frac{1}{\sqrt{-\left\{\left(x-\frac{4}{5}\right)^2-\left(\frac{6}{5}\right)^2\right\}}}\,dx$$

$$= -\frac{1}{5}\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\int \frac{1}{\sqrt{\left(\frac{6}{5}\right)^2-\left(x-\frac{4}{5}\right)^2}}\,dx$$

$$= -\frac{1}{5}\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\sin^{-1}\left(x-\frac{4}{5}\right)\cdot\frac{5}{6} + C$$

$$\therefore I = \frac{9}{5\sqrt{5}}\sin^{-1}\frac{(5x-4)}{6} - \frac{1}{5}\sqrt{4+8x-5x^2} + C \text{ Ans.}$$

8. $\int \sqrt{\frac{1+x}{1-x}}\,dx$ **[B.E. 2060]**

Sol^n. $I = \int \sqrt{\frac{1+x}{1-x}}\,dx$

$$= \int \sqrt{\frac{1+x}{1-x}\times\frac{1+x}{1+x}}\,dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}x - \sqrt{1-x^2} + C$$

$$\therefore I = \sin^{-1}x - \sqrt{1-x^2} + C \text{ Ans.}$$

9. $\int \sqrt{18x-65-x^2}\,dx$

Sol^n. Let $I = \int \sqrt{18x-65-x^2}\,dx$

$$= \int \sqrt{-\left(x^2-2x.9+(9)^2-81+65\right)}\,dx$$

$$= \int \sqrt{-\left\{(x-9)^2-(4)^2\right\}}\,dx$$

$$= \int \sqrt{(4)^2-(x-9)^2}\,dx$$

$$= \frac{(x-9)}{2}\cdot\sqrt{(4)^2-(x-9)^2} + \frac{4^2}{2}\sin^{-1}\frac{(x-9)}{4} + C$$

$$\therefore I = \frac{(x-9)}{2}\sqrt{18x-65-x^2} + 8\sin^{-1}\left(\frac{x-9}{4}\right) + C \text{ Ans.}$$

10. $\int \frac{\cos x\, dx}{\sqrt{2\sin^2 x + 3\sin x + 4}}$

Sol^n. Let $I = \int \frac{\cos x}{\sqrt{2\sin^2 x + 3\sin x + 4}}\, dx$ ..... (i)

Put $\sin x = t$ diff. w. r. t. x

$\cos x\, dx = dt$

Now (i) becomes,

$$I = \int \frac{1}{\sqrt{2t^2 + 3t + 4}}\, dt = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{t^2 + \frac{3}{2}t + 2}}\, dt$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{t^2 + 2t\frac{3}{4} - \left(\frac{3}{4}\right)^2 - \frac{9}{16} + 2}}\, dt$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(t + \frac{3}{4}\right)^2 + \left(\frac{23}{4}\right)^2}}\, dt$$

$$= \frac{1}{\sqrt{2}} \sinh^{-1}\left(t + \frac{3}{4}\right) \cdot \frac{4}{23} + C$$

$$= \frac{1}{\sqrt{2}} \sinh^{-1}\left(\frac{4t+3}{23}\right) + C$$

$$\therefore I = \frac{1}{\sqrt{2}} \sinh^{-1}\left(\frac{4\sin x + 3}{23}\right) + C \text{ Ans.}$$

11. $\int \frac{1}{\sqrt{1+\sin x}}\, dx$

Sol^n. Let $I = \int \frac{1}{\sqrt{1+\sin x}}\, dx$

$$= \int \frac{1}{\sqrt{\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right)}}\, dx$$

$$= \int \frac{1}{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}}$$

$$= \int \frac{1}{\sin\frac{x}{2} + \cos\frac{x}{2}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\frac{1}{\sqrt{2}}\sin\frac{x}{2} + \frac{1}{\sqrt{2}}\cos\frac{x}{2}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sin\frac{x}{2}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos\frac{x}{2}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(\frac{x}{2} + \frac{\pi}{2}\right) dx$$

$$= \frac{1}{\sqrt{2}} \times 2 \log\left\{\tan\left(\frac{x}{4} + \frac{\pi}{8}\right)\right\} + C$$

$$\therefore I = \sqrt{2} \log\left\{\tan\left(\frac{x}{4} + \frac{\pi}{8}\right)\right\} + C \text{ Ans.}$$

12. $\int \frac{1}{4+5\cos x}\, dx$

Sol^n. Let $I = \int \frac{1}{4+5\cos x}\, dx$

$$= \int \frac{1}{4 + 5\cos^2\frac{x}{2} - 5\sin^2\frac{x}{2}}\, dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{4\sec^2\frac{x}{2} + 5 - 5\tan^2\frac{x}{2}}\, dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{4 + 4\tan^2\frac{x}{2} + 5 - 5\tan^2\frac{x}{2}}\, dx$$

$$\therefore I = \int \frac{\sec^2\frac{x}{2}}{9 - \tan^2\frac{x}{2}}\, dx \text{ ..... (i)}$$

Put $\tan\frac{x}{2} = t$ diff. w. r. t. 'x'

$\sec^2\frac{x}{2} = 2dt$

Now, (i) becomes

$$I = \int \frac{2}{(3)^2 - t^2}\, dt$$

$$= 2 \cdot \frac{1}{2.3} \log\frac{3+t}{3-t} + C$$

$$\therefore I = \frac{1}{3}\log\left(\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right) + C \text{ Ans.}$$

13. $\int \frac{1}{1+3\sin^2 x}\,dx$

Sol^n. Let $I = \int \frac{1}{1+3\sin^2 x}\,dx$

$$= \int \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x}\,dx$$

$$= \int \frac{\sec^2 x}{1+\tan^2 x + 3\tan^2 x}\,dx$$

$$\therefore I = \int \frac{\sec^2 x}{1+4\tan^2 x}\,dx \text{ ..... (i)}$$

Put, $2\tan x = t$, diff. w. r. t. 'x'

$2\sec^2 x\,dx = dt$

Now (i) becomes

$$I = \frac{1}{2}\int \frac{1}{1+t^2}\,dt$$

$$= \frac{1}{2}\tan^{-1} t + C$$

$$\therefore I = \frac{1}{2}\tan^{-1}(2\tan x) + C \text{ Ans.}$$

14. $\int \frac{(1+\cos x)}{\sin x \cos x}\,dx$

Sol^n. Let $I = \int \frac{(1+\cos x)}{\sin x \cos x}\,dx$

$$= 2\int \frac{1}{2\sin x \cos x}\,dx + \int \frac{\cos x}{\sin x \cos x}\,dx$$

$$= 2\int \frac{1}{\sin 2x}\,dx + \int \operatorname{cosec} x\,dx$$

$$= 2\int \operatorname{cosec} 2x\,dx + \int \operatorname{cosec} x\,dx$$

$$= 2 \cdot \frac{1}{2}\log \tan x + \log\left(\tan\frac{x}{2}\right) + C$$

$$\therefore I = \log\left(\tan x \cdot \tan\frac{x}{2}\right) + C \text{ Ans.}$$

15. $\int \frac{x^2 \tan^{-1} x}{1+x^2}\,dx$ [B. E. 2059]

Sol^n. Let $I = \int \frac{x^2 \tan^{-1} x}{1+x^2}\,dx$ ..... (i)

Put $\tan^{-1} x = t$ diff. (i) w. r. t. 'x'

$$\frac{1}{(1+x^2)}\,dx = dt$$

Now (i) becomes

$$\therefore I = \int t \tan^2 t\,dt$$

$$= \int t\,(\sec^2 t - 1)\,dt$$

$$= \int t \sec^2 t\,dt - \int t\,dt$$

$$= t\tan t - \int \tan t - \frac{t^2}{2} + C$$

$$= t\tan t - \log\sec t - \frac{t^2}{2} + C$$

$$= t\tan t - \log\sqrt{1+\tan^2 t} - \frac{t^2}{2} + C$$

$$= \tan^{-1} x\,\tan(\tan^{-1} x) - \log\sqrt{1+x^2} - \frac{1}{2}(\tan^{-1} x)^2 + C$$

$$\therefore I = x\tan^{-1} x - \frac{1}{2}\log(1+x^2) - \frac{1}{2}(\tan^{-1} x)^2 + C \text{ Ans.}$$

16. $\int \frac{1}{1+\tan x}\,dx$

Sol^n. Let $I = \int \frac{1}{1+\tan x}\,dx$

$$= \int \frac{1}{1+\frac{\sin x}{\cos x}}\,dx$$

$$= \int \frac{\cos x}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) + (\cos x - \sin x)}{(\sin x + \cos x)}\,dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x)}{(\sin x + \cos x)}\,dx + \frac{1}{2}\int\left(\frac{\cos x - \sin x}{\sin x + \cos x}\right)dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\log(\sin x + \cos x) + C$$

$$\therefore I = \frac{1}{2}x + \frac{1}{2}\log(\sin x + \cos x) + C \text{ Ans.}$$

17. $\int \frac{1}{a^2-b^2\cos^2 x}\,dx$, $(a > b)$

Sol^n. Let $I = \int \frac{1}{a^2-b^2\cos^2 x}\,dx$

$$= \int \frac{\sec^2 x}{a^2\sec^2 x - b^2}\,dx$$

$$= \int \frac{\sec^2 x}{a^2+a^2\tan^2 x - b^2}\,dx$$

$$\therefore I = \int \frac{\sec^2 x}{a^2-b^2+a^2\tan^2 x} \quad \text{.....(i)}$$

Since $a > b \Rightarrow a^2 - b^2 > 0$

Put $a\tan x = t$ diff. w. r. t. 'x'

$a\sec^2 x\,dx = dt$

$\sec^2 x\,dx = \frac{1}{a}dt$

Now (i) becomes

$$I = \frac{1}{a}\int \frac{1}{a^2-b^2+t^2}\,dt$$

$$= \frac{1}{a}\int \frac{1}{\left(\sqrt{a^2-b^2}\right)^2+t^2}\,dt$$

$$= \frac{1}{a\sqrt{a^2-b^2}}\tan^{-1}\frac{t}{\sqrt{a^2-b^2}} + C$$

$$\therefore I = \frac{1}{a\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{a\tan x}{\sqrt{a^2-b^2}}\right) + C \text{ Ans.}$$

18. $\int \sqrt{\sec x - 1}\,dx$

Sol^n. Let $I = \int \sqrt{\sec x - 1}\,dx$

$$= \int \sqrt{\frac{1}{\cos x} - 1}\,dx = \int \sqrt{\frac{1-\cos x}{\cos x}}\,dx$$

$$\therefore I = \int \sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}-1}}\,dx = \sqrt{2}\int \frac{\sin\frac{x}{2}}{\sqrt{2\cos^2\frac{x}{2}-1}} \quad \text{.....(i)}$$

Put $\sqrt{2}\cos\frac{x}{2} = t$ diff. w.r.t. 'x'

$$-\sqrt{2}\cdot\frac{1}{2}\sin\frac{x}{2}\,dx = dt \Rightarrow \sin\frac{x}{2}\,dx = -\sqrt{2}\,dt$$

Now (i) becomes

$$I = -\sqrt{2}\times\sqrt{2}\int \frac{1}{t^2-1}\,dt$$

$$= -2\cosh^{-1} t + C$$

$$\therefore I = -2\cosh^{-1}\left(\sqrt{2}\cos\frac{x}{2}\right) + C \text{ Ans.}$$

19. $\int \frac{\cos x}{5-3\cos x}\,dx$

Sol^n. Let $I = \int \frac{\cos x}{5-3\cos x}\,dx$

$$= \int \frac{-\frac{1}{3}(5-3\cos x)+\frac{5}{3}}{5-3\cos x}\,dx$$

$$= -\frac{1}{3}\int dx + \frac{5}{3}\int \frac{1}{5-3\cos x}\,dx$$

$$= -\frac{1}{3}x + \frac{5}{3}\int \frac{1}{5-6\cos^2\frac{x}{2}+3}\,dx$$

$$= -\frac{1}{3}x + \frac{5}{3}\int \frac{1}{8-6\cos^2\frac{x}{2}}\,dx$$

$$= -\frac{1}{3}x + \frac{5}{3}\int \frac{\sec^2\frac{x}{2}}{8\sec^2\frac{x}{2}-6}\,dx$$

$$= -\frac{1}{3}x + \frac{5}{3}\int \frac{\sec^2\frac{x}{2}}{8+8\tan^2\frac{x}{2}-6}\,dx$$

$$= -\frac{1}{3}x + \frac{5}{3}\int \frac{\sec^2\frac{x}{2}}{2+8\tan^2\frac{x}{2}}\,dx$$

$$\therefore I = -\frac{1}{3}x + \frac{5}{6}\int \frac{\sec^2\frac{x}{2}}{1+4\tan^2\frac{x}{2}}\,dx \quad \text{.....(i)}$$

Put $2\tan\frac{x}{2} = t$ diff. w. r. t. 'x'

$2\cdot\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt \Rightarrow \sec^2\frac{x}{2}\,dx = dt$

Now (i) becomes

$$I = -\frac{1}{3}x + \frac{5}{6}\int\frac{1}{1+t^2}\,dt$$

$$= -\frac{1}{3}x + \frac{5}{6}\tan^{-1}t + C$$

$$\therefore I = -\frac{1}{3}x + \frac{5}{6}\tan^{-1}\left(2\tan^{-1}\frac{x}{2}\right) + C \text{ Ans.}$$

20. $\int\frac{1}{1-\cos x+\sin x}\,dx$

Sol^n. Let $I = \int\frac{1}{1-\cos x+\sin x}\,dx$

$$= \int\frac{1}{2\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$$

$$\therefore I = \frac{1}{2}\int\frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+\tan\frac{x}{2}}\,dx \text{ ..... (i)}$$

Put $\tan\frac{x}{2} = t$. diff. w. r. t. 'x'

$$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt \Rightarrow \sec^2\frac{x}{2}\,dx = 2\,dt$$

Now (i) becomes,

$$I = \frac{1}{2}\cdot 2\int\frac{1}{t^2+t}\,dt$$

$$= \int\frac{1}{t(t+1)}\,dt = \int\left(\frac{1}{t}-\frac{1}{1+t}\right)dt$$

$$= \log t - \log(1+t) + C$$

$$= \log\left(\frac{t}{1+t}\right) + C$$

$$\therefore I = \log\left(\frac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right) + C$$

$$= -\log\left(\frac{1+\tan\frac{x}{2}}{\tan\frac{x}{2}}\right) + C$$

$$= -\log\left(\cot\frac{x}{2}+1\right) + C \text{ Ans.}$$

21. $\int\frac{5\cos x+6}{2\cos x+\sin x+3}\,dx$

Sol^n. Let $I = \int\frac{5\cos x+6}{2\cos x+\sin x+3}\,dx$

$$= \int\frac{2(2\cos x+\sin x+3)+(\cos x-2\sin x)}{2\cos x+\sin x+3}\,dx$$

$$= 2\int dx + \int\frac{\cos x-2\sin x}{2\cos x+\sin x+3}\,dx$$

$$\therefore I = 2x + \log(2\cos x + \sin x + 3) + C \text{ Ans.}$$

22. $\int\frac{(x+\sin x)}{1+\cos x}\,dx$

Sol^n. Let $I = \int\frac{(x+\sin x)}{1+\cos x}\,dx$

$$= \int\frac{x}{1+\cos x}\,dx + \int\frac{\sin x}{1+\cos x}\,dx$$

$$= \int\frac{x}{2\cos^2\frac{x}{2}}\,dx + \int\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int x\sec^2\frac{x}{2}\,dx + \int\tan\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left(x\,.\,2\tan\frac{x}{2} - \int 2\,.\,\tan\frac{x}{2}\,dx\right) + \frac{1}{2}\log\sec\frac{x}{2} + c$$

$$= x\tan\frac{x}{2} - \frac{1}{2}\log\sec\frac{x}{2} + \frac{1}{2}\log\sec\frac{x}{2} + C$$

$$\therefore I = x\tan\frac{x}{2} \text{ Ans.}$$

23. $\int\frac{\cos x}{2\sin x+3\cos x}$

Sol^n. Let $I = \frac{\cos x}{2\sin x+3\cos x}\,dx$

Let cosx = A (denominator) + B (differential of denominator)

cosx = A (2sinx + 3cosx) + B (2cosx – 3 sinx)

∴ cosx = (2A – 3B) . sinx + B (3A + 2B) cosx

Comparing the coefficient of sinx and cosx of both sides, we get

$2A - 3B = 0,\ 3A + 2B = 1$

Solving $A = \frac{1}{13}$, $B = \frac{2}{13}$

$$\therefore \cos x = \frac{3}{13}(2\sin x + 3\cos x) + \frac{2}{13}(2\cos x - 3\sin x)$$

Now, $I = \int \frac{\cos x}{2\sin x + 3\cos x} dx$

$= \frac{3}{13}\int dx + \frac{2}{13}\int \frac{2\cos x - 3\sin x}{2\sin x + 3\cos x} dx$

$= \frac{3}{13}x + \frac{2}{13}\log(2\sin x + 3\cos x) + C$

$\therefore I = \frac{3}{13}x + \frac{2}{13}\log(2\sin x + 3\cos x) + C$ Ans.

24. $\int \frac{e^x(1+\sin x)}{1+\cos x} dx$

Sol^n. Let $I = \int \frac{e^x(1+\sin x)}{1+\cos x} dx$

$= \int \frac{e^x}{1+\cos x} dx + \int \frac{e^x \sin x}{1+\cos x} dx$

$= \int \frac{e^x}{2\cos^2\frac{x}{2}} dx + \int \frac{e^x 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} dx$

$= \frac{1}{2}\int e^x \sec^2\frac{x}{2} dx + \int e^x \tan\frac{x}{2} dx$

$= \frac{1}{2}e^x . 2\tan\frac{x}{2} - \frac{1}{2} . 2\int e^x . \tan\frac{x}{2} dx + \int e^x \tan\frac{x}{2} dx + C$

$= e^x \tan\frac{x}{2} + C$

$\therefore I = e^x \tan\frac{x}{2} + C$ Ans.

25. $\int \frac{1}{\sin x(3+2\cos x)} dx$ **[B. E. 2057, 2062]**

Sol^n. Let $I = \int \frac{1}{\sin x(3+2\cos x)} dx$

$= \int \frac{\sin x}{\sin^2 x(3+2\cos x)} dx$

$I = \int \frac{\sin x}{(1-\cos^2 x)(3+2\cos x)} dx$ ..... (i)

Put $\cos x = t$ diff. w. r. t. 'x'

$-\sin x\, dx = dt$

Now (i) becomes,

$I = -\int \frac{1}{(1-t^2)(3+2t)} dt$

$= -\int \frac{1}{(1+t)(1-t)(3+2t)} dt$

By using simple partial fraction

Now, $\frac{-1}{(1+t)(1-t)(3+2t)} = \frac{A}{(1+t)} + \frac{B}{(1-t)} + \frac{C}{(3+2t)}$

$\therefore -1 = A(1-t)(3+2t) + B(1+t)(3+2t) + C(1+t)(1-t)$

Put $t = 1, \Rightarrow B = \frac{-1}{10}$

Put $t = -1, \Rightarrow A = \frac{1}{2}$

Equating the coefficient of t^2; we get

$-2A + 2B - C = 0$

$-2\left(-\frac{1}{2}\right) + 2\left(\frac{-1}{10}\right) - C = 0$

or, $1 - \frac{1}{5} - C = 0$

or, $\frac{4}{5} - C = 0$

$\Rightarrow C = \frac{4}{5}$

$\therefore I = -\frac{1}{2}\int \frac{1}{1+t} dt - \frac{1}{10}\int \frac{1}{(1-t)} dt + \frac{4}{5}\int \frac{1}{(3+2t)} dt$

$= -\frac{1}{2}\log(1+t) + \frac{1}{10}\log(1-t) + \frac{2}{5}\log(3+2t) + C$

$\therefore I = -\frac{1}{2}\log(1+\cos x) + \frac{1}{10}\log(1-\cos x) + \frac{2}{5}\log(3+2\cos x) + C$

26. $\int \frac{1}{5+4\cos x} dx$

Sol^n. Let $I = \int \frac{1}{5+4\cos x} dx$

$= \int \frac{1}{5\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) + 4\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)} dx$

$= \int \frac{1}{9\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} dx$

$\therefore I = \int \frac{\sec^2\frac{x}{2}}{9 + \tan^2\frac{x}{2}} dx$ ..... (i)

Put $\tan\frac{x}{2} = t$ diff. w. r. t. 'x'

$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

Now (i) becomes

$$I = 2\int\frac{1}{(3)^2+t^2}\,dt$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)+C$$

$$\therefore I = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+C \text{ Ans.}$$

27. $\int\frac{1}{5-4\sin x}\,dx$

Sol^n. Let $I = \int\frac{1}{5-4\sin x}\,dx$

$$= \int\frac{1}{5\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)-8\sin\frac{x}{2}\cdot\cos\frac{x}{2}}\,dx$$

$$\therefore I = \int\frac{\sec^2\frac{x}{2}}{5\tan^2\frac{x}{2}-8\tan\frac{x}{2}+5}\,dx \text{ ..... (i)}$$

Put $\tan\frac{x}{2} = t$ diff. w. r. t. 'x'

$\sec^2\frac{x}{2}\,dx = 2dt$

Now (i) becomes

$$I = 2\int\frac{1}{5t^2-8t+5}\,dt$$

$$= \frac{2}{5}\int\frac{1}{t^2-\frac{8}{5}t+1}\,dt$$

$$= \frac{2}{5}\int\frac{1}{t^2-2t\frac{4}{5}+\left(\frac{4}{5}\right)^2-\frac{16}{25}+1}\,dt$$

$$= \frac{2}{5}\int\frac{1}{\left(t-\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}\,dt$$

$$= \frac{2}{5}\cdot\frac{5}{3}\tan^{-1}\left(\left(t-\frac{4}{5}\right)\cdot\frac{5}{3}\right)+C$$

$$\therefore I = \frac{2}{3}\tan^{-1}\left\{\frac{(5t-4)}{3}\right\}+C$$

$$\text{or, } I = \frac{2}{3}\tan^{-1}\left\{\frac{\left(5\tan\frac{x}{2}-4\right)}{3}\right\}+C \text{ Ans.}$$

28. $\int\frac{\sin x}{\sqrt{a^2\cos^2 x+b^2\sin^2 x}}\,dx,\ a<b$

Sol^n. Let $I = \int\frac{\sin x}{\sqrt{a^2\cos^2 x+b^2\sin^2 x}}\,dx$

$$= \int\frac{\sin x}{\sqrt{a^2\cos^2 x+b^2(1-\cos^2 x)}}\,dx$$

$$\therefore I = \int\frac{\sin x}{\sqrt{b^2-(b^2-a^2)\cos^2 x}} \text{ ..... (i)}$$

Put $\sqrt{b^2-a^2}\cos x = t$ diff. w. r. t. 'x'

$-\sqrt{b^2-a^2}\sin x\,dx = dt$

$\sin x\,dx = -\frac{1}{\sqrt{b^2-a^2}}\,dt$

Now (i) becomes,

$$I = -\frac{1}{\sqrt{b^2-a^2}}\int\frac{dt}{\sqrt{b^2-t^2}} = -\frac{1}{\sqrt{b^2-a^2}}+\sin^{-1}\frac{t}{b}+C$$

$$\therefore I = -\frac{1}{\sqrt{b^2-a^2}}\sin^{-1}\left\{\frac{\sqrt{b^2-a^2}}{b}\cos x\right\}+C \text{ Ans.}$$

29. $\int\frac{(1+\sin x)}{\sin x(1+\cos x)}\,dx$

Sol^n. $I = \int\frac{(1+\sin x)}{\sin x(1+\cos x)}\,dx$

$$= \int\frac{1}{\sin x(1+\cos x)}\,dx + \int\frac{\sin x}{\sin x(1+\cos x)}\,dx$$

$$= \int\frac{\sin x}{\sin^2 x(1+\cos x)}\,dx + \int\frac{1}{2\cos^2\frac{x}{2}}\,dx$$

$$\therefore I = \int\frac{\sin x}{(1-\cos^2 x)(1+\cos x)}\,dx + \frac{1}{2}\cdot\int\sec^2\frac{x}{2}\,dx \text{ ..... (i)}$$

Put $\cos x = t$ in the first integrate of (i)

diff. w. r. t. 'x'

$-\sin x\,dx = dt$

Now (i) becomes,

$$I = -\int \frac{1}{(1-t^2)(1+t)}\,dt + \frac{1}{2}\cdot\tan\frac{x}{2}$$

Now by using the simple partial fraction,

$$\frac{-1}{(1-t)(1+t)^2} = \frac{A}{1-t} + \frac{B}{(1+t)} + \frac{C}{(1+t)^2}$$

or, $-1 = A(1+t)^2 + B(1-t)(1+t) + C(1-t)$

Put $t = 1 \Rightarrow A = -\frac{1}{4}$

Put $t = -1 \Rightarrow C = \frac{1}{2}$

Equating the coefficient of t^2 we get,

$A - B = 0$

$\Rightarrow B = A = -\frac{1}{4}$

$$\therefore I = -\frac{1}{4}\int\frac{1}{1-t}\,dt - \frac{1}{4}\int\frac{1}{(1+t)}\,dt + \frac{1}{2}\int\frac{1}{(1+t)^2}\,dt + \tan\frac{x}{2}$$

$$= \frac{1}{4}\log\left(\frac{1-t}{1+t}\right) - \frac{1}{2}\cdot\frac{1}{(1+t)} + \tan\frac{x}{2} + C$$

$$\therefore I = \frac{1}{4}\log\left(\frac{1-\cos x}{1+\cos x}\right) + \frac{1}{2}\cdot\frac{1}{1+\cos x} + \tan\frac{x}{2} + C$$

$$= \frac{1}{4}\log\left\{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\right\} + \frac{1}{2}\frac{1}{2\cos^2\frac{x}{2}} + \tan\frac{x}{2} + C$$

$$= \frac{1}{4}\log\left\{\tan^2\frac{x}{2}\right\} + \frac{1}{4}\sec^2\frac{x}{2} + \tan\frac{x}{2} + C$$

$$\therefore I = \frac{1}{2}\log\left(\tan\frac{x}{2}\right) + \frac{1}{4}\sec^2\frac{x}{2} + \tan\frac{x}{2} + C \text{ Ans.}$$

30. $\int\sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$

Sol^n. Let $I = \int\sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$

$$= \int\sqrt{\frac{\sin x\cos\alpha - \cos x\sin\alpha}{\sin x\cos\alpha + \cos x\sin\alpha}}\,dx$$

$$= \int\sqrt{\frac{(\sin x\cos\alpha - \cos x\sin\alpha)}{(\sin x\cos\alpha + \cos x\sin\alpha)}\times\frac{(\sin x\cos\alpha - \cos x\sin\alpha)}{(\sin x\cos\alpha - \cos x\sin\alpha)}}\,dx$$

$$= \int\frac{\sin x\cos\alpha - \cos x\sin\alpha}{\sqrt{\sin^2\alpha\cos^2\alpha - \cos^2 x\sin^2\alpha}}\,dx$$

$$= \int\frac{\sin x\cos\alpha}{\sqrt{\sin^2 x\cos^2\alpha - \cos^2 x\sin^2\alpha}} - \int\frac{\cos x\sin\alpha}{\sin^2 x\cos^2\alpha - \cos^2\alpha\sin^2\alpha}\,dx$$

$$= \cos\alpha\int\frac{\sin x}{\sqrt{\sin^2 x\cos^2\alpha - \cos^2 x\sin^2\alpha}}\,dx - \sin\alpha\int\frac{\cos x}{\sqrt{\sin^2 x\cos^2\alpha - \cos^2\sin^2\alpha}}$$

$$I = \cos\alpha\int\frac{\sin x}{\sqrt{\cos^2\alpha - \cos^2\alpha}}\,dx - \sin\alpha\int\frac{\cos x}{\sqrt{\sin^2 x - \sin^2\alpha}}\,dx \quad \text{..... (i)}$$

$\therefore \sin^2 x\cos^2\alpha - \cos^2\alpha\sin^2\alpha = (1-\cos^2 x)\cos^2\alpha - \cos^2 x(1-\cos^2\alpha)$

$= \cos^2\alpha - \cos^2 x\,.\,\cos^2\alpha - \cos^2 x + \cos^2 x\,.\,\cos^2\alpha$

$\therefore \sin^2 x\cos^2\alpha - \cos^2 x\sin^2\alpha = \cos^2\alpha - \cos^2 x$

Similarly $\sin^2 x\cos^2\alpha - \cos^2 x\sin^2\alpha = \sin^2 x - \sin^2\alpha$

Put $\cos x = u$ in the first integral

$\sin x\,dx = du$

and $\sin x = v$ in the second integral

$\cos x\,dx = dv$

Now (i) becomes,

$$I = -\cos\alpha\int\frac{1}{\sqrt{\cos^2\alpha - u^2}}\,du - \sin\alpha\int\frac{1}{\sqrt{v^2 - \sin^2\alpha}}\,du$$

$$I = -\cos\alpha\left\{\sin^{-1}\frac{u}{\cos\alpha}\right\} - \sin\alpha.\log\left\{v + \sqrt{v^2 + \sin^2\alpha}\right\} + C$$

$$I = -\cos\alpha\sin^{-1}\{\cos x\sec\alpha\} - \sin\alpha\,.\,\log\left\{\sin x + \sqrt{\sin^2 x - \sin^2\alpha}\right\} + C$$

31. $\int\frac{\cos x}{(1+\sin x)(2+\sin x)}\,dx$

Sol^n. Let $I = \int\frac{\cos x}{(1+\sin x)(2+\sin x)}\,dx$ ..... (i)

Put $\sin x = t$ diff. w. r. t. 'x'

$\cos x\,dx = dt$

Now, (i) becomes

$$I = \int\frac{1}{(1+t)(2+t)}\,dt$$

By using the simple partial fraction.

$$I = \int\left[\frac{1}{1+t} - \frac{1}{2+t}\right]dt$$

$$= \int\frac{1}{1+t}\,dt - \int\frac{1}{2+t}\,dt$$

$= \log(1+t) - \log(2+t) + C$

$= \log\left(\frac{1+t}{2+t}\right) + C$

$\therefore I = \log\left(\frac{1+\sin x}{2+\sin x}\right) + C$ Ans.

32. $\int \frac{1}{x+\sqrt{x}}\,dx$

Sol^n. Let $I = \int \frac{1}{x+\sqrt{x}}\,dx$

$= \int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\,dx$ ..... (i)

Put $\sqrt{x}+1 = t$ diff. w. r. t. 'x'

$\frac{2}{\sqrt{x}}\,dx = dt$

Now (i) becomes,

$I = 2\int \frac{1}{t}\,dt = 2\log t + C$

$\therefore I = 2\log(\sqrt{x}+1) + C$ Ans.

33. $\int \frac{x^2}{x^6-6x^3+5}\,dx$

Sol^n. Let $I = \int \frac{x^2}{x^6-6x^3+5}\,dx$ ..... (i)

Put $x^3 = t$ diff. w. r. t. 'x'

$3x^2dx = dt$

Now (i) becomes

$I = \frac{1}{3}\int \frac{1}{t^2-6t+5}\,dt = \frac{1}{3}\int \frac{1}{(t-1)(t-5)}\,dt$

By using the simple partial fraction

$= \frac{1}{3}\cdot\frac{1}{4}\int\left[\frac{1}{(t-5)} - \frac{1}{(t-1)}\right]dt$

$= \frac{1}{12}\{\log(t-5) - \log(t-1)\} + C$

$= \frac{1}{12}\log\left(\frac{t-5}{t-1}\right) + C$

$\therefore I = \frac{1}{12}\log\left(\frac{x^3-5}{x^3-1}\right) + C$ Ans.

34. $\int x^2\sqrt{x^6-1}\,dx$

Sol^n. Let $I = \int x^2\sqrt{x^6-1}\,dx$ ..... (i)

Put $x^3 = 6$ diff. w. r. t. 'x'

$3x^3\,dx = dt$

Now, (i) becomes,

$\therefore I = \frac{1}{3}\int \sqrt{t^2-1}\,dt$

$= \frac{1}{3}\frac{t\sqrt{t^2-1}}{2} - \frac{1}{2}\log\left\{t+\sqrt{t^2-1}\right\} + C$

$\therefore I = \frac{1}{6}\left\{x^3\sqrt{x^3-1} - \log\left(x^3+\sqrt{x^6-1}\right)\right\} + C$ Ans.

35. $\int \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx$

Sol^n. Let $I = \int \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx$

$\therefore I = \frac{1}{(b^2-a^2)}\left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right)dx$

$= \frac{1}{(b^2-a^2)}\left[\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{1}{b}\tan^{-1}\frac{x}{b}\right] + C$

$\therefore I = \frac{1}{(a^2-b^2)}\left[\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right] + C$ Ans.

36. $\int \frac{1}{\sqrt{(x-\alpha)(x-\beta)}}\,dx,\ \alpha > \beta$

Sol^n. Let $I = \int \frac{1}{\sqrt{(x-\alpha)(x-\beta)}}\,dx$ ..... (i)

Put $x-\alpha = t^2 \Rightarrow x = t^2+\alpha$

$dx = 2t\,dt$

Now (i) becomes

$I = \int \frac{1}{\sqrt{t^2(t^2+\alpha-\beta)}}\,2t\,dt$

$= 2\int \frac{1}{\sqrt{(t^2+\alpha-\beta)}}\,dt$

$= 2\int \frac{1}{t^2+\left(\sqrt{\alpha-\beta}\right)^2}\,dt$

$$= 2\log\left(t+\sqrt{t^2+\alpha-\beta}\right)$$

$$\therefore I = 2\log\left(\sqrt{x-\alpha}+\sqrt{x-\beta}\right)+C \text{ Ans.}$$

37. $\int \frac{xe^x}{(x+1)^2}dx$ **[B. E. 2061]**

Sol^n. Let $I = \int \frac{xe^x}{(x+1)^2}dx = \int \frac{(x+1-1)\,e^x}{(x+1)^2}dx$

$$= \int \frac{(x+1)\,e^x}{(x+1)^2}dx - \int \frac{e^x}{(x+1)^2}dx = \int \frac{e^x}{(x+1)^2}dx - \int \frac{e^x}{(x+1)^2}dx$$

$$= \frac{1}{(x+1)^2}e^x + \int \frac{1}{(x+1)^2}e^x\,dx - \int \frac{e^x}{(x+1)^2}dx$$

$$\therefore I = \frac{e^x}{(x+1)^2}+C \text{ Ans.}$$

38. $\int \frac{x^2}{(x+1)^2(x+2)}dx$

Sol^n. Let $I = \int \frac{x^2}{(x+1)^2(x+2)}dx$ ..... (i)

Now, by using the simple partial fraction.

$$\frac{x^2}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}$$

$x^2 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

Put $x = -1$ then $1 = 0 + B + 0 \Rightarrow B = 1$

Put $x = -2$ then $4 = 0 + 0 + C \Rightarrow C = 4$

Equating the coefficient of x^2 we have,

$A + C = 1$

$C = 1 - 4 \Rightarrow C = -3$

$$\therefore \frac{x^2}{(x+1)^2(x+2)} = \frac{-3}{(x+1)} + \frac{1}{(x+1)^2} + \frac{4}{(x+2)}$$

Now (i) becomes,

$$I = \int \frac{-3}{(x+1)}dx + \int \frac{1}{(x+1)^2}dx + \int \frac{4}{(x+2)}dx$$

$$= -3\log x - \frac{1}{(x+1)} + 4\log(x+2) + C$$

$$\therefore I = \frac{-1}{(x+1)} - 3\log x + 4\log(x+2) + C \text{ Ans.}$$

39. $\int \frac{x^3}{x^4-x^2-12}dx$

Sol^n. Let $I = \int \frac{x^3}{x^4-x^2-12}dx$ ..... (i)

Put $x^2 = t$

$2x\,dx = dt$

Now (i) becomes,

$$I = \frac{1}{2}\int \frac{t\,dt}{t^2-t-12}$$

$$= \frac{1}{2}\int \frac{\frac{1}{2}(2t-1)+\frac{1}{2}}{t^2-t-12}dt$$

$$= \frac{1}{4}\int \frac{2t-1}{t^2-t-12}dt + \frac{1}{4}\int \frac{1}{t^2-t-12}dt$$

$$= \frac{1}{4}\log(t^2-t-12) + \frac{1}{4}\int \frac{1}{(t+3)(t-4)}dt$$

$$= \frac{1}{4}\log(t^2-t-12) + \frac{1}{4}\left(\frac{-1}{7}\right)\left(\frac{1}{t+3}-\frac{1}{t-4}\right)dt$$

$$= \frac{1}{4}\log(t^2-t-12) - \frac{1}{28}\{\log(t+3)-\log(t-4)\}$$

$$= \frac{1}{4}\log\{(t+3)(t-4)\} - \frac{1}{28}\log(t+3) + \frac{1}{28}\log(t-4) + C$$

$$= \frac{1}{4}\log(t+3) + \frac{1}{4}\log(t-4) - \frac{1}{28}\log(t+3)$$

$$+ \frac{1}{28}\log(t-4) + C$$

$$= \frac{6}{28}\log(t+3) + \frac{8}{28}\log(t-4) + C$$

$$= \frac{3}{14}\log(t+3) + \frac{2}{7}\log(t-4) + C$$

$$\therefore I = \frac{3}{14}\log(x^2+3) + \frac{2}{7}\log(x^2-4) + C \text{ Ans.}$$

40. $\int \frac{1}{1+x^4}dx$

Sol^n. Let $I = \int \frac{1}{1+x^4}dx$

$$= \frac{1}{2}\int\left(\frac{x^2+1}{1+x^4} - \frac{x^2-1}{1+x^4}\right)dx$$

$$= \frac{1}{2}\int \frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(x^2+\frac{1}{x^2}\right)}dx - \frac{1}{2}\int \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(x^2+\frac{1}{x^2}\right)}dx$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}dx - \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}dx = \frac{1}{2}(I_1 - I_2) \text{ say}$$

In I_1 put $x - \frac{1}{x} = t$

Then, $\left(1+\frac{1}{x^2}\right)dx = dt$

$$\therefore I_1 = \int \frac{1}{\left(x-\frac{1}{x}\right)^2+2}dt = \int \frac{1}{t^2+2}dt$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)$$

In I_2 put $x + \frac{1}{x} = z$

Then $\left(1-\frac{1}{x^2}\right)dx = dz$

$$\therefore I_2 = \int \frac{1}{\left(x+\frac{1}{x}\right)^2-2}dz = \int \frac{1}{z^2-2}dz$$

$$= \frac{1}{2\sqrt{2}}\log\left(\frac{z-\sqrt{2}}{z+\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}\log\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right)$$

$$\therefore I = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x^2-1}{\sqrt{2}x} - \frac{1}{4\sqrt{2}}\log\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right) + C \text{ Ans.}$$

Definite Integral

8

If f(x) is an integral of F(x). Then we defined

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

$\int_a^b f(x)\,dx$ is said to be definite intergal of F(x) between the limit a and b, b is called upper limit and a is lower limt.

Exercise - 8

Integrate the following

1. $\int_0^a \frac{x+a}{a^2+x^2}dx$

Sol^n. Let $I = \int_0^a \frac{x+a}{a^2+x^2}dx$

$$= \int_0^a \frac{\frac{1}{2}(2x)+a}{a^2+x^2}dx$$

$$= \frac{1}{2}\int_0^a \frac{2x}{a^2+x^2}dx + a\int_0^a \frac{1}{a^2+x^2}dx$$

$$= \frac{1}{2}\left[\log(a^2+x^2)\right]_0^a + a\cdot\frac{1}{a}\left[\tan^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{1}{2}(\log 2a^2 - \log a^2) + \frac{\pi}{4}$$

$$\therefore I = \frac{1}{2}\log 2 + \frac{\pi}{4} \text{ Ans.}$$

2. $\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Sol. Let $I = \int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ ——— (i)

Put $\sin^{-1}x = t$

$\frac{1}{\sqrt{1-x^2}}\,dx = dt$

When $x \to 0$ then $t \to 0$

When $x \to 1$ then $t \to \frac{\pi}{2}$

Now (i) becomes

$I = \int_0^{\frac{\pi}{2}} t\,dt = \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{8}$

$\therefore I = \frac{\pi^2}{8}$ Ans.

3. $\int_0^{\frac{\pi}{4}} \cos^4 d\,dx$

Sol. Let $I = \int_0^{\frac{\pi}{4}} \cos^4 x\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{(1+\cos 2x)^2}{2}\,dx$

$= \frac{1}{4}\int_0^{\frac{\pi}{4}} (1+2\cos 2x+\cos^2 2x)\,dx$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} dx + \frac{1}{4}\cdot\int_0^{\frac{\pi}{2}} \cos 2x + \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1+\cos 4x}{2}\,dx$

$= \frac{1}{4}[x]_0^{\frac{\pi}{2}} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} + \frac{1}{8}[x]_0^{\frac{\pi}{2}} + \frac{1}{8}[\sin 4x]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{8}+0+\frac{\pi}{16}+0 = \frac{3\pi}{16}$

$\therefore I = \frac{3\pi}{16}$ Ans.

4. $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\,dx$

Sol. Let $I = \int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\,dx$

$= \int_\alpha^\beta \sqrt{-x^2+(\alpha+\beta)x-\alpha\beta}\,dx$

$= \int_\alpha^\beta \sqrt{\left(\frac{\alpha+\beta}{2}\right)^2-\alpha\beta-\left[x^2-(\alpha+\beta)x+\left(\frac{\alpha+\beta}{2}\right)^2\right]}\,dx$

$= \int_\alpha^\beta \sqrt{\left(\frac{\alpha-\beta}{2}\right)^2-\left\{x-\left(\frac{\alpha+\beta}{2}\right)\right\}^2}$

Put $\frac{\alpha-\beta}{2} = a$ and $x-\frac{\alpha+\beta}{2} = y$

$dx = dy$

When $x \to \alpha$, then $y \to \alpha - \frac{\alpha+\beta}{2} = \frac{\alpha-\beta}{2} = a$

when $x \to \beta$ then $y \to \beta - \frac{\beta+\alpha}{2} = \frac{\beta-\alpha}{2} = -a$

$\therefore I = \int_{-a}^{a} \sqrt{a^2-y^2}\,dy$

$= \left[y\frac{\sqrt{a^2-y^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_{-a}^{a}$

$= \left[0+\frac{a^2}{2}\sin^{-1}(1)\right]-\left[0+\frac{a^2}{2}\sin^{-1}(-1)\right]$

$= \frac{a^2}{2}\left(\frac{\pi}{2}\right)-\frac{a^2}{2}\left(-\frac{\pi}{2}\right)$

$= \frac{\pi a^2}{4}+\frac{\pi a^2}{4}$

$= \frac{2\pi a^2}{4} = \frac{\pi a^2}{2} = \frac{\pi}{2}\left(\frac{\alpha-\beta}{2}\right)^2 = \frac{\pi}{8}(\alpha-\beta)^2$

$\therefore I = \frac{\pi}{8}(\beta-\alpha)^2$ Ans.

5. $\int_0^a x\sqrt{a^2-x^2}\,dx$

Sol. Let $I = \int_0^a x\sqrt{a^2-x^2}\,dx$ ..... (i)

Put $a^2-x^2 = t^2$

$-2x\,dx = 2t\,dt$

$\therefore x\,dx = -t\,dt$

When $x \to 0$. Then $t \to a$

When $x \to a$ then $t \to 0$

Now (i) becomes,

$$I = -\int_a^0 t.t\,dt = -\int_a^0 t^2\,dt$$

$$= -\left[\frac{t^3}{3}\right]_a^0 = -\left(\frac{-a^3}{3}\right) = \frac{a^3}{3}$$

$\therefore I = \frac{a^3}{3}$ Ans.

6. $\int_0^1 \frac{1}{(1+x)\sqrt{x^2+2x}}\,dx$ **[B. E. 2059]**

Sol^n. Let $I = \int_0^1 \frac{1}{(1+x)\sqrt{x^2+2x}}\,dx$

$$I = \int_0^1 \frac{1}{(1+x)\sqrt{(x+1)^2-1}}\,dx \text{ ..... (i)}$$

Put $1+x = t$

$dx = dt$

When $x \to 0$ then $t \to 1$

When $x \to 1$ then $t \to 2$

Now (i) becomes,

$$I = \int_1^2 \frac{dt}{t\sqrt{t^2-1}} = \left[\sec^{-1} t\right]_1^2 = \left(\frac{\pi}{3} - 0\right) = \frac{\pi}{3}$$

$\therefore I = \frac{\pi}{3}$ Ans.

7. $\int_0^1 \frac{1}{(1+x^2)\sqrt{1-x^2}}\,dx$

Sol^n. Let $I = \int_0^1 \frac{1}{(1+x^2)\sqrt{1-x^2}}\,dx$ ..... (i)

Put $x = \sin\theta$

$dx = \cos\theta\,d\theta$

When $x \to 0$ then $\theta \to 0$

When $x \to 1$ then $\theta \to \frac{\pi}{2}$

Now (i) becomes,

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{(1+\sin^2\theta)} \frac{\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{\sec^2\theta + \tan^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{1+\tan^2\theta+\tan^2\theta}\,d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{2+\tan^2\theta+1}\,d\theta \text{ ..... (ii)}$$

Put $\tan\theta = t$ then $\sec^2\theta\,d\theta = dt$

When $\theta \to 0$ then $t \to 0$

When $\theta \to \frac{\pi}{2}$ then $t \to \infty$

Now (ii) becomes,

$$I = \int_0^\infty \frac{1}{2t^2+1}\,dt = \frac{1}{2}\int_0^\infty \frac{1}{t^2+\left(\frac{1}{\sqrt{2}}\right)^2}\,dt$$

$$= \frac{1}{2}.\frac{\sqrt{2}}{1}\left[\tan^{-1}\sqrt{2}\,t\right]_0^\infty$$

$$= \frac{1}{\sqrt{2}}.\frac{\pi}{2} = \frac{\pi}{2\sqrt{2}}$$

$\therefore I = \frac{\pi}{2\sqrt{2}}$ Ans.

8. $\int_0^{\frac{1}{2}} \frac{1}{(1-2x^2)\sqrt{1-x^2}}\,dx$

Sol^n. Let $I = \int_0^{\frac{1}{2}} \frac{1}{(1-2x^2)\sqrt{1-x^2}}\,dx$ ..... (i)

Put $x = \sin\theta$ then $dx = \cos\theta\,d\theta$

When $x \to 0$ then $\theta \to 0$

When $x \to \frac{1}{2}$ then $\theta \to \frac{\pi}{6}$

Now (i) becomes,

$$I = \int_0^{\frac{\pi}{6}} \frac{\cos\theta}{(1-2\sin^2\theta)\sqrt{(1-\sin^2\theta)}}\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{(1-2\sin^2\theta)}\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\cos 2\theta}\,d\theta = \int_0^{\frac{\pi}{6}} \sec 2\theta\,d\theta$$

$$= \frac{1}{2}\log\left[\sec 2\theta + \tan 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left\{\log\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - \log(\sec 0 + \tan 0)\right\}$$

$$= \frac{1}{2}\{\log(2+\sqrt{3}) - \log 1\}$$

$\therefore I = \frac{1}{2}\log(2+\sqrt{3})$ Ans.

9. $\int_1^e \frac{1}{x\sqrt{1-(\log x)^2}}dx$

Sol^n. Let $I = \int_1^e \frac{1}{x\sqrt{1-(\log x)^2}}dx$ ..... (i)

Put $\log x = t$ diff. w. r. t. 'x'

$\frac{1}{x}dx = dt$

When $x \to 1$, then $t = 0$

when $x \to e$, then $t = 1$

Now (i) becomes,

$$I = \int_0^1 \frac{1}{\sqrt{1-t^2}}dt = \left[\sin^{-1}t\right]_0^1 = \left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}$$

$\therefore I = \frac{\pi}{2}$ Ans.

10. $\int_0^1 \frac{1-x}{1+x}dx$ [B. E. 2061]

Sol^n. Let $I = \int_0^1 \frac{1-x}{1+x}dx = \int_0^1 \frac{-(1+x)+2}{(1+x)}dx$

$$= \int_0^1 \frac{-(1+x)}{(1+x)}dx + 2\int_0^1 \frac{1}{(1+x)}dx$$

$$= -\left[x\right]_0^1 + 2\left[\log(1+x)\right]_0^1$$

$$= -1 + 2(\log 2 - \log 1)$$

$$= -1 + 2\log 2$$

$\therefore I = 2\log 2 - 1$ Ans.

11. $\int_1^2 \frac{1}{x(1+x^4)}dx$

Sol^n. Let $I = \int_1^2 \frac{1}{x(1+x^4)}dx$

$$= \int_1^2 \left[\frac{1}{x} - \frac{x^3}{1+x^4}\right]dx$$

$$= \left[\log x - \frac{1}{4}\log(1+x^4)\right]_1^2$$

$$= \log 2 - \frac{1}{4}\log 17 - \log 1 + \frac{1}{4}\log 2$$

$$= \log 2 - \frac{1}{4}\log 17 + \frac{1}{4}\log 2$$

$$= \log 2 - \frac{1}{4}\log\frac{2}{17}$$

$$= \frac{1}{4}\left(4\log 2 - \log\frac{2}{17}\right)$$

$$= \frac{1}{4}\log 16 \times \frac{2}{17} = \frac{1}{4}\log\frac{32}{17}$$

$\therefore I = \frac{1}{4}\log\frac{32}{17}$ Ans.

12. $\int_0^1 \frac{1-4x+2x^2}{\sqrt{2x-x^2}}dx$

Sol^n. Let, $I = \int_0^1 \frac{1-4x+2x^2}{\sqrt{2x-x^2}}dx$

$$= \int_0^1 \frac{-2(2x-x^2)+1}{\sqrt{2x-x^2}}dx$$

$$= -2\int_0^1 \sqrt{2x-x^2}\,dx + \int_0^1 \frac{1}{\sqrt{2x-x^2}}dx$$

$$= -2\int_0^1 \sqrt{1-(1-x)^2}\,dx + \int_0^1 \frac{1}{\sqrt{1-(1-x)^2}}dx$$

$$= -2\left[\frac{(1-x)\sqrt{2x-x^2}}{2} + \frac{1}{2}\sin^{-1}(1-x)\right]_0^1 + \left[\sin^{-1}(1-x)\right]_0^1$$

$$= -2[0+0] + 2\left[0 + \frac{1}{2}\cdot\frac{\pi}{2}\right] + \left(0 - \frac{\pi}{2}\right)$$

$$= +\frac{\pi}{2} - \frac{\pi}{2} = 0$$

$\therefore I = 0$ Ans.

13. $\int_0^1 \frac{1-x^2}{1+x^2}dx$

Sol^n. Let $I = \int_0^1 \frac{-(1+x^2)+1+1}{(1+x^2)}dx$

$= -\int_0^1 \frac{1+x^2}{1+x^2}dx + 2\int_0^1 \frac{1}{1+x^2}dx$

$= -[x]_0^1 + 2.\left[\tan^{-1}x\right]_0^1$

$= -1 + 2.\frac{\pi}{4}$

$= -1 + \frac{\pi}{2}$

$\therefore I = \frac{\pi}{2} - 1$ Ans.

14. $\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}dx$

Sol^n. Let $I = \int_0^a \frac{1}{x+\sqrt{a^2-x^2}}dx$ ..... (i)

Let $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

When $x \to 0$ then $\theta \to 0$

When $x \to a$ then $\theta \to \frac{\pi}{2}$

Now (i) becomes,

$I = \int_0^{\frac{\pi}{2}} \frac{1}{a\sin\theta + \sqrt{a^2 - a^2\sin^2\theta}}.\, a\cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}d\theta$ ..... (ii)

Now $\cos\theta = l(\sin\theta + \cos\theta) + m(\cos\theta - \sin\theta)$

$\cos\theta = (l - m)\sin\theta + (l + m)\cos\theta$

Equating the coefficient of $\sin\theta$ and $\cos\theta$

We get,

$l - m = 0$ and $l + m = 1$

Solving these $l = \frac{1}{2}$ and $m = \frac{1}{2}$

Now equation (ii) becomes,

$I = \int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}(\sin\theta+\cos\theta) + \frac{1}{2}(\cos\theta - \sin\theta)}{\sin\theta+\cos\theta}d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(\sin\theta+\cos\theta)}{\sin\theta+\cos\theta}d\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}d\theta$

$= \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} + \frac{1}{2}[\log(\sin\theta+\cos\theta)]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) + \frac{1}{2}(\log 1 - \log 1)$

$= \frac{\pi}{4}$

$\therefore I = \frac{\pi}{4}$

15. $\int_1^2 \frac{1}{(x+1)\sqrt{x^2-1}}dx$

Sol^n. Let $I = \int_1^2 \frac{1}{(x+1)\sqrt{x^2-1}}dx$ ..... (i)

Put $x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

When $x \to 1$ then $\theta \to 0$

When $x \to 2$ then $\theta \to \frac{\pi}{3}$

Now (i) becomes,

$I = \int_0^{\frac{\pi}{3}} \frac{1}{(\sec\theta+1)\sqrt{\sec^2\theta-1}}.\sec\theta\tan\theta\, d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{1}{(\sec\theta+1)\tan\theta}d\theta.\sec\theta\tan\theta\, d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{\sec\theta}{(\sec\theta+1)} = \int_0^{\frac{\pi}{3}} \frac{1}{1+\sec\theta}d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{1}{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{1}{2\cos^2\frac{\theta}{2}}d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}}\sec^2\frac{\theta}{2}\, d\theta = \frac{1}{2}.\frac{2}{1}\left[\tan\frac{\theta}{2}\right]_0^{\frac{\pi}{3}} = \frac{1}{\sqrt{\ }} - 0$

$\therefore I = \frac{1}{\sqrt{3}}$ Ans.

16. $\int_0^1 \tan^{-1} x \, dx$

Sol^n. Let $I = \int_0^1 1 . \tan^{-1} x \, dx$

$$= \left[\tan^{-1} x . x\right]_0^1 - \int_0^1 \frac{1}{1+x^2} . x \, dx$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\log(1+x^2)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}[\log 2 - \log 1]$$

$\therefore I = \frac{\pi}{4} - \frac{1}{2}\log 2$ Ans.

17. $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta} \sin^3\theta \, d\theta$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta} \sin^3\theta \, d\theta$

$$= \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{1}{2}} (1 - \cos^2\theta) \sin\theta \, d\theta \text{ ..... (i)}$$

Put $\cos\theta = t \Rightarrow -\sin\theta \, d\theta = dt$

When $x \to 0$ then $t \to 1$

When $x \to \frac{\pi}{2}$ then $t = 0$

Now (i) becomes,

$$\therefore I = -\int_1^0 t^{\frac{1}{2}} (1 - t^2) \, dt$$

$$= -\int_1^0 t^{\frac{1}{2}} \, dt + \int_1^0 t^{\frac{5}{2}} \, dt$$

$$= -\frac{2}{3}\left[t^{\frac{3}{2}}\right]_1^0 + \frac{2}{7}\left[t^{\frac{7}{2}}\right]_1^0$$

$$= -\frac{2}{3}(0-1) + \frac{2}{7}(0-1)$$

$$= \frac{2}{3} - \frac{2}{7}$$

$$= \frac{14-6}{21} = \frac{8}{21}$$

$\therefore I = \frac{8}{21}$ Ans.

18. $\int_0^{\frac{\pi}{2}} e^x (\sin x + \cos x) \, dx$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}} e^x (\sin x + \cos x) \, dx$

$$= \left[e^x \sin x\right]_0^{\frac{\pi}{2}} \text{ since } . \int e^x [f(x) + f'(x)] \, dx = e^x f(x)$$

$$= \left[e^{\frac{\pi}{2}} . 1 - 1 . 0\right]$$

$\therefore I = e^{\frac{\pi}{2}}$ Ans.

19. $\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^3\theta \, d\theta$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}} \sin^5\theta \cos^3\theta \, d\theta$

$$= \int_0^{\frac{\pi}{2}} \sin^5\theta (1 - \sin^2\theta) \cos\theta \, d\theta \text{ ..... (i)}$$

Put $\sin\theta = t \Rightarrow \cos\theta \, d\theta = dt$

When $\theta \to 0$ then $t \to 0$

When $\theta \to \frac{\pi}{2}$ then $t \to 1$

Now (i) becomes,

$$I = \int_0^1 t^5 (1 - t^2) \, dt = \int_0^1 t^5 \, dt - \int_0^1 t^7 \, dt$$

$$= \left[\frac{t^5}{6}\right]_0^1 - \left[\frac{t^8}{8}\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{8}$$

$$= \frac{4-3}{24} = \frac{1}{24}$$

$\therefore I = \frac{1}{24}$ Ans.

20. $\int_0^{\frac{\pi}{2}} \frac{1}{1 + 2\cos x} \, dx$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{1 + 2\cos x} \, dx$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\cos^2\frac{x}{2} - 2\sin^2\frac{x}{2}} \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{3 - \tan^2\frac{x}{2}}\,dx \text{ ..... (i)}$$

Put $\tan\frac{x}{2} = t \Rightarrow$ then $\sec^2\frac{x}{2}\,dx = 2dt$

When $x \to 0$ then $t \to 0$

When $x \to \frac{\pi}{2}$ then $t \to 1$

Now (i) becomes

$$I = 2\int_0^1 \frac{1}{\left(\sqrt{3}\right)^2 - t^2}\,dt$$

$$= 2 \cdot \frac{1}{2.\sqrt{3}}\left[\log\frac{\sqrt{3}+t}{\sqrt{3}-t}\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\left\{\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \log\left(\frac{\sqrt{3}}{\sqrt{3}}\right)\right\}$$

$$= \frac{1}{\sqrt{3}}\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \frac{1}{\sqrt{3}}\log 1$$

$$= \frac{1}{\sqrt{3}}\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

$$= \frac{1}{\sqrt{3}}\log\left\{\frac{\left(\sqrt{3}\right)^2 + 2.\sqrt{3}.1 + 1}{3-1}\right\}$$

$$= \frac{1}{\sqrt{3}}\log\left\{\frac{\left(4+2\sqrt{3}\right)}{2}\right\}$$

$$= \frac{1}{\sqrt{3}}\log\left\{\frac{2\left(2+\sqrt{3}\right)}{2}\right\}$$

$\therefore I = \frac{1}{\sqrt{3}}\log\left(2+\sqrt{3}\right)$ Ans.

21. $\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\,dx$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\,dx$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4\sin^2\frac{x}{2} + 4\cos^2\frac{x}{2} + 5\cos^2\frac{x}{2} - 5\sin^2\frac{x}{2}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{9\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{9 - \tan^2\frac{x}{2}}\,dx \text{ ..... (i)}$$

Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}\,dx = 2dt$

When $x \to 0$ then $t \to 0$

When $x \to \frac{\pi}{2}$ then $t \to 1$

Now (i) becomes

$$I = 2\int_0^1 \frac{1}{9-t^2}\,dt = 2 \cdot \frac{1}{2.3}\left[\log\frac{3+t}{3-t}\right]_0^1 = \frac{1}{2}\left[\log\frac{4}{2} - \log\frac{3}{3}\right]$$

$\therefore I = \frac{1}{3}\log 2$ Ans.

22. $\int_0^{\pi} \frac{1}{3+2\sin x+\cos x}\,dx$

Sol^n. Let $I = \int_0^{\pi} \frac{1}{3+2\sin x+\cos x}\,dx$

$$= \int_0^{\pi} \frac{1}{3\sin^2\frac{x}{2} + 3\cos^2\frac{x}{2} + 4\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\,dx$$

$$= \int_0^{\pi} \frac{1}{4\cos^2\frac{x}{2} + 4\sin\frac{x}{2}\cos\frac{x}{2} + 2\sin^2\frac{x}{2}}\,dx$$

$$I = \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{4 + 4\tan\frac{x}{2} + 2\tan^2\frac{x}{2}}\,dx \text{ ..... (i)}$$

Put $\tan\frac{x}{2} = t \Rightarrow$ then $\sec^2\frac{x}{2}\,dx = 2dt$

When $x \to 0$ then $t \to 0$

When $x \to 0$ then $t \to \infty$

Now (i) becomes

$$I = 2\int_0^1 \frac{1}{4+4t+2t^2}\,dt$$

$$= 2\cdot\frac{1}{2}\int_0^1 \frac{1}{2+2t+t^2}\,dt$$

$$= \int_0^1 \frac{1}{t^2+2.t.1+(1)^2-(1)^2+2}\,dt$$

$$= \int_0^1 \frac{1}{(t+1)^2+1}\,dt$$

$$= \left[\tan^{-1}(t+1)\right]_0^1 = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$$

$\therefore I = \frac{\pi}{4}$ Ans.

23. $\int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)}{(9+16\sin 2x)}\,dx$ **[B. E. 2061]**

Sol^n. Let $I = \int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)}{(9+16\sin 2x)}\,dx$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16(1-1+\sin 2x)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16-16(1-2\sin x\cos x)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{25-16(\sin^2 x+\cos^2 x-2\sin x\cos x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{25-16(\sin x-\cos x)^2}\,dx \text{ ..... (i)}$$

Put $\sin x-\cos x = t \Rightarrow (\cos x+\sin x)\,dx = dt$

When $x \to 0$ then $t \to -1$

When $x \to \frac{\pi}{4}$ then $t \to 0$

Now (i) becomes

$$I = \int_{-1}^{0} \frac{1}{25-16t^2}\,dt$$

$$= \frac{1}{16}\int_{-1}^{0} \frac{1}{\left(\frac{5}{4}\right)^2-t^2}\,dt$$

$$= \frac{1}{16.2.\frac{5}{4}}\left[\log\left(\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right)\right]_{-1}^{0}$$

$$= \frac{1}{40}\left[\log\left(\frac{5+4t}{5-4t}\right)\right]_{-1}^{0}$$

$$= \frac{1}{40}\left[\log\frac{5}{5}-\log\frac{1}{9}\right]$$

$$= \frac{1}{40}\left(-\log\frac{1}{9}\right) = \frac{1}{40}\log 9 = \frac{1}{40}\times 2\log 3$$

$\therefore I = \frac{1}{20}\log 3$ Ans.

24. $\int_0^{\frac{\pi}{2}}\left(\sqrt{\tan x}+\sqrt{\cos x}\right)dx$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}}\left(\sqrt{\tan x}+\sqrt{\cos x}\right)dx$

$$= \int_0^{\frac{\pi}{2}}\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}\,dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sqrt{1-1+2\sin x\cos x}}\,dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{(\sin x+\cos x)}{\sqrt{1-\left(\sin^2 x+\cos^2 x-2\sin x\cos x\right)}}\,dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{(\sin x+\cos x)}{\sqrt{1-(\sin x-\cos x)^2}}\,dx \text{ ..... (i)}$$

Put $\sin x-\cos x = t \Rightarrow (\cos x+\sin x)\,dx = dt$

When $x \to 0$ then $t \to -1$

When $x \to \frac{\pi}{2}$ then $t \to 1$

Now (i) becomes,

$$I = \sqrt{2}\int_{-1}^{1} \frac{1}{1-t^2}\,dt$$

$$= \sqrt{2}\left[\sin^{-1} t\right]_{-1}^{1} = \sqrt{2}\left[\sin^{-1}(1)-\sin^{-1}(1)\right]$$

$$= \sqrt{2}\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\} = \pi\sqrt{2}$$

$\therefore I = \pi\sqrt{2}$ Ans.

25. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2}\,dx$

Sol^n. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2}\,dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^4 x}{(a^2 + b^2\tan^2 x)^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,(1+\tan^2 x)}{(a^2 + b^2\tan^2 x)^2}\,dx$$

Put $b\tan x = a\tan\theta$

$b\sec^2 x\,dx = a\sec^2\theta\,d\theta$

When $x \to 0$ then $\theta \to 0$

When $x \to \frac{\pi}{2}$ then $\theta \to \frac{\pi}{2}$

Now (i) becomes,

$$I = \int_0^{\frac{\pi}{2}} \frac{\frac{a}{b}\sec^2\theta\left(1 + \frac{a^2}{b^2}\tan^2\theta\right)}{(a^2 + b^2\tan^2\theta)^2}\,d\theta$$

$$= \frac{a}{b}\cdot\frac{1}{b^2}\cdot\frac{1}{a^4}\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\left(b^2 + a^2\tan^2\theta\right)}{\left(1+\tan^2\theta\right)^2}\,d\theta$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\left(b^2 + a^2\tan^2\theta\right)}{\sec^4\theta}\,d\theta$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} \frac{\left(b^2 + a^2\tan^2\theta\right)}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} (b^2\cos^2\theta + a^2\sin^2\theta)\,d\theta$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} \left[b^2\left(\frac{1+\cos 2\theta}{2}\right) + a^2\left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} \left[\left(\frac{b^2+a^2}{2}\right) + \frac{(b^2-a^2)}{2}\cos 2\theta\right]d\theta$$

$$= \frac{1}{a^3b^3}\cdot\frac{a^2+b^2}{2}\int_0^{\frac{\pi}{2}} d\theta + \frac{1}{a^3b^3}\cdot\frac{b^2-a^2}{2}\int_0^{\frac{\pi}{2}} \cos 2\theta\,d\theta$$

$$= \frac{a^2+b^2}{2a^3b^3}\left[\theta\right]_0^{\frac{\pi}{2}} - \frac{(a^2-b^2)}{2a^3b^3}\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2+b^2}{2a^3b^3}\cdot\frac{\pi}{2} - \frac{(a^2-b^2)}{2a^3b^3}(0-0)$$

$$= \pi\frac{(a^2+b^2)}{4a^3b^3} - 0$$

$\therefore I = \dfrac{\pi(a^2+b^2)}{4a^3b^3}$ Ans.

26. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)(2+\sin x)}\,dx$

Sol^n. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)(2+\sin x)}\,dx$ ..... (i)

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

When $x \to 0$ then $t \to 0$

When $x \to \frac{\pi}{2}$ then $t \to 1$

Now (i) becomes,

$$I = \int_0^1 \frac{1}{(1+t)(2+t)}\,dt$$

By using the simpleal partial fraction

$$I = \int_0^1 \left(\frac{1}{1+t} - \frac{1}{2+t}\right)dt$$

$$= \left[\log(1+t) - \log(2+t)\right]_0^1$$

$$= (\log 2 - \log 3 - \log 1 + \log 2)$$

$$= \log 4 - \log 3$$

$$= \log\frac{4}{3}$$

$\therefore I = \log\dfrac{4}{3}$ Ans.

Properties of Definite Integrals

If $\int_a^b f(x)\,dx = f(x)$ then $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.

Property 1 : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Property 2 : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Property 3 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ as $a < c < b$

In general,

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x) + \ldots\ldots \int_{c_n}^{b} f(x)\,dx.$$

as $a < c_1 < c_2 < \ldots\ldots < c_n < b$

Property 4 : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Property 5 : $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{when } f(2a-x) = f(x) \\ 0 & \text{when } f(2a-x) = -f(x) \end{cases}$

Even and odd function

A function f(x) is said to be even function.

if $f(-x) = f(x)$

For example, x^2, cosx, secx, etc.

A function f(x) is said to be odd function

if $f(-x) = -f(x)$

For example, x^3, tanx, sinx, etc.

Property 6 : $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$

Exercise - 9

Show that,

1. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$

Solⁿ. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$ ..... (i)

By using the properties of definite integral,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx \text{ ..... (ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)}{(\sin x + \cos x)}\,dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}}$$

$\therefore 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$ Proved.

2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \frac{\pi}{4}$

Solⁿ. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ ..... (i)

By using the properties of definite integral.

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \text{ ..... (ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\right) dx$$

$$2I = \int_0^{\frac{\pi}{2}} dx \quad [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$\therefore I = \frac{\pi}{4}$ Proved.

3. $\int_0^{\pi} x \sin^2 x \, dx = \frac{\pi^2}{4}$

Sol^n. Let $I = \int_0^{\pi} x \sin^2 x \, dx$ ..... (i)

By using the properties of definite integral,

$I = \int_0^{\pi} (\pi - x) \{\sin(\pi - x)\}^2 \, dx$

$I = \int_0^{\pi} (\pi - x) \sin^2 x \, dx$

$= \pi \int_0^{\pi} \sin^2 x \, dx - \int_0^{\pi} \sin^2 x \, dx$

$\therefore I = \pi \int_0^{\pi} \frac{(1 - \cos 2x)}{2} dx - I$ by using (i)

or, $2I = \frac{\pi}{2} \left[x - \frac{\sin x}{2} \right]_0^{\pi}$

$= \frac{\pi}{2} . (\pi - 0)$

$\therefore I = \frac{\pi^2}{4}$ Proved.

4. $\int_0^{\pi} \log(\tan x) \, dx = 0$

Sol^n. Let $I = \int_0^{\pi} \log(\tan x) \, dx$ ..... (i)

By the properties of definite integral,

$I = \int_0^{\pi} \log \tan\left(\frac{\pi}{2} - x\right) dx$

$I = \int_0^{\pi} \log \cot x \, dx$ ..... (ii)

Adding (i) and (ii)

$2I = \int_0^{\pi} (\log \tan x + \log \cot x) \, dx$

$= \int_0^{\pi} \log(\tan x . \cot x) \, dx$

$= \int_0^{\pi} \log 1 \, dx$

$2I = 0$

$\therefore I = 0$ Proved.

5. $\int_0^{a} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx = \frac{a}{2}$

$\sqrt{a-x}$

Sol^n. Let $I = \int_0^{a} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx$ ..... (i)

By using the properties of definite integral,

$I = \int_0^{a} \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{a - (a - x)}} dx$

$I = \int_0^{a} \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} dx$ ..... (ii)

Adding (i) and (ii)

$2I = \int_0^{a} \left(\frac{\sqrt{a - x} + \sqrt{x}}{\sqrt{a - x} + x} \right) dx = \int_0^{a} dx = [x]_0^{a}$

or, $2I = a \Rightarrow I = \frac{a}{2}$ Proved.

6. $\int_0^{\frac{\pi}{2}} \sin 2x \log(\tan x) \, dx = 0$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}} \sin 2x \log(\tan x) \, dx$ ..... (i)

By using the properties of definite integral,

$\sin 2\left(\frac{\pi}{2} - x\right) \log \tan(\pi - x)$

$= \sin 2x \log \cot x$

$2I = \int_0^{\pi} \sin x \log 1$

$I = \int_0^{\frac{\pi}{2}} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx$

$I = \int_0^{\frac{\pi}{2}} \sin 2x \log \cot x \, dx$ ..... (ii)

Adding (i) and (ii)

$2I = \int_0^{\frac{\pi}{2}} \sin 2x (\log \tan x + \log \cot x) \, dx$

$= \int_0^{\frac{\pi}{2}} \sin 2x \log(\tan x \cot x) \, dx$

$= \int_0^{\frac{\pi}{2}} \sin 2x \log(1) \, dx$

$2I = 0 \Rightarrow I = 0$ Proved.

7. $\int_0^{1} \frac{\log x}{\sqrt{1 - x^2}} dx = \frac{\pi}{2} \log \frac{1}{2}$ [B. E. 2059, 2060]

Sol^n. Let $I = \int_0^{1} \frac{\log x}{\sqrt{1 - x^2}} dx$

Put $x = \sin^2\theta$

$- \frac{\log \sin\theta}{\cos\theta} \cdot \cos\theta \, d\theta$

$\log$

Put $x = \sin\theta$

$dx = \cos x\theta \, d\theta$

When $x \to 0$ then $\theta \to 0$

When $x \to 1$ then $\theta \to \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \frac{\log \sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta \, d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \log \sin\theta \, d\theta \text{ ..... (i)}$$

By using the properties of definite integral,

$$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2}-\theta\right) d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \log \cos\theta \, d\theta \text{ ..... (ii)}$$

Adding (i) an (ii)

$$2I = \int_0^{\frac{\pi}{2}} (\log \sin\theta + \log \cos\theta) \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \log \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \log 2\sin\theta \cos\theta \cdot \frac{1}{2} \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2\theta \, d\theta + \int_0^{\frac{\pi}{2}} \log \frac{1}{2} \, d\theta$$

Put $2\theta = t \Rightarrow d\theta = \frac{1}{2} dt$

When $\theta \to 0$ then $t \to 0$

When $\theta \to \frac{\pi}{2}$ then $t \to \pi$

$$\therefore 2I = \frac{1}{2} \int_0^{\pi} \log \sin t \, dt + \log \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin\theta \, d\theta + \log \frac{1}{2} \left[\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \cdot 2 \int_0^{\frac{\pi}{2}} \log \sin\theta \, d\theta + \log \frac{1}{2} \left(\frac{\pi}{2}-\theta\right)$$

$$2I = I + \frac{\pi}{2} \log \frac{1}{2}$$

$$\therefore I = \frac{\pi}{2} \log \frac{1}{2} \text{ Proved.}$$

8. $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sqrt{\tan x}} dx = \frac{\pi}{4}$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sqrt{\tan x}} dx$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \text{ ..... (i)}$$

By using the properties of definite integral,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)} + \sqrt{\sin\left(\frac{\pi}{2}-x\right)}} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \text{ ..... (ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\frac{\pi}{2}} \frac{\left(\sqrt{\cos x} + \sqrt{\sin x}\right)}{\left(\sqrt{\sin x} + \sqrt{\cos x}\right)} dx = \int_0^{\frac{\pi}{2}} dx$$

or, $2I = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

$\therefore I = \frac{\pi}{4}$ Proved.

9. $\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8} \log 2$ [B.E. 2062, 061, 059]

Sol^n. Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Put $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

When $x \to 0$ then $\theta \to 0$

When $x \to 1$ then $\theta \to \frac{\pi}{4}$

$$\therefore I \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} \cdot \sec^2\theta \, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta) \, d\theta \text{ ..... (i)}$$

By using the properties of definite integral,

$$I = \int_0^{\frac{\pi}{4}} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log \frac{(1+\tan\theta+1-\tan\theta)}{1+\tan\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log 2\, d\theta - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\, d\theta$$

$$= \log 2 \int_0^{\frac{\pi}{4}} d\theta - I$$

$$2I = \log 2\, [\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\log 2$$

$\therefore I = \frac{\pi}{8}\log 2$ Proved.

10. $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x} dx = \frac{\pi}{2}(\pi - 2)$

Sol^n. $\int_0^{\pi} \frac{x\sin x}{1+\sin x} dx$ ..... (i)

By using the properties of definite integral,

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\sin(\pi - x)} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x}$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx - \int_0^{\pi} \frac{x\sin x}{1+\sin x} dx$$

$$= \pi \int_0^{\pi} \frac{1+\sin x - 1}{1+\sin x} dx - I \text{ By using (i)}$$

$$2I = \pi \int_0^{\pi} dx - \pi \int_0^{\pi} \frac{1}{1+\sin x} dx$$

$$= \pi [x]_0^{\pi} - \pi \int_0^{\pi} \frac{1}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} dx$$

$$2I = \pi^2 - \pi \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2} + 1 + 2\tan\frac{x}{2}} dx$$

Put $\tan\frac{x}{2} = t$ then, $\sec^2\frac{x}{2}\, dx = 2dt$

When $x \to 0$ then $t \to 0$

When $x \to \pi$ then $t \to \infty$

$$\therefore 2I = \pi^2 - \pi \int_0^{\infty} \frac{1}{t^2 + 2t + 1} dt$$

$$= \pi^2 - 2\pi \int_0^{\infty} \frac{1}{(t+1)^2} dt$$

$$= \pi^2 - 2\pi \left[-\frac{1}{t+1}\right]_0^{\infty}$$

$$2I = \pi^2 - 2\pi(0+1) = \pi^2 - 2\pi$$

$\therefore I = \frac{\pi}{2}(\pi - 2)$ Proved.

11. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log\left(\sqrt{2} + 1\right)$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$ ..... (i)

By using the properties of definite integral.

$$I = \int_0^{\frac{\pi}{2}} \frac{\left\{\sin\left(\frac{\pi}{2} - x\right)\right\}^2}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x} dx \text{ ..... (ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\sin\left(x + \frac{\pi}{4}\right)} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}}\left[\log\left\{\operatorname{cosec}\left(x+\frac{\pi}{4}\right)-\cot\left(x+\frac{\pi}{4}\right)\right\}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\operatorname{cosec}\frac{3\pi}{4}-\cot\frac{3\pi}{4}\right)-\log\left(\operatorname{cosec}\frac{\pi}{4}-\cot\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log(\sqrt{2}+1)-\log(\sqrt{2}-1)\right]$$

$$= \frac{1}{\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \frac{1}{\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$2I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2$$

$$\therefore I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1) \text{ Proved.}$$

12. $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$

Sol^n. Let $I = \int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$ ..... (i)

By using the properties of definite integral,

$$I = \int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\{\cos(\pi-x)\}^2}$$

$$= \int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$$

$$= \pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx - \int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}$$

$$I = \pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx - I$$

$$2I = \pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

Put $\cos x = t$

$-\sin x\,dx = dt$

When $x \to 0$ then $t \to 1$

When $x \to \pi$ then $t \to -1$

$$\therefore 2I = -\pi\int_1^{-1}\frac{1}{1+t^2}dt$$

$$= \pi\int_{-1}^{1}\frac{1}{1+t^2}dt$$

$$= 2\pi\int_0^1\frac{1}{1+t^2}dt = 2\pi\left[\tan^{-1}t\right]_0^1$$

$$\text{or, } 2I = 2\pi\{\tan^{-1}(1)-\tan^{-1}(0)\}$$

$$= 2\pi\cdot\frac{\pi}{4}$$

$$\Rightarrow I = \frac{\pi^2}{4} \text{ Proved.}$$

13. $\int_0^{\pi} x\log(\sin x)\,dx = \frac{\pi^2}{2}\log\frac{1}{2}$

Sol^n. Let $I = \int_0^{\pi} x\log(\sin x)\,dx$ ..... (i)

By using the properties of definite integral

$$I = \int_0^{\pi}(\pi-x)\log\sin(\pi-x)\,dx$$

$$= \int_0^{\pi}(\pi-x)\log(\sin x)\,dx$$

$$= \pi\int_0^{\pi}\log(\sin x)\,dx - \int_0^{\pi} x\log(\sin x)\,dx$$

$$I = \pi\cdot\int_0^{\pi}\log(\sin x)\,dx - I$$

$$2I = \pi\int_0^{\pi}\log(\sin x)\,dx$$

$$\int_0^{\pi/2}\log(\sin x) = \frac{\pi}{2}\log\frac{1}{2}$$

$$= \pi\cdot 2\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx$$

Using $\int_0^{\frac{\pi}{2}}\log(\sin\theta)\,d\theta = \frac{\pi}{2}\log\frac{1}{2}$

$$2I = 2\pi\cdot\left(\frac{\pi}{2}\log\frac{1}{2}\right)$$

$$\therefore I = \frac{\pi^2}{2}\log\frac{1}{2} \text{ Proved.}$$

14. $\int_0^{\frac{\pi}{2}}\frac{x}{\sin x+\cos x}dx = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}}\frac{x}{\sin x+\cos x}dx$ ..... (i)

By using the properties of definite integral,

$$I = \int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)}{\cos x + \sin x} dx$$

$$I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx - \int_0^{\frac{\pi}{2}} \frac{x}{\sin x + \cos x} dx$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx - I$$

$$2I = \frac{\pi}{2} \cdot \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x} dx$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}} dx$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\sin\left(x + \frac{\pi}{4}\right)} dx$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx$$

$$= \frac{\pi}{2\sqrt{2}} \left[\log\left\{\operatorname{cosec}\left(x + \frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right)\right\}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2\sqrt{2}} \left[\log\left(\operatorname{cosec}\frac{3\pi}{4} - \cot\frac{3\pi}{4}\right) - \log\left(\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi}{2\sqrt{2}} \left[\log\left(\sqrt{2}+1\right) - \log\left(\sqrt{2}-1\right)\right]$$

$$= \frac{\pi}{2\sqrt{2}} \log\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{\pi}{2\sqrt{2}}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$= \frac{\pi}{2\sqrt{2}} \log\left(\sqrt{2}+1\right)^2$$

$$2I = \frac{2\pi}{2\sqrt{2}} \log\left(\sqrt{2}+1\right)$$

$$\therefore I = \frac{2\pi}{2\sqrt{2}} \log\left(\sqrt{2}+1\right) \text{ Proved.}$$

15. $\int_0^1 \cot^{-1}(1 - x + x^2)\, dx = \frac{\pi}{2} - \log 2$

Sol^n. Let $I = \int_0^1 \cot^{-1}(1 - x + x^2)\, dx$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right) dx$$

$$= \int_0^1 \tan^{-1}\left\{\frac{x + (1-x)}{1 - x(1-x)}\right\} dx$$

$$I = \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x)\, dx$$

By using the properties of definite integral in the second integral only.

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}\{1 - (1-x)\}\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1} x\, dx$$

$$= 2\int_0^1 1 \cdot \tan^{-1} x\, dx$$

$$= 2\left\{\left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot x\, dx\right\}$$

$$= 2\left\{\frac{\pi}{4} - \frac{1}{2}\left[\log(1+x^2)\right]_0^1\right\}$$

$$= \frac{\pi}{2} - (\log 2 - \log 1)$$

$$\therefore I = \frac{\pi}{2} - \log 2 \text{ Proved.}$$

16. $\int_0^{\pi} \frac{x}{a^2\sin^2 x + b^2\cos^2 x} dx,\ a, b > 0 = \frac{\pi^2}{2ab}$

Sol^n. Let $I = \int_0^{\pi} \frac{x}{a^2\sin^2 x + b^2\cos^2 x} dx,\ a, b > 0$ ..... (i)

By using the properties of definite integral

$$I = \int_0^{\pi} \frac{(\pi - x)}{a^2\{\sin(\pi - x)\}^2 + b^2\{\cos(\pi - x)\}^2} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)}{a^2\sin^2 x + b^2\cos^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{1}{a^2\sin^2 x + b^2\cos^2 x} dx - \int_0^{\pi} \frac{x}{a^2\sin^2 x + b^2\cos^2 x} dx$$

$$I = \pi \int_0^{\pi} \frac{\sec^2 x}{a^2\tan^2 x + b^2} dx - I$$

$$2I = 2 \cdot \pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2\tan^2 x + b^2} dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

When $x \to 0$ then $t \to 0$

When $x \to \frac{\pi}{2}$ then $t \to \infty$

$$I = \pi \int_0^\infty \frac{1}{a^2t^2+b^2}\,dt$$

$$= \frac{\pi}{a^2} \int_0^\infty \frac{1}{t^2+\left(\frac{b}{a}\right)^2}\,dx$$

$$= \frac{\pi}{a^2} \cdot \frac{a}{b}\left[\tan^{-1}\frac{at}{b}\right]_0^\infty$$

$$= \frac{\pi}{ab}\left(\frac{\pi}{2}-0\right)$$

$\therefore I = \frac{\pi^2}{2ab}$ Proved.

17. $$\int_0^\infty \frac{x}{(a^2\sin^2x+b^2\cos^2x)^2}\,dx = \frac{\pi^2(a^2+b^2)}{4a^3b^3}$$

Solⁿ. Let $I = \int_0^\infty \frac{x}{\left(a^2\sin^2x+b^2\cos^2x\right)^2}\,dx$ ..... (i)

By using the properties of definite integral.

$$I = \int_0^\infty \frac{(\pi-x)}{\left[a^2\{\sin(\pi-x)\}^2+b^2\{\cos(\pi-x)\}^2\right]^2}\,dx$$

$$= \int_0^\infty \frac{(\pi-x)}{\left(a^2\sin^2x+b^2\cos^2x\right)^2}\,dx$$

$$= \pi \cdot \int_0^\infty \frac{1}{\left(a^2\sin^2x+b^2\cos^2x\right)^2}\,dx - \int_0^\infty \frac{x}{\left(a^2\sin^2x+b^2\cos^2x\right)^2}\,dx$$

$$I = \pi \int_0^\infty \frac{1}{\left(a^2\sin^2x+b^2\cos^2x\right)^2}\,dx - I$$

$$2I = \pi \int_0^\infty \frac{1}{\left(a^2\sin^2x+b^2\cos^2x\right)^2}\,dx$$

$$= \pi \cdot 2\int_0^{\frac{\pi}{2}} \frac{1}{\left(a^2\sin^2x+b^2\cos^2x\right)^2}\,dx$$

$$I = \pi \cdot \int_0^{\frac{\pi}{2}} \frac{\sec^4x}{\left(a^2\tan^2x+b^2\right)^2}\,dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2x\left(1+\tan^2x\right)}{\left(a^2\tan^2x+b^2\right)^2}\,dx$$

Put $a\tan x = b\tan\theta$

$\sec^2x\,dx = b\sec^2\theta\,d\theta$

When $x \to 0$ then $\theta \to 0$

When $t \to \frac{\pi}{2}$ then $\theta \to \frac{\pi}{2}$

$$\therefore I = \pi \int_0^{\frac{\pi}{2}} \frac{\frac{b}{a}\sec^2\theta\left(1+\frac{b^2}{a^2}\tan^2\theta\right)}{\left(b^2\tan^2\theta+b^2\right)^2}\,d\theta$$

$$= \frac{\pi b}{a^3b^4} \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\,(a^2+b^2\tan^2\theta)}{\left(\tan^2\theta+1\right)^2}\,d\theta$$

$$= \frac{\pi}{a^3b^3} \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\,(a^2+b^2\tan^2\theta)}{\sec^4\theta}\,d\theta$$

$$= \frac{\pi}{a^3b^3} \int_0^{\frac{\pi}{2}} (a^2\cos^2\theta+b^2\sin^2\theta)\,d\theta$$

$$= \frac{\pi}{a^3b^3} \int_0^{\frac{\pi}{2}} \left[a^2\left(\frac{1+\cos2\theta}{2}\right)+b^2\left(\frac{1-\cos2\theta}{2}\right)\right]d\theta$$

$$= \frac{\pi}{a^3b^3}\left[\frac{\left(a^2+b^2\right)}{2}\int_0^{\frac{\pi}{2}} d\theta + \frac{\left(a^2-b^2\right)}{2}\int_0^{\frac{\pi}{2}} \cos2\theta\,d\theta\right]$$

$$= \frac{\pi}{a^3b^3}\left\{\frac{\left(a^2+b^2\right)}{2}\cdot[\theta]_0^{\frac{\pi}{2}} + \frac{\left(a^2+b^2\right)}{2}\left[\frac{\sin2\theta}{2}\right]_0^{\frac{\pi}{2}}\right\}$$

$$= \frac{\pi}{a^3b^3}\left\{\frac{\left(a^2+b^2\right)}{2}\cdot\frac{\pi}{2} + \frac{\left(a^2-b^2\right)}{2}(0-0)\right\}$$

$\therefore I = \frac{\pi^2\left(a^2+b^2\right)}{4a^3b^3}$ Proved.

Definite Integral as the Limit of a Sum

Some Important Formula

1. $\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh)$

$= \lim_{h\to 0} h \sum_{r=1}^{n-1} f(a+rh)$ where $nh = b-a$

2. $\lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

3. $\int_a^b f(x)\,dx = \sum_{r=1}^{n} f(z_i)\,\delta i$

4. $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

5. $1+2+3+4+\ldots\ldots n = n\,\frac{(n+1)}{2}$

6. $1^2+2^2+3^2+4^2+\ldots\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

7. $1^3+2^3+3^3+\ldots\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

8. $\cos\alpha + \cos(\alpha+h) + \cos(2+2h) + \ldots\ldots \cos(\alpha+nh-h)$

$= \frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}} \cos\frac{1}{2}$ [Angle of first term + angle of last term]

9. $\sin\alpha + \sin(\alpha+h) + \sin(\alpha+2h) - \ldots\ldots \sin(\alpha+nh-h)$

$= \frac{\sin nh}{\sin\frac{h}{2}} \sin\frac{1}{2}$ [Angle of first term + Angle of last term]

Exercise - 10

Evaluate the following:

1. $\lim_{n\to\infty}\left[\frac{n}{n^2+1} + \frac{n}{n^2+2^2} + \ldots\ldots \frac{n}{2n^2}\right]$

Sol^n. Let $A = \lim_{n\to\infty}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots\ldots \frac{n}{2n^2}\right]$

$\therefore A = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2}$

$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\frac{1}{n}}{1+\frac{r^2}{n^2}} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1+\left(\frac{r}{n}\right)^2}$

But we know $\lim_{n\to\infty} \frac{1}{n} \Sigma f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

$\therefore A \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4}$

$\therefore A = \frac{\pi}{4}$ Ans.

2. $\lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \ldots\ldots \frac{1}{n}\right)$

Sol^n. Let $A = \lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2} + \ldots\ldots \frac{1}{n}\right)$

$\therefore A = \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{\frac{1}{n}+\frac{r}{n^2}}{1+\frac{r^2}{n^2}}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{\left(1+\frac{r}{n}\right)}{1+\frac{r^2}{n^2}}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1+\left(\frac{r}{n}\right)}{1+\left(\frac{r}{n}\right)^2}$

But we know, $\lim_{n\to\infty} \frac{1}{n} \sum f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

$\therefore A = \int_0^1 \frac{1+x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x\,dx}{1+x^2} + \int_0^1 \frac{1}{1+x^2}\,dx$

$= \frac{1}{2}\left[\log\left(1+x^2\right)\right]_0^1 + \left[\tan^{-1}x\right]_0^1$

$$= \frac{1}{2}(\log 2 - \log 1) + \left(\frac{\pi}{4} - 0\right)$$

$$\therefore A = \frac{1}{2}\log\frac{1}{2} - \frac{\pi}{4} \text{ Ans.}$$

3. $\lim\limits_{n\to\infty} \frac{r^3}{r^4+n^4}$

Sol^n. Let $A = \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \frac{r^3}{r^4+n^4} = \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \left(\frac{\frac{r^3}{n^4}}{\frac{r^4}{n^4}+1} \right)$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{\frac{r^3}{n^3}}{\frac{r^4}{n^4}+1} \right) = \lim_{n\to\infty} \frac{1}{n} \sum \left\{ \frac{\left(\frac{r}{n}\right)^3}{\left[\left(\frac{r}{n}\right)^4+1\right]} \right\}$$

But we know $\lim\limits_{n\to\infty} \frac{1}{n} \Sigma f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

$$\therefore A = \int_0^1 \frac{x^3}{x^4+1}dx = \frac{1}{4}\int_0^1 \frac{4x^3}{x^4+1}dx = \frac{1}{4}\left[\log\left(x^4+1\right)\right]_0^1 = \frac{1}{4}(\log 2 - \log 1)$$

$$\therefore A = \frac{1}{4}\log 2 \text{ Ans.}$$

4. $\lim\limits_{n\to\infty} \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)+\ldots\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$

Sol^n. Let $A = \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)+\ldots\ldots\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$

Taking log on both sides,

$$\log A = \frac{1}{n}\log\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)+\ldots\ldots\left(1+\frac{n}{n}\right)\right]$$

$$= \frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\ldots+\log\left(1+\frac{n}{n}\right)\right] = \frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

Taking lim on both sides an $n \to \infty$ we get,

$$\therefore \lim_{n\to\infty} \log A = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

Since $\lim\limits_{n\to\infty} \frac{1}{n}\Sigma f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

$$\therefore \lim_{n\to\infty} \log A = \int_0^1 \log(1+x)\,dx$$

$$= \left[x\log(1+x)\right]_0^1 - \int_0^1 \frac{1}{1+x}\cdot x\,dx = (\log 2 - 0) - \int_0^1 \frac{1+x-1}{1+x}dx$$

$$= \log 2 - \int_0^1 dx + \int_0^1 \frac{1}{1+x}dx = \log 2 - \left[x\right]_0^1 + \left[\log(1+x)\right]_0^1$$

$$= \log 2 - 1 + \log 2 = 1\log 2 - 0$$

$$= \log 4 - 1 = \log 4 - \log e = \log\frac{4}{e}$$

$$\therefore \lim_{n\to\infty} A = e^{\log\frac{4}{2}} = \frac{4}{e} \text{ Ans.}$$

5. $\lim\limits_{n\to\infty} \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$

Sol^n. Let $A = \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$

Taking log on both sides,

$$\log A = \frac{1}{n}\log\frac{n!}{n^n}$$

$$\therefore \log A = \frac{1}{n}\log\left\{\frac{1.2.3\ldots\ldots n}{n.n.n\ldots\ldots n}\right\}$$

$$= \frac{1}{n}\log\left\{\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\ldots\ldots\left(\frac{n}{n}\right)\right\}$$

$$= \frac{1}{n}\left[\log\left(\frac{1}{n}\right)+\log\left(\frac{2}{n}\right)+\ldots\ldots\log\left(\frac{n}{n}\right)\right]$$

$$\therefore \log A = \frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$$

$$\therefore \lim_{n\to\infty} \log A = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$$

But we know, $\lim\limits_{n\to\infty} \frac{1}{n}\sum f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

$$\therefore \lim_{n\to\infty} \log A = \int_0^1 \log x\,dx = \left[x\log x - x\right]_0^1 = 0 - 1 = -1$$

$$\therefore \lim_{n\to\infty} A = e^{-1} = \frac{1}{e} \text{ Ans.}$$

Improper Integrals

An integral $\int_a^b f(x)\,dx$ either the range is infinite or the integrand f(x) has an infinite discontinuous in the range. Then the integral $\int_a^b f(x)\,dx$ is said to be improper integral. For example:

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx,\ \int_0^{\infty}\frac{1}{a^2}dx$$

$$\int_0^{\infty}\frac{1}{x^2+1}dx,\ \int_0^{b}\frac{0}{x(a-x)}dx \text{ etc.}$$

To evaluate the value of infinite integrals, the integral can be defined as follows.

1. $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$; provided the limit exists.

2. $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to\infty}\int_a^b f(x)\,dx$; provided the limit exists

3. $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$

   $= \lim_{a\to\infty}\int_{-a}^{0} f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)$; provided both the limit exists

4. If $f(x)\to\infty$ as $x\to a$ then

   $\int_a^b f(x)\,dx = \lim_{h\to 0}\int_{a+h}^{b} f(x)\,dx,\ h>0$

5. If $f(x)\to\infty$ as $x\to b$ then,

   $\int_a^b f(x)\,dx = \lim_{h\to 0}\int_a^{b-h'} f(x)\,dx,\ h'>0$

6. If $f(x)\to\infty$ as $x\to c,\ a<c<b$ then

   $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

   $\lim_{h\to 0}\int_a^{c-h} f(x)\,dx + \lim_{h'\to 0}\int_{c+h}^{b} f(x)\,dx,\ h>0, h'>0$

Exercise - 11

Evaluate if possible, the following improper integrals:

1. $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$

Sol^n. $I = \int_{-\infty}^{0}\frac{1}{1+x^2}dx + \int_0^{\infty}\frac{1}{1+x^2}dx$

$= \lim_{a\to\infty}\int_{-a}^{0}\frac{1}{1+x^2}dx + \lim_{b\to\infty}\int_0^b\frac{1}{1+x^2}dx$

$= \lim_{a\to\infty}\left[\tan^{-1}x\right]_{-a}^{0} + \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b$

$= \lim_{a\to\infty}\left[\tan^{-1}0 - \tan^{-1}(-a)\right] + \lim_{b\to\infty}\left[\tan^{-1}b - \tan^{-1}b\right]$

$= \left[0+\frac{\pi}{2}+\frac{\pi}{2}-0\right]$

$\therefore I = \pi$ Ans.

2. $\int_0^{\infty}\frac{1}{9+x^2}dx$

Sol^n. Let $I = \int_0^{\infty}\frac{1}{9+x^2}dx$

$= \lim_{b\to\infty}\int_0^b\frac{1}{3^2+x^2}dx$

$= \lim_{b\to\infty}\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^b$

$= \lim_{b\to\infty}\frac{1}{3}\left\{\tan^{-1}\frac{b}{3} - \tan^{-1}0\right\}$

$= \frac{1}{3}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{6}$

$\therefore I = \frac{\pi}{6}$ Ans.

3. $\int_1^{\infty}\frac{\log x}{x^2}dx$

Sol^n. Let $I = \int^{\infty}\frac{\log x}{x^2}dx$

$= \lim_{b\to\infty}\int^{b}\frac{\log x}{x^2}dx$

$$= \lim_{b\to\infty}\left[\log x\cdot\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \frac{1}{x}\times\left(-\frac{1}{x}\right)dx$$

$$= \lim_{b\to\infty}\left[-\frac{\log b}{b}+\log 1\right] + \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{b}\right) + \lim_{b\to\infty}\left(-\frac{1}{b}+1\right)$$

$$= 0+0+1 = 1$$

$\therefore I = 1$ Ans.

4. $\int_0^\infty \frac{x}{\left(1+x^2\right)^2}dx$

Sol^n. Let $I = \int_0^\infty \frac{x}{\left(1+x^2\right)^2}dx$

$$= \lim_{b\to\infty}\int_0^b \frac{x}{\left(1+x^2\right)^2}dx \text{ ..... (i)}$$

Now, $\int \frac{x}{\left(1+x^2\right)^2}dx$, Put $1+x^2 = t$; $2x\,dx = dt \Rightarrow x dx = \frac{1}{2}dt$

$$\therefore \int \frac{x}{\left(1+x^2\right)^2}dx = \frac{1}{2}\int \frac{1}{t^2}dt = -\frac{1}{2t} = -\frac{1}{2\left(1+x^2\right)}$$

Now (i) becomes,

$$I = \lim_{b\to\infty}\left[-\frac{1}{2\left(1+x^2\right)}\right]_1^b$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2(1+b^2)}+\frac{1}{2(1+1)}\right]$$

$$= \left(0+\frac{1}{4}\right)$$

$\therefore I = \frac{1}{4}$ Ans.

5. $\int_1^\infty \frac{1}{x^3}dx$

Sol^n. Let $I = \int_1^\infty \frac{1}{x^3}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}dx$

$$= \lim_{b\to\infty}\left[\frac{1}{-2x^2}\right]_1^b$$

$$= \lim_{b\to\infty}\left(\frac{1}{-2b^2}+\frac{1}{2}\right) = \frac{1}{2}$$

$\therefore I = \frac{1}{2}$ Ans.

6. $\int_0^\infty \frac{1}{\sqrt{x}}dx$

Sol^n. Let $I = \int_0^\infty \frac{1}{\sqrt{x}}dx = \lim_{b\to\infty}\int_0^b x^{-\frac{1}{2}}dx$

$$= \lim_{b\to\infty}\left[2x^{\frac{1}{2}}\right]_1^b = \lim_{b\to\infty}\left(2b^{\frac{1}{2}}-2\right) = \infty - 2 = \infty$$

Thus the integral does not exist.

7. $\int_{-1}^1 \frac{1}{x^2}dx$

Sol^n. Let $I = \int_{-1}^1 \frac{1}{x^2}dx$

Here, $\frac{1}{x^2} \to \infty$ as $x \to 0$

So, $I = \int_{-1}^0 \frac{1}{x^2}dx + \int_0^1 \frac{1}{x^2}dx$

$$= \lim_{h\to 0}\int_{-1}^{-h}\frac{1}{x^2}dx + \lim_{h'\to 0}\int_{h'}^1 \frac{1}{x^2}dx \text{ as } h>0,\ h'>0$$

$$= \lim_{h\to 0}\left[-\frac{1}{x}\right]_{-1}^{-h} + \lim_{h'\to 0}\left[-\frac{1}{x}\right]_{h'}^1$$

$$= \lim_{h\to 0}\left(\frac{1}{h}-1\right) + \lim_{h'\to 0}\left[-1+\frac{1}{h'}\right]$$

$$= -1 + \lim_{h\to 0}\frac{1}{h} - 1 + \lim_{h'\to 0}\frac{1}{h'}$$

$$= -2 + \lim_{h\to 0}\frac{1}{h} + \lim_{h'\to 0}\frac{1}{h'}$$

Here, both the limit doesnot exists.
If $h = h'$ then

$$I = -2 + \lim_{h\to 0}\left(\frac{1}{h}+\frac{1}{h}\right) = -2 + \lim_{h\to 0}\left(\frac{2}{h}\right)$$

$$= -2 + \infty = \infty$$

Thus the integral doesnot exists.

8. $\int_0^\infty \frac{x}{x^2+4}\,dx$

Sol^n. Let $I = \int_0^\infty \frac{x}{x^2+4}\,dx$

$= \lim_{b\to\infty} \int_0^b \frac{x}{x^2+4}\,dx$ ..... (i)

Now, $\int \frac{x}{x^2+4}\,dx$

Put $x^2+4 = t \Rightarrow 2x\,dx = dt$

or, $x\,dx = \frac{1}{2}\,dt$

$\therefore \int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\log t = \frac{1}{2}\log(x^2+4)$

Now, (i) becomes,

$I = \lim_{b\to\infty} \frac{1}{2}\left[\log\left(x^2+4\right)\right]_0^b$

$= \lim_{b\to\infty} \frac{1}{2}\left[\log(b^2+4) - \log 4\right]$

$= \lim_{b\to\infty} \frac{1}{2}\left\{\log\left(\frac{b^2+4}{4}\right)\right\}$

$= \frac{1}{2} \times \infty = \infty$

Thus the integral doesnot exist.

9. $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$ **[B. E. 2062]**

Sol^n. Let $I = \int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$

$= \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_0^{\infty} \frac{1}{1+e^{2x}}\,dx$

$= \lim_{a\to\infty} \int_{-a}^{0} \frac{e^x}{1+e^{2x}}\,dx + \lim_{b\to\infty} \int_0^b \frac{1}{1+e^{2x}}\,dx$ ..... (i)

Now, $\int \frac{e^x}{1+\left(e^x\right)^2}\,dx$

Put, $e^x = t$, $e^x\,dx = dt$

$\therefore \int \frac{e^x}{1+\left(e^x\right)^2}\,dx = \int \frac{1}{1+t^2}\,dt = \tan^{-1} t = \tan^{-1}\{e^x\}$

Now (i) becomes,

$I = \lim_{a\to\infty} \left[\tan^{-1} e^x\right]_{-a}^{0} + \lim_{b\to\infty} \left[\tan^{-1} e^x\right]_0^b$

$= \lim_{a\to\infty} \{\tan^{-1}(1) - \tan^{-1} e^{-a}\} + \lim_{b\to\infty} \{\tan^{-1} e^b - \tan^{-1} 1\}$

$\therefore \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}$

$\therefore I = \frac{\pi}{2}$ Ans.

10. $\int_0^{\pi} \frac{\sin x}{\cos^2 x}\,dx$ **[B. E. 2055, 061]**

Sol^n. Let $I = \int_0^{\pi} \frac{\sin x}{\cos^2 x}\,dx$

Put $\cos x = t$

$-\sin x\,dx = dt$

When $x \to 0$ then $t \to 1$; when $x \to \pi$ then $t \to -1$

$\therefore I = \int_1^{-1} \frac{1}{t^2}\,dt$

$= \int_{-1}^{1} \frac{1}{t^2}\,dt$

$I = 2\int_0^1 \frac{1}{t^2}\,dt$

Here, $\frac{1}{t^2} \to \infty$ as $t \to 0$

So, $I = 2\lim_{h\to 0} \int_h^1 \frac{1}{t^2}\,dt$

$= 2\lim_{h\to 0} \left[\frac{1}{-t}\right]_h^1$

$= 2\lim_{h\to 0} \left(-1 + \frac{1}{h}\right)$

$= 2(-1+\infty) = \infty$

Thus the integral doesnot exist.

11. $\int_0^a \sqrt{\frac{a-x}{x}}\,dx$

Sol^n. Let $I = \int_0^a \sqrt{\frac{a-x}{x}}\,dx$

Here, $\frac{a-x}{x} \to \infty$ as $x \to 0$

So, $I = \lim_{h\to 0} \int_h^a \sqrt{\frac{a-x}{x}}\,dx$

$$= \lim_{h\to 0} \int_h^a \frac{a-x}{\sqrt{ax-x^2}}\,dx$$

$$= \lim_{h\to 0} \int_h^a \frac{\frac{1}{2}(a-2x)+\frac{1}{2}a}{\sqrt{ax-x^2}}\,dx$$

$$= \lim_{h\to 0} \left[\frac{1}{2}\int_h^a \frac{a-x}{\sqrt{ax-x^2}}\,dx + \frac{a}{2}\int_h^a \frac{1}{\sqrt{\left(\frac{a}{2}\right)^2-\left(x-\frac{a}{2}\right)^2}}\,dx\right]$$

$$= \lim_{h\to 0} \left[\frac{1}{2}\times 2\sqrt{ax-x^2}\right]_h^a + \frac{a}{2}\lim_{h\to 0}\left[\sin^{-1}\left(\frac{2x-a}{a}\right)\right]_h^a$$

$$= \lim_{h\to 0} \left\{0-\sqrt{ah-h^2}\right\} + \frac{a}{2}\lim_{h\to 0}\left\{\sin^{-1}(1)-\sin^{-1}\left(\frac{2h-a}{a}\right)\right\}$$

$$= 0 + \frac{a}{2}\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{-a}{a}\right)\right] = \frac{a}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{a\pi}{2}$$

$\therefore I = \frac{a\pi}{2}$ Ans.

12. $\int_0^2 \frac{1}{(1-x)^2}\,dx$

Sol^n. Let $I = \int_0^2 \frac{1}{(1-x)^2}\,dx$

Here, $\frac{1}{(1-x^2)} \to \infty$ as $x \to 1$

So, $I = \int_0^1 \frac{1}{(1-x)^2}\,dx + \int_1^2 \frac{1}{(1-x)^2}\,dx$

$$= \lim_{h\to 0}\int_0^{1-h} \frac{1}{(1-x)^2}\,dx + \lim_{h'\to 0}\int_{1+h}^2 \frac{1}{(1-x)^2}\,dx$$

$$= \lim_{h\to 0}\left[\frac{1}{(1-x)}\right]_0^{1-h} + \lim_{h'\to 0}\left[\frac{1}{(1-x)}\right]_{1+h'}^2$$

$$= \lim_{h\to 0}\left(\frac{1}{h}-1\right) + \lim_{h'\to 0}\left(-1+\frac{1}{h'}\right)$$

$$= -2 + \lim_{h\to 0}\left(\frac{1}{h}\right) + \lim_{h'\to 0}\left(\frac{1}{h'}\right)$$

Here both the limit doesnot exists.
If $h = h'$ then,

$$I = -2 + \lim_{h\to 0}\left(\frac{1}{h}+\frac{1}{h}\right)$$

$$I = -2 + \lim_{h\to 0}\left(\frac{2}{h}\right)$$

Thus the integral doesnot exists.

13. $\int_{-1}^2 \frac{1}{x^3}\,dx$ [B. E. 2058]

Sol^n. Let $I = \int_{-1}^2 \frac{1}{x^3}\,dx$

Here, $\frac{1}{x^3} \to \infty$ as $x \to 0$

So, $I = \int_{-1}^0 \frac{1}{x^3}\,dx + \int_0^2 \frac{1}{x^3}\,dx$

$$= \lim_{h\to 0}\int_{-1}^{-h} \frac{1}{x^3}\,dx + \lim_{h'\to 0}\int_{h'}^2 \frac{1}{x^3}\,dx$$

$$= \lim_{h\to 0}\left[-\frac{1}{2x^2}\right]_{-1}^{-h} + \lim_{h'\to 0}\left[-\frac{1}{2x^2}\right]_{h'}^2$$

$$= \lim_{h\to 0}\left(-\frac{1}{2h^2}+\frac{1}{2}\right) + \lim_{h'\to 0}\left(-\frac{1}{8}+\frac{1}{2h'^2}\right)$$

$$= \frac{1}{2}-\frac{1}{8} + \lim_{h\to 0}\left(-\frac{1}{2h^2}\right) + \lim_{h'\to 0}\left(\frac{1}{2h'^2}\right)$$

Both the limit does not exists,
But if $h = h'$ then,

$$I = \frac{4-1}{8} + \lim_{h\to 0}\left(-\frac{1}{2h^2}+\frac{1}{2h^2}\right) = \frac{3}{8}$$

$\therefore I = \frac{3}{8}$ is the principal value.

14. $\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx$

Sol^n. Let $I = \int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx$

Here, $\frac{1+x}{1-x} \to \infty$ as $x \to 1$

So, $I = \lim_{h\to 0}\int_0^{1-h} \sqrt{\frac{1+x}{1-x}}\,dx$

$$= \lim_{h\to 0}\int_0^{1-h} \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$\int \frac{1}{\sqrt{1-x^2}} + \int \frac{x}{\sqrt{}}$

$x - x^2$ $-$

$$= \lim_{h\to 0} \int_0^{1-h} \frac{-\frac{1}{2}(1-2x)+1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{h\to 0} \int_0^{1-h} \frac{-\frac{1}{2}(1-2x)}{\sqrt{1-x^2}}\,dx + \lim_{h\to 0} \int_0^{1-h} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{h\to 0} \left[-\frac{1}{2}\times 2\sqrt{1-x^2}\right]_0^{1-h} + \lim_{h\to 0} \left[\sin^{-1}x\right]_0^{1-h}$$

$$= \lim_{h\to 0} \left\{-\sqrt{1-(1-h)^2}+1\right\} + \lim_{h\to 0} \left\{\sin^{-1}(1-h)-0\right\}$$

$$= 0+1+\frac{\pi}{2}-0 \qquad \therefore I = \frac{\pi}{2}+1 \text{ Ans.}$$

15. Show that $\int_0^\infty \frac{\log(1+x^2)}{1+x^2}dx = \pi \log 2$.

Solⁿ. Let $I = \int_0^\infty \frac{\log(1+x^2)}{1+x^2}dx$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

When $x \to 0$ then $\theta \to 0$

When $x \to \infty$ then $\theta \to \frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\log\left(1+\tan^2\theta\right)}{\left(1+\tan^2\theta\right)} \times \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \log \sec^2\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \log \sec\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \log \frac{1}{\cos\theta}\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} [\log 1 - \log\cos\theta]\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \log 1\, d\theta - 2\int_0^{\frac{\pi}{2}} \log\cos\theta\, d\theta$$

$$= 2\times 0 - 2\times \frac{\pi}{2}\log\frac{1}{2}$$

Using $\int_0^{\frac{\pi}{2}} \log\cos\theta\, d\theta = \frac{\pi}{2}\log\frac{1}{2} = -\pi\log\frac{1}{2}$

$\therefore I = \pi \log 2$ Ans.

Differentiation under Integral Sign

Differentiation under integral sign is one of the strongest methods, which helps us to integrate the functions which are not easily integrable. Firstly, the integrand of definite integral is differentiated with respect a quantity of which the limits of integration are independent.

Leibnitz's Rule

If $f(x, \alpha)$ and $\frac{\partial}{\partial\alpha} f(x, \alpha)$ are continuous function of x and α, then

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx$$

Rule of differentiation under the integral sign when the limits of integration are functions of parameter.

If $f(x, \alpha)$ and $\frac{\partial}{\partial\alpha} f(x, \alpha)$ are continuous function of x and α. Then,

$$\frac{d}{d\alpha}\left[\int_{\phi(\alpha)}^{\Psi(\alpha)} f(x,\alpha)\,dx\right] = \int_{\phi(\alpha)}^{\Psi(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha) - \frac{d\phi}{d\alpha} f[\phi(\alpha),\alpha] + \frac{d\Psi}{d\alpha} f[\phi(\alpha),\alpha]$$

Important IOE extra questions

If $f(x,\alpha)$ & $\frac{\partial f(x,\alpha)}{\partial x}$ be continuous func. of x & α

1. $\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}dx$

Solⁿ. Let $I = \int_0^\infty \frac{\log(1+a^2x^2)}{(1+b^2x^2)}dx$ .....(1)

diff. (1) w.r.t a

$\frac{\partial f}{\partial a}$

$$\frac{dI}{da} = \frac{d}{da}\int_0^\infty \frac{\log(1+a^2x^2)}{(1+b^2x^2)}dx$$

$$= \int_0^\infty \frac{\partial}{\partial a}\left\{\frac{\log(1+a^2x^2)}{(1+b^2x^2)}\right\}dx$$

$$= \int_0^\infty \frac{2ax^2}{(1+a^2x^2)(1+b^2x^2)}dx$$

$$= 2a\int_0^\infty \frac{x^2}{(1+a^2x^2)(1+b^2x^2)}dx$$

$$= \frac{2a}{(a^2-b^2)}\int_0^\infty \left(\frac{1}{(1+b^2x^2} - \frac{1}{1+a^2x^2}\right)dx$$

$$= \frac{2a}{(a^2-b^2)}\left[\frac{1}{b}\tan^{-1}bx - \frac{1}{a}\tan^{-1}ax\right]_0^\infty$$

$$= \frac{2a}{(a^2-b^2)}\left(\frac{1}{b}-\frac{1}{a}\right)\frac{\pi}{2}$$

$$\frac{dI}{da} = \frac{\pi}{b(a+b)}$$

$\int dI = \int \frac{\pi}{b(a+b)} da$ Integrating

$$I = \frac{\pi}{b}\log(a+b) + c \text{ .....(2)}$$

When $a = 0$ from (1). we get $I = 0$

using $I = 0$; when $a = 0$ in (2)

$$0 = \frac{\pi}{b}\log b + c$$

or, $c = -\frac{\pi}{b}\log b$

Hence again from (2)

$$I = \frac{\pi}{b}\log(a+b) - \frac{\pi}{b}\log b$$

$$= \frac{\pi}{b}\log\left(\frac{a+b}{b}\right) \text{ Ans.}$$

2. $\int_0^{\frac{\pi}{2}} \log(1+\cos\theta\cos x).\frac{dx}{\cos x}$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}} \log(1+\cos\theta\cos x).\frac{dx}{\cos x}$ .....(1)

diff. (1) w.r.t. θ

$$\frac{dI}{d\theta} = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial\theta}\{\log(1+\cos\theta\cos x)\}\frac{dx}{\cos x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{-\sin\theta\cos x}{(1+\cos\theta\cos x)}.\frac{dx}{\cos x}$$

$$= -\sin\theta\int_0^{\frac{\pi}{2}} \frac{1}{(1+\cos\theta\cos x)}dx$$

$$= -\sin\theta\int_0^{\frac{\pi}{2}} \frac{1}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+\cos\theta(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}dx$$

$$= -\sin\theta\int_0^{\frac{\pi}{2}} \frac{1}{(1-\cos\theta)\sin^2\frac{x}{2}+(1+\cos\theta)\cos^2\frac{x}{2}}dx$$

$$= -\sin\theta\int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{(1-\cos\theta)\tan^2\frac{x}{2}+(1+\cos\theta)}dx$$

Put $\tan\frac{x}{2} = t$

$\sec^2\frac{x}{2}dx = 2dt$

When $x \to \left(0, \frac{\pi}{2}\right)$ Then $t \to (0, 1)$

$$\therefore \frac{dI}{d\theta} = -\sin\theta.2\int_0^1 \frac{1}{(1-\cos\theta)t^2+(1+\cos\theta)}dt$$

$$= -\sin\theta.\frac{2}{1-\cos\theta}\int_0^1 \frac{1}{t^2+\left(\frac{1+\cos\theta}{1-\cos\theta}\right)}dt$$

$$= -\sin\theta.\frac{2}{2\sin^2\frac{\theta}{2}}\int_0^1 \frac{1}{t^2+\cot^2\frac{\theta}{2}}dt$$

$$= -\sin\theta.\frac{1}{\sin^2\frac{\theta}{2}.\cot\frac{\theta}{2}}\left[\tan^{-1}\frac{t}{\cot\frac{\theta}{2}}\right]_0^1$$

$$= -\sin\theta.\frac{1}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\left[\tan^{-1}\frac{1}{\cot\frac{\theta}{2}} - 0\right]$$

$$= -\sin\theta.\frac{1}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}.\frac{\theta}{2}$$

$$= -\sin\theta.\frac{\theta}{\sin\theta} = -\theta$$

$$\therefore \frac{dI}{d\theta} = -\theta$$

or, $\int dI = -\int\theta d\theta$ Integrating

$$I = \frac{-\theta^2}{2} + c \text{ .....(ii)}$$

When $\theta = \frac{\pi}{2}$ from (1) $I = 0$

using $I = 0$ when $\theta = \frac{\pi}{2}$ in (ii)

$$0 = -\frac{\pi^2}{8} + c \Rightarrow c = \frac{\pi^2}{8}$$

Hence, eq^n (4) gives

$$I = \frac{-\theta^2}{2} + \frac{\pi^2}{8} \text{ Ans.}$$

3. **Evaluate** $\int_0^\infty \frac{x^\gamma \log x}{1+x^2} dx$.

Sol^n. $I = \int_0^\infty \frac{x^\gamma \log x}{1+x^2} dx$

Integrating both sides w.r.t. γ we get

$$\int I d\gamma = \int \left\{ \int_0^\infty \frac{x^\gamma \log x}{(1+x^2)\log x} dx \right\} d\gamma$$

$$= \int_0^\infty \left[\int \frac{x^\gamma \log x}{(1+x^2)} d\gamma \right] dx$$

$$= \int_0^\infty \left[\frac{\log x}{(1+x^2)} \int x^\gamma d^\gamma \right] dx$$

$$= \int_0^\infty \frac{\log x}{(1+x^2)} \cdot \frac{x^\gamma}{\log x} dx$$

$$\int I d\gamma = \int_0^\infty \frac{x^\gamma}{(1+x^2)} dx$$

Put $x^2 = t$ we get $2xdx = dt$

or, $xdx = \frac{1}{2x} dt = \frac{1}{2} t^{\frac{1}{2}} dt$.

When $x \to (0, \infty)$ Then $t \to (0, \infty)$

$$\therefore \int I d\gamma = \int_0^\infty \frac{t^{\frac{\gamma}{2}}}{1+t} \frac{1}{2} t^{\frac{-1}{2}} dt$$

$$= \frac{1}{2} \int_0^\infty \frac{t^{\frac{\gamma-1}{2}}}{1+t} dt = \frac{1}{2} \int_0^\infty \frac{t^{\frac{\gamma+1}{2}-1}}{1+t} dt$$

$$= \frac{1}{2} \pi \operatorname{cosec}\left(\frac{\gamma+1}{2}\right)\pi \quad \because \int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$$

$$\int I d\gamma = \frac{1}{2} \pi \sec \frac{\gamma\pi}{2}$$

Differentiating w.r.t γ we get

$$I = \frac{\pi}{2} \cdot \frac{\pi}{2} \sec \frac{\gamma\pi}{2} \cdot \tan \frac{\gamma\pi}{2}$$

Hence, $I = \frac{\pi^2}{4} \sec \frac{\gamma\pi}{2} \cdot \tan \frac{\gamma\pi}{2}$ Ans.

4. **Evaluate** $\int_0^\infty e^{-a^2x^2} \cos(2bx) dx$

Sol^n. Let $I = \int_0^\infty e^{-a^2x^2} \cos(2bx) dx$ .....(1)

diff. (1) w.r.t 'b' we get

$$\frac{dI}{db} = \int_0^\infty \frac{\partial}{\partial b} \left\{ e^{-a^2x^2} \cos(2bx) dx \right\}$$

$$= \int_0^\infty e^{-a^2x^2} (-2x) \sin 2bx$$

$$= \frac{1}{a^2} \int_0^\infty \left[e^{-a^2x^2} (-2a^2x) \right] \sin(2bx) dx$$

using Integrating by parts

$$\frac{dI}{db} = \frac{1}{a^2} \left[\left\{ \sin(2bx) \frac{e^{-a^2x^2}}{a^2} \right\}_0^\infty - \int_0^\infty 2b \cos 2bx \frac{e^{-a^2x^2}}{a^2} dx \right]$$

$$= \frac{1}{a^2} [0 - 2bI] = \frac{-2b}{a^2} I$$

$$\int \frac{dI}{I} = \int \frac{-2b}{a^2} db \text{ Integrating}$$

$$\log I = \frac{-b^2}{a^2} + \log e$$

$$\text{or, } I = ce^{\frac{-b^2}{a^2}} \quad .....(2)$$

When $b = 0$ from (1) $I = \int_0^\infty e^{-a^2x^2} dx$

i.e. $I = \frac{\sqrt{\pi}}{2a}$ [from gamma function; standard form]

using $I = \frac{\sqrt{\pi}}{2a}$; when $b = 0$ in (ii)

$$\frac{\sqrt{\pi}}{2a} = ce^0 \Rightarrow c = \frac{\sqrt{\pi}}{2a}$$

Hence (2) gives

$I = \frac{\sqrt{\pi}}{2a} e^{\frac{-x^2}{a^2}}$ Ans.

5. **Evaluate** $\int_0^1 \frac{x^{\alpha} + x^{-\alpha} - 2}{(1-x)\log x} dx$

Sol^n. Let $I = \int_0^1 \frac{x^{\alpha} + x^{-\alpha} - 2}{(1-x)\log x} dx$ .....(1)

Differentiating (1) w.r.t. α

$$\frac{dI}{d\alpha} = \int_0^1 \frac{(x^{\alpha} - x^{-\alpha})}{(1-x)\log x} \log x dx$$

$$\text{or, } \frac{dI}{d\alpha} = \int_0^1 \frac{(x^{\alpha} - x^{-\alpha})}{1-x} dx \text{ .....(2)}$$

We know,

$$\pi\cot(\pi\alpha) = \int_0^{\infty} \frac{x^{\alpha-1}}{1-x} dx = \int_0^1 \frac{x^{\alpha-1}}{1-x} dx + \int_1^{\infty} \frac{x^{\alpha-1}}{1-x} dx \text{ .....(3)}$$

Let $I_1 = \int_1^{\infty} \frac{x^{\alpha-1}}{1-x} dx$

Put $x = \frac{1}{y}; dx = -\frac{1}{y^2} dy$

When $x \to (1, \infty) \Rightarrow y \to (1, 0)$

$$\therefore I_1 = \int_1^0 y^{-\alpha+1} \frac{y}{y-1}\left(\frac{-dy}{y^2}\right) = \int_0^1 \frac{y^{-\alpha}}{y-1} dy$$

$$= \int_0^1 \frac{x^{-\alpha}}{x-1} dx \quad \text{[Replacing y by x; property of definite integral]}$$

Hence from (3)

$$\pi\cot(\pi\alpha) = \int_0^1 \frac{x^{\alpha-1}}{1-x} dx + \int_0^1 \frac{x^{-\alpha}}{x-1} dx$$

$$= \int_0^1 \frac{x^{\alpha-1} - x^{-\alpha}}{1-x} dx$$

$$= \int_0^1 \frac{x^{\alpha-1} - x^{\alpha} + x^{\alpha} - x^{-\alpha}}{1-x} dx$$

$$= \int_0^1 \frac{x^{\alpha}(x^{-1} - 1) + (x^{\alpha} - x^{-\alpha})}{(1-x)} dx$$

$$= \int_0^1 \frac{x^{\alpha}}{x} dx + \int_0^1 \frac{x^{\alpha} - x^{-\alpha}}{1-x} dx$$

$$= \int_0^1 x^{\alpha-1} dx + \frac{dI}{d\alpha} \text{ [from 2]}$$

$$\pi\cot(\pi\alpha) = \left[\frac{x^{\alpha-1+1}}{\alpha}\right]_0^1 + \frac{dI}{d}\alpha$$

$$= \frac{1}{\alpha} + \frac{dI}{d\alpha}$$

$$\int dI = \int\left(\pi\cot(\pi\alpha) - \frac{1}{\alpha}\right) d\alpha; \text{ Integrating}$$

$$I = \pi\frac{1}{\pi}\log\sin(\pi\alpha) - \log\alpha + \log c$$

$$I = \log\left(c\frac{\sin(\pi\alpha)}{\alpha}\right) \text{.....(4)}$$

When $\alpha = 0$ from (1) $I = 0$

using $I = 0$ when $\alpha = 0$ in (4)

$$0 = \log\left\{e \lim_{\alpha \to 0} \frac{\sin(\pi\alpha)}{\alpha}\right\} = \log \pi c$$

$$\Rightarrow \pi c = e^0 = 1 \text{ or } c = \frac{1}{\pi}$$

Hence (4) gives

$$I = \log\left(\frac{1}{\pi\alpha}\sin\pi\alpha\right) \text{ Ans.}$$

6. **Evaluate** $\int_0^1 \left(\frac{x^{\alpha-1} - x^{-\alpha}}{1+x}\right)\frac{1}{\log x} dx$

Sol^n. Let $I = \int_0^1 \left(\frac{x^{\alpha-1} - x^{-\alpha}}{1+x}\right)\frac{1}{\log x} dx$ .....(1)

diff (1) w.r.t. α

$$\frac{dI}{d\alpha} = \int_0^1 \left(x^{\alpha-1} + x^{-\alpha}\right) . \log x . \frac{1}{(1+x)\log x} dx$$

$$\frac{dI}{d\alpha} = \int_0^1 \frac{(x^{\alpha-1} + x^{-\alpha})}{1+x} dx \text{ .....(2)}$$

We know,

$$\pi\operatorname{cosec}(\pi\alpha) = \int_0^{\infty} \frac{t^{\alpha-1}}{1+t} dt = \int_0^1 \frac{t^{\alpha-1}}{1+t} dt + \int_1^{\infty} \frac{t^{\alpha-1}}{1+t} dt \text{.....(3)}$$

Let $I_1 = \int_1^{\infty} \frac{t^{\alpha-1}}{1+t} dt$

Put $t = \frac{1}{y} \Rightarrow dt = -\frac{1}{y^2} dy$

When $t \to (1, \infty)$ Then $y \to (1, 0)$

$$\therefore I_1 = \int_0^1 \frac{y y^{-\alpha+1}}{y+1}\left(\frac{dy}{-y^2}\right)$$

$$= \int_0^1 \frac{t^{\alpha}}{1+t}dt \text{ (Changing } y \leftrightarrow t \text{ properties of definite integral)}$$

$\therefore$ From (3)

$$\pi \operatorname{cosec}(\pi\alpha) = \int_0^1 \frac{t^{\alpha-1}}{1+t}dt + \int_0^1 \frac{t^{-\alpha}}{1+t}dt$$

$$= \int_0^1 \frac{t^{\alpha-1}+t^{-\alpha}}{1+t}dt$$

$$\pi \operatorname{cosec}(\pi\alpha) = \frac{dI}{d\alpha} \text{ from(2)}$$

$$\int dI = \pi \int \operatorname{cosec}(\alpha\pi)d\alpha \text{ Integrating}$$

$$I = \pi \cdot \frac{1}{\pi} \log \tan \frac{(\pi\alpha)}{2} + c$$

$$I = \log \tan\left(\frac{\pi\alpha}{2}\right) + c \text{ .....(4)}$$

When $\alpha = \frac{1}{2}$; from(1) $I = 0$

Using $I = 0$ When $\alpha = \frac{1}{2}$ in (4)

$$0 = \log \tan\left(\frac{\pi}{\alpha}\right) + c \Rightarrow c = 0$$

Hence (4) gives

$$I = \log \tan\left(\frac{\pi\alpha}{2}\right) \text{ Ans.}$$

7. **Evaluate;** $\int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}dx$

Sol^n. Let $I = \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}dx$ .....(1)

diff. (1) w.r.t. 'a'

$$\frac{dI}{da} = \int_0^{\infty} \frac{\partial}{\partial a}\left\{\frac{\tan^{-1}(ax)}{x(1+x^2)}\right\}dx$$

$$= \int_0^{\infty} \frac{1}{x(1+x^2)} \cdot \frac{x}{(1+a^2x^2)}dx$$

$$\frac{dI}{da} = \int_0^{\infty} \frac{1}{(1+x^2)(1+a^2x^2)}dx$$

$$= \frac{1}{(1-a^2)}\int_0^{\infty}\left[\frac{1}{(1+x^2)} - \frac{a^2}{(1+a^2x^2)}\right]dx \text{ (by using partial fraction)}$$

$$= \frac{1}{(1-a^2)}\left[\tan^{-1}x - a\tan^{-1}(ax)\right]_0^{\infty}$$

$$= \frac{1}{(1-a^2)}(1-a)\frac{\pi}{2} = \frac{\pi}{2(1+a)}$$

$$\therefore \int dI = \int \frac{\pi}{2(1+a)}da; \text{ Integrating}$$

$$I = \frac{\pi}{2}\log(1+a) + c \text{ .....(2)}$$

When $a = 0$ from (1) $I = 0$

using $I = 0$ when $a = 0$ in (2)

$$0 = \frac{\pi}{2}\log 1 + c \Rightarrow c = 0$$

Hence (2) gives

$$I = \frac{\pi}{2}\log(1+a) \text{ Ans.}$$

8. **Prove that** $\int_0^{\infty} \frac{e^{-ax}\sin bx}{x}dx = \tan^{-1}\left(\frac{b}{a}\right)$

Hence deduce that $\int_0^{\infty} \frac{\sin bx}{x}dx = \begin{cases} \frac{\pi}{2} \text{ if } b > 0 \\ \frac{-\pi}{2} \text{ if } b < 0 \end{cases}$

Sol^n. Let $I = \int_0^{\infty} \frac{\partial}{\partial b}\left\{\frac{e^{-ax}\sin bx}{x}\right\}dx$ .....(1)

diff. (1) w.r.t. 'b'

$$= \int_0^{\infty} \frac{\partial}{\partial b}\left\{\frac{e^{-\partial x}\sin bx}{x}\right\}dx$$

$$\frac{dI}{db} = \int_0^{\infty} \frac{\partial}{\partial b}\left\{\frac{e^{-ax}.x\cos bx}{x}\right\}dx = \int_0^{\infty} e^{-ax}\cos bx dx$$

$$= \left[\frac{e^{-ax}(-a\cos bx + b\sin bx)}{a^2+b^2}\right]_0^{\infty}$$

$$\frac{dI}{db} = \frac{a}{a^2+b^2}$$

or, $\int dI = \int \frac{a}{a^2+b^2} db$ Integrating,

$I = a.\frac{1}{a}\tan^{-1}\frac{b}{a} + c$

or, $I = \tan^{-1}\frac{b}{a} + c$ .....(2)

When $b = 0$ from (1) $I = 0$

Using $I = 0$ when $b = 0$ in (2)

$0 = \tan^{-1} 0 + c \Rightarrow c = 0$

Hence (2) gives,

$I = \tan^{-1}\frac{b}{a}$ Proved.

Deduction part

We obtained that;

$$\int_0^\infty \frac{e^{-ax}\sin bx}{x} dx = \tan^{-1}\left(\frac{b}{a}\right)$$

Taking $a = 0$ then,

$$\int_0^\infty \frac{\sin bx}{x} dx = \begin{cases} \frac{\pi}{2} \text{ if } b > 0 \\ -\frac{\pi}{2} \text{ if } b < 0 \end{cases}$$ Proved.

9. **Evaluate $\int_0^{\frac{\pi}{2}} \log(a^2\cos^2\theta + b^2\sin^2\theta) d\theta$ a, b > 0**

Sol^n. Let $I = \int_0^{\frac{\pi}{2}} \log(a^2\cos^2\theta + b^2\sin^2\theta) d\theta$ .....(1)

diff (1) w.r.t. 'a' we get

$$\frac{dI}{da} = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial a}\left\{\log(a^2\cos^2\theta + b^2\sin^2\theta)\right\} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{(a^2\cos^2\theta + b^2\sin^2\theta)} . 2a\cos 2\theta d\theta$$

$$\frac{dI}{da} = 2a\int_0^{\frac{\pi}{2}} \frac{\cos^2\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)} d\theta$$

$$= 2a\int_0^{\frac{\pi}{2}} \frac{1}{a^2 + b^2\tan^2\theta)} d\theta$$

Put $\tan\theta = t$

$\Rightarrow \sec^2\theta\, d\theta = dt \Rightarrow d\theta = \frac{1}{(1+t^2)} dt$

When $\theta \to \left(0, \frac{\pi}{2}\right)$ then $t \to (0, \infty)$

$$\frac{dI}{da} = 2a\int_0^\infty \frac{1}{\left(a^2 + b^2t\right)\left(1+t^2\right)} dt$$

$$= 2a\int_0^\infty \left\{\frac{b^2}{a^2+b^2t^2} - \frac{1}{1+t^2}\right\} dt; \text{ (By partial fraction)}$$

$$= \frac{2a}{b^2-a^2}\left[\frac{b^2}{a.b}\tan^{-1}\left(\frac{bt}{a}\right) - \tan^{-1} t\right]_0^\infty$$

$$= \frac{2a}{b^2-a^2}\left(\frac{b}{a} - 1\right)\frac{\pi}{2} = \frac{\pi}{a+b}$$

or, $\int dI = \int \frac{\pi}{(a+b)} da$ Integrating

$I = \pi \log(a+b) + c$ .....(2)

When $a = b$ from (1) $I = \int_0^{\frac{\pi}{2}} \log a^2 d\theta$

$= 2 \log a . \frac{\pi}{2} = \pi \log a$

using $I = \pi \log a$ when $a = b$ in (2)

$\pi \log a = \pi \log 2a + c$

$\Rightarrow c = \pi \log\left(\frac{1}{2}\right) = -\pi \log 2$

Hence (2) gives

$I = \pi \log(a+b) - \pi \log 2$

$= \pi \log \frac{(a+b)}{2}$ Ans.

10. **Evaluate $\int_0^\pi \frac{\log(1 + a\cos x)}{\cos x} dx$**

Sol^n. Let $I = \int_0^\pi \frac{\log(1 + a\cos x)}{\cos x} dx$ .....(1)

Differentiating (1) w.r.t ' a' we get

$$\frac{dI}{da} = \int_0^\pi \frac{\partial}{\partial a}\left\{\frac{\log(1 + a\cos x)}{\cos x}\right\} dx$$

$$= \int_0^\pi \frac{1}{(1 + a\cos x)} . \cos x . \frac{1}{\cos x} dx$$

$$= \int_0^\pi \frac{1}{(1 + a\cos x)} dx$$

$$= \int_0^{\pi} \frac{1}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + a\cos^2\frac{x}{2} - a\sin^2\frac{x}{2}} dx$$

$$= \int_0^{\pi} \frac{1}{(1-a)\sin^2\frac{x}{2} + (1+a)\cos^2\frac{x}{2}} dx$$

$$= \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{(1-a)\tan^2\frac{x}{2} + (1+a)} dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2} dx = 2dt$

When $x \to (0, \pi)$ Then $t \to (0, \infty)$

$$\therefore \frac{dI}{da} = 2\int_0^{\infty} \frac{1}{(1-a)t^2 + (1+a)} dt$$

$$\frac{dI}{da} = \frac{2}{(1-a)} \int_0^{\infty} \frac{1}{t^2 + \left(\frac{1+a}{1-a}\right)} dt$$

$$= \frac{2}{(1-a)} \cdot \frac{1}{\sqrt{\frac{1+a}{1-a}}} \left[\tan^{-1}\left\{\frac{t}{\sqrt{\frac{1+a}{1-a}}}\right\}\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{1-a^2}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{1-a^2}}$$

or, $\frac{dI}{da} = \frac{\pi}{\sqrt{1-a^2}} \Rightarrow \int dI = \int \frac{\pi}{\sqrt{1-a^2}} da$ Integrating

$I = \pi \sin^{-1} a + c$ .....(2)

When $a = 0$ from (1) $I = 0$

Using $I = 0$ when $a = 0$ in (1) $c = 0$

Hence (2) gives, $I = \pi \sin^{-1} a$ Ans.

11. Evaluate, $\int_0^{\frac{\pi}{2}} \frac{1}{a^2\sin^2 x + b^2\cos^2 x} dx$ and hence deduce that

$$\int_0^{\frac{\pi}{2}} \frac{1}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^2} dx = \frac{\pi(a^2+b^2)}{4a^3b^3}$$

Sol^n. Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^2} dx$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 s\tan^2 x + b^2} dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

When $x \to \left(0, \frac{\pi}{2}\right)$ Then $t \to (0, \infty)$

$$\therefore I = \int_0^{\infty} \frac{1}{a^2t^2 + b^2} dt$$

$$= \frac{1}{ab}\left[\tan^{-1}\frac{at}{b}\right]_0^{\infty} = \frac{\pi}{2ab}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{1}{(a^2\sin^2 x + b^2\cos^2 x)} dx = \frac{\pi}{2ab}$$

i.e. $\int_0^{\frac{\pi}{2}} \frac{1}{(a^2\sin^2 x + b^2\cos^2 x)} dx = \frac{\pi}{2ab}$ .....(1)

Differentiating (1) partially w.r.t. 'a' we get

$$\int_0^{\frac{\pi}{2}} \frac{\partial}{\partial a}\left[a^2\sin^2 x + b^2\cos^2 x\right]^{-1} dx = \frac{\partial}{\partial a}\left[\frac{\pi}{2ab}\right]$$

or, $\int_0^{\frac{\pi}{2}} -(a^2\sin^2 x + b^2\cos^2 x)^{-2} \cdot 2a\sin^2 x dx = \frac{\pi}{2b}(-a^2)$

or, $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2} dx = \frac{\pi}{4a^3b}$ .....(2)

Similarly differentiating (1) partically w.r.t 'b' we get

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2} dx = \frac{\pi}{4ab^3} \text{ .....(3)}$$

Adding (2) and (3)

$$\int_0^{\frac{\pi}{2}} \frac{1}{(a^2\sin^2 x + b^2\cos^2 x)^2} dx = \frac{\pi}{4ab^3} + \frac{\pi}{4ab^3}$$

$$= \frac{\pi(a^2+b^2)}{4a^3b^3} \text{ Proved.}$$

12. Evaluate: $\int_0^1 \frac{x^{\alpha} - 1}{\log} dx$

Sol^n. Let $I = \int_0^1 \frac{x^{\alpha} - 1}{\log} dx$ ..... (i)

Differentiating (1) w. r. t. 'α'

$$\frac{dI}{d\alpha} = \int_0^1 \frac{\partial}{\partial\alpha}\left\{\frac{(x^\alpha - 1)}{\log x}\right\}dx$$

$$= \int_0^1 \frac{x^\alpha \log x}{\log x}dx = \int_0^1 x^\alpha = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1$$

$$= \frac{1}{\alpha+1}$$

$\therefore \int dI = \int \frac{1}{\alpha+1} d\alpha$ Integrating

$I = \log(\alpha + 1) + c$ .....(2)

When $\alpha = 0$ from (1) $I = 0$

using $I = 0$ when $\alpha = 0$ in (2)

$0 = \log(1) c \Rightarrow c = 0$

Hence (2) gives

$I = \log(\alpha + 1)$ Ans.

Exercise - 12

Evaluate, by using the rule of differentiation under the sign of integration

1. $\int_0^\infty \frac{1-e^{-ax}}{x} e^{-x} dx$

Sol^n. Let $I = \int_0^\infty \left(\frac{1-e^{-ax}}{x}\right) e^{-x} dx$ .....(1)

differentiating (1) with respect to a under the sign of integration we get,

$$\frac{dI}{da} = \frac{d}{da}\int_0^\infty \left(\frac{1-e^{-ax}}{x}\right) e^{-x} dx$$

$$= \int_0^\infty \frac{\partial}{\partial a}\left(\frac{1-e^{-ax}}{x}\right) e^{-x} dx$$

$$= \int_0^\infty \left\{\frac{0-(-x)e^{-ax}}{x}\right\} e^{-x} dx$$

$$= \int_0^\infty e^{-ax} e^{-x} dx$$

$$= \int_0^\infty e^{-(a+1)x} dx$$

$$\frac{dI}{da} = \lim_{t\to\infty} \int_0^t e^{-(a+1)x} dx$$

$$= \lim_{t\to\infty}\left[\frac{e^{-(a+1)x}}{-(a+1)}\right]_0^t = \lim_{t\to\infty}\left[\frac{e^{-(a+1)t}}{-(a+1)} + \frac{1}{(a+1)}\right]$$

$\frac{dI}{da} = \frac{1}{(a+1)} \Rightarrow \int dI = \int \frac{1}{(a+1)} da$ Integrating

$I = \log(a+1) + c$ .....(2)

From (1), when $a = 0$ then $I = 0$

using $I = 0$ when $a = 0$ in (2)

$0 = \log(1 + 0) + c \therefore c = 0$

Hence (2) gives

$I = \log(a + 1)$ Ans.

2. $\int_0^\infty \frac{e^{-ax}\sin x}{x} dx$ **also deduce that** $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

Sol^n. Let $I = \int_0^\infty \frac{e^{-ax}\sin x}{x} dx$ .....(1)

differentiating (1) with respect to a;

$$\frac{dI}{da} = \frac{d}{da}\int_0^\infty \frac{e^{-ax}\sin x}{x} dx$$

$$= \int_0^\infty \frac{\partial}{\partial x}\frac{e^{-ax}\sin x}{x} dx$$

$$= \int_0^\infty \frac{-xe^{-ax}\sin x}{x} dx$$

$$= -\int_0^\infty e^{-ax}\sin x dx$$

$$= -\lim_{t\to\infty}\int_0^t e^{-ax}\sin x dx$$

$$= -\lim_{t\to\infty}\left[\frac{-ae^{-ax}\sin x - \cos x}{1+a^2}\right]_0^t$$

$$= -\lim_{t\to\infty}\left[\frac{-ae^{-ax}(\sin t - \cos t)}{1+a^2} + \frac{1}{1+a^2}\right]$$

$\frac{dI}{da} = -\frac{1}{1+a^2}$ using $\lim_{t\to\infty} e^{-t} = 0$

or, $\int dI = -\int \frac{1}{1+a^2} da$ Integrating

$I = -\tan^{-1} a + c$ .....(2)

From (1) when $a = \infty$ Then $I = 0$

using $I = 0$ when $a = \infty$ in (2)

$0 = -\frac{\pi}{2} + c \Rightarrow c = \frac{\pi}{2}$

Hence, (2) gives

$I = -\tan^{-1} a + \frac{\pi}{2}$

$= \frac{\pi}{2} - \tan^{-1} a$

$I = \cot^{-1} a$

Deduction part

We obtained that $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx = \cot^{-1} a$

Taking $a = 0$, we find,

$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ proved

3. $\int_0^\infty \frac{\cos \lambda x}{x}\left(e^{-ax} - e^{-bx}\right) dx$

Sol^n. Let $I = \int_0^\infty \frac{\cos \lambda x}{x}\left(e^{-ax} - e^{-bx}\right) dx$ .....(1)

diff. (1) w.r.t 'a'

$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left\{\frac{\cos \lambda x}{x}\left(e^{-ax} - e^{-bx}\right)\right\} dx$

$= \int_0^\infty \frac{\cos \lambda x}{x}(-x)e^{-ax} dx$

$= -\int_0^\infty e^{-ax} \cos \lambda x \, dx$

$\frac{dI}{da} = -\frac{a}{a^2 + \lambda^2}$ [using standard formula $\int_0^\infty e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2}$]

$\int dI = -\int \frac{a}{a^2 + \lambda^2} da$ Integrating

$I = -\frac{1}{2}\log(a^2 + \lambda^2) + c$ .....(2)

When $a = b$ from (1) $I = 0$

Using $I = 0$ when $a = b$ in (2)

$0 = -\frac{1}{2}\log\left(b^2 + \lambda^2\right) + c$

$\Rightarrow c = \frac{1}{2}\log(b^2 + \lambda^2)$

Hence (2) gives,

$I = -\frac{1}{2}\log(a^2 + \lambda^2) + \frac{1}{2}\log(b^2 + \lambda^2)$

$= \frac{1}{2}\log\frac{\left(b^2 + \lambda^2\right)}{\left(a^2 + \lambda^2\right)}$ Ans.

4. $\int_0^1 \frac{x^a - x^b}{\log x} dx$

Sol^n. Let $I = \int_0^1 \frac{x^a - x^b}{\log x} dx$ .....(1)

diff. (1) w.r.t. 'a'

$\frac{dI}{da} = \int_0^1 \frac{\partial}{\partial a}\left\{\frac{(x^a - x^b)}{\log x}\right\} dx$

$= \int_0^1 \frac{x^a \log x - 0}{\log x} dx$

$= \int_0^1 x^a dx$

$= \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1}$

$\therefore \int dI = \int \frac{1}{a+1} da$ Integrating

$I = \log(a + 1) + c$ .....(2)

When $a = b$ from (1) $I = 0$

Using $I = 0$ when $a = b$ in (2)

$0 = \log(b + 1) + c$

$\Rightarrow c = -\log(b + 1)$

Hence (2) gives

$I = \log(a + 1) - \log(b + 1)$

$= \log\left(\frac{a+1}{b+1}\right)$ Ans.

5. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$

Sol^n. Let $I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$ .....(1)

diff (1) w.r.t. 'a'

$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left\{\frac{e^{-ax} - e^{-bx}}{x}\right\} dx$

$$= \int_0^\infty \frac{(-x)e^{-ax} - 0}{x} dx$$

$$= -\int_0^\infty e^{-ax} dx$$

$$= -\left[\frac{e^{-ax}}{-a}\right]_0^\infty$$

$\frac{dI}{da} = -\frac{1}{a}$ or, $\int dI = -\int \frac{1}{a} da$ Integrating

$I = -\log a + c$ .....(2)

When $a = b$ from (1) $I = 0$

Using $I = 0$ when $a = b$ in (2)

$0 = -\log b + c$

$\Rightarrow c = \log b$

Hence (2) gives,

$I = -\log a + \log b$

$= \log \frac{b}{a}$ Ans.

6. $\int_0^{\frac{\pi}{2}} \log(a^2\cos^2\theta + b^2\sin^2\theta) d\theta \ (a > 0, b > 0)$

Sol^n. See important IOE extra question Q. N. 9 page (182)

7. $\int_0^\infty \frac{\log(1 + a^2x^2)}{(1 + b^2x^2)} dx$

Sol^n. See important IOE extra question Q. N. 1, page (180)

8. $\int_0^\pi \frac{\log(1 + \cos a \cos x)}{\cos x} dx$

Sol^n. See important IOE extra question Q. N. 2 page (181)

9. $\int_0^\pi \frac{\log(1 + a\cos x)}{\cos x} dx$

Sol^n. See important IOE extra question Q. N. 10 page (182)

Reduction Formula and Beta & Gamma Functions

9

Reduction formula

The formula of the given integral which is connected with some integrals of lower order is called Reduction formula. In most of the cases, the reduction formula is obtained by the method of integration by parts.

Some reduction formula

1. If $I_n = \int x^n e^{ax} dx$, then $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$

2. If $I_n = \int \sin^n x \, dx$, then $I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$

3. If $I_n = \int \cos^n x \, dx$, then $I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$

4. If $I_n = \int \tan^n x \, dx$, then $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

5. If $I_n = \int \cot^n x \, dx$, hen $I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$

6. If $I_n = \int \sec^n x \, dx$, then $I_n = \frac{\sec^{n-2} \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

7. If $I_n = \int \operatorname{cosec}^n x \, dx$, then $I_n = -\frac{\operatorname{cosec}^{n-2} x \cos tx}{n-1} + \frac{n-2}{n-1} . I_{n-2}$

8. If $I_{m,n} = \int \cos^m x \sin nx \, dx$, then

$$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-n,n-1}$$

9. If $I_{m,n} = \cos^m x \cos x \, dx$ then, $I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$

Cosm

Exercise - 13

1. **Obtain a reduction formula for $\int_0^{\frac{\pi}{2}} \cos^n x\, dx$ and hence evaluate $\int_0^{\frac{\pi}{2}} \cos^{10} x\, dx$.** **[B.E. 2060, 2062]**

Sol^n. Let $J_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x\, dx$$

Using integration by parts

$$= \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1) \cos^{n-2} x\,(-\sin x) \sin x\, dx$$

$$= 0 + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x\,(1-\cos^2 x)\, dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx - (n-1) \int_0^{\frac{\pi}{2}} \cos^n x\, dx$$

$\therefore J_n = (n-1) J_{n-2} - (n-1) J_n$

or $J_n + (n-1) J_n = (n-1) J_{n-2}$

$(1+n-1) J_n = (n-1) J_{n-2}$

$nJ_n = (n-1) J_{n-2}$

$\therefore J_n = \frac{(n-1)}{n} J_{n-2}$ which is he reduction formula for $\int_0^{\frac{\pi}{2}} \cos^n x\, dx$

Next part

$\int_0^{\frac{\pi}{2}} \cos^{10} x\, dx = J_{10}$

$$= \frac{(10-1)}{10} J_8 = \frac{9}{10} \cdot \frac{(8-1)}{8} J_6$$

$$= \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{(6-1)}{6} J_4$$

$$= \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{(4-1)}{4} J_2$$

$$= \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{(2-1)}{2} J_0$$

$$= \frac{63}{256} \int_0^{\frac{\pi}{2}} dx$$

$$= \frac{63}{256} [x]_0^{\frac{\pi}{2}}$$

$$= \frac{63}{256}\left(\frac{\pi}{2} - 0\right) = \frac{63}{512} \frac{\pi}{2}$$

$\therefore \int_0^{\frac{\pi}{2}} \cos^{10} x\, dx = \frac{63\pi}{512}$ Ans.

2. **Obtain the reduction formula for $\int \sec^n x\, dx$ and find $\int \sec^6 x\, dx$.**

Sol^n. Let $I_n = \int \sec^n x\, dx$

$= \int \sec^{n-2} x \sec^2 x\, dx$

Using integration by part

$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x\,[\sec x \tan x] \tan x\, dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x\,(\sec^2 x - 1)\, dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x\, dx + (n-2) \int \sec^{n-2} x\, dx$

$\therefore I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

or, $I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$(1+n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$$I_n = \frac{\sec^{n-2} x \tan x}{(n-1)} + \frac{(n-2)}{(n-1)} I_{n-2}$$

Which is the reduction formula for $\int \sec^n x\, dx$

Next part

$\int \sec^6 x\, dx = I_6$

$$= \frac{\sec^4 x\, d\tan x}{5} + \frac{4}{5} I_4$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{5}\left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2\right]$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{15} \sec^2 x \tan x + \frac{8}{15} \int \sec^2 x\, dx$$

$$I_6 = \frac{\sec^4 \tan x}{5} + \frac{4}{15} \sec^2 x \tan x + \frac{8}{15} \tan x$$

3. **If $I_n = \int_0^{\frac{\pi}{4}} \tan^x x\, dx$. Show that $I_n + I_{n-2} = \frac{1}{n-1}$ and hence deduce the values of I_5.**

Sol^n. Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x\, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,(\sec^2 x - 1)\,dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x\,dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,d(\tan x) - I_{n-2}$$

$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$\therefore I_n + I_{n-2} = \frac{1}{n-1} \text{ Proved.}$$

Next part

We know, $I_n = \frac{1}{n-1} - I_{n-2}$

$$\therefore I_5 = \frac{1}{(5-1)} - I_3$$

$$= \frac{1}{4} - \frac{1}{(3-1)} + I_1$$

$$= \frac{1}{4} - \frac{1}{2} + \int_0^{\frac{\pi}{4}} \tan x\,dx$$

$$= \frac{1-2}{4} + \left[\log \sec x\right]_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{4} + \left\{\log\sqrt{2} - \log 1\right\}$$

$$= -\frac{1}{4} + \log 2^{\frac{1}{2}} - 0$$

$$= -\frac{1}{4} + \frac{1}{2}\log 2$$

$$\therefore I_5 = \frac{1}{2}\log 2 - \frac{1}{4} \text{ Ans.}$$

4. If $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx\,dx$, prove that

$$I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

Sol^n. We have, $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx\,dx$

Using integrating by parts,

$$I_{m,n} = \left[\cos^m x \frac{(-\cos nx)}{n}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} m\cos^{m-1} x \sin x \frac{\cos x}{n}\,dx$$

$$= \left(0 + \frac{1}{n}\right) - \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^{m-1} x \sin x \cos nx\,dx$$

$$= \frac{1}{n} - \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^{m-1} x\,[\sin nx \,.\, \cos x - \sin(n-1)x]\,dx$$

Since, $\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$

$$\Rightarrow \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$$

$$= \frac{1}{n} - \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^m x \sin nx\,dx + \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^{m-1} x \sin(n-1)x\,.\,dx$$

$$\therefore I_{m,n} = \frac{1}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$\left(1 + \frac{m}{n}\right) I_{m,n} = \frac{1}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\therefore I_{m,n} = \frac{n}{m+n}\cdot\frac{1}{n} + \frac{n}{m+n}\cdot\frac{m}{n} I_{m-1,n-1}$$

$$\therefore I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1} \text{ Proved.}$$

5. Prove that $\int_0^{\frac{\pi}{2}} \cos^5 x \sin 3x\,dx = \frac{1}{3}$

Sol^n. We know the reduction formula,

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx\,dx = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

Now, $\int_0^{\frac{\pi}{2}} \cos^5 x \,\text{sn}\, 3x\,dx = I_{5,3}$

$$= \frac{1}{(5+3)} + \frac{5}{(5+3)} I_{4,2}$$

$$= \frac{1}{8} + \frac{5}{8}\left[\frac{1}{(4+2)} + \frac{4}{(4+2)} I_{3,1}\right]$$

$$= \frac{1}{8} + \frac{5}{48} + \frac{20}{48}\left[\frac{1}{(3+1)} + \frac{3}{(3+1)} I_{2,0}\right]$$

$$= \frac{1}{8} + \frac{5}{48} + \frac{20}{48}\cdot\frac{1}{4} + \frac{20}{48}\cdot\frac{3}{4}\int_0^{\frac{\pi}{2}} \cos^2 x.\sin 0^\circ\,dx$$

$$= \frac{1}{8} + \frac{5}{48} + \frac{5}{16}\cdot 0$$

$$= \frac{6+5+5}{48} = \frac{1}{3}$$

$\therefore \int_0^{\frac{\pi}{2}} \cos^5 x \sin 3x \, dx = \frac{1}{3}$ Proved.

6. **Prove that** $\int_0^{\frac{\pi}{2}} \cos^m x \sin mx \, dx = \frac{1}{2^{m+1}} \left(2 + \frac{2^2}{2} + \frac{2^3}{3} + \ldots \frac{2^m}{m} \right)$

Solⁿ. We know that reduction formula.

$$\int \cos^m x \sin nx \, dx = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} \int \cos^{m-1} x \, . \sin (n-1) x \, dx$$

Putting n = m

$$\therefore \int \cos^m x \sin mx \, dx = -\frac{\cos^m x \cos mx}{2n} + \frac{m}{2n} \int \cos^{m-1} x \, . \sin (m-1) x \, dx$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^m x \sin mx \, dx = \left[-\frac{\cos^m x \cos mx}{2m} \right]_0^{\frac{\pi}{2}}$$

$$+ \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{m-1} x \, . \sin (m-1) x \, dx$$

Let $I_m = \int_0^{\frac{\pi}{2}} \cos^m x \sin mx \, dx$

Then $I_m = \frac{1}{2m} + \frac{1}{2} I_{m-1}$

Now, $I_1 = \int_0^{\frac{\pi}{2}} \cos x \sin x = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2x \, dx$

$$= -\frac{1}{4} \left[\cos 2x \right]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{4} [-1-1] = -\frac{1}{2^2} [-2] = \frac{2}{2^2}$$

Again, $I_2 = \frac{1}{2.2} + \frac{1}{2} I_1$

$$= \frac{1}{2.2} + \frac{1}{2} \cdot \frac{1}{2^2} [2] = \frac{1}{2^3} (2+2) = \frac{1}{2^3} \left(2 + \frac{2^2}{2} \right)$$

$I_3 = \frac{1}{2.3} + \frac{1}{2} I_2$

$$= \frac{1}{2.3} + \frac{1}{2} \cdot \frac{1}{2^3} \left(2 + \frac{2^2}{2} \right)$$

$$= \frac{1}{2^4} \left(\frac{2^3}{3} + 2 + \frac{2^2}{2} \right)$$

$$= \frac{1}{2^4} \left[2 + \frac{2^2}{2} + \frac{2^3}{3} \right]$$

Similarly,

$I_4 = \frac{1}{2.4} + \frac{1}{2} I_3$

$$= \frac{1}{2.4} + \frac{1}{2} \cdot \frac{1}{2^4} \left(2 + \frac{2^2}{2} + \frac{2^3}{3} \right)$$

$$= \frac{1}{2^5} \left(\frac{2^4}{4} + 2 + \frac{2^2}{2} + \frac{2^3}{3} \right)$$

$$I_4 = \frac{1}{2^5} \left(2 + \frac{2^2}{2} + \frac{2^3}{3} + \frac{2^4}{4} \right)$$

Thus by inductiona

$$I_m = \frac{1}{2^{m+1}} \left(2 + \frac{2^2}{2} + \frac{2^3}{3} + \frac{2^4}{4} + \ldots \frac{2^m}{m} \right) \text{ Ans.}$$

Beta and Gamma functions

Definition: An integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m>0$, $n>0$ is said to be Beta function or first Eulerian integral. It is denoted by $\beta(m, n)$.

Thus $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ $m>0, n>0$

An integral $\int_0^\infty e^{-x}x^{n-1}dx$, $n>0$ is said to be Gamma function or second Eulerian integral. It is denoted by $\Gamma(n)$ Thus $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$, $n>0$

Properties of Beta and Gamma function

(1) $\beta(m, n) = \beta(n,m)$

(2) $\beta(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

(3) $\Gamma(1) = 1$

(4) $\Gamma(n+1) = n\Gamma(n)$ when n is positive number.

(5) $\Gamma(n+1) = n!$ when n is positive integer,

(6) $\Gamma(m)\Gamma(1-m) = \dfrac{\pi}{\sin m\pi}$ as $0<m<1$

(7) $\beta(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

(8) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(9) $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta d\theta = \dfrac{1}{2}\dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$; $m>-1, n>-1$

(10) $\int_0^{\frac{\pi}{2}} \sin^m\theta d\theta = \dfrac{\sqrt{\pi}}{2}\dfrac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$

(11) $\int_0^{\frac{\pi}{2}} \cos^n\theta d\theta = \dfrac{\sqrt{\pi}}{2}\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$

(12) $\int_0^\infty e^{-x^2}dx = \dfrac{\sqrt{\pi}}{2}$

(13) $\beta(m,n) = \int_0^\infty \dfrac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \dfrac{x^{n-1}}{(1+x)^{m+n}}dx$

Gaurab Shrestha

Exercise - 14

1. **Find the value of**

(i) $\Gamma\left(\frac{7}{2}\right)$

Solⁿ. $\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{3}{2}+1\right)$

$= \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$= \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$

$\therefore \Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$

(ii) $\dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(5)}$

Solⁿ. $\dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(5)} = \dfrac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(4+1)}$

$= \dfrac{\frac{1}{2}\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{4!} = \dfrac{3\pi}{8\times4\times3\times2\times1} = \dfrac{\pi}{2^6}$

$\therefore \dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(5)} = \dfrac{\pi}{64}$ Ans.

2. **Evaluate :**

(i) $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x\, dx$

Solⁿ. Let $I = \int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x \cdot dx$

$= \dfrac{1}{2}\dfrac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{\Gamma\left(\frac{4+2+2}{2}\right)} = \dfrac{1}{2}\dfrac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)} = \dfrac{1}{2}\cdot\dfrac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{3!}$

Gaurab Shrestha

$$= \frac{\pi}{2^5}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \sin^4 \cos^2 x \, dx = \frac{\pi}{32} \text{ Ans.}$$

(ii) $\int_0^{\frac{\pi}{4}} \left(1-2\sin^2\theta\right)^{\frac{3}{2}} \cos\theta \, d\theta$

Solⁿ. Let $I = \int_0^{\frac{\pi}{4}} \left(1-2\sin^2\theta\right)^{\frac{3}{2}} \cos\theta \, d\theta$

Put $\sqrt{2}\sin\theta = \sin t$

$\cos\theta \, d\theta = \frac{1}{\sqrt{2}} \cos t \, dt$

When $\theta \to 0$ then $t \to 0$

When $\theta \to \frac{\pi}{4}$ then $t \to \frac{\pi}{2}$

$$\therefore I = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \left(1-\sin^2 t\right)^{\frac{3}{2}} \cos t \, dt$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \cos^3 t \cos t \, dt$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \cos^4 t \, dt$$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{4+1}{2}\right)}{\Gamma\left(\frac{4+2}{2}\right)}$$

$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{5}{2}\right)}{2\sqrt{2}\,\Gamma 3} = \frac{\sqrt{\pi}\,\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\sqrt{2}\,2\times 1} = \frac{3\pi}{16\sqrt{2}}$$

$$\therefore I = \frac{3\pi}{16\sqrt{2}} \text{ Ans.}$$

3. **Use Beta function to evaluate**

(i) $\int_0^a x^2\left(a^2-x^2\right)^{\frac{3}{2}} dx$

Solⁿ. Let $I = \int_0^a x^2\left(a^2-x^2\right)^{\frac{3}{2}} dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta \, d\theta$

When $x \to 0$ then $\theta \to 0$.

When $x \to a$ then $\theta \to \frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} (a\sin\theta)^2 \left(a^2 - a^2\sin^2\theta\right)^{\frac{3}{2}} a\cos\theta \, d\theta$$

$$= a^6 \int_0^{\frac{\pi}{2}} \sin^2\theta \,.\, \cos^4\theta \, d\theta$$

$$= a^6 \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{2+4+2}{2}\right)} = a^6 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma(4)}$$

$$= \frac{a^6 \frac{1}{2}\cdot\sqrt{\pi}\,\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2\times 3\times 2\times 1} = \frac{a^6\pi}{2^5} = \frac{a^6\pi}{32}$$

$$\therefore I = \frac{a^6\pi}{32} \text{ Ans.}$$

(ii) $\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}} dx$

$x - \sin^2\theta \quad \cos^2\theta$

Solⁿ. Let $I = \int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}} dx$

Put $x = \sin^2\theta \to dx = 2\sin\theta\cos\theta d\theta$

When $x \to (0, 1)$ Then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{\frac{3}{2}} (1-\sin^2\theta)^{\frac{3}{2}} \, 2\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^4\theta d\theta$$

$$= 2\frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{4+4+2}{2}\right)} = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma 5}$$

$$= \frac{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{4\times 3\times 2\times 1} = \frac{3\pi}{2^7}$$

$$\therefore I = \frac{3\pi}{128}$$

(iii) $\int_0^{2a} x^5\sqrt{2ax-x^2}\,dx$

$x^5 \sqrt{(2a-x)x}$

$\sin^4\theta$

$2a$

$4a^2(x - x^2)$

Solⁿ. Let $I = \int_0^{2a} x^5\sqrt{2ax-x^2}\,dx$

Put $x = 2a\sin^2\theta \Rightarrow dx = 4a\sin\theta\cos\theta d\theta$ When $x \to (0, 2a)$ then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = \int_0^{\frac{\pi}{2}} (2a\sin^2\theta)^5 (2a.2a\sin^2\theta - 4a^2\sin^4\theta)^{\frac{1}{2}} \, 4a\sin\theta\cos\theta d\theta$$

$$= 2^5 \times a^5 \times 2a \times 4a \int_0^{\frac{\pi}{2}} \sin^{12}\theta\cos^2\theta d\theta$$

$$= 2^8 a^7 \frac{\Gamma\left(\frac{12+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{12+2+2}{2}\right)} = \frac{2^8 a^7 \Gamma\left(\frac{13}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(8)}$$

$$= \frac{2^8 a^7 \frac{11}{2}.\frac{9}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}}{2.7.6.5.4.3.2.1}$$

$$= \frac{2^8 a^7 33\pi}{2^{12}} = \frac{33a^7\pi}{2^4}$$

$$\therefore I = \frac{33a^7\pi}{16}$$

(iv) $\int_0^{2a} x^{\frac{9}{2}}(2a-x)^{\frac{-1}{2}} dx$

Sol^n. Let $I = \int_0^{2a} x^{\frac{9}{2}}(2a-x)^{\frac{-1}{2}} dx$

Put $x = 2a\sin^2\theta \Rightarrow dx = 4a\sin\theta\cos\theta d\theta$

when $x \to (0, 2a)$ Then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = \int_0^{\frac{\pi}{2}} \left(2a\sin^2\theta\right)^{\frac{9}{2}} \left(2a - 2a\sin^2\theta\right)^{\frac{-1}{2}} 4a\sin\theta\cos\theta d\theta$$

$$= (2a)^{\frac{9}{2}}.(2a)^{\frac{-1}{2}}.4a\int_0^{\frac{\pi}{2}} \sin^9\theta.(\cos\theta)^{-1}\sin\theta\cos\theta d\theta$$

$$= (2a)^{\frac{9-1}{2}}.4a\int_0^{\frac{\pi}{2}} \sin^{10}\theta d\theta$$

$$= (2a)^4.4a\frac{\sqrt{\pi}\Gamma\left(\frac{10+1}{2}\right)}{2\Gamma\left(\frac{10+2}{2}\right)} = \frac{2^6 a^5\sqrt{\pi}\Gamma\left(\frac{11}{2}\right)}{2\Gamma(6)}$$

$$= \frac{2^6 a^5\sqrt{\pi}\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{2\times5.4.3.2.1}$$

$$= \frac{2^6 a^5\pi\times63}{2^9} = \frac{63\pi a^5}{2^3} = = \frac{63\pi a^5}{8}$$

$$\therefore I = = \frac{63\pi a^5}{8}$$

(v) $\int_0^a x^4\sqrt{a^2-x^2}dx$

Sol^n. Let $I = \int_0^a x^4\sqrt{a^2-x^2}dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

when $x \to (0, a)$ Then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = \int_0^{\frac{\pi}{2}} (a\sin\theta)^4\sqrt{a^2 - a^2\sin^2\theta}\, .a\cos\theta d\theta$$

$$= a^4.a.a\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta d\theta$$

$$= a^6\frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{4+2+2}{2}\right)} = \frac{a^6\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)}$$

$$= \frac{a^6\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\frac{1}{2}.\sqrt{\pi}}{2\times3.2.1} = \frac{a^6\pi}{2^5} = \frac{a^6\pi}{32}$$

$$\therefore I = \frac{a^6\pi}{32}$$

(vi) $\int_0^1 \frac{x^6}{\sqrt{1-x^2}}dx$

Sol^n. Let $I = \int_0^1 \frac{x^6}{\sqrt{1-x^2}}dx$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

when $x \to (0, 1)$ Then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^6}{\sqrt{1-\sin^2\theta}}.\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^6\theta d\theta$$

$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{6+2}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{7}{2}\right)}{2\Gamma(4)} = \frac{\sqrt{\pi}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{2\times3\times2\times1} = \frac{5\pi}{2^5}$$

$$\therefore I = \frac{5\pi}{32}$$

(vii) $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}dx$

Sol^n. Let, $I = \int_0^a \frac{x^4}{\sqrt{a^2-x^2}}dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

when $x \to (0, a)$ Then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{(a\sin\theta)^4}{\sqrt{a^2-a^2\sin^2\theta}} a\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{a^4\sin^4\theta\, a\cos\theta}{a\cos\theta} d\theta$$

$$= a^4 \int_0^{\frac{\pi}{2}} \sin^4\theta d\theta$$

$$= \frac{a^4\sqrt{\pi}\,\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{4+2}{2}\right)} = \frac{a^4\sqrt{\pi}\,\Gamma\left(\frac{5}{2}\right)}{2\Gamma(3)}$$

$$= \frac{a^4\sqrt{\pi}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{2\times2\times1} = \frac{3\pi a^4}{2^4} = \frac{3\pi a^4}{16}$$

$$\therefore I = \frac{3\pi a^4}{16}$$

(viii) $\int_0^a x^3\left(a^2-x^2\right)^{\frac{5}{2}}dx$

Sol^n. Let $I = \int_0^a x^3\left(a^2-x^2\right)^{\frac{5}{2}}dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

When $x \to (0, a)$ Then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = \int_0^{\frac{\pi}{2}} (a\sin\theta)^3 (a^2 - a^2\sin^2\theta)^{\frac{5}{2}} a\cos\theta d\theta$$

$$= a^3.a^5.a \int_0^{\frac{\pi}{2}} \sin^3\theta\cos^6\theta d\theta$$

$$= a^9 \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{3+6+2}{2}\right)} = \frac{a^9\Gamma(2)\Gamma\left(\frac{7}{2}\right)}{2\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{a^9.1.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{2.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}} = \frac{2a^9}{63}$$

$$\therefore I = \frac{2a^9}{63}$$

5. **Using Beta Gamma function, Show that**

(i) $\int_0^{\frac{\pi}{6}} \cos^4 3\theta \sin^2 6\theta d\theta = \frac{5\pi}{192}$ **[B.E. 2059/058]**

Sol^n. Let $I = \int_0^{\frac{\pi}{6}} \cos^4 3\theta \sin^2 6\theta d\theta$

$$= \int_0^{\frac{\pi}{6}} \cos^4 3\theta \{\sin 2.3\theta\}^2 d\theta$$

$$= \int_0^{\frac{\pi}{6}} \cos^4 3\theta\, (2\sin3\theta\cos3\theta)^2 d\theta.$$

$$= 4\int_0^{\frac{\pi}{6}} \cos^5 3\theta \sin^2 3\theta\, d\theta$$

Put $3\theta = t \Rightarrow d\theta = \frac{1}{3}dt$

When $\theta \to \left(0, \frac{\pi}{6}\right)$ Then $t \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = \frac{4}{3}\int_0^{\frac{\pi}{2}} \sin^2 t\cos^6 t dt$$

$$= \frac{4}{3}\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{2+6+2}{2}\right)} = \frac{4\times\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{2}\right)}{3\times2\Gamma(5)}$$

$$= \frac{4\times\frac{1}{2}.\sqrt{\pi}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{6\times4.3.2.1} = \frac{5\pi}{192}$$

$$\therefore I = \int_0^{\frac{\pi}{6}} \cos^4 3\theta \sin^2 6\theta d\theta = \frac{5\pi}{192} \text{ Proved.}$$

(ii) $\int_0^{\frac{\pi}{6}} \cos^2 6\theta \sin^4 3\theta d\theta = \frac{7\pi}{192}$

Sol^n. Let $I = \int_0^{\frac{\pi}{6}} \cos^2 6\theta \sin^4 3\theta d\theta$

$$= \int_0^{\frac{\pi}{6}} (\cos 2.3\theta)^2 \sin^4 3\theta d\theta$$

$$= \int_0^{\frac{\pi}{6}} (1-2\sin^2 3\theta)^2 \sin^4 3\theta d\theta$$

Put $3\theta = t \Rightarrow d\theta = \frac{1}{3}dt$

When $\theta \to \left(0, \frac{\pi}{6}\right)$ Then $t \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = \frac{1}{3}\int_0^{\frac{\pi}{2}} (1-2\sin^2 t)^2 \sin^4 t dt$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} (1-4\sin^2 t + 4\sin^4 t)\sin^4 t dt$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} (\sin^4 t - 4\sin^6 t + 4\sin^8 t) dt.$$

$$= \frac{1}{3}\left[\int_0^{\frac{\pi}{2}} \sin^4 t dt - 4\int_0^{\frac{\pi}{2}} \sin^6 t dt + 4\int_0^{\frac{\pi}{2}} \sin^8 t dt\right]$$

$$= \frac{1}{3}\left[\frac{\sqrt{\pi}\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{4+2}{2}\right)} - \frac{4\sqrt{\pi}\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{6+2}{2}\right)} + \frac{4\sqrt{\pi}\Gamma\left(\frac{8+1}{2}\right)}{2\Gamma\left(\frac{8+2}{2}\right)}\right]$$

$$= \frac{1}{3}\left[\frac{\sqrt{\pi}\Gamma\left(\frac{5}{2}\right)}{2\Gamma(3)} - \frac{4\sqrt{\pi}\Gamma\left(\frac{7}{2}\right)}{2\Gamma(4)} + \frac{4\sqrt{\pi}\Gamma\left(\frac{9}{2}\right)}{2\Gamma(5)}\right]$$

$$= \frac{1}{3}\left[\frac{\sqrt{\pi}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{2\times 2.1} - \frac{4\sqrt{\pi}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{2\times 3.2.1} + \frac{4\sqrt{\pi}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{2\times 4.3.2.1}\right]$$

$$= \frac{1}{3}\left[\frac{3\pi}{16} - \frac{5\pi}{8} + \frac{35\pi}{64}\right] = \frac{1}{3}\left[\frac{12\pi - 40\pi + 35\pi}{64}\right] = \frac{7\pi}{192}$$

$\therefore I = \int_0^{\frac{\pi}{6}} \cos^2 6\theta \sin^4 3\theta d\theta = \frac{7\pi}{192}$ proved.

(iii) $\int_0^{\pi} \sin^6 \frac{x}{2} \cos^8 \frac{x}{2} dx = \frac{5\pi}{2^{11}}$

Sol^n. Let $I = \int_0^{\pi} \sin^6 \frac{x}{2} \cos^8 \frac{x}{2} dx$

Put $\frac{x}{2} = t \Rightarrow dx = 2dt$

When $x \to (0, \pi)$ Then $t \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = 2\int_0^{\frac{\pi}{2}} \sin^6 t \cos^8 t dt$$

$$= \frac{2\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{8+1}{2}\right)}{2\Gamma\left(\frac{6+8+2}{2}\right)} = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{9}{2}\right)}{\Gamma(8)}$$

$$= \frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi} \times \frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{7\times 6\times 5\times 4\times 3\times 2\times 1}$$

$$= \frac{5\pi}{2^{11}}$$

$\therefore \int_0^{\pi} \sin^6 \frac{x}{2} \cos^8 \frac{x}{2} dx = \frac{5\pi}{2^{11}}$ proved.

(iv) $\int_0^{\frac{\pi}{4}} \sin^4 x \cos^2 x dx = \frac{3\pi - 4}{192}$

Sol^n. Let $I = \int_0^{\frac{\pi}{4}} \sin^4 x \cos^2 x dx$

$$= \int_0^{\frac{\pi}{4}} \sin^2 x.\cos^2 x.\sin^2 x dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} (2\sin x \cos x)^2 \sin^2 x dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} (\sin 2x)^2 \left(\frac{1-\cos 2x}{2}\right) dx$$

Put $2x = t \Rightarrow dx = \frac{1}{2}dt$

When $x \to \left(0, \frac{\pi}{4}\right)$ Then $t \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = \frac{1}{4} \times \frac{1}{2} \times \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2 t(1-\cos t) dt$$

$$= \frac{1}{16}\left[\int_0^{\frac{\pi}{2}} \sin^2 t dt - \int_0^{\frac{\pi}{2}} \sin^2 t \cos t dt\right]$$

$$= \frac{1}{16}\left[\frac{\sqrt{\pi}\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{2+2}{2}\right)} - \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{1+1}{2}\right)}{2\Gamma\left(\frac{2+1+2}{2}\right)}\right]$$

$$= \frac{1}{16}\left[\frac{\sqrt{\pi}\Gamma\left(\frac{3}{2}\right)}{2\Gamma(2)} - \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(2)}{2\Gamma\left(\frac{5}{2}\right)}\right]$$

$$= \frac{1}{16}\left[\frac{\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}}{2\times 1} - \frac{\frac{1}{2}\sqrt{\pi}.1}{2.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}\right]$$

$$= \frac{1}{16}\left[\frac{\pi}{4} - \frac{1}{3}\right]$$

$$= \frac{1}{16}\times\frac{(3\pi-4)}{12} = \frac{3\pi-4}{192}$$

$$\therefore I = \int_0^{\frac{\pi}{4}} \sin^4 x\cos^2 x dx = \frac{3\pi-4}{192} \text{ proved}$$

6. **Prove that** $\int_0^\infty x^2 e^{-x^4} dx \times \int_0^\infty e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$ **[B.E. 2062]**

Sol^n. L.H.S. Let $I = \int_0^\infty x^2 e^{-x^4} dx \times \int_0^\infty e^{-x^4} dx$

Put $x^4 = t \Rightarrow x = t^{\frac{1}{4}}$

$4x^3 dx = dt \Rightarrow dx = \frac{1}{4t^{\frac{3}{4}}} dt$

When $x \to (0, \infty)$ then $t \to (0, \infty)$.

$$\therefore I = \frac{1}{4}\int_0^\infty \frac{t^{\frac{1}{2}}e^{-t}}{t^{\frac{3}{4}}} dt \times \frac{1}{4}\int_0^\infty e^{-t}.\frac{1}{t^{\frac{3}{4}}} dt$$

$$= \frac{1}{4}\int_0^\infty t^{\frac{-1}{4}} e^{-t} dt \times \frac{1}{4}\int_0^\infty e^{-t} t^{\frac{-3}{4}} dt$$

$$= \frac{1}{16}\int_0^\infty e^{-t} t^{\frac{3}{4}-1} dt \times \int_0^\infty e^{-t} t^{\frac{1}{4}-1} dt$$

$$= \frac{1}{16}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{16}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{16}\frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi}{16\times\frac{1}{\sqrt{2}}} = \frac{\pi}{8\sqrt{2}} \text{ R.H.S. Proved.}$$

7. **Prove that** $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}$

Sol^n. We know,

$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$ as $0<m<1$

Put $m = \frac{1}{3}$; $\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) = \frac{\pi}{\sin\frac{\pi}{3}}$

or, $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}$ proved.

8. **Show that**

$\beta(m, n)\,\beta(m+n, \ell) = \beta(n, \ell)\,\beta(n+\ell, m)$

Sol^n. L.H.S $= \beta(m, n)\,\beta(m+n, \ell) = \frac{\Gamma(m)\Gamma(m)}{\Gamma(m+n)}.\frac{\Gamma(m+n)\Gamma(\ell)}{\Gamma(m+n+\ell)}$

$$= \frac{\Gamma(m)\Gamma(n)\Gamma(\ell)}{\Gamma(m+n+\ell)}$$

R.H.S $= \beta(n, \ell)\,\beta(n+\ell, m) = \frac{\Gamma(n)\Gamma(\ell)}{\Gamma(n+\ell)}.\frac{\Gamma(n+\ell)\Gamma(m)}{\Gamma(n+\ell+m)}$

$$= \frac{\Gamma(m)\Gamma(n)\Gamma(\ell)}{\Gamma(m+n+\ell)}$$

L.H.S = R.H.S proved

9. **Show that,** $\int_0^\infty e^{-x^2} x^\alpha dx = \frac{1}{2}\Gamma\left(\frac{\alpha+1}{2}\right)$

Sol^n. Let $I = \int_0^\infty e^{-x^2} x^\alpha dx$

put $x^2 = t \Rightarrow x = t^{\frac{1}{2}}$

$2x\,dx = dt \Rightarrow dx = \frac{1}{2t^{\frac{1}{2}}} dt$

When $x \to (0, \infty)$ Then $t \to (0, \infty)$

$$\therefore I = \frac{1}{2}\int_0^\infty \frac{e^{-t} t^{\frac{a}{2}}}{t^{\frac{1}{2}}} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{a-1}{2}} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{a+1}{2}-1} dt$$

$$= \frac{1}{2}\Gamma\left(\frac{a+1}{2}\right)$$

$$\therefore I = \int_0^\infty e^{-x^2} x^a dx = \frac{1}{2}\Gamma\left(\frac{a+1}{2}\right) \text{ proved}$$

10. **Prove that**

$$\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}; \; m > -1, n > -1.$$

Sol^n. Let $I = \int_a^b (x-a)^m (b-x)^n dx$

Put $(x-a) = (b-a)t \Rightarrow x = (b-a)t + a$

$\therefore dx = (b-a)\,dt$

When $x \to a$ then $t \to 0$

When $x \to b$ then $t \to 1$

$$\therefore I = \int_0^1 \{(b-a)t\}^m (b - bt + at - a)^n (b-a) dt$$

$$= \int_0^1 (b-a)^m t^m (b-a)^n (1-t)^n (b-a) dt$$

$$= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n dt$$

$$= (b-a)^{m+n+1} \int_0^1 t^{m+1-1} (1-t)^{n+1-1} dt$$

$$= (b-a)^{m+n+1} \beta(m+1, n+1)$$

$$= (b-a)^{m+n+1} \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$$

$$\therefore I = \int_a^b (x-a)^m (b-x)^n dx = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} \text{ Proved.}$$

In particular, $a = -1, b = 1$ then

$$\int_{-1}^1 (x+1)^m (1-x)^n dx = 2^{m+n+1} \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} \text{ Proved.}$$

Quadrature

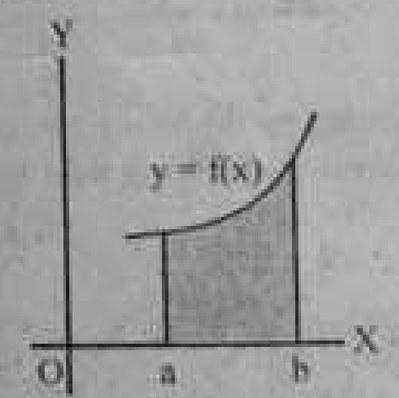
11

Quadrature: Quadrature is the method for finding out the area bounded by any portion of a plane curve. The definite integral $\int_a^b f(x)dx = \int_a^b y dx$ represents the area bounded by the curve $y = f(x)$, the x-axis and the two fixed ordinates $x = a$ and $x = b$.

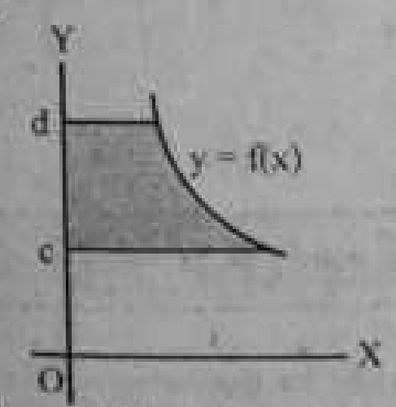
Some Important formula

1. If the region bounded by the curve $y = f(x)$, x – axis with two ordinate $x = a$ and $x = b$ then Area is defined by

$$A = \int_a^b y dx = \int_a^b f(x)dx$$

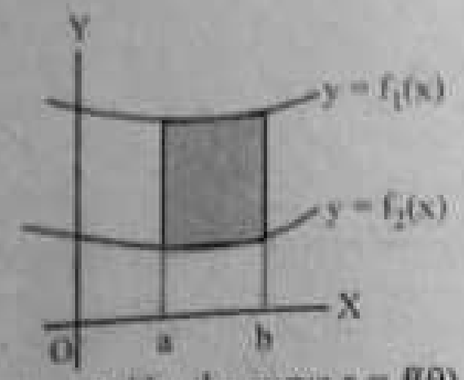


2. If the region bounded by the curve $x = f(y)$, y – axis, with two lines $y = c$ and $y = d$, then Area is defined by $A = \int_c^d x dy = \int_c^d f(y)dy$



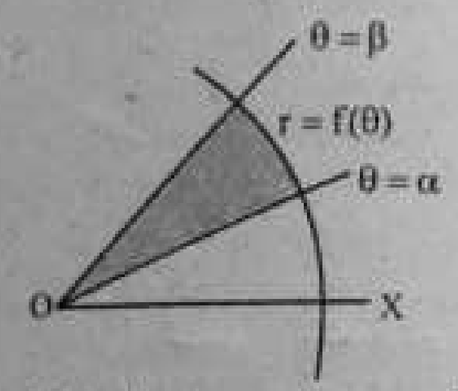
3. If the region is bounded by the curves $y = f_1(x)$ and $y = f_2(x)$, x – axis with two ordinates $x = a$ to $x = b$ then the Area is defined by

$$A = \int_a^b [f_1(x) - f_2(x)]dx$$

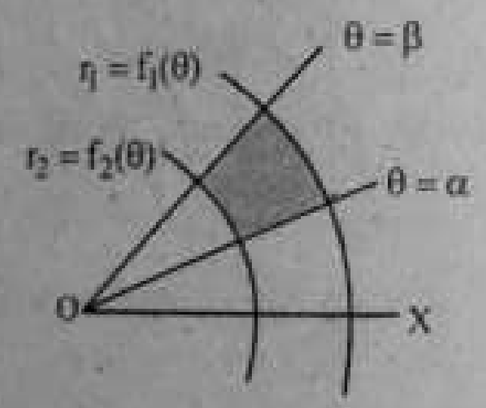


4. If the region is bounded by the curve $r = f(\theta)$, the two radii vector $\theta = \alpha$ to $\theta = \beta$ then the Area is defined by

$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$



5. If the region is bounded by the two curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ with two radii vectors $\theta = \alpha$ to $\theta = \beta$ then the area is defined by



$$A = \frac{1}{2}\int_\alpha^\beta (r_1^2 - r_2^2)d\theta$$

Exercise - 15

1. **Find the area bounded by the curve $y^2 = x^3$ and the line $y = 2x$**

Sol. Given curve $y^2 = x^3$ .....(1) and line $y = 2x$ .....(2)

From (1) & (2)

$4x^2 = x^3$

$4x^2 - x^3 = 0$

$9x^2 = x^3$

$x^2(x^2 - 4)$

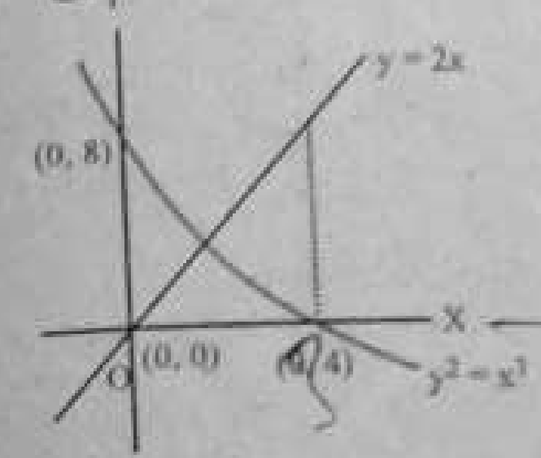
$x = 0, 9$

$x^2(4 - x) = 0$

$x = 0, x = 4$

When $x = 0$ then $y = 0$

When $x = 4$ then $y = 8$



$$\text{Required Area} = \int_0^4 \left(2x - x^{\frac{3}{2}}\right)dx$$

$$= \left[x^2 - \frac{2}{5}x^{\frac{5}{2}}\right]_0^4$$

$$= \left(16 - \frac{2}{5}4^{\frac{5}{2}}\right) = \left(16 - \frac{64}{5}\right) = \frac{80 - 64}{5}$$

$$= \frac{16}{5} \text{ square unit}$$

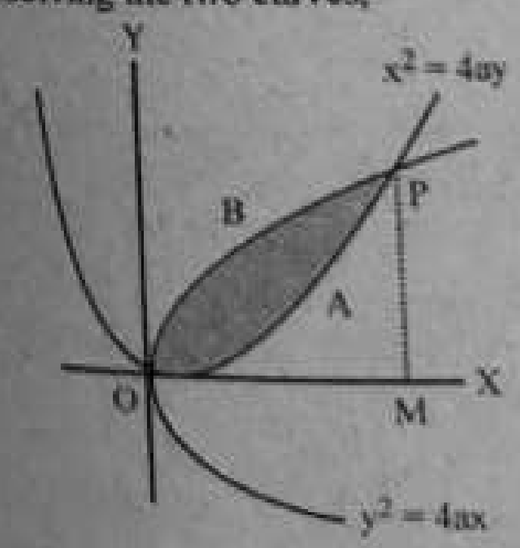
2. **Show that the area bounded by the curves $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{167}{3}a^2$**

Sol. Given curves are;

$y^2 = 4ax$ and $x^2 = 4ay$

Here we have to find the area OAPBO

Solving the two curves,



$y^2 = 4ax$ and $x^2 = 4ay$

we have, $4ay^3 = 4ax^3$

or $y = x$

$\therefore x^2 = 4ax$ or $x(x-4a) = 0$

$\therefore x = 0$ or $4a$

Thus $x = 0$ at 0 and $4a$ at p

Required Area OAPBO

= area OBPMO – area OAPMO

$$= \int_0^{4a}\left(\sqrt{4ax} - \frac{x^2}{4a}\right)dx$$

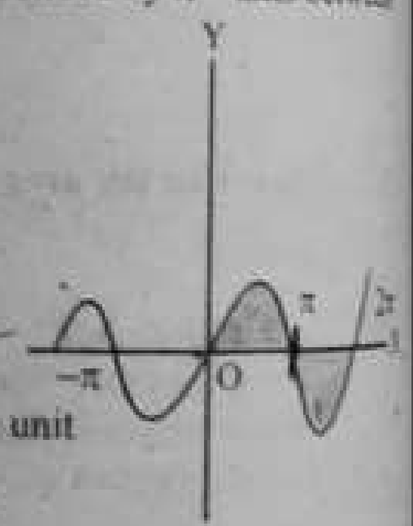
$$= \left[2a^{\frac{1}{2}}\cdot\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4a}\cdot\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4a^{\frac{1}{2}}}{3}(4a)^{\frac{3}{2}} - \frac{1}{12a}(4a)^3$$

$$= \frac{32a^2}{3} - \frac{16}{3}a^2 = \frac{16a^2}{3} \text{ Square unit.}$$

3. **Find the area of the region bounded by the curve $y = \sin x$ and x - axis between $x = 0$ and $x = 2\pi$**

Sol^n. Given curve $y = \sin x$ and the region bounded by x – axis between $x = 0$ and $x = 2\pi$



Required Area $= 2\int_0^{\pi} y\,dx$

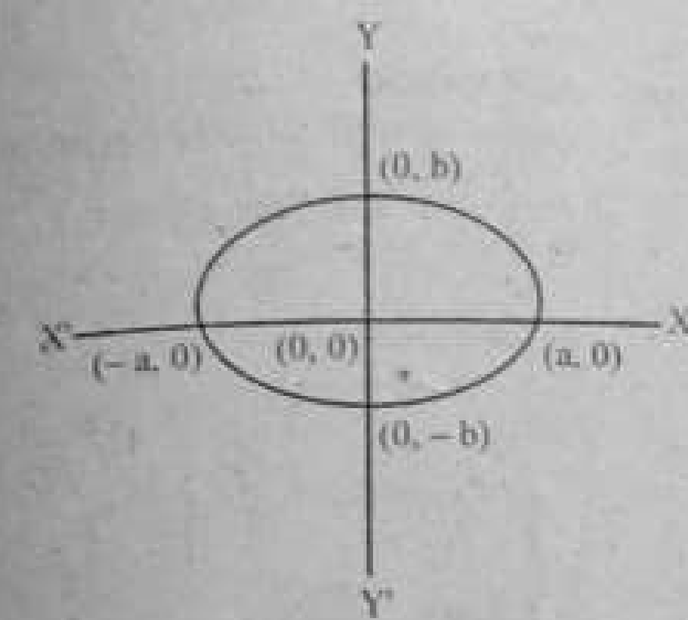
$$= 2\int_0^{\pi} \sin x\,dx$$

$$= 2\left[-\cos x\right]_0^{\pi}$$

$$= 2(1+1) = 4 \text{ square unit}$$

4. **Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**

Sol^n. Given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$\therefore$ Required Area $= 4\int_0^a y\,dx$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

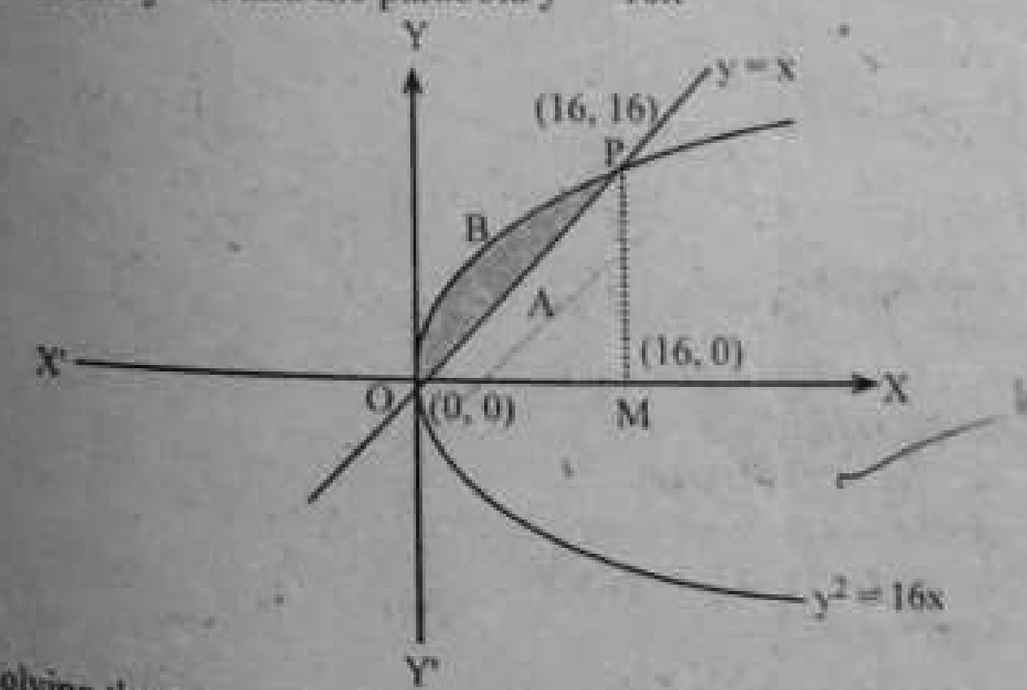
$$= \frac{4b}{a}\left[x\frac{\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left(\frac{a^2}{2}\sin^{-1}1\right) = \frac{4b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \pi ab$$

$\therefore$ Area of the ellipse $= \pi ab$ square unit.

5. **Find the area enclosed between the line $y = x$ and the parabola $y^2 = 16x$**

Sol^n. Given line $y = x$ and the parabola $y^2 = 16x$



Solving these

$x = 0, 16$

When $x = 0$; $y = 0$

When $x = 16, y = 16$

Required Area = Area OAPBO

= Area OMPBO – Area OMPAO

$$= \int_0^{16}\left(\sqrt{16x} - x\right)dx$$

$4x^{1/2}$

$\frac{1}{2}$

$$= \left[4.\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^{16}$$

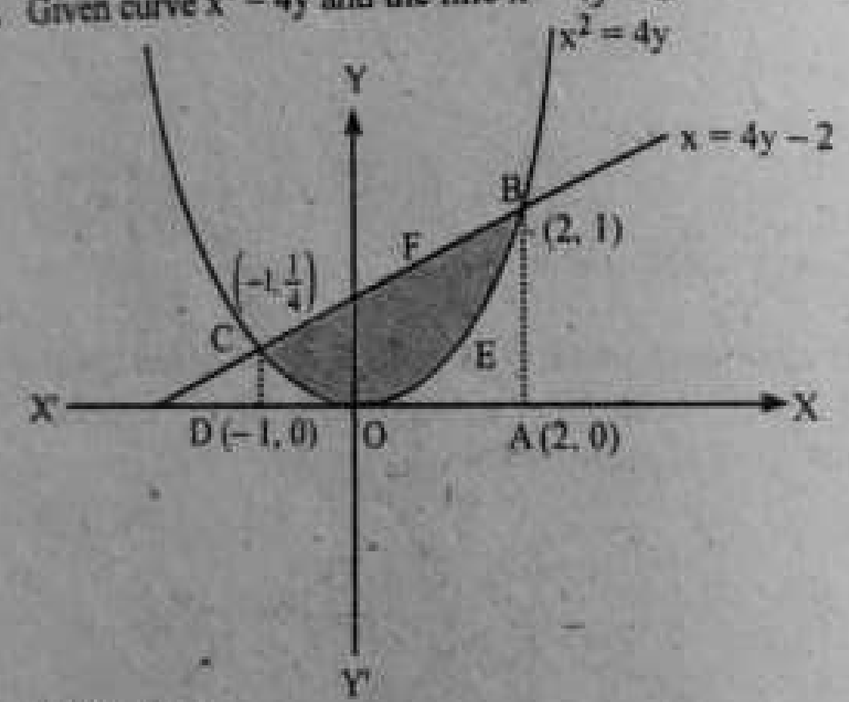
$$= \left[4.\frac{2}{3}(16)^{\frac{3}{2}} - \frac{(16)^2}{2}\right]$$

$$= \left[\frac{8}{3}\times 64 - \frac{256}{2}\right]$$

$$= \frac{512}{3} - 128 = \frac{128}{3} \text{ square unit.}$$

6. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$

Solⁿ. Given curve $x^2 = 4y$ and the line $x = 4y - 2$



Solving these,

$(4y - 2)^2 = 4y$

or, $16y^2 - 16y + 4 - 4y = 0$

or, $16y^2 - 20y + 4 = 0$

or, $4(4y^2 - 5y + 1) = 0$

or, $(4y^2 - 4y - y + 1) = 0$

or, $4y(y - 1) - 1(y - 1) = 0$

or, $(y - 1)(4y - 1) = 0$

$\therefore y = 1, y = \frac{1}{4}$

When $y = 1$ then $x = 2$

When $y = \frac{1}{4}$ then $x = -1$,

$\therefore$ Required Area = Area of BFCO

= Area DOABFCD – Area DOABFCD

$$= \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx$$

$$= \left[\frac{x^2}{8} + \frac{2x}{4} - \frac{x^3}{12}\right]_{-1}^{2}$$

$$= \left(\frac{4}{8} + \frac{4}{4} - \frac{8}{12} - \frac{1}{8} + \frac{2}{4} - \frac{1}{12}\right)$$

$$= \left(\frac{1}{2} + 1 - \frac{2}{3} - \frac{1}{8} + \frac{1}{2} - \frac{1}{12}\right)$$

$$= 2 - \frac{2}{3} - \frac{1}{8} - \frac{1}{12}$$

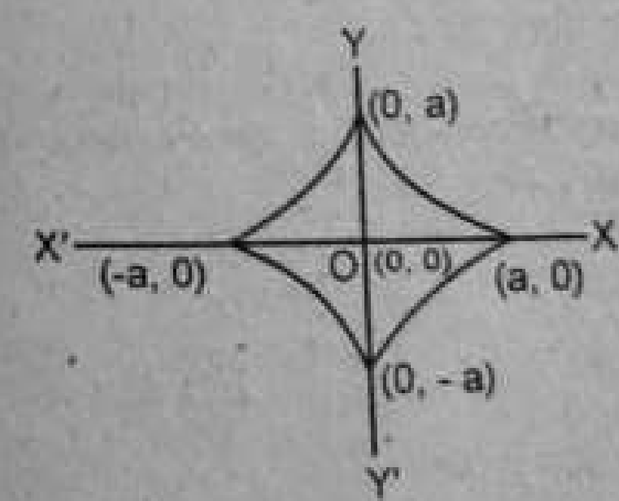
$$= \frac{48 - 16 - 3 - 2}{24} = \frac{48 - 21}{24} = \frac{27}{24}$$

$$= \frac{9}{8} \text{ square unit.}$$

7. Show that the area of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $\frac{3}{8}\pi a^2$

Solⁿ. Given astroid

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$



$\therefore$ Required Area $= 4\int_0^a y\,dx$

$$= 4\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} dx$$

put $x = a\sin^3\theta$

$dx = 3a\sin^2\theta\cos\theta d\theta$

When $x \to (0, a)$ then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$= 4 \times 3a \int_0^{\frac{\pi}{2}} \left\{ a^{\frac{2}{3}} - \left(a \sin^3\theta\right)^{\frac{2}{3}} \right\}^{\frac{3}{2}} \sin^2\theta \cos\theta d\theta$$

$$= 12a \times a \int_0^{\frac{\pi}{2}} \cos^3\theta \sin^2\theta \cos\theta d\theta$$

$$= 12a^2 \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^4\theta d\theta$$

$$= 12a^2 \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{2+4+2}{2}\right)}$$

$$= 12a^2 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma(4)} = 12a^2 \frac{\frac{1}{2}.\sqrt{\pi}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{2 \times 3!}$$

$$= \frac{12a^2\pi}{32} = \frac{3a^2\pi}{8} \text{ square unit.}$$

8. Find the area bounded by curve $x^2y = a^2(a - y)$ and the x – axis [B.E. 2059]

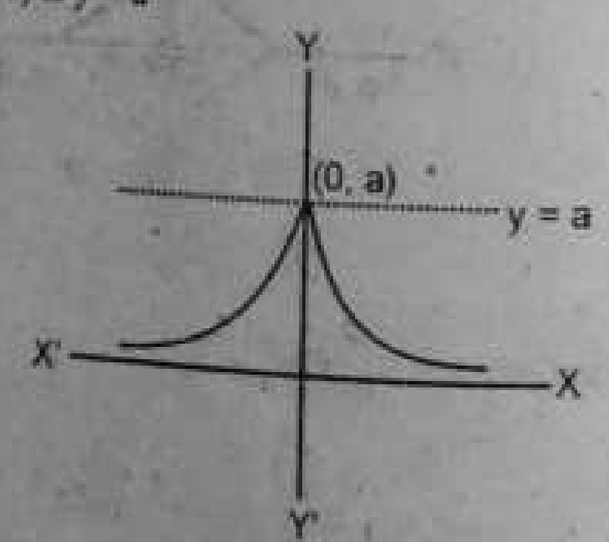
Sol^n. Given curve

$x^2y = a^2(a - y)$ .....(1)

and bounded by x – axis

when $y = a$ from (1) $x = 0$

The curve cuts the y – axis at (0, a) and asymptote is $y = 0$, so the x–axis is asymptote. Also x–becomes imaginary for $y > a$. Also the tangent at (0, a) is $y = a$



$\therefore$ Required Area $= 2\int_0^a x dy$

$$= 2\int_0^a a\sqrt{\frac{a-y}{y}}dy = 2a\int_0^a \sqrt{\frac{a-y}{y}}dy$$

put $y = a\sin^2\theta$

$dy = 2a\sin\theta\cos\theta d\theta$

When $y \to (0, a)$ Then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$= 2a \times 2a \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta} \sin\theta\cos\theta d\theta$$

$$= 4a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$$

$$= 4a^2 \frac{\sqrt{\pi}\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{2+2}{2}\right)} = \frac{4a^2\sqrt{\pi}\Gamma\left(\frac{2}{3}\right)}{2\Gamma(2)}$$

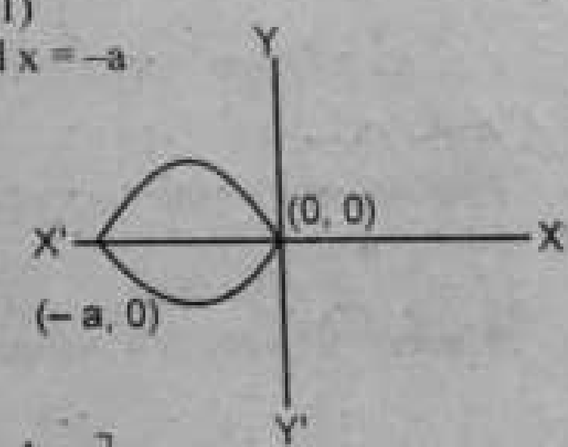
$$= \frac{4a^2\sqrt{\pi}\frac{1}{2}.\sqrt{\pi}}{(1)!}$$

$= \pi a^2$ square unit.

9. Find the area of the loop of the curve $y^2 = x^2(x + a)$

Sol^n. Given curve, $y^2 = x^2(x + a)$ .....(1)

When $y = 0$ from (1) $x = 0$, and $x = -a$



$\therefore$ Required area $= 2\int_{-a}^0 y dx$

$$= 2\int_{-a}^0 x\sqrt{x+a}dx$$

$$= 2\int_{-a}^0 (x+a-a)(x+a)^{\frac{1}{2}}dx$$

$$= 2\left[\int_{-a}^0 (x+a)^{\frac{3}{2}}dx - a\int_{-a}^0 (x+a)^{\frac{1}{2}}dx\right]$$

$$= \left[2 \times \frac{5}{2}(x+a)^{\frac{5}{2}} - 2a \times \frac{2}{3}(x+a)^{\frac{3}{2}}\right]_{-a}^0$$

$$= \frac{4}{5}a^{\frac{5}{2}} - \frac{4a}{3}a^{\frac{3}{2}}$$

$$= \frac{4}{5}a^{\frac{5}{2}} - \frac{4a^{\frac{5}{2}}}{3}$$

$$= \frac{4a^{\frac{5}{2}}(3-5)}{15}$$

$$= \frac{-8a^{\frac{5}{2}}}{15} = \frac{8a^{\frac{5}{2}}}{15} \text{ square unit.}$$

10. Find the area of the curve $y^2 = x(x-1)^2$

Sol^n. Given curve $y^2 = x(x-1)^2$ .....(1)

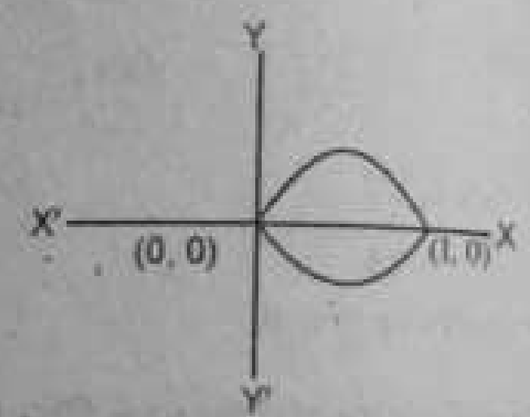
When $y = 0$ from (1) $x = 0$ and $x = 1$

$\therefore$ Required Area $= 2\int_0^1 y\,dx$

$$= 2\int_0^1 \sqrt{x(x-1)^2}\,dx$$

$$= 2\int_0^1 x^{\frac{1}{2}}(x-1)\,dx$$

$$= 2\int_0^1 \left(x^{\frac{3}{2}} - x^{\frac{1}{2}}\right)dx$$



$$= \left[2\times\frac{2}{5}x^{\frac{5}{2}} - 2\times\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$$

$$= \frac{4}{5} - \frac{4}{3} = 4\left(\frac{3-5}{15}\right) = -\frac{8}{15} = \frac{8}{15} \text{ square unit.}$$

11. Find the area of the loop of the curve $y^2(a+x) = x^2(a-x)$

Sol^n. Given curve is

$y^2(a+x) = x^2(a-x)$ .....(1)

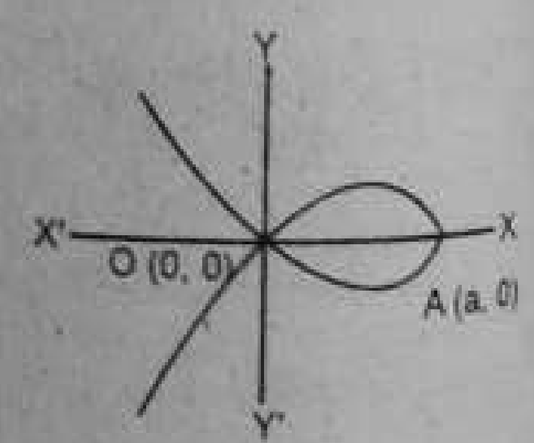
When $y = 0$ from (1) $x = 0$, and $x = a$

$\therefore$ Required Area $= 2\int_0^a y\,dx$

$$= 2\int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx$$

$$= 2\int_0^a \frac{ax - x^2}{\sqrt{(a^2 - x^2)}}\,dx$$

$$= 2\int_0^a \frac{(a^2 - x^2) + (ax - a^2)}{\sqrt{a^2 - x^2}}\,dx$$



$$= 2\left[\int_0^a \sqrt{a^2 - x^2}\,dx + \int_0^a \frac{ax - a^2}{\sqrt{a^2 - x^2}}\,dx\right]$$

$$= 2\left[\int_0^a \sqrt{a^2 - x^2}\,dx + \int_0^a \frac{\left(-\frac{a}{2}\right)(-2x) - a^2}{\sqrt{a^2 - x^2}}\,dx\right]$$

$$= 2\left[\int_0^a \sqrt{a^2 - x^2}\,dx - \frac{a}{2}\int_0^a \frac{-2x}{\sqrt{a^2 - x^2}}\,dx - a^2\int_0^a \frac{1}{\sqrt{a^2 - x^2}}\,dx\right]_0^a$$

$$= 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$- \left[2\cdot\frac{a}{2}\cdot 2\sqrt{a^2 - x^2}\right]_0^a - a^2\left[\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 2\times\frac{a^2}{2}\cdot\frac{\pi}{2} + 2a^2 - a^2\cdot\frac{\pi}{2}$$

$$= 2a^2 - \frac{a^2\pi}{2}$$

$$= a^2\left(2 - \frac{\pi}{2}\right) \text{ square unit.}$$

12. Find the area between each of the following curve and its asymptotes.

(i) $y^2(a-x) = x^3$

Sol^n. Given curve $y^2(a-x) = x^3$ .....(1)

When $y = 0$ from (1) $x = 0$ so the curve passes through the origion.

Since equation contains even power of y . so the curve symmetrical about x – axis.

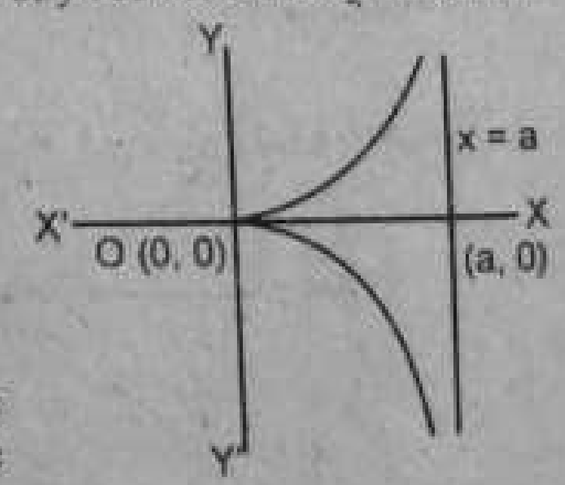
tangent at orgion is

$ay^2 = 0 \rightarrow y = 0$

i.e. x = axis equating the coefficient of y^2 with zero. Then we get

$a - x = 0 \rightarrow x = a$.

$\therefore$ x = a is an asymtote. Which is parallel to y–axis so the curve satisfies $a \le x \le a$.



$\therefore$ Required Area

$= 2\int_0^a y\,dx$

$$= 2\int_0^a \frac{x^{\frac{3}{2}}}{\sqrt{a-x}}dx$$

Put $x = a\sin^2\theta$

$dx = 2a\sin\theta\cos\theta\, d\theta$ When $x \to (0, a)$ Then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$= 2\times 2a\int_0^{\frac{\pi}{2}} \frac{\left(a\sin^2\theta\right)^{\frac{3}{2}}}{\sqrt{a-a\sin^2\theta}}.\sin\theta\cos\theta\, d\theta$$

$$= 4a\times a\int_0^{\frac{\pi}{2}} \sin^4\theta d\theta$$

$$= 4a^2\frac{\sqrt{\pi}\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{4+2}{2}\right)} = \frac{4a^2\sqrt{\pi}\,\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{2.2.1}$$

$= \frac{3\pi a^2}{4}$ square unit.

(ii) $a^2x^2 = y^2(a^2-x^2)$

Sol^n. Given curve $a^2x^2 = y^2(a^2-x^2)$ .....(1)

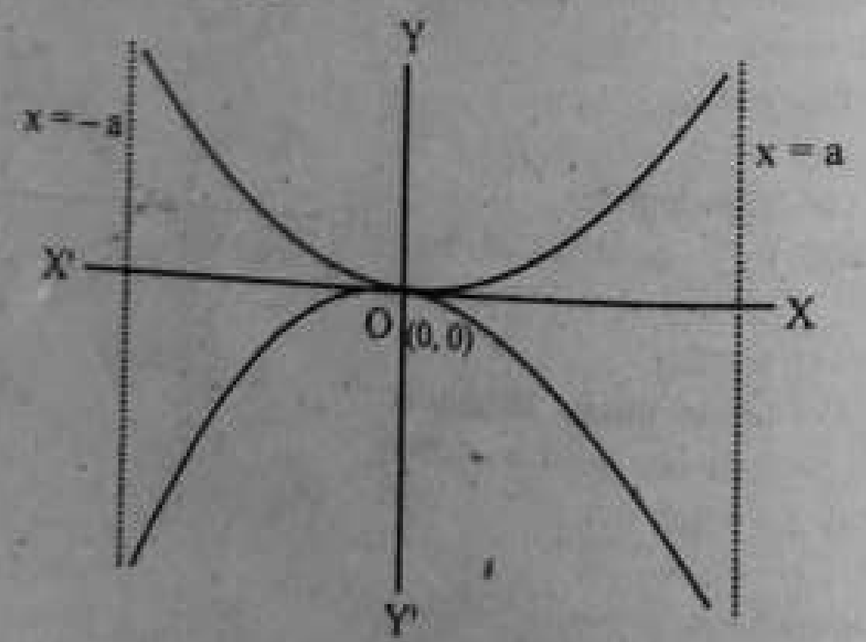
When $y = 0$ from (1) $x = 0$ so the curve passes through the origin since equation contains even power of x & y so the curve symmetrical about both axis.

Tangent at origin is

$a^2y^2 = 0 \Rightarrow y = 0$

i.e. x – axis equating the coefficient of y^2 with zero. Then we get $a^2 - x^2 = 0 \Rightarrow x = \pm a$.

$\therefore$ $x = \pm a$ is an asymptote. Which is parallel to y – axis so teh curve satisfies $-a \le x \le a$.



$\therefore$ Required Area $= 4\int_0^a ydx$

$$= 4\int_0^a \frac{ax}{\sqrt{a^2-x^2}}dx$$

$$= 4\times \frac{-a}{2}\left[2\sqrt{a^2-x^2}\right]_0^a$$

$= -2a\,(0 - 2a) = 4a^2$ square unit.

(iii) $x^2y^2 + a^2b^2 = a^2y^2$

Sol^n. Given curve, $x^2y^2 + a^2b^2 = a^2y^2$

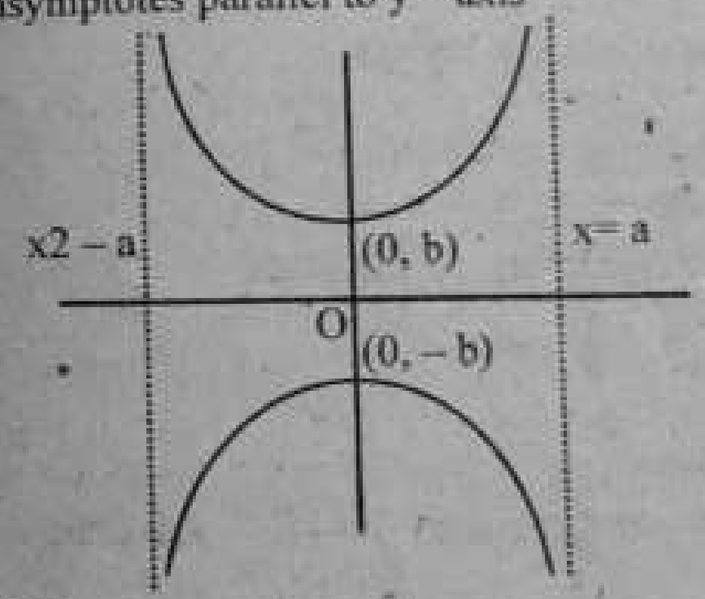
or, $y^2 = \frac{a^2b^2}{(a^2-x^2)}$ .....(1)

Since the equation contains even powers of x and y so the curve is symmetrical about both axis

When $x = 0$ then from from (i) $y = \pm b$ so the curve passes tanough (0, b) and (0, – b)

When $x = \pm a$ from (1) $y^2 = \infty$

$\therefore x = \pm a$ are asymptotes parallel to y – axis



$\therefore$ Required area

$= 4\int_0^a ydx$

$$= 4\int_0^a \frac{ab}{\sqrt{a^2-x^2}}dx$$

$$= 4ab\left[\sin^{-1}\frac{x}{a}\right]_0^a = 4ab.\frac{\pi}{2}$$

$= 2\pi ab$ square unit

(iv) $a(y^2-x^2) = x(x^2+y^2)$

Sol^n. Given curve,

$a(y^2-x^2) = x(x^2+y^2)$

$\therefore y^2(a-x) = x^2(a+x)$

$\therefore y^2 = \dfrac{x^2(a+x)}{(a-x)}$ .....(1)

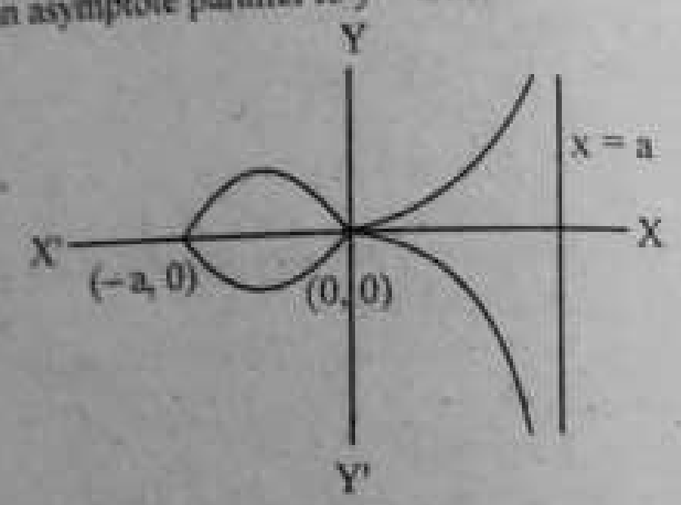
Since the equation contains only even power of y. So it is symmetrical about x – axis.

When y = 0 from (1) x = 0 and x = – a so the curve passes through (0, 0) and (– a, 0).

Equating the coefficient of y^2 with zero then we get,

$a - x = 0 \Rightarrow x = a$

$\therefore$ x = a is an asymptote parallel to y – axis.



$\therefore$ Required Area $= 2\int_0^a y\,dx$

$= 2\int_0^a \dfrac{x\sqrt{a+x}}{\sqrt{a-x}}$

Put $x = a\cos\theta \Rightarrow dx = -2a\sin\theta d\theta$

If $x = 0, \theta = \dfrac{\pi}{4}$ and

If $x = a, \theta = 0$

$\therefore$ Area $= -2\int_{\frac{\pi}{4}}^{0} a\cos 2\theta \sqrt{\dfrac{2\cos^2\theta}{2\sin^2\theta}}\; 2a\sin 2\theta d\theta$

$= 4a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta \cdot \dfrac{\cos\theta}{\sin\theta} \cdot 2\sin\theta\cos\theta \cdot d\theta$

$= 4a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta\,(1 + \cos 2\theta)\,d\theta$

$= 4a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta + 4a^2 \int_0^{\frac{\pi}{4}} \cos^2 2\theta\, d\theta$

$= 4a^2 \left[\dfrac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} + 2a^2 \int_0^{\frac{\pi}{4}} (1 + \cos 4\theta)\, d\theta$

$= 2a^2(1) + 2a^2 \left[\theta + \dfrac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{4}}$

$= 2a^2 + 2a^2\dfrac{\pi}{4}$

$= 2a^2\left(1 + \dfrac{\pi}{4}\right)$ Square unit.

(v) $x^2(x^2 + y^2) = a^2(y^2 - x^2)$

Sol^n. Given curve

$x^2(x^2 + y^2) = a^2(y^2 - x^2)$

$x^4 + x^2y^2 = a^2y^2 - a^2x^2$

$x^4 + a^2x^2 = (a^2 - x^2)\,y^2$

or, $y = x\sqrt{\dfrac{(x^2 + a^2)}{a^2 - x^2}}$ .....(1)

Since eq^n contains even power of x and y so the curve is symmetrical about both axis.

When y = 0 from (1) x = 0, and x = ± a so the curve passes through (0, 0) , (a, a) and (– a, 0)

Tangent at origin is y = ± x

The equation of its asymptotes is x = ± a

So the required Area

$= 4\int_0^a y\,dx$

$= 2\int_0^a x\sqrt{\dfrac{a^2 + x^2}{a^2 - x^2}}\,dx$

Put $x^2 = a^2\cos 2\theta$

$2x\,dx = -2a^2\sin 2\theta\, d\theta$

$x\,dx = a^2\sin 2\theta d\theta$

When $x \to (0, a)$ Then $\theta \to \left(\dfrac{\pi}{4}, 0\right)$

$= 4\int_{\frac{\pi}{4}}^{0} \sqrt{\dfrac{a^2(1 + \cos 2\theta)}{a^2(1 - \cos^2\theta)}}\,(a^2\sin 2\theta)d\theta$

$= 4a^2\int_0^{\frac{\pi}{4}} \sqrt{\dfrac{2\cos^2\theta}{2\sin^2\theta}}\,.2\sin\theta\cos\theta d\theta$

$= 8a^2\int_0^{\frac{\pi}{4}} \dfrac{\cos\theta}{\sin\theta} \times \sin\theta\cos\theta d\theta$

$= 8a^2\int_0^{\frac{\pi}{4}} \cos 2\theta d\theta$

$= 8a^2\int_0^{\frac{\pi}{4}} \dfrac{(1 + \cos 2\theta)}{2}\,d\theta$

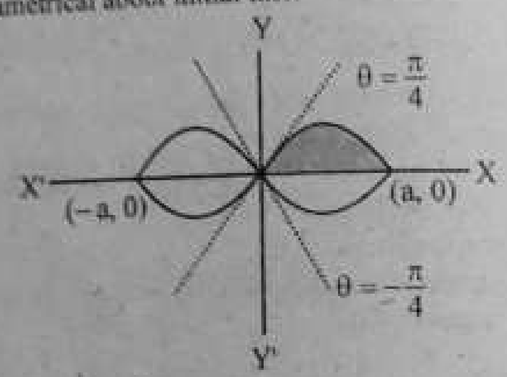
$$= 4a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= 4a^2\left(\frac{\pi}{4} + \frac{1}{2} - 0\right) = a^2(\pi + 2) \text{ square unit.}$$

13. Find the area of a loop of the curve $r^2 = a^2\cos 2\theta$.

Solⁿ. Given curve $r^2 = a^2\cos 2\theta$ .....(1)

Since the eqⁿ (1) is unchanged when θ is replaced by $-\theta$ So the curve is symmetrical about initial line.



∴ Required Area of one loop

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{4}} a^2\cos 2\theta d\theta$$

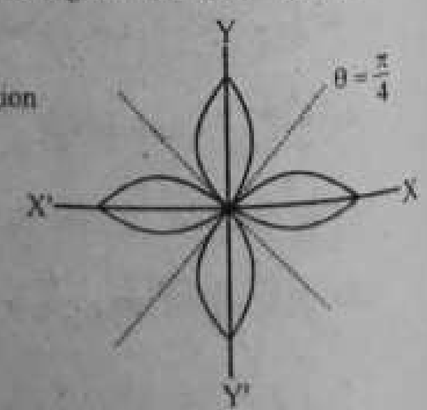
$$= a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= a^2 \cdot \frac{1}{2} = \frac{a^2}{2} \text{ square unit.}$$

14. Find the area of the entire region bounded by the curves $r = a\cos 2\theta$.

Solⁿ. Given curve $r = a\cos 2\theta$

∴ Required Area of entire region



$$= 8\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta$$

$$= \frac{8}{2}\int_0^{\frac{\pi}{4}} a^2\cos^2 2\theta d\theta$$

$$= 4a^2\int_0^{\frac{\pi}{4}} \frac{(1+\cos 4\theta)}{2} d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{4}} (1+\cos 4\theta) d\theta$$

$$= 2a^2\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{4}}$$

$$= 2a^2\left[\frac{\pi}{4} + 0 - 0 - 0\right]$$

$$= \frac{\pi a^2}{2} \text{ Square unit.}$$

15. Find the area bounded by the curve $r = a(1 - \cos\theta)$.

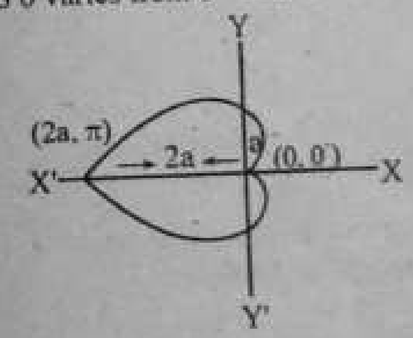
Solⁿ. Given curve is

$r = a(1 - \cos\theta)$.....(1)

Since the eqⁿ (1) is unchanged when θ is replaced by $-\theta$ So the curve is symmetrical about initial line.

Also, if $r = 0$, $\theta = 0$ if

$r = 2a$, $\Rightarrow \theta = \pi$ so θ varies from $\theta = 0$ to $\theta = \pi$



∴ Required Area $= 2\int_0^{\pi} \frac{1}{2} r^2 d\theta$

$$= \int_0^{\pi} \{a(1-\cos\theta)\}^2 d\theta$$

$$= a^2\int_0^{\pi} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= a^2\left[\int_0^{\pi} d\theta - 2\int_0^{\pi} \cos\theta + \frac{1}{2}\int_0^{\pi} (1+\cos 2\theta)\right] d\theta$$

$$= a^2\left[\theta - 2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi}$$

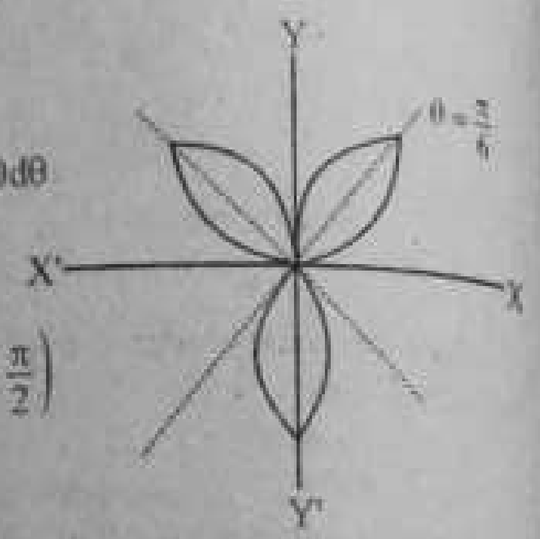
$$= a^2\left[(\pi - 0) - 2(0 - 0) + \frac{1}{2}(\pi + 0 - 0 - 0\right]$$

$$= a^2\left[\pi + \frac{\pi}{2}\right] = \frac{3\pi a^2}{2} \text{ square unit}$$

16. **Find the area of a loop $r = a \sin 3\theta$.** **[BE. 057/060]**

Solⁿ. Given loop is $r = a \sin 3\theta$

The curve consists of three equal loops. One loops is obtained when θ increases from 0 to 60.



$\therefore$ Required Area $= 2\int_0^{\frac{\pi}{6}} \frac{1}{2} r^2 d\theta$

$$= \int_0^{\frac{\pi}{6}} a^2 \sin^2 3\theta d\theta$$

Put $3\theta = t$ then $d\theta = \frac{1}{3}dt$

When $\theta \to \left(0, \frac{\pi}{6}\right)$ then $t \to \left(0, \frac{\pi}{2}\right)$

$$= \int_0^{\frac{\pi}{2}} a^2 \sin^2 t \frac{dt}{3}$$

$$= \frac{a^2}{3}\int_0^{\frac{\pi}{2}} \sin^2 dt = \frac{a^2}{3}\frac{\sqrt{\pi}\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{2+2}{2}\right)} = \frac{a^2}{3}\frac{\sqrt{\pi}\Gamma\left(\frac{3}{2}\right)}{2\Gamma(2)}$$

$$= \frac{a^2}{3}\frac{\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{2} = \frac{a^2\pi}{12} \text{ square unit.}$$

17. **Find the area common to the circle $r = a$ and the cardioid $r = a(1 + \cos\theta)$ [B.E. 2062]**

Solⁿ. Given circle $r = a$ and

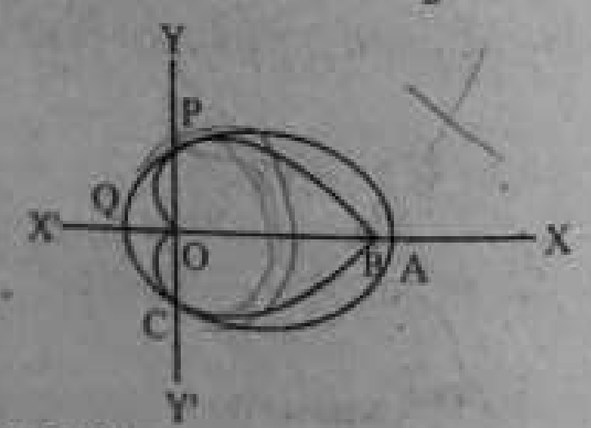
cardioids $r = a(1 + \cos\theta)$

Solving these $a = a(1 + \cos\theta)$

$\cos\theta = 0$

or, $\theta = \frac{\pi}{2}$

$\therefore$ the two curves intersects at p where $\theta = \frac{\pi}{2}$



Required area OCBPQO

$= 2$(Area OBP + Area POQ)

$$= 2\left[\frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} r^2 d\theta\right]$$

$$= 2 \times \frac{1}{2}\int_0^{\frac{\pi}{2}} a^2 d\theta + 2 \times \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} a^2(1+\cos\theta)^2 d\theta$$

$$= a^2 \cdot \frac{\pi}{2} + a^2\int_{\frac{\pi}{2}}^{\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2\pi}{2} + a^2\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]$$

$$= \frac{a^2\pi}{2} + a^2\left[\left(\pi - \frac{\pi}{2}\right) + 2(0-1) + \frac{1}{2}\left(\pi - \frac{\pi}{2}\right) + \frac{1}{2}(0-0)\right]$$

$$= \frac{a^2\pi}{2} + a^2\left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right)$$

$$= \frac{a^2\pi}{2} + a^2\left(\frac{3\pi}{4} - 2\right)$$

$$= \frac{5a^2\pi}{4} - 2a^2$$

$$= a^2\left(\frac{5\pi}{4} - 2\right) \text{ square unit.}$$

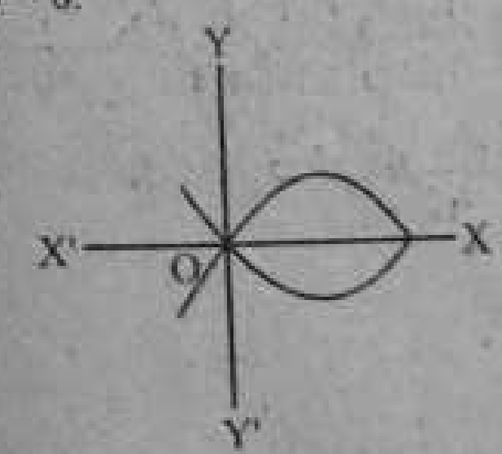
18. **Find the area enclosed by the following curves.**

(i) $x = a(1 - t^2)$, $y = at(1 - t^2)$, $-1 \le t \le 1$.

Solⁿ. Given curves

$x = a(1 - t^2)$ and $y = at(1 - t^2)$, $-1 \le t \le 1$.

When $t = -1$, $x = y = 0$.



Required Area $= \frac{1}{2}\int_{-1}^{1}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$

$$= \frac{1}{2}\int_{-1}^{1}\left[at(1-t^2)a(1-3t^2) - at(1-t^2)a(-2t)\right]dt$$

$$= \frac{1}{2}a^2\int_{-1}^{1}\left(1 - 2t + t^4\right)dt$$

$$= \frac{a^2}{2}\left[t - \frac{2t^3}{3} + \frac{t^5}{5}\right]_{-1}^{1}$$

$$= \frac{1}{2}a^2\left(1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= \frac{1}{2}a^2 \times \frac{16}{15} = \frac{8a^2}{15} \text{ square unit}$$

(ii) $x = \frac{1-t^2}{1+t^2}, y = t\left(\frac{1-t^2}{1+t^2}\right); -1 \le t \le 1$.

Sol. Given curves

$$x = \frac{1-t^2}{1+t^2}, y = t\left(\frac{1-t^2}{1+t^2}\right); -1 \le t \le 1$$

When $t = \pm 1$; $x = 0$ and $y = 0$

Also, y vanishes for $t = 0$

Hence the loops formed between $t = 0$ to $t = 1$

$$\therefore \text{Required Area (A)} = \frac{1}{2}\int_{-1}^{1}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt \quad .....(1)$$

$$\text{Now, } \frac{dx}{dt} = \frac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2}$$

$$= \frac{-2t(1+t^2+1-t^2)}{(1+t^2)^2} = \frac{4t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{(1+3t^2)(1+t^2) - (t-3t^2)(2t)}{(1+t^2)^2} = \frac{1-4t^2-t^4}{(1+t^2)^2}$$

$$\therefore x\frac{dy}{dt} - y\frac{dx}{dt} = \frac{(1-t^2)}{(1+t^2)}\cdot\frac{(1-4t^2-t^4)}{(1+t^2)^2} - \frac{(t-t^3)(4t)}{(1+t^2)(1+t^2)^2}$$

$$= \frac{1-4t^2-t^4-t^2+4t^4+t^6+4t^2-4t^4}{(t^2+1)^3}$$

$$= \frac{t^6-t^4-t^2+1}{(t^2+1)^3}$$

$$= \frac{(t^2-1)^2(t^2+1)}{(t^2+1)^3} = \frac{(t^2-1)^2}{(t^2+1)^2}$$

$\therefore$ (i) becomes

$$A = \frac{1}{2}\int_{-1}^{1}\left(\frac{t^2-1}{t^2+1}\right)^2 dt$$

$$= \frac{1}{2} \times 2\int_{0}^{1}\left(\frac{t^2-1}{t^2+1}\right)^2 dt$$

Put $t = \tan\theta$

$dt = \sec^2\theta\, d\theta$.

When $t \to (0, 1) \Rightarrow \theta \to \left(0, \frac{\pi}{4}\right)$

$$\therefore A = \int_0^{\frac{\pi}{4}} \frac{\left(\tan^2\theta - 1\right)^2}{\sec^4\theta}\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\left(\sec^2\theta - 2\right)^2}{\sec^2\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\left(\sec^4\theta - 4\sec^2\theta + 4\right)}{\sec^2\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^2\theta d\theta - 4\int_0^{\frac{\pi}{4}} d\theta + 4\int_0^{\frac{\pi}{4}} \cos^2\theta d\theta$$

$$= \left[\tan\theta\right]_0^{\frac{\pi}{4}} - 4[\theta]_0^{\frac{\pi}{4}} + 4\int_0^{\frac{\pi}{4}}\left(\frac{1+\cos 2\theta}{2}\right)$$

$$= 1 - \pi + 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= 1 - \pi + 2\left(\frac{\pi}{4} + \frac{1}{2}\right)$$

$$= 1 - \pi + \frac{\pi}{2} + 1 = 2 - \frac{\pi}{2} \text{ square unit.}$$

$$A = \int_{t_1}^{t_2} \frac{1}{2}\int x\frac{dy}{dt} -$$

Rectification

12

The process of finding the Arc- length of the plane curves whose equations are given in Cartesian, parametric or polar form is called rectification.

Some Important formula

1. If the plane curves is $y = f(x)$ then the derivative of Arc -length

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$ so the Arc-length is defined by

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

Where the Arc-length s is measured from $x = a$ to $x = b$ similarly, if the plane curves is $x = f(y)$ then the derivative of Arc-length

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

if follows that the Arc-length s is defined by

$$s = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

where s is measured from $y = c$ to $y = d$.

2. If the plane curve is parametric $x = f(t)$ and $y = g(t)$ then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

If follows that Arc-length is defined by

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt.$$

Where s measure from $t = t_1$ to $t = t_2$.

3. If the equation of the curve is $r = f(\theta)$ then the derivative of Arc length is

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

Arc- lengths is measured from $\theta = \alpha$ to $\theta = \beta$ defined by

$$s = \int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

4. If the equation of the curve is $\theta = f(r)$ then the derivative of Arc-length is

$$\frac{ds}{dr} = \sqrt{1+\left(\frac{rd\theta}{dr}\right)^2}$$

Arc-lengths is measured from $r = r_1$ to $r = r_2$ is defined by

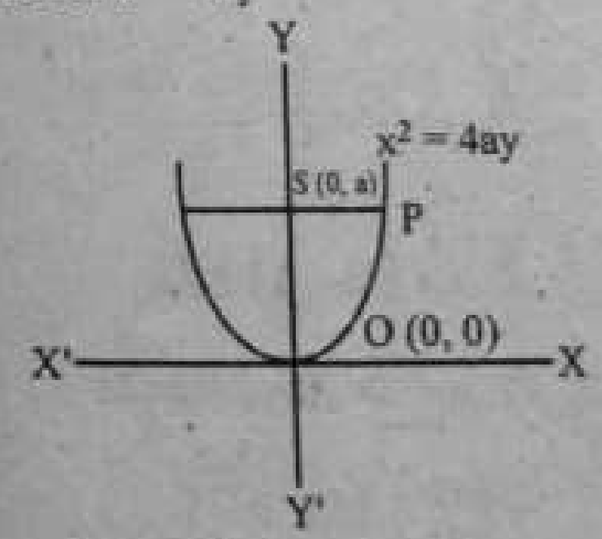
$$s = \int_{r_1}^{r_2} \sqrt{1+\left(\frac{rd\theta}{dr}\right)^2}\, dr$$

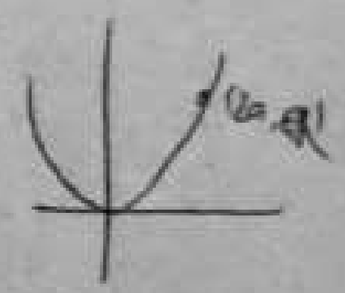
Exercise - 16

1. Find the length of the Arc of the parabola the $x^2 = 4ay$ from the vertex to an extremity of the latus rectum.

Sol^n. Given parabola $x^2 = 4ay$ ..... (i)

Let 0(0, 0) be the vertex s (0, a) be the focus and PQ be the latus rectum of the parabola $x^2 = 4ay$





Then Arc OP = $\int_0^a \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

diff(1) w.r.t. y

$2x\frac{dx}{dy} = 4a$

$\frac{dx}{dy} = \frac{2a}{x}$

$$\therefore \left(\frac{dx}{dy}\right)^2 = \frac{4a^2}{x^2} = \frac{4a^2}{4ay} = \frac{a}{y}$$

$$\therefore \text{Arc OP} = \int_0^a \sqrt{1+\frac{a}{y}}\,dy = \int_0^a \sqrt{\frac{y+a}{y}}\,dy$$

$$= \int_0^a \frac{y+a}{\sqrt{y(y+a)}}\,dy = \frac{1}{2}\int_0^a \frac{2y+a+a}{\sqrt{y^2+ay}}\,dy$$

$$= \frac{1}{2}\left[2(y^2+ay)^{\frac{1}{2}}\right]_0^a + \frac{a}{2}\int_0^a \frac{dy}{\sqrt{y^2+2y\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}}}$$

$$= (a^2+a^2)^{\frac{1}{2}} + \frac{a}{2}\int_0^a \frac{1}{\sqrt{\left(y+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2}}\,dy$$

$$= \left(2a^2\right)^{\frac{1}{2}} + \frac{a}{2}\left[\log\left(y+\frac{a}{2}\right)+\sqrt{\left(y+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2}\right]_0^a$$

$$= \sqrt{2}.a + \frac{a}{2}\left[\log\left(y+\frac{a}{2}\right)+\sqrt{y^2+ay}\right]_0^a$$

$$= \sqrt{2}.a + \frac{a}{2}\left[\log\left(\frac{3a}{2}+\sqrt{2a^2}\right)-\log\left(\frac{a}{2}\right)\right]$$

$$= \sqrt{2}.a + \frac{a}{2}\log\left(\frac{\frac{3a}{2}+\sqrt{2a}}{\frac{a}{2}}\right)$$

$$= \sqrt{2}.a + \frac{a}{2}\log\left(3+2\sqrt{2}\right)$$

$$= \sqrt{2}.a + \frac{a}{2}\log(1+2+2\sqrt{2})$$

$$= \sqrt{2}.a + \frac{a}{2}\log(1+\sqrt{2})$$

$$\sqrt{2}.a + a\log(1+\sqrt{2})$$

$$= a\left[\sqrt{2}+\log(1+\sqrt{2})\right] \text{ unit.}$$

2. Find the arc length of the following

(i) $x^2+y^2=a^2$ circle.

Sol^n. Given eq^n of the circle

$x^2+y^2=a^2$

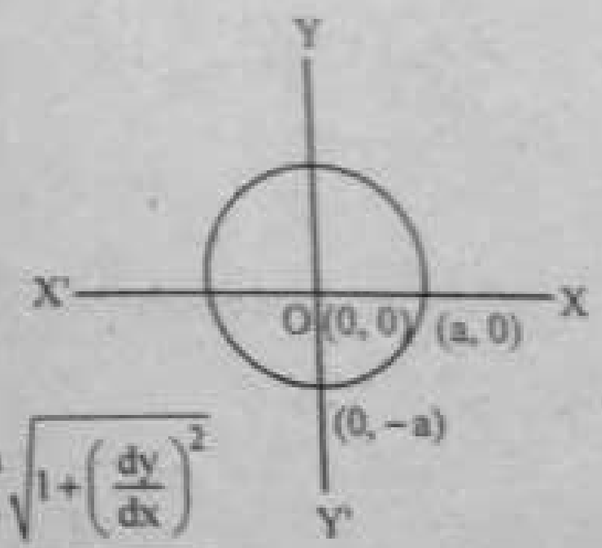
$y^2=a^2-x^2$.....(1)

diff (1) w.r.t 'x'

$2y\frac{dy}{dx}=-2x$

$\frac{dy}{dx}=\frac{-x}{y}$

$\left(\frac{dy}{dx}\right)^2=\frac{-x^2}{y^2}=\frac{x^2}{a^2-x^2}$



$\therefore$ Required Arc length $= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$= 4\int_0^a \sqrt{1+\frac{x^2}{a^2-x^2}}\,dx = 4\int_0^a \sqrt{\frac{a^2-x^2+x^2}{a^2-x^2}}$$

$$= 4\int_0^a \sqrt{\frac{a^2}{a^2-x^2}}\,dx = 4a\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$= 4a\left[\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4a\left[\frac{\pi}{2}-0\right]$$

$$= 2\pi a \text{ unit.}$$

(ii) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ (Astroid)

Sol^n. Given Astroid

$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

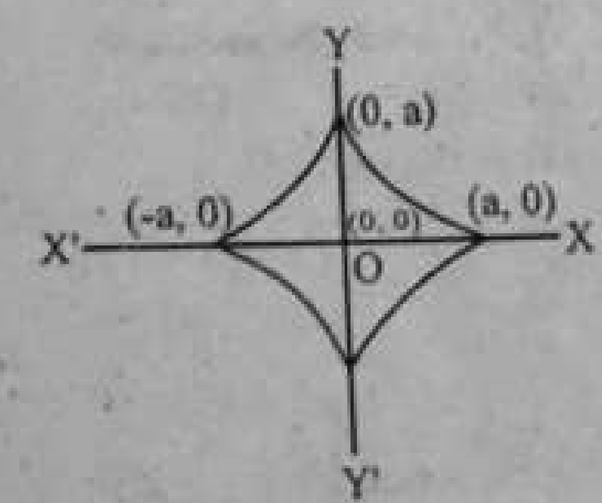
$y^{\frac{2}{3}}=a^{\frac{2}{3}}-x^{\frac{2}{3}}$ .....(1)

diff. (1) w.r.t x

$\frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx}=-\frac{2}{3}x^{\frac{-1}{3}}$

$\therefore \frac{dy}{dx}=\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}$



$$= \frac{\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)}{x^{\frac{2}{3}}}$$

$\therefore$ Required Arc length $= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$= 4\int_0^a \sqrt{1+\frac{\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)}{x^{\frac{2}{3}}}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{x^{\frac{2}{3}} + a^{\frac{2}{3}} - x^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\,dx$$

$= 4$

$$= 4\int_0^a \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}\,dx = 4a^{\frac{1}{3}}\int_0^a x^{\frac{-1}{3}}\,dx$$

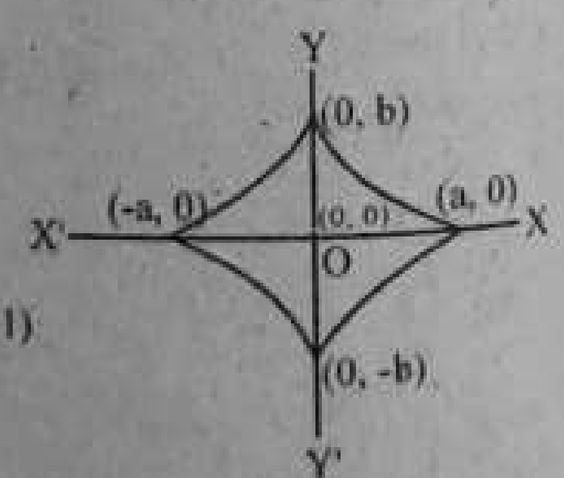
$$= 4a^{\frac{1}{3}}\left[\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right]_0^a = 6a^{\frac{1}{3}}.a^{\frac{2}{3}}$$

$= 6a$ unit.

(iii) $\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1$ **(Hypocycloid)**

Sol. Given Hypocycloid.

$$\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1$$



or, $\frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1 - \frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}}$ .....(1)

diff. (1) w.r.t. x

$$\frac{2}{3b^{\frac{2}{3}}} y^{\frac{-1}{3}} \frac{dy}{dx} = -\frac{2}{3a^{\frac{2}{3}}} x^{\frac{-1}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{b^{\frac{2}{3}} y^{\frac{1}{3}}}{a^{\frac{4}{3}} x^{\frac{1}{3}}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{b^{\frac{4}{3}} y^{\frac{2}{3}}}{a^{\frac{4}{3}} x^{\frac{2}{3}}}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{b^{\frac{4}{3}} \frac{b^{\frac{2}{3}}\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)}{a^{\frac{2}{3}}}}{a^{\frac{4}{3}} x^{\frac{2}{3}}} \text{ (from 1)}$$

$$= \frac{b^2\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)}{a^2 x^{\frac{2}{3}}}$$

$\therefore$ Required Arc length $= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= 4\int_0^a \sqrt{1+\frac{b^2\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)}{a^2 x^{\frac{2}{3}}}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{a^2 x^{\frac{2}{3}} + b^2 a^{\frac{2}{3}} - b^2 x^{\frac{2}{3}}}{a^2 x^{\frac{2}{3}}}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{\left(a^2 - b^2\right) x^{\frac{2}{3}} + b^2 a^{\frac{2}{3}}}{a^2 x^{\frac{2}{3}}}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{\left(a^2 - b^2\right)\left(\frac{x}{a}\right)^{\frac{2}{3}} + b^2}{a^2 \left(\frac{x}{a}\right)^{\frac{2}{3}}}}$$

Put $(a^2 - b^2)\left(\frac{x}{a}\right)^{\frac{2}{3}} + b^2 = t$

$$\frac{2}{3}(a^2 - b^2)\left(\frac{x}{a}\right)^{-\frac{1}{3}} \cdot \frac{1}{a}\,dx = dt$$

$$\Rightarrow \frac{1}{a\left(\frac{x}{a}\right)^{\frac{1}{3}}}dx = \frac{3}{2}\frac{1}{\left(a^2-b^2\right)}dt$$

when $x \to (0, a)$ Then $t \to (b^2, a^2)$

$$= 4\int_{b^2}^{a^2}\frac{3}{2}\frac{\sqrt{t}}{a^2-b^2}dt$$

$$= \frac{6}{a^2-b^2}\int_{b^2}^{a^2}t^{\frac{1}{2}}dt = \frac{6}{a^2-b^2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_{b^2}^{a^2}$$

$$= \frac{6}{a^2-b^2}(a^3-b^3) = \frac{4(a^2+ab+b^2)}{(a+b)} \text{ unit.}$$

3. If s be the length of an Arc of the curve $3ay^2 = x(x-a)^2$ measured from the origin to the point (x, y) Show that $3s^2 = 4x^2 + 3y^2$. Show also that the entire length of the loop is $\frac{4}{3}\sqrt{3}$

[B.E. 2061]

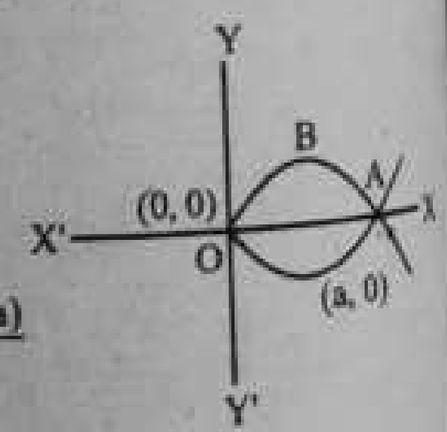
Sol. Given curve $3ay^2 = x(x-a)^2$ .....(1)

since the curve has only even power of y so the curve is symmetrical about x – axis.

When y = 0 from (1) x = 0, x = a so the curve passes through (0, 0) and (a, 0) so the loop is formed between x = 0 and x = a

Also for x < 0, y becomes imaginary, so the curve can not extend the left t of x = 0

Now, diff. (1) w.r.t x



$$6ay\frac{dy}{dx} = (x-a)^2 + 2x(x-a)$$

$$\frac{dy}{dx} = \frac{(x-a)^2+2x(x-a)}{6ay}$$

$$= \frac{(x-a)(x-a+2x)}{6ay} = \frac{(x-a)(3x-a)}{6ay}$$

we know,

$$\therefore S = \int_0^x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \int_0^x\sqrt{1+\frac{(x-a)^2(3x-a)^2}{36a^2y^2}}$$

$$= \int_0^x\sqrt{1+\frac{(x-a)^2(3x-a)^2}{12ax(x-a)^2}}$$

$$= \int_0^x\sqrt{1+\frac{(3x-a)^2}{12ax}} = \int_0^x\sqrt{\frac{12ax+9x^2-6ax+a^2}{12ax}}$$

$$= \int_0^x\frac{3x+a}{2\sqrt{3ax}} \text{ .....(2)}$$

$$= \frac{1}{2\sqrt{3a}}\int_0^x\left(3\sqrt{x}+\frac{a}{\sqrt{x}}\right)dx$$

$$= \frac{1}{2\sqrt{3a}}\left[3\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{ax^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^x$$

$$\therefore s = \frac{1}{2\sqrt{3a}}\left[2x^{\frac{3}{2}}+2ax^{\frac{1}{2}}\right]$$

$$s = \frac{\sqrt{x}}{\sqrt{3a}}(x+a) \text{ squaring on both sides.}$$

$$\therefore s^2 = \frac{x}{3a}(x+a)^2 \Rightarrow 3as^2 = x(x+a)^2 \text{ proved.}$$

Also for the entire length of the loop

$$= 2\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

$$= 2\int_0^a\frac{3x+a}{2\sqrt{3ax}}dx \text{ from (2)}$$

$$= 2.\frac{1}{2\sqrt{3a}}\int_0^a\left[3x^{\frac{1}{2}}+\frac{a}{x^{\frac{1}{2}}}\right]dx$$

$$= \frac{1}{\sqrt{3a}}\left[\frac{3x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{ax^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^a$$

$$= \frac{1}{\sqrt{3a}}\left[2x^{\frac{3}{2}}+2ax^{\frac{1}{2}}\right]_0^a$$

$$= \frac{1}{\sqrt{3a}}\left[2a^{\frac{3}{2}}+2a.\,a^{\frac{1}{2}}\right]$$

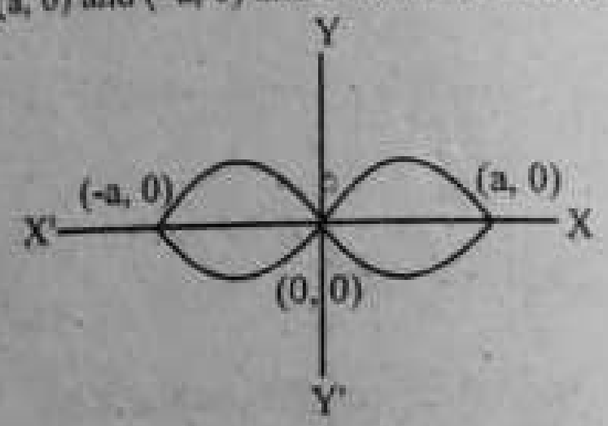
$$= \frac{1}{\sqrt{3a}} 4a^{\frac{3}{2}}$$

$$= \frac{4}{\sqrt{3}} a = \frac{4}{3}\sqrt{3a} \text{ unit}$$

4. Show that the Arc length of the curve $x^2(a^2 - x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$

Solⁿ. Given curve $x^2(a^2 - x^2) = 8a^2y^2$ .....(1)

When $y = 0$ from (1) $x = 0$, and $x = \pm a$ so the curve cut s the x-axis at (0, 0), (a, 0) and (–a, 0) and I t consists of two loops



$\therefore$ Required Arc length $= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Diff. (I) w.r.t x,

$$2a^2x - 4x^3 = 16a^2y\frac{dy}{dx}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{4x^2(a^2-2x^2)^2}{256a^4x^2(a^2-x^2)} = \frac{\left(a^2-2x^2\right)^2}{8a^2(a^2-x^2)}$$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{a^4-4a^2x^2+4x^4}{8a^4-8a^2x^2}$$

$$= \frac{9a^4-12a^2x^2+4x^4}{8a^2(a^2-x^2)} = \frac{\left(3a^2-2x^2\right)^2}{8a^2(a^2-x^2)}$$

$$\therefore S = 4\int_0^a \frac{3a^2-2x^2}{2\sqrt{2}\,.\,a\sqrt{a^2-x^2}}dx$$

$$= 4\int_0^a \frac{2\left(a^2-x^2\right)+a^2}{2\sqrt{2}\,a\sqrt{a^2-x^2}}dx$$

$$= \frac{2\sqrt{2}}{a}\int_0^a \sqrt{a^2-x^2}dx + \sqrt{2}.a\int_0^a \frac{1}{\sqrt{a^2-x^2}}dx$$

$$= \frac{2\sqrt{2}}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a + \sqrt{2}.a\left[\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{2\sqrt{2}}{a}.\frac{a^2}{2}.\frac{\pi}{2}+\sqrt{2}.a\frac{\pi}{2}$$

$$= \frac{\pi a\sqrt{2}}{2}+\frac{\pi a\sqrt{2}}{2}$$

$= \pi a\sqrt{2}$ unit. Proved.

5. Find the Arc length of the parabola $y^2 = 4x$ cut of by the line $y = 2x$

Solⁿ. Given eqⁿ of the parabola $y^2 = 4x$ and line $y = 2x$.

Solving these

$4x^2 = 4x \Rightarrow 4x(x-1) = 0$

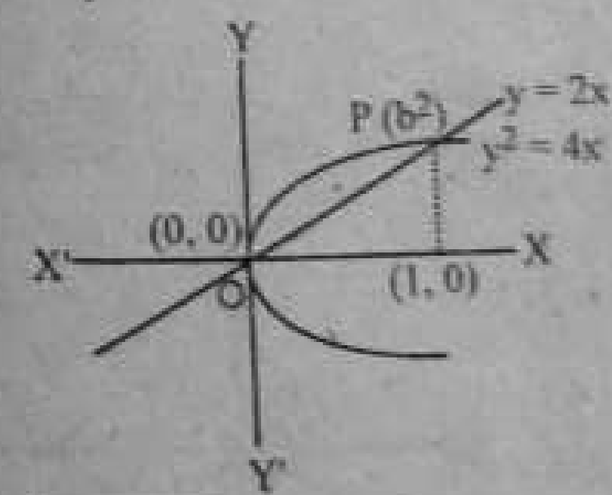
$\Rightarrow x = 0, x = 1$

$\therefore$ The parabola and line intersect at 0(0, 0) and P (1, 2).

$\therefore$ Arc length OP $= \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

Diff. $y^2 = 4x$ w.r.t 'x'

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$



$$\therefore \text{Arc op} = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^1 \sqrt{1+\frac{4}{y^2}}dx$$

$$= \int_0^1 \sqrt{1+\frac{4}{4x}}dx = \int_0^1 \sqrt{\frac{x+1}{x}}dx$$

Put $x = t^2$ $\therefore$ $dx = 2tdt$

When $x \to (0, 1)$ Then $t \to (0, 1)$

$$= \int_0^1 \sqrt{\frac{t^2+1}{t^2}}\, 2t\,dt = 2\int_0^1 \sqrt{t^2+1}\,dt$$

$$= 2\left[\frac{t\sqrt{t^2+1}}{2} + \frac{1}{2}\log t + \sqrt{t^2+1}\right]_0^1$$

$$= \sqrt{2} + \log(1+\sqrt{2}) - \log(1)$$

$$= \sqrt{2} + \log(1+\sqrt{2}) \text{ unit.}$$

$1 - t^2$

6. Prove that the loop of the curve $x = t^2, y = t - \frac{t^3}{3}$ is of length $4\sqrt{3}$.

$2t$

$4t^2 + (1-t^2)^2$

$= (t^2+1)^2$

$2\left[\frac{t^3}{3} + t\right]$

Sol^n. Given curve $x = t^2$

$$y = t - \frac{t^3}{3}$$

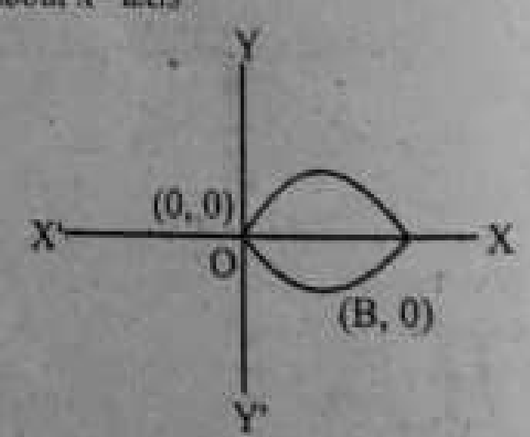
$$\therefore y = t\left(1 - \frac{t^2}{3}\right)$$

$$\Rightarrow y^2 = t^2\left(1 - \frac{t^2}{3}\right)^2$$

$$y^2 = x\left(1 - \frac{x}{3}\right)^2$$

or, $9y^2 = x(3-x)^2$ .....(1)

When $y = 0$ from (1) $x = 0$, $x = 3$ so the curve cuts the x – axis at (0, 0) and (3, 0) since the curve only even power of y so the curve is symmetrical about x– axis



The loop is formed between $x = 0$ and $x = 3$

$\therefore x = t^2$

i.e. loop is formed between $t = 0$ and $t = \sqrt{3}$

$$\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (2t)^2 + (1-t^2)^2$$

$$= 4t^2 + 1 - 2t^2 + t^4$$

$$= t^4 + 2t^2 + 1$$

$$= (t^2+1)^2$$

$$\therefore \text{Required Arc length} = 2\int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= 2\int_0^{\sqrt{3}} (t^2+1)\,dt$$

$$= 2\left[\frac{t^3}{3} + t\right]_0^{\sqrt{3}}$$

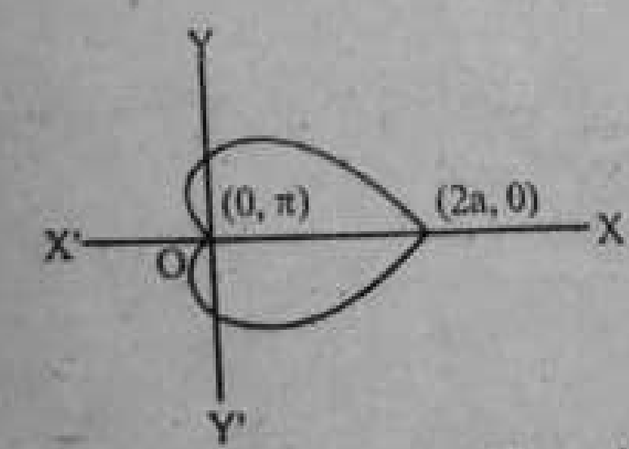
$$= 2\left(\frac{3^{\frac{3}{2}}}{3} + \sqrt{3}\right) = 2\left(\sqrt{3} + \sqrt{3}\right) = 4\sqrt{3} \text{ unit. Proved.}$$

7. Find the perimeter of the cardioid $r = a(1 + \cos\theta)$ and show that the Arc of upper half is bisected by $\theta = \frac{\pi}{3}$

Sol^n. Given cardioid $r = a(1 + \cos\theta)$ .....(1)

diff (1). w.r.t. θ

$$\frac{dr}{d\theta} = -a\sin\theta$$



The curve is symmetrical about the initial line and the limits for the upper half are o and π.

$$\therefore \text{Length of upper half} = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

$$= \int_0^{\pi} a\sqrt{2(1+\cos\theta)}\,d\theta$$

$$= \int_0^{\pi} 2a\cos\frac{\theta}{2}\,d\theta = 4a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 4a$$

$\therefore$ The entire length $= 2 \times 4a = 8a$ unit

If the take the limits of integration o and $\frac{\pi}{3}$ the part of the upper half.

intercepted between $\therefore \theta = 0$ and $\theta = \frac{\pi}{3}$ is found to be

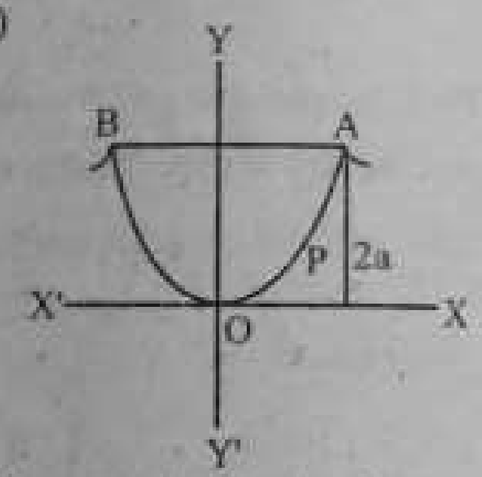
$$4a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 4a \times \frac{1}{2} = 2a$$

which is half of 4a.

$\therefore$ The upper half of the circle is bisected by $\theta = \frac{\pi}{3}$

8. In the cycloid $x = a(\theta + \sin\theta)$ $y = a(1 - \cos\theta)$ show that $s^2 = 8ay$, s being measuyred from the vertex to any point. [B. E. 2069]

Sol^n. Given cycloid $x = a(\theta + \sin\theta)$



$y = a(1 - \cos\theta)$

diff. w.r.t θ

$\frac{dx}{d\theta} = a(1 + \cos\theta)$

and $\frac{dy}{d\theta} = a\sin\theta$

At the vertex 0, $\theta = 0$

$\therefore$ length of the are from $\theta = 0$ to any point θ is given by

$$s = \int_0^{\theta}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{\theta}\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

$$= a\int_0^{\theta}\sqrt{2(1+\cos\theta)}\,d\theta$$

$$= a\int_0^{\theta} 2\cos\frac{\theta}{2}\,d\theta = 4a\sin\frac{\theta}{2}$$

$$\therefore s^2 = 16a^2\sin^2\frac{\theta}{2} = 8aa(1 - \cos\theta) = 8ay$$

At A, $y = 2a$, and hence $\theta = \pi$
$\therefore$ The whole length of the cycloid

$s = 4a\sin\frac{\theta}{2}$

$s^2 = 16a^2\sin^2\frac{\theta}{2}$

$\cancel{16a^2(1-\cos^2\frac{\theta}{2})}$

$s^2 = 8a^2\, 2\sin^2\frac{\theta}{2}$

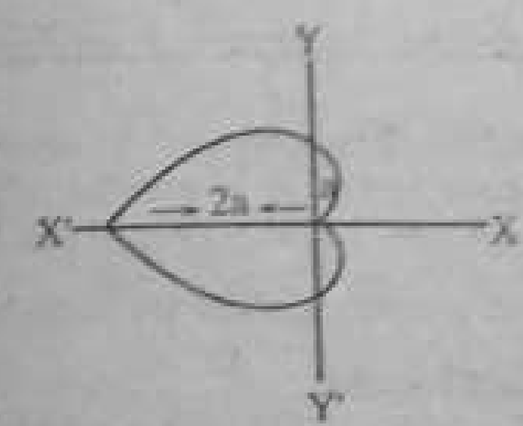
$= 8a^2(1-\cos\theta)$

$= 8ay$

$$2\,\text{Arc OA} = 2.4a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 8a \text{ unit}$$

9. Find the perimeter of the cardioid $r = a(1 - \cos\theta)$ and show that the arc of the upper half is bisected by $\theta = \frac{2\pi}{3}$.

Sol^n. Given curve $r = a(1 - \cos\theta)$ .....(1)



diff(1) w.r.t θ

$\frac{dr}{d\theta} = a\sin\theta$

$$\therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1 - \cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]$$

$$= a^2(2 - 2\cos\theta) = 2a^2(1 - \cos\theta)$$

$$= 2a^2.\,2\sin^2\frac{\theta}{2} = 4a^2\sin^2\frac{\theta}{2}$$

$\therefore$ perimeter of the cardioid $= 2\int_0^{\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$$= 2\int_0^{\pi} 2a\sin\frac{\theta}{2}\,d\theta = 4a\left[-2\cos\frac{\theta}{2}\right]_0^{\pi} = -8a\left[\cos\frac{\theta}{2}\right]_0^{\pi}$$

$= -8a(0 - 1) = 8a.$

$\therefore$ Length of the Arc of upper half is 4a also the length of the are from $\theta = 0$ to $\theta = \frac{2\pi}{3}$ is Arc

$$S = \int_0^{\frac{2\pi}{3}} 2a\sin\frac{\theta}{2}\,d\theta = 2a\left[-2\cos\frac{\theta}{2}\right]_0^{\frac{2\pi}{3}}$$

$$= -4a\left(\cos\frac{\pi}{3} - \cos 0\right) = -4a\left(\frac{1}{2} - 1\right) = -4a\left(\frac{-1}{2}\right) = 2a$$

$\therefore$ The Arc of the upper half is bisected by $\theta = \frac{2\pi}{3}$ unit

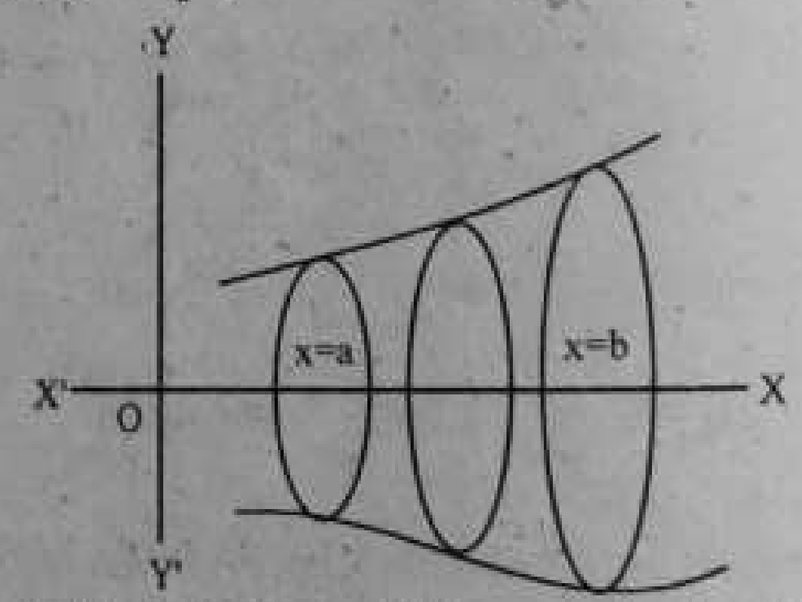
Volume and Surface

13

Volume of solid of Revolution:- If the equation of curve $y = f(x)$ is continuous in the interval (a, b) then the volume of the solid formed when the area bounded by the curve, the x–axis and the ordinates at $x = a$ and $x = b$ is revolved about the x – axis is $\int_a^b \pi y^2 dx$

Some standard Formula

1. If the area of a curve $y = f(x)$ from $x = a$ to $x = b$ is rotated through one revolution about the x–axis then, it generated a solid with plane circular ends as shown in the figure.
Thus the required volume is
defined by $v = \int_a^b \pi y^2 dx$



2. The volume of solid generated by revolution about y–axis of the area bounded by the curve $x = f(y)$, the y– axis and abscissa $y = c$ y $= d$ is defined by
$V = \int_c^d \pi x^2 dy$

3. If the curve is $r = f(\theta)$ and it revolves about initial line, the volume generated is defined by $v = \int_a^b \pi y^2 dx = \pi \int_{\alpha}^{\beta} (r\sin\theta)^2 d(r\cos\pi\theta)$

4. If the area bounded buy the curve $r = f(\theta), \theta = \theta_1, \theta = \theta_2$ is rotated about the initial line then the volume is generated is defined by

$v = \frac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3 \sin\theta d\theta$

5. If the curve $y = f(x)$ is rotated about x – axis from $x = a$ to $x = b$ then the whole surface is defined by
$S = \int_a^b 2\pi y.ds = \int_a^b 2\pi y \frac{ds}{dx} dx$
$= \int_a^b 2\pi f(x) \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$

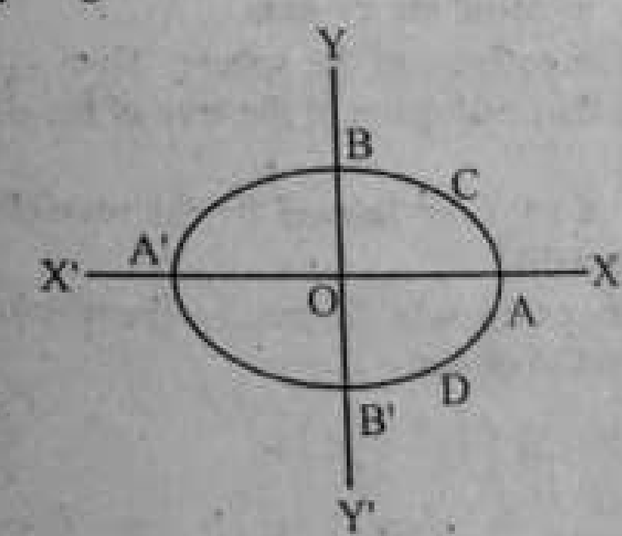
6. If the curve $x = f(y)$ is rotated about y – axis from $y = c$ to $y = d$ then the whole surface is defined by
$S = \int_c^d 2\pi x.ds = \int_c^d 2\pi x \frac{ds}{dy} dy = \int_c^d 2\pi f(y) \sqrt{1+\left(\frac{dx}{dy}\right)^2} dy$

7. If the curve is polar form $r = f(\theta)$ then the surface is defined by
$S = \int_a^b 2\pi y.ds = \int_a^b 2\pi r\sin\theta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} d\theta$

Exercise - 17

1. **Find the volume of ellipsoid formed by the revolution of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the x–axis.**

Solⁿ. Given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



The volume of the solid formed by the revolution of the ellipse about the x – axis.
= Volume formed by the revolution of two portion ABA'
= 2 (volume formed by the revolution of the portion ACB)

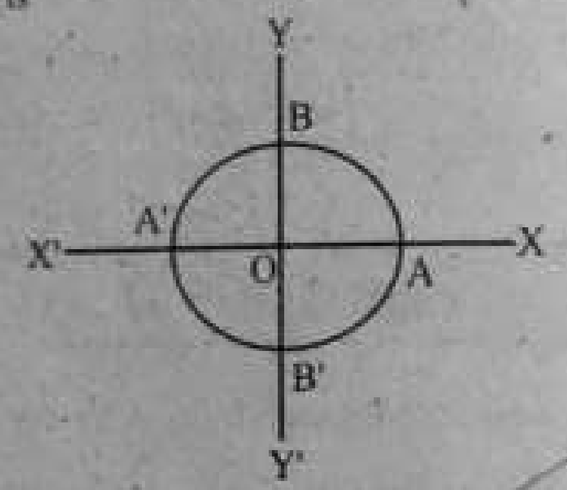
$$=2\int_0^a \pi y^2 dx = 2\pi\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$=2\pi b^2\left[x-\frac{x^3}{3a^2}\right]_0^a$$

$$=2\pi b^2\left(a-\frac{a}{3}\right)=\frac{4\pi ab^2}{3}$$

2. **Prove that the volume and surface of a sphere of radius a is $\frac{4}{3}\pi a^3$ and $4\pi a^2$ respectively.**

Solⁿ. Let o be the centre and a be the radius of the circle ABA^1B^1 so that its equation is



$x^2+y^2=a^2$

$\Rightarrow y=\sqrt{a^2-x^2}$

The sphere is formed by the revolution of the area of the circle $x^2+y^2=a^2$ about the x – axis.

∴ If v is the volume of the sphere. Then v = volume of the solid formed by the revoluation of the area of the circle $x^2+y^2=a^1$ about x – axis.

= volume of the solid formed by the revolution of the area of the semi-circle ABA^1

= 2 (Volume of the solid formed by the revolution of the area of the quadrant ABOA about

$$=2\int_0^a \pi y^2 dx = 2\pi\int_0^a\left(a^2-x^2\right)dx$$

$$=2\pi\left[a^2x-\frac{x^3}{3}\right]_0^a=2\pi\left(a^3-\frac{a^3}{3}\right)$$

$$=2\pi\left[\frac{2a^3}{3}\right]=\frac{4\pi a^3}{3}$$

If S be the surface area of the sphere. Then S = 2(Surface of the solid formed by the revolution by the are AB about x – axis.

$$=2\times 2\pi\int_0^a y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

But $\frac{dy}{dx}=\frac{1}{2\sqrt{a^2-x^2}}(-2x)=\frac{-x}{\sqrt{a^2-x^2}}$

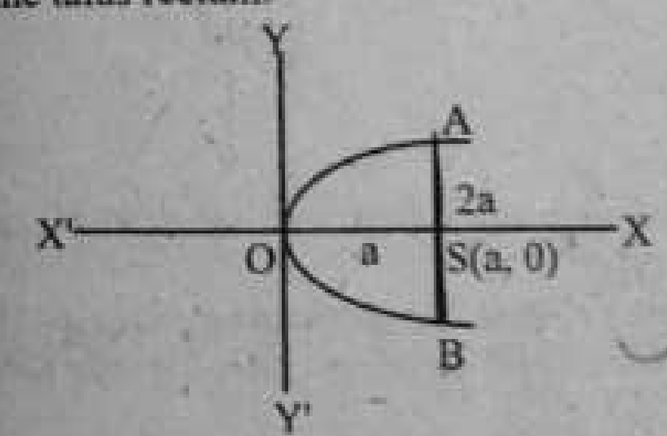
$$1+\left(\frac{dy}{dx}\right)^2=\frac{a^2-x^2+x^2}{a^2-x^2}=\frac{a^2}{a^2-x^2}$$

$$\therefore S=4\pi\int_0^a\sqrt{a^2-x^2}\,\frac{a}{\sqrt{a^2-x^2}}dx$$

$$=4\pi a\int_0^a dx=4\pi a^2$$

3. **An Arc of a parabola is bounded at both ends by the latus rectum of length 4a. Find the volume generated when the are is rotated about the latus rectum.**

Solⁿ. The parabola with latus rectum 4a is $y^2=4ax$. The ends of the latus rectum are, (a, 2a) and (a, 2a) and (a, – 2a) since the axis of revolution is the talus rectum.



So, v = 2 (volume generated by the area OASO about AB)

$$=2\int_0^{2a}\pi(a-x)^2dy$$

$$=2\pi\int_0^{2a}\left(a-\frac{y^2}{4a}\right)^2dy$$

$$=\frac{2\pi}{16a^2}\int_0^{2a}\left(4a^2-y^2\right)dy$$

$$=\frac{\pi}{8a^2}\int_0^{2a}(16a^4-8a^2y^2+y^4)\,dy$$

$$=\frac{\pi}{8a^2}\left[16a^4y-8a^2\frac{y^3}{3}+\frac{y^5}{5}\right]_0^{2a}$$

$$= \frac{\pi}{8a^2}\left[32a^5 - \frac{64a^5}{3} + \frac{3a^5}{5}\right]$$

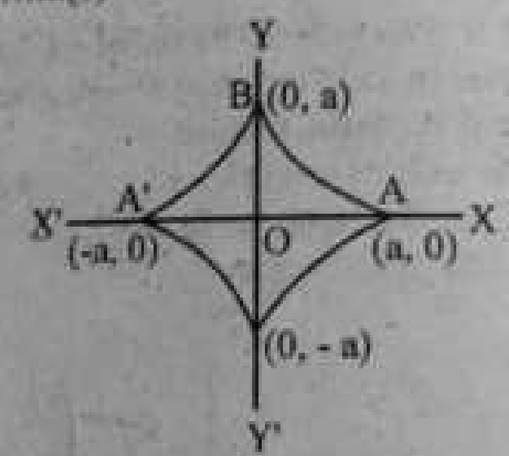
$$= \frac{32a^5\pi}{8a^2}\left[1 - \frac{2}{3} + \frac{1}{5}\right]$$

$$= 4\pi a^3\left[\frac{15-10+3}{15}\right] = 4\pi a^3 \cdot \frac{8}{15} = \frac{32a^3\pi}{15}$$

4. Find the volume and surface area of the solid generated by revolving the asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the axis of x.

Sol^n. Given asteroid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \quad \text{.....(1)}$$



If v be the volume of the solid formed by the revolution of the asteroid. Then

V = volume of the solid generated by the area $ABA'A$

= 2(Volume of the solid generated by the area ABOA)

$$\therefore V = 2\int_0^a \pi y^2 dx = 2\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx$$

Put $x = a\sin^3\theta$

$dx = 3a\sin^2\theta\cos\theta\, d\theta$

When $x \to (0, a)$ Then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$\therefore V = 2\int_0^{\frac{\pi}{2}} a^2(1-\sin^2\theta)^3\, 3a\sin^2\theta\cos\theta d\theta$$

$$= 6a^3\pi \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^7\theta\, d\theta$$

$$= 6a^3\pi \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{7+1}{2}\right)}{2\Gamma\left(\frac{2+7+2}{2}\right)} = \frac{6a^3\pi\Phi\left(\frac{2}{3}\right)\Gamma\left(\frac{8}{2}\right)}{2\Gamma\left(\frac{\pi}{2}\right)}$$

$$= \frac{6a^3\pi \cdot \frac{1}{2}\cdot\sqrt{\pi}\, 3.2.1}{2\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}$$

$$= \frac{2^5 a^3\pi}{105}$$

$$= \frac{32a^3\pi}{105}$$

Next part

If s be the surface area generated by the asteroid then

$$S = 2\int_b^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Now diff. (1) w.r.t x

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\text{or, } \frac{dy}{dx} = \frac{-y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}$$

$$\therefore S = 4\pi\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}} dx$$

Put $x = a\sin^3\theta$

$dx = 3a\sin^2\theta\cos\theta d\theta$

When $x \to (0, a)$ Then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$\therefore S = 4\pi\int_0^{\frac{\pi}{2}} a(1-\sin^2\theta)^{\frac{3}{2}} \frac{a^{\frac{1}{3}}}{a^{\frac{1}{3}}\sin\theta} 3a\sin^2\theta\cos\theta d\theta$$

$$= 12\pi a^2 \int_0^{\frac{\pi}{2}} \sin\theta\cos^4\theta d\theta$$

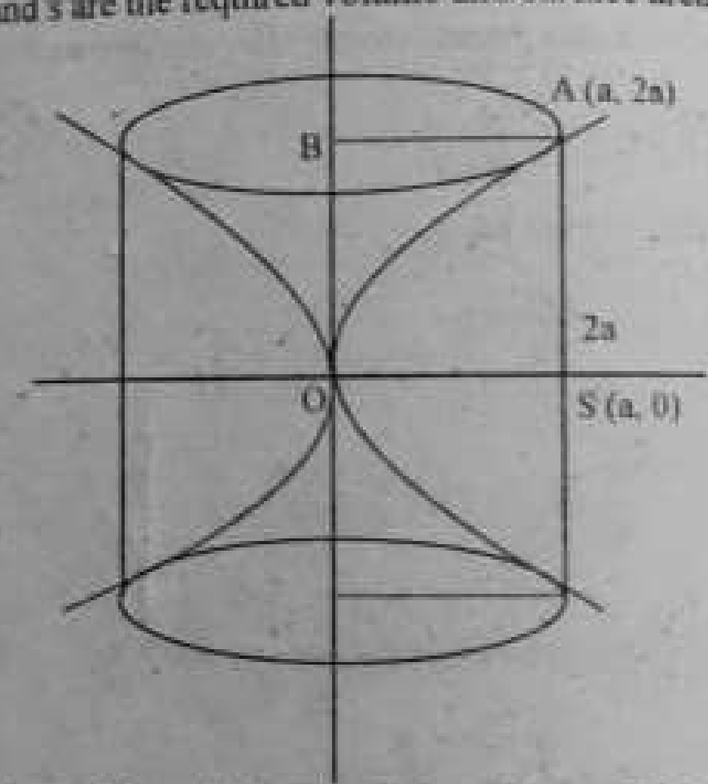
$$= \frac{12\pi a^2 \Gamma\left(\frac{1+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{1+4+2}{2}\right)} = \frac{12\pi a^2 \Gamma\left(\frac{2}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{12\pi a^2 . 1 . \frac{3}{2} . \frac{1}{2}\sqrt{\pi}}{2 . \frac{5}{2} . \frac{3}{2} . \frac{1}{2}\sqrt{\pi}} = \frac{12}{5}\pi a^2$$

5. **The part of the parabola $y^2 = 4ax$ bounded by the latus rectum revolves about the tangent at the vertex. Find the volume and the area of the curved surface thus generated.**

Sol^n. Given parabola $y^2 = 4ax$ .....(1)

The ends of the latus rectum are (a, 2a) and (a, − 2a). Since the tangent at the vertex is y − axis. So the axis of revolution is y-axis. If v and s are the required volume and surface area then



V = 2 (volume of the solid generated by the area OBAO about the y − axis).

$$= 2\int_0^{2a} \pi x^2 dy$$

$$= 2\pi \int_0^{2a} \frac{y^4}{16a^2} dy$$

$$= \frac{2\pi}{16a^2}\left[\frac{y^5}{5}\right]_0^{2a} = \frac{\pi}{8a^2} \cdot \frac{32a^5}{5} = \frac{4\pi a^3}{5}$$

Again; s = 2 (surface area generated by the are OA about OY).

$$= 2\int_0^{2a} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2} dy$$

Now diff. (1) w.r.t y

$$\frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{y^2}{4a^2}}$$

$$\therefore S = 4\pi \int_0^{2a} \frac{y^2}{4a}\sqrt{1+\frac{y^2}{4a^2}} dy$$

$$= \frac{4\pi}{8a^2}\int_0^{2a} y^2\sqrt{4a^2+y^2} dy$$

Put $y = 2a \tan\theta$

$dy = 2a\sec^2\theta d\theta$

When $y \to (0, 2a)$ then $\theta \to \left(0, \frac{\pi}{4}\right)$

$$\therefore S = \frac{\pi}{2a^2}\int_0^{\frac{\pi}{4}} 4a^2\tan^2\theta\sqrt{4a^2(1+\tan^2\theta)}.2a\sec^2\theta d\theta$$

$$= \frac{\pi}{2a^2}.16a^4\int_0^{\frac{\pi}{4}} \tan^2\theta.\sec^3\theta d\theta$$

$$= 8\pi a^2\int_0^{\frac{\pi}{4}}\left(\sec^2\theta - 1\right)\sec^3\theta d\theta$$

$$\therefore S = 8\pi a^2\left[\int_0^{\frac{\pi}{4}}\sec^5\theta d\theta - \int_0^{\frac{\pi}{4}}\sec^3\theta d\theta\right] .....(2)$$

Now, $\int_0^{\frac{\pi}{4}}\sec^5\theta d\theta = \int_0^{\frac{\pi}{4}}\sec^3\theta\sec^2\theta d\theta$

$$= \left[\sec^3\theta\tan\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 3\sec^3\theta\tan\theta.\tan\theta d\theta$$

$$= \left(\sqrt{2}\right)^3 - \int_0^{\frac{\pi}{4}} 3\sec^3\theta(\sec^2\theta - 1)d\theta$$

$$= 2\sqrt{2} - 3\int_0^{\frac{\pi}{4}}\sec^5\theta d\theta + 3\int_0^{\frac{\pi}{4}}\sec^3\theta d\theta.$$

$$\therefore 4\int_0^{\frac{\pi}{4}}\sec^5\theta d\theta = 2\sqrt{2} + 3\int_0^{\frac{\pi}{4}}\sec^3\theta d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{4}}\sec^5\theta d\theta = \frac{1}{\sqrt{2}} + \frac{3}{4}\int_0^{\frac{\pi}{4}}\sec^3\theta d\theta .....(3)$$

Now, $\int_0^{\frac{\pi}{4}}\sec^3 d\theta = \left[\sec\theta.\tan\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\sec\theta\tan\theta\tan\theta d\theta$

$$= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta + \int_0^{\frac{\pi}{4}} \sec\theta d\theta$$

$$\Rightarrow 2\int_0^{\frac{\pi}{4}} \sec^3\theta d\theta = \sqrt{2} + \left[\log\sec\theta + \tan\theta\right]_0^{\frac{\pi}{4}}$$

or, $\int_0^{\frac{\pi}{4}} \sec^3\theta d\theta = \frac{1}{\sqrt{2}} + \frac{1}{2}\log(\sqrt{2}+1)$ .....(4)

from (2) (3) and (4) we get

$$\therefore S = 8\pi a^2\left[\frac{1}{\sqrt{2}} + \frac{3}{4}\right]_0^{\frac{\pi}{4}} \sec^3\theta d\theta - \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta$$

$$= 8\pi a^2\left[\frac{1}{\sqrt{2}} - \frac{1}{4}\int_0^{\frac{\pi}{4}} \sec^3\theta d\theta\right]$$

$$= 8\pi a^2\left[\frac{1}{\sqrt{2}} - \frac{1}{4}\left\{\frac{1}{\sqrt{2}} + \frac{1}{2}\log\left(\sqrt{2}+1\right)\right\}\right]$$

$$= 8\pi a^2\left[\frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}} - \frac{1}{8}\log\left(\sqrt{2}+1\right)\right]$$

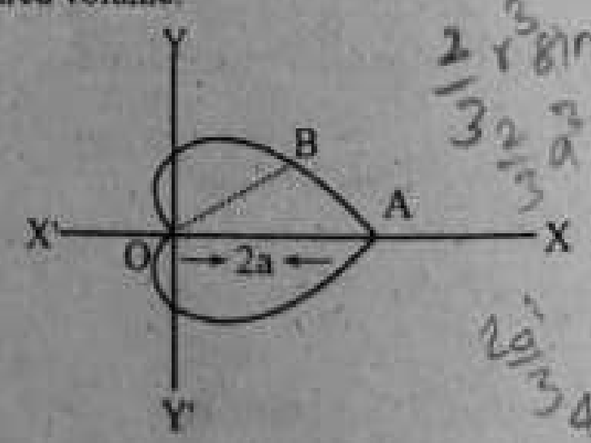
$$= 8\pi a^2\left[\frac{1}{\sqrt{2}}\cdot\frac{3}{4} - \frac{1}{8}\log\left(\sqrt{2}+1\right)\right]$$

$$\therefore S = \pi a^2\left[3\sqrt{2} - \log\left(\sqrt{2}+1\right)\right]$$

6. Find the volume of the solid formed by the revolution of the cordioid $r = a(1 + \cos\theta)$ about the initial line.

Solⁿ. Given cordioid $r = a(1 + \cos\theta)$ .....(1)

If v is the required volume.



Then,

V = volume of the solid generated by the area ABO BA about initial line. Here x –varies from x = 0 to x = 2a.

$$\therefore V = \int_0^{2a} \pi y^2 dx$$

Put $x = r\cos\theta$ then from (1)

$x = a(1 + \cos\theta)\cos\theta$

$= a(\cos\theta + \cos^2\theta)$

$\therefore dx = -a(\sin\theta + 2\cos\theta\sin\theta)d\theta$

$= -a\sin\theta(1 + 2\cos\theta)d\theta$

Put $y = r\sin\theta$ then from (1)

$y = a(1 + \cos\theta)\sin\theta$.

When $x \to (0, 2a)$ Then $\theta \to (\pi, 0)$

$$\therefore V = \int_{\pi}^{0} \pi a^2(1+\cos\theta)^2\sin^2\theta a(\sin\theta + 2\sin\theta\cos\theta)d\theta$$

$$= \pi a^3\int_0^{\pi} \sin^3\theta(1+\cos\theta)^2(1+2\cos\theta)d\theta$$

$$= \pi a^3\int_0^{\pi} 8\sin^3\frac{\theta}{2}.\cos^3\frac{\theta}{2}4\cos^4\frac{\theta}{2}\left(1+2-4\sin^2\frac{\theta}{2}\right)d\theta$$

$$= 32\pi a^3\int_0^{\pi} \sin^3\frac{\theta}{2}\cos^7\frac{\theta}{2}\left(3-4\sin^2\frac{\theta}{2}\right)d\theta$$

$$= 96\pi a^3\int_0^{\pi} \sin^3\frac{\theta}{2}\cos^7\frac{\theta}{2}d\theta - 128\pi a^3\int_0^{\pi} \sin^5\frac{\theta}{2}\cos^7\frac{\theta}{2}d\theta$$

Put $\frac{\theta}{2} = t$

$d\theta = 2dt$

When $\theta \to (0, \pi)$ then $t \to \left(0, \frac{\pi}{2}\right)$

$$\therefore V = 192\pi a^3\int_0^{\frac{\pi}{2}} \sin^3 + \cos^7 t dt - 256\pi a^3\int_0^{\pi} \sin^5 t\cos^7 t dt$$

$$= 192\pi a^3\frac{\Gamma\left(\frac{4}{2}\right)\Gamma\left(\frac{8}{2}\right)}{2\Gamma\left(\frac{12}{2}\right)} - 256\pi\frac{a^3\Gamma\left(\frac{6}{2}\right)\Gamma\left(\frac{8}{2}\right)}{2\Gamma\left(\frac{14}{2}\right)}$$

$$= 192\pi a^3\frac{1\times3\times2\times1}{2\times5\times4\times3\times2\times1} - 256\pi a^3\frac{2\times1\times3\times2\times1}{2\times6\times5\times4\times3\times2\times1}$$

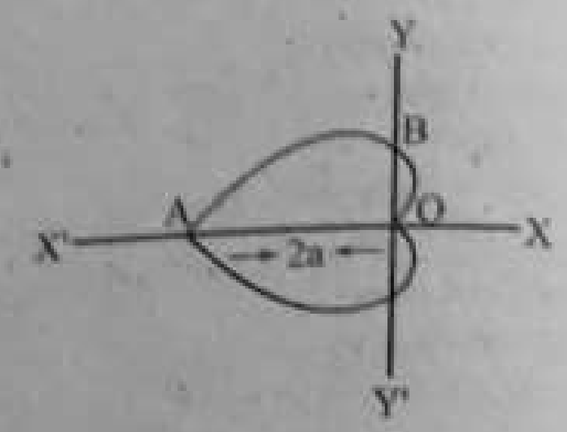
$$= \frac{24\pi a^3}{5} - \frac{32\pi a^3}{15}$$

$$= \frac{8}{5}\pi a^3\left[3 - \frac{4}{3}\right] = \frac{8\pi a^3}{5}\times\frac{5}{3} = \frac{8\pi a^3}{3}$$

7. Find the area of the surface of the solid generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about the initial line.

Solⁿ. Given cardioid $r = a(1 - \cos\theta)$ .....(1)

let s be thte surface generated by the are OBA



Then $S = \int_0^{\pi} 2\pi y\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ .....(2)

Put $y = r\sin\theta$ from (1)

$y = a(1-\cos\theta)\sin\theta$

diff. (1) w.r.t. θ

$\frac{dr}{d\theta} = a\sin\theta$

Now $\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}$

$= a\sqrt{1-2\cos\theta+1}$

$= a\sqrt{2(1-\cos\theta)}$

$= a\sqrt{4\sin^2\frac{\theta}{2}} = 2a\sin\frac{\theta}{2}$

from (2)

$S = 2\pi\int_0^{\pi} a(1-\cos\theta)\sin\theta.2a\sin\frac{\theta}{2}d\theta$

$= 4a^2\pi\int_0^{\pi} 2\sin^2\frac{\theta}{2}.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.\sin\frac{\theta}{2}d\theta$

$= 16a^2\pi\int_0^{\pi}\sin^4\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$

put $\frac{\theta}{2} = t \Rightarrow d\theta = 2dt$

When $\theta \to (0,\pi)$ then $t \to \left(0,\frac{\pi}{2}\right)$

$\therefore S = 32a^2\pi\int_0^{\frac{\pi}{2}}\sin^4 t\cos t\,dt$

$= 32a^2\pi\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{2}{2}\right)}{2\Gamma\left(\frac{7}{2}\right)}$

$= 32a^2\pi\frac{\frac{3}{2}.\frac{1}{2}\sqrt{\pi}.1}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}} = \frac{32a^2\pi}{5}$

7. **Find the volume of the solids formed by the revolution of the curve $y^2 = x^2(a-x)$ about the x – axis**

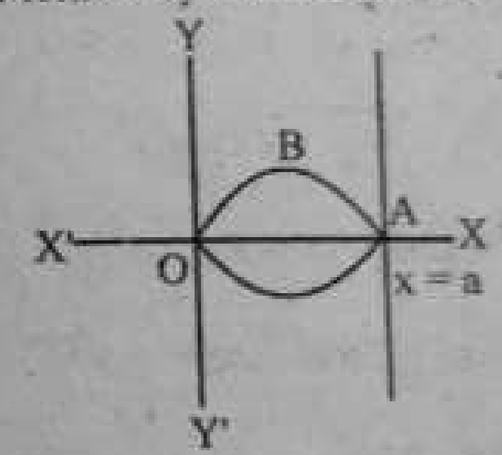
Sol. Given curve $y^2 = x^2(a-x)$ ..... (i)

Since the curve (1) is even power of y only so the curve is symmetrical about x – axis.

When $y = 0$ form (1) $x = 0$ and $x = a$.

So the loop is formed between $x = 0$ and $x = a$

Let v be the volume formed by the area OABO.



$\therefore V = \int_0^a \pi y^2 dx$

$= \pi\int_0^a x^2(a-x)dx$

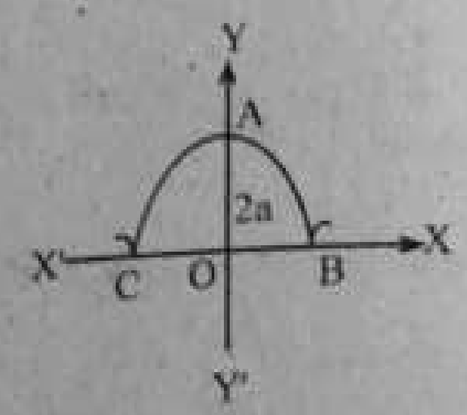
$= \pi\int_0^a\left(ax^2-x^3\right)dx = \pi\left[\frac{ax^3}{3}-\frac{x^4}{4}\right]_0^a$

$= \pi\left[\frac{a^4}{3}-\frac{a^4}{4}\right] = \pi.\frac{a^4}{12} = \frac{\pi a^4}{12}$

8. **Find the surface area of solid generated by revolving the cycloid $x = (\theta + \sin\theta)$, $y = a(1+\cos\theta)$ about its base.**

Sol. Given cyaloid $x = a(\theta+\sin\theta)$, $y = a(1+\cos\theta)$

The surface area required = 2 time the area of the surface generated by the revolution of the portion AOB.



$$= 2\int_0^{\pi} 2\pi y \frac{ds}{d\theta} d\theta$$

But $\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$

$$= \sqrt{a^2(1+\cos\theta)^2 + a^2(-\sin\theta)^2}$$

$$= a\sqrt{2(1+\cos\theta)} = 2a\cos\frac{\theta}{2}$$

$\therefore$ Surface $= 4\pi\int_0^{\pi} a(1+\cos\theta)2a\cos\frac{\theta}{2} d\theta$

$$= 8\pi a^2 \int_0^{\pi} 2\cos^3\frac{\theta}{2} d\theta$$

put $\frac{\theta}{2} = t$ Then $d\theta = 2dt$

When $\theta \to (0, \pi)$ then $t \to \left(0, \frac{\pi}{2}\right)$

$\therefore$ Surface $= 16\pi a^2 \int_0^{\frac{\pi}{2}} 2\cos^3\theta d\theta$

$$= 32\pi a^2 \cdot \frac{\sqrt{\pi}\Gamma\left(\frac{3+1}{2}\right)}{2\Gamma\left(\frac{3+2}{2}\right)}$$

$$= 16\pi a^2 \frac{\sqrt{\pi}\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)} = 16\pi \frac{a^2\sqrt{\pi}.1}{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{64\pi a^2}{3}$$

10. Find the volume of the solid formed by revolving the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base **[B.E. 2058/2062]**

Sol^n. Given cycloid

$x = a(\theta + \sin\theta)$ and

$y = a(1 + \cos\theta)$

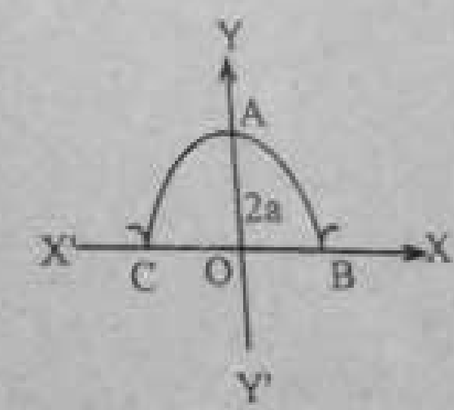
diff. w.r.t θ

$\frac{dx}{d\theta} = a(1+\cos\theta)$

$dx = a(1+\cos\theta)d\theta$

$\frac{dy}{d\theta} = -a\sin\theta$

$dy = -a\sin\theta d\theta$



The required volume obtained by revolving the cycloid about its base is defined by

$$V = \pi\int_{\pi}^{\pi} y^2 dx$$

$$= \pi\int_{\pi}^{\pi} a^2(1+\cos\theta)^2 a(1+\cos\theta)d\theta$$

$$= \pi a^2 . 2\int_0^{\pi}(1+\cos\theta)^3 d\theta$$

$$= 2\pi a^3 \int_0^{\pi}\left\{2\cos^2\frac{\theta}{2}\right\}^3 d\theta$$

$$= 2\pi a^3 . 8\int_0^{\pi}\cos^6\frac{\theta}{2} d\theta$$

Put $\frac{\theta}{2} = t$

$d\theta = 2tdt$

When $\theta \to (0, \pi)$ then $t \to \left(0, \frac{\pi}{2}\right)$

$$v = 32\pi a^3 \int_0^{\frac{\pi}{2}} \cos^6 t dt$$

$$= 32\pi a^2 \frac{\sqrt{\pi}\Gamma\frac{(6+1)}{2}}{2\Gamma\left(\frac{6+2}{2}\right)} = \frac{16\pi a^3\sqrt{\pi}\Gamma\left(\frac{7}{2}\right)}{\Gamma(4)}$$

$$= \frac{16\pi a^3\sqrt{\pi}\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{3\times2\times1} = 5\pi^2 a^3$$

Unit - III
Ordinary Differential Equation and its Applications

Chapter - 14 Ordinary Differential Equations

Chapter - 15 First Order and First Degree Differential Equations

Chapter - 16 First Order but not First Degree Differential Equations

Chapter - 17 Second order Linear differential Equations with Constant Coefficients

Chapter - 18 Application of Differential Equation

Ordinary Differential Equations

14

Definitions : An equation which involves differentials or differential coefficients is called the differential equation.

Example

(i) $\frac{dy}{dx} = \cos x$

(ii) $x^3 \frac{d^3y}{dx^3} + x\frac{dy}{dx} - y = x$

(iii) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3 = 0$

Differential equations are two types

(i) **Ordinary differential equations :** The differential equations which involves only one independent variable is called ordinary differential equations.

(ii) **Partial differential equations :** The differential equation which involves two or more than two independent variables is called partial differential equations.

For example : $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 5z$ is a partial differential equations.

Order : The order of the differential equation is the order of the highest derivative in the equation.

Example :

$$\left.\begin{aligned} &\frac{dy}{dx} = \sin x \\ &\left(\frac{dy}{dx}\right)^2 = px^2 + qx + c \end{aligned}\right\} \text{ First order}$$

$\frac{d^2y}{dx^2} = x^2$ Second order

Degree : The degree of differential equation is the degree of derivative of highest order. It has been made free from radicals of the derivatives.

Exercise - 18

1. Determine the order and degree of each of the following differential equations.

(i) $(x+3y-2)\,dx+(2x-3y+5)\,dy=0$

Solⁿ. Given differential equation is,

$(x+3y-2)\,dx+(2x-3y+5)\,dy=0$

or, $\frac{dy}{dx}+\frac{x+3y-2}{2x-3y+5}=0$

Which is first order and first degree differential equation.

(ii) $y=x\frac{d^2y}{dx^2}+\frac{k}{\frac{d^2y}{dx^2}}$

Solⁿ. Given differential equation is,

$$y=x\frac{d^2y}{dx^2}+\frac{k}{\frac{d^2y}{dx^2}}$$

$$\text{or, } y\frac{d^2y}{dx^2}=x\left(\frac{d^2y}{dx^2}\right)^2+k$$

$$\text{or, } x\left(\frac{d^2y}{dx^2}\right)^2-y\frac{d^2y}{dx^2}+k=0$$

Which is second degree and second order differential equation.

(iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}=k\frac{d^2y}{dx^2}$

Solⁿ. Given differential equation is,

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}=k\frac{d^2y}{dx^2}$$

Squaring both sides we get,

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=k^2\left(\frac{d^2y}{dx^2}\right)^2$$

Which is second order and second degree differential equation.

(iv) $x^2\frac{d^2y}{dx^2}+2xy\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^3=0$

Solⁿ. Give, differential equation is,

$$x^2\frac{d^2y}{dx^2}+2xy\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^3=0$$

Which is second order and fist degree differential equation.

2. From the differential equations from the following equations.

(i) $y=a\log x+b$

Solⁿ. Given equation is,

$y=a\log x+b$

Differentiating both sides w.r. to x we get,

$$\frac{dy}{dx}=a\cdot\frac{1}{x}$$

Again differentiating both sides w. r. to x we get,

$$\frac{d^2y}{dx^2}=-\frac{a}{x^2}$$

$$\text{or, } x\frac{d^2y}{dx^2}=-\frac{a}{x}=-\frac{dy}{dx}$$

$$\therefore x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

Which is the required differential equation.

(ii) $xy=a+bx$

Solⁿ. Given equation is,

$$xy=a+bx\Rightarrow y=\frac{a}{x}+b$$

Differentiating both sides w. r. to x we get,

$$\frac{dy}{dx}=-\frac{a}{x^2}$$

Again differentiating both sides w. r. to x we get,

$$\frac{d^2y}{dx^2}=-\frac{2a}{x^3}$$

$$\text{or, } x\frac{d^2y}{dx^2}=-\frac{2a}{x^2}=-2\frac{dy}{dx}$$

$$\therefore x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$

Which is the required differential equation.

(iii) $xy = Ae^x + Be^{-x}$

Sol^n. Given equation is,

$xy = Ae^x + Be^{-x}$

Differentiating both sides w. r. to x we get,

$$x\frac{dy}{dx} + y \cdot 1 = Ae^x - Be^{-x}$$

Again, differentiating both sides w. r. to x we get,

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 1 + \frac{dy}{dx} = Ae^x + Be^{-x}$$

$$\text{or, } x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$$

$$\text{or, } \frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = y$$

$$\therefore \frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - y = 0$$

Which is the required differential equation.

(iv) $y = ax^3 + bx^2$

Sol^n. Given equation is,

$y = ax^3 + bx^2$

Differentiating both sides w. r. to x we get,

$$\frac{dy}{dx} = 3ax^2 + abx \text{ ..... (i)}$$

Again differentiating both sides w. r. to x we get,

$$\frac{d^2y}{dx^2} = 6ax + 2b \text{ ..... (ii)}$$

From (i), $x\frac{dy}{dx} = 3ax^3 + 2bx^2$ ..... (iii)

From (ii), $x^2\frac{d^2y}{dx^2} = 6ax^3 + 2bx^2$ ..... (iv)

From (iii) and (iv)

$$4x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2} = 12ax^3 + 8bx^2 - 6ax^3 - 2bx^2$$

$$\text{or, } 4x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2} = 6ax^3 + 6bx^2$$

$$\text{or, } 4x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2} = 6(ax^3 + bx^2) = 6y$$

$$\therefore x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0$$

Which is the required differential equation.

(v) $a\cos(\log x) + b\sin(\log x)$

Sol^n. Given equation is,

$y = a\cos(\log x) + b\sin(\log x)$

Differentiating both sides w. r. to x we get,

$$\frac{dy}{dx} = -a\sin(\log x) \cdot \frac{1}{x} + b\cos(\log x) \cdot \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{-a}{x}\sin(\log x) + \frac{b}{x}\cos(\log x) \text{ ..... (i)}$$

Again differentiating both sides w. r. to x we get,

$$\frac{d^2y}{dx^2} = \frac{-a}{x}\cos(\log x) \cdot \frac{1}{x} + \frac{a}{x^2}\sin(\log x)$$

$$-\frac{b}{x}\sin(\log x) \cdot \frac{1}{x} - \frac{b}{x^2}\cos(\log x)$$

$$\text{or, } \frac{d^2y}{dx^2} = \frac{-a}{x}\cos(\log x) + \frac{a}{x^2}\sin(\log x)$$

$$-\frac{b}{x^2}\sin(\log x) - \frac{b}{x^2}\cos(\log x)$$

$$x\frac{d^2y}{dx^2} = -\frac{a}{x}\cos(\log x) + \frac{a}{x}\sin(\log x) - \frac{b}{x}\sin(\log x)$$

$$-\frac{b}{x}\cos(\log x) \text{ ..... (ii)}$$

From (i) and (ii)

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{a}{x}\cos(\log x) - \frac{b}{x}\sin(\log x)$$

$$\text{or, } \frac{xd^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}[a\cos(\log x) + b\sin(\log x)] = -\frac{1}{x}y$$

$$\therefore x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Which is the required differential equation.

3. **Obtain the differential equation of all circles of radius a and centre (h, k).**

Sol^n. We have the equation of circle of raidous a and centre (h, k) is

$(x-h)^2 + (y-k)^2 = a^2$ ..... (i)

Differential equation (i) w. r. t. 'x' we get, $(x-h) + (y-k)y_1 = 0$ ..... (ii)

Again differential equation $1 + y_1^2 + (y - k) y_2 = 0$

or, $y - k = \dfrac{-\left(1+y_1^2\right)}{y_2}$ ..... (iii)

Put the value of $y - k$ in (ii)

$x - h = -(y - k) y_1 = \dfrac{\left(1+y_1^2\right)}{y_2} . y_1$ ..... (iv)

Using (iii) and (iv) in (i) we get,

$$\left[\frac{\left(1+y_1^2\right)}{y_2} y_1\right]^2 + \left[\frac{\left(1+y_1^2\right)}{y_2}\right]^2 = a^2$$

$$\Rightarrow \frac{\left(1+y_1^2\right)^2}{y_2^2} (1 + y_1^2) = a^2$$

or, $(1 + y_1^2)^3 = a^2 y_2^2$

$$\therefore \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2 \left(\frac{d^2y}{dx}\right)^2$$

which is the required differential equation.

4. From a differential equation of simple harmonic motion given by $x = A\cos(nt + \alpha)$

Solⁿ. Given equation is,

$x = A\cos(nt + \alpha)$

Differentiating both sides w. r. to 't' we get,

$$\frac{dx}{dt} = -A \sin(nt + \alpha) . n$$

Again differentiating both sides w.r. to t,

$$\frac{d^2x}{dt^2} = -A \cos(nt + \alpha) . n^2 = -n^2 x$$

$$\therefore \frac{d^2x}{dt^2} + n^2 x = 0$$

Which is the required differential equation.

First Order and First Degree Differential Equation

15

Exercise - 19

Solve the following differential equations

1. $x\sqrt{1+y^2}\, dx + y\sqrt{1+x^2}\, dy = 0$

Solⁿ. Given differential equation is,

$$x\sqrt{1+y^2}\, dx + y\sqrt{1+x^2}\, dy = 0$$

Separating the variables,

$$x\frac{dx}{\sqrt{1+x^2}} + y\frac{dy}{\sqrt{1+y^2}} = 0$$

$$\text{or, } 2x\frac{dx}{\sqrt{1+x^2}} + \frac{2y\, dy}{\sqrt{1+y^2}} = 0$$

Integrating we get,

$\sqrt{1+x^2} + \sqrt{1+y^2} = C$ Ans.

2. $(x^2+1)\dfrac{dy}{dx} = 1$

Solⁿ. Given differential equation is,

$$(x^2 + 1)\frac{dy}{dx} = 1$$

Separating the variables,

$$dy = \frac{dx}{x^2+1}$$

Integrating we get,

$y = \tan^{-1}x + C$ Ans.

3. $y\,dx = (e^x + 1)\,dy$

Sol^n. Given differential equation is,

$y\,dx = (e^x + 1)\,dy$

Separating the variables,

$$\frac{dx}{e^x+1} = \frac{dy}{y}$$

$$\text{or, } -\frac{e^{-x}}{1+e^{-x}} = \frac{dy}{y}$$

Integrating we get,

$\log(1+e^{-x}) = -\log y - \log C$

or, $\log y + \log(1+e^{-x}) + \log C = 0$

$\therefore y(1+e^{-x})C = 0$ Ans.

4. $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$

Sol^n. Given differential equation is,

$(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$

or, $x(y^2+1)\,dx + y(x^2+1)\,dy = 0$

Separating the variables,

$$x\frac{dx}{(x^2+1)} + y\frac{dy}{(y^2+1)} = 0$$

$$\text{or, } 2x\frac{dx}{(x^2+1)} + \frac{2y\,dy}{(y^2+1)} = 0$$

Integrating we get,

$\log(x^2+1) + \log(y^2+1) = \log C$

$\therefore (x^2+1)(y^2+1) = C$ Ans.

5. $\tan y\,dx + \tan x\,dy = 0$

Sol^n. Given differential equation is,

$\tan y\,dx + \tan x\,dy = 0$

Separating the variables,

$$\frac{dx}{\tan x} + \frac{dy}{\tan y} = 0$$

or, $\cot x\,dx + \cot y\,dy = 0$

$$\text{or, } \frac{\cos x}{\sin x}dx + \frac{\cos y}{\sin y}dy = 0$$

Integrating we get,

$\log(\sin x) + \log(\sin y) = \log C$

$\therefore \sin x \sin y = C$ Ans.

6. $\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$

Sol^n. Given differential equation is,

$$\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$$

$$\text{or, } y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$$

$$\text{or, } y(1-ay) = (a+x)\frac{dy}{dx}$$

Separating the variables,

$$\frac{dx}{(a+x)} = \frac{dy}{y(1-ay)}$$

$$\text{or, } \frac{dx}{(a+x)} = \frac{1}{y}\,dy - \frac{-a\,dy}{1-dy}$$

Integrating we get,

$\log(a+x) = \log y - \log(1-ay) + \log C$

or, $\log(a+x) + \log(1-ay) = \log yC$

$\therefore (a+x)(1-ay) = yC$ Ans.

7. $(1+x)(1+y^2)\,dx + (1+y)(1+x^2)\,dy = 0$

Sol^n. Given differential equation is,

$(1+x)(1+y^2)\,dx + (1+y)(1+x^2)\,dy = 0$

Separating the variables,

$$\frac{(1+x)\,dx}{(1+x^2)} + \frac{(1+y)\,dy}{(1+y^2)} = 0$$

$$\text{or, } \frac{dx}{1+x^2} + x\frac{dx}{1+x^2} + \frac{dy}{1+y^2} + y\frac{dy}{1+y^2} = 0$$

$$\text{or, } \frac{dx}{1+x^2} + \frac{1}{2}\cdot\frac{2x\,dx}{1+x^2} + \frac{dy}{1+y^2} + \frac{1}{2}\frac{2y\,dy}{1+y^2} = 0$$

Integrating we get,

$$\tan^{-1}x + \frac{1}{2}\log(1+x^2) + \tan^{-1}y + \frac{1}{2}\log(1+y^2) = \log C$$

$$\text{or, } \tan^{-1}x + \tan^{-1}y + \log\left(1+x^2\right)^{\frac{1}{2}} + \log\left(1+y^2\right)^{\frac{1}{2}} = \log C$$

$$\therefore \tan^{-1}x + \tan^{-1}y + \log\sqrt{1+x^2}\,\sqrt{1+y^2} = \log C \text{ Ans.}$$

8. $(e^x + 1)\,y\,dy = (y+1)\,e^x\,dx$

Sol^n. Given differential equation is,

$(e^x + 1)\,y\,dy = (y+1)\,e^x\,dx$

Separating the variables,

$$\frac{y\,dy}{(1+y)} = \frac{e^x dx}{(e^x+1)}$$

or, $\left(1-\frac{1}{1+y}\right)dy = \frac{e^x}{e^x+1}\,dx$

Integrating we get,

$y - \log(1+y) = \log(e^x+1) + \log C$

or, $y = \log(1+y) + \log(e^x+1) + \log C$

$\therefore y = \log(1+y)(e^x+1)\,C$ Ans.

9. $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$

Sol^n. Given differential equation is,

$3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$

Separating the variables,

$$\frac{3e^x dx}{1-e^x} + \frac{\sec^2 y\,dy}{\tan y} = 0$$

or, $-3\left[\frac{e^x dx}{1-e^x}\right] + \frac{\sec^2 y\,dy}{\tan y} = 0$

Integrating we get,

$-3\log(1-e^x) + \log\tan y = \log C$

or, $\log\tan y = \log C + \log(1-e^x)^3$

$\therefore \tan y = C(1-e^x)^3$ Ans.

10. $e^{x-y}\,dx + e^{y-x}\,dy = 0$

Sol^n. Given differential equation is,

$e^{x-y}\,dx + e^{y-x}\,dy = 0$

or, $e^x . e^{-y}\,dx + e^y e^{-x}\,dy = 0$

Separating the variables,

or, $\frac{e^x}{e^{-x}}dx + \frac{e^y dy}{e^{-y}} = 0$

or, $e^{2x}dx + e^{2y}dy = 0$

Integrating we get,

$$\frac{e^{2x}}{2} + \frac{e^{2y}}{2} = \frac{C}{2}$$

$\therefore e^{2x} + e^{2y} = C$ Ans.

11. $(a^2+y^2)\,x\,dx + y(x^2-a^2)\,dy = 0$

Sol^n. Given differential equation is,

$(a^2+y^2)\,x\,dx + y(x^2-a^2)\,dy = 0$

Separating the variables,

$$\frac{x\,dx}{(x^2-a^2)} + \frac{y\,dy}{(a^2+y^2)} = 0$$

or, $\frac{2x\,dx}{(x^2-a^2)} + \frac{2y\,dy}{(a^2+y^2)} = 0$

Integrating we get,

$\log(x^2-a^2) + \log(a^2+y^2) = \log C$

$\therefore (x^2-a^2)(a^2+y^2) = C$ Ans.

12. $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

Sol^n. Given differential equation is,

$(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

Separating the variables,

$$\frac{\cos x\,dx}{\sin x} + \frac{e^y dy}{(e^y+1)} = 0$$

Integrating we get,

$\log(\sin x) + \log(e^y+1) = \log C$

$\therefore \sin x\,(e^y+1) = C$ Ans.

13. Find the particular solution of $2xy' = 3y$ given that $y = 4$ when $x = 1$

Sol^n. Given differential equation is,

$2xy' = 3y$

i.e. $2x\frac{dy}{dx} = 3y$

Separating the variables,

$2\frac{dy}{dx} = 3\frac{dx}{x}$

Integrating both sides we get,

$2\log y = 3\log x + \log C$

or, $\log y^2 = \log x^3 + \log C$

or, $\frac{y^2}{x^3} = C$

When $y = 4$, $x = 1$ then,

$\frac{16}{1} = C \Rightarrow C = 16$

$\therefore \frac{y^2}{x^3} = 16$

or, $y^2 = 16x^3$

$\therefore y = 4x^{\frac{3}{2}}$ Ans.

14. Find the particular solution of $y' = \sec y$, given that $y = 0$ when $x = 0$.

Sol^n. Given differential equation is,

$y' = \sec y$

i.e. $\frac{dy}{dx} = \sec y$

Separating the variables,

$\frac{dy}{\sec y} = dx \Rightarrow \cos y\, dy = dx$

Integrating both sides we get,

$\sin y = x + C$

when $y = 0, x = 0$ then

$\sin 0 = 0 + C$

$\Rightarrow C = 0$

$\therefore \sin y = x$ Ans.

15. Find the particular solution of $\frac{dy}{dx} = e^{x+y}$ and it is given that for $x = 1, y = 1$ find y as $x = -1$

Sol^n. Given differential equation is,

$\frac{dy}{dx} = e^{x+y}$

or, $\frac{dy}{dx} = e^x . e^y$

or, $\frac{dy}{e^y} = e^x\, dx$

or, $\int e^{-y}\, dy = \int e^x\, dx$ Integrating

$-e^{-y} = e^x + C$

$-e^x - e^{-y} = C$ ..... (i)

Using $x = 1, y = 1$ in (i)

$-e - e^{-1} = C$ ..... (ii)

Again using $x = -1$ in (i)

$-e^{-1} - e^{-y} = C$ ..... (iii)

From (ii) and (iii)

$-e - e^{-1} = -e^{-1} - e^{-y}$

$\Rightarrow e = e^{-y}$

Taking log on both sides,

$\log e = \log e^{-y}$

$-y = 1$

$\Rightarrow y = -1$ Ans.

Also, from (i) and (ii)

$-e^x - e^{-y} = -e - e^{-1}$ is the particular solution of the given differential equation.

16. Find the equation of the curve which passes through the point (1, 2) and has at every point, $\frac{dy}{dx} = \frac{-2xy}{x^2+1}$

Sol^n. Given differential equation is,

$\frac{dy}{dx} = \frac{-2xy}{x^2+1}$

$\int \frac{dy}{y} = -\int \frac{2x}{x^2+1}$; Integrating

$\log y = -\log(x^2+1) + \log C$

$\log y = \log \frac{c}{(x^2+1)} \Rightarrow y = \frac{C}{(x^2+1)}$

or, $y(x^2+1) = C$ ..... (i)

Since equation (i) passes through the point (1, 2)

So, $2(1^2+1) = C \Rightarrow C = 4$

Hence (i) gives,

$y(x^2+1) = 4$ Ans.

17. Find the particular solution of $y(1-x^2)\frac{dy}{dx} + x(1-y^2) = 0$ given that $y = 1$ when $x = 0$

Sol^n. Given differential equation is,

$y(1-x^2)\frac{dy}{dx} + x(1-y^2) = 0$

or, $\int \frac{y}{1-y^2} dy = -\int \frac{x}{1-x^2} dx$; Integrating

$-\log(1-y^2) = \log(1-x^2) \log C$

or, $\log(1-y^2)^{-1} = \log \frac{(1-x^2)}{C}$

$\Rightarrow \frac{1}{(1-y^2)} = \frac{(1-x^2)}{C}$

or, $(1-x^2)(1-y^2) = C$ ..... (i)

Using $x = 0$ and $y = 1$, in (i) we get

$C = 0$

Hence (i) gives,

$(1-x^2)(1-y^2) = 0$ Ans.

18. **Find the equation of the curve represented by $(y - yx)\,dx + (x + xy)\,dy = 0$ and passes the point (1, 1)**

Solⁿ. Given differential equation is,

$(y - yx)\,dx + (x + xy)\,dy = 0$

or, $y(1 - x)\,dx + x(1 + y)\,dy = 0$

or, $\frac{(1-x)}{x}dx = -\frac{(1+y)}{y}dy$

or, $\int\left(\frac{1}{x}-1\right)dx = \int\left(\frac{1}{y}+1\right)dy$ Integrating

$\log x - x = -\log y - y + C$

or, $\log x + \log y = x - y + C$

$\log xy = x - y + C$ ..... (i)

Hence equation (i) passes through the point (1, 1)

$\log 1 = 1 - 1 + C \Rightarrow C = 0$

Hence (i) givens,

$\log xy = x - y$ Ans.

Change a Variable

If the differential equation is of the form $\frac{dy}{dx} = f(ax + by + c)$ ..... (i)

Where a, b, c are constant

Then we put $ax + by + c = v$

$a + b\frac{dy}{dv} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{1}{b}\frac{dv}{dx} - \frac{a}{b}$

Now, equation (i) becomes,

$\frac{1}{b}\frac{dv}{dx} - \frac{a}{b} = f(v)$

Separating the variables and integrating we get the required solution.

Exercise - 20

Solve the following differential equations

1. $(x + y)^2 \frac{dy}{dx} = a^2$

Solⁿ. Given differential equation is,

$(x + y)^2 \frac{dy}{dx} = a^2$

$\therefore \frac{dy}{dx} = \frac{a^2}{(x+y)^2}$ ..... (i)

Put $x + y = v$

$\therefore 1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Now (i) becomes,

$\frac{dv}{dx} - 1 = \frac{a^2}{v^2}$

or, $\frac{dv}{dx} = \frac{a^2}{v^2} + 1 = \frac{a^2 + v^2}{v^2}$

or, $\int\frac{v^2}{a^2 + v^2}dv = \int dx$ Integrating

or, $\int\left(1 - \frac{a^2}{a^2 + v^2}\right)dv = \int dx$

or, $v - a^2 \cdot \frac{1}{a} \tan^{-1} \frac{v}{a} = x + C$

or, $v - a \tan^{-1}\left(\frac{v}{a}\right) = x + C$

Restoring the value of v we get,

$x + y - a\tan^{-1}\left(\frac{x+y}{a}\right) = x + C$

or, $y - a \tan^{-1} \frac{(x+y)}{a} = C$

$\Rightarrow y - C = a \tan^{-1}\left(\frac{x+y}{a}\right)$

or, $\frac{y-C}{a} = \tan^{-1}\left(\frac{x+y}{a}\right)$

$\Rightarrow x + y = a\tan^{-1}\left(\frac{y-C}{a}\right)$ is the required solution.

2. $\cos(x+y)\,dy = dx$

Sol^n. Given differential equations is,

$\cos(x+y)\,dy = dx$

or, $\frac{dy}{dx} = \frac{1}{\cos(x+y)}$ ..... (i)

Put $x + y = v$

$\therefore 1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dv} = \frac{dv}{dx} - 1$

Now (i) becomes,

$\frac{dv}{dx} - 1 = \frac{1}{\cos v}$

or, $\frac{dv}{dx} = \frac{1}{\cos v} + 1$

or, $\frac{dv}{dx} = \frac{1+\cos v}{\cos v}$

or, $\int \frac{\cos v}{1+\cos v} dv = \int dx$; Integrating

or, $\int \frac{\cos v + 1 - 1}{1+\cos v} dv = \int dx$

or, $\int \left(1 - \frac{1}{1+\cos v}\right) dv = \int dx$

or, $\int dv - \int \frac{1}{\sin^2\frac{v}{2} + \cos^2\frac{v}{2} + \cos^2\frac{v}{2} - \sin^2\frac{v}{2}} dv = \int dx$

or, $\int dv - \int \frac{1}{2\cos^2\frac{v}{2}} dv = \int dx$

or, $\int dv - \frac{1}{2}\int \sec^2\frac{v}{2}\, dv = \int dx$

or, $v - \frac{1}{2}\, 2\tan\frac{v}{2} = x + C$

Restoring the value of v

$x + y - \tan\frac{x+y}{2} = x + C$

$y - \tan\frac{(x+y)}{2} = C \Rightarrow y - C = \tan\left(\frac{x+y}{2}\right)$

or, $\tan\left(\frac{x+y}{2}\right) = y - C$ is the required solution.

3. $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Sol^n. Given differential equation is,

$\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

or, $\frac{dy}{dx} = \sin(x+y)$ ..... (i)

Put $x + y = v$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Now equation (i) becomes,

$\frac{dv}{dx} - 1 = \sin v$

or, $\frac{dv}{dx} = \sin v + 1$

or, $\int \frac{1}{\sin v + 1} dv = \int dx$; Integrating

or, $\int \frac{(1-\sin\theta)}{1-\sin^2 v} dv = \int dx$

or, $\int \left(\frac{1-\sin v}{\cos^2 v}\right) dv = \int dx$

or, $\int \sec^2 v\, dv - \int \sec v \tan v\, dv = \int dx$

or, $\tan v - \sec v = x + C$

Restoring the value of v we get,

$\tan(x+y) - \sec(x+y) = x + C$ is the required solution.

4. $\frac{dy}{dx} + 1 = e^{x+y}$

Solⁿ. Given differential equations

$\frac{dy}{dx} + 1 = e^{x+y}$

or, $\frac{dy}{dx} = e^{x+y} - 1$ ..... (i)

Put $x + y = v$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Now (i) becomes,

$\frac{dv}{dx} - 1 = e^{v} - 1$

or, $\frac{dv}{dx} = e^{v}$

or, $\frac{1}{e^{v}} dv = dx \Rightarrow \int e^{-v} dv = \int dx$; Integrating

or, $-e^{-v} = x + C$

Restoring the value of v we get,

$-e^{-(x+y)} = x + C$

or, $-1 = (x + C) e^{(x+y)}$

or, $(x + C) e^{x+y} + 1 = 0$

or, $(x + C) e^{x} . e^{y} = -1$

or, $(x + C) e^{x} = -e^{-y}$

or, $(x + C) e^{x} + e^{-y} = 0$ is the required solution.

5. $\frac{dy}{dx} + 1 = e^{x-y}$

Solⁿ. Given differential equations is,

$\frac{dy}{dx} + 1 = e^{x-y}$

or, $\frac{dy}{dx} = e^{x-y} - 1$ ..... (i)

Put $x - y = v$

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = 1 - \frac{dv}{dx}$

Now (i) becomes,

$1 - \frac{dv}{dx} = e^{v} - 1$

or, $\frac{dv}{dx} = -e^{v} + 2$

or, $\int \frac{1}{2 - e^{v}} dv = \int dx$; Integrating

or, $\int \frac{e^{-v}}{2e^{-v} - 1} dv = \int dx$

or, $\frac{1}{2} \int \frac{2e^{-v}}{2e^{-v} - 1} dv = \int dx$

or, $-\frac{1}{2} \log (2e^{-v} - 1) = x + C$

or, $\log (2e^{-v} - 1) = -2 (x + C)$

or, $(2e^{-v} - 1) = e^{-2(x+C)}$

Restoring the values of v we get,

$(2e^{-x+y} - 1) = e^{-2(x+C)}$

$2e^{-x} . e^{y} - 1 = e^{-2x} . e^{-2C}$

$e^{y} = \left(\frac{e^{-2x} . e^{-2C}}{2 e^{-x}}\right) + 1$

$= \left(\frac{e^{-2C}}{2}\right) e^{-x} + \frac{1}{2} e^{x}$

$e^{y} = \frac{1}{2} e^{x} + k e^{-x}$ where $K = \frac{e^{-2C}}{2}$ is the required solution.

6. $\frac{dy}{dx} - x \tan (y - x) = 1$

Solⁿ. Given differential equation is,

$\frac{dy}{dx} - x \tan (y - x) = 1$ ..... (i)

Put $y - x = v$

$\frac{dy}{dx} - 1 = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \frac{dv}{dx} + 1$

Now (i) becomes,

$\frac{dv}{dx} + 1 - x \tan v = 1$

or, $\frac{dv}{dx} = x \tan v$

or, $\frac{1}{\tan v} dv = x\, dx$

or, $\int \cot v\, dv = \int x\, dx$; Integrating

$$\log \sin y = \frac{x^2}{2} + \log C$$

Restoring the value of v we get,

$$\log \sin (y - x) = \frac{x^2}{2} + \log C$$

$\Rightarrow \sin (y - x) = C\, e^{\frac{x^2}{2}}$ is the required solution.

7. $\frac{dy}{dx} = (4x + y + 1)^2$

Sol^n. Given differential equation is,

$$\frac{dy}{dx} = (4x + y + 1)^2 \text{ ..... (i)}$$

Put $4x + y + 1 = v$ differentiating w. r. t. 'x' we get,

$$4 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\text{or, } \frac{dy}{dx} = \frac{dv}{dx} - 4$$

Now (i) becomes

$$\frac{dv}{dx} - 4 = v^2$$

$$\text{or, } \frac{dv}{dx} = v^2 + 4$$

$$\text{or, } \int \frac{1}{v^2 + 4}\, dv = \int dx; \text{ Integrating}$$

$$\frac{1}{2} \tan^{-1} \frac{v}{2} = x + C$$

Restoring the value of v we get,

$$\frac{1}{2} \tan^{-1} \left(\frac{4x + y + 1}{2} \right) = (x + C)$$

$$\text{or, } \tan^{-1} \left(\frac{4x + y + 1}{2} \right) = 2\,(x + C)$$

$$\Rightarrow \left(\frac{4x + y + 1}{2} \right) = \tan^{-1} \{2\,(x + C)\}$$

or, $(4x + y + 1) = 2 \tan^{-1} (2x + 2C)$

$(4x + y + 1) = 2 \tan^{-1} (2x + k)$; where $k = 2C$ is the required solution.

8. $(x^2 + y^2 + 2xy + 1)\, dy = (x + y)\, dx$

Sol^n. Given differential equation is,

$$(x^2 + y^2 + 2xy + 1)\, dy = (x + y)\, dx$$

$$\text{or, } \frac{dy}{dx} = \frac{(x + y)}{(x + y)^2 + 1} \text{ ..... (i)}$$

Put $x + y = v$; differentiating w. r. t. x we get,

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\text{or, } \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Now (i) becomes,

$$\frac{dv}{dx} - 1 = \frac{v}{v^2 + 1}$$

$$\text{or, } \frac{dv}{dx} = \frac{v}{v^2 + 1} + 1$$

$$\text{or, } \frac{dv}{dx} = \frac{v + v^2 + 1}{v^2 + 1}$$

$$\text{or, } \int \frac{v^2 + 1}{v^2 + v + 1}\, dv = \int dx; \text{ Integrating}$$

$$\text{or, } \int \frac{(v^2 + v + 1) - v}{(v^2 + v + 1)}\, dv = \int dx$$

$$\text{or, } \int dv - \int \frac{v}{v^2 + v + 1}\, dv = \int dx$$

$$\text{or, } \int dv - \int \frac{\frac{1}{2}(2v + 1) - \frac{1}{2}}{(v^2 + v + 1)}\, dv = \int dx$$

$$\text{or, } \int dv - \frac{1}{2} \int \frac{(2v + 1)}{(v^2 + v + 1)}\, dx + \frac{1}{2} \int \frac{1}{v^2 + 2v.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \frac{1}{4} + 1}\, dv = \int dx$$

$$\text{or, } \int dv - \frac{1}{2} \int \frac{(2v + 1)}{(v^2 + v + 1)}\, dx + \frac{1}{2} \int \frac{1}{\left(v + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dv = \int dx$$

$$\text{or, } v - \frac{1}{2} \log (v^2 + v + 1) + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \left\{ \frac{v + 1}{2} \cdot \frac{2}{\sqrt{3}} \right\} = x + C$$

$$\text{or, } v - \frac{1}{2} \log (v^2 + v + 1) + \frac{1}{\sqrt{3}} \tan^{-1} \left\{ \frac{2v + 1}{\sqrt{3}} \right\} = x + C$$

Restoring the value of v we get,

$$x + y - \frac{1}{2} \log \{(x + y)^2 + x + y + 1\}$$

$$+\frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{2(x+y)+1}{\sqrt{3}}\right\}=x+C$$

or, $y-\frac{1}{2}\log(x+y)^2+x+y+1\}+\frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{2x+2y+1}{\sqrt{3}}\right\}=$

is the required solution.

9. $(x+y+1)\frac{dy}{dx}=1$

Solⁿ. Given differential equation is,

$(x+y+1)\frac{dy}{dx}=1$

or, $\frac{dy}{dx}=\frac{1}{x+y+1}$ ..... (i)

Put $x+y+1=v$ differentiating w. r. t. 'x' we get,

or, $1+\frac{dy}{dx}=\frac{dv}{dx}$

or, $\frac{dy}{dx}=\frac{dv}{dx}-1$

Now (i) becomes,

$\frac{dv}{dx}-1=\frac{1}{v}$ or, $\frac{dv}{dx}=\frac{1}{v}+1=\frac{1+v}{v}$

or, $\int\frac{v}{1+v}dv=\int dv$; Integrating

or, $\int\left(1-\frac{1}{1+v}\right)dv=\int dx$

or, $v-\log(1+v)=x+C$

Restoring the value of v we get,

$x+y+1-\log(1+x+y+1)=x+C$

or, $y+1-\log(x+y+2)=C$

or, $y-\log(x+y+2)=C-1=K$ where $K=c-1$

is the required solution.

10. $\frac{dy}{dx}=\sqrt{y-x}$

Solⁿ. Given differential equation is,

$\frac{dy}{dx}=\sqrt{y-x}$ ..... (i)

Put $y-x=v^2$ differentiating w. r. t. 'x'

$\frac{dy}{dx}-1=2v\frac{dv}{dx}$

or, $\frac{dy}{dx}=2v\frac{dv}{dx}+1$

Now (i) becomes,

$2v\frac{dv}{dx}+1=v$

or, $\frac{dv}{dx}=\frac{v-1}{2v}$

or, $\int\frac{2v}{v-1}dv=\int dx$; Integrating

or, $2\int\left(1+\frac{1}{v-1}\right)dv=\int dx$

or, $2v+2\log(v-1)=x+C$

Restoring the value of v we get,

$2\sqrt{y-x}+2\log\left(\sqrt{y-x}-1\right)=x+C$

or, $\sqrt{y-x}+\log\left(\sqrt{y-x}-1\right)=\frac{x}{2}+\frac{C}{2}$

or, $\sqrt{y-x}+\log\left(\sqrt{y-x}-1\right)=\frac{x}{2}+K$ where $K=\frac{C}{2}$

is the required solution.

11. $x^2(xdx+y\,dy)+2y(x\,dy-y\,dx)=0$

Solⁿ. Given differential equation is,

$x^2(xdx+y\,dy)+2y(x\,dy-y\,dx)=0$ ..... (i)

Put $x=r\cos\theta$ and $y=r\sin\theta$

Then, $x^2+y^2=r^2$ and $\frac{y}{x}=\tan\theta$

$\frac{xdx-ydx}{x^2}=\sec^2\theta$

and $x\,dx+y\,dy=r\,dr$

Now equation (i) becomes

$x\,dx+y\,dy+2y\frac{xdy-ydx}{x^2}=0$

$rdr+2r\sin\theta\,.\,\sec^2\theta=0$

$\int dr+2\int\tan\theta\sec\theta\,d\theta=\int 0$; Integrating

$r+2\sec\theta=C$

Restoring the value of r and θ we get,

$\sqrt{x^2+y^2}+2\sqrt{1+\frac{y^2}{x^2}}=C$

or, $\sqrt{x^2+y^2}+2\frac{\sqrt{x^2+y^2}}{x}=C$

or, $\sqrt{x^2+y^2}\left(1+\frac{2}{x}\right)=C$

or, $(x+2)\sqrt{x^2+y^2}=Cx$ is the required solution.

Homogeneous Differential Equation of First Order

If the differential equation is of the form $\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$ where f(x, y) and ϕ(x, y) are homogeneous function of the same degree n.

Then it can be solved by putting y = vx.

Differentiating w. r. t. 'x' we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So the differential equation reduce to

$$v + x\frac{dv}{dx} = f(v)$$

Then separating the variables and integrating we get the required solution.

Exercise - 21

Solve the following differential equations

1. $x + y\frac{dy}{dx} = 2y$

Solⁿ. Given differential equation is,

$$x + y\frac{dy}{dx} = 2y$$

$$\text{or, } \frac{dy}{dx} = \frac{2y - x}{y} \quad \text{.....(i)}$$

Equation (i) is homogeneous differential equation

So, Put y = vx

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{2vx - x}{vx} = \frac{x(2v-1)}{vx}$$

$$\therefore v + x\frac{dv}{dx} = \frac{2v-1}{v}$$

$$x\frac{dv}{dx} = \frac{2v-1}{v} - v$$

$$= \frac{2v - 1 - v^2}{v} = -\frac{(v^2 - 2v + 1)}{v}$$

$$\int \frac{v}{(v^2 - 2v + 1)}\,dv = -\int \frac{1}{x}\,dx; \text{ Integrating}$$

$$\text{or, } \int \frac{\frac{1}{2}(2v-2)+1}{v^2 - 2v + 1}\,dv = -\int \frac{1}{x}\,dx$$

$$\text{or, } \frac{1}{2}\int \frac{2v-2}{v^2-2v+1}\,dv + \int \frac{1}{(v-1)^2}\,dv = -\int \frac{1}{x}\,dx$$

$$\text{or, } \frac{1}{2}\log(v^2 - 2v + 1) - \frac{1}{(v-1)} = -\log x + C$$

$$\text{or, } \frac{1}{2}\log(v-1)^2 - \frac{1}{(v-1)} = -\log x + C$$

$$\text{or, } \log(v-1) - \frac{1}{(v-1)} = -\log x + C$$

Restoring the value of $v = \frac{y}{x}$ we get,

$$\log\left(\frac{y}{x} - 1\right) - \frac{1}{\left(\frac{y}{x} - 1\right)} = -\log x + C$$

$$\text{or, } \log\left(\frac{y-x}{x}\right) - \frac{x}{y-x} = -\log x + C$$

$$\text{or, } \log(y-x) - \log x - \frac{x}{y-x} = -\log x + C$$

$$\Rightarrow \log(y-x) = C + \frac{x}{y-x} \text{ is the required solution.}$$

2. $(x^2 - y^2)\,dx + 2xy\,dy = 0$

Solⁿ. Given differential equation is,

$$(x^2 - y^2)\,dx + 2xy\,dy = 0$$

$$\text{or, } \frac{dy}{dx} = \frac{-(x^2 - y^2)}{2xy} \quad \text{.....(i)}$$

Equation (i) is homogeneous differential equation

So, put y = vx

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{-(x^2 - x^2v^2)}{2x \cdot vx} = -\frac{(1 - v^2)}{2v}$$

or, $x\frac{dv}{dx} = \frac{-1+v^2}{2v} - v = \frac{-1+v^2-2v^2}{2v}$

or, $x\frac{dv}{dx} = \frac{-1-v^2}{2v} = \frac{-(v^2+1)}{2v}$

$\int \frac{2v}{v^2+1} dv = -\int \frac{1}{x} dx$; Integrating

or, $\log(v^2+1) = -\log x + \log C$

$\log(v^2+1) = \log\left(\frac{c}{x}\right)$

or, $v^2+1 = \frac{C}{x}$

or, $x(v^2+1) = C$

Restoring the value of $v = \frac{y}{x}$ we get,

$x\left(\frac{y^2}{x^2}+1\right) = C$

or, $x\frac{(y^2+x^2)}{x^2} = C$

or, $\frac{y^2+x^2}{x} = C \Rightarrow x^2+y^2 = Cx$ is the required solution.

3. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Sol^n. Given differential equation is,

$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

or, $\frac{dy}{dx} = \frac{y^2}{x^2} - \frac{y}{x}$ ..... (i)

Equation (i) is homogeneous differential equation

So, put $y = vx$

then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{v^2x^2}{x^2} - \frac{vx}{x}$

or, $v + x\frac{dv}{dx} = v^2 - v$

or, $x\frac{dv}{dx} = v^2 - 2v$

or, $\int \frac{1}{v^2-2v} dv = \int \frac{1}{x} dx$; Integrating

or, $\int \frac{1}{v(v-2)} dv = \int \frac{1}{x} dx$

or, $\frac{1}{2}\int\left(\frac{-1}{v} + \frac{1}{(v-2)}\right) dv = \int \frac{1}{x} dx$

or, $\frac{1}{2}\{(-\log v + \log(v-2)\} = \log x + \log C$

or, $\frac{1}{2}\log\frac{(v-2)}{v} = \log x C$

or, $\sqrt{\frac{v-2}{v}} = xC \Rightarrow \frac{v-2}{v} = x^2C^2$

Restoring the value of $v = \frac{y}{x}$ we get

or, $\frac{\frac{y}{x}-2}{\frac{y}{x}} = x^2C^2 \Rightarrow \frac{y-2x}{y} = x^2C^2$

or, $y - 2x = x^2y\,C^2$

or, $2x - y = (-C^2)\,x^2y$

or, $2x - y = K\,x^2y^2$ where $K = -C^2$

is the required solution.

4. $(x^2+y^2)\,dx = (x^2+xy)\,dy$

Sol^n. Given differential equation is,

$(x^2+y^2)\,dx = (x^2+xy)\,dy$

or, $\frac{dy}{dx} = \frac{(x^2+y^2)}{(x^2+xy)}$ ..... (i)

Equation (i) is homogeneous differential equation

So, put $y = vx$ then

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{x^2+x^2v^2}{x^2+x^2v}$

or, $v + x\frac{dv}{dx} = \frac{1+v^2}{1+v}$

or, $x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v$

or, $x\frac{dv}{dx} = \frac{1+v^2-v-v^2}{1+v}$

or, $x\frac{dv}{dx} = \frac{1-v}{1+v}$

or, $\int\frac{1+v}{1-v}dv = \int\frac{1}{x}dx$; Integrating

or, $\int\left(-1+\frac{2}{1-v}\right)dv = \int\frac{1}{x}dx$;

$-v-2\log(1-v) = \log x + \log C$

$-v-\log(1-v)^2 = \log x\,C$

or, $-v = \log xC(1-v)^2$

$\Rightarrow xC(1-v)^2 = e^{-v}$

Restoring the value of $v = \frac{y}{x}$ we get,

or, $xC\left(1-\frac{y}{x}\right)^2 = e^{-\frac{y}{x}}$

or, $xC\frac{(x-y)^2}{x^2} = e^{-\frac{y}{x}}$

or, $C(x-y) = Cx\,e^{-\frac{y}{x}}$ is the required solution.

5. **$x(x-y)\,dy = y(x+y)\,dx$**

Sol^n. Given differential equation is,

$x(x-y)\,dy = y(x+y)\,dx$

$\therefore \frac{dy}{dx} = \frac{y(x+y)}{x(x-y)}$ ..... (i)

Equation (i) is homogeneous differential equation

So, put $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{vx(x+vx)}{x(x-vx)}$

or, $v + x\frac{dv}{dx} = \frac{v(1+v)}{(1-v)}$

or, $x\frac{dv}{dx} = \frac{v+v^2}{1-v} - v$

or, $x\frac{dv}{dx} = \frac{v+v^2-v+v^2}{1-v} = \frac{2v^2}{1-v}$

or, $x\frac{dv}{dx} = \frac{2v^2}{1-v}$

$\int\frac{1-v}{2v^2}dv = \int\frac{1}{x}dx$; Integrating

or, $\int\left(\frac{1}{v^2}-\frac{1}{v}\right)dv = 2\int\frac{1}{x}dx$

or, $-\frac{1}{v} + \log v = 2\log x + \log C$

or, $-\frac{1}{v} = \log x^2Cv$

or, $x^2Cv = e^{-\frac{1}{v}}$

Restoring the value of $v = \frac{y}{x}$, we get,

$x^2C\cdot\frac{y}{x} = e^{-\frac{x}{y}}$

or, $Cxy = e^{-\frac{x}{y}} \Rightarrow e^{-\frac{x}{y}} = C\,xy$ is the required solution.

6. **$(x^2+y^2)\,dy = xy\,dx$**

Sol^n. Given differential equation is,

$(x^2+y^2)\,dy = xy\,dx$

or, $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ ..... (i)

Equation (i) is homogeneous differential equation

So, put $y = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{x\cdot vx}{x^2+v^2x^2}$

or, $v + x\frac{dv}{dx} = \frac{v}{1+v^2}$

or, $\frac{xdv}{dx} = \frac{v}{1+v^2} - v$

or, $x\frac{dv}{dx} = \frac{v-v-v^3}{1+v^2}$

or, $\frac{xdv}{dx} = \frac{-v^3}{1+v^2}$

or, $\int \frac{1+v^2}{v^3} dv = \int \frac{1}{x} dx$; Integrating

or, $\int \left(\frac{1}{v^3} + \frac{1}{v}\right) dv = -\int \frac{1}{x} dx$

or, $-\frac{1}{2v^2} + \log v = -\log x + \log C$

or, $\log v + \log x - \log C = \frac{1}{2v^2}$

or, $\log \frac{vx}{C} = \frac{1}{2v^2}$

or, $\frac{vx}{C} = e^{\frac{1}{2v^2}}$

or, $vx = C\, e^{\frac{1}{2v^2}}$

Restoring the value of $v = \frac{y}{x}$ we get,

$\frac{y}{x} \cdot x = C\, e^{\frac{x^2}{2y^2}}$

or, $y = Ce^{\frac{x^2}{2y^2}}$ is the required solution.

7. $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ **[B.E. 2059]**

Sol^n. Given differential equation is,

$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ ..... (i)

Equation (i) is homogeneous differential equation

So, put $y = vx$

So, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{vx}{x} + \tan\frac{vx}{x}$

or, $v + x\frac{dv}{dx} = v + \tan v$

or, $x\frac{dv}{dx} = \tan v$

or, $\int \frac{1}{\tan v} dv = \int \frac{1}{x} dx$; Integrating

or, $\int \cot v\, dv = \int \frac{1}{x} dx$

or, $\log \sin v = \log x + \log C$

or, $\log \sin v = \log xC$

or, $\sin v = xC$

Restoring the value of $v = \frac{y}{x}$ we get,

$\sin\frac{y}{x} = xC$ is the required solution.

8. $x^2dy + y(x+y)\,dx = 0$

Sol^n. Given differential equation is,

$x^2dy + y(x+y)\,dx = 0$

or, $\frac{dy}{dx} = -\frac{y(x+y)}{x^2}$ ..... (i)

Equation (i) is homogeneous differential equation

So, put $y = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{-vx(x+vx)}{x^2} = -v(1+v)$

or, $x\frac{dv}{dx} = -v - v^2 - v$

or, $x\frac{dv}{dx} = -(v^2 + 2v)$

or, $\int \frac{1}{v^2+2v} dv = -\int \frac{1}{x} dx$; Integrating

or, $\int \frac{1}{v(v+2)} dv = -\int \frac{1}{x} dx$

or, $\frac{1}{2}\int \left[\frac{1}{v} - \frac{1}{(v+2)}\right] dv = -\int \frac{1}{x} dx$

or, $\int \left(\frac{1}{v} - \frac{1}{v+2}\right) dv = -2\int \frac{1}{x} dx$

or, $\log v - \log(v+2) = -2\log x + \log C$

or, $\log\left(\frac{v}{v+2}\right) = \log\frac{C}{x^2}$

or, $v = (y+2)\frac{C}{x^2}$

Restoring the value of $v = \frac{y}{x}$ we get,

$$\frac{y}{x} = \left(\frac{y}{x}+2\right)\frac{C}{x^2}$$

or, $y = \frac{(y+2x)}{x^2}C$

$\Rightarrow x^2y = C(y+2x)$ is the required solution.

9. $x\frac{dy}{dx} = y - \sqrt{x^2+y^2}$

Sol^n. Given differential equation is,

$$x\frac{dy}{dx} = y - \sqrt{x^2+y^2}$$

or, $\frac{dy}{dx} = \frac{y-\sqrt{x^2-y^2}}{x}$ ..... (i)

Equation (i) is homogeneous differential equation

So put $y = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{vx-\sqrt{x^2+x^2v^2}}{x}$$

or, $v + x\frac{dv}{dx} = v - \sqrt{1+v^2}$

or, $x\frac{dv}{dx} = -\sqrt{1+v^2}$

or, $\int\frac{1}{\sqrt{1+v^2}}dv = -\int\frac{1}{x}dx$; Integrating

or, $\log\left(v+\sqrt{1+v^2}\right) = -\log x + \log C$

or, $v + \sqrt{1+v^2} = \frac{C}{x}$

Restoring the value of $v = \frac{y}{x}$ we get,

$$\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = \frac{c}{x}$$

or, $y + \sqrt{x^2+y^2} = C$ is the required solution.

10. $x\sin\frac{y}{x}\,dy = \left(y\sin\frac{y}{x} - x\right)dx$

Sol^n. Given differential equations is,

$$x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\left(\frac{y}{x}\right)-x\right)dx$$

or, $\frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right)-x}{x\sin\left(\frac{y}{x}\right)}$ ..... (i)

Equation (i) is homogeneous differential equation

So, put $y = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{vx\sin\left(\frac{vx}{x}\right)-x}{x\sin\left(\frac{vx}{x}\right)}$$

or, $v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$

or, $x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v$

or, $x\frac{dv}{dx} = \frac{v\sin v - 1 - v\sin v}{\sin v}$

or, $x\frac{dv}{dx} = -\frac{1}{\sin v}$

or, $\int \sin v\,dv = -\int\frac{1}{x}dx$; Integrating

or, $-\cos v = -\log x + C$

or, $\log x = \cos v + C$

Restoring the value of $v = \frac{y}{x}$ we get,

$\log x = \cos\left(\frac{y}{x}\right) + C$ is the required solution.

11. $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

Sol^n. Given differential equation is,

$$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$$

$$\text{or, } x\,dy = y\,dx + \sqrt{x^2+y^2}\,dx$$

$$\text{or, } x\,dy = \left(y+\sqrt{x^2+y^2}\right)dx$$

$$\text{or, } \frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \quad \text{.....(i)}$$

Equation (i) is homogeneous differential equation

So, put $y = vx$

then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{vx+\sqrt{x^2+v^2x^2}}{x}$$

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$\text{or, } x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\text{or, } \int\frac{1}{\sqrt{1+v^2}}dv = \int\frac{1}{x}\,dx; \text{ Integrating}$$

$$\log\left(v+\sqrt{1+v^2}\right) = \log x + \log C$$

$$\text{or, } v + \sqrt{1+v^2} = xC$$

Restoring the value of $v = \frac{y}{x}$ we get,

$$\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = xC$$

$$\text{or, } y + \sqrt{x^2+y^2} = x^2C$$

$$\text{or, } \sqrt{x^2+y^2} = x^2C - y$$

Squaring on both sides,

$x^2+y^2 = (Cx^2-y)^2$ is the required solution.

12. $(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

Sol^n. Given differential equation is,

$$(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$$

$$\text{or, } \frac{dx}{dy} = \frac{-e^{x/y}\left(1-\frac{x}{y}\right)}{\left(1+e^{\frac{x}{y}}\right)} \quad \text{.....(i)}$$

Equation (i) is homogeneous differential equation

So, put $x = vy$

Then $\frac{dx}{dy} = v + \frac{dv}{dy}$

Now (i) becomes,

$$v + \frac{dv}{dy} = \frac{-e^{\frac{vy}{y}}\left(1-\frac{vy}{y}\right)}{\left(1+e^{\frac{vy}{y}}\right)}$$

$$\text{or, } v + \frac{dv}{dy} = \frac{-e^{v}(1-v)}{\left(1+e^{v}\right)} \qquad \text{or, } v + \frac{dv}{dy} = \frac{-e^{v}+ve^{v}}{1+e^{v}} - v$$

$$\text{or, } \frac{dv}{dy} = \frac{-e^{v}+ve^{v}-v-ve^{v}}{1+e^{v}} \qquad \text{or, } \frac{dv}{dy} = \frac{-e^{v}-v}{1+e^{v}}$$

$$\text{or, } \int\frac{1+e^{v}}{v+e^{v}}dv = -\int\frac{1}{y}\,dy; \text{ Integrating}$$

$$\text{or, } \log(v+e^{v}) = -\log y + \log C$$

$$\text{or, } \log(v+e^{v}) = \log\frac{C}{y}$$

$$\text{or, } v+e^{v} = \frac{C}{y} \quad \text{or, } y(v+e^{v}) = C$$

Restoring the value of $v = \frac{x}{y}$ we get,

$$y\left(\frac{x}{y}+e^{\frac{x}{y}}\right) = C$$

$$\text{or, } \left(x + y\,e^{\frac{x}{y}}\right) = C \text{ is the required solution.}$$

13. $\frac{dy}{dx} = \frac{3xy+y^2}{3x^2}$

Sol^n. Given differential equation is,

$$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2} \quad \ldots\ldots (i)$$

Equation (i) is homogeneous differential equation

So, put $y = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{3x \cdot vx + v^2x^2}{3x^2} = \frac{3v + v^2}{3}$$

or, $x\frac{dv}{dx} = \frac{3v + v^2}{3} - v$ or, $x\frac{dv}{dx} = \frac{3v + v^2 - 3v}{3}$

or, $x\frac{dv}{dx} = \frac{v^2}{3}$ or, $\int \frac{1}{v^2} dv = \int \frac{1}{3x} dx$; Intgrating

or, $-\frac{1}{v} = \frac{1}{3} \log x + C$

Restoring the value of $v = \frac{y}{x}$ we get,

$$-\frac{x}{y} = \frac{1}{3} \log x + C$$

or, $-3x = y \log x + 3Cy$ or, $3x + y \log x = (-3C)y$

or, $3x + y \log x = Ky$ where $K = -3C$ is the required solution.

14. $(x^2 + 2xy^2)\,dx + (2x^2y + y^2)\,dy = 0$

Sol^n. Given differential equation is,

$(x^2 + 2xy^2)\,dx + (2x^2y + y^2)\,dy = 0$

or, $\frac{dy}{dx} = -\frac{(x^2 + 2xy^2)}{(2x^2y + y^2)}$ or, $2x^2y\,dy + y^2dy = -x^2dx - 2xy^2\,dx$

or, $x^2\,dx + y^2dy + 2xy^2\,dx + 2yx^2\,dy = 0$

or, $x^2dx + y^2dy = -(2xy^2dx + 2yx^2dy)$

or, $\int d\left(\frac{x^3}{3}\right) + \int d\left(\frac{y^3}{3}\right) = -\int d\,(x^2y^2)$; Integrating

or, $\frac{x^3}{3} + \frac{y^3}{3} = -x^2y^2 + C$

or, $x^3 + y^3 + 3x^2y^2 = 3C$

or, $x^3 + y^3 + 3x^2y^2 = K$ where $K = 3C$ is the required solution.

First Order Differential Equations Reducible to Homogenous Form

If the differential equation is of the form $\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$, then it can be solved by the following ways.

1. If the differential equation is of the form,

   $\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$ and $\frac{a}{A} = \frac{b}{B} = \frac{1}{\ell}$

   Then the equation reduced to the form

   $\frac{dy}{dx} = \frac{(ax + by + c)}{\ell\,(ax + by) + C}$

   and put $ax + by = v$ then $a + b\frac{dy}{dx} = \frac{dv}{dx}$

   Separating the variable and integrating we get, the required solution.

2. If the differential equation is of the form

   $\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$ and $\frac{a}{A} \neq \frac{b}{B}$ then the equation reduced to the homogeneous form by putting $x = X + h$ and $y = Y + K$. Where h, K are constant which will be interms of h and K, the equation are to be chose in such a way that the differential equation should be homogeneous.

Exercise - 22

Solve the following differential equations

1. $\frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}$ **[B. E. 2058]**

Sol^n. Given differential equation is,

$$\frac{dy}{dx} = \frac{x + y + 1}{x + y - 1} \quad \ldots\ldots (i)$$

Put $x + y = v$

Then, $1 + \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Equation (i) becomes,

$$\frac{dv}{dx} - 1 = \frac{v+1}{v-1}$$

$$\text{or, } \frac{dv}{dx} = \frac{v+1}{v-1} + 1$$

$$\text{or, } \frac{dv}{dx} = \frac{v+1+v-1}{v-1} = \frac{2v}{v-1}$$

$$\text{or, } \int \frac{v-1}{2v} dv = \int dx; \text{ Integrating}$$

$$\text{or, } \int \left(\frac{1}{2} - \frac{1}{2v}\right) dv = \int dx$$

$$\text{or, } \frac{1}{2}v - \frac{1}{2}\log v = x + C$$

Restoring the value of v we get,

$$\frac{1}{2}(x+y) - \frac{1}{2}\log(x+y) = x + C$$

$$\text{or, } x + y - \log(x+y) = 2x + 2C$$

$$\text{or, } y - x - \log(x+y) = K \text{ where } K = 2C \text{ is the required solution.}$$

2. $(x+y+1)\,dx - (2x+2y+1)\,dy = 0$

Sol^n. Given differential equation is,

$$(x+y+1)\,dx - (2x+2y+1)\,dy = 0$$

$$\text{or, } \frac{dy}{dx} = \frac{(x+y+1)}{2(x+y)+1} \quad \text{..... (i)}$$

Put $x + y = v$ then $1 + \frac{dy}{dx} = \frac{dv}{dx}$

$$\text{or, } \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Now (i) becomes,

$$\frac{dv}{dx} - 1 = \frac{v+1}{2v+1}$$

$$\text{or, } \frac{dv}{dx} = \frac{v+1}{2v+1} + 1 = \frac{v+1+2v+1}{2v+1} = \frac{3v+2}{2v+1}$$

$$\text{or, } \int \frac{2v+1}{3v+2} dv = \int dx; \text{ Integrating}$$

$$\int \frac{\frac{2}{3}(3v+2) - \frac{4}{3} + 1}{3v+2} dv = \int dx$$

$$\text{or, } \frac{2}{3}\int \frac{3v+2}{3v+2} dv - \frac{1}{3}\int \frac{1}{3v+2} dv = \int dx$$

$$\text{or, } \frac{2}{3}v - \frac{1}{3}\cdot\frac{1}{3}\log(3v+2) = x + C$$

$$2v - \frac{1}{3}\log(3v+2) = 3x + 3C$$

Restoring the value of v we get,

$$2x + 2y - \frac{1}{3}\log(3x+3y+2) = 3x + 3C$$

$$2y - x - \frac{1}{3}\log(3x+3y+2) = 3C$$

$$\text{or, } 6y - 3x - \log(3x+3y+2) = 6C$$

$$\text{or, } 6y - 3x = \log(3x+3y+2) + K \text{ where } K = 6C \text{ is the required solution.}$$

3. $(4x+6y+5)\,dy = (3y+2x+4)\,dx$

Sol^n. Given differential equation is,

$$(4x+6y+5)\,dy = (3y+2x+4)\,dx$$

$$\text{or, } \frac{dy}{dx} = \frac{(3y+2x+4)}{(4x+6y+5)}$$

$$\text{or, } \frac{dy}{dx} = \frac{(3y+2x+4)}{2(2x+3y)+5} \quad \text{..... (i)}$$

Put $2x + 3y = v$ then,

$$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$$

$$\text{or, } \frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$$

Now (i) becomes,

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{v+4}{2v+5}$$

$$\text{or, } \frac{dv}{dx} - 2 = \frac{3v+12}{2v+5}$$

$$\text{or, } \frac{dv}{dx} = \frac{2v+12}{2v+5} + 2$$

$$\frac{dv}{dx} = \frac{3v+12+4v+10}{2v+5} = \frac{7v+22}{2v+5}$$

$$\text{or, } \int \left(\frac{2v+5}{7v+22}\right) dv = \int dx; \text{ Integrating}$$

$$\text{or, } \int \frac{\frac{2}{7}(7v+22) - \frac{44}{7} + 5}{7v+22} dv = \int dx$$

$$\text{or, } \frac{2}{7}\int \frac{(7v+22)}{(7v+22)} dv - \frac{39}{7}\int \frac{1}{7v+22} dv = \int dx$$

$$\text{or, } \frac{2}{7}v - \frac{9}{7}\cdot\frac{1}{7}\log(7v+22) = x + C$$

Gaurab Sanc

or, $2v - \frac{9}{7}\log(7v+22) = 7x + 7C$

Restoring the value of v we get,

or, $4x + 6y - \frac{9}{7}\log(14x+21y+22) = 7x + 7C$

or, $6y - 3x - \frac{9}{7}\log(14x+21y+22) = 7C$

or, $42y - 21x - 9\log(14x+21y+22) = 49C$

or, $42y - 21x - 9\log(14x+21y+22) = K$

Where $K = 49C$

or, $7(6y - 3x) - 9\log(14x + 21y + 22) = K$ is the require solution.

4. **$(2x + 2y + 3)\,dy - (x + y + 1)\,dx = 0$** [B.E. 2057/2060]

Sol^n. Given differential equation is,

$(2x + 2y + 3)\,dy - (x + y + 1)\,dx = 0$

or, $\frac{dy}{dx} = \frac{x+y+1}{2(x+y)+3}$ ..... (i)

Put $x + y = v$ then $1 + \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Now (i) becomes,

$\frac{dv}{dx} - 1 = \frac{v+1}{2v+3}$

or, $\frac{dv}{dx} = \frac{v+1}{2v+3} + 1$

$\frac{dv}{dx} = \frac{v+1+2v+3}{2v+3} = \frac{3v+4}{2v+3}$

or, $\int\frac{2v+3}{3v+4}dv = \int dx$; Integrating

$\int\frac{\frac{2}{3}(3v+4) - \frac{8}{3} + 3}{3v+4}dv = \int dx$

or, $\frac{2}{3}\int\frac{3v+4}{3v+4}dv + \frac{1}{3}\int\frac{1}{3v+4}dv = \int dx$

or, $\frac{2}{3}v + \frac{1}{3}\cdot\frac{1}{3}\log(3v+4) = x + C$

or, $2v + \frac{1}{3}\log(3v+4) = 3x + 3C$

Restoring the value of v

$2x + 2y + \frac{1}{3}\log(3x+3y+4) = 3x + 3C$

or, $2y - x + \frac{1}{3}\log(3x+3y+4) = 3C$

or, $6y - 3x + \log(3x+3y+4) = 9C$

or, $6y - 3x + \log(3x+3y+4) = K$ where $K = 9C$ is the required solution.

5. $\frac{dy}{dx} = \frac{x+y}{x+y-2}$

Sol^n. Given differential equation is,

$\frac{dy}{dx} = \frac{x+y}{x+y-2}$ ..... (i)

Put $x + y = v$ then,

$1 + \frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Now (i) becomes,

$\frac{dv}{dx} - 1 = \frac{v}{v-2}$

or, $\frac{dv}{dx} = \frac{v}{v-2} + 1$

$\frac{dv}{dx} = \frac{v+v-2}{v-2} = \frac{2v-2}{v-2} = \frac{2(v-1)}{v-2}$

or, $\int\frac{v-2}{v-1}dv = 2\int dx$; Integrating

or, $\int\frac{v-1-1}{v-1}dv = 2\int dx$

or, $\int 1\,dv - \int\frac{1}{v-1}dv = 2\int dx$

or, $v - \log(v-1) = 2x + C$

Restoring the value of v we get,

$x + y - \log(x+y-1) = 2x + C$

or, $y = x - \log(x+y-1) = C$ is the required solution.

6. **$(6x - 5y + 4)\,dy + (y - 2x - 1)\,dx = 0$**

Sol^n. Given differential equation is,

$(6x - 5y + 4)\,dy + (y - 2x - 1)\,dx = 0$

or, $\frac{dy}{dx} = \frac{(y-2x-1)}{(6x-5y+4)}$ ..... (i)

Put $x = X + h$, and $y = Y + K$, where h and K are constant

So that $dx = dX$ and $dy = dY$

Then $\frac{dy}{dx} = \frac{dY}{dX}$

Now equation (i) becomes,

$$\frac{dY}{dX} = \frac{-(Y+K)-2(X+h)-1}{6(X+h)-5(Y+k)+4} \quad \text{..... (ii)}$$

$$= -\frac{(Y-2X)+(K-2h-1)}{(6X-5Y)+(6h-5K+4)}$$

Let us choose h and K such that

$K-2h-1=0$ and $Ch-5K+4=0$

Solving these, we get, $h=-\frac{1}{4}, k=\frac{1}{2}$

Now equation (ii) reduces to

$$\frac{dY}{dX} = -\frac{(Y-2X)}{(6X-5Y)} \quad \text{..... (iii)}$$

Equation (iii) is homogeneous differential equation

So, put $Y = vX$ then

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

Now (iii) becomes,

$$V + X\frac{dv}{dX} = \frac{-(Xv-2X)}{(6X-5vX)} = -\frac{(V-2)}{(6-5v)}$$

$$\text{or, } x\frac{dv}{dx} = -\frac{(v-2)}{(6-5v)} - V$$

$$= \frac{-v+2-6v+5v^2}{6-5V} \text{ or } \frac{dv}{dx} = \frac{5v^2-7v+2}{6-5v}$$

$$\text{or, } \int\frac{6-5v}{5v^2-7v+2}dv = \int\frac{1}{X}dX \text{; Integrating}$$

$$\text{or, } \int\frac{-\frac{1}{2}(10v-7)-\frac{7}{2}+6}{5v^2-7v+2}dv = \int\frac{1}{X}dX$$

$$\text{or, } \frac{-1}{2}\int\frac{10v-7}{5v^2-7v+2}dv - \frac{5}{2.5}\int\frac{1}{v^2-2.v.\frac{7}{10}+\left(\frac{7}{10}\right)^2+\frac{2}{5}}dv$$

$$= \int\frac{1}{X}dX$$

$$\text{or, } \frac{-1}{2}\int\frac{10v-7}{5v^2-7v+2}dv - \frac{1}{2}\int\frac{1}{\left(v-\frac{7}{10}\right)^2-\left(\frac{3}{10}\right)^2}dv = \int\frac{1}{X}dX$$

$$\text{or, } -\frac{1}{2}\log(5v^2-7v+2) - \frac{1}{2}\cdot\frac{1}{2.\frac{3}{10}}\log\frac{\left(v-\frac{7}{10}-\frac{3}{10}\right)}{\left(v-\frac{7}{10}+\frac{3}{10}\right)} = \log X$$

$+\log C$

$$\text{or, } \log(5v^2-7v+2) - \frac{5}{3}\log\left(\frac{v-1}{5v-2}\right) + \log X = -2\log C$$

$$\text{or, } \log\left\{X^2\left(5v^2-7v+2\right)\left(\frac{5v-2}{v-1}\right)^{\frac{5}{3}}\right\} = \log\frac{1}{C^2}$$

$$\text{or, } X^2(5v^2-7v+2)\frac{(5v-2)^{\frac{5}{3}}}{(v-1)^{\frac{5}{3}}} = \frac{1}{C^2}$$

$$\text{or, } X^2\frac{(5v-2)(v-1)(5v-2)^{\frac{5}{3}}}{(v-1)^{\frac{5}{3}}} = \frac{1}{C^2}$$

$$\text{or, } X^2\frac{(5v-2)^{\frac{8}{3}}}{(v-1)^{\frac{2}{3}}} = \frac{1}{C^2}$$

$$\text{or, } X^2\left(5v-2\right)^{\frac{8}{3}} = \left(v-1\right)^{\frac{2}{3}}\cdot\frac{1}{C^2}$$

Restoring the value of $V = \frac{Y}{X}$ we get,

$$X^2\left(\frac{5Y}{X}-2\right)^{\frac{8}{3}} = \left(\frac{Y}{X}-1\right)^{\frac{2}{3}}\cdot\frac{1}{C^2}$$

$$\text{or, } \frac{X^2(5Y-2X)^{\frac{8}{3}}}{X^2.X^{\frac{2}{3}}} = \frac{(Y-X)^{\frac{2}{3}}}{X^{\frac{2}{3}}}\cdot\frac{1}{C^2}$$

$$\text{or, } (5Y-2X)^{\frac{8}{3}} = \frac{(Y-X)^{\frac{2}{3}}}{C^2}$$

$$\text{or, } (5Y-2X)^8 = \frac{(Y-X)^2}{C^6}$$

$$\text{or, } (5Y-2X)^8 = (Y-X)^2\cdot\frac{1}{C^2}$$

or, $\left[5\left(y-\frac{1}{2}\right)-2\left(x+\frac{1}{4}\right)\right]^4 = \frac{1}{C^2}\left[\left(y-\frac{1}{2}\right)-\left(x+\frac{1}{4}\right)\right]$

or, $(5x-2x-3)^4 = K(4y-4x-3)$ is the required solution.

7. $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Solⁿ. Given differential equation is,

$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ ..... (i)

Put $x = X + h$, $y = Y + K$ so that $dx = dX$, $dy = dY$ where h and k are constant.

Then $\frac{dy}{dx} = \frac{dY}{dX}$

Hence equation (i) will reduces to

$\frac{dY}{dX} = \frac{(x+h)+2(Y+K)-2}{2(X+h)+(Y+K)-3} = \frac{(X+2Y)+(h+2K-3)}{(2X+Y)+(2h+K-3)}$ ..... (ii)

Let us choose h and K such that,

$h + 2K - 3 = 0$ and $2h + K - 3 = 0$

Solving these,

$h = 1, K = 1$

Now equation (ii) becomes,

$\frac{dY}{dX} = \frac{X-2Y}{2X+Y}$ ..... (iii)

Equation (iii) is homogenous differential equation

So, put $Y = vX$

Then, $\frac{dY}{dX} = v + X\frac{dv}{dX}$

Hence, equation (iii) becomes,

$v + X\frac{dv}{dX} = \frac{X+2vX}{2X+vX}$

or, $v + X\frac{dv}{dX} = \frac{1+2v}{2+v}$

or, $X\frac{dv}{dX} = \frac{1+2v}{2+v} - v = \frac{1+2v-2v-v^2}{2+v}$

or, $X\frac{dv}{dX} = \frac{1-v^2}{2+v}$

or, $\int\frac{dX}{X} = \int\frac{2+v}{1-v^2}dv$ Integrating

$= \int\frac{2+v}{(1-v)(1+v)}dv = \int\frac{1}{2}\left[\frac{3}{1-v}+\frac{1}{1+v}\right]dv$

(By using the partial function)

$\therefore \log X = \frac{1}{2}[-3\log(1-v)+\log(1+v)] + \frac{1}{2}\log K$

or, $2\log X = -3\log(1-v)+\log(1+v)+\log K$

or, $\log X^2 = \log(1-v)^{-3}+\log(1+v)+\log K$

or, $\log X^2 = \log(1-v)^{-3}(1+v)K$

$\Rightarrow X^2 = (1-v)^{-3}(1+v)K$

$\Rightarrow X^2 = \frac{K(1+v)}{(1-v)^3}$

or, $X^2(1-v)^3 = K(1+v)$

Restoring the value of $v = \frac{Y}{X}$ we get,

$\Rightarrow X^2\left(1-\frac{Y}{X}\right)^3 = k\left(1+\frac{Y}{X}\right)$

$\frac{(X-Y)^3}{X} = K\frac{(X+Y)}{X}$

$\Rightarrow (X-Y)^3 = K(X+h)$ ..... (iv)

But $x = X + h = X + 1$ and $y = Y + K = y + 1$

$\therefore X = x - 1, Y = y - 1$

Hence, equation (iv) becomes,

$\{(x-1)-(y-1)^3\} = k(x-1+y-1)$

$\Rightarrow (x-y)^3 = k(x+y-2)$

$\Rightarrow x+y-2 = C(x-y)^3$

Where $C = \frac{1}{K}$ which is the required solution.

8. $\frac{dy}{dx} + \frac{2x+3y+1}{3x+5y-1}$

Solⁿ. Given differential equation is,

$\frac{dy}{dx} = \frac{2x+3y+1}{3x+5y-1} = 0$ ..... (i)

Put $x = X + h$ and $y = Y + K$ where h and K are constant.

So, that $dx = dX$ and $dy = dY$

$\therefore \frac{dy}{dx} = \frac{dY}{dX}$

Now equation (i) becomes,

$\frac{dY}{dX} = \frac{+2(X+h)+3(Y+x)+1}{3(X+h)+5(Y+x)-1} = 0$

or, $\frac{dY}{dX} = -\frac{(2X+3Y)+(2h+3k+1)}{(3X+5Y)+(3h+5K-1)}$ ..... (ii)

Let us choose h and k such that

$2h + 3K + 1 = 0$ and $3h + 5K - 1 = 0$ solving these

We get, $h = -8$, $K = 5$

Now (ii) becomes,

$$\frac{dY}{dX} = \frac{-(2X+3Y)}{(3X+5Y)} \quad \text{.....(iii)}$$

Equation (iii) is homogenous differential equation,

So, put $Y = vX$

then $\frac{dY}{dX} = v + X\frac{dv}{dX}$

From (iii)

$$\therefore v + X\frac{dv}{dX} = -\frac{(2X+3vX)}{(3X+5vX)} = -\frac{(2+3v)}{(3+5v)}$$

$$\text{or, } X\frac{dv}{dX} = \frac{-2-3v}{3+5v} - v$$

$$\text{or, } X\frac{dv}{dX} = \frac{-2-3v-3v-5v^2}{3+5v} = \frac{-5v^2-5v-2}{3+5v}$$

$$\text{or, } \int\frac{(3+5v)}{(5v^2+5v+1)}dv = -\int\frac{1}{X}dX; \text{ Integrating}$$

$$\frac{1}{2}\log(5v^2+6v+2) = -\log X + \log C$$

$$\log(5v^2+6v+2) = -2\log X + 2\log C$$

$$\text{or, } \log(5v^2+6v+2) = -\log X^2 + \log C^2$$

$$\text{or, } \log(5v^2+6v+2)X^2 = \log C^2$$

$$\text{or, } (5v^2+6v+2)x^2 = C^2$$

Restoring the value of $v = \frac{Y}{X}$ we get,

$$\left(5\frac{Y^2}{X^2}+\frac{6Y}{X}+2\right)X^2 = C^2$$

$$\text{or, } (5Y^2+6XY+2X^2) = C^2$$

$$\text{or, } 5(y-5)^2+6(x+8)(y-5)+2(x+8)^2 = C^2$$

$$\text{or, } 5(y^2-10y+25)+6(xy-5x+8y-40)+2(x^2+16x+64) = C^2$$

$$\text{or, } 5y^2-50y+125+6xy-30x+48y-240+2x^2+32x+128 = C^2$$

$$\text{or, } 5y^2+2x^2-2y+2x+6xy+13 = C^2$$

$\text{or, } 2x^2+5y^2+6xy-2y+2x = K$ where $K = C^2-13$ is the required solution.

9. **$(x-3y+4)\,dy + (7y-5x)\,dx = 0$**

Sol^n. Given differential equation is,

$$(x-3y+4)\,dy + (7y-5x)\,dx = 0$$

$$\text{or, } \frac{dy}{dx} = -\frac{(7y-5x)}{(x-3y+4)} \quad \text{.....(i)}$$

Put $y = X + h$ and $y = Y + K$ where h and K are constant

So that $dy = dX$ and $dy = dY$

$$\therefore \frac{dy}{dx} = \frac{dY}{dX}$$

Now equation (i) becomes,

$$\frac{dY}{dX} = -\frac{7(Y+K)-5(X+h)}{(X+h)-3(Y+K)+4}$$

$$\frac{dY}{dX} = \frac{(5X-7Y)+(5h-7K)}{(X-3Y)+(h-3K+4)} \quad \text{.....(ii)}$$

Let us choose h and K such that

$5h - 7K = 0$ and $h - 3K + 4 = 0$

Solving these we get,

$h = \frac{7}{2}$ and $K = \frac{5}{2}$

Now, equation (ii) reduce to

$$\frac{dY}{dX} = \frac{(5X-7Y)}{X-3Y} \quad \text{.....(iii)}$$

Equation (iii) is homogenous differential equation

So, put $Y = vX$

Then $dY = v + X\frac{dv}{dX}$

Now equation (iii) becomes,

$$v + X\frac{dv}{dX} = \frac{(5X-7vX)}{(X-3vX)} = \frac{(5-7v)}{(1-3v)}$$

$$X\frac{dv}{dX} = \frac{5-7v}{1-3v} - v$$

$$= \frac{5-7v-v+3v^2}{1-3v} = \frac{3v^2-8v+5}{1-3v}$$

$$\text{or, } \int\frac{1-3v}{3v^2+8v+5}dv = \int\frac{1}{X}dX; \text{ Integrating}$$

$$\text{or, } \int\frac{(1-3v)}{(v-1)(3v-5)}dv = \int\frac{1}{X}dX$$

$$\text{or, } \int\left(\frac{1}{v-1}-\frac{6}{3v-5}\right)dv = \int\frac{1}{X}dX$$

$$\text{or, } \log(v-1) - 2\log(3v-5) = \log X + \log C$$

$$\text{or, } \frac{(v-1)}{(3v-5)^2} = XC$$

$$\text{or, } (v-1) = (3v-5)^2 XC$$

Restoring the value of $v = \frac{Y}{X}$ then, we get,

$$\left(\frac{Y}{X}-1\right)=\left(\frac{3Y}{X}-5\right)^2 XC$$

or, $(Y-X)=(3Y-5X)^2 C$

or, $\left[\left(y-\frac{5}{2}\right)-\left(x-\frac{7}{2}\right)\right]=\left[3\left(y-\frac{5}{2}\right)-5\left(x-\frac{7}{x}\right)\right]^2 C$

or, $(y-x+1)=\frac{(6y-10x-15+35)^2}{2}C$

$$=\frac{(6y-10x+20)^2}{2}C$$

$$=\left\{\frac{2(3y-5x+10}{2}\right\}^2 C$$

$\Rightarrow (3y-5x+10)^2 C=(y-x+1)$ is the required solution.

18. $(x-y)\,dy-(x+y+1)\,dx=0$

Sol^n. Given differential equation is,

$(x-y)\,dy-(x+y+1)\,dx=0$

or, $\frac{dy}{dx}=\frac{(x+y+1)}{(x-y)}$ ..... (i)

Put $x=X+h$ and $y=Y+K$

Where h and K are constant,

So, that $dx=dX$ and $dy=dY$

$\therefore \frac{dy}{dx}=\frac{dY}{dX}$

Now equation (i) becomes,

$\frac{dY}{dX}=\frac{X+h+Y+K+1}{X+h-Y-K}=\frac{X+Y+(h+K+1)}{X-Y+(h-K)}$ ..... (ii)

Let us choose h and K such that,

$h+k+1=0$ and $h-k=0$

Solving these $h=-\frac{1}{2}$ and $K=-\frac{1}{2}$

Hence equation (ii) becomes,

$\frac{dY}{dX}=\frac{X+Y}{X-Y}$ ..... (iii)

Equation (iii) is homoenous differential equation

So, put $Y=vX$

Then, $\frac{dY}{dX}=v+X\frac{dv}{dX}$

Now equation (iii) becomes,

$v+X\frac{dv}{dX}=\frac{X+vX}{X-vx}=\frac{1+v}{1-v}$

or, $X\frac{dv}{dX}=\frac{1+v}{1-v}-v=\frac{1+v-v+v^2}{1-v}=\frac{1+v^2}{1-v}$

or, $\int\frac{1-v}{1+v^2}dv=\int\frac{1}{X}\,dX$; Integrating

or, $\int\frac{1}{1+v^2}dv-\int\frac{v}{1+v^2}dv=\int\frac{1}{X}dX$

or, $\tan^{-1}v-\frac{1}{2}\log(1+v^2)=\log X+\log C$

or, $\tan^{-1}v=\log\left(\sqrt{1+v^2}\right)+\log XC$

$\tan^{-1}v=\log\left\{\sqrt{1+v^2}XC\right\}$

Restoring the value of $v=\frac{Y}{X}$ we get,

$\tan^{-1}\left(\frac{Y}{X}\right)=\log\left\{\sqrt{1+\frac{Y^2}{X^2}}.XC\right\}$

or, $\tan^{-1}\left(\frac{Y}{X}\right)=\log\left\{\sqrt{X^2+Y^2}.C\right\}$

Put $Y=y-K=y+\frac{1}{2}$ and $X=x-h+x+\frac{1}{2}$

Then $\tan^{-1}\left(\frac{y+\frac{1}{2}}{x+\frac{1}{2}}\right)=\frac{1}{2}\log\left\{C^{\frac{1}{2}}\left[\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2\right]\right\}$

or, $\tan^{-1}\left(\frac{2y+1}{2x+1}\right)=\frac{1}{2}\log\left[K\left\{\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2\right\}\right]$

Where $K=C^{\frac{1}{2}}$ is the required solution.

First Order Linear Differential Equation

An equation is of the form $\frac{dy}{dx} + py = Q$ where P and Q are function of x or constant is called first order linear differential equation.

To solve the linear differential equation. We multiply $e^{\int pdx}$ on the both sides of the given differential equation.

So the equation becomes,

$$e^{\int pdx}\frac{dy}{dx} + py\,e^{\int pdx} = Qe^{\int pdx}$$

or, $\frac{d}{dx}\left(y \times e^{\int pdx}\right) = Q^{\int pdx}$ Integrating

or, $\int d\left(y \times e^{\int pdx}\right) = \int Qe^{\int pdx}$ Integrating

or, $y \times e^{\int pdx} = \int Qe^{\int pdx}\,dx + C$

or, $y \times e^{\int pdx} = \int Qe^{\int pdx} + C$ is the required solution.

So its formula is remembered as $y \times e^{\int pdx} = \int Qe^{\int pdx}dx + c$

The factor $e^{\int pdx}$ is called Integrating factor i.e. I.F. = $e^{\int pdx}$

So, $Y \times I.F = \int (Q \times I.F)\,dx + C$

Similarly the first ordor differential equation is of the form $\frac{dx}{dy} + px = Q$ Where p and Q are function of y or constant is called linear differential equation in x and its I. F = $e^{\int pdy}$ Then its solution is $x \times I.F = \int (Q \times I.F.)\,dy + c$.

Exercise - 23

Solve the following differential equation.

1. $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$

Solution:

Given differential equation is

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2} \quad \text{.....(1)}$$

eq^n (1) is linear differential eq^n form

$$P = \frac{2x}{1+x^2}, Q = \frac{1}{(1+x^2)^2}$$

$$\therefore I.F = e^{\int pdx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = (1+x^2)$$

Multiplying eq^n (1) by I.F. we get

$$y.(x^2+1) = \int \frac{1}{(1+x^2)^2}.(1+x^2) + c$$

$$= \int \frac{1}{(1+x^2)}dx + c$$

$y.(x^2+1) = \tan^{-1}x$ is the required solution.

2. $\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^2}$

Sol^n. Given differential equation is

$$\frac{dy}{dx} + \frac{y}{x^2}y = \frac{1}{x^2} \quad \text{.....(1)}$$

eq^n(1) is linear differential eq^n form,

$$P = \frac{1}{x^2}, Q = \frac{1}{x^2}$$

$$\therefore I.F = e^{\int \frac{1}{x^2}dx} = e^{-\frac{1}{x}}$$

multiplying eq^n (1) by I.F. we get

$$y.e^{-\frac{1}{x}} = \int \frac{1}{x^2}e^{-\frac{1}{x}}dx + C$$

$$= -\int e^{-\frac{1}{x}}d\left(\frac{1}{x}\right) + c$$

$$ye^{-\frac{1}{x}} = e^{-\frac{1}{x}} + c$$

or, $y = 1 + ce^{\frac{1}{x}}$ is the required solution.

3. $(1-x^2)\frac{dy}{dx} - xy = 1$

Sol^n. Given differential eq^n is

$(1-x^2)\frac{dy}{dx}-xy=1$

or, $\frac{dy}{dx}-\frac{x}{1-x^2}.y=\frac{1}{1-x^2}$ .....(1)

eq^n(1) is linear differential eq^n form

$p=\frac{-x}{1-x^2}, Q=\frac{1}{1-x^2}$

$\therefore$ I.F. $= e^{-\int\frac{x}{1-x^2}dx} = e^{\frac{1}{2}\log(1-x^2)} = \sqrt{(1-x^2)}$

Multiplying eq^n (1) by I.F. we get

$y\sqrt{(1-x^2)} = \int\frac{1}{(1-x^2)}.\sqrt{(1-x^2)}dx+c$

$= \int\frac{1}{\sqrt{1-x^2}}dx+c$

$y\sqrt{1-x^2}=\sin^{-1}x+c$ is the required solution.

4. $\frac{dy}{dx}+2y=4x$

Sol^n. Given differential equation is

$\frac{dy}{dx}+2y=4x$ .....(1)

eq^n (1) is linear differential equation form

$p=2, Q=4x$

$\therefore$ I.F. $= e^{\int 2dx} = e^{2x}$

Multiplying eq^n (1) my I.F. we get

$y.e^{2x} = \int 4x.e^{2x}dx.+c$

$= 4x.\frac{e^{2x}}{2}-4\int 1.\frac{e^{2x}}{2}dx+c$

$= 2xe^{2x}-2\int e^{2x}dx+c$

$= 2x\,e^{2x}-2\frac{e^{2x}}{2}+c$

or, $y.e^{2x}=2xe^{2x}-e^{2x}+c$

or, $y=2x-1+ce^{-2x}$ is the required sol^n

5. $(1+x)\frac{dy}{dx}-xy=1-x$

Sol^n. Given differential equation is

$(1+x)\frac{dy}{dx}-xy=1-x$

or, $\frac{dy}{dx}-\frac{x}{1+x}.y=\frac{1-x}{1+x}$ .....(1)

eq^n (1) is linear differential equation form

$P=\frac{-x}{1+x}, Q=\frac{1-x}{1+x}$

$\therefore$ I.F $= e^{\int\frac{-x}{1+x}dx} = e^{\int\left(-1+\frac{1}{1+x}\right)dx}$

$= e^{-x+\log x} = e^{-x}.e^{\log(1+x)} = e^{-x(1+x)}$

Multiplying eq^n (1) by I.F. we get

or, $y.e^{-x}(1+x) = \int\frac{(1-x)}{1+x)}(1+x)e^{-x}dx+c$

$= \int(1-x)e^{-x}dx+c$

$= \int e^{-x}dx-\int xe^{-x}dx+c$

$= -e^{-x}-\frac{xe^{-x}}{-1}+\int 1.\frac{e^{-x}}{-1}dx+c$

$y.e^{-x}(1+x) = -e^{-x}+xe^{-x}+e^{-x}+c$

or, $ye^{-x}(1+x) = xe^{-x}+c$

or, $y(1+x) = x+ce^{x}$ is the required solution.

6. $(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$

Sol^n. Given differential equation is

$(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$

or, $\frac{dy}{dx}+\frac{1}{1+x^2}.y=\frac{e^{\tan^{-1}x}}{1+x^2}$ .....(1)

eq^n (1) is linear differential equation form

$P=\frac{1}{1+x^2}, Q=\frac{e^{\tan^{-1}x}}{1+x^2}$

I.F. $= e^{\int pdx} = e^{\int\frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$

Multiplying eq^n (1) by I.F. we get

$y.\,e^{\tan^{-1}x} = \int\frac{e^{\tan^{-1}x}}{1+x^2}.e^{\tan^{-1}x}dx+c$

$= \int\frac{e^{2\tan^{-1}x}}{(1+x^2)}dx+c$

Put $\tan^{-1}x=t$

Then $\left(\frac{1}{1+x^2}\right)dx = dt$

$\therefore y e^{\tan^{-1}x} = \int e^{2t}dt + c = \frac{1}{2}e^{2t} + c = \frac{1}{2}e^{2\tan^{-1}x} + c$

or, $y e^{\tan^{-1}x} = \frac{1}{2}e^{2\tan^{-1}x} + c$

or, $2 y e^{\tan^{-1}x} = e^{2\tan^{-1}x} + 2C$

$2y e^{\tan^{-1}x} = e^{2\tan^{-1}x} + K$ Where $K = 2c$ is the required solution.

7. $\mathbf{x\cos x \frac{dy}{dx} + y(x\sin x + \cos x) = 1}$

Solⁿ. Given differential equation is

$x \cos x \frac{dy}{dx} + y(x \sin x + \cos x) = 1$

or, $\frac{dy}{dx} + \frac{(x\sin x + \cos x)}{x\cos x}.y = \frac{1}{x\cos x}$ .....(1)

eqⁿ (1) is linear differential equation form

$P = \frac{(x\sin x + \cos x)}{x\cos x}, Q = \frac{1}{x\cos x}$

$\therefore I.F. = e^{\int pdx} = e^{\int \frac{(x\sin x + \cos x)}{x\cos x}dx}$

$= e^{\int\left(\tan x + \frac{1}{x}\right)dx} = e^{(\log\sec x + \log x)} = e^{\log x\sec x}$

$= x\sec x$

Multiplying eqⁿ (1) by I. F we get

$y . x\sec x = \int \frac{1}{x\cos x}.x\sec x dx + c$

$= \int \sec^2 x dx + C$

or, $xy \sec x = \tan x + c$ is the required solution

8. $\mathbf{\cos^2 x \frac{dy}{dx} + y = \tan x}$

solⁿ. Given differential equation is $\cos^2 x \frac{dy}{dx} + y = \tan x$

or, $\frac{dy}{dx} + \frac{1}{\cos^2 x}.y = \frac{\tan x}{\cos^2 x}$

or, $\frac{dy}{dx} + \sec^2 x.y = \tan x \sec^2 x$ .....(1)

eqⁿ (1) is linear differential eqⁿ form, $p = \sec^2 x$. $Q = \sec^2 x$

$\therefore I. F = e^{\int pdx} = e^{\int \sec^2 x dx} = e^{\tan x}$

Multiplying eqⁿ (1) by I.F. we get

$y. e^{\tan x} = \int \tan x. \sec^2 x . e^{\tan x} dx + c$

Put $\tan x = t$ then $\sec^2 dx = dt$

$\therefore y. e^{\tan x} = \int te^t dt + c$

$= te^t - e^t + c$

or, $ye^{\tan x} = \tan x\, e^{\tan x} - e^{\tan x} + c$

or, $(y - \tan x + 1) e^{\tan x} = c$ is the required solution.

9. $\mathbf{\sin x \frac{dy}{dx} + y\cos x = x\sin x}$

Solⁿ. Given differential equation is

$\sin x \frac{dy}{dx} + y\cos x = x\sin x$

or, $\frac{dy}{dx} + \frac{\cos x}{\sin x}.y = \frac{x\sin x}{\sin x}$

or, $\frac{dy}{dx} + \cot x y = x$ .....(1)

eqⁿ (1) is linear differential equation form

$P = \cot x$, $Q = x$

$\therefore I. F = e^{\int pdx} = e^{\int \cot x\, dx} = \log \sin x$

$= \sin x$

Multiplying eqⁿ (1) by I. F. we get

$y. \sin x = \int x \sin x\, dx + c$

$= -x\cos x + \int \cos x\, dx + c$

$y\sin x = -x\cos x + \sin x + c$

or, $y\sin x + x\cos x - \sin x = c$ is the required solution.

10. $\mathbf{\frac{dy}{dx} + y\cot x = 2\cos x}$

Solⁿ. Given differential equation is

$\frac{dy}{dx} + y\cot x = 2\cos x$ .....(1)

eqⁿ (1) is linear differential equation

$P = \cot x$, $Q = 2\cos x$

$\therefore I.F = e^{\int pdx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$

Multiplying eqⁿ (1) by I.F. we get

$y \times \sin x = \int 2\cos x \sin x\, dx + c$

or, $y\sin x = \int \sin 2x\, dx + c$

$= -\frac{\cos 2x}{2} + c$

or, $2y\sin x + \cos 2x + 2c$

$2y\sin x = k - \cos 2x$ where $K = 2c$ is the required solution

11. $\frac{dy}{dx} + y\tan x = \sec x$

Sol^n. Given differential equation is

$\frac{dy}{dx} + y\tan x = \sec x$ .....(1)

eq^n (1) is linear differential equation form

$P = \tan x,\ Q = \sec x$

$\therefore I.F = e^{\int p dx}$

$= e^{\int \tan x\, dx} = e^{\log \sec x}$

$= \sec x$

Multiplying eq^n (1) by I.F. we get

$y.\sec x = \int \sec x.\sec x\, dx + c$

$= \int \sec^2 x\, dx + c$

or, $y\sec x = \tan x + C$

$y = \sin x + c\cos x$ is the required solution.

12. $x\log x\frac{dy}{dx} + y = 2\log x$ **[B.E. 2061]**

Sol^n. Given differential equation is

$x\log x\frac{dy}{dx} + y = 2\log x$

or, $\frac{dy}{dx} + \frac{1}{x\log x}.y = \frac{2\log x}{x\log x}$

or, $\frac{dy}{dx} + \frac{1}{x\log x}.y = \frac{2}{x}$ .....(1)

eq^n (1) is linear differential equation form

$P = \frac{1}{x\log x}.\ Q = \frac{2}{x}$

$\therefore I.F = e^{\int p dx} = e^{\int \frac{1}{x\log x}dx}$

Put $\log x = t$ Then $\frac{1}{x}dx = dt$

$\therefore I.F. = e^{\int \frac{1}{t}dt} = e^{\log t} = t = \log x$

Multiplying eq^n (1) by I.F. we get,

$y.\log x = \int \frac{2}{x}\log x dx + c$

$= 2\int \frac{\log x}{x}dx + c$

Put $\log x = t$ Then $\frac{1}{x}dx = dt$

$y\log x = 2\int t\, dt + c$

$= \frac{2t^2}{2} + c$

or, $y\log x = (\log x)^2 + c$ is the required solution.

13. $\frac{dy}{dx} + y = \cos x$

Sol^n. Given differential equation is

$\frac{dy}{dx} + y = \cos x$ ....(1)

eq^n (1) is linear differential equation form,

$P = 1,\ Q = \cos x$

$\therefore I.F = e^{\int p dx} = e^x$

Multiplying eq^n (1) by I.F. we get

$ye^x = \int \cos x\, e^x dx + c$

$= \int e^x \cos x\, dx$

$ye^x = e^x\frac{(\cos x + \sin x)}{2} + c$

$2y = \cos x + \sin x + 2ce^{-x}$

or, $2y = \cos x + \sin x + ke^{-x}$ where $K = 2c$ is the required solution.

14. $(1 + y^2)\, dx = (\tan^{-1}y - x)\, dy$

Sol^n. Given differential equation is

$(1 + y^2)\, dx = (\tan^{-1}y - x)\, dy$

or, $\frac{dx}{dy} = \frac{(\tan^{-1}y - x)}{1+y^2}$

or, $\frac{dy}{dx} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$

or, $\frac{dy}{dx} + \frac{1}{1+y^2}.x = \frac{\tan^{-1}y}{1+y^2}$ .....(1)

eq^n (1) is linear differential equation of the form $\frac{dx}{dy} + px = Q$

Here, $P = \frac{1}{1+y^2}.\ Q = \frac{\tan^{-1}y}{1+y^2}$

$\therefore I.F. = e^{\int p dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

Multiplying eq^n (1) by I.F. we get,

of, $e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y} dy + c$ put $\tan^{-1}y = t$ Then

$\frac{1}{1+y^2} dy = dt$

or, $xe^{\tan^{-1}y} = \int te^t dt + c$
$= te^t - e^t + c$

$xe^{\tan^{-1}y} = \tan^{-1}y\ e^{\tan^{-1}y}$ is the required solution.

15. $(1+y^2) + \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = 0$

Solⁿ. Given differential equation is

$(1+y^2) + \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = 0$

or, $\left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = -(1+y^2)$

$\frac{dy}{dx} = \frac{-(1+y^2)}{(x - e^{\tan^{-1}}y)}$

$\therefore \frac{dx}{dy} = \frac{-(x - e^{\tan^{-1}}y)}{(1+y^2)}$

or, $\frac{dx}{dy} = -\frac{x}{1+y^2} + \frac{e^{\tan^{-1}}y}{1+y^2}$

or, $\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1}}y}{1+y^2}$ .....(1)

eqⁿ (1) is linear differential equation is of the form $\frac{dx}{dy} + px = Q$

$P = \frac{1}{1+y^2}, Q = \frac{e^{\tan^{-1}}y}{1+y^2}$

$\therefore$ I. F $= e^{\int pdy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

Multiplying eqⁿ (1) by I.F. we get,

$x.e^{\tan^{-1}}y = \int \frac{e^{\tan^{-1}}y}{1+y^2} e^{\tan^{-1}} y\, dy + c$

$= \int \frac{e^{2\tan^{-1}}y}{1+y^2} dy + c$

Put $\tan^{-1}y = t$ Then $\frac{1}{1+y^2}dy = dt$

$\therefore xe^{\tan^{-1}}y = \int e^{2t}dt + c$
$= \frac{1}{2}e^{2t} + c$

or, $xe^{\tan^{-1}}y = \frac{1}{2}e^{2\tan^{-1}}y + c$

or, $2xe^{\tan^{-1}}y = e^{2\tan^{-1}y} + 2c$

or, $2xe^{\tan^{-1}}y = e^{2\tan^{-1}y} + K$

Where $K = 2c$ is the required solution.

16. Solve: $\frac{dy}{dx} + \frac{y}{x} = x^2$ **If = 1, When x = 1**

Solⁿ: Given differential equation is

$\frac{dy}{dx} + \frac{1}{x}y = x^2$ .....(1)

eqⁿ(1) is linear differential equation form

$p = \frac{1}{x}, Q = x^2$

$\therefore$ I. F. $= e^{\int pdx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Multiplying eqⁿ (1) by I. F. we get,

$y.x = \int x^2 x dx + c$

or, $xy = \int x^3 dx + c$

$xy = \frac{x^4}{4} + c$

$4xy = x^4 + 4c$ .....(2)

When $x = 1, y = 1$

$4 = 1 + 4c$

$\Rightarrow c = \frac{3}{4}$

Now (2) becomes

$4xy = x^4 + 4.\frac{3}{4}$

or, $4xy = x^4 + 3$ is the required solution.

Bernoulli's equation

An equation is of the form $\frac{dy}{dx} + py = Qy^n$ Where P and Q are function of x or constant is called Bernoulli's equation. It can be solved by reducing it to linear equation.

For this $\frac{dy}{dx} + py = Qy^n$ .....(1)

Dividing both sides by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{y}{y^n}P = Q$$

or, $\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}p = Q$

Put $\frac{1}{y^{n-1}} = v$

$y^{-n+1} = v$ differentiating it w.r. to x

$$(-n+1)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$$

or, $\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(-n+1)}\frac{dv}{dx}$

So the eq^n (1) reduces to

$$\frac{1}{(-n+1)}\frac{dv}{dx} + pv = v$$

$$\frac{dv}{dx} + P(-n+1)v = (-n+1)Q \text{ .....(2)}$$

Thus eq^n (2) is linear differential eq^n of first order in v so its integrating factor (I. F.) = $e^{\int p(-n+1)dx}$

and its general solution is

$v \times I.F. = \int(-n+1)Q \times (I.F)\,dx + c$

Exercise - 24

Solve the following differential equations

1. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Sol^n. Given differential equation is

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$$

Dividing both sides by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\cdot\frac{1}{x} = \frac{1}{x^2} \text{ .....(1)}$$

Put $\frac{1}{y} = v$ Then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{1}{y^2}\frac{dy}{dx} = -\frac{dv}{dx}$

Now eq^n (1) becomes

$$-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x^2}$$

or, $\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x^2}$ .....(2)

eq^n (2) is linear differential eq^n. form

$P = -\frac{1}{x}, Q = -\frac{1}{x^2}$

$$I.F. = e^{\int pdx} = e^{-\int\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Multiplying eq^n (2) by I.F. we get

$$v.\frac{1}{x} = -\int\frac{1}{x^2}\cdot\frac{1}{x}dx + c$$

$$= -\int\frac{1}{x^3}dx + c$$

$$\frac{v}{x} = \frac{1}{2x^2} + c$$

Restoring the value of $v = \frac{1}{y}$ we get

$$\frac{1}{xy} = \frac{1}{2x^2} + c$$

or, $2x = y + 2x^2y\,.\,c$

or, $2x = y + 2cx^2y$ is the required solution.

2. $(1-x^2)\frac{dy}{dx} + xy = xy^2$

Sol^n. Given differential equation is

$$(1-x^2)\frac{dy}{dx} + xy = xy^2$$

or, $\frac{dy}{dx}+\frac{x}{1-x^2}y=\frac{x}{1-x^2}.y^2$

Dividing both sides by y^2

$\frac{1}{y^2}\frac{dy}{dx}+\frac{x}{1-x^2}.\frac{1}{y}=\frac{x}{1-x^2}$ .....(1)

Put $\frac{1}{y}=v$ Then $-\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$

Now eq^n (1) becomes

$-\frac{dv}{dx}+\frac{x}{1-x^2}v=\frac{x}{1-x^2}$ .....(2)

or, $\frac{dv}{dx}-\frac{x}{1-x^2}v=-\frac{x}{1-x^2}$ .....(2)

eq^n (2) is lineal differential equation form.

Here, $P=-\frac{x}{1-x^2}, Q=-\frac{x}{1-x^2}$

$\therefore$ I. F. $=e^{\int pdx}=e^{\int-\frac{x}{1-x^2}dx}=e^{\frac{1}{2}\log(1-x^2)}$

$=e^{\log\left(\sqrt{1-x^2}\right)}=\sqrt{1-x^2}$

Multiplying eq^n (2) by I.F. we get,

$v.\sqrt{1-x^2}=-\int\frac{x}{(1-x^2)}\sqrt{1-x^2}dx+c$

$-\int\frac{x}{\sqrt{1-x^2}}dx+c$

$v\sqrt{1-x^2}=\sqrt{1-x^2}+c$

Restoring the value of $v=\frac{1}{y}$ we get

$\frac{1}{y}\sqrt{1-x^2}=\sqrt{1-x^2}+c$

$\sqrt{1-x^2}=y\sqrt{1-x^2}-cy$

$\left(\sqrt{1-x^2}\right)(1-y)=cy$ is the required solution.

3. $\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$

Sol^n. Given differential equation is

$\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$

Dividing both sides by e^y

$\frac{1}{e^y}\frac{dy}{dx}+\frac{1}{e^y}.\frac{1}{x}=\frac{1}{x^2}$ .....(1)

Put $\frac{1}{e^y}=v$

$e^{-y}=v$ Then $-e^{-y}\frac{dy}{dx}=\frac{dv}{dx}$

or, $\frac{1}{e^y}\frac{dy}{dx}=\frac{-dv}{dx}$

Now (1) becomes

$-\frac{dv}{dx}+\frac{1}{x}v=\frac{1}{x^2}$

or, $\frac{dv}{dx}-\frac{1}{x}v=\frac{-1}{x^2}$ .....(2)

eq^n (2) is linear differential equation form

$P=-\frac{1}{x}, Q=-\frac{1}{x^2}$

$\therefore$ I.F $=e^{\int pdx}=e^{\int-\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$

Multiplying eq^n (2) by I.F. we get,

$v.\frac{1}{x}=\int-\frac{1}{x^2}.\frac{1}{x}dx+c$

$=-\int\frac{1}{x^3}dx+c$

or, $\frac{v}{x}=\frac{1}{2x^2}+c$

Restoring the value of $v=\frac{1}{e^y}$ we get

$\frac{1}{xe^y}=\frac{1}{2x^2}+c$

or, $2x=(1+2cx^2)e^y$

or, $2x=(1+kx^2)e^y$ where $K=2c$ is the required solution.

4. $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$

Sol^n. Given differential equation is

$\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$

dividing both sides by $y(\log y)^2$

$$\frac{1}{y(\log y)^2}\frac{dy}{dx}+\frac{1}{(\log y)}\frac{1}{x}=\frac{1}{x^2} \quad \text{.....(1)}$$

Put $\frac{1}{\log y}=Q$

or, $\frac{-1}{(\log y)^2}\cdot\frac{1}{y}\frac{dy}{dx}=\frac{dv}{dx}$

or, $\frac{1}{y(\log y)^2}\frac{dy}{dx}=-\frac{dv}{dx}$

Now eq^n (1) becomes\

$$\frac{-dv}{dx}+\frac{1}{x}v=\frac{1}{x^2}$$

or, $\frac{dv}{dx}-\frac{1}{x}v=\frac{-1}{x^2}$ .....(2)

eq^n (2) is linear differential equation form ;

Here, $P=\frac{-1}{x}, Q=-\frac{1}{x^2}$

$\therefore$ I. F. $=e^{\int pdx}=e^{-\int\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$

multiplying eq^n (2) by I.F. = we get

$$v.\frac{1}{x}=\int-\frac{1}{x^2}\cdot\frac{1}{x}dx+c$$

$$=-\int\frac{1}{x^3}dx+c$$

$$\frac{v}{x}=\frac{1}{2x^2}+c$$

Restoring the value of $v=\frac{1}{(\log y)}$ we get

$$\frac{1}{x\log y}=\frac{1}{2x^2}+c$$

or, $x=\log y\left(\left(\frac{1}{2}+cx^2\right)\right.$ is the required solution.

5. $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$ **[B. E. 2061]**

Sol^n. Given differential equation is

$$\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$$

Dividing both sides by $\cos^2 y$ we get,

$$\frac{1}{\cos^2 y}\frac{dy}{dx}+2x.\tan y=x^3 \quad \text{.....(1)}$$

Put $\tan y=v$

Then $\sec^2 y\frac{dy}{dx}=\frac{dv}{dx}$

or, $\frac{1}{\cos^2 y}\frac{dy}{dx}=\frac{dv}{dx}$

Now eq^n (1) becomes

$$\frac{dv}{dx}+2xv=x^3 \quad \text{.....(2)}$$

eq^n (2) is linear differential equation form;

Here, $p=2x, Q=x^3$

$\therefore$ I.F. $=e^{\int pdx}=e^{\int 2xdx}=e^{x^2}$

Multiplying eq^n (2) by I.F. we get,

$$v.e^{x^2}=\int x^3e^{x^2}dx+c$$

$$=\int x.x^2e^{x^2}dx+c$$

put $x^2=t$ then $2xdx=dt$ or, $xdx=\frac{1}{2}dt$

$$\therefore ve^{x^2}=\frac{1}{2}\int te^t dt$$

$$=\frac{1}{2}\left[te^t-e^t\right]+c$$

$$ve^{x^2}=\frac{1}{2}\left(x^2e^{x^2}-e^{x^2}\right)+c$$

Restoring the value of $v=\tan y$ we get

$$\tan y\, e^{x^2}=\frac{1}{2}\left(x^2e^{x^2}-e^{x^2}\right)+c$$

or, $e^{x^2}\tan y=\frac{1}{2}e^{x^2}(x^2-1)+c$ is the required solution.

6. $\cos x\frac{dy}{dx}=y(\sin x-y)$

Sol^n. Given differential equation is

$$\cos x\frac{dy}{dx}=y(\sin x-y)$$

or, $\frac{dy}{dx}=\frac{(y\sin x-y^2)}{\cos x}=y\frac{\sin x}{\cos x}-\frac{y^2}{\cos x}$

or, $\frac{dy}{dx} = y\tan x - \frac{y^2}{\cos x}$

or, $\frac{dy}{dx} - y\tan x = \frac{-y^2}{\cos x}$ .....(1)

dividing both sides by $-y^2$

or, $\frac{-1}{y^2}\frac{dy}{dx} + \frac{1}{y}\tan x = \frac{1}{\cos x}$

Put $\frac{1}{y} = v$ Then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

Now eq^n (1) becomes

$\frac{dv}{dx} + \tan v = \frac{1}{\cos x}$ .....(2)

eq^n (2) is linear differential eq^n form $p = \tan x$, $Q = \frac{1}{\cos x}$

$\therefore$ I.F. $= e^{\int p dx} = e^{\int \tan x dx} = e^{\log \sec x}$

$= \sec x$

Multiplying eq^n (2) by I.F. we get

$v.\sec x = \int \frac{1}{\cos x}.\sec x dx + c$

$= \int \sec^2 x dx + c$

or, $v.\sec x = \tan x + c$

Restaring the value of $v = \frac{1}{y}$ we get

$\frac{1}{y}\sec x = \tan x + c$

$\sec x = y\tan x + cy$ is the required solution.

7. $x\frac{dy}{dx} + y = y^2 \log x$

Sol^n. Given differential equation is

$x\frac{dy}{dx} + y = y^2 \log x$

Dividing both sides by xy^2 we get,

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}.\frac{1}{y} = \frac{\log x}{x}$ .....(1)

put $\frac{1}{y} = v$ Then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{1}{y^2}\frac{dy}{dx} = -\frac{dv}{dx}$

Now eq^n (1) becomes

$-\frac{dv}{dx} + \frac{1}{x}.v = \frac{\log x}{x}$

or, $\frac{dv}{dx} - \frac{1}{x}v = \frac{-\log x}{x}$ .....(2)

eq^n (2) is linear differential equation form

$p = \frac{-1}{x}$, $Q = \frac{\log x}{x}$

$\therefore$ I.F $= e^{\int p dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$

Multiplying eq^n (2) by I.F. we get

$v.\frac{1}{x} = -\int \frac{\log x}{x}.\frac{1}{x}dx + c$

or, $\frac{v}{x} = -\int \frac{1}{x^2}\log x dx + c$

$= -\left\{\log x\left(\frac{-1}{x}\right) - \int \frac{1}{x}.\left(\frac{-1}{x}\right)dx\right\} + c$

$= \frac{\log x}{x} - \int \frac{1}{x^2}dx + c$

$\frac{v}{x} = \frac{\log x}{x} + \frac{1}{x} + c$

Restoring the value of $v = \frac{1}{y}$ we get

$\frac{1}{xy} = \frac{\log x}{x} + \frac{1}{x} + c$

or, $1 = y(\log x + 1) + cxy$

or, $cxy + y(\log x + 1) = 1$ is the required solution.

8. $\frac{dy}{dx} = y\tan x - y^2 \sec x$ **[B.E. 2060]**

Sol^n. Given differential equation is

$\frac{dy}{dx} = y\tan x - y^2 \sec x$

or, $\frac{dy}{dx} - y\tan x = -y^2 \sec x$

Dividing both sides by $-y^2$ we get

$\frac{-1}{y^2}\frac{dy}{dx} + \tan x.\frac{1}{y} = \sec x$ .....(1)

Put $\frac{1}{y} = v$ then

$$\frac{-1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$$

Now eq^n (1) becomes,

$$\frac{dv}{dx}+\tan x.v=\sec x \text{ .....(2)}$$

eq^n (2) is limear differential equation form $p=\tan x$, $Q=\sec x$

$$\therefore \text{I. F.}=e^{\int p dx}=e^{\int \tan x dx}=e^{\log \sec x}=\sec x$$

Multiplying eq^n (2) by I.F. we get,

$$v.\sec x=\int \sec x.\sec x dx+c$$

$$=\int \sec^2 x dx+c$$

$$v.\sec x=\tan x+c$$

Restoring the value of $v=\frac{1}{y}$ we get,

$$\frac{\sec x}{y}=\tan x+c$$

or, $\sec x = y\tan x + cy$ is the required solution.

9. $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x \sec y.$

Sol^n. Given differential equation is

$$\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x \sec y.$$

dividing both sides by secy

$$\text{or, } \frac{1}{\sec y}\frac{dy}{dx}-\frac{1}{1+x}\sin y=(1+x)e^x \text{ .....(1)}$$

Put siny = v then cosy $\frac{dy}{dx}=\frac{dv}{dx}$

$$\text{or, } \frac{1}{\sec y}\frac{dy}{dx}=\frac{dv}{dx}$$

Now eq^n (1) becomes

$$\frac{dv}{dx}-\frac{1}{1+x}v=(1+x)e^x \text{ .....(2)}$$

eq^n (2) is linear differential equation form

$$p=\frac{-1}{1+x}, Q=(1+x)e^x$$

$$\therefore \text{I. F.}=e^{\int p dx}=e^{\int -\frac{1}{1+x}dx}=e^{-\log(1+x)}$$

$$=e^{\log(1+x)^{-1}}=(1+x)^{-1}=\frac{1}{(1+x)}$$

Multiplying eq^n (2) by I. F. we get

$$v.\frac{1}{(1+x)}=\int e^x(1+x).\frac{1}{(1+x)}dx+c=\int e^x dx+c$$

$$\frac{v}{(1+x)}=e^x+c$$

Restoring the value of v = siny we get

$$\frac{\sin y}{(1+x)}=e^x+c$$

or, siny = (1 + x) (e^x + c) is the required solution

10. $\frac{dy}{dx}+\frac{1}{x}\sin 2y=x^2\cos^2 y$

Sol^n. Given differential equation is

$$\frac{dy}{dx}+\frac{1}{x}\sin 2y=x^3\cos^2 y$$

dividing both sides by $\cos^2 y$

$$\frac{1}{\cos^2 y}\frac{dy}{dx}+\frac{2}{x}\tan y=x^3 \text{ ..... (1)}$$

Put tany = v Then $\sec^2 y\ \frac{dy}{dx}=\frac{dv}{dx}$

$$\text{or, } \frac{1}{\cos^2 y}\frac{dy}{dx}=\frac{dv}{dx}$$

Now eq^n (1) becomes

$$\frac{dv}{dx}+\frac{2}{x}v=x^3 \text{ .....(2)}$$

Equation (2) is linear differential equation form

$$p=\frac{2}{x} \text{ and } Q=x^3$$

$$\therefore \text{I. F.}=e^{\int p dx}=e^{\int \frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$$

Multiplying eq^n (2) by I.F. we get,

$$v.x^2=\int x^3.x^2 dx+c$$

$$=\int x^5 dx+c$$

$$vx^2=\frac{x^6}{6}+c$$

Restoring the value of v = tany we get

$$\tan y.\ x^2=\frac{x^6}{6}+c$$

or, $6x^2 \tan y = x^6 + 6c$

or, $6x^2 \tan y = x^6 + K$ where K = 6c is the required solution.

Exact Differential equation

An equation $Mdx + Ndy = 0$ where M and N are function of x and y is said to be exact differential equation if there exist a function $U(x, y)$ such that $Mdx + Ndy = d\{u(x, y)\}$
i.e. If $Mdx + Ndy$ is perfect differential

Some exact differential equation

1. $xdy + y\,dx = d(xy)$
2. $\dfrac{xdy + ydx}{xy} = d\{\log(xy)\}$
3. $(xy)^n(xdy + ydx) = \dfrac{1}{(n+1)}d(xy)^{n+1}$
4. $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$
5. $\dfrac{xdy - ydx}{y^2} = -d\left(\dfrac{x}{y}\right)$
6. $\dfrac{xdy - ydx}{xy} = d\left\{\log\left(\dfrac{y}{x}\right)\right\}$
7. $\dfrac{xdy - ydx}{x^2 + y^2} = d\tan^{-1}\left(\dfrac{y}{x}\right)$
8. $\dfrac{ydx - xdy}{x^2 + y^2} = d\tan^{-1}\left(\dfrac{x}{y}\right)$
9. $\dfrac{xdx + ydy}{2} = \dfrac{1}{2}d(x^2 + y^2)$
10. $\dfrac{xdx + ydy}{x^2 + y^2} = \dfrac{1}{2}\left\{d\log(x^2 + y^2)\right\}$
11. $\dfrac{dx + dy}{x + y} = d\{\log(x + y)\}$
12. $\dfrac{2xydx - x^2dy}{y^2} = d\left(\dfrac{x^2}{y}\right)$
13. $\dfrac{2xydy - y^2dx}{x^2} = d\left(\dfrac{y^2}{x}\right)$

Exercise - 25

Solve the following equations

1. $(2ax + by)\,y\,dx + (ax + 2by)\,x\,dy = 0$

Solⁿ. Given differential equation is
$(2ax + by)\,y\,dx + (ax + 2by)\,x\,dy = 0$
or, $2axy\,dx + by^2\,dx + ax^2dy + 2bxydy = 0$
or, $2axydx + ax^2dy + by^2dx + 2bxy\,dy = 0$
or, $a\int d(x^2y) + b\int d(y^2x) = \int odx$ Integrating
or, $ax^2y + by^2x = c$ is the required solution.

2. $(x^2 - ay)dx - (ax - y^2)\,dy = 0$

Solⁿ. Given differential equation is
$(x^2 - ay)dx - (ax - y^2)\,dy = 0$
or, $x^2dx - aydx - axdy + y^2dy = 0$
or, $x^2dx + y^2dy - a(ydx + xdy) = 0$
or, $\int x^2dx + \int y^2dy - a\int d(xy) = \int odx$ Integrating
or, $\dfrac{x^3}{3} + \dfrac{y^3}{3} - axy = c$
or, $x^3 + y^3 - 3axy = 3c$
or, $x^3 + y^3 - 3axy = K$ where K is constant is the required solution.

3. $\dfrac{dy}{dx} = \dfrac{2x - y + 1}{x + 2y - 3}$

Solⁿ. Given differential equation is
$\dfrac{dy}{dx} = \dfrac{2x - y + 1}{x + 2y - 3}$
or, $xdy + 2ydy - 3dy = 2x\,dx - ydx + dx$
or, $xdy + ydx = 2x\,dx - 2ydy + dx + 3dy$
or, $\int d(xy) = \int 2xdx - \int 2ydy + \int dx + \int 3dy$; Integrating
$xy = \dfrac{2x^2}{2} \dfrac{-2y^2}{2} + x + 3y + c$
or, $xy = x^2 - y^2 + x + 3y + c$
or, $y^2 - x^2 + xy - 3y - x = c$ is the required solution.

4. $(x + y)\,dy + (y - x)\,dx = 0$

Solⁿ. Given differential equation is
$(x + y)\,dy + (y - x)\,dx = 0$
or, $xdy + ydy + ydx - xdx = 0$
or, $xdy + ydx = xdx - ydy$
or, $\int d\,(xy) = \int xdx - \int ydy$ Integrating

or, $xy = \frac{x^2}{2} - \frac{y^2}{2} + c$

or, $2xy = x^2 - y^2 + 2c$

or, $x^2 - y^2 - 2xy = -2c$

or, $x^2 - y^2 - 2xy = k$ where $k = -2c$ is the required solution.

5. $\mathbf{2xydx - (x^2 - y^2)\,dy = 0}$

Sol^n. Given differential equation is

$2xydx - (x^2 - y^2)\,dy = 0$

or, $2xydx = (x^2 - y^2)\,dy$

or, $2xydx = x^2dy - y^2dy$

or, $2xydx - x^2dy = -y^2dy$

Dividing both sides by y^2

$$\frac{2xydx - x^2dy}{y^2} = -dy$$

or, $\int d\left(\frac{x^2}{y}\right) = -\int dy$ Integrating

or, $\frac{x^2}{y} = -y + c$

$x^2 + y^2 = yc$ is the required solution.

6. $\mathbf{(x^2 + y^2 + 2x)dx + xy\,dy = 0}$

Sol^n. Given differential equation is

$(x^2 + y^2 + 2x)dx + xy\,dy = 0$

or, $x^2dx + y^2dx + 2xdx + xydy = 0$

or, $x^2dx + 2xdx + y^2dx + xydy = 0$

Multiplying both sides by x

$x^3dx + 2x^2dx + xy^2dx + x^2ydy = 0$

$\int x^3dx + \int 2x^2dx + \frac{1}{2}\int d(x^2y^2) = \int 0dx$ Integrating

or, $\frac{x^4}{4} + \frac{2x^3}{2} + \frac{1}{2}x^2y^2 = c$

or, $3x^4 + 8x^3 + 6x^2y^2 = 12c$

or, $3x^4 + 8x^3 + 6x^2y^2 = K$ where $K = 12c$ is the required solution

7. $\mathbf{x\frac{dy}{dx} + y = y^2\log x}$

Sol^n. Given differential equation is

$x\frac{dy}{dx} + y = y^2\log x$

or, $xdy + ydx = y^2\log x\,dx$

Dividing both sides by x^2y^2 we get,

or, $\left(\frac{1}{xy^2}\right)dy + \left(\frac{1}{x^2y}\right)dx = \left(\frac{1}{x^2}\log x\right)dx$

or, $-\left\{\frac{1}{x}\left(-\frac{1}{y^2}\right)dy + \frac{1}{y}\left(\frac{-1}{x^2}\right)dx\right\} = \left(\frac{1}{x^2}\log x\right)dx$

or, $-\int d\left(\frac{1}{x}\cdot\frac{1}{y}\right) = \int\frac{\log x}{x^2}dx$. Integrating

$\frac{1}{xy} = \int\frac{\log x}{x^2} + c$ .....(1)

or, $-\frac{1}{xy} = \log x\left(-\frac{1}{x}\right) - \int\frac{1}{x}\left(-\frac{1}{x}\right)dx + c$

or, $-\frac{1}{xy} = -\frac{\log x}{x} + \int\frac{1}{x^2}\,dx + c$

or, $\frac{-1}{xy} = -\frac{\log x}{x} - \frac{1}{x} + c$

or, $-1 = -y(\log x + 1) + cxy$

or, $y(\log x + 1) = cxy + 1$ is the required solution.

8. $\mathbf{xdy - ydx + a(x^2 + y^2)\,dx = 0}$

Sol^n. Given differential equation is

$xdy - ydx + a(x^2 + y^2)\,dx = 0$

or, $\frac{xdy - ydx}{x^2 + y^2} + adx = 0$

or, $\frac{\frac{xdy - ydx}{x^2}}{1 + \left(\frac{y}{x}\right)^2} + adx = 0$

or, $\int\frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} + a\int dx = \int 0dx$ Integrating

or $\tan^{-1}\left(\frac{y}{x}\right) + ax = c$ is the required solution.

9. $\mathbf{xdx + ydy + (x^2 + y^2)\,dy = 0}$

Sol^n. Given differential equation is

$xdx + ydy + (x^2 + y^2)\,dy = 0$

or, $\frac{xdx + ydy}{x^2+y^2} + dy = 0$

or, $\frac{1}{2}\int\frac{d(x^2+y^2)}{(x^2+y^2)} + \int dy = \int 0dx$ Integrating

or, $\frac{1}{2}\log(x^2+y^2) + y = 0$

or, $\log(x^2+y^2) + 2y = 2c$

or, $\log(x^2+y^2) + 2y = k$ where $k = 2c$ is the required solution.

10. $(1 + xy)\, y\, dx + (1 - xy)\, xdy = 0$

Sol^n. Given differential equation is

$(1 + xy)\, y\, dx + (1 - xy)\, xdy = 0$

or, $ydx + xy^2dx + xdy - x^2y\, dy = 0$

or, $ydx + xdy + xy^2dx - x^2ydy = 0$

dividing both sides by x^2y^2 we get.

$\frac{1}{x^2y}dx + \frac{1}{xy^2}dy + \frac{1}{x}dx - \frac{1}{y}dy = 0$

or, $-\int d\left(\frac{1}{x}\cdot\frac{1}{y}\right) + \int\frac{1}{x}dx = \int\frac{1}{y}dy$; Integrating

or, $-\frac{1}{xy} + \log x = \log y + c$

or, $-\frac{1}{xy} + \log\left(\frac{x}{y}\right) = c$

or, $\log\left(\frac{x}{y}\right) = c + \frac{1}{xy}$ is the required solution.

11. $\sin x\, dy - y\cos x dx + y^2dx = 0$.

Sol^n. Given differential equation is

$\sin x\, dy - y\cos x dx + y^2dx = 0$

or, $y^2dx = y\cos x\, dx - \sin x\, dy$

dividing both sides by y^2 we get,

$dx = \frac{y\cos x dx - \sin x dy}{y^2}$

or, $\int dx = \int d\left(\frac{\sin x}{y}\right)$; Integrating

or, $x = \frac{\sin x}{y} + c$

or, $xy = \sin x + cy$ is the required solution.

12. $x\frac{dy}{dx} = y + x^2\log x$

Sol^n. Given differential equation is

or, $x\frac{dy}{dx} = y + x^2\log x$

or, $xdy = (y + x^2\log x)\, dx$

or, $xdy = ydx + x^2\log x\, dx$

or, $xdy - ydx = x^2\log x\, dx$

Dividing both sides by x^2 we get

or, $\frac{xdy - ydx}{x^2} = \log x dx$

or, $\int d\left(\frac{y}{x}\right) = \int\log x dx$ Integrating

or, $\frac{y}{x} = x\log x - x + c$

or, $y = x^2\log x - x^2 + cx$ is the required solution.

13. $x\cos\left(\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)(xdy - ydx)$

Sol^n. Given differential equation is

$x\cos\left(\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)(xdy - ydx)$

Dividing both sides by 'x' we get

$\cos\left(\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)\frac{(xdy - ydx)}{x}$

or, $\cos\left(\frac{y}{x}\right)(ydx + xdy) = xy\sin\left(\frac{y}{x}\right)\frac{(xdy - ydx)}{x^2}$

or, $\cos\left(\frac{y}{x}\right)(ydx + xdy) = xy\sin\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$

or, $\cos\left(\frac{y}{x}\right)d(xy) = (xy)\sin\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$

or, $\int\frac{1}{xy}d(xy) = \int\frac{\sin\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right)}d\left(\frac{y}{x}\right)$; Integrating

$\log(xy) = -\log\cos\left(\frac{y}{x}\right) + \log c$

$\log(xy) + \log\cos\left(\frac{y}{x}\right) = \log c$

or, $\log(xy)\cos\left(\frac{y}{x}\right) = \log c$

or, $(xy)\cos\left(\frac{y}{x}\right) = c$ is the required sol^n.

14. Cosx (cosx – siny) dx + cosy (cosy – sinx) dy = 0

Sol^n. Given differential equation is

cosx (cosx – siny) dx + cosy (cosy – sinx)dy = 0

or, $\cos^2 x dx - \sin y \cos x\, dx + \cos^2 y\, dy - \cos y \sin x\, dy = 0$

or, $\frac{(1+\cos 2x)}{2}dy + \left(\frac{1+\cos 2y}{2}\right)dy = \cos y \sin x dy + \sin y \cos dx$

or, $\int\frac{(1+\cos x)}{2}dx + \int\left(\frac{1+\cos 2y}{2}\right)dy = \int d(\sin x \sin y)$; integrating

or, $\frac{1}{2}x + \frac{\sin 2x}{4} + \frac{1}{2}y + \frac{\sin 2y}{4} = \sin x \sin y + c$

or, $2x + 2y + \sin 2x + \sin 2y - 4\sin x \sin y = 4c$

or, $2x + 2y + \sin 2x + \sin 2y - 4\sin x \sin y = K$ where $k = 4c$ is the required solution.

15. $(x + 2y^3)\, dy = ydx$

Sol^n. Given differential equation is

$(x + 2y^3)\, dy = ydx$ or, $xdy + 2y^3 dy = ydx$

or, $-2y^3 dy = ydx - xdy$

Dividing both sides by y^2

$2ydy = \frac{ydx - xdy}{y^2}$

or, $\int 2ydy = \int d\left(\frac{x}{y}\right)$ Integrating

or, $\frac{2y^2}{2} = \frac{x}{y} + c$

or, $y^2 = \frac{x}{y} + c$

or, $y^3 = x + cy$ is the required soltion.

16. $x^2y^3\, dx + 3x^2y\, dy + 2ydx = 0$

Sol^n. Given differential equation is

$x^2y^3\, dx + 3x^2y\, dy + 2ydx = 0$

or, $x^2y^3 dx + 2ydx + 3x^2ydy = 0$

or, $y(x^2y^2 + 2)\, dx = -3x^2y\, dy$

or, $\frac{dy}{dx} = -\frac{y(x^2y^2+2)}{3x^2y}$ .....(1)

Put $xy = v$. Then

$y + x\frac{dy}{dx} = \frac{dv}{dx}$

or, $\frac{dy}{dx} = \left(\frac{dv}{dx} - y\right)\frac{1}{x}$

Now eq^n (1) becomes

$\left(\frac{dv}{dx} - y\right)\frac{1}{x} = \frac{-y(x^2y^2+2)}{2xy.x}$

Using $xy = v$ and $y = \frac{v}{x}$

or, $\left(\frac{dv}{dx} - \frac{v}{x}\right)\frac{1}{x} = -\frac{v(v^2+2)}{x3vx}$

or, $\left(\frac{dv}{dx} - \frac{v}{x}\right)\frac{1}{x} = \frac{-(v^2+2)}{3x^2}$

or, $\frac{dv}{dx} = \frac{v}{x} - \frac{1}{3x}(v^2+2)$

or, $\frac{dv}{dx} = \frac{1}{x}\left[v - \frac{v^2+2}{3}\right]$

or, $\frac{dv}{dx} = \frac{1}{x}\frac{(3v - v^2 + 2)}{3}$

or, $\frac{3dv}{3v - v^2 + 2} = \frac{dx}{x}$

or, $\int\frac{3dv}{v^2 - 3v + 2} = -\int\frac{dx}{x}$ Integrating

or, $\int\frac{3dv}{v^2 - 2.\frac{3}{2}v + \left(\frac{3}{2}\right)^2 - \frac{1}{4}} = -\int\frac{dx}{x}$

or, $\int\frac{3dv}{\left(v - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = -\int\frac{dx}{x}$

or, $3.\frac{1}{2.\frac{1}{2}}\log\left\{\frac{\left(v - \frac{3}{2}\right) - \frac{1}{2}}{\left(v - \frac{3}{2}\right) + \frac{1}{2}}\right\} = -\log x + \log c$

or, $3\log\left(\frac{v-2}{v-1}\right) = -\log x + \log c$

or, $\log\left(\frac{v-2}{v-1}\right)^3 + \log x = \log c$

or, $\log\left(\frac{v-2}{v-1}\right)^3 x = \log c$ or, $\frac{(v-2)^3}{(v-1)^3}x = c$

or, $x(v-2)^3 = c(v-1)^3$

But $xy = v$

$\therefore x(xy-2)^3 = c(xy-1)^3$ is the required solution.

17. $\frac{xdx+ydy}{xdy-ydx} = \frac{\sqrt{a^2-x^2-y^2}}{x^2+y^2}$

Sol^n. Given differential equation is

$$\frac{xdx+ydy}{xdy-ydx} = \frac{\sqrt{a^2-x^2-y^2}}{x^2+y^2}$$

or, $\frac{xdx+ydy}{\sqrt{a^2-(x^2+y^2)}} = \frac{xdy-ydx}{x^2+y^2}$

or, $\frac{1}{2}\int\frac{d(x^2+y^2)}{\sqrt{a^2-\left(\sqrt{x^2+y^2}\right)^2}} = \int d\tan^{-1}\left(\frac{y}{x}\right)$; Integrating

or, $\sin^{-1}\frac{\sqrt{x^2+y^2}}{a} = 2\tan^{-1}\left(\frac{y}{x}\right)+c$

$$\frac{\sqrt{x^2-y^2}}{a} = \sin\left\{2\tan^{-1}\left(\frac{y}{x}\right)+c\right\}$$

or, $\sqrt{x^2-y^2} = a\sin\left\{2+\tan^{-1}\left(\frac{y}{x}\right)+c\right\}$

is the required solution.

18. $(x^2+y^2+2x)\,dx + 2ydy = 0$

Sol^n. Given differential equation is

$(x^2+y^2+2x)\,dx + 2ydy = 0$

or, $(x^2+y^2)\,dx = -2(xdx+ydy)$

or, $\int dx = -2\int\frac{(xdx+ydy)}{(x^2+y^2)}$ Integrating

or, $x = -\log(x^2+y^2)+c$

or, $x + \log(x^2+y^2) = c$ is the required solution

19. $x^2dy + xydx + 2\sqrt{1-x^2y^2}\,dx = 0$

Sol^n. Given differential equation is

$x^2dy + xydx + 2\sqrt{1-x^2y^2}\,dx = 0$

Dividing both sides by $\sqrt{1-x^2y^2}$ we get

$$\frac{x^2dy+xydx}{\sqrt{1-(xy)^2}} = -2dx$$

or, $\frac{x(xdy+ydx)}{\sqrt{1-(xy)^2}} = -2dx$ or, $\frac{xdy+ydx}{\sqrt{1-(xy)^2}} = -\frac{2}{x}dx$

$\int\frac{d(xy)}{\sqrt{1-(xy)^2}} = -\int\frac{2}{x}dx$ Integrating

or, $\sin^{-1}(xy) = -2\log x + c$ is the required solution.

20. $xdx + ydy = \frac{a^2(xdy-ydx)}{x^2+y^2}$

Sol^n. Given differential equation is

$$xdx + ydy = \frac{a^2(xdy-ydx)}{x^2+y^2}$$

or, $\int xdx + \int ydy = a^2\int d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$ Integrating

or, $\frac{x^2}{2}+\frac{y^2}{2} = a^2\tan^{-1}\left(\frac{y}{x}\right)+c$

or, $x^2+y^2 = 2a^2\tan^{-1}\left(\frac{y}{x}\right)+c$ is the required solution.

21. $(x^2+y^2)\,dx - 2xy\,dy = 0$

Sol^n. Given differential equation is

$(x^2+y^2)\,dx - 2xy\,dy = 0$

$x^2dx + y^2dx - 2xydy = 0$

or, $y^2dx - 2xy\,dy = -x^2\,dx$

Dividing both sides by $-x^2$

$$\frac{2xydy-y^2dx}{x^2} = dx$$

or, $\int d\left(\frac{y^2}{x}\right) = \int dx$ Integrating or, $\frac{y^2}{x} = x+c$

or, $y^2 = x^2+cx$ is the required solution.

First Order but not First Degree Differential Equations

16

An equation is of the form $f(x, y, p) = 0$ where $p = \frac{dy}{dx}$ is called the first order but not a first degree differential equation. The solution of such equations contrains only one arbitary constant.

Solvable for p

If the differential equation $f(x, y, p) = 0$ can be factorized into linear factor seen as $\{p - f_1(x, y)\}\{p - f_2(x, y)\} \ldots$ $\{p - f_n(x, y)\}$, then it is called solvable for p.

Each factor equated to zero, we get the solution of the form.

$F_1(x, y, c) = 0, F_2(x, y, c) = 0, \ldots\ldots F_n(x, y, c_n) = 0$

It general solution is,

$F_1(x, y, c), F_2(x, y, c) \ldots\ldots F_n(x, y, c) = 0$

Exercise - 26

Solve the following equations

1. $p^2 + p - 6 = 0$

Sol^n. Given differential equation is,

$p^2 + p - 6 = 0$

or, $p^2 + 3p - 2p - 6 = 0$

or, $p(p + 3) - 2(p + 3) = 0$

or, $(p + 3)(p - 2) = 0$

Either $p + 3 = 0$

$\Rightarrow \frac{dy}{dx} = -3$

$\int dy = -\int 3dx$ Integrating

$y = -3x + C$

$\Rightarrow y + 3x + C = 0$

Hence the general solution is,

$(y + 3x + C)(y - 2x + C) = 0$

or, $p - 2 = 0$

$\Rightarrow \frac{dy}{dx} = 2$

$\int dy = \int 2dx$; Integrating

$y = 2x + C$

$\Rightarrow y - 2x + C = 0$

2. $p^2 + 2px - 3x^2 = 0$

Sol^n. Given differential equation is,

$p^2 + 2px - 3x^2 = 0$

or, $p^2 + 3px - px - 3x^2 = 0$

or, $p(p + 3x) - x(p + 3x) = 0$

or, $(p + 3x)(p - x) = 0$

either $p + 3x = 0$

$\Rightarrow \frac{dy}{dx} = -3x$

$\int dy = -\int 3x\, dx$; Integration

or, $y = \frac{3x^2}{2} + C$

or, $\left(y - \frac{3x^2}{2} + C\right) = 0$

or, $(2y - 3x^2 - 2C) = 0$

or, $(2y - 3x^2 - K) = 0$

Where $K = 2C$

Hence the general solution is,

$(2y - 3x^2 - K)(2y - x^2 - K) = 0$

or, $p - x = 0$

$\Rightarrow \frac{dy}{dx} = x$

$\int dy = \int x\, dx$; Integration

or, $y = \frac{x^2}{2} + C$

or, $2y - x^2 = 2C$

or, $2y - x^2 - 2C = 0$

$(2y - x^2 - K) = 0$

where $K = 2C$

3. $p^2 - p(e^x + e^{-x}) + 1 = 0$

Sol^n. Given differential equation is,

$p^2 - p(e^x + e^{-x}) + 1 = 0$

or, $p^2 - pe^x - pe^{-x} + 1 = 0$

or, $p(p - e^x) - e^{-x}(p - e^x) = 0$

or, $(p - e^x)(p - e^{-x}) = 0$

either $p - e^x = 0$

or, $\frac{dy}{dx} = e^x$

or, $\int dy = \int e^x dx$; Integration

or, $y = e^x + C$

or, $(y - e^x - C) = 0$

Hence the general solution is,

$(y - e^x + C)(y + e^{-x} + C) = 0$

or, $p - e^{-x} = 0$

or, $\frac{dy}{dx} = e^{-x}$

or, $\int dy = \int e^{-x} dx$; Integration

or, $y = -e^{-x} + C$

or, $y + e^{-x} - C = 0$

4. $p(p^2 + xy) = p^2(x + y)$

Sol^n. Given differential equation is,

$p(p^2 + xy) = p^2(x + y)$

or, $p^3 + pxy - p^2x - p^2y = 0$

or, $p(p^2 + xy - px - py) = 0$

or, $p(p^2 - px + xy - py) = 0$

or, $p\{p(p - x) + y(x - p)\} = 0$

or, $p\{p(p-x)-y(p-x)\}=0$

or, $p(p-x)(p-y)=0$

either $p=0$

or, $\frac{dy}{dx}=0$

or, $\int dy=\int dx$; Int.

or, $y=x+C$

$\Rightarrow y-x-C=0$

or, $p-x=0$

or, $\frac{dy}{dx}=x$

or, $\int dy=\int x\,dx$; int.

or, $y=\frac{x^2}{2}+\frac{C}{2}$

or, $2y=x^2+C$

$\Rightarrow (2y-x^2-C)=0$

or, $p-y=0$

or, $\frac{dy}{dx}=y$

or, $\int\frac{1}{y}dy=\int dx$, int.

or, $\log y=x+\log C$

or, $\log\frac{y}{C}=x$

$\Rightarrow y-Ce^x=0$

Hence, general solution is,

$(y-x-C)(2y-x^2-C)(y-Ce^x)=0$

5. $yp^2+(x-y)p-x=0$ y?

Sol^n. Given differential equation is,

$yp^2+(x-y)p-x=0$

or, $yp^2+xp-yp-x=0$

or, $yp^2-yp+xp-x=0$

or, $yp(p-x)+x(p-x)=0$

or, $(p-x)(yp+x)=0$

either, $p-x=0$

or, $\frac{dy}{dx}=x$

or, $\int dy=\int x\,dx$; integration

or, $y=\frac{x^2}{2}+\frac{C}{2}$

or, $2y-x^2-C=0$

or, $yp+x=0$

or, $y\frac{dy}{dx}=-x$

or, $\int y\,dy=-\int x\,dx$; Integration

or, $\frac{y^2}{2}=-\frac{x^2}{2}+\frac{C}{2}$

or, $y^2=-x^2+C$

or, $y^2+x^2-C=0$

Hence, the general solution is,

$(2y-x^2-C)(y^2+x^2-C)=0$

6. $p^2+2px+py+2xy=0$

Sol^n. Given differential equation is,

$p^2+2px+py+2xy=0$

or, $p(p+2x)+y(p+2x)=0$

or, $(p+2x)(p+y)=0$

either $p+2x=0$

or, $\frac{dy}{dx}=-2x$

or, $\int dy=-\int 2x\,dx$; Integration

or, $p+y=0$

or, $\frac{dy}{dx}=-y$

or, $\int\frac{1}{y}dy=-\int dx$ Integration

or, $y=-x^2+C$

or, $y+x^2-C=0$

or, $\log y=-x+\log C$

or, $\log\frac{y}{C}=-x$

or, $y=Ce^{-x}$

or, $y-Ce^{-x}=0$

Hence, the general solution is,

$(y+x^2-C)(y-Ce^{-x})=0$

$p^3+2xp^2-y^2p^2-2xy^2p=0$

7. Sol^n. Given, Differential equation is,

$p^3+2xp^2-y^2p^2-2xy^2p=0$

or, $p^2(p+2x)-y^2p(p+2x)=0$

or, $(p+2x)(p^2-y^2p)=0$

or, $p(p+2x)(p-y^2)=0$

either $p=0$

$\frac{dy}{dx}=0$

or, $\int dy=\int 0.\,dx$; Int.

or, $y=C$

or, $p+2x=0$

or, $\frac{dy}{dx}=-2x$

or, $\int dy=\int -2x\,dx$; Int.

or, $y=-x^2+C$

or, $p-y^2=0$

or, $\frac{dy}{dx}=y^2$

$\int\frac{1}{y^2}dy=\int dx$; Int.

or, $-\frac{1}{y}=x+C$

or, $-1=xy+Cy$

or, $xy+1+Cy=0$

Hence, the general solution.

$(y-C)(y+x^2-C)(xy+1+Cy)=0$

8. $p^3-p(x^2+xy+y^2)+x^2y+xy^2=0$

Sol^n. Given differential equation is,

$p^3-p(x^2+xy+y^2)+x^2y+xy^2=0$

or, $p^3-px^2-pxy-py^2+x^2y+xy^2=0$

or, $p^3-px^2-pxy+x^2y-py^2+xy^2=0$

or, $p(p^2-x^2)-xy(p-x)-y^2(p-x)=0$

or, $p(p+x)(p-x)-xy(p-x)-y^2(p-x)=0$

or, $(p-x)\{p(p+x)-xy-y^2\}=0$

or, $(p-x)(p^2+px-xy-y^2)=0$

or, $(p-x)(p^2-y^2+px-xy)=0$

or, $(p-x)\{1(p^2-y^2)+x(p-y)\}=0$

or, $(p-x)\{(p+y)(p-y)+x(p-y)\}=0$

or, $(p-x)(p-y)(p+y+x)=0$

either $p-x=0$

or, $\frac{dy}{dx}=x$

or, $\int dy=\int x\,dx$; Int.

or, $p-y=0$

or, $\frac{dy}{dx}=y$

or, $\int\frac{1}{y}dy=\int dx$; Int.

or, $p+y+x=0$

or, $\frac{dy}{dx}+y=-x$

which is linear differential form

$1-x-y=e^x$

$\frac{1}{1-x-y}=e^{-x}$

e^{-x}

$v=-(x+y)$

$\frac{dv}{dx}=-1-\frac{dy}{dx}$

$\frac{dy}{dx}=-1-\frac{dv}{dx}$

$-1-\frac{dv}{dx}=v$

$\frac{dv}{1+v}=-dx$

$\frac{dv}{dx}=-(1+v)$

$x+\log(1+v)=\log c$

$\left(\frac{1+v}{c}\right)=e^x$

$1+v=ce^x$

$x+\log(1+v)=c$

or, $y = \frac{x^2}{2} + \frac{C}{2}$

or, $2y = x^2 + C$

or, $2y - x^2 - C = 0$

or, $\log \frac{y}{C} = x$

or, $y = C e^x$

or, $y - C e^x = 0$

$\therefore$ I.F. $= e^{\int dx} = e^{\log x} = e^x$

Multiplying it by I.F. we get

$y.e^x = \int -x e^x dx + C$

or, $ye^x = -x e^x + e^x + C$

or, $y = -x + 1 + C e^{-x} = 0$

or, $y + x - 1 - C e^{-x} = 0$

Hence, $(2y - x^2 - C)(y - C e^x)(y + x - 1 - C e^{-x}) = 0$

9. $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$

Sol^n. Given differential equation is,

$x^2p^2 + xyp - 6y^2$

$x^2y^2 + 3 \quad 2$

$$x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$$

or, $x^2p^2 + xyp - 6y^2 = 0$ where $p = \frac{dy}{dx}$

or, $x^2p^2 + 3xyp - 2xyp - 6y^2 = 0$

or, $xp(xp + 3y) - 2y(xp + 3y) = 0$

or, $(xp + 3y)(xp - 2y) = 0$

either $xp + 3y = 0$

or, $x\frac{dy}{dx} = -3y$

or, $\int\frac{1}{y}dy = -\int\frac{3}{x}dx$; Integration

or, $\log y = -3\log x + \log C$

or, $\log y + \log x^3 = \log C$

or, $\log yx^3 = \log C$

or, $yx^3 = C$

or, $yx^3 - C = 0$

or, $xp - 2y = 0$

or, $x\frac{dy}{dx} = 2y$

or, $\int\frac{1}{y}dy = \int\frac{2}{x}dx$; Integration

or, $\log y = 2\log x + \log C$

or, $\log y = \log x^2 + \log C$

or, $\log y = \log x^2C$

or, $y = Cx^2$

or, $y - Cx^2 = 0$

Hence the general solution is,

$(yx^3 - C)(y - Cx^2) = 0$

10. $\left(\frac{dy}{dx}\right)^2 + 2y\cot x\frac{dy}{dx} = y^2$

Sol^n. Given differential equation is,

$$\left(\frac{dy}{dx}\right)^2 + 2y\cot x\frac{dy}{dx} = y^2$$

or, $p^2 + 2y\cot x\, p - y^2 = 0$ where $\frac{dy}{dx} = p$.

Which is quadratic in p So,

$$p = \frac{-2y\cot x \pm \sqrt{(2y\cot x)^2 - 4.1.y^2}}{2.1}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x - 4y^2}}{2}$$

$$= \frac{-2y\cot x \pm 2y\sqrt{\cot^2 x - 1}}{2}$$

$$= y(-\cot x \pm \operatorname{cosec} x) = y\left(-\frac{\cos x}{\sin x} \pm \frac{1}{\sin x}\right)$$

$$= y\frac{(-\cos x \pm 1)}{\sin x}$$

Taking (+ ve) sign,

$p = y\frac{(-\cos x + 1)}{\sin x}$ or, $\frac{dy}{dx} = y\frac{(-\cos x + 1)}{\sin x}$

$\int\frac{1}{y}dy = \int\frac{(1-\cos x)(1+\cos x)}{\sin x(1+\cos x)}dx$; Integrating

or, $\log y = \int\frac{\sin^2 x}{\sin x(1+\cos x)}dx + \log C$

or, $\log y = -\int\frac{\sin x}{1+\cos x}dx + \log C$

or, $\log y = -\log(1 + \cos x) + \log C$

or, $\log y + \log(1 + \cos x) = \log C$

or, $\log y(1 + \cos x) = \log C$

or, $y(1 + \cos x) = C$

or, $(y + y\cos x - C) = 0$

Taking (– ve) sign,

$p = y\frac{(-\cos x - 1)}{\sin x}$

or, $\frac{dy}{dx} = -y\frac{(1+\cos x)(1-\cos x)}{\sin x(1-\cos x)}$ or, $\frac{dy}{dx} = -y\frac{\sin^2 x}{\sin x(1-\cos x)}$

or, $\frac{dy}{dx} = -y\frac{\sin x}{(1-\cos x)}$ or, $\int\frac{1}{y}dy = -\int\frac{\sin x}{(1-\cos x)}dx$; Integrating

or, $\log y = -\log(1 - \cos x) + \log C$

or, $\log y = \log(1 - \cos x) = \log C$

or, $\log y(1 - \cos x) = \log C$

or, $y(1 - \cos x) = C$

or, $y - y\cos x - C = 0$

Hence, the general solution is

$(y + y\cos x - C)(y - y\cos x - C) = 0$

Solvable for y

If an equation of the form $y = f(x, p)$ then it can be solved by the method of solvable for y.

Let $y = f(x, y)$ ..... (i)

Differentiating with respect to x,

$$\frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$$

$$\text{or, } p = \phi\left(x, p, \frac{dp}{dx}\right) \text{ ..... (ii)}$$

Which is differential equation in the two variables x and p. Suppose its solution is $f(x, p, c) = 0$ ..... (iii)

Eliminating p from (i) and (iii) gives the required solution. If p can not be eliminated from given two equations (i) and (iii), then the equations (i) and (iii) together give required solution.

Solvable for x

If an equation of the form $x = f(y, p)$ then it can be solved by the method of solvable for x.

Let $x = f(y, p)$ ..... (i)

Differentiating with respect to y,

$$\frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

$$\text{or, } \frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \text{ ..... (ii)}$$

Which is a differential equation of two variables y and p. Suppose its solution be $f(y, p, c)$

Eliminating p from (i) and (iii) gives the required solution.

If p can not be eliminated from the given two equations (i) and (iii), then the equation (i) and (iii) together given required solution.

Exercise - 27

Solve the following equations

1. $p^3x - p^2y - 1 = 0$

Sol^n. Given differential equation is,

$p^3x - p^2y - 1 = 0$

or, $p^2y = p^3x - 1$

$p^2y = p^3x - 1$

$y = px - \frac{1}{p^2}$

$p = p + x\frac{dp}{dx} + 2p^{-3}$

$$\text{or, } y = \frac{p^3x - 1}{p^2} = px - \frac{1}{p} \text{ ..... (i)}$$

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{p^2}\frac{dp}{dx}$$

$$\text{or, } p = p + x\frac{dp}{dx} + \frac{1}{p^2}\frac{dp}{dx}$$

$$\text{or, } \left(x + \frac{1}{p^2}\right)\frac{dp}{dx} = 0$$

Either $\frac{dp}{dx} = 0$

or, $\int dp = \int 0 \, . \, dx$; Integrating

$p = C$ ..... (ii)

From (i) and (ii)

$C^3x - C^2y - 1 = 0$ is the required solution.

2. $y = 2px + p^3y^2$

Sol^n. Given differential equation is,

$y = 2px + p^3y^2$

or, $2px = y - p^3y^2$

$$\text{or, } 2x = \frac{y}{p} - p^2y^2 \text{ ..... (i)}$$

Differential equation (i) w. r. t. 'y'

$$\text{or, } 2\frac{dx}{dy} = \frac{p.1 - y\frac{dp}{dx}}{p^2} - 2p^2y - 2y^2p\frac{dp}{dy}$$

$$\text{or, } \frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2p^2y - 2y^2p\frac{dp}{dy}$$

$$\text{or, } \frac{2}{p} - \frac{1}{p} + 2p^2y = -\frac{dp}{dy}\left(\frac{y}{p^2} + 2y^2p\right)$$

$$\text{or, } \left(\frac{2 - 1 + 2yp^3}{p}\right) = \frac{-\left(y + 2y^2p^3\right)}{p^2}\frac{dp}{dy}$$

$$\text{or, } \frac{\left(1 + 2yp^3\right)}{p} = \frac{-y\left(1 + 2yp^3\right)}{p^2}\frac{dp}{dy}$$

$$\text{or, } 1 = \frac{-y}{p}\frac{dp}{dy}$$

or, $\int \frac{1}{y} dy = -\int \frac{1}{p} dp$; Integrating

$\log y = -\log p + \log C$

or, $\log yp = \log C \Rightarrow yp = C$

or, $p = \frac{C}{y}$ ..... (ii)

From (i) and (ii)

$$y = 2 . \frac{C}{y} x + \left(\frac{C}{y}\right)^3 y^2$$

$y^2 = 2Cx + C^3$ is the required solution.

3. $x = 4p + 4p^3$

Solⁿ. Given differential equation is,

$x = 4p + 4p^3$ ..... (i)

Differential equation (i) w. r. t. 'y'

$$\frac{dx}{dy} = 4\frac{dp}{dy} = 4 . 3p^2 \frac{dp}{dy}$$

or, $\frac{1}{p} = (4 + 12p^2) \frac{dp}{dy}$

or, $\int dy = \int (4p + 12p^3)\, dp$ Integrating

$y = 2p^2 + 3p^4 + C$ .....(ii)

From (i) and (ii)

$x = 4p + 4p^3$ and $y = 2p^2 + 3p^4 + C$ is the required solution.

4. $\sin y \cos px - \cos y \sin px = p$

Solⁿ. Given differential equation is,

$\sin y \cos px - \cos y \sin px = p$

or, $\sin (y - px) = p$

or, $y - px = \sin^{-1} (p)$

or, $y = px + \sin^{-1} (p)$ ..... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}} \frac{dp}{dx}$$

$$\text{or, } p = p + x\frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}} \frac{dp}{dx}$$

$$\text{or, } \frac{dp}{dx}\left(x + \frac{1}{\sqrt{1-p^2}}\right) = 0$$

$p = p + x\frac{dp}{dx} + \frac{1}{\sqrt{}}$

$\left(x + \frac{1}{\sqrt{1-p^2}}\right)\frac{dp}{}$

$y = cx + \sin^{-1}$

Either $\frac{dp}{dx} = 0$

or, $\int dp = \int 0 . dx$; Integrating

$p = C$ ..... (ii)

From (i) and (ii)

$y = Cx + \sin^{-1}C$

$y - Cx = \sin^{-1}C$

or, $\sin (y - Cx) = C$ is the required solution.

5. $y + px = x^4p^2$

Solⁿ. Given differential equation is,

$y = x^4p^2 - px$ ..... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = 2x^4p \frac{dp}{dx} + 4x^3p^2 - p - x \frac{dp}{dx}$$

$$\text{or, } p - 4x^3p^2 + p = (2x^4 p - x) \frac{dp}{dx}$$

$$\text{or, } (2p - 4x^3 p^2) = (2x^4p - x) \frac{dp}{dx}$$

$$\text{or, } 2p (1 - 2x^3p) = x (2x^3 p - 1) \frac{dp}{dx}$$

$$\text{or, } 2p (1 - 2x^3p) = -x (1 - 2x^3P) \frac{dp}{dx}$$

$$\text{or, } 2p = -x \frac{dp}{dx}$$

or, $2\int \frac{dx}{x} = -\int \frac{1}{p} dp$; Integrating

$2 \log x = -\log p + \log C$

$\log x^2 + \log p = \log C$

or, $\log x^2p + \log C \Rightarrow x^2p = C$

or, $p = \frac{C}{x^2}$ ..... (ii)

From (i) and (ii)

$$y = x^4 . \frac{C^2}{x^4} - \frac{C}{x^2} . x$$

or, $y = C^2 - \frac{C}{x}$

or, $xy = C^2 x - C \Rightarrow xy + C = C^2 x$ is the required solution.

6. $p^2 - py + x = 0$

Solⁿ. Given differential equations,

$p^2 - py + x = 0$

or, $yp = x + p^2$

or, $y = \frac{x}{p} + p$ ..... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = \frac{1}{p} - \frac{x}{p^2}\frac{dp}{dx} + \frac{dp}{dx}$$

$$\text{or, } p - \frac{1}{p} = \frac{dp}{dx}\left(1 - \frac{x}{p^2}\right)$$

$$\text{or, } \frac{\left(p^2-1\right)}{p} = \frac{dp}{dx}\,\frac{\left(p^2-x\right)}{p^2}$$

$$\text{or, } (p^2-1) = \frac{\left(p^2-x\right)}{p}\,\frac{dp}{dx}$$

$$\frac{dx}{dp} = \frac{\left(p^2-x\right)}{p\left(p^2-1\right)} = \frac{p}{p^2-1} - \frac{x}{p\left(p^2-1\right)}$$

$$\text{or, } \frac{dx}{dp} + \frac{1}{p\left(p^2-x\right)}x = \frac{p}{p^2-1} \quad \text{..... (ii)}$$

Equation (ii) is linear in x so,

$$\text{I.F.} = e^{\int \frac{1}{p(p^2-1)}dp} = e^{\int\left(\frac{-1}{p} - \frac{p}{p^2-1}\right)dp} = e^{-\int \log p - \frac{1}{2}\log\left(p^2-1\right)}$$

$$= e^{-\left(\log p + \log\sqrt{p^2-1}\right)} = e^{-\log\frac{p}{\sqrt{p^2-1}}} = e^{\log\frac{\sqrt{p^2-1}}{p}}$$

$$\text{I.F.} = \frac{\sqrt{p^2-1}}{p}$$

Multiplying (ii) by I. F. we get,

$$x \cdot \frac{\sqrt{p^2-1}}{p} = \int \frac{p}{p^2-1} \cdot \frac{\sqrt{p^2-1}}{p}dp + C$$

$$= \int \frac{1}{\sqrt{\left(p^2-1\right)}}dp + C$$

$$\text{or, } \frac{x\sqrt{p^2-1}}{p} = \log\left(p + \sqrt{p^2-1}\right) + C$$

$$\text{or, } x = \frac{p\left\{\log\left(p + \sqrt{p^2-1}\right) + C\right\}}{\sqrt{p^2-1}} \quad \text{..... (iii)}$$

From equation (i) and (iii)

$$\left.\begin{aligned} x &= \frac{p\left\{\log p + \sqrt{p^2-1} + C\right\}}{\sqrt{p^2-1}} \\ \text{and } x &= py - p^2 \end{aligned}\right\} \text{ is the required solution.}$$

7. $e^y - p^3 - p = 0$

Sol^n. Given differential equation is,

$e^y - p^3 - p = 0$

or, $e^y = p^3 + p$

or, $y = \log(p^3 + p)$ ..... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = \frac{1}{\left(p^3+p\right)}(3p^2+1)\frac{dp}{dx}$$

$$\text{or, } p = \frac{\left(3p^2+1\right)}{p\left(p^2+1\right)}\frac{dp}{dx}$$

$$\text{or, } \int dx = \int \frac{3p^2+1}{p^2\left(p^2+1\right)}dp; \text{ Integrating}$$

$$\text{or, } \int dx = \int\left(\frac{2}{1+p^2} + \frac{1}{p^2}\right)dp$$

$$\text{or, } x = 2\tan^{-1}p - \frac{1}{p} + C \quad \text{..... (ii)}$$

From equation (i) and (ii)

$x = 2\tan^{-1}p - \frac{1}{p} + C$ and $y = \log(p^3 + p)$ is the required solution.

8. $4(xp^2 + yp) = y^4$

Sol^n. Given differential equation is,

$4(xp^2 + yp) = y^4$

or, $4xp^2 + 4yp = y^4$

$4xp^2 = y^4 - 4yp$

$$\text{or, } 4x = \frac{y^4}{p^2} - \frac{4y}{p} \quad \text{..... (i)}$$

Differential equation (i) w. r. t. 'y'

$$4\frac{dy}{dx} = \frac{p^2.4y^3 - y^4.2p\frac{dp}{dy}}{p^4} - \frac{4p - 4y\frac{dp}{dy}}{p^2}$$

$$\text{or, } \frac{4}{p} + \frac{4}{p} - \frac{4y^3}{p^2} = \left(\frac{4y}{p^2} - \frac{2y^4}{p^3}\right)\frac{dp}{dy}$$

$$\text{or, } \left(\frac{8p - 4y^3}{p^2}\right) = \frac{\left(4py - 2y^4\right)}{p^3}\frac{dp}{dy}$$

$$\text{or, } \frac{4\left(4p - y^3\right)}{p^2} = \frac{2y\left(2p - y^3\right)}{p^3}\frac{dp}{dy}$$

$$\text{or, } 2 = \frac{y}{p}\frac{dp}{dy}$$

$$\text{or, } \int\frac{2}{y}dy = \int\frac{1}{p}dp; \text{ Integrating}$$

$$2\log y = \log p = \log C$$

$$\text{or, } y^2 = pC \Rightarrow p = \frac{y^2}{C} \text{ ..... (ii)}$$

From (i) and (ii)

$$4x = \frac{y^4}{y^4} \times C^2 - \frac{4y}{y^2}.C$$

$$\text{or, } 4x = C^2 - \frac{4}{y}C \text{ is the required solution.}$$

9. $x + \frac{p}{\sqrt{1-p^2}} = a$

Solⁿ. Given differential equation is,

$$x + \frac{p}{\sqrt{1-p^2}} = a$$

$$\text{or, } x = a - \frac{p}{\sqrt{1-p^2}} \text{ ..... (i)}$$

Differential equation (i) w. r. t. 'y' we get,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-p^2}}\frac{dp}{dy} - p.\frac{(-1)}{\left(1-p^2\right)^{-\frac{3}{2}}}.\frac{1}{2}.(-2p)\frac{dp}{dy}$$

$$\frac{1}{p} = \left[\frac{-1}{\sqrt{1-p^2}} - \frac{p^2}{\left(1-p^2\right)^{-\frac{3}{2}}}\right]\frac{dp}{dy}$$

$$\text{or, } \frac{1}{p} = -\left[\frac{1-p^2+p^2}{\left(1-p^2\right)^{-\frac{3}{2}}}\right]\frac{dp}{dy}$$

$$\text{or, } \frac{1}{p} = -\frac{1}{\left(1-p^2\right)^{-\frac{3}{2}}}\frac{dp}{dy}$$

$$\text{or, } \int dy = -\int\frac{p}{\left(1-p^2\right)^{-\frac{3}{2}}}dp; \text{ Integrating}$$

Put $1 - p^2 = t$

$-2p\,dp = dt$

$-p\,dp = \frac{1}{2}dt$

$$\text{or, } \int dp = \frac{1}{2}\int\frac{1}{t^{-\frac{3}{2}}}dt$$

$$\text{or, } y = -\frac{1}{t^{\frac{1}{2}}} - C$$

$$\text{or, } y + C = \frac{-1}{\sqrt{1-p^2}} \text{ Squaring on both side}$$

$$(y+C)^2 = \frac{1}{\left(1-p^2\right)}$$

$$1 - p^2 = \frac{1}{(y+C)^2} \text{ ..... (ii)}$$

$$p^2 = 1 - \frac{1}{(y+C)^2} \text{ ..... (iii)}$$

Also form equation (i) $(x-a)^2 = \frac{p^2}{1-p^2}$ ..... (iv)

From (ii) and (iii) putting the value of p^2 and $1 - p^2$ in (iv) we get,

$$(x-a)^2 = \frac{1-\frac{1}{(y+C)^2}}{\frac{1}{(y+C)^2}}$$

or, $(x-a)^2 = (y+C)^2 - 1$

or, $(y+C)^2 - (x-a)^2 = 1$ is the required solution.

10. $xp^3 = a + bp$

Solⁿ. Given differential equation is,

$xp^3 = a + bp$

or, $x = \frac{a}{p^3} + \frac{b}{p^2}$ ..... (i)

Differential equation (i) w. r. t. 'y'

$$\frac{dx}{dy} = -\frac{3a}{p^4}\frac{dp}{dy} - \frac{2b}{p^3}\frac{dp}{dy}$$

or, $\frac{1}{p} = -\left(\frac{3a}{p^4} + \frac{2b}{p^3}\right)\frac{dp}{dy}$

or, $\frac{1}{p} = -\frac{(3a+2bp)}{p^4}\frac{dp}{dy}$

or, $1 = -\left(\frac{3a}{p^3} + \frac{2b}{p^2}\right)\frac{dp}{dy}$

or, $\int dy = -\int\left(\frac{3a}{p^3} + \frac{2b}{p^2}\right)dp$; Integrating

or, $y = -\left\{\frac{3a}{-2p^2} + \frac{2b}{-p}\right\}$

or, $y = \frac{3a}{2p^2} + \frac{2b}{p}$ ..... (ii)

From (i) and (ii)

$y = \frac{3a}{2p^2} + \frac{2b}{p}$ and $x = \frac{a}{p^3} + \frac{b}{p^2}$ is the required solution.

11. $y = 2px + p^2$

Solⁿ. Given differential equation is,

$y = 2px + p^2$ ..... (i)

differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or, $p - 2p = (2x + 2p)\frac{dp}{dx}$

or, $-p = 2(x+p)\frac{dp}{dx}$

or, $\frac{dp}{dx} = \frac{2(x+p)}{-p}$

or, $\frac{dp}{dx} + \frac{2}{p}.x = -2$ ..... (ii)

Which is linear in p so,

I. F. $= e^{\int\frac{2}{p}dp} = e^{2\log p} = e^{\log p^2} = p^2$

Multiplying (ii) by I. F. we get,

or, $xp^2 = -2\int p^2 dp + C$

or, $xp^2 = -\frac{2}{3}p^3 + C$

$x = -\frac{2}{3}p + \frac{C}{p^2}$ ..... (iii)

From equation (i) and (iii)

$y = 2px + p^2$ and $x = -\frac{2}{3}p + \frac{C}{p^2}$ is the required solution.

12. $p^3 - p(y+3) + x = 0$

Solⁿ. Given differential equation is,

$p^3 - p(y+3) + x = 0$

or, $x = p(y+3) - p^3$ ..... (i)

Differential equation (i) w. r. t. 'y'

$$\frac{dx}{xy} = p + (y+3)\frac{dp}{dx} - 3p^2\frac{dp}{dy}$$

or, $\frac{1}{p} - p = (y + 3 - 3p^2)\frac{dp}{dy}$

or, $\frac{(1-p^2)}{p} = (y + 3 - 3p^2)\frac{dp}{dy}$

or, $\frac{dp}{dy} = \frac{(y+3-3p^3)}{(1-p^2)}p$

or, $\frac{dp}{dy} - \frac{p}{1-p^2}.y = \frac{(3-3p^2)p}{(1-p^2)}$ ..... (ii)

Equation (ii) is linear in y so,

$$\text{I.F.} = e^{\int \frac{p}{1-p^2}dp} = e^{\frac{1}{2}\log(1-p^2)} = e^{\log\sqrt{1-p^2}} = \sqrt{1-p^2}$$

Multiplying equation (ii) by I. F. we get,

$$y \cdot \sqrt{1-p^2} = \int \frac{3p\left(1-p^2\right)}{\left(1-p^2\right)} \cdot \sqrt{1-p^2}\ dp + C$$

$$y\sqrt{1-p^2} = 3\int p\sqrt{1-p^2}\ dp + C$$

Put $1-p^2 = t$

$$-2p\,dp = dt \Rightarrow p\,dp = -\frac{1}{2}\,dt$$

$$\text{or, } y\sqrt{1-p^2} = -\frac{3}{2}\int t^{\frac{1}{2}}\,dt + C$$

$$= -\frac{3}{2}\cdot\frac{2}{3}\,t^{\frac{3}{2}} + C$$

$$y\sqrt{1-p^2} = -\left(1-p^2\right)^{\frac{3}{2}} + C$$

$$\text{or, } y\sqrt{1-p^2} + \left(1-p^2\right)^{\frac{3}{2}} = C \text{ ..... (iii)}$$

From (i) and (iii)

$y\sqrt{1-p^2} + \left(1-p^2\right)^{\frac{3}{2}} = C$ and $x = p\,(y+3) - p^3$ is the requirel solution.

13. $x + yp = ap^2$

Solⁿ. Given differential equation is,

$x + yp = ap^2$

$yp = -x + ap^2$

$$\text{or, } y = \frac{-x}{p} + ap \text{ ..... (i)}$$

Differential equation (i) w. r. t. 'x' we get,

$$\frac{dy}{dx} = -\left[\frac{1}{p} + x\left(\frac{-1}{p^2}\right)\frac{dp}{dx}\right] + \frac{adp}{dx}$$

$$\text{or, } p + \frac{1}{p} = \left(\frac{x}{p^2} + a\right)\frac{dp}{dx}$$

$$\text{or, } \frac{\left(p^2+1\right)}{p} = \frac{\left(x+ap^2\right)}{p^2}\ \frac{dp}{dx}$$

$$\frac{dx}{dp}\left(\frac{p^2+1}{p}\right) = \frac{\left(x+ap^2\right)}{p^2}$$

$$\text{or, } \frac{dx}{dp} = \frac{\left(x+ap^2\right)}{p^2}\cdot\frac{p}{\left(p^2+1\right)}$$

$$= \frac{\left(x+ap^2\right)}{p\left(p^2+1\right)} = \frac{1}{p\left(p^2+1\right)}x + \frac{ap}{\left(p^2+1\right)}$$

$$\text{or, } \frac{dx}{dp} - \frac{1}{p\left(p^2+1\right)}x = \frac{ap}{p^2 1} \text{ ..... (ii)}$$

Which is linear in p so,

$$\text{I.F.} = e^{-\int \frac{1}{p\left(p^2+1\right)}dp} = e^{\int\left[-\frac{1}{p}+\frac{p}{p^2+1}\right]dp}$$

$$= e^{-\log p + \frac{1}{2}\log\left(p^2+1\right)} = e^{-\log p + \log\sqrt{p^2+1}} = e^{\log\sqrt{\left(\frac{p^2+1}{p}\right)}} = \frac{\sqrt{p^2+1}}{p}$$

Multiplying (i) by I. F,. we get,

$$x \cdot \frac{\sqrt{p^2+1}}{p} = \int \frac{ap}{\left(p^2+1\right)} \cdot \frac{\sqrt{p^2+1}}{p}dp + C$$

$$= a\int \frac{1}{\sqrt{p^2+1}}\,dp + C$$

$$\text{or, } \frac{x\sqrt{p^2+1}}{p} = a\log\left(p+\sqrt{p^2+1}\right) + C$$

$$\text{or, } x\left(p^2+1\right)^{\frac{1}{2}} = P\left[a\log\left(p+\sqrt{p^2+1}\right)\right] + C \text{ ..... (iii)}$$

From (i) and (iii)

$x + yp = ap^2$ and $x\left(p^2+1\right)^{\frac{1}{2}} = p\left[a\log\left(p+\sqrt{p^2+1}\right)\right] + C$ is the required solution.

14. $y = \sin p - p\cos p$

Solⁿ. Given differential equation is,

$y = \sin p - p\cos p$. ..... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = \cos p\frac{dp}{dx} + p\sin p\frac{dp}{dx} - \cos p\frac{dp}{dx}$$

or, $p = (\cos p + p\sin p - \cos p)\frac{dp}{dx}$

or, $p = p\sin p\frac{dp}{dx}$

or, $dx = \frac{p\sin p}{p}dp$

or, $\int dx = \int \sin p\, dp$; Integrating

$x = -\cos p + C$

or, $x = C - \cos p$ ..... (ii)

from (i) and (ii)

$y = \sin p - p\cos p$ and $x = c - \cos p$ is the required solution.

15. $p^2 - 2px + 1 = 0$

Solⁿ. Given differential equation is,

$p^2 - 2px + 1 = 0$

or, $2px = p^2 + 1$

or, $2x = p + \frac{1}{p}$ ..... (i)

Differential (i) w. r. t. 'y'

$2\frac{dx}{dy} = \frac{dp}{dy} - \frac{1}{p^2}\frac{dp}{dy}$

or, $\frac{2}{p} = \left(1 - \frac{1}{p^2}\right)\frac{dp}{dy}$

or, $\frac{2}{p} = \left(\frac{p^2 - 1}{p^2}\right)\frac{dp}{dy}$

or, $2 = \frac{(p^2 - 1)}{p}\frac{dp}{dy}$

or, $\int dy = \frac{1}{2}\int\left(p - \frac{1}{p}\right)dp$; Integrating

$y = \frac{1}{2}\left(\frac{p^2}{2} - \log p\right) + C$

or, $y = \frac{p^2}{4} - \frac{1}{2}\log p + C$ ..... (ii)

From (i) and (ii)

$x = \frac{1}{2}\left(p + \frac{1}{2}\right)$ and $y = \frac{p^2}{4} - \frac{1}{2}\log p + C$ is the required solution.

16. $y = (1 + p)x + ap^2$

Solⁿ. Given differential equation is,

$y = (1 + p)x + ap^2$ ..... (i)

Differential equation (i) w. r. t. 'x'

$\frac{dy}{dx} = (1 + p) + x\frac{dp}{dx} + a \cdot 2p\frac{dp}{dx}$

or, $p = 1 + p + (x + 2ap)\frac{dp}{dx}$

or, $-1 = (x + 2ap)\frac{dp}{dx}$

or, $\frac{dx}{dp} = -(x + 2ap)$

or, $\frac{dx}{dp} + x = -2ap$ ..... (ii)

Equation (ii) is linear differential equation

So, I.F. $= e^{\int 1 dp} = e^p$

Multiplying (ii) by I. F. we get,

$x \cdot e^p = -2a\int pe^p\, dp + C$

$= -2a(pe^p - e^p) + C$

$xe^p = -2ae^p(p - 1) + C$

or, $x = -2a(p - 1) + ce^{-p}$

or, $x = 2a(1 - p) + ce^{-p}$ ..... (iii)

From (i) and (iii)

$x = 2a(1 - p) + Ce^{-p}$ and $y = (1 + p)x + ap^2$ is the required solution.

Clairaut's Equation

An equation is of the form $y = px + f(p)$ is called clairaut's equation. If can be solved by the following method.

Let $y = px + f(p)$ ..... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\text{or, } p = p + [x + f'(p)]\frac{dp}{dx}$$

$$\text{or, } (x + f'(p))\frac{dp}{dx} = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow \int dp = \int 0 \,.\, dx$, Integrating

$p = C$ ..... (ii)

From (i) and (ii)

$y = Cx + f(c)$ is the required general solution.

or, $[x + f'(p)] = 0 \Rightarrow f'(p) = -x$ ..... (iii)

Eliminating p from (i) and (iii) gives the required singular solution.

Exercise - 28

Solve the following equations

1. $y = px + p - p^2$

Sol^n. Given differential equation is,

$y = px + p - p^2$ ..... (i)

Differential equation (i) w. r. t. 'x'.

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\text{or, } (x + 1 - 2p)\frac{dp}{dx} = 0$$

Either $\frac{dp}{dx} = 0$

$\Rightarrow p = C$ ..... (ii)

From (i) and (ii)

$y = Cx + C - C^2$ is the required general solution.

or, $x + 1 - 2p = 0$

or, $2p = x + 1$

or, $p = \frac{1}{2}(x + 1)$ ..... (iii)

From (i) and (iii)

$$y = \left(\frac{x+1}{2}\right).x + \left(\frac{x+1}{2}\right) - \left(\frac{x+1}{2}\right)^2$$

$$= \left(\frac{x+1}{2}\right)\left\{x + 1 - \frac{x+1}{2}\right\}$$

$$= \left(\frac{x+1}{2}\right)\left(\frac{2x+2-x-1}{2}\right)$$

$$= \left(\frac{x+1}{2}\right)\left(\frac{x+1}{2}\right) = \left(\frac{x+1}{2}\right)^2$$

$4y = (x + 1)^2$ is the singular solution.

2. $y = px + \frac{a}{p}$

Sol^n. Given differential equation is,

$y = px + \frac{a}{p}$ ..... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

$$\text{or, } p = p + \left(x - \frac{a}{p^2}\right)\frac{dp}{dx}$$

$$\text{or, } \left(x - \frac{a}{p^2}\right)\frac{dp}{dx} = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow p = C$ ..... (ii)

From (i) and (ii)

$y = Cx + \frac{a}{C}$ is the required general solution.

$$\text{or, } x - \frac{a}{p^2} = 0$$

or, $xp^2 = a$

$$\text{or, } p^2 = \frac{a}{x} \Rightarrow p = \sqrt{\frac{a}{x}} \text{ ..... (iii)}$$

From (i) and (iii)

$$y = x \cdot \sqrt{\frac{a}{x}} + \frac{a}{\sqrt{\frac{a}{x}}}$$

$$y = \sqrt{ax} - \sqrt{ax}$$

or, $y = 2\sqrt{ax}$ squaring on both sides

$y^2 = 4ax$ is the required singular solution.

3. $py = p^2(x - b) + a$

Sol^n. Given differential equation is,

$$py = p^2(x - b) + a$$

$$\text{or, } y = px + \frac{a}{p} - bp \text{ ..... (i)}$$

Differential equation w. r.t , 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{a}{p}\frac{dp}{dx} - b\frac{dp}{dx}$$

$$\text{or, } p = p + \left(x - \frac{a}{p^2} - b\right)\frac{dp}{dx}$$

$$\text{or, } \left(x - \frac{a}{p^2} - b\right)\frac{dp}{dx} = 0$$

$$\text{Either } \frac{dp}{dx} = 0 \Rightarrow p = c \text{ ..... (ii)}$$

From (i) and (ii)

$yC = C^2(x - b) + a$ is the required general solution.

$$\text{or, } x - \frac{a}{p^2} - b = 0$$

$$\text{or, } \frac{a}{p^2} = (x - b)$$

$$p = \sqrt{\frac{a}{x - b}} \text{ ..... (ii)}$$

From (i) and (iii)

$$\sqrt{\frac{a}{x-b}} \cdot y = \frac{a}{(x-b)}(x - b) + a$$

$$\text{or, } y\frac{\sqrt{a}}{\sqrt{x-b}} = a + a = 2a$$

Squaring on both sides,

$y^2a = 4a^2(x - b)$

or, $y^2 = 4a(x - b)$ is the required singular solution.

4. $y = px + ap - ap^2$

Sol^n. Given differential equation is,

$$y = px + ap - ap^2 \text{ ..... (i)}$$

Differential equation w. r. t. 'x' we get,

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + a\frac{dp}{dx} - 2ap\frac{dp}{dx}$$

$$\text{or, } p = p + x\frac{dp}{dx} + a\frac{dp}{dx} - 2ap\frac{dp}{dx}$$

$$\text{or, } 0 = (x + a - 2ap)\frac{dp}{dx}$$

$$\text{or, } (x + a - 2ap)\frac{dp}{dx} = 0$$

$$\text{Either } \frac{dp}{dx} = 0 \Rightarrow p = C \text{ ..... (ii)}$$

From (i) and (ii)

$y = Cx + aC - aC^2$ is the required general solution.

or, $x + a - 2ap = 0$

or, $2ap = x + a$

$$p = \frac{x + a}{2a} \text{ ..... (iii)}$$

From (i) and (iii)

$$y = x \cdot \frac{(x+a)}{2a} + a\frac{(x+a)}{2a} - a\frac{(x+a)^2}{4a^2}$$

$$\text{or, } y = \frac{(x+a)}{2a}\left[x + a - \frac{a(x+a)}{2a}\right]$$

$$= \frac{(x+a)}{2a}\frac{(2ax + 2a^2 - ax - a^2)}{2a}$$

$$= \frac{(x+a)}{2a}\frac{(ax + a^2)}{2a}$$

$$= \frac{(x+a)}{2a}\frac{(x+a)}{2}$$

$4ay = (x + a)^2$ is the required singular solution.

5. $(y - px)^2(1 + p^2) = a^2p^2$

Sol^n. Given differential equation is,

$$(y - px)^2(1 + p^2) = a^2p^2$$

$$\text{or, } (y - px)^2 = \frac{a^2p^2}{(1 + p^2)}$$

$$\Rightarrow y - px = \frac{ap}{\sqrt{1 + p^2}}$$

or, $y = px + \frac{ap}{\sqrt{1+p^2}}$ ..... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \left[\frac{\sqrt{1+p^2}\cdot a\frac{dp}{dx} - aP\cdot\frac{2p}{2\sqrt{1+p^2}}\cdot\frac{dp}{dt}}{(1+p^2)}\right]$$

$$p = p + x\frac{dp}{dx} + \frac{dp}{dx}\left[\frac{a\sqrt{1+p^2}}{(1+p^2)} - \frac{ap^2}{(1+p^2)\sqrt{1+p^2}}\right]$$

$$\text{or, } 0 = \frac{dp}{dx}\left[x + a\left\{\frac{1+p^2-p^2}{(1+p^2)}\right\}\right]$$

$$\text{or, } \frac{dp}{dx}\left[x + \frac{a}{(1+p^2)\left(\sqrt{1+p^2}\right)}\right] = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow p = C$ ..... (ii)

From (i) and (2)

$(y - Cx)^2 (1 + C^2) = a^2C^2$ is the required general solution.

$$\text{or, } x + \frac{a}{\left(1+p^2\right)\sqrt{1+p^2}} = 0$$

$$\text{or, } x = -\frac{a}{\left(1+p^2\right)^{\frac{3}{2}}}$$

$$\text{or, } x^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}}{(1+p^2)} \text{ ..... (iii)}$$

Also, from $y = px + \frac{ap}{\sqrt{1+p^2}}$

$$y = -\frac{pa}{\left(1+p^2\right)^{\frac{3}{2}}} + \frac{ap}{\sqrt{1+p^2}} = \frac{ap[-1+1+p^2]}{\left(1+p^2\right)^{\frac{3}{2}}}$$

$$\text{or, } y = \frac{ap^3}{\left(1+p^2\right)^{\frac{3}{2}}}$$

$$\text{or, } y^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}p^2}{\left(1+p^2\right)} \text{ ..... (iv)}$$

From (iii) and (iv)

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}}{\left(1+p^2\right)} + \frac{a^{\frac{2}{3}}\cdot p^2}{1+p^2} = \frac{a^{\frac{2}{3}}(1+p^2)}{\left(1+p^2\right)}$$

$\therefore x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is the required singular solution.

6. **$p = \log(px - y)$**

Sol^n. Given differential equation is,

$p = \log(px - y)$

$\Rightarrow px - y = e^p$

or, $y = px - e^p$ ..... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - e^p\frac{dp}{dx}$$

$$p = p + (x - e^p)\frac{dp}{dx}$$

$$\text{or, } \frac{dp}{dx}(x - e^p)\frac{dp}{dx}$$

$$\text{or, } \frac{dp}{dx}(x - e^p) = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow p = C$ ..... (ii)

From (i) and (ii)

$y = Cx - e^C$ is the required general solution.

or, $x - e^p = 0$

or, $x = e^p$

$\Rightarrow p = \log x$ ..... (iii) from equation (i) and (iii)

$y = x\log x - e^{\log x}$

or, $y = x\log x - x$ is the required singular solution.

7. **$(y + 1)p - xp^2 + 2 = 0$**

Sol^n. Given differential equation is,

$(y + 1)p - xp^2 + 2 = 0$

or, $yp + p = xp^2 - 2$

or, $yp = xp^2 - p - 2$

or, $y = px - 1 - \frac{2}{p}$ ..... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{2}{p^2}\frac{dp}{dx}$$

or, $p = p + \left(x + \frac{2}{p^2}\right)\frac{dp}{dx}$

or, $\frac{dp}{dx}\left(x + \frac{2}{p^2}\right) = 0$

Either $\frac{dp}{dx} = 0 \Rightarrow p = C$ ..... (ii)

From equation (i) and (ii)

$(y+1)C - xC^2 + 2 = 0$ is the required general solution.

or, $x + \frac{2}{p^2} = 0$

or, $p^2 = -\frac{2}{x}$

Using $p^2 = -\frac{2}{x}$ in $(y+1)p = xp^2 - 2$

or, $(y+1)^2 p^2 = x^2(p^2-2)^2$

or, $(y+1)^2\left(-\frac{2}{x}\right) = x^2\left[\left(-\frac{2}{x}\right)(-2)\right]^2$

or, $-\frac{2}{x}(y+1)^2 = x^2\left(\frac{4}{x}\right)^2$

or, $-\frac{2}{x}(y+1) = 16$

or, $(y+1)^2 = -\frac{16x}{2}$

or, $(y+1)^2 + 8x = 0$ is the required singular solution.

8. $(xp - y)^2 = p^2 - 1$

Sol^n. Given differential equation is,

$(xp - y)^2 = p^2 - 1$

or, $xp - y = \sqrt{p^2 - 1}$

$y = xp + \sqrt{p^2 - 1}$ ..... (i)

Differential equation (i) w. r. t. 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{2}\cdot\frac{2p}{\sqrt{p^2-1}}\frac{dp}{dx}$$

or, $p = p + \left(x + \frac{p}{\sqrt{p^2-1}}\right)\frac{dp}{dx}$

or, $\frac{dp}{dx}\left(x + \frac{p}{\sqrt{p^2-1}}\right) = 0$

Either $\frac{dp}{dx} = 0 \Rightarrow p = C$ ..... (ii)

From (i) and (ii)

$(xC - y)^2 = C^2 - 1$ is the required general solution.

or, $x + \frac{p}{\sqrt{p^2-1}} = 0$

or, $x = -\frac{p}{\sqrt{p^2-1}}$ ..... (iii)

From (i) and (iii)

$$y = -\frac{p}{\sqrt{p^2-1}}p + \sqrt{p^2-1} = -\frac{p^2+p^2-1}{\sqrt{p^2-1}} = \frac{-1}{\sqrt{p^2-1}}$$

or, $y = -\frac{1}{\sqrt{p^2-1}}$ squaring on both sides

$y^2 = \frac{1}{\sqrt{p^2-1}}$

or, $p^2 - 1 = \frac{1}{y^2}$

or, $p^2 = 1 + \frac{1}{y^2} = \frac{y^2+1}{y^2}$ ..... (iv)

From (iii) and (iv)

$$x^2 = \frac{p^2}{1-p^2} = \frac{\frac{y^2+1}{y^2}}{1-\frac{y^2+1}{y^2}} = \frac{y^2+1}{y^2}\cdot\frac{y^2}{\left(y^2-y^2+1\right)}$$

or, $x^2 = y^2 + 1$

or, $x^2 - y^2 = 1$ is the required singular solution.

Second Order Linear Differential Equation with Constant Coefficient

17

An equation of the form $\frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = Q$

Where p_1 and p_2 are constants and Q is function of x or constant is called second order linearm differential equation with constant coefficient.

Note : If we write $\frac{d}{dx} = D$ and $\frac{d^2}{dx^2} = D^2$

Then second order linear differential equation with constant coefficient can be written as,

$$(D^2 + p_1 D + p_2)\, y = 0 \text{ and}$$
$$(D^2 + p_1 D + p_2)\, y = Q$$

Theorem : 1

If $y = f_1(x)$ and $y = f_2(x)$ are two independent solutions of the equation

$$(D^2 + p_1 D + p_2)\, y = 0$$

i.e. $f(D)\, y = 0$ then $y = C_1 f_1(x) + C_2 f_2(x)$

Will be the general solution of the equation.

Solution of $\frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = 0$

or, $(D^2 + p_1 D + p_2)\, y = 0$ ..... (i)

Where, $\frac{d}{dx} = D$ and $\frac{d^2}{dx^2} = D^2$

Let $y = e^{mx}$ be a solution of (i) then (i) becomes,

$$m^2 e^{mx} + p_1 m e^{mx} + p_2 e^{mx} = 0$$
$$\text{or, } e^{mx}(m^2 + p_1 m + p_2) = 0$$

Since, $e^{mx} \neq 0$,

So, $m^2 + p_1 m + p_2 = 0$

Which is called Auxillary equation (A. E.) of the given differential equation and it is quadratic equation in m. So m has two roots.

Case - 1

If m has two different real roots, say $m = \alpha$ and $m = \beta$ then the general solution of (i) is

$$y = C_1 e^{\alpha x} + C_2 e^{\beta x}$$

Case - 2

If m has two real and equal roots, say $m = \alpha, \alpha$ then the general solution of (i) is $y = (C_1 + C_2 x)\, e^{\alpha x}$

Case - 3

If m has two imaginary roots say $m = \alpha + i\beta$ and $\alpha - i\beta$ then the general solution of (i) is,

$$y = e^{\alpha x}(A \cos\beta x + B \sin\beta x)$$

Exercise - 29

Solve the following equations.

1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

$(D^2 + 3D + 2)y = 0$

Putting $y = e^{mx}$

its AE

$m^2 + 3m +$

Solⁿ. Given differential equation is,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$
$$\text{or, } (D^2 + 3D + 2)\, y = 0$$

So, It's A. E. is,

$$m^2 + 3m + 2 = 0$$
$$\text{or, } m^2 + 2m + m + 2 = 0$$
$$\text{or, } m(m + 2) + 1(m + 2) = 0$$
$$\text{or, } (m + 2)(m + 1) = 0$$
$$\therefore m = -1, -2$$

Thus, $y = C_1 e^{-x} + C_2 e^{-2x}$ is the required general solution.

2. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$

Solⁿ. Given differential equation is,

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$$
$$\text{or, } (D^2 + 5D + 4)\, y = 0$$

So, It's A. E. is,

$$m^2 + 5m + 4 = 0$$
$$\text{or, } m^2 + 4m + m + 4 = 0$$
$$\text{or, } m(m + 4) + 1(m + 4) = 0$$
$$\text{or, } (m + 4)(m + 1) = 0$$
$$\therefore m = -1, -4$$

Thus, $y = C_1 e^{-x} + C_2 e^{-4x}$ is the required general solution.

3. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

Solⁿ. Given differential equation is,

$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

or, $(D^2+2D+1)\,y=0$

So, It's A. E. is,

$m^2+2m+1=0$

or, $(m+1)^2=0$

$\Rightarrow m=-1,-1$

This, $y=(C_1+C_2x)\,e^{-x}$ is the required general solution.

4. $(D^2+D)\,y=0$

Solⁿ. Given differential equation is,

$(D^2+D)\,y=0$

So, It's A. E. is;

$m^2+m=0$

or, $m\,(m+1)=0$

$\Rightarrow m=0, m=-1$

Thus, $y=C_1+C_2\,e^{-x}$ is the required general solution.

5. $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+25y=0$

Solⁿ. Given differential equation is,

$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+25y=0$

or, $(D^2+6D+25)\,y=0$

So, It's A. E. is the $m^2+6m+25=0$

$$\text{or, } m=\frac{-6\pm\sqrt{(6)^2-4.1.25}}{2.1}$$

$$=\frac{-6\pm\sqrt{36-100}}{2}=\frac{-6\pm\sqrt{-64}}{2}=\frac{-6\pm 8i}{2}=-3\pm 4i$$

Thus, $y=e^{-3x}$ (A cos 4x + B sin 4x) is the required general solution.)

6. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13\,y=0$

Solⁿ. Given differential equation is,

$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0$

or, $(D^2+4D+13)\,y=0$

So, it's A. E. is,

$m^2+4m+13=0$

$$\text{or, } m=\frac{-4\pm\sqrt{4^2-4.1.13}}{2.1}=\frac{-4\pm 6i}{2}$$

$\therefore m=-2\pm 3i$

Thus, $y=e^{-2x}$ (A cos 3x + B sin 3x) is the required general solution.

7. $16\frac{d^2y}{dx^2}+24\frac{dy}{dx}+9y=0$

Solⁿ. Given differential equation is,

$16\frac{d^2y}{dx^2}+24\frac{dy}{dx}+9y=0$

or, $(16D^2+24D+9)\,y=0$

So, it's A. E. is,

$16m^2+24m+9=0$

or, $16m^2+12m+12m+9=0$

or, $4m\,(4m+3)+3\,(4m+3)=0$

or, $(4m+3)\,(4m+3)=0$

$\Rightarrow m=-\frac{3}{4},-\frac{3}{4}$

Thus, $y=(C_1+C_2\,x)\,.\,e^{-\frac{3}{4}x}$ is the required general solution.

8. $(D+3)^2\,y=0$

Solⁿ. Given differential equation is,

$(D+3)^2\,y=0$

So, it's, A. E. is;

$(m+3)^2=0$

$\Rightarrow m=-3,-3$

Thus, $y=(C_1+C_2x)\;e^{-3x}$ is the required general solution.

9. $(D^2+3aD-4a^2)\,y=0$

Solⁿ. Given differential equation is,

$(D^2+3aD-4a^2)\,y=0$

So, it's, A. E. is,

$m^2+3am-4a^2=0$

or, $m^2+4am-am-4a^2=0$

or, $m\,(m+4a)-a\,(m+4a)=0$

or, $(m + 4a)(m - a) = 0$

$\Rightarrow m = a, -4a$

Thus, $y = C_1 e^{ax} + C_2 e^{-4ax}$ is the required general solution.

10. Solve $\frac{d^2x}{dt^2} + \mu x = 0, \mu > 0$ given that $x = a$ and $\frac{dx}{dt} = 0$ when $t = \frac{\pi}{2\sqrt{\mu}}$.

Sol^n. Given differential equation is,

$$\frac{d^2x}{dt^2} + \mu x = 0$$

or, $(D^2 + \mu)x = 0$ where $D = \frac{d}{dt}$ and $D^2 = \frac{d^2}{dt^2}$

So, it's A. E. is;

$m^2 + \mu = 0$

or, $m = \pm i\sqrt{\mu}$

Thus, $x = A\cos\sqrt{\mu}\, t + B\sin\sqrt{\mu}\, t$ ..... (i)

Differential equation (i) w. r. t. 't'

$\frac{dx}{dt} = -A\sqrt{\mu}\sin\sqrt{\mu}\, t + B\sqrt{\mu}\cos\sqrt{\mu}\, t$ ..... (ii)

Using, when $t = \frac{\pi}{2\sqrt{\mu}}$ given,

$x = a$ and $\frac{dx}{dt} = 0$ in (i) and (ii) respectively.

We get,

$a = 0 + B \Rightarrow B = a$

and $0 = -A\sqrt{\mu} \Rightarrow \mu > 0$

So, $A = 0$

Hence, (i) becomes,

$x = a\sin\sqrt{\mu}\, t$ is the required particular solution.

11. $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$ given that $x = 1$ when $t = 0$ and $\frac{dx}{dt} = 1$ when $t = 0$.

Sol^n. Given differential equation is,

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$$

or, $(D^2 - 3D + 2)x = 0$ where, $D = \frac{d}{dt}$ and $D^2 = \frac{d^2}{dt^2}$

So, it's A. E. is;

$m^2 - 3m + 2 = 0$

or, $m^2 - 2m - m + 2 = 0$

or, $m(m - 2) - 1(m - 2) = 0$

or, $(m - 1)(m - 2) = 0$

$\Rightarrow m = 1, m = 2$

Thus, $x = C_1 e^t + C_2 e^{2t}$ ..... (i)

Differential equation (i) w. r. t. 't'

$\frac{dx}{dt} = C_1 e^t + 2C_2 e^{2t}$ ..... (ii)

Using, when $t = 0$ given that $x = 1$ and $\frac{dx}{dt} = 0$ in (i) and (ii) respectively we get,

$1 = C_1 + C_2$ and $0 = C_1 + 2C_2$

or, $C_1 = -2C_2$

$\therefore 1 = -2C_2 + C_2$

$\Rightarrow C_2 = -1$ and $C_1 = 2$

Hence, (i) becomes,

$x = 2e^t - e^{2t}$ is the required particular solution.

12. $\frac{d^2x}{dt^2} + y = 0$ given that $y = 4$ when $x = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

Sol^n. Given differential equation is,

$$\frac{d^2x}{dt^2} + y = 0$$

or, $(D^2 + 1)y = 0$

So, it's A. E. is;

$m^2 + 1 = 0 \Rightarrow m = \pm i$

Thus, $y = A\cos x + B\sin x$ ..... (i)

Differential equation (i) w. r. t. 'x'

$\frac{dy}{dx} = -A\sin x + B\cos x$ ..... (ii)

Using when $x = 0$ given $y = 4$ and $\frac{dy}{dx} = 0$ in (i) and (ii) respectively we get,

$4 = A$ and $0 = B$

Hence, (i) becomes,

$y = 4\cos x$ is the required particular solution.

Particular Integral

An equation is of the form $(D^2 + P_1D + P_2)\,y = Q$ i.e. $f(D)\,y = Q$ is called second order linear differential equation with constant coefficient.

Where, P_1, P_2 are constant and Q is function of x or constant.

Clearly, this equations is satisfied by,

$$y = \frac{1}{f(D)} \cdot Q$$

So that $\frac{1}{f(D)} \cdot Q$ is the particular Integral (P. I.) of the given differential equation.

$$\therefore \text{P. I.} = \frac{1}{f(D)} \cdot Q$$

The general solution of $(D^2 + P_1D + P_2)\,y = 0$ is called the complementary function (C. F.)

Theorem : II

If $y = f(x)$ is the complete solution of $f(D)\,y = 0$ and $y = g(x)$ is a particular solution of the equation $f(D)y = Q$ then the complete solution of the equation $f(D)\,y = Q$ is $y = f(x) + g(x)$.

Theorem : III

If Q be function of x then $\frac{1}{D}$ Q operates the integration of Q with respect to x.

$$\text{i.e. } \frac{1}{D}Q = \int Q\,dx.$$

Theorem : IV

$\frac{1}{(D-a)}Q = e^{ax}\int Q.e^{-ax}\,dx$ where a is any constant.

Working Rules for Finding Particular Integral (P.I.)

For $\text{P. I.} = \frac{1}{f(D)} \cdot Q$

1. When $Q = e^{ax}$ where a is any constant.

$$\text{P. I.} = \frac{1}{f(D)}\,e^{ax} = \frac{1}{f(a)}\,e^{ax}, \text{ if } f(a) \neq 0$$

Note : (i) If $f(a) = 0$ then,

$$\text{P.I.} = x.\ \frac{1}{f'(D)}\,e^{ax} = x.\ \frac{1}{f'(a)}\,e^{ax}, \text{ if } f'(a) \neq 0$$

(ii) If $f'(a) = 0$ then,

$$\text{P.I.} = x^2\ \frac{1}{f''(D)}\,e^{ax} = x^2.\ \frac{1}{f''(a)}\,e^{ax}, \text{ if } f''(a) \neq 0 \text{ and so on.}$$

2. When $Q = \sin ax$. or $\cos ax$

$$\text{P. I.} = \frac{1}{f(D^2)}\ \sin ax = \frac{1}{f\left(-a^2\right)}\ \sin ax \text{ if } f(-a^2) \neq 0.$$

Note : (i) If $f(-a^2) = 0$ then,

$$\text{P. I.} = x.\ \frac{1}{f'(D^2)}\ \sin ax = x.\ \frac{1}{f'(a^2)}\ \sin ax \text{ if } f'(-a^2) \neq 0$$

(ii) If $f'(-a^2) = 0$ then,

$$\text{P. I.} = x^2\ \frac{1}{f''(D^2)}\ \sin ax \text{ if } f''(-a^2) \neq 0 \text{ and so on.}$$

Similarly, $\text{P. I.} = \frac{1}{f\left(D^2\right)}\ \cos ax = \frac{1}{f\left(-a^2\right)}\cos ax$ if $f(-a^2) \neq 0$

Note : (i) If $f(-a^2) = 0$; then,

$$\text{P. I.} = x\ \frac{1}{f'(D^2)}\ \cos ax = x\ \frac{1}{f'(-a^2)}\ \cos ax \text{ if } f'(-a^2) \neq 0$$

(ii) $f'(-a^2) = 0$ then,

$$\text{P. I.} = x^2.\ \frac{1}{f''\left(D^2\right)}\ \cos ax = x^2\ \frac{1}{f''\left(-a^2\right)}\ \cos ax,$$

if $f''(-a^2) \neq 0$ and so on.

3. If $Q = x^m$, where m is positive integer.

$$\text{Then, P. I.} = \frac{1}{f(D)}Q = \frac{1}{f(D)}\ x^m$$

$$= \frac{1}{\left(D^2 + P_1D + P_2\right)}x^m = \frac{1}{p^2}\left(1 + \frac{D^2 + P_1D}{P_2}\right)^{-1} x^m$$

$[f(D)]^{-1}$ can be expanded in ascending power of D and then operate on x^m with each term of the expansions.

It can be expanded by the Binomial Theorem as follows:

(i) $(1 + D)^{-1} = 1 - D + D^2 - D^3 + \ldots\ldots\ldots$

(ii) $(1 - D)^{-1} = 1 + D + D^2 + D^3 + \ldots\ldots\ldots$

(iii) $(1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \ldots\ldots\ldots$

(iv) $(1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \ldots\ldots\ldots$

4. If $Q = e^{ax}\,V$ where V is the function of x, then

$$\text{P. I.} = \frac{1}{f(D)}Q = \frac{1}{f(D)}\ e^{ax}\,V = e^{ax}\ \frac{1}{f(D+a)}.\ V$$

5. If $Q = x\,V$ where V is function of x. Then

$$P.I. = \frac{1}{f(D)} Q = \frac{1}{f(D)} x\,V = x \frac{1}{f(D)} V - \frac{f'(D)}{[F(D)]^2} V$$

6. If $Q = x^m \cos ax$ and $Q = x^m \sin ax$. Then,

$$P.I. = \frac{1}{f(D)} Q = \frac{1}{f(D)} x^m \cos ax = \frac{1}{f(D)} [\text{Real part of } x^m e^{iax}]$$

and

$$P.I. = \frac{1}{f(D)} Q = \frac{1}{f(D)} x^m \sin ax$$

$$= \frac{1}{f(D)} [\text{Imaginary part of } x^m e^{iax}]$$

Note : General solution $(D^2y + P_1D + P_2)\,y = Q$ is $y = C.F. + P.I.$

Exercise - 30

Solve the following equation.

1. $(D^2 - 1)\,y = 5e^{2x}$

Sol^n. Given differential equation is,

$(D^2 - 1)\,y = 5e^{2x}$.

So, It's A. E. is,

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

So, $C.F. = C_1e^x + C_2e^{-x}$

and $P.I. = \dfrac{1}{(D^2-1)} 5e^{2x}$

$= \dfrac{1}{(2^2-1)} 5e^{2x} = \dfrac{5}{3} e^{2x}$

Thus, $y = C.F. + P.I.$

or, $y = C_1e^x + C_2e^{-x} + \dfrac{5}{3} e^{2x}$ is the required general solution.

2. $(D-2)^2 y = e^{4x}$

Sol^n. Given differential equation is,

$(D-2)^2 y = e^{4x}$.

So, it's A. E. is, $(m-2)^2 = 0 \Rightarrow m = 2, 2$

So, $C.F. = (C_1 + C_2x)\,e^{2x}$

and $P.I. = \dfrac{1}{(D-2)^2} e^{4x} = \dfrac{1}{(4-2)^2} e^{4x} = \dfrac{1}{4} e^{4x}$

Thus, $y = C.F. + P.I.$

or, $y = (C_1 + C_2x)\,e^{2x} + \dfrac{e^{4x}}{4}$ is the required general solution.

3. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 2e^{3x}$

Sol^n. Given differential equation is,

$\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 2e^{3x}$

or, $(D^2 - 4D + 3)\,y = 2e^{3x}$

So, It's A. E. is,

$m^2 - 4m + 3 = 0$

or, $m^2 - 3m - m + 3 = 0$

or, $m(m-3) - 1(m-3) = 0$

or, $(m-1)(m-3) = 0$

$\Rightarrow m = 1, 3$

So, $C.F. = C_1e^x + C_2e^{3x}$

and $P.I. = \dfrac{1}{(D^2 - 4D + 3)} 2 \cdot e^{3x}$

$= x \cdot \dfrac{1}{(2D-4)} 2 \cdot e^{3x} = x \cdot \dfrac{1}{(6-4)} \cdot 2e^{3x} = x \cdot \dfrac{1}{2} \cdot 2e^{3x} = xe^{3x}$

Thus, $y = C.P + F.I.$

$y = C_1e^x + C_2e^{3x} + xe^{3x}$ is the required general solution.

4. $\dfrac{d^2y}{dx^2} + 4y = \sin 2x$

Sol^n. Given differential equation is,

$\dfrac{d^2y}{dx^2} + 4y = \sin 2x$

or, $(D^2 + 4)\,y = \sin 2x$

So, it's A. E. is, $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

So, $C.F. = A\cos 2x + B\sin 2x$

and $P.I. = \dfrac{1}{(D^2+4)} \cdot \sin 2x$

$= x \cdot \dfrac{1}{2D} \sin 2x = \dfrac{x}{2} \int \sin 2x\, dx = -\dfrac{x}{4} \cos 2x$

Thus, $y = C.F. + P.I.$

or, $y = A\cos 2x + B\sin 2x - \dfrac{x}{4} \cos 2x$ is the required general solution.

5. $(D^2 + 16) = \cos 4x$

Solⁿ. Given differential equation is,

$(D^2 + 16)\, y = \cos 4x$

So, Its, A. E. is; $m^2 + 16 = 0 \Rightarrow m = \pm 4i$

So, C. F. = $A \cos 4x + B \sin 4x$

and P. I. = $\dfrac{1}{(D^2+16)} \cos 4x$

or, P. I. = $x \cdot \dfrac{1}{2D} \cos 4x = \dfrac{x}{2} \int \cos 4x \, dx$

or, P.I. = $\dfrac{x}{8} \sin 4x$

Thus, y = C. F. + P. I.

or, $y = A \cos 4x + B \sin 4x + \dfrac{x}{8} \sin 4x$ is the required general solution.

6. $\dfrac{d^2y}{dx^2} + y = \cos^2 x$

Solⁿ. Given differential equation is,

$\dfrac{d^2y}{dx^2} + y = \cos^2 x$

or, $(D^2 + 1)\, y = \cos^2 x$

or, It's A. E. is; $m^2 + 1 = 0 \Rightarrow m = \pm i$

So, C. F. = $A \cos x + B \sin x$

and P. I. = $\dfrac{1}{(D^2+1)} \cos^2 x = \dfrac{1}{(D^2+1)} \dfrac{(1+\cos 2x)}{2}$

$= \dfrac{1}{2}\left[\dfrac{1}{(D^2+1)} e^{0x} + \dfrac{1}{(D^2+1)} \cos 2x\right]$

$= \dfrac{1}{2}\left[1 + \dfrac{1}{(-2^2+1)} \cos 2x\right] = \dfrac{1}{2}\left(1 - \dfrac{\cos 2x}{3}\right)$

or, P. I. = $\dfrac{1}{2} - \dfrac{1}{6} \cos 2x$

Thus, y = C. F. + P. I.

or, $y = A \cos x + B \sin x + \dfrac{1}{2} - \dfrac{1}{6} \cos 2x$ is the required general solution.

7. $(D^2 - D - 2)\, y = \sin 2x + e^x$

Solⁿ. Given differential equation is,

$(D^2 - D - 2) y = \sin 2x + e^x$

So, it's A. E. is $m^2 - m - 2 = 0$

or, $m^2 - 2m + m - 2 = 0$

or, $m(m-2) + 1(m-2) = 0$

or, $(m+1)(m-2) = 0$

$\therefore m = -1, 2$

So, C. F. = $C_1 e^{-x} + C_2 e^{2x}$

and P. I. = $\dfrac{1}{(D^2-D-2)} (\sin 2x + e^x)$

$= \dfrac{1}{(D^2-D-2)} \sin 2x + \dfrac{1}{(D^2-D-2)} e^x$

$= \dfrac{1}{(-2^2-D-2)} \sin 2x + \dfrac{1}{1-1-2} e^x$

$= \dfrac{1}{-(D+6)} \sin 2x - \dfrac{1}{2} e^x$

$= \dfrac{-(D-6)}{(D^2-36)} \sin 2x - \dfrac{1}{2} e^x$

$= \dfrac{-(D-6)}{(-2^2-36)} \sin 2x - \dfrac{1}{2} e^x$

$= \dfrac{2}{40} \cos 2x + \dfrac{6}{40} \sin 2x - \dfrac{1}{2} e^x$

$= \dfrac{1}{20} \cos 2x + \dfrac{3}{20} \sin 2x - \dfrac{1}{2} e^x$

$= \dfrac{1}{20} [\cos 2x + 3 \sin 2x] - \dfrac{1}{2} e^x$

Thus, y = C. F. + P. I.

$y = C_1 e^{-x} + C_2 e^{2x} - \dfrac{1}{2} e^x + \dfrac{1}{20} (\cos 2x + 3 \sin 2x)$ is the required general solution.

8. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{2x} \sin x$

Solⁿ. Given differential equation is,

$\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{2x} \sin x$

or, $(D^2 + 3D + 2)\, y = e^{2x} \sin x$

So, It,s A. E. is;

$$m^2 + 3m + 2 = 0$$

or, $m^2 + 2m + m + 2 = 0$

or, $m(m+2) + 1(m+2) = 0$

or, $(m+2)(m+1) = 0$

$\therefore m = -1, -2$

So, C. F. $= C_1 e^{-x} + C_2 e^{-2x}$

and P. I. $= \dfrac{1}{\left(D^2+3D+2\right)} e^{2x} \sin x$

$$= e^{2x} \frac{1}{\left[(D+2)^2 + 3(D+2) + 2\right]} \sin x$$

$$= e^{2x} \frac{1}{\left(D^2+4D+4+3D+6+2\right)} \sin x$$

$$= e^{2x} \frac{1}{\left(D^2+7D+12\right)} \sin x$$

$$= e^{2x} \frac{1}{\left(-1^2+7D+12\right)} \sin x$$

$$= e^{2x} \frac{1}{(7D+1)} \sin x$$

$$= e^{2x} \frac{(7D-11)}{\left(49D^2-121\right)} \sin x$$

$$= e^{2x} \frac{(7D-11)}{(-49-121)} \sin x$$

$$= e^{2x} \left(\frac{7}{-170}\cos x + \frac{11}{70}\sin x\right)$$

$$= \frac{e^{2x}}{170}(11 \sin x - 7 \cos x)$$

Thus y = C. F. + P. I.

or, $y = C_1 e^{-x} + C_2 e^{-2x} + \dfrac{e^{2x}}{170}(11 \sin x - 7 \cos x)$ is the required general solution.

9. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2 e^{3x}$.

Solⁿ. Given differential equation is,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^{3x}$$

or, $(D^2 - 2D + 1)\, y = x^2 e^{3x}$

So, it's A. E.:

$$m^2 - 2m + 1 = 0$$

or, $(m-1)^2 = 0 \Rightarrow m = 1, 1$

So, C. F. $= (C_1 + C_2 x)\, e^x$

and P. I. $= \dfrac{1}{\left(D^2-2D+1\right)} x^2 e^{3x}$

$$= e^{3x} \frac{1}{\left[(D+3)^2 - 2(D+3) + 1\right]} x^2$$

$$= e^{3x} \frac{1}{\left[D^2+6D+9-2D-6+1\right]} x^2$$

$$= e^{3x} \frac{1}{\left(D^2+4D+4\right)} x^2$$

$$= \frac{e^{3x}}{4} \frac{1}{\left[1+\left(\frac{D^2+4D}{4}\right)\right]} x^2$$

$$= \frac{e^{3x}}{4} \left[1+\left(\frac{D^2+4D}{4}\right)\right]^{-1} \cdot x^2$$

$$= \frac{e^{3x}}{4} \left[1-\left(\frac{D^2+4D}{4}\right)+\left(\frac{D^2+4D}{4}\right)^2 - \ldots\right] x^2$$

$$= \frac{e^{3x}}{4} \left[x^2 - \left(\frac{2+8x}{4}\right) + \left(\frac{2}{4}\right)^2\right]$$

$$= \frac{e^{3x}}{4} \left[x^2 - \frac{1}{2} - 2x + \frac{1}{4}\right]$$

$$= \frac{e^{3x}}{4} \left(x^2 - 2x + \frac{3}{2}\right)$$

$$= \frac{e^{3x}}{8} (2x^2 - 4x + 3)$$

Thus, y = C. E. + P. I.

or, $y = (C_1 + C_2x)\,e^x + \frac{e^{3x}}{8}(2x^2 - 4x + 3)$ is the required general solution.

10. $(D^2+2)\,y = \sin\sqrt{2}\,x$

Sol^n. Given differential equation is,

$(D^2+2)\,y = \sin\left(\sqrt{2}\,x\right)$

So, it's A. E. is, $m^2+2=0 \Rightarrow m = \pm\sqrt{2}\,i$

So, C. F. $= A\cos\sqrt{2}\,x + B\sin\sqrt{2}\,x$

and P. I. $= \frac{1}{\left(D^2+2\right)}\sin\left(\sqrt{2}\,x\right)$

$= x\,.\,\frac{1}{2D}\sin\left(\sqrt{2}\,x\right)$

$= \frac{x}{2}\,.\int\sin\left(\sqrt{2}\,x\right)dx$

$= -\frac{x}{2\sqrt{2}}\cos\sqrt{2}\,x$

Thus, y = C. F. + P. I.

or, $y = A\cos(\sqrt{2}\,x) + B\sin(\sqrt{2}\,x) - \frac{x}{2\sqrt{2}}\cos(\sqrt{2}\,x)$ is the required general solution.

11. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x\cos x$

Sol^n. Given differential equation is,

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x\cos x$

or, $(D^2 - 2D + 4)\,y = e^x\cos x$

or, It's A. E. is, $m^2 - 2m + 4 = 0$

or, $m = \frac{-(-2)\pm\sqrt{(-2)^2 - 4.1.4}}{2.1} = \frac{2\pm i\sqrt{12}}{2} = 1\pm\sqrt{3}\,i$

So, C. E. $= e^x\left(A\cos\sqrt{3}\,x + B\sin\sqrt{3}\,x\right)$

and P. I. $= \frac{1}{\left(D^2-2D+4\right)}e^x\cos x$

$= e^x\frac{1}{\left[(D+1)^2 - 2(D+1) + 4\right]}\cos x$

$= e^x\frac{1}{\left(D^2+2D+1-2D-2+4\right)}\cos x$

$= e^x\frac{1}{\left(D^2+3\right)}\cos x$

$= e^x\frac{1}{(-1+3)}\cos x$

$= \frac{e^x\cos x}{2}$

Thus, y = C. F. + P. I.

or, $y = e^x\left(A\cos\sqrt{3}\,x + B\sin\sqrt{3}\,x\right) + \frac{e^x\cos x}{2}$ is the required general solution.

12. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$

Sol^n. Given differential equation is,

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$

or, $(D^2 + D - 2)\,y = x + \sin x$

So, It's A. E. is;

$m^2 + m - 2 = 0$

or, $m^2 + 2m - m - 2 = 0$

or, $m(m+2) - 1(m+1) = 0$

or, $(m-1)(m+2) = 0$

or, $m = 1, 2$

So, C. F. $= C_1e^x + C_2e^{2x}$

and P. I. $= \frac{1}{\left(D^2+D-2\right)}\,.\,(x + \sin x)$

$= \frac{1}{\left(D^2+D-2\right)}x + \frac{1}{\left(D^2+D-2\right)}\sin x$

$= \frac{-1}{2}\left[1 - \left(\frac{D^2+D}{2}\right)\right]^{-1}x + \frac{1}{(-1+D-2)}\sin x$

$= \frac{-1}{2}\left[1 + \left(\frac{D^2+D}{2}\right) + \ldots\ldots\right]x + \frac{1}{(D-3)}\sin x$

$$= \frac{-1}{2}\left(x+\frac{1}{2}\right)+\frac{(D+3)}{(D^2-9)}\sin x$$

$$= -\frac{x}{2}-\frac{1}{4}+\frac{(D+3)}{(-1-9)}\sin x$$

$$= -\frac{x}{2}-\frac{1}{4}-\frac{\cos x}{10}-\frac{3}{10}\sin x$$

or, P. I. $= -\frac{x}{2}-\frac{1}{4}-\frac{1}{10}(\cos x + 3\sin x)$

Thus, $y = C.F. + P.I.$

or, $y = C_1e^x + C_2e^{2x} - \frac{x}{2}-\frac{1}{4}-\frac{1}{10}(\cos x + 3\sin x)$

or, $y = C_1e^x + C_2e^{2x} - \frac{1}{10}(\cos x + 3\sin x) - \frac{1}{4}(2x+1)$

is the required general solution.

13. $(D^2 - 4D + 4)y = x^2 + e^{2x}$

Sol^n. Given differential equation is,

$(D^2 - 4D + 4)y = x^2 + e^{2x}$

So, its, A. E. is,

$m^2 - 4m + 4 = 0$

or, $(m-2) = 0$

or, $m = 2, 2$

So, C. F. $= (C_1 + C_2x)e^{2x}$

and P. I. $= \frac{1}{(D^2-4D+4)}.(x^2+e^{2x})$

$$= \frac{1}{(2-D)^2}x^2 + \frac{1}{(D^2-4D+4)}e^{2x}$$

$$= \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}x^2 + x.\frac{1}{(2D-4)}e^{2x}$$

$$= \frac{1}{4}\left[1+2D+3D^2+\ldots\ldots\right]x^2 + x.\frac{1}{(2D-4)}e^{2x}$$

$$= \frac{1}{4}\left(x^2+\frac{4x}{2}+\frac{6}{4}\right)+x^2\frac{1}{2}e^{2x}$$

$$= \frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)+\frac{x^2e^{2x}}{2}$$

Thus, $y = C.F. + P.I.$

$$y = (C_1+C_2x)e^{2x} + \frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)+\frac{x^2e^{2x}}{2}$$

is the required general solution.

14. $(D-2)^2 y = x^2 e^{2x}$

Sol^n. Given differential equation is,

$(D-2)^2 y = x^2 e^{2x}$

So, It's A. E. is, $(m-2)^2 = 0 \Rightarrow m = 2, 2$

So, C. E. $= (C_1 + C_2x)e^{2x}$

and P. I. $= \frac{1}{(D-2)^2}x^2e^{2x}$

$$= e^{2x}\frac{1}{\{(D+2)-2\}^2}x^2$$

$$= e^{2x}\frac{1}{D^2}x^2 = e^{2x}\frac{1}{D}\int x^2\,dx = e^{2x}\frac{1}{D}\frac{x^3}{3} = \frac{e^{2x}}{3}\int x^3\,dx$$

$$= \frac{e^{2x}}{3}.\frac{x^4}{4} = \frac{e^{2x}x^4}{12}$$

Thus, $y = C.F. + P.I.$

or, $y = (C_1 + C_2x)e^{2x} + \frac{e^{2x}x^4}{12}$ is the required general solution.

15. $(D^2 - 3D + 2)y = \cosh x$

Sol^n. Given differential equation is,

$(D^2 - 3D + 2)y = \cosh x$

So, A. E. is;

$m^2 - 3m + 2 = 0$

or, $m^2 - 2m - m + 2 = 0$

or, $m(m-2) - 1(m-2) = 0$

or, $(m-2)(m-1) = 0$

$\Rightarrow m = 1, 2$

So, C. F. $= C_1e^x + C_2e^{2x}$

and P. I. $= \frac{1}{(D^2-3D+2)}\cosh x = \frac{1}{(D^2-3D+2)}\frac{(e^x+e^{-x})}{2}$

$$= \frac{1}{2}\left[\frac{1}{(D^2-3D+2)}e^x + \frac{1}{(D^2-3D+2)}e^{-x}\right]$$

$$= \frac{1}{2}\left[x.\frac{1}{(2D-3)}e^x + \frac{1}{(-1)^2-3(-1)+2}e^{-x}\right]$$

$$= \frac{1}{2}\left(x.\frac{1}{2.1-3}e^x + \frac{1}{6}e^{-x}\right)$$

$$= -\frac{1}{2}xe^{x} + \frac{1}{12}e^{-x}$$

Thus, $y = C.F. + P.I.$

or, $y = C_1e^{x} + C_2e^{2x} + \frac{1}{12}e^{-x} - \frac{1}{2}x\,e^{x}$ is the required general solution.

16. $(D^2 - 1)\,y = \sinh x$

Sol^n. Given differential equation is,

$(D^2 - 1)\,y = \sin hx$

or, It's A. E. is $m^2 - 1 = 0 \Rightarrow m = 1, 1$

So, C. F. $= (C_1 + C_2x)\,e^{x}$

and P. I. $= \frac{1}{(D^2-1)}\sin hx$

$$= \frac{1}{(D^2-1)}\frac{(e^{x}-e^{-x})}{2}$$

$$= \frac{1}{2}\left[\frac{1}{(D^2-1)}e^{x} - \frac{1}{(D^2-1)}e^{-x}\right]$$

$$= \frac{1}{2}\left[x\frac{1}{2D}e^{x} - x\frac{1}{2D}e^{-x}\right]$$

$$= \frac{1}{4}x\,[\int e^{x}\,dx - \int e^{-x}\,dx]$$

$$= \frac{1}{4}x\,(e^{x} + e^{-x})$$

$$= \frac{x}{2}\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$= \frac{x}{2}\cosh x$$

Thus, $y = C.F. + P.I.$

or, $y = (C_1 + C_2x)\,e^{x} + \frac{x}{2}\cosh x$ is the required general solution.

17. $(D^2 + 4)\,y = x\sin^2 x$

Sol^n. Given differential equation is,

$(D^2 + 4)\,y = x\sin^2 x$

So, it's A. E. is, $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

So, C. F. $= A\cos 2x + B\sin 2x$

and P. I. $= \frac{1}{(D^2+4)}x\sin^2 x$

$$= \frac{1}{(D^2+4)}x\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(D^2+4)}x - \frac{1}{(D^2+4)}x\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x - \frac{1}{(D^2+4)}\boxed{\text{Real Part of } xe^{i2x}}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}\left(1-\frac{D^2}{4}+\ldots\right)x - \text{Real part of}\left\{e^{i2x}\frac{1}{\left[(D+2i)^2+4\right]}.x\right\}\right]$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of}\left[e^{i2x}\frac{1}{(D^2+4Di-4+4)}.x\right]$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of}\left\{e^{i2x}\frac{1}{(D^2+4Di)}.x\right\}$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of}\left\{e^{i2x}\frac{1}{4Di}\left(1+\frac{D}{4i}\right)^{-1}x\right\}$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of}\left\{e^{i2x}\frac{1}{4Di}\left(1-\frac{D}{4i}+\left(\frac{D}{4i}\right)^2-\ldots\right)x\right\}$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of}\left\{e^{i2x}.\frac{1}{4Di}\left(x-\frac{1}{4i}\right)\right\}$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of}\left[e^{i2x}\left(\frac{x^2}{8i}+\frac{x}{16}\right)\right]$$

$$= \frac{1}{8}x - \frac{1}{2}\text{ Real part of}\left\{\cos 2x + i\sin 2x\left[-\frac{ix^2}{8}+\frac{x}{16}\right]\right\}$$

$$= \frac{x}{8} + \frac{1}{32}\text{ real part of }\{\cos 2x + i\sin 2x\,(2ix^2 - x)\}$$

$$= \frac{x}{8} + \frac{1}{32}(-2x^2\sin 2x - x\cos 2x)$$

$\therefore$ P. I. $= \frac{x}{8} - \frac{x}{32}\cos 2x - \frac{1}{16}x^2 \sin 2x$

Thus, $y =$ C. F. + P. I.

$y = A\cos 2x + B\sin 2x + \frac{x}{8} - \frac{x}{32}\cos 2x - \frac{1}{16}x^2 \sin 2x$ is the required solution.

18. $(D^2 - 4)\, y = x \sin hx$

Sol^n. Given differential equation is,

$(D^2 - 4)\, y = x \sin hx$

So, It's A. E. is $m^2 - 4 = 0 \Rightarrow m = \pm 2$

So, C. F. $= C_1 e^{2x} + C_2 e^{-2x}$

and P. I. $= \frac{1}{(D^2-4)} x \sin hx$

$$= \frac{1}{(D^2-4)} x \cdot \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(D^2-4)} x e^x - \frac{1}{(D^2-4)} x e^{-x}\right]$$

$$= \frac{1}{2}\left\{e^x \frac{1}{\left[(D+1)^2-4\right]} x - e^{-x} \frac{1}{\left[(D-1)^2-4\right]} x\right\}$$

$$= \frac{1}{2}\left\{e^x \frac{1}{(D^2+2D+1-4)} x - e^{-x} \frac{1}{(D^2-2D+1-4)} x\right\}$$

$$= \frac{1}{2}\left\{e^x \frac{1}{(D^2+2D-3)} x - e^{-x} \frac{1}{(D^2-2D-3)} x\right\}$$

$$= \frac{1}{2}\left\{\frac{e^x}{-3}\left[1 - \frac{(D^2+2D)}{3}\right]^{-1} x + \frac{e^{-x}}{3}\left[1 - \frac{D^2-2D}{3}\right]^{-1} x\right\}$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left\{1 + \frac{D^2+2D}{3} + \ldots\right\} x + \frac{e^{-x}}{3}\left\{1 + \frac{D^2-2D}{3} + \ldots\right\} x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left(x + \frac{2}{3}\right) + \frac{e^{-x}}{3}\left(x - \frac{2}{3}\right)\right]$$

sin2x

$(-a^2)$

$x^2 \sin 2x$

$(D+1)^2 + (D+1)$

$$= -\frac{x}{3}\frac{(e^x - e^{-x})}{2} - \frac{2}{18} \times 2\frac{(e^x + e^{-x})}{2}$$

P. I. $= -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

This, $y =$ C. F. + P. I.

or, $y = C_1 e^{2x} + C_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$ is the required general solution.

19. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

Sol^n. Given differential equation is,

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$

or, $(D^2 - 2D + 1)\, y = x e^x \sin x$

So, its, A. E. is

$m^2 - 2m + 1 = 0$

or, $(m-1)^2 = 0 \Rightarrow m = 1, 1$

So, C. F. $= (C_1 + C_2 x)\, e^x$

and P. I. $= \frac{1}{(D^2 - 2D + 1)} x e^x \sin x$

$$= e^x \frac{1}{\left[(D+1)^2 - 2(D+1) + 1\right]} x \sin x$$

$$= e^x \frac{1}{(D^2 + 2D + 1 - 2D - 2 + 1)} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \left[x \cdot \frac{1}{D^2}\sin x - \frac{2D}{(D^2)^2}\sin x\right]$$

$= e^x [x(-\sin x) - 2\cos x]$

$= -e^x (x \sin x + 2\cos x)$

Thus, $y =$ C. E. + P. I.

or, $y = (C_1 + C_2 x)\, e^x - e^x (x \sin x + 2\cos x)$ is the required general solution.

20. $(D^2 + 2D + 1)\, y = x \cos x$

Sol^n. Given differential equation is,

$(D^2 + 2D + 1)\, y = x \cos x$

So, it's A. E. is

$m^2 + 2m + 1 = 0$

or, $(m + 1)^2 = 0 \Rightarrow m = -1, -1$

So, C. F. $= (C_1 + C_2 x)\, e^{-x}$

And P. I. $= \dfrac{1}{(D^2+2D+1)}(x \cos x)$

$$= x \frac{1}{(D^2+2D+1)} \cos x - \frac{2D+2}{(D^2+2D+1)^2} \cos x$$

$$= x \frac{1}{(-1+2D+1)} \cos x - \frac{(2D+2)}{(-1+2D+1)^2} \cos x$$

$$= \frac{x}{2} \cdot \frac{1}{D} \cos x - \frac{2(D+1)}{4D^2} \cos x$$

$$= \frac{x}{2} \sin x - \frac{1}{2D^2} (D+1) \cos x$$

$$= \frac{x}{2} \sin x - \frac{1}{2D^2} (\sin x + \cos x)$$

$D^2 = -a^2$

$$= \frac{x}{2} \sin x - \frac{1}{2} (\sin x - \cos x)$$

Thus, y = C. F. + P. I.

or, $y = (C_1 + C_2 x)\, e^{-x} + \dfrac{x}{2} \sin x - \dfrac{1}{2} (\sin x - \cos x)$ is the required general solution.

Homogeneous Linear Differential Equation

An equation is of the form $x^2 \dfrac{d^2y}{dx^2} + P_1 x \dfrac{dy}{dx} + P_2 y = Q$

or, $(x^2D^2 + P_1 xD + P_2)\, y = Q$ where P_1 and P_2 are constant and Q is function of x or constant is called second order homogeneous linear differential equation.

Than can be reduced to the linear differential equation with constant coefficient by the substitution.

$x = e^z$ or $z = \log x$ and $\dfrac{dz}{dx} = \dfrac{1}{x}$

Exercise - 31

Solve the following differential equations.

1. $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} + 2y = \dfrac{1}{x}$

Sol^n. Given differential equation is,

$$x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = \frac{1}{x}$$

or, $(x^2D^2 - 2xD + 2)\, y = \dfrac{1}{x}$ ..... (i)

Equation (i) is homogeneous differential equation.

So, put $x = e^z \Rightarrow z = \log x$

$xDy = \delta y$, $x^2D^2y = (\delta^2 - \delta)\, y$ where $\delta = \dfrac{d}{dz}$ in equation (i)

Now equation (i) becomes,

$(\delta^2 - \delta - 2\delta + 2)\, y = \dfrac{1}{e^z}$

or, $(\delta^2 - 3\delta + 2)\, y = -e^z$

So, it's A. E. is;

$m^2 - 3m + 2 = 0$

or, $m^2 - 2m - m + 2 = 0$

or, $m(m-2) - 1(m-2) = 0$

or, $(m-1)(m-2) = 0$

$\therefore m = 1, m = 2$

Hence, C. F. $= C_1 e^z + C_2 e^{2z}$

and P. I. $= \dfrac{1}{\left(\delta^2 - 3\delta + 2\right)} e^{-z}$

$= \dfrac{1}{(-1)^2 - 3(-1) + 2} e^{-z} = \dfrac{1}{6} e^z$

Thus, y = C. F. + P. I.

or, $y = C_1 e^z + C_2 e^{2z} + \dfrac{1}{6} e^z$

or, $y = C_1 e^{\log x} + C_2 e^{2\log x} + \dfrac{1}{6x}$

or, $y = C_1 x + C_2 x^2 + \dfrac{1}{6x}$

is the required general solution.

2. $(x^2D^2 + xD - 1)\, y = x^2$

Sol^n. Given differential equation is,

$(x^2D^2 + xD - 1)\, y = x^2$ ..... (i)

Equation (i) is homogeneous differential equation

So, put $x = e^z \Rightarrow \log x = z$

$xDy = \delta y,\ x^2D^2y = (\delta^2 - \delta)\, y$ where $\delta = \dfrac{d}{dz}$ in (i)

Now (i) becomes,

$(\delta^2 - \delta + \delta - 1)\, y = e^{2z}$

or, $(\delta^2 - 1)\, y = e^{2z}$

So, it's A. E. is, $m^2 - 1 = 0 \Rightarrow m = \pm 1$

Hence, C. F. $= C_1 e^z + C_1 e^{-z}$

and P. I. $= \dfrac{1}{\left(\delta^2 - 1\right)} e^{2z} = \dfrac{1}{\left(2^2 - 1\right)} e^{2z} = \dfrac{e^{2z}}{(4-1)} = \dfrac{e^{2z}}{3}$

Thus, y = C. F. + P. I.

or, $y = C_1 e^z + c_2 e^{-z} + \dfrac{e^{2z}}{3}$

or, $y = C_1 e^{\log x} + C_2 e^{-\log x} + \dfrac{e^{2\log x}}{3}$

or, $y = C_1 x + \dfrac{C_2}{x} + \dfrac{x^2}{3}$ is the required solution.

3. $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} + 2y = 4x^3$

Sol^n. Given differential equation is,

$x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} + 2y = 4x^3$

or, $(x^2D^2 - 2xD + 2)\, y = 4x^3$ ..... (i)

Equation (i) is homogeneous linear differential equation.

So, put $x = e^z \Rightarrow \log x = z$

$xDy = \delta y,\ x^2D^2y = (\delta^2 - \delta)\, y$ whre, $\delta = \dfrac{d}{dz}$ in equation (i)

Now, (i) becomes,

$(\delta^2 - \delta - 2\delta + 2)\, y = 4e^{3z}$

or, $(\delta^2 - 3\delta + 2)\, y = 4e^{3z}$

So, it's A. E. is,

$m^2 - 3m + 2 = 0$

or, $m^2 - m - 2m + 2 = 0$

or, $m(m-1) - 2(m-1) = 0$

or, $(m-1)(m-2) = 0$

$\Rightarrow m = 1, 2$

Hence, C. F. $= C_1 e^z + C_2 e^{2z}$

and P. I. $= \dfrac{1}{\left(\delta^2 - 3\delta + 2\right)} 4e^{3z}$

$= \dfrac{1}{\left(3^2 - 3.3 + 2\right)} 4e^{3z} = \dfrac{4}{2} e^{3z} = 2e^{3z}$

Thus, y = C. F + P. I.

or, $y = C_1 e^z + C_2 e^{2z} + 2e^{3z}$

or, $y = C_1 e^{\log x} + C_2 e^{2\log x} + 2e^{3\log x}$

or, $y = C_1 x + C_2 x^2 + 2x^3$ is the required solution.

4. $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 4y = x^6$

Sol^n. Given differential equation is,

$x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 4y = x^6$

or, $(x^2D^2 - 3xD + 4)\, y = x^6$ ..... (i)

Equation (i) in homogeneous differential equation,

So, put $x = e^z \Rightarrow \log x = z$

$xDy = \delta y,\ x^2D^2y = (\delta^2 - \delta)\, y$

Where, $\delta = \dfrac{d}{dz}$ in (i)

Now, (i) becomes,

$(\delta^2 - \delta - 3\delta + 4)\, y = e^{6z}$

or, $(\delta^2 - 4\delta + 4)\, y = e^{6z}$

or, So its, A. E. is

$m^2 - 4m + 2 = 0$

or, $(m-2)^2 = 0 \Rightarrow m = 2, 2$

Hence, C. F. $= (C_1 + C_2 z)\, e^{2z}$

and P. I. $= \dfrac{1}{(\delta^2 - 4\delta + 4)} e^{6z}$

$= \dfrac{1}{(6^2 - 4.6 + 4)} e^{6z} = \dfrac{1}{(36 - 24 + 4)} e^{6z} = \dfrac{1}{16} e^{6z}$

Thus, y = C. F. + P. I.

$= (C_1 + C_2 z)\, e^{2z} + \dfrac{1}{16} e^{6z}$

$= (C_1 + C_2 \log x)\, e^{2\log x} + \dfrac{1}{16} e^{6\log x}$

or, $y = (C_1 + C_2 \log x)\, x^2 + \dfrac{1}{16} x^6$ is the required solution.

5. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 6x$

Sol^n. Given differential equation is,

$x\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 6x$

Multiplying both sides by x we get,

$x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 6x^2$

or, $(x^2D^2 + 2xD)\, y = 6x^2$ ..... (i)

Equation (i) is homogeneous differential equation.

So, put $x = e^z \Rightarrow \log x = z$

$xDy = \delta y;\ x^2D^2y = (\delta^2 - \delta)\, y$

When, $\delta = \dfrac{d}{dz}$ in (i)

Now, (i) becomes,

$(\delta^2 - \delta + 2\delta)\, y = 6e^{2z}$

or, $(\delta^2 + \delta)\, y = 6e^{2z}$

So, it A. E. is,

$m^2 + m = 0 \Rightarrow m(m+1) = 0$

or, $m = 0, -1$

Hence, C. F. $= C_1 + C_2 e^{-z}$

and P. I. $= \dfrac{1}{\delta^2 + \delta} 6e^{2z} = \dfrac{1}{2^2 + 2} 6e^{2z} = \dfrac{6}{6} e^{2z} = e^{2z}$

Thus, y = C. F. + P. I.

$= C_1 + C_2 e^{-z} + e^{2z}$

$= C_1 + C_2 e^{-\log x} + e^{2\log x}$

or, $y = C_1 + \dfrac{C_2}{x} + x^2$ is the required solution.

6. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$

Sol^n. Given differential equation is,

$x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$

or, $(x^2D^2 - 2xD - 4)\, y = x^4$ ..... (i)

Equation (i) is homogeneous differential equation.

So, put $x = e^z \Rightarrow \log x = z$

$xDy = \delta y;\ x^2D^2y = (\delta^2 - \delta)\, y$

Where, $\delta = \dfrac{d}{dz}$ in equation (i)

Now, equation (i) becomes,

$(\delta^2 - \delta - 2\delta - 4)\, y = e^{4z}$

or, $(\delta^2 - 3\delta - 4)\, y = e^{4z}$

So, it's A. E. is,

$m^2 - 3m - 4 = 0$

or, $m^2 - 4m + m - 4 = 0$

or, $m(m-4) + 1(m-4) = 0$

or, $(m+1)(m-4) = 0$

$\Rightarrow m = -1, 4$

Hence, C. F. $= C_1 e^{-z} + C_2 e^{4z}$

and P.I. $= \dfrac{1}{(\delta^2 - 3\delta - 4)} \cdot e^4 = \dfrac{z}{(2\delta - 3)} e^{4z} = \dfrac{3}{(2.4 - 3)} e^{4z} = \dfrac{ze^{4z}}{5}$

Thus, y = C. F. + P. I.

$= C_1 e^{-z} + C_2 e^{4z} + \dfrac{z}{5} e^{4z}$

or, $C_1 e^{-\log x} + C_2 e^{4\log x} + \dfrac{\log x}{5} e^{4\log x}$

or, $y = \dfrac{C_1}{x} + C_2 x^4 + \dfrac{\log x}{5} x^4$ is the required solution.

7. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 4y = x^2$

Sol^n. Given differential equation is,

$x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 4y = x^2$

or, $(x^2D^2 + xD - 4)\, y = x^2$ ..... (i)

Equation (i) in homogeneous differential equations

So, put $= e^z \Rightarrow \log x = z$

$xDy = \delta y$; $x^2D^2y = (\delta^2 - \delta)\, y$

Where, $\delta = \frac{d}{dz}$ in equation (i)

Now, (i) becomes,

$(\delta^2 - \delta + \delta - 4)\, y = e^{2z}$

or, $(\delta^2 - 4)\, y = e^{2z}$

So, it A. E. is $m^2 - 4 = 0 \Rightarrow m = \pm 2$

Hence, C. F. $= C_1e^{2z} + C_2e^{-2z}$

and P.I. $= \frac{1}{(\delta^2 - 4)} e^{2z} = \frac{z}{2z} e^{2z} = \frac{3}{4} e^{2z}$

Thus, y = C. F. + P. I.

$= C_1e^{2z} + C_2e^{-2z} + \frac{3}{4} e^{2z}$

or, $y = C_1e^{2\log x} + C_2e^{-2\log x} + \frac{\log x}{4} e^{2\log x}$

or, $y = C_1x^2 + \frac{C_2}{x^2} + \frac{x^2}{4} \log x$ is the required solution.

8. $\mathbf{x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x}$

Sol^n. Given differential equation is,

$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

or, $(x^2D^2 - xD + 1)y = \log x$ ..... (i)

Equation (i) is homogeneous differential equation is,

So, put, $x = e^z \Rightarrow \log x = 3$

$xDy = \delta y$; and $x^2D^2y = (\delta^2 - \delta)\, y$

Where, $\delta = \frac{d}{dz}$ in equation (i)

Now (i) becomes,

$(\delta^2 - \delta - \delta + 1)\, y = z$

or, $(\delta^2 - 2\delta + 1)\ y = z$

so, it's A. E. is, $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

Hence, C. F. $= (C_1 + C_2z)\, e^z$

and P. I. $= \frac{1}{(\delta^2 - 2\delta + 1)} z = \frac{1}{(1-\delta)^2} z = [1 - \delta]^{-2} z$

$= [1 + 2\delta + 3\delta^2 + \ldots\ldots] z$

P. I. $= (z + 2)$

Thus, y = C. F. + P. I.

or, $y = (C_1 + C_2z)\, e^z + z + 2$

or, $y = (C_1 + C_2 \log x)\, e^{\log x} + \log x + 2$

or, $y = (C_1 + C_2\log x) + \log x + 2$ is the required solution.

9. $\mathbf{x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x}$

Sol^n. Given differential equation is,

$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$

or, $(x^2D^2 + 4xD + 2)\, y = e^x$ ..... (i)

Equation (i) is homogeneous differential equation,

So, put, $x = e^z \Rightarrow \log x = z$

$xDy = \delta y$; $x^2D^2y = (\delta^2 - \delta)\, y$

Where, $\delta = \frac{d}{dz}$ in equation (i)

Now (i) becomes,

$(\delta^2 - \delta + 4\delta + 2)\, y = e^{e^z}$

or, $(\delta^2 + 3\delta + 2)\, y = e^{e^z}$

So, It's A. E. is

$m^2 + 3m + 2 = 0$

or, $m^2 + 2m + m + 2 = 0$

or, $(m + 2) + 1\,(m + 2) = 0$

or, $(m + 1)\,(m + 2) = 0$

$\Rightarrow m = -1, -2$

Hence, C. F. $= (C_1e^{-z} + C_2e^{-2z})$

and P. I. $= \frac{1}{(\delta^2 + 3\delta + 2)} e^{e^z} = \frac{1}{(\delta+1)(\delta+2)} e^{e^z}$

$= \left[\frac{1}{(\delta+1)} - \frac{1}{(\delta+2)}\right] e^{e^z}$ (by using partial fraction)

P.I. $= \frac{1}{(\delta+1)} e^{e^z} - \frac{1}{(\delta+2)} e^{e^z}$

We know, $\frac{X}{D-a} = e^{ax} \int e^{-ax} : X\, dx$

$\therefore$ P.I. $= \left(\frac{e^{e^z}}{\delta+1} - \frac{e^{e^z}}{\delta+2}\right) = e^{-z}\int e^z e^{e^z}\, dz - e^{-2z}\int e^{2z} e^{e^z}\, dz$ ..... (ii)

Now, $\int e^z e^{e^z}\, dz$

Put $e^z = t \Rightarrow e^z\, dz = dt$

$= \int e^t dt = e^t = e^{e^z}$

Also, $\int e^{2z} e^{e^z}\, dz$

Put, $e^z = t \Rightarrow e^z\,dz = dt$

$$= \int e^z \cdot e^z e^{e^z}\,dz$$
$$= \int t \cdot e^t\,dt$$
$$= t e^t - e^t = e^z e^{e^z} - e^{e^z}$$

$$\therefore \text{P. I.} = e^{-z} e^z - e^{-2z}[e^z e^{e^z} - e^{e^z}]$$
$$= e^{-z} e^z - e^{-z} e^z + e^{-2z} e^z$$

$$\text{P.I.} = e^{-2z} e^{e^z}$$

Thus, y = C. F. + P. I.

$$= C_1 e^{-z} + C_2 e^{-2z} + e^{-2z} e^{e^z}$$

or, $y = C_1 e^{-\log x} + C_2 e^{-2\log x} + e^{-2\log x} \cdot e^{\log x}$

or, $y = \dfrac{C_1}{x} + \dfrac{C_2}{x^2} + \dfrac{1}{x^2} e^x$ is the required solution.

10. $(x^2D^2 + xD + 1)\,y = \sin(\log x^2)$

Sol^n. Given differential equation is,

$(x^2D^2 + xD + 1)\,y = \sin(\log x^2)$

Equation (i) is homogeneous differential equation,

So, put, $x = e^z \Rightarrow \log x = z$

$xDy = \delta y;\ x^2D^2y = (\delta^2 - \delta)\,y$

Where, $\delta = \dfrac{d}{dz}$ in equation (i)

Now (i) becomes,

$(\delta^2 - \delta + \delta + 1)\,y = \sin 2z$

or, $(\delta^2 + 1)\,y = \sin 2z$

So, It's A. E. is $m^2 + 1 = 0 \Rightarrow m = \pm i$

Hence, C. F. = Acosz + B sinz

and P. I. $= \dfrac{1}{(\delta^2+1)} \sin 2z = \dfrac{1}{-2^2+1} \sin 2z = -\dfrac{1}{3} \sin 2z$

Thus, y = C. F. + P. I.

$= A\cos z + B\sin z - \dfrac{1}{3} \sin 2z$

$= A\cos(\log x) + B\sin(\log x) - \dfrac{1}{3} \sin(\log x^2)$ is the required solution.

11. $(x^2D^2 - 2)\,y = x^2 + \dfrac{1}{x}$

Sol^n. Given differential equation is,

$(x^2D^2 - 2)\,y = x^2 + \dfrac{1}{x}$ ..... (i)

Equation (i) is homogeneous differential equaton.

So, Put, $x = e^z \Rightarrow \log x = z$

$xDy = \delta y;\ x^2D^2y = (\delta^2 - \delta)\,y$

Gaurab Shrestha

Where, $\delta = \dfrac{d}{dz}$ in equation (i)

Now (i) becomes,

$(\delta^2 - \delta - 2)y = e^{2z} + e^{-z}$

So, It's A. E. = $m^2 - m - 2 = 0$

or, $m^2 - 2m + m - 2 = 0$

or, $m(m-2) + 1(m-2) = 0$

or, $(m+1)(m-2) = 0$

or, $m = -1, 2$

Hence, C. F. $= C_1e^{-z} + C_2e^{2z}$

and P. I. $= \dfrac{1}{(\delta^2 - \delta - 2)} e^{2z} + e^{-z}$

$$= \frac{1}{(\delta^2 - \delta - 2)} e^{2z} + \frac{1}{(\delta^2 - \delta - 2)} e^{-z}$$

$$= \frac{z}{2\delta - 1} e^{2z} + \frac{z}{2\delta - 1} e^{-z}$$

$$= \frac{z}{3} e^{2z} + \frac{z}{-3} e^{-z}$$

$$= \frac{z}{3}(e^{2z} - e^{-z})$$

Thus, y = C. F. + P. I.

or, $y = C_1e^{-z} + C_2e^{2z} + \dfrac{z}{3}(e^{2z} - e^{-z})$

or, $y = C_1e^{-\log x} + C_2e^{2\log x} + \dfrac{\log x}{3}(e^{2\log x} - e^{-\log x})$

or, $y = \dfrac{C_1}{x} + C_2x^2 + \dfrac{\log x}{3}\left(x^2 - \dfrac{1}{x}\right)$ is the required solution.

12. $(x+p)^2 \dfrac{d^2y}{dx^2} - 4(x+p)\dfrac{dy}{dx} + 6y = x$

Sol^n. Given differential equation is,

$(x+p)^2 \dfrac{d^2y}{dx^2} - 4(x+p)\dfrac{dy}{dx} + 6y = x$

or, $(x+p)^2 \dfrac{d^2y}{dx^2} - 4(x+p)\dfrac{dy}{dx} + 6y = x$

or, $\{(x+p)^2 D^2 - 4(x+p)D + 6\}\,y = x$ ..... (i)

equation (i) is homogeneous differential equation is,

So, put, $(x+p) = e^z \Rightarrow z = \log(x+p)$

$(x+p)\,Dy = \delta y,\ (x+p)^2D^2y = (\delta^2 - \delta)\,y$

Which $\delta = \dfrac{d}{dz}$ in equation (i)

Gaurab. Shrestha

Now, equation (i) becomes,

$(\delta^2 - \delta - 4\delta + 6)\, y = (e^z - p)$

or, $(\delta^2 - 5\delta + 6)\, y = e^z - p$

So, It's, A. E. is

$m^2 - 5m + 6 = 0$

or, $m^2 - 2m - 3m + 6 = 0$

or, $m(m-2) - 3(m-2) = 0$

or, $(m-2)(m-3) = 0$

or, m (2, 3

Hence, C. F. $= C_1e^{2z} + C_2e^{3z}$

and P.I. $= \dfrac{1}{\left(\delta^2 - 5\delta + 6\right)}(e^z - p)$

$= \dfrac{1}{\left(\delta^2 - 5\delta + 6\right)} e^z - \dfrac{z}{\left(\delta^2 - 5\delta + 6\right)} e^{0z}$

P.I. $= \dfrac{1}{2} e^z - \dfrac{p}{6}$

Thus, y = C. F. + P. I.

$= C_1e^{2z} + C_2e^{3z} + \dfrac{1}{2} e^z - \dfrac{p}{6}$

or, $y = C_1e^{2\log(x+p)} + C_2e^{3\log(x+p)} + \dfrac{1}{2} e^{\log(x+p)} - \dfrac{p}{6}$

or, $y = C_1(x+p)^2 + C_2(x+p)^3 + \dfrac{1}{2}(x+p) - \dfrac{p}{6}$

or, $y = C_1(x+p)^2 + C_2(x+p)^3 + \dfrac{(3x+2p)}{6}$ is the requirel solution.

13. $(x+1)^2\dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} = (2x+3)(2x+4)$

Sol^n. Given differential equation is,

$(x+1)^2\dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} = (2x+3)(2x+4)$

or, $\{(x+1)^2 D^2 + (x+1) D\}\, y = (2x+3)(2x+4)$ .....(i)

Equation (i) is homogeneous differential equation.

So, put $(x+1) = e^z \Rightarrow z = \log(x+1)$

or, $(x+1) Dy = \delta y$; $(x+1)^2 D^2y = (\delta^2 - \delta)\, y$

Where, $\delta = \dfrac{d}{dz}$ in equation (i)

Now equation (i) becomes,

$(\delta^2 - \delta + \delta)\, y = \{2(e^z - 1) + 3\}\{2(e^z - 1) + 4\}$

or, $\delta^2 y = (2e^z + 1)(2e^z + 2)$

$= 2(2e^z + 1)(e^z + 1)$

$= 2(2e^{2z} + 3e^z + 1)$

$= 4(e^{2z} + 6e^z + 2)$

It.s A. E. is, $m^2 = 0 \Rightarrow m = 0, 0$

Hence, C. F. $= C_1 + C_2z$

and P. I. $= \dfrac{1}{\delta^2}(4e^{2z} + 6e^z + 2) = \dfrac{4e^{2z}}{4} + 6e^z + z^2$

Thus, y = C. F. + P. I.

or, $y = C_1 + C_2z + e^{2z} + 6e^z + z^2$

or, $y = C_1 + C_2\log(x+1) + e^{2\log(x+1)} + 6e^{\log(x+1)} + \{\log(x+1)\}^2$

or, $y = C_1 + C_2\log(x+1) + (x+1)^2 + 6(x+1) + \{\log(x+1)\}^2$

or, $y = C_1 + C_2\log(x+1) + x^2 + 2x + 1 + 6x + 6 + \{\log(x+1)\}^2$

or, $y = C_1 + C_2\log(x+1) + \{\log(x+1)\}^2 + 8x + 7$ is the required solution.

14. $(2x+3)^2\dfrac{d^2y}{dx^2} + 2(2x+3)\dfrac{dy}{dx} - 4y = 8x$

Sol^n. Given differential equation is,

$(2x+3)^2\dfrac{d^2y}{dx^2} + 2(2x+3)\dfrac{dy}{dx} - 4y = 8x$

or, $(2x+3)^2D^2 + 2(2x+3)D - 4\}\, y = 8x$ ..... (i)

Equation (i) is homogeneous differential equation.

So, put $(2x+3) = e^z \Rightarrow z = \log(2x+3)$

and $(2x+3) Dy = 2\delta y$; $(2x+3) D^2y = 4(\delta^2 - \delta)$

Where $\delta = \dfrac{d}{dz}$ in equation (i)

Now equation (i) becomes,

$\{4(\delta^2 - \delta) + 4\delta - 4\}\, y = \dfrac{8}{2}(e^z - 3)$

or, $(4\delta^2 - 4\delta + 4\delta - 4)\, y = 4(e^z - 3)$

or, $(4\delta^2 - 4)y = 4(e^z - 3)$

So, Its, A. E. is,

$4m^2 - 4 = 0 \Rightarrow m = \pm 1$

Hence, C. F. $= C_1e^z + C_2e^{-z}$

and P. I. $= \dfrac{1}{\left(4\delta^2 - 4\right)} \cdot 4(e^z - 3)$

$= \left[\dfrac{1}{\left(\delta^2 - 1\right)} e^z - \dfrac{z}{\left(\delta^2 - 1\right)}\right]$

$= \left[\dfrac{z}{2\delta} e^z + 3\right]$

$= \frac{z}{2} e^z + 3$

Thus, $y = C.F. + P.I.$

$= (C_1 e^z + C_2 e^{-z}) + \frac{z}{2} e^z + 3$

or, $y = C_1 e^{\log(2x+3)} + C_2 e^{-\log(2x+3)} + \frac{\log(2x+3)}{2} e^{\log(2x+3)} + 3$

or, $y = C_1 (2x+3) + \frac{C_2}{(2x+3)} + \frac{(2x+3)}{2} \log(2x+3) + 3$ is the required solution.

15. $(2x+1)^2 \frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$

Sol^n. Given differential equation is,

$(2x+1)^2 \frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$

or, $(2x+1)^2 D^2y - 6(2x+1) Dy + 16y = 8(2x+1)^2$

or, $\{(2x+1)^2 D^2 - 6(2x+1) D + 16\} y = 8(2x+1)^2$ ..... (i)

Equation (i) is homogeneous differential equation

So, put, $(2x+1) = e^z \Rightarrow z = \log(2x+1)$

$(2x+1) Dy = 2\delta$; $(2x+1)^2 D^2y = 4(\delta^2 - \delta)$

Where, $\delta = \frac{d}{dz}$ in (i)

Now, equation (i) becomes,

$\{4(\delta^2 - \delta) - 12\delta + 16\} y = 8e^{2z}$

or, $(4\delta^2 - 4\delta - 12\delta + 16) y = 8e^{2z}$

or, $(4\delta^2 - 16\delta + 16) y = 8e^{2z}$

or, $(\delta^2 - 4\delta + 4) = 2e^{2z}$

So, It's A. E. is $m^2 - 4m + 4 = 0$

or, $(m-2)^2 = 0 \Rightarrow m = 2, 2$

Hence, C. F. $= (C_1 + C_2 z) e^{2z}$

and P. I. $= \frac{1}{(\delta^2 - 4\delta + 4)} 2e^{2z} = \frac{z}{(2z-4)} 2e^{2z} = \frac{z^2}{2} . 2e^{2z}$

P. I. $= z^2 e^{2z}$

Thus, $y = C.F. + P.I.$

$= (C_1 + C_2 z) e^{2z} + z^2 e^{2z}$

or, $y = \{C_1 + C_1 \log(2x+1)\} e^{2\log(2x+1)} + \{\log(2x+1)\}^2 e^{2\log(2x+1)}$

or, $y = C_1 + C_1 \log(2x+1)\} (2x+1)^2 + \{\log(2x+1)\}^2 (2x+1)^2$

$= (2x+1)^2 [\{\log(2x+1)\}^2 + C_1 + C_2 \log(2x+1)]$ is the required general solution.

Application of Differential Equation in Engineering field 18

Important IOE Extra Questions

1. **A particle of mass 'm' is projected vertically upwards under gravity, the resistance of ain being mk times the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}[\lambda - \log(1+\lambda)]$, where V is the greatest velocity which the above mass will attain when it falls freely and λv is the initivelocity.**

Sol^n. Let u = velocity of the particle at time 't'

x = height attained at any instant.

Now, accelerating force on the particle = – [its weight acting vertically down ward + resistance of air vertically down ward)

$= -[mg + mku]$

∴ By Newto'ns second Law of motion, the equation of motion of the partical is given by,

$mu\frac{du}{dx} = -(mg + mku)$

or, $u\frac{du}{dx} = -g - ku$ ..... (i)

When the particle falls freely under gravity, then equation (i) becomes,

$u\frac{du}{dx} = g - ku$ (by changing 'g' by – g) ..... (ii)

When the particle attains the greatest velocity v, its acceleration vanishes.

∴ from (ii), $0 = g - kv$ or, $k = \frac{g}{v}$

Putting this value of k in (i), we have $u\frac{du}{dx} = -g - \frac{g}{v}u = -\frac{g}{v}(v+u)$

or, $\frac{u}{v+u} du = -\frac{g}{v} dx$

Integrating, $\int \frac{u}{v+u} du = -\frac{g}{v} \int dx$

or, $\int \left(1 - \frac{v}{v+u}\right) du = -\frac{g}{v} x + C$

or, $u - v \log (V + u) = -\frac{g}{v} x + C$ ..... (iii)

Initially for $x = 0, y = \lambda v$ (given)

$\therefore$ From (iii), $\lambda v - v \log (v + \lambda v) = C$

or, $C = v [\lambda - \log 1 (1 + \lambda)]$

Substituting C is (iii), we get,

$u - v \log (v + u) = -\frac{g}{v} x + v [\lambda - \log (v + v\lambda)]$ ..... (iv)

Let h = greatest height attained by the particle. So when $u = 0, x = h$

$\therefore$ from (iv), $-v\log v = -\frac{g}{v} h + v (\lambda - \log v (1 + \lambda)]$

or, $\frac{g}{v} h = v\lambda + v \log v - v \log v (1 + \lambda) = v\lambda + v \log \frac{v}{v(1+\lambda)}$

or, $\frac{g}{v} h = v\lambda + v \log (1 + \lambda)^{-1}$

or, $h = \frac{v^2}{g} [\lambda - \log(1+\lambda)]$

2. **A body of mass m, falling from rest, is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e. kv^2). g it falls through a distance x and possesses a velocity v at that instant, prove that $\frac{2kx}{m} = \log \frac{a^2}{a^2 - v^2}$; where $mg = ka^2$.**

Sol^n. The force acting on the body are,

(i) its weight mg (acting on the body) vertically downeards

(ii) the air resistance kv^2 acting vertically upwards.

$\therefore$ Accelarating force on the body

$= mg - kv^2 = ka^2 - kv^2 = k (a^2 - v^2)$

By Newton's second law, the equation of motion of the body is,

$mv \frac{dv}{dx} = k (a^2 - v^2)$

or, $\frac{v}{a^2 - v^2} dv = \frac{k}{m} dx$

Integrating, $\int \frac{v}{a^2 - v^2} dv = \frac{k}{m} \int dx$

or, $-\frac{1}{2} \log \left(a^2 - v^2\right) = \frac{k}{m} x + C$ ..... (i)

Initially, when $x = 0, v = 0$

$\therefore$ from (i), $-\frac{1}{2} \log (a^2 - 0) = 0 + C$

$\therefore C = -\frac{1}{2} \log a^2$

Hence, again from (i)

$-\frac{1}{2} \log (a^2 - v^2) = \frac{k}{m} x - \frac{1}{2} \log a^2$

or, $\frac{2kx}{m} = \log a^2 - \log (a^2 - v^2) = \log \frac{a^2}{a^2 - v^2}$ Proved.

3. **A constant emf E volts is applied to a circuit containing a constant resistance R ohms in series and a constant reductance L henries. If the initial current is zero, show that the current builds up to half its theoretical maximum in (L log 2) / R seconds.**

Sol^n. Let i = current in the circuit at any time T.

By kirchhoff's law, we have $L \frac{di}{dt} + Ri = E$

or, $\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$ ..... (i)

This is a linear equation

$I.F. = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} t}$

Hence, solution of (i) is,

$i \times (I.F.) = \int \frac{E}{L} . e^{\frac{R}{L} t} dt$

or, $i . e^{\frac{R}{L} t} = \frac{E}{L} . \frac{L}{R} e^{\frac{Rt}{L}} + C$

or, $i = \frac{E}{R} + C e^{-\frac{R}{L} t}$ ..... (ii)

Initially, when $t = 0, i = 0$ so that $C = -\frac{E}{R}$

Thus (ii) becomes,

$i = \frac{E}{R} \left(1 - e^{-\frac{R}{L} t}\right)$ ..... (iii)

This gives the current i in any time T

Clearly maximum value of current $i = \frac{E}{R}$ (from iii)

Let the current i attains its half theoretical maximum after time T.

Then, from (iii)

$$\frac{1}{2}\frac{E}{R} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}}\right)$$

$$\text{or, } e^{-\frac{R}{L}t} = \frac{1}{2}$$

$$\text{or, } -\frac{R}{L}T = \log\frac{1}{2} = -\log 2$$

$$\therefore R = (L \log 2) / R$$

4. The equation of emf in terms of current i for an electrical circuit having resistance R and a condenser of capacity C, in series, is $E = Ri + \int \frac{i}{c} dt$. Find the current i at any time t, when $E = E_0 \sin\omega t$.

Sol^n. The given equation can be written as,

$$Ri + \int \frac{i}{c} dt = E_0 \sin\omega t$$

Differentiating both sides w.r.t 't', we have,

$$R\frac{di}{dt} + \frac{i}{C} = \omega E_0 \cos\omega t$$

$$\text{or, } \frac{di}{dt} + \frac{i}{RC} \cos\omega t \text{ ..... (i)}$$

This is a linear equation.

Now, integrating factor I. F. $= e^{\int \frac{1}{RC} dt} = e^{\frac{t}{RC}}$

$\therefore$ Solution of equation (i) is,

$$i \times (\text{I. F.}) = \int \frac{\omega E_0}{R} \cos\omega t \, (\text{I. F.}) \, dt$$

$$\text{or, } i \,.\, e^{\frac{t}{RC}} = \frac{\omega E_0}{R} \int e^{\frac{t}{RC}} \cos\omega t \, dt$$

$$= \frac{\omega E_0}{R}\left[\frac{e^{\frac{t}{RC}}\left\{\left(\frac{1}{RC}\right)\cos\omega t + -\omega \sin\omega t\right\}}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}\right] + k$$

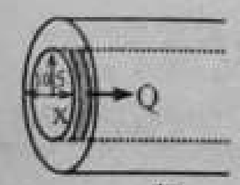
$$= \frac{\omega E_0}{R} e^{\frac{t}{RC}} \frac{1}{RC} \frac{\left[\cos\omega t + \omega RC \sin\omega t\right]}{\sqrt{1 + R^2C^2\omega^2} \cdot \frac{1}{RC}} + k$$

$$= \frac{\omega E0}{R} \frac{\left[\cos\omega t + \omega Rc \sin\omega t\right]}{\sqrt{1 + R^2c^2\omega^2}} + ke^{-\frac{t}{Rc}}$$

5. A long hollow pipe has inner diameter of 10cm and outer diameter of 20cm. The inner surface is kept at 200°C and the outer surface at 50°C. The thermal conductivity is 0.12. How much heat is lost per minute from a potion of the pipe 20 meter long ? Find the temperature at a distance x = 7.5 cm from the centre of the pipe.

Sol^n. The thermal surfaces are cylinders, the axis of each one of them is the axis of the pipe. Consider one such cylinder of radius x cm of unit length. Then the surface area of this cylinder = $2\pi x$ sq. cm.

Let Q = quantity of heat in calori flowing across this surface per second.



$$\text{Then } Q = -KA\frac{dT}{dx} = -k \,.\, 2\pi x \frac{dT}{dx}$$

$$\left[\begin{array}{l} T = \text{temperature at any instant} \\ k = \text{thermal conductivity} = 0.12 \end{array}\right]$$

$$dT = -\frac{Q}{2\pi k}\frac{dx}{x}$$

$$\text{Integrating, } T = -\frac{Q}{2\pi k}\log x + C \text{ ..... (i)}$$

Given that when x = 5, T = 200

$$\therefore 200 = -\frac{Q}{2\pi k}\log 5 + C \text{ ..... (ii)}$$

Also, when x = 10, T = 50

$$\therefore 50 = -\frac{Q}{2\pi k}\log 10 + C \text{ ..... (iii)}$$

Subtracting (iii) from (ii),

$$150 = \frac{Q}{2\pi k}(\log 10 - \log 5) = \frac{Q}{2\pi k}\log 2 \text{ ..... (iv)}$$

$$\therefore Q = \frac{2\pi k \times 150}{\log 2} = \frac{2\pi \times 0.12 \times 150}{\log 2} = 163 \text{ cal/sec}$$

Hence, the heat lost per minute through 20 meter long of pipe $= 20 \times 100 \times 60\ Q = 120000 \times 163 = 1956000$ cal

Now, Let T = t, when x = 7.5

$$\therefore \text{From (i), } t = -\frac{Q}{2\pi k}\log\ (7.5) + C \text{ ..... (v)}$$

Subtracting (ii) from (v),

$$t - 200 = -\frac{Q}{2\pi k}(\log 7.5 - \log 5)$$

$$= -\frac{Q}{2\pi k}(\log(1.5))$$

Dividing (6) by (4); we get,

$$\frac{t-200}{150} = -\frac{\log(1.5)}{\log_e 2} \Rightarrow t = 113$$

$\therefore$ When $x = 7.5$ cm, $T = 113°C$

6. **Uranium disintegrates at a rate proportional to the amount pressure at any instant. If M_1 and M_2 grams of Uranium are present at times T_1 and T_2 respectively, show that the half life of uranium is $\frac{(T_2-T_1)\log 2}{\log(M_1/M_2)}$.**

Sol^n. Let M be the mass of uranium present at any instant t in grams.
The equation of disintegration of uranium is,

$$\frac{dM}{dt} = -KM; \text{ where K is a constant}$$

or, $\frac{dM}{M} = -kdt$

Interating $\log M = -kt + C$ ..... (i)

Initially, when $t = 0$, $M = M_0$ (say)

$\therefore$ from (i), $\log M_0 = C$

Again, from (i), $\log M = -kt + \log M_0$

or, $kt = \log M_0 - \log M$ ..... (ii)

When, $t = T_1$, $M = M_1$ and when $t = T_2$, $M = M_2$

$\therefore$ from (ii)

$$KT_1 = \log M_0 - \log M_1 \text{ ..... (iii)}$$

$$kT_2 = \log M_0 - \log M_2 \text{ ..... (iv)}$$

Subtracting (iii) from (iv), we get,

$$k(T_2 - T_1) = \log M_1 - \log M_2$$

or, $k = \frac{\log M_1/M_2}{T_2 - T_1}$ ..... (v)

Let T = half life of uranium. Then when $t = T$; $M = \frac{1}{2}M_0$

$\therefore$ From (ii), we get, $KT = \log M_0 - \log\frac{M_0}{2}$

or, $KT = \log 2$

or, $T = \frac{\log 2}{k} = \frac{\log 2}{\frac{\log(M_1/M_2)}{(T_1-T_2)}} = \frac{(T_1-T_2)\log 2}{\log(M_1/M_2)}$ Proved.

7. **Newton's law of colling states that the temperature of an object changes at a rate proportional to the difference of a temperature between the object and the surroundings**

Supposing water at a temperature 100°C cools to 80°C in 10 minutes, in a room maintained at a temperature of 30°C, Find when the temperature of water will become 40°C. [B.E. 2067]

Sol^n. Let the unit of time be minutes and T = temperature of water at time t. Then by Newton's law of colling we have,

$$\frac{dT}{dt} = -k(T-30)$$

or, $\frac{dT}{T-30} = -kdt$ $\because$ [K = Proportionality constant]

[room temperature = 30°]

Integrating,

$\log(T-30) = -kt + C$ ..... (i)

Given that when $t = 0$, $T = 100$

$\therefore$ From (i), $\log 70 = C$

Again from (i), $\log(T-30) = -kt + \log 70$

or, $kt = \log\frac{70}{T-30}$ ..... (ii)

Also, when, $t = 10$, $T = 80°$

$\therefore k(10) = \log\frac{70}{80-30} = \log\frac{70}{50}$

$k = \frac{1}{10}\log\frac{70}{50}$

Putting value of k in (ii), we get,

$$\left(\frac{1}{10}\log\frac{70}{50}\right)t = \log\frac{70}{T-30}$$

or, $\frac{t}{10} = \frac{\log\frac{70}{T-30}}{\log\frac{70}{50}}$ ..... (iii)

Now, when $T = 40$,

$$\frac{t}{10} = \frac{\log\frac{70}{40-30}}{\log\frac{70}{50}} = \frac{\log\frac{70}{10}}{\log\frac{70}{50}} = \frac{\log(7)}{\log(1.4)} = 5.78$$

$\therefore$ Required time $t = 57.8$ minutes.

8. **A tank contains 1000 litres of fresh water. Salt water, which contains 150 gms of salt per litre, runs into it at the rateof 5 litre per minute and well-strirred mixture runs out of it at the same rate. When will the tank contain 5000 grams of salt.**

Sol^n. Let Q = quantity of salt in gm present in the tank at time 't'.

Then, $\frac{dQ}{dt}$ = rate at which the salt content is changing

= rate of salt entering the tank - rate of salt leowing the tank

Now, the rate at which salt increase due to the inflow = 150 × 5 = 750 gm/ minute

Let C gm be the concentration of salt at time 't'.

The rate at which the salt content decreases due to the outflow = C × 5 = 5C gm/ min

Since the rate of inflow is the same as the rate of outflow, there is no change in the volume of wate at any instant.

$$\Rightarrow C = \frac{Q}{1000}$$

The rate of decrease of salt content $= 5 \times \frac{Q}{1000} = \frac{Q}{200}$ gm/mm

$$\therefore \frac{dQ}{dt} = 750 - \frac{Q}{200} \quad \text{or,} \quad \frac{dQ}{dt} = \frac{150000 - Q}{200} \quad \text{or,} \quad \frac{dQ}{150000 - Q} = \frac{dt}{200}$$

Integrating, $-\log(150000 - Q) = \frac{t}{200} + C$ ..... (i)

Initially, when $t = 0$, $Q = 0$

$\therefore$ from (i), $C = -\log 150000$

$\therefore$ From (i), again, $-\log(150000 - Q) = \frac{t}{200} - \log(150000)$

or, $\frac{t}{200} = \log\frac{150000}{150000 - Q}$ ..... (ii)

Let tank will contain 5000 gms of salt at time $t = T$.

Then from (ii),

$$\frac{T}{200} = \log\frac{150000}{150000 - 5000}$$

$$\therefore T = 200 \times \log\frac{150000}{145000} = 6.78 \text{ minutes.}$$

Unit - IV
Plane Analytic Geometry

Chapter - 19 Transformations of Axes

Chapter - 20 The Ellipse

Chapter - 21 The Hyperbola

Chapter - 22 General Equation of Conic Section

Transformations of Axes 19

Exercise - 32

1. Transform the equation $x^2 - 3y^2 + 4x + 6y = 0$ by transferring the origin to the point (– 2, 1) and the coordinate axes remaining parallel.

Solⁿ. Here, the given equation is $x^2 - 3y^2 + 4x + 6y = 0$ and the origin is transformed to (– 2, 1) to are streman's parallel, so, put $x = x - 2$ and $y = y + 1$ in given equation, so, we get,

$(x-2)^2 - 3(y+1)^2 + 4(x-2) + 6(y+1) = 0$

or, $x^2 - 4x + 4 - 3(y^2 + 2y + 1) + 4x - 8 + 6y + 6 = 0$

or, $x^2 - 4x + 4 - 3y^2 - 6y - 3 + 4x - 8 + 6y + 6 = 0$

or, $x^2 - 3y^2 = 1$ is the required equation referred to new ones.

2. Transform the equation $x^2 + 3y^2 + 3x - 40 = 0$ to parallel axes through (4, – 1).

Solⁿ. Here, the given equation is $x^2 + 3y^2 + 3x - 40 = 0$ and the origin is transformed to (4, – 1) having the axes parallel, so,

Put $x = x + 4$ and $y = y - 1$

Then, $(x+4)^2 + 3(y-1)^2 + 3(x+4) - 40 = 0$

or, $x^2 + 8x + 16 + 3y^2 - 6y + 3 + 3x + 12 - 40 = 0$

or, $x^2 + 3y^2 + 11x - 6y - 9 = 0$

Hence, the equation referred to new axes is $x^2 + 3y^2 + 11x - 6y - 9 = 0$

3. Transform the equation $x^2 + y^2 - 5xy + 4 = 0$ to parallel axes through the point (1, 3).

Solⁿ. The given equation is $x^2 + y^2 - 5xy + 4 = 0$ and the axes are parallel whereas the origin is transformed to (1, 3).

For this put $x = x + 1$ and $y = y + 2$ in given equation, so that,

$(x+1)^2 + (y+3)^2 - 5(x+1)(y+3) + 4 = 0$

or, $x^2 + 2x + 1 + y^2 + 6y + 9 - 5(xy + 3x + y + 3) + 4 = 0$

or, $x^2 + 2x + 1 + y^2 + 6y + 9 - 5xy - 15x - 5y - 19 = 0$

or, $x^2 + y^2 - 5xy - 13x + y - 1 = 0$

Hence, the equation referred to new axis is;

$x^2 + y^2 - 5xy - 13x + y = 1$

4. Transform the equation $2x^2 + 4xy + 5y^2 - 4x - 22y + 7 = 0$ to parallel axes through (– 2, 3).

Solⁿ. The given equation is $2x^2 + 4xy + 5y^2 - 4x - 22y + 7 = 0$ and the origin is transformed to (– 2, 3) to substitute $x = x - 2$ and $y = y + 3$ in a given equation.

Thus,

$2(x-2)^2 + 4(x-2)(y+3) + 5(y+3)^2 - 4(x-2) - (y+3) + 7 = 0$

or, $2(x^2 - 4x + 4) + 4(xy + 3x - 2y - 6) + 5(y^2 + 6y + 9)$
$-4x + 8 - 22y - 66 + 7 = 0$

or, $2x^2 - 8x + 8 + 4xy + 12x - 8y - 24 + 5y^2 + 30y + 45$
$-4x + 8 - 22y - 66 + 7 = 0$

or, $2x^2 + 5y^2 + 4xy - 22 = 0$

Hence the required equation referred to new axis, is,

$2x^2 + 5y^2 + 4xy - 22 = 0$

5. What does the equation $x^2 + 2\sqrt{3}\,xy - y^2 = 2a^2$ become when the axes are turned through an angle 30° to the original axes.

Solⁿ. Here, the given equation is $x^2 + 2\sqrt{3}\,xy - y^2 = 2a^2$ and it is turned through one angle $\theta = 30°$ to the original axis. So,

Put $x = x\cos\theta - y\sin\theta$

$y = y\cos\theta + x\sin\theta$

Hence,

$x = x\cos 30° - y\sin 30°$

or, $x = \dfrac{\sqrt{3}\,x}{2} - \dfrac{y}{3} = \dfrac{\sqrt{3}\,x - y}{2}$

and $y = y\cos 30° + x\sin 30°$

$= \dfrac{\sqrt{3}\,y}{2} + \dfrac{x}{2} = \dfrac{x + \sqrt{3}\,y}{2}$

Thus, substituting the values of x and y in given equation, fields

$$\left(\frac{\sqrt{3}x - y}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}x - y}{2}\right)\left(\frac{x + \sqrt{3}y}{2}\right) - \left(\frac{x + \sqrt{3}\,y}{2}\right)^2 = 2a^2$$

$$\text{or, } \frac{3x^2 - 2\sqrt{3}\,xy + y^2}{4} + 2\sqrt{3}\left(\frac{x^2\sqrt{3} + 3xy - xy - \sqrt{3}\,y^2}{4}\right)$$

$$-\left(\frac{x^2 + 2\sqrt{3}\,xy + 3y^2}{4}\right) = 2a^2$$

or, $3x^2 - 2\sqrt{3}\,xy + y^2 + 6x^2 + 6\sqrt{3}\,xy - 2\sqrt{3}\,xy$
$-6y^2 - x^2 - 2\sqrt{3}\,xy - 3y^2 = 8a^2$

or, $8x^2 - 8y^2 = 8a^2$

or, $x^2 - y^2 = a^2$ is the required equation referred to new axis.

6. What does the equation $2x + \sqrt{3y} = \sqrt{2}$ become when the axes are turned through an angle 45° to the original axes.

Sol^n. The given equation is $2x + \sqrt{3}\,y = \sqrt{2}$ and the axis are turned through an angle 45° with axis, so, for the new required equation.

We substitude,

$x = x\cos\theta - y\sin\theta$

$= x\cos 45° - y\sin 45°$

$= \dfrac{x-y}{\sqrt{2}}$

And $y = 1 = y\cos\theta + x\sin\theta$

$= \dfrac{x+y}{\sqrt{2}}$

Hence, the equation $2x + 3y = \sqrt{2}$ is reduced in to

$\dfrac{2(x-y)}{\sqrt{2}} + 3\dfrac{(y+x)}{\sqrt{2}} = \sqrt{2}$

or, $2(x-y) + 3(y+x) = 2$

or, $2x - 2y + 3y + 3x = 2$

or, $y + 5x = 2$

Hence, $y + 5x = 2$ is the required turned equations.

7. What does the equation $3x^2 + 3y^2 + 2xy = 2$ become when the axes are turned through an angle 45° to the original axes.

Sol^n. The given equation is $3x^2 + 3y^2 + 2xy = 2$ and it is turned through an angle $\theta = 45°$ with original axes, so, for this substitute,

$x = x\cos\theta - y\sin\theta$

$x = x\cos 45° - y\sin 45°$

$= \dfrac{1}{\sqrt{2}}x - \dfrac{1}{\sqrt{2}}y = \dfrac{x-y}{\sqrt{2}}$

and $y = y\cos\theta + x\sin\theta$

$= y\cos 45° + x\sin 45°$

$= \dfrac{x+y}{\sqrt{2}}$

Thus, the equation $3x^2 + 3y^2 + 2xy = 2$ is changed in to,

$$3\left(\frac{x-y}{\sqrt{2}}\right)^2 + 3\left(\frac{x+y}{\sqrt{2}}\right)^2 + 2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) = 2$$

$$\text{or, } 3\frac{(x^2 - 2xy + y^2)}{2} + 3\frac{(x^2 + 2xy + y^2)}{2} + \frac{2(x^2 - y^2)}{2} = 2$$

or, $3x^2 - 6xy + 3y^2 + 3x^2 + 6xy + 3y^2 + 2x^2 - 2y^2 = 4$

or, $8x^2 + 4y^2 = 4$

or, $2x^2 + y^2 = 1$

or, $2x^2 + y^2 = 1$, is the required equation turned under given $\theta = 45°$.

8. Through what angle should the axes be rotated so that the equation. $9x^2 - 2\sqrt{3}\,xy + 7y^2 = 10$ may be changed to $3x^2 + 5y^2 = 5$.

Sol^n. The given equation is $9x^2 - 2\sqrt{3}\,xy + 7y^2 = 10$ and is required to change in to $3x^2 + 5y^2 = 5$ for this by question rotation is allowed.

Let θ be the required rotation,

So, $x = x\cos\theta - y\sin\theta$

$y = y\cos\theta + x\sin\theta$

Then,

$9x^2 + 7y^2 - 2\sqrt{3}\,xy = 10$ is reduced into

$9(x\cos\theta - y\sin\theta)^2 + 7(y\cos\theta + x\sin\theta)^2$

$-2\sqrt{3}\,(x\cos\theta - y\sin\theta)(y\cos\theta + x\sin\theta) = 10$

or, $9[x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin^2\theta]$

$+ 7[y^2\cos^2\theta + 2xy\cos\theta\sin\theta + x^2\sin^2\theta]$

$-2\sqrt{3}\,[xy\cos^2\theta + x^2\sin\theta\cos\theta - y^2\sin\theta\cos\theta - xy\sin^2\theta] = 10$

or, $9x^2\cos^2\theta - 18\,xy\sin\theta\cos\theta + 9y^2\sin\theta + 7y^2\cos^2\theta$

$+ 14xy\cos\theta\sin\theta + 7x^2\sin^2\theta - 2\sqrt{3}\,xy\cos^2\theta - 23\,x^2\sin\theta\cos\theta$

$+ 2\sqrt{3}\,y^2\sin\theta\cos\theta + 2\sqrt{3}\,xy\sin^2\theta = 10$

or, $x^2[9\cos^2\theta + 7\sin^2\theta - 2\sqrt{3}\sin\theta\cos\theta] + y^2[9\cos^2\theta + 7\cos^2\theta]$

$+ 2\sqrt{3}\,\sin\theta\cos\theta] + xy[-18\sin\theta\cos\theta + 14\sin\theta\cos\theta$

$+ 2\sqrt{3}\,\sin^2\theta - 2\sqrt{3}\,\sin^2\theta] = 10$

For coefficient $x^2 = 3$

$9\cos^2\theta + 7\sin^2\theta - 2\sqrt{3}\,\sin\theta\cos\theta = 3$ ..... (i)

For coefficient $y^2 = 5$

$9\sin^2\theta + 7\cos^2\theta + 2\sqrt{3}\,\sin\theta\cos\theta = 5$ ..... (ii)

and coefficient $xy = 0$

$14\sin\theta\cos\theta - 18\sin\theta\cos\theta + 2\sqrt{3}\,\sin^2\theta - 2\sqrt{3}\,\cos^2\theta = 0$

or, $-4\sin\theta\cos\theta = -2\sqrt{3}\,\sin^2\theta + 2\sqrt{3}\,\cos^2\theta$

or, $-2\sin\theta\cos\theta + 2 = 2\sqrt{3}\,\cos^2\theta$

or, $\tan^2\theta = -\sqrt{3} = \tan 120°$

Thus, $2\theta = 120°$

$$\theta = \frac{120^0}{2} = 60^0$$

Hence $\theta = 60^0$.

9. Find the angle through which the axes may be turned so that the equation $x + 2y + 5 = 0$ may be reduced to the form $x = c$ and determine the value of c.

Sol^n. Let θ be the required angle of rotation, so that $x + 2y + 5 = 0$ is reduced in to the form $x = c$ so substituate

$x = x\cos\theta - y\sin\theta$

$y = y\cos\theta + x\sin\theta$

Thus,

$(x\cos\theta - y\sin\theta) + 2(y\cos\theta + x\sin\theta) + 5 = 0$

$x[\cos\theta + 2\sin\theta] + y[2\cos\theta - \sin\theta] + 5 = 0$

For coefficient, $y = 0$

$2\cos\theta - \sin\theta = 0$

$\sin\theta = 2\cos\theta \Rightarrow \tan\theta = 2$

Thus, $\theta = \tan^{-1}2$

Hence,

$x(\cos\theta + 2\sin\theta) + y(2\cos\theta - \sin\theta) + 5 = 0$ is reduced in to

$x(\cos\tan^{-1}2 + 2\sin\tan^{-1}2) + 5 = 0$

Since, $\tan\theta = \frac{2}{1} = \frac{P}{b}$

$\theta\, b = \sqrt{2^2 + 1^2} \Rightarrow b = \sqrt{5}$

So, $\cos\tan^{-1}2 = \frac{1}{\sqrt{5}}$ $\sin\tan^{-1}2 = \frac{2}{\sqrt{4}}$

or, $x\left(\frac{1}{\sqrt{5}} + 2\times\frac{2}{\sqrt{5}}\right) + 5 = 0$

or, $\frac{5x}{\sqrt{5}} = -5$

Hence, $C = -\sqrt{5}$

Thus, $\theta = \tan^{-1}2$ and $C = -\sqrt{5}$

10. What does the equation of the straight line $7x^2 + 4xy + 4y^2 = 0$ become when the axes are the bisectors of the angles between them.

Sol^n. The equation of the straight lines is $7x^2 + 4xy + 4y^2 = 0$ and of the axis are so rotated that they will bisect the angle between the original axis,

So, $\theta = \frac{\pi}{4}$ is given,

Hence, $x = x\cos\theta - y\sin\theta$

$= x\cos 45^0 - y\sin 45^0$

$= \frac{x-y}{\sqrt{2}}$

and $y = y\cos\theta + x\sin\theta$

$= y\cos 45^0 + x\sin 45^0$

$= \frac{y}{\sqrt{2}} + \frac{x}{\sqrt{2}}$

$= \frac{y+x}{\sqrt{2}}$

Thus, substitute the values of x and y be $\frac{x-y}{\sqrt{2}}$ and $\frac{x+y}{\sqrt{2}}$ in given equation yields,

$$7\left(\frac{x-y}{\sqrt{2}}\right)^2 + 4\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right)^4 + \left(\frac{x+y}{\sqrt{2}}\right)^2 = 0$$

$$\text{or, } 7\left(\frac{x^2 - 2xy + y^2}{2}\right) + 4\left(\frac{x^2 - y^2}{2}\right) + 4\left(\frac{x^2 + 2xy + y^2}{2}\right) = 0$$

$$\text{or, } \frac{7(x^2 - 2xy + y^2) + 4(x^2 - y^2) + 4(x^2 + 2xy + y^2)}{2} = 0$$

or, $7x^2 - 14xy + 7y^2 + 4x^2 - 4y^2 + 4x^2 + 4y^2 + 8xy = 0$

or, $15x^2 - 6xy + 7y^2 = 0$

11. Through what angle must the axes be rotated to remove the term containing xy in $3x^2 + 2xy + 3y^2 - \sqrt{2}\,x = 0$. Also find the transformed equation.

Sol^n. Here, given equation is $3x^2 + 2xy + 3y^2 - \sqrt{2}\,x = 0$. let θ be the required angle of rotation is,

Put $x = x\cos\theta - y\sin\theta$

$y = y\cos\theta + x\sin\theta$

Then,

$3x^2 + 2xy + 3y^2 - \sqrt{2}\,x = 0$ becomes,

$3(x\cos\theta - y\sin\theta)^2 + 2(x\cos\theta - y\sin\theta)(y\cos\theta + x\cos\theta)$

$+ 3(y\cos\theta + x\sin\theta)^2 - \sqrt{2}\,(x\cos\theta - y\sin\theta) = 0$

or, $3[x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin^2\theta] + 2[xy\cos^2\theta$

$+ x^2\sin\theta\cos\theta - y^2\sin\theta\cos\theta - xy\sin^2\theta] + 3[y^2\cos^2\theta$

$+ xy\sin\theta\cos\theta + x^2\sin^2\theta] - \sqrt{2}\,[x\cos\theta - y\sin\theta] = 0$

Here, coefficient xy is,

$-\sin\theta\cos\theta + 2\cos^2\theta - 2\sin^2\theta + 6\sin\theta\cos\theta$

By question to remove the coefficient of xy it must be zero.

$\sin\theta\cos\theta = 2\cos^2\theta - 2\sin^2\theta - 3\sin\theta\cos\theta$

or, $3\sin\theta\cos\theta = \cos^2\theta - 3\sin\theta\cos\theta$

or, $\cos 2\theta = 0 = \cos 90^\circ$

or, $2\theta = 90^\circ$

or, $\theta = \frac{\pi}{4} = 45^\circ$

Hence, the rotation must be 45° to remove xy term. Then the equation will be when it rotated by 45° as;

$$x = x\cos 45^\circ - y\sin 45^\circ = \frac{x-y}{\sqrt{2}}$$

$$y = x\cos 45^\circ + y\sin 45^\circ = \frac{x+y}{\sqrt{2}}$$

Thus, equation is,

$$3\left(\frac{x-y}{\sqrt{2}}\right)^2 + 3\left(\frac{x+y}{\sqrt{2}}\right)^2 + 2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) - \sqrt{2}\left(\frac{x-y}{\sqrt{2}}\right) = 0$$

$$\text{or, } 3\frac{(x^2-2xy+y^2)}{2} + 3\frac{(x^2+2xy+y^2)}{2} + 2\frac{(x^2-y^2)}{2} - (x-y) = 0$$

or, $3x^2 - 6xy + 3y^2 + 3x^2 + 6xy + 3y^2 + 2x^2 - 2y^2 - x + y = 0$

or, $6x^2 + 6y^2 - x + y = 0$ is the required equation obtained by roateting given axis with 45°.

12. Transform the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$ by translating the axes into an equation with linear term missing.

Sol^n. The given equation is,

$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$

Which is to be transformed in such a way that the linear terms are missing. Let the required transform is to the point (h, K) so,

$x = x + h$

$y = y + K$

Then,

$12(x+h)^2 - 10(x+h)(y+K) + 2(y+K)^2 + 11(x+h) - 5(y+K) + 2 = 0$

i.e.

$12(x^2 + 2xh + h^2) - 10(xy + xK + hy + hK) + 2(y^2 + 2yk + k^2) + 11x + 11h - 5y - 5k + 2 = 0$

$12x^2 + 2y^2 + x(24h - 10k + 11) + y(-10h + 4k - 5) + xy(-10) + 12h^2 - 10hk + 2k^2 + 11h - 5k + 2 = 0$

For coefficient x = 0, Coefficient y = 0

We have,

$24h - 10k + 11 = 0$ ..... (i)

$10h - 4k + 5 = 0$ ..... (ii)

So,

$48h - 20k + 22 = 0$

$50h - 20k + 25 = 0$

From (i) $\left(24 \times \left(\frac{-3}{2}\right) + 11 = 10k\right)$

$\therefore K = -\frac{25}{10} = -\frac{5}{2}$

Hence, $(h, K) = \left(-\frac{3}{2}, -\frac{5}{2}\right)$

Thus, equation reduced to,

$$12x^2 + 2y^2 + 12\left(-\frac{3}{2}\right)^2 - 10 \times \left(-\frac{3}{2}\right) \times \left(-\frac{5}{2}\right) - 10xy + 2\left(-\frac{5}{2}\right)^2 + 11 \times \left(-\frac{3}{2}\right) - 5 \times \frac{-5}{2} + 2 = 0$$

$$\text{or, } 12x^2 + 2y^2 + 12 \times \frac{9}{4} - \frac{150}{4} + \frac{50}{4} - \frac{33}{2} + \frac{25}{4} + 2 = 0$$

$$\text{or, } 12x^2 + 2y^2 - 10xy + \frac{108}{4} - \frac{150}{4} + \frac{50}{4} - \frac{66}{4} + \frac{50}{4} + \frac{8}{4} = 0$$

or, $12x^2 + 2y^2 - 10xy = 0$

or, $6x^2 - 5xy + 2y^2 = 0$ is the required equation after transforming the equation in such a way that the linear terms are missing.

13. The equation $x^2 - y^2 = a^2$ is transferred to $xy = c^2$ by a change of rectangular axes. Find the inclination of new axes axes to the original axes and the value of c^2.

Sol^n. Since, $x^2 - y^2 = a^2$ is transferred

So, Let the axis is rotated by θ

So, $x = x\cos\theta - y\sin\theta$

$y = y\cos\theta + x\sin\theta$

$(x\cos\theta - y\sin\theta)^2 - (y\cos\theta + x\sin\theta)^2 = a^2$

or, $x^2\cos^2\theta - 2xy\sin\theta + y^2\sin^2\theta - y^2\cos^2\theta - 2xy\sin\theta\cos\theta - x^2\sin^2\theta = a^2$

or, $(\cos^2\theta - \sin^2\theta) + y^2(\sin^2\theta - \cos^2\theta) + xy[-2\sin\theta\cos\theta - 2\sin\theta\cos\theta] = a^2$

Since, $x^2 - y^2$ terms are missing,

So, $\cos^2\theta - \sin^2\theta = 0$

$\sin^2\theta - \cos^2\theta = 0$

Hence,

$\cos^2\theta - \sin^2\theta = 0$

or, $\cos^2\theta = 0 = \cos 90^\circ$

or, $2\theta = 90^\circ$

or, $\theta = 45^\circ$

Hence, $xy[-4\sin\theta\cos\theta] = a^2$

or, $xy[-2\sin 2\theta] = a^2$

or, $xy[-2\sin 90^\circ] = a^2$

or, $xy(-2) = a^2$

or, $xy = -\frac{a^2}{2}$

Comparing it with $xy = C^2$ yields

$C^2 = -\frac{a^2}{2}$ and the rotation is $\theta = 45^\circ$

14. If (x, y) and (x_1, y_1) be the coordinates of the same point referred to two sets of rectangular axes with same origin and if $ux + vy$ where u and v are independent x, y, become $u_1x_1 + v_1y_1$. Show that $u^2 + v^2 = u_1^2 + v_1^2$.

Sol^n. By questions,

Thus, axes have same origin

So, let Q be the rotation then

$x = x_1\cos\theta - y_1\sin\theta$

and $y = y_1\cos\theta + x_1\sin\theta$

Then, by questions,

$ux + vy = u_1x_1 + v_1y_1$

i.e., $u(x_1\cos\theta - y_1\sin\theta) + y_1(v\cos\theta - u\sin\theta)$

$= u_1x_1 + v_1y_1$

Thus, $u_1 = u\cos\theta + v\sin\theta$

$v_1 = v\cos\theta - u\sin\theta$

Thus,

$u_1^2 + v_1^2 = (u\cos\theta + v\sin\theta)^2 + (v\cos\theta - u\sin\theta)^2$

$= u^2\cos^2\theta + v^2\sin^2\theta + v^2\cos^2\theta + v^2\sin^2\theta - 2uv\sin\theta\cos\theta$

$+ 2uv\sin\theta\cos\theta$

$= u^2(\cos^2\theta + \sin^2\theta) + v^2(\sin^2\theta + \cos^2\theta)$

$= u^2 + v^2$

15. Transform the equation $x^2 - 2xy + y^2 + x - 3y = 0$ to axes through the point $(-1, 0)$ parallel to the lines bisecting the angles between original axes.

Sol^n. The given equation is,

$x^2 - 2xy + y^2 + x - 3y = 0$

So, the angle between the axes is 90, so 17 bisects is $\theta = 45^\circ$.

Hence, $x = x\cos\theta - y\sin\theta + h$

$x\cos\frac{\pi}{4} - y\sin\frac{\pi}{4} - 1$

$= \frac{x - y - \sqrt{2}}{\sqrt{2}}$

$y = y\cos\theta + x\sin\theta + 0$

$= \frac{y + x}{\sqrt{2}}$

Thus,

Substituting yelds,

$$\left(\frac{x-y}{\sqrt{2}} - 1\right)^2 - 2\left(\frac{x-y}{\sqrt{2}} - 1\right)\left(\frac{y+x}{\sqrt{2}}\right) + \left(\frac{y+x}{\sqrt{2}}\right)^2 + \frac{x-y-\sqrt{2}}{\sqrt{2}}$$

$$-3\left(\frac{y+x}{\sqrt{2}}\right) = 0$$

or, $$\left(\frac{x-y}{\sqrt{2}}\right)^2 - 2\left(\frac{x-y}{\sqrt{2}}\right) + 1 - 2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) + 2\left(\frac{x+y}{\sqrt{2}}\right)$$

$$+\left(\frac{x+y}{\sqrt{2}}\right)^2 + \frac{x-y}{\sqrt{2}} - 1 - \frac{3(x+y)}{\sqrt{2}} = 0$$

or, $$\frac{x^2}{2} - \frac{2xy}{2} + \frac{y^2}{2} - 2\left(\frac{xy}{\sqrt{2}}\right) + 1 - 2\left(\frac{x^2}{2} - \frac{y^2}{2}\right) + 2\left(\frac{x+y}{\sqrt{2}}\right)$$

$$+\frac{x^2}{2} + \frac{y^2}{2} + \frac{2xy}{2} + \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} - 1 - \frac{3x}{\sqrt{2}} - \frac{3y}{\sqrt{2}} = 0$$

or, $$x^2 + y^2 - x^2 + y^2 + \frac{2x}{\sqrt{2}} + \frac{2y}{\sqrt{2}} + \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} - 1 - \frac{3x}{\sqrt{2}} - \frac{3y}{\sqrt{2}} = 0$$

or, $$2y^2 + 2\left(\frac{x+y}{\sqrt{2}}\right) - \frac{2x}{\sqrt{2}} - \frac{4y}{\sqrt{2}} - \frac{2x}{\sqrt{2}} + \frac{2y}{\sqrt{2}} = 0$$

or, $$2y^2 + \sqrt{2}(x+y) - \frac{2x - 4y}{\sqrt{2}} = 1$$

or, $2\sqrt{2}\,y^2 + 2(x+y) - (2x - 4y) = \sqrt{2}$

or, $2\sqrt{2}\,y^2 + 6y - \sqrt{2} = 0$ is the required solution.

The Ellipse

20

Exercise - 33

1. **Find the equation of ellipse.**

(i) **whose foci (± 2, 0) and eccentricity $\frac{1}{2}$.**

Sol^n. Here, foci (± 2, 0) and $e = \frac{1}{2}$

Here, the major axis is along x-axis

Given that, $ae = 2$, $e = \frac{1}{2}$

So, $a \times \frac{1}{2} = 2$

$\therefore a = 4$

Now, $b^2 = a^2 - a^2e^2$, $a > b$

or, $b^2 = 16 - 4$

or, $b^2 = 12$

$\therefore b = \sqrt{12}$

Hence, the equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or, $\frac{x^2}{16} + \frac{y^2}{12} = 1$

or, $\frac{3x^2 + 4y^2}{48} = 1$

or, $3x^2 + 4y^2 = 48$

is the required equation under given condition.

(ii) **whose vertices (± 5, 0) and foci (± 4, 0)**

Sol^n. Here, vertices (± 5, 10) and foci = (± 4, 10)

Here, the major axis is along x-axis

Hence, veetices a = 5 and the foci ae = 4 thus, $e = \frac{4}{5}$

Since, the major axis is along x-axis,

$b^2 = a^2(1 - e^2)$ for $a > b$

or, $b^2 = 25\left(1 - \frac{16}{25}\right)$

or, $b^2 = 25 \times \frac{5}{25} = 9$

or, $b = \sqrt{9}$

Hence, the equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or, $\frac{x^2}{25} + \frac{y^2}{9} = 1$

or, $9x^2 + 25y^2 = 225$ is the required equation.

(iii) **focus (2, 4) eccentricity $\frac{1}{\sqrt{13}}$ and directrix $x - y + 13 = 0$**

Sol^n. Here, focus (2, 4), eccentricity = $\frac{1}{\sqrt{13}}$ and directrix is

$x - y + 13 = 0$.

We know that, if (x, y) be any point on ellipse then the e times perpendicular distance from (x, y) to $x - y + 13 = 0$ must equal the distance of (x, y) from focus, so,

$$(x-2)^2 + (y-4)^2 = e^2\left(\frac{x - y + 13}{\sqrt{2}}\right)^2$$

or, $x^2 - 4x + 4 + y^2 - 8y + 16 = \frac{1}{13}\frac{(x - y + 13)^2}{2}$

or, $26x^2 + 26y^2 - 104x - 208y + 520 = x^2 + y^2 - 2xy + 26x - 26y + 169$

or, $25x^2 + 25y^2 + 2xy - 130x - 182y + 451 = 0$

is the required equation of ellipse under given condition.

(iv) **focus (– 1, 1) eccentricity $\frac{1}{2}$ and directrix, $x - y + 3 = 0$**

Sol^n. Here, focus (– 1, 1), $e = \frac{1}{2}$ and the directrix is $x - y + 3 = 0$

By detinition, we know that the distance from any point (x, y) of the ellipse to the focus (– 1, 1) is equal to e times the perpendicular distance from the point to the directrix. So

$$(x+1)^2+(y-1)^2=\frac{1}{4}\left[\frac{x-y+3}{\sqrt{2}}\right]^2$$

or, $x^2+2x+1+y^2-2y+1=\frac{1}{8}[x^2-2xy+y^2+6x-6y+9]$

or, $8x^2+8y^2+16x-16y=x^2-2xy+y^2+6x-6y+9$

or, $7x^2+7y^2+2xy+10x-10y+7=0$ is the required equation of ellipse.

2. **Find the equation of the ellipse whose centre is the origin, whose axes are the axes of coordinates and which passes through**

(i) **the points (1, 4) and (−3, 2)**

Sol^n. Here, the equation of ellipse having centre as origin and axis of coordinate is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ..... (i)

By question (i) passess through (1, 4)

So, $\frac{1}{a^2}+\frac{16}{b^2}=1$ ..... (ii)

and since it passess through (−3, 2), So, is

$\frac{9}{a^2}+\frac{4}{b^2}=1$ ..... (iii)

From (ii) and (iii) solving yields,

$$\frac{9}{a^2}+\frac{144}{b^2}=9$$
$$\frac{9}{a^2}+\frac{4}{b^2}=1$$
$$- \quad - \quad -$$
$$\frac{140}{b^2}=8$$

or, $8b^2=140 \Rightarrow b^2=\frac{140}{8}$ or, $b^2=\frac{70}{4}=\frac{35}{2}$

Similarly, $\frac{1}{a^2}+\frac{16}{b^2}=1$

$$\frac{36}{a^2}+\frac{16}{b^2}=1$$
$$\frac{1}{a^2}+\frac{16}{b^2}=4$$
$$- \quad - \quad -$$
$$-\frac{35}{a^2}=-3$$

or, $a^2=\frac{35}{3}$

Hence, equation is $\frac{3x^2}{35}+\frac{2y^2}{35}=1$

Thus, $3x^2+2y^2=25$ is the required equation of ellipse.

(ii) **the points (2, 1) and (1, −3)**

Sol^n. Here, Let the equation of ellipse is

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ..... (i)

Since, it passess through (2, 1) and (1, −3), these coordinates must satisfy (i) so,

$\frac{4}{a^2}+\frac{1}{b^2}=1$ ..... (ii)

or, $\frac{1}{a^2}+\frac{9}{b^2}=1$ ..... (iii)

Solving yields,

$$\frac{4}{a^2}+\frac{1}{b^2}=1$$
$$\frac{4}{a^2}+\frac{36}{b^2}=4$$
$$- \quad - \quad -$$
$$\frac{-35}{b^2}=-3$$

$\therefore b^2=\frac{35}{3}$

Similarly, $a^2=\frac{35}{8}$

Hence the required equation of ellipse is,

$$\frac{x^2}{\frac{35}{8}}+\frac{y^2}{\frac{35}{3}}=1$$

or, $8x^2+3y^2=35$ is the required equation of ellipse.

3. **Find the equation of ellipse whose latus rectum 5 eccentricity $\frac{2}{3}$ and its axes as the axes of coordinates.**

Sol^n. Here, the equation of ellipse is

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ..... (i)

By the questions,

Ecentricity $e = \frac{2}{3}$ and

Latus – rectum $\frac{2b^2}{a} = 5$ ..... (ii)

Since, $b^2 = a^2(1-e^2)$

or, $b^2 = a^2\left(1-\frac{4}{9}\right)$

or, $\frac{5a}{2} = a^2 \times \frac{5}{9}$

or, $9a = 2a^2$

or, $a = \frac{9}{2}$

Thus,

$b^2 = a^2(1-e^2)$

$= \frac{9}{2} \times \frac{9}{2}\left(1-\frac{4}{9}\right) = \frac{81}{4} \times \frac{5}{9} = \frac{45}{4}$

$\therefore b^2 = \frac{45}{4}$

Hence, the required equation of ellipse is,

$\frac{x^2}{\frac{81}{4}} + \frac{y^2}{\frac{45}{4}} = 1$

or, $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$

or, $\frac{20x^2 + 36y^2}{405} = 1$

i.e. $20x^2 + 36y^2 = 405$ is the required ellipse.

4. Find the eccentricity, latus rectum, directrix and foci of the ellipse

(i) $3x^2 + 4y^2 = 12$

Sol^n. Here, $3x^2 + 4y^2 = 12$

or, $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Here, $a > b$

Thus, eccentricity is obtained from

$b^2 = a^2(1-e^2)$

or, $3 = 4(1-e^2)$

or, $\frac{3}{4} = 1 - e^2$

or, $e^2 = 1 - \frac{3}{4} = \frac{1}{4}$

or, $e = \frac{1}{2}$

Latus -rectum $= \frac{2b^2}{a} = \frac{2 \times 3}{2} = 3$

Directrix is $x = \pm\frac{a}{e} = \pm\frac{2}{\frac{1}{2}} = \pm 4$

Foci $= (\pm ae, 0) = (\pm 1, 0)$

(ii) $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Sol^n. Here, a . b, thus for eccentricity

$a^2 = b^2(1-e^2)$

or, $9 = 16(1-e^2)$

or, $\frac{9}{16} = 1 - e^2$

or, $e^2 = 1 - \frac{9}{16} = \frac{7}{16}$

So, $e = \frac{\sqrt{7}}{4}$

We have, Latus -rectum $= \frac{2a^2}{b} = \frac{2 \times 9}{4} = \frac{9}{2}$

$\therefore$ The equation of directrix $y = \pm\frac{b}{e}$

$= \pm\frac{4}{\frac{\sqrt{7}}{4}} = \pm\frac{16}{\sqrt{7}}$

$\therefore$ The coordinates of foci $= (0, \pm be)$

$= \left(0, \pm 4 \times \frac{\sqrt{7}}{4}\right) = \left(0, \pm\sqrt{7}\right)$

(iii) $4x^2 + y^2 - 8x + 2y + 1 = 0$

Sol^n. Here, $4x^2 + y^2 - 8x + 2y + 1 = 0$ can be written as,

$4(x^2-2)^2 + y^2 + 2y + 1 = 0$

or, $4(x^2 - 2x + 1) + y^2 + 2y + 1 - 4 = 0$

or, $4(x-1)^2 + (y+1)^2 = 4$

or, $\frac{(x-1)^2}{1} + \frac{(y+1)^2}{4} = 1$

Here, $b > a$

Thus, for eccentricity,

$a^2 = b^2(1-e^2)$

or, $1 = 4(1-e^2)$

or, $\frac{1}{4} = 1 - e^2$

or, $e^2 = 1 - \frac{1}{4} = \frac{3}{4}$

Here, $e = \frac{\sqrt{3}}{2}$

The length of latus -rectum is $\frac{2a^2}{b} = \frac{2\times 1}{2} = 1$

The equation of directix is,

$y = K \pm \frac{b}{e} \Rightarrow -1 \pm \frac{2}{\frac{\sqrt{3}}{2}} = -1 \pm \frac{4}{\sqrt{3}}$

The coordinate of foci = (h, k ± be)

$= \left(1, -1 \pm 2\times\frac{\sqrt{3}}{2}\right) = \left(1, -1 \pm \sqrt{3}\right)$

(iv) $7x^2 + 6y^2 - 42x - 24y + 86 = 0$

Sol^n. Here, $7x^2 + 6y^2 - 42x - 24y + 86 = 0$ can be written as,

$7(x^2 - 6x) + 6(y^2 - 4y) + 86 = 0$

or, $7(x^2 - 6x + 9) + 6(y^2 - 4y + 4) + 86 = 63 + 24$

or, $7(x-3)^2 + 6(y-2)^2 = 1$

or, $\frac{(x-3)^2}{\frac{1}{7}} + \frac{(y-2)^2}{\frac{1}{6}} = 1$

Since, b > a, so for eccentricity,

$a^2 = b^2(1-e^2)$

or, $\frac{1}{7} = \frac{1}{6}(1-e^2)$

or, $\frac{6}{7} = 1 - e^2$

or, $e^2 = 1 - \frac{6}{7} = \frac{1}{7}$

So, $e = \frac{1}{\sqrt{7}}$

The latus - rectum $= \frac{2a^2}{b}$

or, $\frac{2\times\frac{1}{7}}{\frac{1}{\sqrt{6}}} = \frac{2\sqrt{6}}{7}$

The equation of directrix is,

$y = K \pm \frac{b}{e} = 2 \pm \frac{\sqrt{7}}{\sqrt{6}}$

The coordinates of foci = (h, k ± be)

$= \left(3, 2 \pm \frac{1}{\sqrt{42}}\right)$

(v) $8(x-1)^2 + 6(y+1)^2 = 1$

Sol^n. Here, The equation of ellipse is,

$8(x-1)^2 + 6(y+1)^2 = 1$

Which can be written as,

$\frac{(x-1)^2}{\frac{1}{8}} + \frac{(y+1)^2}{\frac{1}{6}} = 1$

Here, b > a, so, for eccentricity we use,

$a^2 = b^2(1-e^2)$

or, $\frac{1}{8} = \frac{1}{6}(1-e^2)$

or, $\frac{3}{4} = 1 - e^2$

or, $e^2 = \frac{1}{2}$

The latusrectum $\frac{2a^2}{b} = \frac{2\times\frac{1}{8}}{\sqrt{\frac{1}{6}}} = \frac{2}{8}\times\frac{\sqrt{6}}{1} = \frac{\sqrt{6}}{4}$

The equation of latus rectum is,

$y = k \pm \frac{b}{e}$

$= -1 \pm \frac{1}{\sqrt{6}} \times \frac{2}{1}$

$= -1 \pm \frac{2}{\sqrt{6}}$

The coordinate of foci are h, k ± be

$= \left(1, -1 \pm \frac{1}{\sqrt{6}} \times \frac{1}{2}\right) = \left(1, -1 \pm \frac{1}{2\sqrt{6}}\right)$

5. **Find the centre, length of axes, eccentricity and directrix of the ellipse**

(i) $2x^2 + 3y^2 - 4x + 5y + 4 = 0$

Sol^n. Here, The given equation of ellipse

$2x^2 + 3y^2 - 4x + 5y + 4 = 0$ which can be written as,

$$2(x^2 - 2x) + 3\left(y^2 + \frac{5}{3}y\right) + 4 = 0$$

$$\text{or, } 2(x^2 - 2x + 1) + 3\left(y^2 + 2y\frac{5}{6} + \frac{25}{36}\right) - 2 - 3 \times \frac{25}{36} + 4 = 0$$

$$\text{or, } 2(x-1)^2 + 3\left(y + \frac{5}{6}\right)^2 = \frac{75}{36} - 2 = \frac{3}{36} = \frac{1}{12}$$

$$\text{or, } \frac{(x-1)^2}{\frac{1}{24}} + \frac{\left(y + \frac{5}{6}\right)^2}{\frac{1}{36}} = 1$$

Thus, can be is $(h, k) = \left(1, -\frac{5}{6}\right)$

Length of axis are;

$$\text{Major axis} = 2 \times \frac{1}{\sqrt{24}} = \frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}$$

$$\text{Minor axis} = 2 \times \frac{1}{\sqrt{36}} = \frac{2}{6} = \frac{1}{3}$$

For eccentricity, since $a > b$

$b^2 = a^2(1 - e^2)$

$$\text{or, } \frac{1}{36} = \frac{1}{24}(1 - e^2)$$

$$\text{or, } \frac{2}{3} = 1 - e^2$$

$$\therefore e^2 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$e = \frac{1}{\sqrt{3}}$$

The equation of directrix is,

$$x = h \pm \frac{a}{e}$$

$$= 1 \pm \frac{1}{\sqrt{24}} \times \frac{\sqrt{3}}{1}$$

$$= 1 \pm \frac{1}{\sqrt{8}}$$

$$= 1 \pm \frac{1}{2\sqrt{2}}$$

(ii) $2x^2 + 3y^2 - 4x - 12y + 13 = 0$

Sol^n. Here, the given equation of ellipse is,

$2x^2 + 3y^2 - 4x - 12y + 13 = 0$ which can be written as,

$2(x^2 - 2x + 1) + 3(y^2 - 4y + 4) - 2 - 12 + 13 = 0$

$$\text{or, } \frac{(x-1)^2}{\frac{1}{2}} + \frac{(y-2)^2}{\frac{1}{3}} = 1$$

Since, $a > b$ = So,

For eccentricing we use,

$b^2 = a^2(1 - e^2)$

$$\text{or, } \frac{1}{3} = \frac{1}{2}(1 - e^2)$$

$$\text{or, } \frac{2}{3} = 1 - e^2$$

$$\text{or, } e^2 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\text{or, } e = \frac{1}{\sqrt{3}}$$

Centre is $(h, k) = (1, 2)$

$$\text{Length of major asxis} = 2a = 2 \times \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\text{The length of miner axis} = 2 \times \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

The equation of directrix is,

$$x = h \pm \frac{a}{e}$$

$$= 1 \pm \sqrt{\frac{3}{2}}$$

6. **Find the coordinates of the second focus and the equation of second directrix of the ellipse whose one focus is S (1, 2) and the corresponding directrix is the line $x - y = 5$ and $e = \frac{1}{2}$.**

Sol^n. The coordinate of focus and the equation of directrix is required to find one focus is S (1, 2), and the corresponding directrix is,

$x - y = 5$ and the eccentricity $e = \frac{1}{2}$

Since the equation of directrix is $x - y = 5$ ..... (i)

Whose slope is 1

So, the major axis has slope -1

The equation of major axis passess through focus $(1, 2)$ to equation with slope -1 is,

$y-2=-1(x-1)$

$y-2=-x+1$

or, $y+x-3=0$ ..... (ii)

Solving (i) and (ii) gives,

$x=4$

$y=-1$

Thus, we have,

$$\frac{SA}{AZ}=\frac{1}{2}$$

Hence A divides SZ internally.

$$\text{So, } A=\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$$

$$=\left(\frac{1\times4+2\times1}{1+2}, \frac{1\times(-1)+2\times2}{3}\right)$$

$=(2, 1)$

and $A'=(-2, 5)$

Smce, C is centre So, $(0, 3)$

C is the mid-point of zz^1 as well So

Let, $z=(h, k)$, then,

or, $\frac{h+4}{2}=0$

$\therefore (h=-4$

and $\frac{k-1}{2}=3$

or, $k-1=6$

or, $k=7$

Hence, $z^1=(-4, 7)$

The directrix is parallel to $z-y-5=0$ is $x-y+k=0$ which passess through $(-4, 7)$ is,

$-4-7+k=0 \Rightarrow k=11$

So, the required directrix is, $x-y+11=0$

To find focus, let (h^1, k^1) be focus,

$$\text{So, } \left(\frac{h^1+1}{2}, \frac{k^1+2}{2}\right)=(0, 3)$$

or, $\frac{h^1+1}{2}=$ or, $h^1=-1$ and d $\frac{k^1+2}{2}=3$

or, $k^1=4$ Thus, focus is $(-1, 4)$

By transferring the origin to the point (2, 3) and turning the axes through an angle $\frac{\pi}{4}$ prove that $3x^2+2xy+3y^2-18x-22y+50=0$ represents an ellipse.

Sol: If the axis is turned through $\frac{\pi}{4}$ and orign is transformed to (2, 3), then the equation is changed and is obtained by substituting,

$x=x\cos\theta-y\sin\theta+h$

$y=y\cos\theta+x\sin\theta+k$

i.e., $x=x\cos\frac{\pi}{4}-y\sin\frac{\pi}{4}+2$

$=\frac{x}{\sqrt{2}}-\frac{y}{\sqrt{2}}+2$

$=\frac{x-y}{\sqrt{2}}+2$

$y=\frac{y}{\sqrt{2}}+\frac{x}{\sqrt{2}}+k=\frac{x+y}{\sqrt{2}}+3$

$$\text{or, } 3\left(\frac{x-y}{\sqrt{2}}+2\right)^2+2\left(\frac{x+y}{\sqrt{2}}+3\right)\left(\frac{x-y}{\sqrt{2}}+2\right)+3\left(\frac{x+y}{\sqrt{2}}+3\right)^2$$

$$-18\left(\frac{x-y}{\sqrt{2}}+2\right)-22\left(\frac{x+y}{\sqrt{2}}+3\right)+50=0$$

$$\text{or, } \frac{3}{2}(x-y)^2+\frac{12}{\sqrt{2}}(x-y)+12+x^2-y^2+\frac{6}{\sqrt{2}}(x-y)$$

$$+12+\frac{4}{\sqrt{2}}(x+y)+\frac{3}{2}(x+y)^2+\frac{18}{\sqrt{2}}(x+y)+27$$

$$-\frac{18}{\sqrt{2}}(x-y)-36-\frac{22}{\sqrt{2}}(x+y)-66+50=0$$

$$\text{or, } \frac{3}{2}(x-y)^2+\frac{3}{2}(x+y)^2+x^2-y^2-1=0$$

or, $3[x^2-2xy+y^2+x^2+2xy+y^2]+2x^2-2y^2-2=0$

$\therefore 8x^2+4y^2-2=0$

$\therefore 4x^2+2y^2=1$

Hence, transformed equation is $4x^2+2y^2=1$

i.e., $\frac{x^2}{\frac{1}{4}}+\frac{y^2}{\frac{1}{2}}=1$ which is the equation of ellipse.

Show that the point $x=a.\frac{1-t^2}{1+t^2}$ and $y=b.\frac{2t}{1+t^2}$ where to is a variable parameter, lies on an ellipse.

Sol^n. Here, $x = a\frac{(1-t^2)}{1+t^2}$ and $y = b\left(\frac{2t}{1+t^2}\right)$ where to show that, these points lies on the equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

Substituting values of x and y yields,

$$\frac{a^2}{a^2}\left(\frac{1-t^2}{1+t^2}\right)^2 + b^2\left(\frac{2t}{1+t^2}\right)^2$$

$$= \frac{\left(1-t^2\right)^2}{\left(1+t^2\right)^2} + \frac{4t^2}{\left(1+t^2\right)} = \frac{\left(1-t^2\right)^2+4t^2}{\left(1+t^2\right)^2} = \frac{\left(1+t^2\right)^2}{\left(1+t^2\right)} = 1$$

9. **A point P moves in such a way that the sum of its distance from S and S_1 is 10 and $SS_1 = 8$. Show that the locus of P is an ellipse. Find its equation, its eccentricity and length of latus rectum.**

Sol^n. Let p (x, y) be any point on locus and S and S' be considered in such a way that SS' = B . So, let S (– 4, 0) and S' (4, 0) be standard co-ordinates on x-axis. By question,

$PS + PS' = 10$

or, $\sqrt{(x+4)^2+y^2} + \sqrt{(x-4)^2+y^2} = 10$

or, $\sqrt{(x+4)^2+y^2} - 10 = -\sqrt{(x-4)^2+y^2}$

Squaring both sides, we get,

$(x+4)^2+y^2-20\sqrt{(x+4)^2+y^2} + 100$

$= x^2-8x+16+y^2$

or, $x^2+8x+16+y^2-20\sqrt{(x+4)^2+y^2} + 100$

$= x^2-8x+16+y^2$

or, $16x-100 = 20\sqrt{(x-4)^2+y^2}$

Again squaring both sides, we get,

$(16x-100)^2 = 400(x^2-8x+16+y^2)$

or, $16(4x-25)^2 = 400(x^2-8x+16+y^2)$

or, $(4x-25)^2 = 25(x^2-8x+16+y^2)$

or, $16x^2-200x+625 = 25x^2-200x+400+25y^2$

or, $9x^2+25y^2 = 225$

or, $\frac{x^2}{25}+\frac{y^2}{9} = 1$ which is the required equation of ellipse.

10. **Find the equation of the ellipse whose foci are at (– 2, 4) and (4, 4) and major axis is 10. Also find eccentricity of the ellipse.**

Sol^n. The foci of ellipse are (– 2, 4) and (4, 4) and major axis is 10.

$(h-ae, k) = (-2, 4)$

$(h+ae, k) = (4, 4)$

and major axis,

$2a = 10$

$a = 5$

Thus,

$h-ae = -2$

$h+ae = 4$

$2h = 2$

$h = 1$

$\therefore h-ae = -2$

$1-5\times 2 = -2$

$-5e = -3$

$e = \frac{3}{5}$

We know that,

$b^2 = a^2(1-e^2)$

or, $b^2 = 5^2\left(1-\frac{9}{25}\right)$

or, $b^2 = 25\left(\frac{16}{25}\right)$

$\therefore b^2 = 16$

Again h = 1 and k = 4, so centre is (1, 4)

So, the equation is,

$\frac{(x-1)^2}{25}+\frac{(y-4)^2}{16} = 1$

or, $16(x^2-2x+1)+25(y^2-8y+16) = 400$

or, $16x^2-32x+16+25y^2-200y+400 = 400$

or, $16x^2+25y^2-32x-200y+16 = 0$ is the required equation of ellipse cender given condtion.

11. **Find the eccentric angle of a point on the ellipse $\frac{x^2}{4}+\frac{y^2}{5} = 2$ whose distance from the centre is $\frac{\sqrt{34}}{2}$.**

Sol^n. The equation of ellipse is

$\frac{x^2}{4}+\frac{y^2}{5} = 2$

i.e. $\frac{x^2}{8}+\frac{y^2}{10} = 1$

Let ϕ be the eccentric angle of point P on the ellipse $\frac{x^2}{8}+\frac{y^2}{10}=1$

The point P is,

$P\left(\sqrt{8}\cos\phi, \sqrt{10}\sin\phi\right)$

The centre of $\frac{x^2}{8}+\frac{y^2}{10}=1$ is 0 (0, 0)

By equation,

The distance OP $= \frac{\sqrt{34}}{2}$

So, $8\cos^2\phi + 10\sin^2\phi = \frac{34}{4}$

or, $16\cos^2\phi + 20\sin^2\phi - 17 = 0$

or, $16\cos^2\phi + 20 - 20\cos^2\phi - 17 = 0$

or, $-4\cos^2\phi + 3 = 0$

or, $4\cos^2\phi = 3$

or, $\cos\phi = \pm\frac{\sqrt{3}}{2}$ the required eccentric angles.

Thus, $\phi = \pm\frac{\pi}{3}$ or, $\pm\frac{4\pi}{3}$ is the required eccentric angles.

Exercise - 34

1. **Fine the equation of tangent and normal**
(i) **at the point (3, – 2) of the ellipse $4x^2 + 9y^2 = 36$**

Sol^n. The given equation of ellipse is,

$4x^2 + 9y^2 = 36$ which is of the form $\frac{x^2}{9}+\frac{y^2}{4}=1$ and we have to find the equation of tangent to it at (3, – 2), so by the formula, equation of tangent is,

$\frac{x.3}{9}+\frac{y(-2)}{4}=1$

i.e. $\frac{3x}{9}-\frac{2y}{4}=1$

or, $12x - 18y = 36$

or, $2x - 3y = 6$ is the required equation of tangent under given condition

The given equation of ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$ and at (3, – 2)

We have to find the equation of normal

The equation of normal at (3, – 2) to $\frac{x^2}{9}+\frac{y^2}{4}=1$ is;

$\frac{x-3}{\frac{3}{9}}+\frac{y+2}{\frac{-2}{4}}$

or, $3(x-3) = -2(y+2)$

or, $3x - 9 = -2y - 4$

or, $3x + 2y - 5 = 0$ is the required equation of normal.

(ii) **at the point $\left(1, \frac{4}{3}\right)$ of the ellipse $4x^2 + 9y^2 = 20$**

Sol^n. The given equation of ellipse is

$4x^2 + 9y^2 = 20$ which is of the form $\frac{x^2}{5}+\frac{y^2}{\frac{20}{9}}=1$ and we have to find the equation of tangent at $\left(1, \frac{4}{3}\right)$

So, by formula the equation of tangent is,

$$\frac{x \cdot 1}{5} + \frac{y \cdot \frac{4}{3}}{\frac{20}{9}} = 1$$

or, $\frac{x}{5} + \frac{12y}{20} = 1$

or, $4x + 12y = 20$

$x + 3y = 5$ is required equation of tangent

The equation of normal is always perpendicular to tangent so is,

$3x - y + k = 0$

But it passess through $\left(1, \frac{4}{3}\right)$. So, it must satisfy;

$3x - y + k = 0$

or, $3 \times 1 - \frac{4}{3} + k = 0$

or, $\frac{9-4}{3} = -k$

or, $K = -\frac{5}{3}$

or, $3x - y - \frac{5}{3} = 0$

or, $9x - 3y = 5$ is the required equation of normal.

(iii) at the ends of the latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol^n. The equation of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so the end of latus-rectum are $\left(-ae, \frac{b^2}{a}\right)$ and $\left(-ae, -\frac{b^2}{a}\right)$

Thus, the equation of tangent at $\left(-ae, \frac{b^2}{a}\right)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$

$= 1$ is, $\frac{x(-ae)}{a^2} + \frac{y\left(\frac{b^2}{a}\right)}{b^2} = 1$

or, $-\frac{ae}{a} + \frac{y}{a} = 1$ or $-xe + y = a$

Hence, tangent at $\left(-ae, \frac{b^2}{a}\right)$ is $-xe + y = a$

Similarly, dt $\left(-ae, -\frac{b^2}{a}\right)$ is $-xe - y = a$

i.e. $y + xe = -a$. For normal at $\left(-ae, \frac{b^2}{a}\right)$ norla is perpendicular to tangent

So, $-x - ey + k = 0$

Since it passess through $\left(-ae, \frac{b^2}{a}\right)$, So,

$ae - e \times \frac{b^2}{a} + K = 0$

or, $a^2e - b^2e = -Ka$

or, $e(a^2 - b^2) = -Ka$ ..... (i)

Since, $b^2 = a^2(1 - C^2)$

or, $b^2 - a^2 = -a^2C^2$

Hence, $-Ka = a^2e^2 \times E$

or, $K = -ae^3$

Thus, normal at $\left(-ae, \frac{b^2}{a}\right)$ is,

$-x - ey - ae^3 = 0$

or, $x + ey = ae^3$ similarly for another.

2. Find the equation of the tangents to the ellipse $9x^2 + 16y^2 = 144$ and passes through the point (2, 3).

Sol^n. We know that the equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$y = mx \pm \sqrt{a^2m^2 + b^2}$

So, the equation of tangent to

$\frac{x^2}{16} + \frac{y^2}{9} = 1$ is $y = mx \pm \sqrt{16m^2 + 9}$ ..... (i)

Since it passess through (2, 3), so, it satisfies (i)

$3 = 2m \pm \sqrt{16m^2 + 9}$

or, $3 - 2m = \pm \sqrt{16m^2 + 9}$

Squaring both sides, yields,

$(3 - 2m)^2 = 16m^2 + 9$

or, $9 - 12m + 4m^2 = 16m^2 + 9$

or, $-12m + 4m^2 - 16m^2 = 0$

or, $-12m^2 - 12m = 0$

or, $m = -1$ or $m = 0$

Thus, the equation of tangents are,

$y - 3 = -1(x - 2)$

or, $y - 3 = -x + 2$

or, $y + x = 5$ and $y - 3 = 0(x - 2)$

or, $y = 3$

3. Show that $4x + y + 7 = 0$ is a tangent in the ellipse $x^2 + 3y^2 = 3$ also find the point of contact.

Sol^n. The equation of the ellipse is $x^2 + 3y^2 = 3$ which is of the form

$\frac{x^2}{3} + \frac{y^2}{1} = 1$ and the given line is $4x + y + 7 = 0$

i.e. $y = -4x - 7$

Now, the point of intersection of $y = -4x - y$ and the ellipse $x^2 + 3y^2 = 3$ is;

$x^2 + 3(-4x - 7)^2 = 3$

i.e. $x^2 + 3(16x^2 + 36x + 49) = 3$

or, $x^2 + 48x^2 + 168x + 147 = 3$

or, $49x^2 + 168x + 147 = 3$

$49x^2 + 16xx + 144 = 0$

or, $(7x)^2 + 2 \times 7x \times 12x + (12)^2 = 0$

or, $(7x + 12)^2 = 0$

or, $7x + 12 = 0$

or, $x = \frac{-12}{7}$

$y = 4 \times \frac{12}{7} - 7 = -\frac{1}{7}$

Since, it perfect square

So, $y + 4x + 7 = 0$ is tangent and the point of contact is

$\left(\frac{-12}{7}, \frac{-1}{7}\right)$

4. Show that $x + y = \sqrt{a^2 + b^2}$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and find the coordinates of the point of contact.

Sol^n. The equation of ellipse is,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line is $x + y = \sqrt{a^2 + b^2}$

i.e. $y = \sqrt{a^2 + b^2} - x$ ..... (i)

Thus, their point of contact is,

$b^2x^2 + a^2(\sqrt{a^2 + b^2} - x)^2 = a^2b^2$

i.e. $b^2x^2 + a^2(a^2 + b^2 - 2x\sqrt{a^2 + b^2} + x^2) = a^2b^2$

or, $b^2x^2 + a^4 + a^2b^2 - 2a^2\sqrt{a^2 + b^2}\,x + a^2x^2 = a^2b^2$

or, $(b^2 + a^2)x^2 - 2a + 2\sqrt{a^2 + b^2}\,x + a^4 = 0$

or, $(\sqrt{a^2 + b^2})^2 - 2 \times a \times \sqrt{a^2 + b^2}\,x + (a^2)^2 = 0$

i.e., $(x\sqrt{a^2 + b^2} - a^2)^2 = 0$ which is perfect square so the given line is tangent.

To find point of contact

$x\sqrt{a^2 + b^2} - a^2 = 0$

i.e., $x = \frac{a^2}{\sqrt{a^2 + b^2}}$

From (i),

$y = \sqrt{a^2 + b^2} - \frac{a^2}{\sqrt{a^2 + b^2}} = \frac{(a^2 + b^2) - a^2}{\sqrt{a^2 + b^2}}$

Thus, Point of contact is $\left(\frac{a^2}{\sqrt{a^2 + b^2}}, \frac{b^2}{\sqrt{a^2 + b^2}}\right)$.

5. Find the equation of the tangents to the ellipse $3x^2 + 2y^2 = 1$ perpendicular to the line $x + 3y = 1$.

Sol^n. The given equation of ellipse is

$3x^2 + 2y^2 = 1$ which can be written as,

$\frac{x^2}{\frac{1}{3}} + \frac{y^2}{\frac{1}{2}} = 1$ and the line is $x + 3y = 1$

Hence, the tangent perpendicular to it is,

$3x - y + k = 0$ ..... (i)

If it is tangent to $3x^2 + 2y^2 = 1$ then their point of intersection is;

$3x^2 + 2(3x + k)^2 = 1$

or, $3x^2 + 2(9x^2 + 6kx + k^2) = 1$

or, $3x^2 + 18x^2 + 12kx + 2k^2 - 1 = 0$

or, $21x^2 + 12kx + 2k^2 - 1 = 0$

If the (i) tangent so;

$B^2 - 4AC = 0$

or, $144k^2 - 84(2k^2 - 1) = 0$

or, $144k^2 - 168k^2 + 84 = 0$

or, $-24k^2 = -84$

or, $k^2 = \frac{7}{2}$

This,

$k = \pm\frac{7}{2}$

Thus, the required equation of tangent perpendicular to $3y + x = 1$ are

$3y - y \pm \frac{7}{2} = 0$

6. **Find the equation of tangent to the ellipse $x^2 + 3y^2 = 3$ which are perpendicular to the straight line $x - 4y - 5 = 0$**

Sol^n. The equation of lines which are perpendicular to $x - 4y - 5 = 0$ are $-4x - y + k = 0$ it if tangent to $x^2 + 3y^2 = 3$

So, their point of intersection is,

$x^2 + 3(k - 4x)^2 = 3$

or, $x^2 + 3(k^2 - 8kx + 16x^2) = 3$

or, $x^2 + 3k^2 - 24kx + 48x^2 = 3$

or, $49x^2 - 24kx + 3k^2 - 3 = 0$

Thus, by condition of tangentcy $B^2 - 4AC = 0$

Gives,

$(-24k)^2 - 4 \times 49 \times (3k^2 - 3) = 0$

or, $576k^2 - 588k^2 + 588 = 0$

or, $-12k^2 = -588$

or, $k^2 = 49$

or, $k = \pm 7$

Thus, required equation of tangent is,

$-4x - y \pm 7 = 0$

i.e. $4x + y \pm 7 = 0$

7. **Find the equation is to the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the straight line $3x - y + 7 = 0$, also find the point of contact.**

Sol^n. The given equation of line is $3x - y + 7 = 0$ and the line parallel to it are $3x - y + k = 0$ and it is tangent to $4x^2 + 3y^2 = 5$ then, their point of intersection is obtained by,

$4x^2 + 3(3x + k)^2 = 5$

or, $4x^2 + 3(9x^2 + 6kx + k^2) = 5$

or, $4x^2 + 27x^2 + 18kx + 3k^2 - 5 = 0$

or, $31x^2 + 18kx + 3k^2 - 5 = 0$

If the line is tangent So,

$B^2 - 4AC = 0$ gives,

$(18k^2) - 4 \times 31 \times (3k^2 - 5) = 0$

or, $324k^2 - 372k^2 + 620 = 0$

or, $-48k^2 = -620$

or, $k^2 = \frac{620}{48} = \frac{310}{24} = \frac{155}{12}$

or, $k = \pm\sqrt{\frac{155}{12}} \pm \frac{1}{2}\sqrt{\frac{155}{3}}$

Hence, tangents are $y = 3x \pm \frac{1}{2}\sqrt{\frac{155}{3}}$ ..... (i)

To find the point of contact, we solve (i) and the ellipse.

Thus, $4x^2 + 3\left(3x \pm \frac{1}{2}\sqrt{\frac{155}{3}}\right)^2 = 5$

or, $4x^2 + 3\left(9x^2 \pm 3x\sqrt{\frac{155}{3}} + \frac{155}{6}\right) = 5$

or, $4x^2 + 27x^2 \pm 9x\sqrt{\frac{155}{3}} + \frac{155}{2} = 5$

or, $31x^2 \pm 9x\sqrt{\frac{155}{3}} + \frac{145}{2} = 0$

Thus, $x = \frac{9\sqrt{\frac{155}{3}} \pm \sqrt{9^2 + \frac{155}{3} - 4 \times 31 \times \frac{145}{2}}}{2 \times 31}$

or, $x = \frac{\sqrt{155 \times 27} \pm \sqrt{27 \times 155 - 31 \times 145}}{62}$

8. **Find the equations of the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which make equal intercepts on the both axes.**

Sol^n. Let the equation of tangent which makes equal intercept k on both axis is,

$\frac{x}{k} + \frac{y}{k} = 1$

i.e. $x + y = k$

If it is tangent to $b^2x^2 + a^2y^2 = a^2b^2$

So, $b^2x^2 + a^2(k - x)^2 = a^2b^2$ is their point of intersection.

or, $b^2x^2 + a^2(k^2 - 2kx + x^2) = a^2b^2$

or, $(b^2 + a^2)x^2 - 2ka^2x + a^2k^2 - a^2b^2 = 0$

If (i) is tangent to ellipse given then,

$B^2 - 4AC = 0$

or, $4k^2a^4 = 4(a^2 + b^2)(a^2k^2 - a^2b^2)$

or, $4k^2a^4 = 4k^2a^4 - 4a^4b^2 + 4a^2b^2k^2 - 4a^2b^4$

or, $4a^4b^2 + 4a^2b^4 = 4a^2b^2k^2$

or, $4a^2b^2(a^2 + b^2) = 4a^2b^2k^2$

$\therefore k^2 = a^2 + b^2$

or, $k = \pm\sqrt{a^2 + b^2}$

Thus, required equation of tangent is;

$x + y = \pm\sqrt{a^2+b^2}$

9. **Find the points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ such the tangents at each of them makes equal angles with the axes. Show that the length of the perpendicular from the centre on either of these tangents is $\sqrt{\frac{a^2+b^2}{2}}$.**

Solⁿ. Any point on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $(a\cos\theta, b\sin\theta)$. the equation of line at this point makes the angle equal to both axis so its slope is 1 or -1.

The equation of tangents at $(a\cos\theta, b\sin\theta)$ to

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is; $\frac{x\,a\cos\theta}{a^2}+\frac{y\,b\sin\theta}{b^2}=1$

or, $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

It's slope $=-\frac{\text{coeff. } x}{\text{coeff. } y}=-\frac{\cos\theta}{a}\times\frac{b}{\sin\theta}=\frac{b}{a}\times(-\cot\theta)=\frac{a}{b}$

$-1 = \frac{b}{a}\times\cot\theta$

$\tan\theta = \frac{b}{a}$

or, $\tan\theta=\frac{b}{a}$

or, $\frac{P}{b}=\frac{b}{a}$ So, $h=\sqrt{a^2+b^2}$

$\frac{mx+gy+c}{\sqrt{m^2+g^2}}$

So, $h=\sqrt{a^2+b^2}$

Thus,

$\sin\theta=\frac{b}{\sqrt{a^2+b^2}}$

$\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$

Thus, coordinate is,

$(a\cos\theta, b\sin\theta)=\frac{a^2}{\sqrt{a^2+b^2}},\frac{b^2}{\sqrt{a^2+b^2}}$

We have to prove that the length of perpendicular from centre $(0, 0)$ upon $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1=0$ is equal to the distance from $(0, 0)$ upon,

$\frac{x}{a}\times\frac{a}{\sqrt{a^2+b^2}}+\frac{y}{b}\frac{b}{\sqrt{a^2+b^2}}-1=0$

i.e., $\frac{x}{\sqrt{a^2+b^2}}+\frac{y}{\sqrt{a^2+b^2}}-1=0$

So, $\frac{1}{\sqrt{\frac{1}{a^2+b^2}+\frac{1}{a^2+b^2}}}=\frac{1}{\sqrt{\frac{2}{a^2+b^2}}}=\sqrt{\frac{a^2+b^2}{2}}$

10. **If the normal at the end of a latus rectum of an ellipse passes through one extremity of the minor axis, then show that the eccentricity of the curve is given by the equation $e^4+e^2-1=0$.**

Sol. Let the equation of ellipse is,

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Suppose; LSL' be a latus –rectum with respectively $\left(ae, \frac{b^2}{a}\right)$ and $\left(ae, -\frac{b^2}{a}\right)$.

Firstly, we take the normal at L $\left(ae, \frac{b^2}{a}\right)$

So, $\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}$

or, $\frac{x-ae}{\frac{ae}{a^2}}=\frac{y-\frac{b^2}{a}}{\frac{b^2}{\frac{a}{b^2}}}$

or, $\frac{a}{e}=a\left(y-\frac{b^2}{a^2}\right)$

or, $\frac{a}{e}(x-ae)=ay-b^2$ ..... (i)

but if this normal passess through the minor axis $(0, -b)$ of ellipse, Then, equation (i) will be,

$\frac{a}{e}(x-ae)=ay-b^2$

or, $-a^2=-(ab+b^2)$

or, $a^2=ab+b^2$

or, $a^2=ab+a^2-a^2e^2$

or, $a^2e^2=ab$

Squaring both sides, we get,

$a^4e^4 = a^2b^2$

or, $a^4e^4 = a^4(1-e^2)$

or, $e^4 = 1 - e^2$

or, $e^4 + e^2 - 1 = 0$

11. Prove that locus of the foot of the perpendicular drawn from the centre on any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$.

Sol^n. Let the given equation of ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{..... (i)}$$

Any tangent to equation (i) is,

$$y = mx \pm \sqrt{a^2m^2+b^2} \quad \text{..... (ii)}$$

Whose slope is m. Hence, slope of normal is $-\frac{1}{m}$. So any line from centre (0, 0) and perpendicular to equation (ii) is,

$$y - 0 = -\frac{1}{m}(x-0)$$

$$\text{or, } y = -\frac{x}{m} \quad \text{..... (iii)}$$

Let (h, k) be the foot of the perpendicular drawn from centre to equation (ii) so, (h, k)
is satisfied by equation (ii) and (iii)

$$K = mh + \sqrt{a^2m^2+b^2} \quad \text{..... (iv)}$$

$$K = -\frac{h}{m} \quad \text{..... (v)}$$

To find the locus of (h, k), we have, to eliminate the variable m between equation (iv) and (v)

Using $m = -\frac{h}{k}$ from equation (v) on (iv) then,

$$k = -\left(\frac{h}{k}\right)k + \sqrt{a^2\left(-\frac{h}{k}\right)^2+b^2}$$

$$\text{or, } k^2 = -h^2 + \sqrt{a^2h^2+b^2k^2}$$

Locus of (h, k) is,

$(x^2+y^2)^2 = a^2x^2 + b^2y^2$

Changing into polar form;

$x = r\cos\theta;\ y = r\sin\theta$

So, $x^2 + y^2 = r^2$

$\therefore (r^2)^2 = a^2r^2\cos^2\theta + b^2r^2\sin^2\theta$

or, $r^4 = r^2(a^2\cos^2\theta + b^2\sin^2\theta)$

$\therefore r^2 = a^2\cos^2\theta + b^2\sin^2\theta$ is the required locus.

12. Find the locus of the foot of the perpendicular drawn upon any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from either focus.

Sol^n. The equation of ellipse is,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{..... (i)}$$

Any tangent to equation (i) is,

$$y = mx + \sqrt{a^2m^2+b^2} \quad \text{..... (ii)}$$

Where slope is m. Hence slope of normal is $\frac{-1}{m}$.

Hence, any line through focus (ae, 0) is,

$$y - 0 = \frac{-1}{m}(x - ae)$$

$$\text{or, } y = \frac{-1}{m}(x-ae) \quad \text{..... (iii)}$$

Let (h, K) be the foot of the perpendicular drown from locus to the tangent (ii)

So, (h, k) satisfies equation (ii) and (iii)

$$k = mh + \sqrt{a^2m^2+b^2} \quad \text{..... (iv)}$$

$$K = -\frac{h}{m} + \frac{ae}{m} \quad \text{..... (v)}$$

To find the locus of (h, K), we have, eliminate the variable m between equation (iv) and (v)

$$\text{So, } K = \frac{-h+ae}{K}$$

$$\text{or, } K = -\left(\frac{h+ae}{m}\right)h + \sqrt{a^2\left(\frac{h+ae}{m}\right)^2+b^2}$$

$$\text{or, } k^2 = -(h+ae)h + \sqrt{a^2(h-ae)^2+b^2k^2}$$

$$\text{or, } K^2 = -h^2 - aeh + \sqrt{a^2(h-ae)^2+b^2k^2}$$

$$\text{or, } (h^2+k^2+ach)^2 = a^2(h-ae)^2 + b^2k^2$$

$$\text{or, } (h^2+k^2)^2 + 2(h^2+k^2)ach + a^2e^2h^2 = a^2(h^2 - 2aeh + a^2e^2 + b^2k^2)$$

Thus, solving gives,

$$(h^2+k^2)^2 = a^4$$

So, $h^2 + k^2 = a^2$

Thus, locus is $x^2 + y^2 = a^2$

13. If Q be the point on the auxiliary circle corresponding to a point P on an ellipse, then show that the normals at P and Q meet on a fixed circle $x^2 + y^2 = (a + b)^2$.

Sol^n. Let Q be any point Q ($a\cos\theta$, $a\sin\theta$) on auxiliary circle and p ($a\cos\theta$, $b\sin\theta$) be the point on ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The equation of normal at p ($a\cos\theta$, $a\sin\theta$) is y has slope;

$$\frac{a\sin\theta}{a\cos\theta - 0} = \tan\theta$$

So, its equation is,

$y - a\sin\theta = \tan\theta\,(x - a\cos\theta)$

or, $y - a\sin\theta = x\tan\theta - a\sin\theta$

So, $y = x\tan\theta$ ..... (i)

Similarly the equation of normal at ($a\cos\theta$, $a\sin\theta$) is;

$$\frac{a^2(x - a\cos\theta)}{a\cos\theta} = \frac{b^2(y - b\sin\theta}{b\sin\theta}$$

$$\text{or, } \frac{ax - a^2\cos\theta}{\cos\theta} = \frac{by - b^2\sin\theta}{\sin\theta}$$

or, $ax\sec\theta - a^2 = by\,\mathrm{cosec}\theta - b^2$

or, $ax\sec\theta - a^2 = by\,\mathrm{cosec}\theta = b^2 - a^2$ ..... (ii)

Solving (i) and (ii) to eliminate $\sec\theta$ and $\mathrm{cosec}\theta$, we get,

$x^2 + y^2 = (a + b)^2$

14. Find the locus of a point from which two tangents can be drawn to an ellipse making angles θ_1 and θ_2 with the major axis such that

(i) $\tan\theta_1 - \tan\theta_2 = d$ (constant)

(ii) $\cot\theta_1 - \cot\theta_2 = 2d$ (constant)

Sol^n. Let the equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ ..... (i)}$$

Any tangent to ellipse (i) is,

$$y = mx + \sqrt{a^2m^2 + b^2} \text{ ..... (ii)}$$

Let (h, k) be a point form which two tangents are drawn. Then (h, k) lies on equation (ii)

$$k = mh + \sqrt{a^2m^2 + b^2}$$

Squaring both sides, we get,

$(k - mh)^2 = a^2m^2 + b^2$

or, $k^2 - 2mkh + m^2h^2 = a^2m^2 + b^2$

or, $m^2(h^2 - a^2) - 2kmh + k^2 - b^2 = 0$ ..... (iii)

This is quadratic in m, so it gives two values of m say m_1 and m_2 corresponding to two slopes m_1, m_2

We get to tangents through p (h, k)

Let θ_1 and θ_2 be the angles of inclination of tangents with x –axis

$\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$

Then by equation (iii) we have,

$$m_1 + m_2 = \frac{2hk}{h^2 - a^2}$$

$$\text{or, } \tan\theta_1 + \tan\theta_2 = \frac{2hk}{h^2 - a^2} \text{ ..... (iv)}$$

$$\text{Product of } m_1 \cdot m_2 = \frac{k^2 - b^2}{h^2 - a^2}$$

$$\text{or, } \tan\theta_1 \cdot \tan\theta_2 = \frac{k^2 - b^2}{h^2 - a^2} \text{ ..... (v)}$$

or, $(\tan\theta_1 - \tan\theta_2) = (\tan\theta_1 + \tan\theta_2)^2 - 4\tan\theta_1\tan\theta_2$

or, $d^2(h^2 - a^2)^2 = 4h^2k^2 - 4(k^2 - h^2)(h^2 - a^2)$

or, $d^2(x^2 - a^2)^2 = 4x^2y^2 - 4(y^2 - x^2)(x^2 - a^2)$

Which is the required locus.

Also,

$\cot\theta_1 \cdot \cot\theta_2 = 2d$

or, $\tan\theta_2 - \tan\theta_1 = 2d\tan^2\theta_1 \cdot \tan^2\theta_2$

or, $(\tan\theta_1 - \tan\theta_2)^2 = 4d^2 + \tan^2\theta_1 \cdot \tan^2\theta_2$

or, $(\tan\theta_1 + \tan\theta_2)^2 - 4\tan\theta_1\tan\theta_2 = 4d^2\tan^2\theta_1\tan^2\theta_2$

$$\text{or, } \frac{4h^2k^2}{(h^2 - a^2)^2} - \frac{4(k^2 - b^2)}{h^2 - a^2} = 4d^2\left(\frac{k^2 - b^2}{h^2 - a^2}\right)^2$$

or, $4h^2k^2 - 4(k^2 - b^2)(h^2 - a^2) = 4d^2(k^2 - b^2)^2$

So the required locus is,

$4x^2y^2 - 4(y^2 - b^2)(x^2 - a^2) = 4d^2(y^2 - b^2)^2$

The Hyperbola

21

Exercise - 35

1. **Find the centre, eccentricity, vertices, foci, directrices, length of axes and latus rectum of the hyperbola.**

(i) $4x^2 - 9y^2 = 36$

Solⁿ. The given equation of hyperbola is,

$4x^2 - 9y^2 = 36$ which can be written as

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

Thus, centre $(h, k) = (0, 0)$

Since $b^2 = a^2(e^2 - 1)$

So, $4 = 9(e^2 - 1)$

or, $\frac{4}{9} = e^2 - 1$

or, $e^2 = 1 + \frac{4}{9} = \frac{13}{9}$

$\therefore e = \pm \frac{\sqrt{13}}{3}$

Vertices $= (\pm a, 0) = (\pm 3, 10)$

Foci $= (\pm ae, 0) = \left(\pm 3 \times \frac{\sqrt{3}}{3}, 0\right) = (\pm\sqrt{3}, 0)$

Directrix is, $x = \pm \frac{a}{e}$

or, $x = \pm \frac{3}{\frac{\sqrt{13}}{3}}$

or, $x = \pm \frac{9}{\sqrt{13}}$

Length of axes, $2a = 2 \times 3 = 6$

$2b = 2 \times 2 = 4$

The length of latus -rectum $= \frac{2b^2}{a} = \frac{2 \times 4}{3} = \frac{8}{3}$

(ii) $16x^2 - 9y^2 = -144$

Solⁿ. The given equation of ellipse is $16x^2 - 9y^2 = -144$ which is of the form $\frac{x^2}{9} - \frac{y^2}{16} = -1$

Thus, centre $(0, 0)$

Ecentricity $a^2 = b^2(e^2 - 1)$

or, $9 = 16(e^2 - 1)$

or, $\frac{9}{16} + 1 = e^2$

or, $e = \frac{5}{4}$

Foci $= (0, \pm be) = \left(0, \pm 4 \times \frac{5}{4}\right) = (0, \pm 5)$

Directrix $y = \pm \frac{b}{e}$

or, $y = \pm \frac{4}{\frac{5}{4}} = \pm \frac{16}{5}$

Length of axis $= 2b = 2 \times 4 = 8$

$= 2a = 2 \times 3 = 6$

$\therefore$ The length of latus -rectum $= \frac{2a^2}{b} = \frac{2 \times 3^2}{4} = \frac{9}{2}$

(iii) $16x^2 - 25y^2 = 400$

Solⁿ. The given equation of ellipse is

$16x^2 - 25y^2 = 400$ which can be written as

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

Thus, Centre $(h, k) = (0, 0)$

For eccentricity $b^2 = a^2(e^2 - 1)$

or, $16 = 25(e^2 - 1)$

or, $\frac{16}{35} = e^2 - 1$

or, $\frac{16}{25} + 1 = e^2$

or, $e = \frac{\sqrt{41}}{5}$

Foci $= (\pm a, 0) = (\pm 5, 0)$

Equation of directrix is;

$x = \pm \frac{a}{e}$

or $x = \pm \frac{5}{\sqrt{\frac{41}{5}}}$

or, $x = \pm \frac{25}{\sqrt{41}}$

$\therefore$ The length of axis; $2a = 2 \times 5 = 10$

$2b = 2 \times 4 = 8$

The length of latus -rection $\frac{2b^2}{a} = \frac{2 \times 16}{5} = \frac{32}{5}$

(iv) **$9x^2 - 16y^2 + 72x - 32y - 16 = 0$**

Sol^n. The given equation of hyperbola is

$9x^2 - 16y^2 + 72x - 32y - 16 = 0$

or, $9(x^2 + 8x) - 16(y^2 + 2y) = 16$

or, $9(x^2 + 2 . x . 4 + 4^2 - 16) - 16(y^2 + 2y + 1 - 1) = 16$

or, $9(x + 4)^2 - 16(y + 1)^2 = 144$

or, $\frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$

Thus, centre $(h, k) = (-4, 1)$

For eccentricity

$b^2 = a^2(e^2 - 1)$

or, $9 = 16(e^2 - 1)$

or, $\frac{9}{16} + 1 = e^2$

or, $e = \frac{5}{4}$

the Foci $= (h \pm ae, k)$

$= (-4 \pm 4 \times \frac{5}{4}, -1)$

$= (-4 \pm 5, -1)$

$(9, -1)$ and $(1, -1)$

Vertices $= (h + a, k) = (-4 \neq 4, -1)$

$(0, -1)$ and $(-8, -1)$

Length of axis;

$2a = 2 \times 4 = 8$

$2b = 2 \times 3 = 6$

The equation of directrix;

$x = h \pm \frac{a}{e}$

or, $x = -4 \pm \frac{4 \times 4}{5}$

or, $x = -4 \pm \frac{16}{5}$

$\therefore x = -\frac{4}{5}$ and $x = \frac{36}{5}$

2. **Find the equation to the hyperbola referred to its axes as axes of coordinates.**

(i) **with distances between the foci is 9, eccentricity is $\sqrt{3}$**

Sol^n. The distance between foci is 9 that is $(\pm ae, 0)$ and $(-ae, 0)$ have distane 9

So, $2ae = 9$

Also, $e = \sqrt{3}$ is given;

So $a = \frac{9}{2\sqrt{3}} = \frac{3\sqrt{3}}{2}$

By using $b^2 = a^2(e^2 - 1)$

or, $b^2 = \frac{27}{4}(3 - 1)$

or, $b^2 = \frac{27}{2}$

Thus, required equation of hyperbola is,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or, $\frac{x^2}{\frac{27}{4}} - \frac{y^2}{\frac{27}{2}} = 1$

or, $4x^2 - 2y^2 = 27$ is required equation of hyperbola.

(ii) **with distances between the foci is 16, eccentricity is $\sqrt{2}$**

Sol^n. By the question,

The distance between foci,

i.e. $2ae = 16$

Also, $e = \sqrt{2}$ is given,

Thus, $a = \frac{16}{2e} = \frac{8}{e} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$

The know that,

$b^2 = a^2(e^2 - 1)$

So, $b^2 = 3^2(2 - 1)$

or, $b^2 = 32$

thus, $a^2 = 32$ and $b^2 = 32$, so the required equation of hyperbola is,

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

or, $\frac{x^2}{32} - \frac{y^2}{32} = 1$ So, $x^2 - y^2 = 32$ is required equation.

(iii) **With length of transverse and conjugate axes are 3 and 4 respectively.**

Sol^n. The length of transverse axis $2a = 3$
and conjugate axis $2b = 4$

So, $a = \frac{3}{2}$ and $b = 2$

Thus, required equation of hyperbola is,

$$\frac{x^2}{\frac{9}{4}} - \frac{y^2}{4} = 1 \quad \text{or, } \frac{4x^2}{9} - \frac{y^2}{4} = 1$$

i.e., $16x^2 - 9y^2 = 36$ is required equation.

(iv) **with length of conjugate axis is 10, passes through the point (2, – 1).**

Sol^n. The standard equation of hyperbola is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{.....(i)}$$

By question; the length of conjugate axis,
$2b = 10$
$\therefore$ b 5 also (i) passess through (2, – 1) thus,

$$\frac{2^2}{a^2} - \frac{(-1)^2}{b^2} = 1 \quad \text{or, } \frac{4}{a^2} - \frac{1}{b^2} = 1$$

Since, $b = 5$ given,

$$\frac{4}{a^2} = 1 + \frac{1}{25} = \frac{26}{25}$$

or, $a^2 = \frac{50}{13}$ Thus form equation (i)

$$\frac{x^2}{\frac{50}{13}} - \frac{y^2}{25} = 1 \quad \text{or, } \frac{13x^2}{50} - \frac{y^2}{25} = 1$$

or, $13x^2 - 2y^2 = 50$ is required of hyperbola under this the given condition.

(v) **with length of conjugate axis 7 and passes through the (3, – 2).**

Sol^n. The equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{.....(i)}$$

By question the length of conjugate axis $2b = 7$ is given So,

$$b = \frac{7}{2}$$

We are given that (i) passess through (3, – 2) So,

$$\frac{9}{a^2} - \frac{4}{b^2} = 1$$

$$\text{or, } \frac{9}{a^2} - \frac{4}{\frac{49}{4}} = 1 \quad \text{or, } \frac{9}{a^2} - \frac{16}{49} = 1 \quad \text{or, } \frac{9}{a^2} = \frac{16}{49} + 1 = \frac{65}{49}$$

$$\text{or, } a^2 = \frac{49 \times 9}{65} = \frac{441}{65}$$

The equation of hyperbola is,

$$\frac{x^2}{\frac{441}{65}} - \frac{y^2}{\frac{59}{4}} = 1$$

or, $65x^2 - 36y^2 = 441$ is the required equation of hyperbola.

3. **Find the equation of the hyperbola whose directrix is $2x + y = 1$ and focus (1, 2) and eccentricity $\sqrt{3}$.**

Sol^n. The equation of the hyperbola is to be determined.
Focus is (1, 2) and $2x + y - 1 = 0$ is directrix.
Thus, by the definition of hyperbola,

$$e = \frac{SP}{pm}$$

Where, p (x, y) be any point of hyperbola,

$$\text{So, } \sqrt{3} = \frac{\sqrt{(x-1)^2 + (y-2)^2}}{\left(\frac{2x+y-1}{\sqrt{5}}\right)}$$

Squaring both sides, we get

$$3 = \frac{(x-1)^2 + (y-2)^2 \times 5}{(2x+y-1)^2}$$

or, $3[4x^2 + 4xy + y^2 - 4x - 2y + 1] = (x^2 - 2x + 1 + y^2 - 4y + 4)5$
or, $12x^2 + 12xy + 3y^2 - 12x - 6y + 3 = 5x^2 + 10x + 5 + 5y^2 - 20y + 20$
i.e. $7x^2 - 2y^2 + 12xy + 14y - 22 = 0$ is the required equation of hyperbola.

4. **Find the equation of the hyperbola with vertices ($\pm$ 5, 0) and foci and ($\pm$ 7, 0).**

Sol^n. The vertex of hyperbola is,
$(\pm a, 0) = (\pm 7, 0)$
So, $a = 5$
Foci $= (\pm ae, 0) = (\pm 7, 0)$
So, $ae = 7$
We know that, $b^2 = a^2(e^2 - 1)$

$$\text{or, } b^2 = 25\left(\frac{49}{25} - 1\right) = 24 \quad \text{or, } b^2 = 29$$

Thus, required equation of hyperbola is,

$$\frac{x^2}{15}-\frac{y^2}{24}=1$$

i.e. $24x^2-25y^2=600$

Hence, required equation is $24x^2-25y^2=600$

5. **Prove that the locus of a point which moves in such way that the difference of its distances from the points (5, 0) and (– 5, 0) is 2 is as hyperbola.**

Solⁿ. Let p (x, y) be the locus of point

By equation, the difference of its distance from A (5, 0) and B (–5, 0) is 2.

i.e. PA – PB = 2

So, $\sqrt{(x-5)^2+(y-0)^2}-\sqrt{(x+5)^2+(y-0)^2}=2$

Squaring gives,

$$\sqrt{x^2+10x+25+y^2}-\sqrt{x^2+10x+25+y^2}=2$$

i.e., $\sqrt{x^2-10x+25+y^2}=2+\sqrt{x^2+10x+25+y^2}$

Squaring both sides, we get,

$$x^2-10x+25+y^2=x^2+10x+25+y^2+4+2\sqrt{x^2+10x+25+y^2}\times 2$$

or, $-20x-4=4\sqrt{x^2+10x+25+y^2}$

or, $-5x-1=\sqrt{x^2+10x+25+y^2}$

Again, squaring both sides, we get,

$25x^2+10x+1=x^2+10x+y^2+25$

or, $24x^2-y^2=24$

or, $\frac{x^2}{1}-\frac{y^2}{24}=1$ which is required equation of hyporbola.

6. **Prove that the lines $\frac{x}{a}+\frac{y}{b}$ K and $\frac{x}{a}-\frac{y}{b}=\frac{1}{k}$ where k is variable parameter always intersect on a hyperbola.**

Solⁿ. The given lines are,

$\frac{x}{a}+\frac{y}{b}=k$ and $\frac{x}{a}-\frac{y}{b}=\frac{1}{k}$

To find their point of intersection,

$\frac{bx+ay}{ab}=k$ ..... (i)

or, $\frac{bx-ay}{ab}=\frac{1}{k}$

i.e. $k=\frac{ab}{bx-ay}$ ..... (ii)

So, from (i) and (ii)

$\frac{bx+ay}{ab}=\frac{ab}{bx-ay}$

or, $b^2x^2-a^2y^2=a^2b^2$ or, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Since, their point of intersection is,

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is hyperbola.

Show that the eccentricity of a hyperbola with transverse axis is 2a and aexs are the axes of coordinates which passes through the point (h, k) is given by $\left(\frac{h^2+k^2-a^2}{h^2-a^2}\right)^{\frac{1}{2}}$.

Solⁿ. The equation of hyperbola in its standard form is,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Since it passess through (h, k), So,

$$\frac{h^2}{a^2}-\frac{k^2}{b^2}=1$$

or, $h^2b^2-a^2k^2=a^2b^2$

or, $h^2b^2-a^2b^2=a^2k^2$

or, $k^2=\frac{b^2(h^2-a^2)}{a^2}$ ..... (i)

We know that $b^2=a^2(e^2-1)$ ..... (ii)

If transverse axis is 2a

So, from equation (i), we have,

$\frac{a^2k^2}{h^2-a^2}=b^2$ (using in equation (ii)

or, $b^2=a^2(e^2-1)$ or, $\frac{a^2k^2}{h^2-a^2}=a^2(e^2-1)$

or, $k^2=(e^2-1)(h^2-a^2)$ or, $k^2=e^2(h^2-a^2)-h^2+a^2$

or, $k^2+h^2-a^2=e^2(h^2-a^2)$

or, $e^2=\frac{k^2+h^2-a^2}{h^2-a^2}$

Thus, $e=\sqrt{\frac{k^2+h^2-a^2}{h^2-a^2}}$

Exercise - 36

1. **Find the equation of tangent, and normal at the point $(5\sqrt{2}, 4)$ to the hyperbola $16x^2 - 25y^2 = 400$.**

Sol^n. The equation of hyperbola is,

$16x^2 - 25y^2 = 400$

i.e. $\frac{x^2}{25} - \frac{y^2}{16} = 1$

We have to find the equation of tangent at $(5\sqrt{2}, 4)$ so, by formula

$\frac{x \cdot 5\sqrt{2}}{25} - \frac{y \cdot 4}{16} = 1$

or, $\frac{x\sqrt{2}}{5} - \frac{y}{4} = 1$

or, $4\sqrt{2}x - 5y = 20$ is required equation of tangent.

The equation of normal is perpendicular to tangent, so, it is perpendicular to tangent, so is,

$-5x - 4\sqrt{2y} + k = 0$ but it passess through $(5\sqrt{2}, 4)$ So,

$-25\sqrt{2} - 16\sqrt{2} + k = 0$

or, $k = 41\sqrt{2}$

Hence, equation of normal is $5x + 4\sqrt{2}y = 41\sqrt{2}$ is the required equation of normal.

2. **Find the equation of the tangent and normal to the hyperbola $2x^2 - 3y^2 = 6$ at the end of its latus rectum in the first quadrant.**

Sol^n. The equation of hyperbola is

$2x^2 - 3y^2 = 6$ which can be written as

$\frac{x^2}{3} - \frac{y^2}{2} = 1$

We know that the end of latus rectum in first quadrant is $\left(ae, \frac{b^2}{a}\right)$

Here, $a = \sqrt{3}$, $b = \sqrt{2}$

So, $b^2 = a^2(e^2 - 1)$

or, $2 = 3(e^2 - 1)$

or, $\frac{2}{3} + 1 = e^2$

or, $e^2 = \frac{5}{3}$

$e = \sqrt{\frac{5}{3}}$

Hence, coordinate is $\left(\sqrt{3} \times \frac{\sqrt{5}}{3}, \frac{2}{\sqrt{3}}\right) = \left(\sqrt{5}, \frac{2}{\sqrt{3}}\right)$

Hence, the equation of tangent is,

$\frac{x\sqrt{5}}{3} - \frac{y \cdot \frac{2}{\sqrt{3}}}{2} = 1$

or, $\frac{x\sqrt{5}}{3} - \frac{y}{\sqrt{3}} = 1$

or, $x\sqrt{5} - 3y = 3\sqrt{3}$

or, $x\sqrt{5} - \sqrt{3}y = 3$ is the required equation of tangent. To find the equation of normal which is always perpendicular to tangent is,

$-\sqrt{3}x - \sqrt{5}y + k = 0$ which passess through $\left(\sqrt{5}, \frac{2}{\sqrt{3}}\right)$, So

or, $-\sqrt{15} - \sqrt{5} \times \frac{2}{\sqrt{3}} + k = 0$

or, $\frac{-\sqrt{45} - \sqrt{20}}{\sqrt{3}} = -k$

or, $k = \frac{\sqrt{65}}{\sqrt{3}}$

Thus, $-\sqrt{3}x - \sqrt{5}y + \frac{\sqrt{65}}{\sqrt{3}} = 0$

or, $-2x - \sqrt{15}y = -\sqrt{65}$

$3x + \sqrt{15}y = \sqrt{65}$ is required equation of normal.

Prove that the straight line $21x + 5y = 116$ is a tangent to the hyperbola $7x^2 - 5y^2 = 232$ and find its point of contact.

Sol^n. The equation of strainght lines given is $21x + 5y = 116$ and given equation of hyperbola is $7x^2 - 5y^2 = 232$.

Solving these equations yields,

$7x^2 - 5\left(\frac{116 - 21x}{5}\right)^2 = 232$

or, $7x^2 - \left(\frac{13246 - 7872x + 441x^2}{5}\right) = 232$

or, $35x^2 - 13456 + 4872x - 441x^2 = 1160$

or, $-406x^2 + 4872x - 14616 = 0$

or, $406x^2 - 4x72x + 14616 = 0$

or, $406x(x-6) - 2436(x-6) = 0$

or, $(406x - 2436)(x-6) = 0$

or, $x = 6$ Then $21 \times 6 + 5y = 16$

or, $x = 5y = -10$

$y = -2$

The point of contant is only one so, it is tangent and point of contant is $(6, -2)$

4. **Find the equation of tangent to the hyperbola $3x^2 - 4y^2 = 1$ which is parallel to the line $2x - y + 4 = 0$**

Sol^n. The equation of line parallel to $2x - y + 4 = 0$ is $2x - y + k = 0$ if $y = 2x +_ k$. Is taxgent to the line $3x^2 - 4y^2 = 1$

So, their point of intersection is obtained.

$3x^2 - 4(2x + k)^2 = 1$

or, $3x^2 - 4[4x^2 + 4xk + k^2] = 1$

or, $3x^2 - 16x^2 + 16xk - 4k^2 - 1 = 0$

or, $-13x^2 - 16xk - 4k^2 - 1 = 0$

or, $13x^2 + 16xk + 4k^2 + 1 = 0$

Since for tangent $B - 4AC = 0$

$(16k)^2 - 4 \times 13 \times (4k^2 + 1) = 0$

or, $256k^2 - 208k^2 - 52 = 0$

or, $48k^2 - 52 = 0$

or, $k^2 = \frac{52}{48} = \frac{13}{12}$

or, $K^2 = \pm\sqrt{\frac{13}{12}} = \pm\frac{\sqrt{13}}{\sqrt{12}} = \frac{\sqrt{13}}{2\sqrt{3}}$

Thus, the equation of tangent is,

$2x - y \pm \frac{\sqrt{13}}{2\sqrt{3}} = 0$

or, $4\sqrt{3}x - 2\sqrt{3}y \pm \sqrt{13} = 0$

5. **Find the equation of tangent to the hyperbola $3x^2 - 4y^2 = 1$ and perpendicular to the line $2x = 3y = 5$.**

Sol^n. The equation of the line which is perpendicular to line $2x + 3y - 5 = 0$ is $3x + 2y + k = 0$

If it is tangent to the hperbola.

$3x^2 - 4\left(\frac{3x + k}{2}\right)^2 - 1 = 0$

or, $3x^2 - 4\left(\frac{9x^2 + 6xk + k^2}{4}\right) - 1 = 0$

or, $3x^2 - 9x^2 - 6xk - k^2 - 1 = 0$

or, $-6x^2 - 6xk - k^2 - 1 = 0$

or, $6x^2 + 6xk + k^2 + 1 = 0$

Since, $B^2 - 4AC$ must be zero.

$(6k)^2 - 4 \times 6(k^2 + 1) = 0$

$36k^2 - 24k^2 - 24 = 0$

or, $12k^2 = 24$

or, $k^2 = 2$

$k = \pm\sqrt{2}$

Thus, the equation of tangents is $3x - 2y \pm \sqrt{2} = 0$

Find the point of intersection of the tangents at t_1 and t_2 on the hyperbola $xy = c^2$.

Sol^n. Any point t_1 is $\left(ct_1, \frac{c}{t_1}\right)$ and t_2 is $\left(ct_2, \frac{c}{t_2}\right)$

The equation of tangent at $\left(ct_1, \frac{c}{t_1}\right)$ to $xy = c^2$ is,

$\frac{x}{ct_1} + \frac{t_1 y}{C} = 2$

or, $x + y t_1^2 = 2ct_1$ ..... (i)

Similarly, the equation of tangent at $\left(ct_2, \frac{c}{t_2}\right)$ is,

$x + y t_2^2 = 2C t_2$ ..... (ii)

Solving (i) and (ii)

$2Ct_1 - y t_1^2 + y t_2^2 = 2c t_2$

or, $2Ct_1 - yt_1^2 - yt_2^2$

or, $2(t_1 - t_2) = y(t_1 - t_2)(t_1 + t_2)$

$\therefore y = \left(\frac{2c}{t_1 + t_2}\right)$

Also, from (i)

$x = 2ct_2 - yt_2^2$

$= 2ct_2 - t_2^2 \times \frac{2c}{t_1 + t_2}$

$= 2ct_2\left[1 - \frac{t_2}{t_1 + t_2}\right]$

$$= 2ct_2\left[\frac{t_1+t_2-t_2}{t_1+t_2}\right]$$

$$= \left[\frac{2ct_1t_2}{t_1+t_2}\right]$$

Hence, the point of intersection of tangent at t_1 and t_2 is,

$$\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$$

7. **Find the equation of a hyperbola whose asymptotes are $2x - y - 3 = 0$ and $3x + y - y = 0$ and passes through the point (1, 1).**

Sol^n. Here, The equations of asymptotes are,

$2x - y - 3 = 0$

and $3x + y - 7 = 0$

We know that, the combined equation of asymptotes and the equation of hyperbola is differ by constant on y. So, μ- it be λ.

So, the equation of hyperbola is,

$(2x - y - 3)(3x + y - 7) + \lambda = 0$

or, $(-2)(-3) + \lambda = 0$

or, $\lambda + 6 = 0$

$\therefore \lambda = -6$

Hence, required equation of hyperbola is,

$(3x + y - 7)(2x - y - 3) - 6 = 0$

or, $6x^2 - 3xy - 9x + 2xy - y^2 - 3y - 14x + 7y + 21 - 6 = 0$

or, $6x^2 - y^2 - xy - 23x + 4y + 15 = 0$ is the required equation of hyperbola.

8. **Prove that the normal at a point 't' of the rectangular hyperbola $xy = c^2$ meets the curve again at a point 't_1' such that $t^3t_1 = -1$.**

Sol^n. The equation of normal at t with the hyperbola $xy = c^2$ is,

By the question, it passess through $\left(ct_1, \frac{c}{t_1}\right)$ So,

$$Ct_1 \times t^2 - \frac{C}{t_1} = Ct^3 - \frac{C}{t}$$

$$\text{or, } t^2t_1 - \frac{1}{t_1} = t^3 - \frac{1}{t}$$

$$\text{or, } \frac{t^2t_1^2 - 1}{t_1} = \frac{t^4 - 1}{t}$$

or, $t^3t_1^2 - t = t^4t_1 - t_1$

or, $t^3t_1^2 - t^4t_1 = t - t_1$

or, $t^3 + t_1(t - t_1) = -(t - t_1)$

$\therefore t^3t_1 = -1$

General Equation of Conic Section

22

Exercise - 37

1. **Show that the conic $9x^2 - 24xy + 16y^2 - 50x - 100y + 225$ represents a parabola. Find the latus rectum, vertex, focus and directrix of the parabola.**

Sol^n. The given equation of conic is,

$9x^2 - 24x + 16y^2 - 50x - 100y + 225 = 0$

Comparing in with $ax^2 + 2hxy + by^2 + 29x + 2fy + c = 0$ yiels,

$b = 16, a = 9, h = -12, g = -25, f = -50, c = 225$

Here,

$abc + 2fgh - af^2 - bg^2 - ch^2$

$= 9 \times 16 \times 225 + 2 \times (-50) \times (-25) \times (-12) - 9 \times (-50)^2 - 16 \times (-25)^2 - 225 \times (-12)^2$

$= 32400 - 30000 - 9 \times 2500 - 16 \times 625 - 225 \times 144$

$= 2400 - 22500 - 10000 - 32400 \neq 0$

and $ab - h^2 = 9 \times 16 - (-12)^2 = 144 - 144 = 0$

So, it is parabola

Alos, $9x^2 - 24xy + 16y^2 - 50x - 100y + 225 = 0$ yields,

$(3x - 4y + \lambda)^2 - 50x - 100y + 225 - 6\lambda x + 8\lambda y + \lambda^2 = 0$

or, $(3x - 4y + \lambda)^2 - x(50 + 6\lambda) - y(100 - 8\lambda) - \lambda^2 + 225 = 0$

or, $(3x - 4y + \lambda)^2 = x(50 + 6\lambda) + y(100 - 8\lambda) + \lambda^2 - 225$

Let us choose λ in such 9 way that $3x - 4y + \lambda = 0$ and $x(50 + 6\lambda) + y(100 - 8\lambda) + \lambda^2 - 225 = 0$ are perpendicular to each other.

$$\text{So, } \frac{3}{4} \times \frac{-50 - 6\lambda}{100 - 8\lambda} = -1$$

or, $-150 - 18\lambda = -400 + 32\lambda$

or, $250 = 50\lambda$

or, $\lambda = 5$

Hence,

$(3x - 4y + 5)^2 = 80x + 60y - 200$

or, $\left(\frac{3x-4y+5}{\sqrt{3^2+4^2}}\right)^2 \times 25 = \frac{80x+60y-200}{\sqrt{(80)^2+(60)^2}} \times 100$

or, $\left(\frac{3x-4y+5}{5}\right)^2 = \frac{100}{25}\left[\frac{18x+60y-200}{100}\right]$

Which is of the form $Y^2 = 4AX$

Length of latus rectum $= 4A = \frac{100}{25} = 4$

For veretex,

3	−4	5	3	−4
8	6	−2	8	6

$\therefore \frac{x}{80-30} = \frac{y}{40+80} = \frac{1}{18+32}$

or, $x = \frac{50}{50} = 1$ or, $y = \frac{100}{50} = 2$

Thus, the vertex is (1, 2)

Focus

We know that focus is the intersection of of equation of latus rectum and axis of parabola the first find the equation of latus-rectum, so

$X = A$

or, $\frac{80x+60y-200}{100} = 1$

or, $80x + 60y - 200 = 100$

or, $80x + 60y - 300 = 0$

or, $8x + 6y - 30 = 0$

or, $4x + 3y - 15 = 0$ ..... (i)

Also the equation of axis is,

$Y = 0$

or, $\frac{3x-4y+5}{5} =$

or, $3x - 4y + 5 = 0$ ..... (ii)

Solving (i) and (ii)

x	y	1		
4	3		4	3
3	−4	5	3	−4

$\therefore \frac{x}{-45} = \frac{y}{-65} = \frac{1}{-25}$

or, $x = \frac{-45}{-25} = \frac{9}{5}$

or, $y = \frac{-65}{-25} = \frac{13}{5}$

Hence, $(x, y) = \left(\frac{9}{5}, \frac{13}{5}\right)$ is focus

For directix

The equation of directrix is $X = -A$ for $Y^2 = 4AX$

So, $\frac{80x+60y-200}{100} = -1$

or, $80x + 60y - 200 = -100$

or, $80x + 60y - 100 = 0$

or, $8x + 6y - 10 = 0$

or, $4x + 3y - 5 = 0$ is the equation directrix.

2. Show that the conic $x^2 - 2xy + y^2 - 2x - 2y + 3 = 0$ represents a parabola. Find latus rectum, vertex, focus and directrix of the parabola.

Solⁿ. The given equation is,

$x^2 - 2xy + y^2 - 2x - 2y + 3 = 0$

Comparing it with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

gives, $a = 1, h = -1, b = 1, g = -1, f = -1, c = 3$

and $\Delta = abc + 2fgh - af^2 - by^2 - ch^2$

$= 1 \times 1 \times 3 + 2 \times (-1) \times (-1) \times (-1) - 1 \times (-1)^2 - 3 \times (-1)^2$

$= 3 - 2 - 1 - 1 - 3$

$= -4 \neq 0$

and $ab - h^2 = 1 \times 1 - (-1)^2 = 1 - 1 = 0$

Since $\Delta \neq 0$ and $ab - h^2 = 0$ so, given equation represents the equation of parabola.

Hence,

$x^2 - 2xy + y^2 - 2x - 2y + 3 = 0$

or, $(x-y)^2 - 2x - 2y + 3 = 0$

or, $(x - y + \lambda)^2 - 2x - 2y + 3 + 2\lambda y - 2\lambda x - \lambda^2 + 3 = 0$

or, $(x - y + \lambda)^2 - x(2\lambda + 2) - y(2 - 2\lambda) + 3 - \lambda^2 = 0$

We choose λ in such a way that,

$x - y + \lambda = 0$ and

or, $x(2\lambda + 2) + y(2 - 2\lambda) + \lambda^2 - 3 = 0$ becomes perpendicular to each other.

So, $1 \times \frac{-(2\lambda+2)}{2-2\lambda} = -1$

i.e. $2\lambda + 2 = 2 - 2\lambda$

Hence,

$(x-y)^2 = 2x + 2y - 3$

or, $\left(\frac{x-y}{\sqrt{2}}\right)^2 \times 2 = \frac{2x+2y-3}{\sqrt{4+4}} \times \sqrt{3}$

or, $\left(\frac{x-y}{\sqrt{2}}\right)^2 = \frac{\sqrt{8}}{2}\left[\frac{2x+2y-3}{\sqrt{8}}\right]$

Which is of the form $y^2 = 4AX$

Hence, length of latubs-rectum $= \frac{\sqrt{8}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

For vertex

x	y	1		
1	−1	0	1	−1
2	2	−3	2	2

or, $\frac{x}{3} = \frac{y}{3} = \frac{1}{4}$

Hence, $x = \frac{3}{4}$ and $y = \frac{3}{4}$

Hence, vertex $= \left(\frac{3}{4}, \frac{3}{4}\right)$

For Focus

We know that the focus is the point of intersection of equation of latus-rectum and axis of parabola. So, we find equation of directrix and the equation of axis and then solve themd. We know that $X = A$ is directrix.

$\frac{2x+2y-3}{\sqrt{8}} = \frac{\sqrt{8}}{8}$

or, $2x + 2y - 3 = 1$

or, $2x + 2y - 4 = 0$

or, $x + y = 2$

Also, the equation of axis is, $Y = 0$

or, $x - y = 0$

Thus, solving

$$\begin{array}{l} x - y = 0 \\ x + y = 2 \\ \hline 2x = 2 \end{array}$$

or, $x = 1$ and $y = 1$

Thus, Focus is $(1, 1)$

For directrix

The equation of directrix is $Y = -A$ for $Y^2 = 4AX$

So, $\frac{2x+2y-3}{\sqrt{8}} = \frac{-\sqrt{8}}{8}$

or, $2x + 2y - 3 = -1$

or, $2x + 2y - 2 = 0$

or, $x + y = 1$

Hence, The equation of directrix is $x + y = 1$

3. **Find the positions and lengths of the axes of the conic $x^2 - 4xy - 2y^2 + 10x + 4y = 0$**

Sol^n. The given conic is,

$x^2 - 4xy - 2y^2 - 10x + 4y = 0$

Comparing it with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

We get,

$a = 1, h = -2, b = -2, g = 5, f = 2, c = 0$

and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 2 \times 2 \times 5 \times (-2) - 1 \times (2)^2 \times 2(5)^2$

$= -40 - 4 + 50$

$= 6 \neq 0$

and $ab - h^2 = 1 \times (-2) - (-2)^2$ $h^2 - ab < 0$

$= -2 - 4 = -6 < 0$

Since $\Delta \neq = 0$

and $ab - h^2 = 1 \times (-2) - (-2)^2 - 2 - 4 = -6$

Hence, given equation is hyperbola.

For length of axis, we have the equations

$\frac{1}{v^4} - \frac{1}{v^2}(A + B) + (AB - H^2) = 0$

For this, we find the centre,

So, $\phi = x^2 - 4xy - 2y^2 + 10x + 4y$

or, $\frac{\delta\phi}{\delta x} = 2x - 4y + 10$

or, $\frac{\delta\phi}{\delta y} = -4x - 4y + 4$

For co-ordinates of centre

$\frac{\delta\phi}{\delta x} = 0$ and $\frac{\delta\phi}{\delta y} = 0$ is solved

or, $2x - 4y + 10 = 0$

or, $-4x - 4y + 4 = 0$

or, $x - 2y = -5$

$-x - y = -1$

or, $y = 3$

and $x = 2 \times 3 - 5 = 1$

Thus, centre is $(1, 3)$

Standard form referred to centre is,

$a\left(\frac{h^2 - ab}{\Delta}\right)x^2 + \frac{2h\left(h^2 - ab\right)}{\Delta}xy + \frac{b\left(b^2 - ab\right)}{\Delta}y^2 = 1$

or, $x^2 - 4xy - 2y^2 = 1$

or, $x^2 - 4xy - 2y^2 - 1 = 0$

or, Comparing it with,

$Ax^2 + 2Hxy + By^2 = 1$

$A = 1, H = -2, B = -2$

Thus, length of axis;

$\frac{1}{r^4} + \frac{1}{r^2} - 6 = 0$

or, $1 + r^2 - 6r^4 = 0$

or, $6r^4 - 3r^2 + 2r^2 - 1 = 0$

or, $3r^2(2r^2 - 1) + 1(2r^2 - 1) = 0$

or, $(3r^2 + 1)(2r^2 - 1) = 0$

or, $2r^2 = 1, r = \frac{1}{\sqrt{2}}$ and $r = -\frac{1}{\sqrt{3}}$

Hence, to find the equation of asymptotes we have that the equation of asymptotes dister by constant compared to hyperbola

So, asymptote is,

$$x^2 - 4xy - 2y^2 + 10x + 4y - \frac{\Delta}{ab - h^2} = 0$$

So, $x^2 - 4xy - 2y^2 + 10x + 4y + 1 = 0$

4. Find the product of semi-axis of the conic $x^2 - 4xy + 5y^2 = 2$

Sol^n. The equation of conic is,

$x^2 - 4xy + 5y^2 = 2$

Comparing it with $ax^2 + 2hxy + by^2 + 3gx + 2fy + c = 0$

We get,

$a = 1, h = -2, b = 5, g = 0, f = 0, c = -2$

Now,

$\Delta = abc + 2fgh - af^2 - by^2 - ch^2$

$= 1 \times 5(-2) + 2 \times 0 \times 0 \times (-2) - 1 \times (1)^2 - 5 \times 0^2 + 2(-2)^2$

$= -10 + 8$

$= -2 \neq 0$

and $ab - h^2 = 1 \times 5 - (-2)^2 = 5 - 4 = 1$

$\therefore h^2 - ab = -1$

Thus,

The equation of conic referred to its centre is,

$$a\frac{(h^2 - ab)}{\Delta}X^2 + 2h\left(\frac{h^2 - ab}{\Delta}\right) \times X + b\left(\frac{h^2 - ab}{\Delta}\right)Y^2 = 1$$

or, $a_1\left(\frac{-1}{-2}\right)X^2 - 4\left(\frac{-1}{-2}\right)XY + 5\left(\frac{-1}{-2}\right)Y^2 = 1$

or, $\frac{X^2}{2} - 2XY + \frac{5}{2}Y^2 = 1$

Comparing it with $AX^2 + 2HXY + BY^2 = 1$

We get,

$A = \frac{1}{2}, H = -1, B = \frac{5}{2}$

So, $\frac{1}{r^4} - \frac{1}{r^2}(A + B) + (AB - H^2) = 0$

or, $\frac{1}{r^4} - \left(\frac{1}{2} + \frac{5}{2}\right)\frac{1}{r^2} + \left(\frac{5}{4} - 1\right) = 0$

or, $\frac{1}{r^4} - \frac{3}{r^2} + \frac{1}{4} = 0$

or, $r^4 - 12r^2 + 4 = 0$

It is quadratic in r^2

So, it has two values.

So, the product of r_1^2 and $r_2^2 = \frac{c}{a}$ in quadratic equation.

$r_1^2 r_2^2 = 4$

or, $r_1 \times r_2 = 2$

The product of axes = 2.

5. Find the centre, lengths of the axes and eccentricity of the conic.

(i) $9x^2 + 4xy + 6y^2 - 22x - 16y + 9 = 0$

Sol^n. The equation of conic is,

$9x^2 + 4xy + 6y^2 - 22x - 16y + 9 = 0$

Comparing it with $ax^2 + 2hxy + by^2 + 29x + 2fy + c = 0$

We get,

$a = 9, h = 2, b = 6, g = -11, f = -8, c = 9$

Now,

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 9 \times 6 \times 9 + 2(-9)(-11) \times 2 - 9(-8)^2 - 6 \times (-11)^2 - 9(2^2)$

$= 486 + 352 - 576 - 726 - 36$

$= -550 \neq 0$

and $ab - h^2 = 9 \times 6 - (2)^2$

$= 54 - 4$

$= 50 > 0$

So, it represents ellipse.

For centre

Let $\phi = 9x^2 + 4xy + 6y^2 - 22x - 16y + 9$

$\therefore \frac{\partial\phi}{\partial x} = 18x + 4y - 22$

$\frac{\partial\phi}{\partial y} = 4x + 12y - 16$

For centre we solve $\frac{\partial\phi}{\partial x} = 0$ and $\frac{\partial\phi}{\partial y} = 0$

So, $9x + 2y - 11 = 0$

or, $2x + 6y - 8 = 0$

or, $\frac{x}{-16+66} = \frac{y}{-22+72} = \frac{1}{54-4}$

or, $\frac{x}{50} = \frac{y}{50} = \frac{1}{50}$

$\therefore x = \frac{50}{50}, y = \frac{50}{50}$

So, centre is (1, 1)

For length of axes

$a\left(\frac{h^2-ab}{\Delta}\right)x^2 + 2h\left(\frac{h^2-ab}{\Delta}\right)xy + b\left(\frac{h^2-ab}{\Delta}\right)y^2 = 1$

or, $9\left(\frac{50}{500}\right)x^2 + 2\times 2\left(\frac{50}{500}\right)xy + 6\left(\frac{50}{500}\right)y^2 = 1$

or, $\frac{9}{10}x^2 + \frac{4}{10}xy + \frac{6}{10}y^2 = 1$

So, $A = \frac{9}{10}, H = \frac{4}{10}, B = \frac{6}{10}$

So, $\frac{1}{r^4} - \frac{1}{r^2}(A+B) + (AB + H^2) = 0$

or, $\frac{1}{r^4} - \frac{1}{r^2}\left(\frac{15}{10}\right) + \left(\frac{54}{100} - \frac{4}{100}\right) = 0$

or, $\frac{1}{r^4} - \frac{3}{3r^2} + \frac{50}{100} = 0$

or, $50r^4 - 150r^2 + 100 = 0$

or, $r^4 - 3r^2 + 2 = 0$

or, $r^4 - r^2 - 2r^2 + 2 = 0$

or, $r^2(r^2 - 1) - 2(r^2 - 1) = 0$

or, $(r^2 - 2)(r^2 - 1) = 0$

Either $r^2 = 2$ or, $r^2 = 1$

$\therefore r = \sqrt{2}$ or, $r = 1$

Major axis = $\sqrt{2}$,

minor axis = 1

For eccentricity

$r_1 = 1, r_2 = \sqrt{2}$

Over,

$r_1^2 = r_2^2 = (1 - e^2)$

or, $1 = 2(1 - e^2)$

or, $\frac{1}{2} = 1 - e^2$

or, $e^2 = 1 - \frac{1}{2}$

or, $e^2 = \frac{1}{2}$

$\therefore e = \frac{1}{\sqrt{2}}$

(ii) $2x^2 + 3y^2 - 4x - 12y + 13 = 0$

Sol^n. The equation of conic is,

$2x^2 + 3y^2 - 4x - 12y + 13 = 0$

Comparing it with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

We get,

$a = 2, h = 0, b = 3, g = -2, f = -6, c = 13$

Now,

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 78 + 2(-6)(-2) \times 0 - 2 \times 36 - 3 \times 4 - 13 \times (0)^2$

$= 78 - 72 - 12$

$= -6 \neq 0$

and $ab - h^2 = 2 \times 3 - 0 = 6 > 0$

So, it represents ellipse,

For centre,

Let $\phi = 2x^2 + 3y^2 - 4x - 12y + 13$

$\therefore \frac{\partial\phi}{\partial x} = 4x - 4, \quad \frac{\partial\phi}{\partial y} = 6y - 12$

$x = 1, y = 2$

Thus, centre (1, 2)

The equation referred to centre is,

$\frac{a(h^2-ab)}{\Delta}x^2 + 2h\frac{(h^2-ab)}{\Delta} + \frac{b(h^2-ab)}{\Delta}y^2 = 1$

or, $2\left(\frac{-6}{6}\right)x^2 + 3\left(\frac{-6}{6}\right)y^2 = 1$

or, $2x^2 + 3y^2 = 1$

So, $A = 2, H = 0, B = 3$

So, we have the equation,

$\frac{1}{r^4} - \frac{1}{r^2}(A+B) + (AB + H^2) = 0$

or, $\frac{1}{r^4} - \frac{1}{r^2}(2+3) + 6 = 0$

or, $\frac{1}{r^4} - \frac{5}{r^2} + 6 = 0$

or, $6r^4 - 5r^2 + 1 = 0$

or, $6r^4 - 3r^2 - 2r^2 + 1 = 0$

or, $3r^2(2r^2-1)-1(2r^2-1)=0$
or, $(3r^2-1)(2r^2-1)=0$
Either $3r^2=1$, or, $2r^2=1$

$\therefore r=\frac{1}{\sqrt{3}}$ or, $r=\frac{1}{\sqrt{2}}$

Thus, $\left(\frac{1}{\sqrt{3}}\right)^2=\left(\frac{1}{\sqrt{2}}\right)^2(1-e^2)$

or, $\frac{1}{3}=\frac{1}{2}(1-e^2)$

or, $e^2=1-\frac{2}{3}=\frac{1}{3}$

$\therefore e=\frac{1}{\sqrt{3}}$

6. Find the asymptotes of the hyperbola $2x^2+5xy+2y^2+4x+5y=0$

Sol^n. The hyperbola is,
$2x^2+5xy+2y^2+4x+5y=0$
Thus, the equation of asymptotes is,

$$2x^2+5xy+2y^2+4x+5y-\frac{\Delta}{ab-h^2}=0$$

where, $\Delta=abc+2fgh-af^2-bg^2-ch^2$

$$=0+2\times\frac{5}{2}\times3\times2\times\frac{5}{2}-2\times\left(\frac{5}{2}\right)^2-2\times2^2=0$$

$$=25-\frac{25}{2}-8=\frac{25}{2}-8=\frac{9}{2}$$

and $ab-h^2=2\times2-\left(\frac{5}{2}\right)^2=4-\frac{25}{4}=\frac{-9}{4}$

$$\therefore 2x^2+5xy+2y^2+4x+5y-\frac{\frac{9}{2}}{\frac{-9}{4}}=0$$

or, $2x^2+5xy+2y^2+4x+5y+2=0$
or, $(2x+y+2)(x+2y+1)=0$
Thus, equation of asymptotes are:
$2x+y+2=0$
and $x+2y+1=0$

7. Find the foci and eccentricity of the conic $x^2+4xy+y^2-2x+2y-6=0$

Sol^n. The equation of conic is
$x^2+4xy+y^2-2x+2y-6=0$

Comparing it with $ax^2+2hxy+by^2+2gx+2fy+c=0$
We get,
$a=1, h=2, b=1, g=-1, f=1, c=-6$
Now,
$\Delta=abc+2fgh-af^2-by^2-ch^2$
$=-6+2\times1(-1)\times2-1(1)^2-1(-1)^2+6\times4$
$=-6-4-1-1+24$
$=12\neq0$
and $ab-h^2=\frac{1}{4}=-3$

So, $a\left(\frac{h^2-ab}{\Delta}\right)x^2+2h\frac{\left(h^2-ab\right)}{\Delta}xy+\frac{b\left(h^2-ab\right)}{\Delta}y^2=1$

or, $1\left(\frac{3}{12}\right)x^2+4\left(\frac{3}{12}\right)xy+1\left(\frac{3}{12}\right)y^2=1$

or, $\frac{x^2}{4}+\frac{4xy}{4}+\frac{1}{4}y^2=1$

So, $A=\frac{1}{4}, H=\frac{1}{2}, B=\frac{1}{4}$

$\therefore$ We have equation,

$\frac{1}{r^4}-\frac{1}{r^2}(A+B)+(AB-H^2)=0$

or, $\frac{1}{r^4}-\frac{2}{4}\frac{1}{r^2}+\frac{1}{16}-\frac{1}{4}=0$

or, $\frac{1}{r^4}-\frac{1}{2r^2}+\frac{3}{16}=0$

or, $-3r^4-8r^2+16=0$
or, $3r^4+8r^2-16=0$
or, $3r^4+12r^2-4r^2-16=0$
or, $3r^2(r^2+4)-4(r^2+4)=0$
or, $(3r^2-4)(r^2+4)=0$

Either $r^2=\frac{4}{3}$ or, $r^2=-4$

For centre, (α, β)
$\phi=x^2+4xy+y^2-2x+2y-6$
$\frac{\partial\phi}{\partial x}=2x+4y-2$, $\frac{\partial\phi}{\partial y}=2y+2+4x$

So, $2x+4y-2=0$
i.e. $x+2y-1=0$
and $4x+2y+2=0$
or, $2x+y+1=0$
Now,

$$\frac{x}{2+1} = \frac{y}{-2-1} = \frac{1}{1-4}$$

or, $x = -1, y = 1$

Thus,

Centre $= (-1, 1)$

$\therefore$ The equation of transverse axis,

$$A - \left(\frac{1}{r_1^2}\right)X + HY = 1$$

Thus, we can find $\tan\theta$ = slope of transverse are as 1 as $x - y = 0$

For eccentricity,

$$4 = \frac{4}{3}(e^2 - 1)$$

or, $3 = e^2 - 1$

or, $e = 2$

or, $r_1 = \frac{2}{\sqrt{3}}$

Thus, foci $= (\alpha \pm er_1\cos\theta - B \pm er_1\sin\theta)$

$$= \left(-1 \pm 2 \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{2}}, 1 \pm 2 \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{2}}\right)$$

$$= \left(-1 \pm \frac{2\sqrt{2}}{\sqrt{3}}, 1 \pm \frac{2\sqrt{2}}{\sqrt{3}}\right)$$

and eccentricity = 2

Exercise - 38

Describe and sketch the graph of the following polar equation of conic section.

1. $r = \frac{2}{1-\cos\theta}$

$\frac{de}{1-e\cos\theta}$

Sol^n. Here, $r = \frac{2}{1-\cos\theta}$

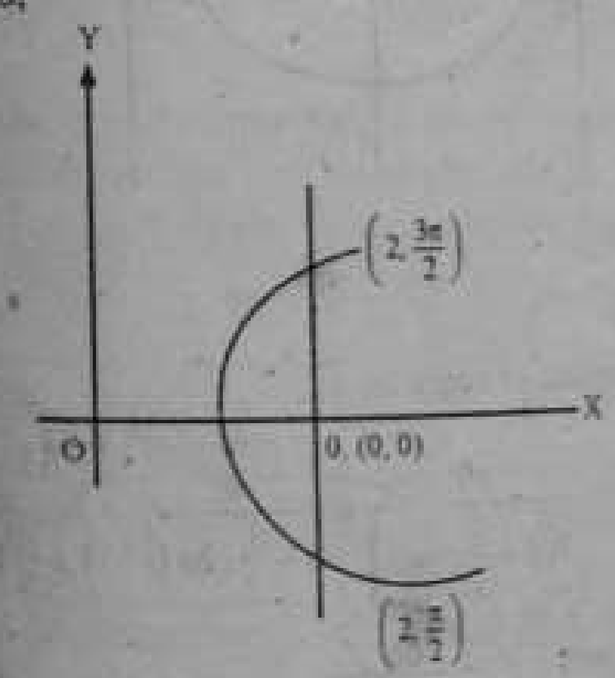
Comparing it with $r = \frac{de}{1-\cos\theta}$

we find $d = 2, e = 1$

therefore, the conic represents parabola with focus as pole. The expression $\cos\theta$ tells us its axis in on the polar axis. The co-ordinates of p (r, θ) on the curve are shown as the table.

θ	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{2}{1-\cos\theta}$	2	1	2
(r, θ)	$\left(2, \frac{\pi}{2}\right)$	$(1, \pi)$	$\left(2, \frac{3\pi}{2}\right)$

The graph is,



2. $r = \frac{10}{3-2\cos\theta}$

Sol^n. Here, the equation of conic is,

$$r = \frac{10}{3-2\cos\theta}$$

or, $r = \dfrac{\frac{10}{3}}{1-\frac{2}{3}\cos\theta} = \dfrac{5\times\frac{3}{2}}{1-\frac{2}{3}\cos\theta}$

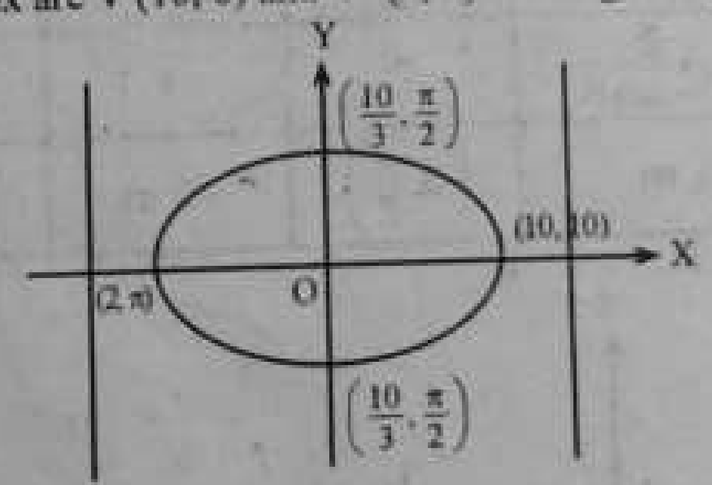
Comparing it with $r = \dfrac{de}{1-\cos\theta}$

We find $d = 5, e = \frac{2}{3} < 1$

Therefore, the conic represents an ellipse with focus as pole. The expression $\cos\theta$ tells us that the major axis is on polar axis. The co-ordinate p (r, θ) on the curve is shown on the table below:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \dfrac{10}{3-2\cos\theta}$	$\frac{10}{1}$	$\frac{10}{3}$	$\frac{10}{5} = 2$	$\frac{10}{3}$
(r, θ)	$(10, 0)$	$\left(\frac{10}{3}, \frac{\pi}{2}\right)$	$(2, \pi)$	$\left(\frac{10}{3}, \frac{3\pi}{2}\right)$

The vertex are V $(10, 0)$ and V' $(2, \pi)$. The figure is,



3. $r = \dfrac{12}{3+\cos\theta}$

Sol^n. Here, the equation of conic is,

$$r = \frac{12}{3+\cos\theta} = \frac{12}{3\left(1+\frac{1}{3}\cos\theta\right)} = \frac{4}{1+\frac{1}{3}\cos\theta} = \frac{12\times\frac{1}{3}}{1+\frac{1}{3}\cos\theta}$$

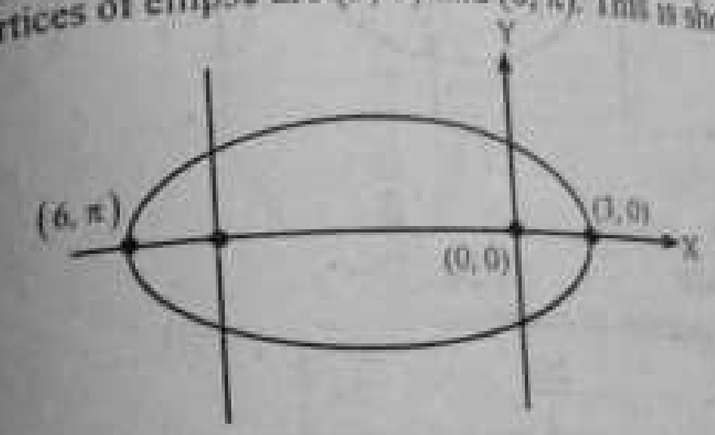
Comparing it with $r = \dfrac{de}{1+e\cos\theta}$

Therefore, the conic represents an ellipse with focus as pole. The expression $\cos\theta$ tells us that its axis is on the polar axis. The coordinate of p (r, θ) on the curve is shown on the table below:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

$r = \dfrac{12}{3+\cos\theta}$	3	4	6	4
(r, θ)	$(3, 0)$	$\left(4, \frac{\pi}{2}\right)$	$(6, \pi)$	$\left(4, \frac{3\pi}{2}\right)$

The vertices of ellipse are $(3, 0)$ and $(6, \pi)$. This is shown as:



4. $r = \dfrac{10}{3+2\cos\theta}$

Sol^n. Here, the equation of conic is,

$$r = \frac{10}{3+2\cos\theta} = \frac{10}{3\left(1+\frac{2}{3}\theta\right)} = \frac{\frac{10}{3}}{\left(1+\frac{2}{3}\cos\theta\right)} = \frac{5\times\frac{2}{3}}{1+\frac{2}{3}\cos\theta}$$

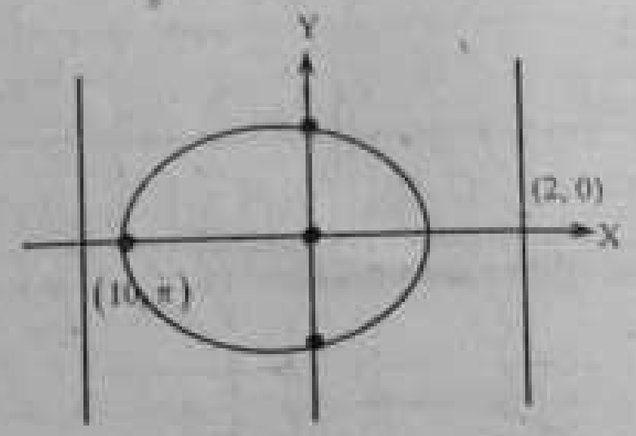
Comparing it with $r = \dfrac{de}{1+e\cos\theta}$

We find $d = 5, e = \frac{2}{3} < 1$

Therefore, the conic represents an ellipse with focus as pole. The expression $\cos\theta$ tells us that the major axis on Polar-axis. the co-ordinate p (r, θ) on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \dfrac{10}{3+2\cos\theta}$	2	$\frac{10}{3}$	10	$\frac{10}{3}$
(r, θ)	$(2, 0)$	$\left(\frac{10}{3}, \frac{\pi}{2}\right)$	$(10, \pi)$	$\left(\frac{10}{3}, \frac{3\pi}{2}\right)$

The vertex of ellipse are V $(2, 0)$ and V' $(10, \pi)$. The figure is shown as,



5. $r = \dfrac{10}{3+2\sin\theta}$

Sol^n. Here, the equation of conic is,

$$r = \frac{10}{3+2\sin\theta} = \frac{\frac{10}{3}}{\left(1+\frac{2}{3}\sin\theta\right)} = \frac{5\times\frac{2}{3}}{1+\frac{2}{3}\sin\theta}$$

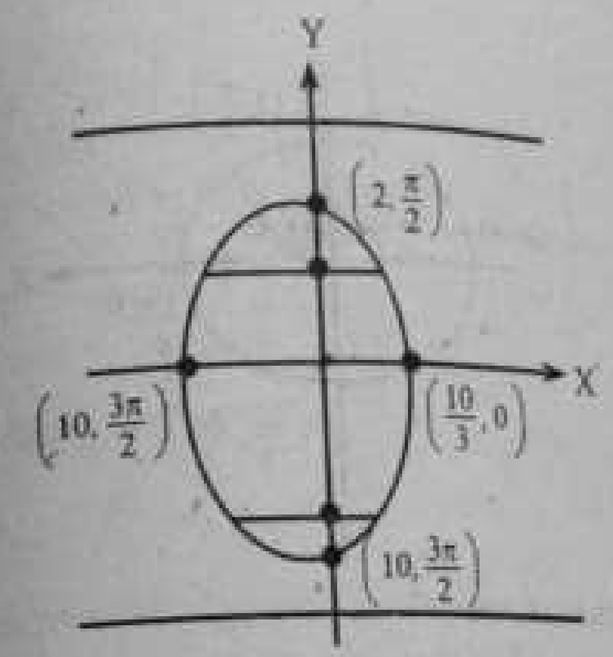
Comparing is with $r = \dfrac{de}{1+e\sin\theta}$

We find $d = 5, e = \dfrac{2}{3} < 1$

Therefore, the conis represents an ellipse with focus as pole. The enpression $\sin\theta$ tells us that the major axis is on the polar axis. The co-ordinate p (r, θ) on the curve is shown on the table bellows.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{10}{3+2\sin\theta}$	$\frac{10}{3}$	2	$\frac{10}{3}$	10
(r, θ)	$\left(\frac{10}{3}, 0\right)$	$\left(2, \frac{\pi}{2}\right)$	$\left(\frac{10}{3}, \pi\right)$	$\left(10, \frac{3\pi}{2}\right)$

The vertex of the major axis of ellipse are V $\left(2, \frac{\pi}{2}\right)$ and V' $\left(10, \frac{3\pi}{2}\right)$. These points together type of conic leads to sketch the graph of ellipse as follow:



6. $r = \dfrac{12}{3-2\sin\theta}$

Sol^n. Here, the equation of conic is,

$$r = \frac{12}{3-2\sin\theta} = \frac{4}{\left(1-\frac{2}{3}\sin\theta\right)} = \frac{6\times\frac{2}{3}}{1-\frac{2}{3}\sin\theta}$$

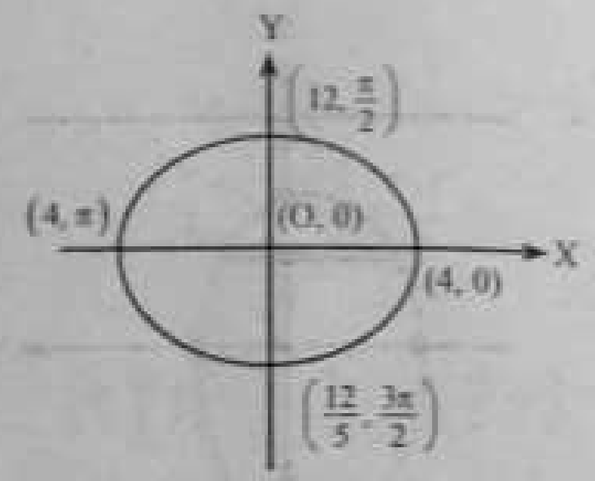
Comparing it with $r = \dfrac{de}{1-e\sin\theta}$

We find, $d = 6, e = \dfrac{2}{3} < 1$

So, it represents ellipse with focus as pole. The expression $\sin\theta$ tells us that the major axis is perpendicular to the polar axis. The co-ordinate p (r, θ) on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{12}{3-2\sin\theta}$	4	12	4	$\frac{12}{5}$
(r, θ)	(4, 0)	$\left(12, \frac{\pi}{2}\right)$	$(4, \pi)$	$\left(\frac{12}{5}, \frac{3\pi}{2}\right)$

The vertex of the major axis of ellipse are V $\left(12, \frac{\pi}{2}\right)$ and V' $\left(\frac{12}{5}, \frac{3\pi}{2}\right)$. The figure is,



7. $r = \dfrac{6}{3+2\sin\theta}$

Sol^n. Here, The equation of conic is,

$$r = \frac{6}{2+3\sin\theta} = \frac{3}{\left(1+\frac{3}{2}\sin\theta\right)} = \frac{\frac{1}{2}\times\frac{3}{2}}{1+\frac{3}{2}\sin\theta}$$

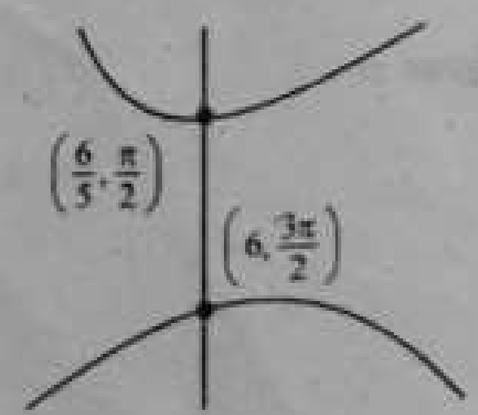
Comparing it with $r = \dfrac{de}{1+e\sin\theta}$

We find $d = \frac{1}{2}, e = \frac{3}{2} < 1$

Therefore, the conic represents a hyperbola with focus as the pole. The expression $\sin\theta$ tells us that. The transverse axis of the hyperbola is perpendicular to polar axis. The co-ordinate p (r, θ) on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \dfrac{6}{2+3\sin\theta}$	3	$\frac{6}{5}$	3	-6
(r, θ)	(3, 10)	$\left(\frac{6}{5}, \frac{\pi}{2}\right)$	$(3, \pi)$	$\left(-6, \frac{3\pi}{2}\right)$

The vertex of the major axis of ellipse are $V\left(\frac{6}{5}, \frac{\pi}{2}\right)$ and V' $\left(-6, \frac{3\pi}{2}\right)$. The figure is,



8. $r = \dfrac{4}{1+3\cos\theta}$

Sol^n. Here, the equation of conic is,

$$r = \frac{4}{1+\cos\theta} = \frac{3\times\frac{4}{3}}{1+\cos\theta}$$

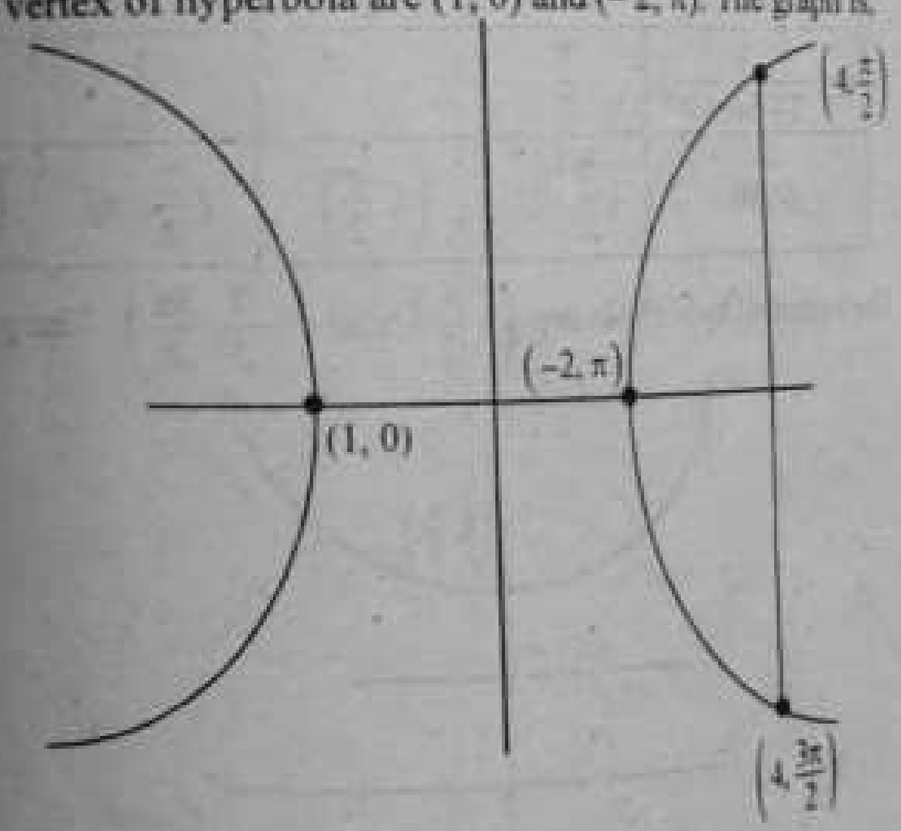
Comparing it with $r = \dfrac{de}{1+e\cos\theta}$

we find, $d = \frac{4}{3}, e = 3 > 1$

So, it represents hyperbola. The expression $\cos\theta$ tellsus that its transverse axis is on the polar axis. The co-ordinate p (r, θ) on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \dfrac{4}{1+\cos\theta}$	1	4	-2	4
(r, θ)	(1, 0)	$\left(4, \frac{\pi}{2}\right)$	$(-2, \pi)$	$\left(4, \frac{3\pi}{2}\right)$

The vertex of hyperbola are (1, 0) and $(-2, \pi)$. The graph is,



9. $r = \dfrac{10\,\text{cosec}\theta}{2\,\text{cosec}\theta + 3}$

Sol^n. Here, the equation of conic is,

$$r = \frac{10\,\text{cosec}\theta}{2\,\text{cosec}\theta + 3}$$

$$= \frac{\frac{10}{\sin\theta}}{\frac{2}{\sin\theta}+3} = \frac{\frac{10}{\sin\theta}}{\frac{2+3\sin\theta}{\sin\theta}} = \frac{10}{2+\sin\theta}$$

Proceed as Q. No. 7.

10. $r = \dfrac{7}{2+5\sin\theta}$

Sol^n. Here, the equation of conic is,

$$r = \frac{7}{2+5\cos\theta} = \frac{\frac{5}{2}\times\frac{14}{5}}{1+\frac{5}{2}\cos\theta}$$

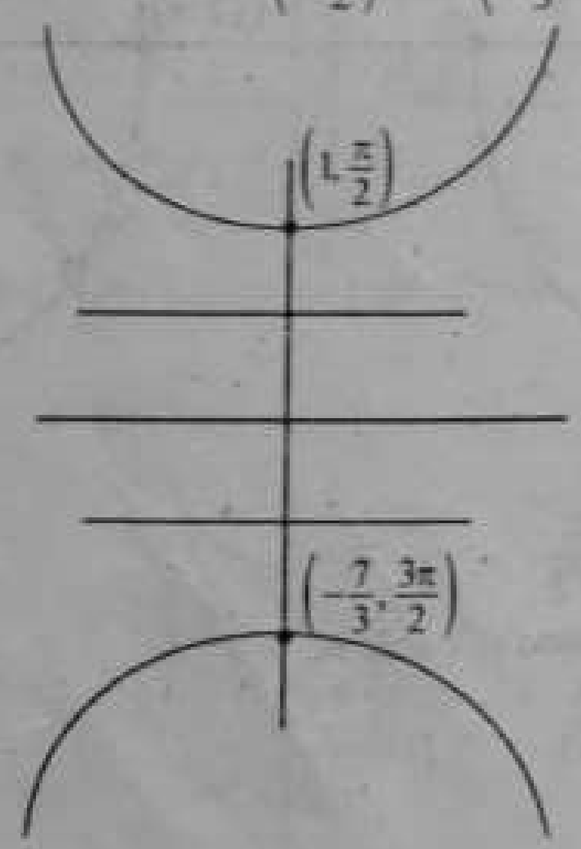
Comparing it with $r = \dfrac{de}{1+e\sin\theta}$

We get $d = \dfrac{14}{5}, e = \dfrac{5}{2} > 1$

So, it represents hyperbola. the expression $\sin\theta$ tells us that the transserse axis of hyperbola is perpendicular to polar axis. The co-ordinate p (r, θ) on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{7}{2+5\cos\theta}$	$\frac{7}{2}$	1	$\frac{7}{2}$	$-\frac{7}{2}$
(r, θ)	$(\frac{7}{2}, 0)$	$\left(1, \frac{\pi}{2}\right)$	$(\frac{7}{2}, \pi)$	$\left(-\frac{7}{3}, \frac{3\pi}{2}\right)$

The vertex of hyperbola are $\left(1, \frac{\pi}{2}\right)$ and $\left(-\frac{7}{3}, \frac{3\pi}{2}\right)$. The graph is,



11. $r = \dfrac{12}{2+4\sin\theta}$

Sol^n. Here, the equation of conic is,

$$r = \frac{12}{2+4\sin\theta} = \frac{6}{2+2\sin\theta}$$

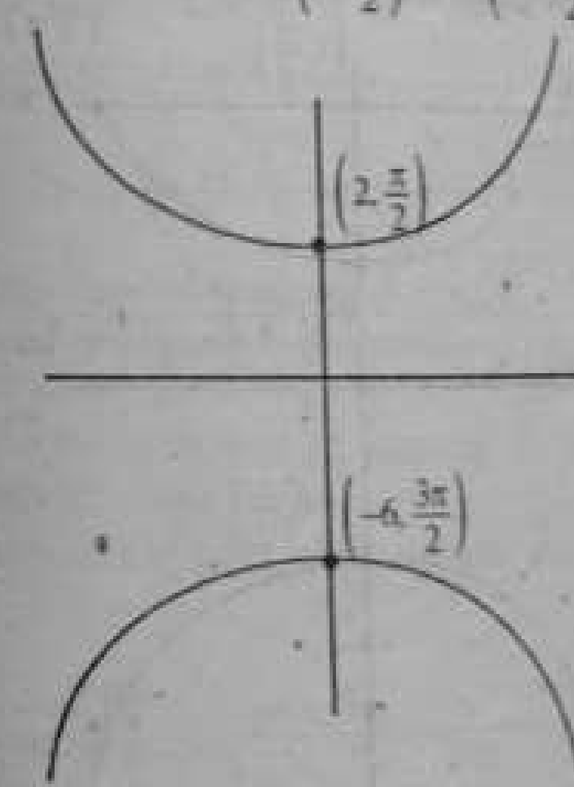
Comparing it with $r = \dfrac{de}{1+e\cos\theta}$

We get, $d = 3, e\ 2 > 1$

So, it represents hyperbola. The expression $\cos\theta$ tells us that the transverse aixs of hyperbola is perpendicular to polar axis. The co-ordinate p (r, θ) on the curve is shown or the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{12}{2+4\sin\theta}$	6	2	6	-6
(r, θ)	$(6, 0)$	$\left(2, \frac{\pi}{2}\right)$	$(6, \pi)$	$\left(-6, \frac{3\pi}{2}\right)$

The vertex of hyperbola are $\left(2, \frac{\pi}{2}\right)$ and $\left(-6, \frac{3\pi}{2}\right)$. The graph is,



12. $r = \dfrac{6\operatorname{cosec}\theta}{2\operatorname{cosec}\theta - 3}$

Sol^n. Here, the equation of conic is,

$$r = \frac{6\operatorname{cosec}\theta}{2\operatorname{cosec}\theta - 3} = \frac{6\times\frac{3}{2}}{1-\frac{3}{2}\sin\theta}$$

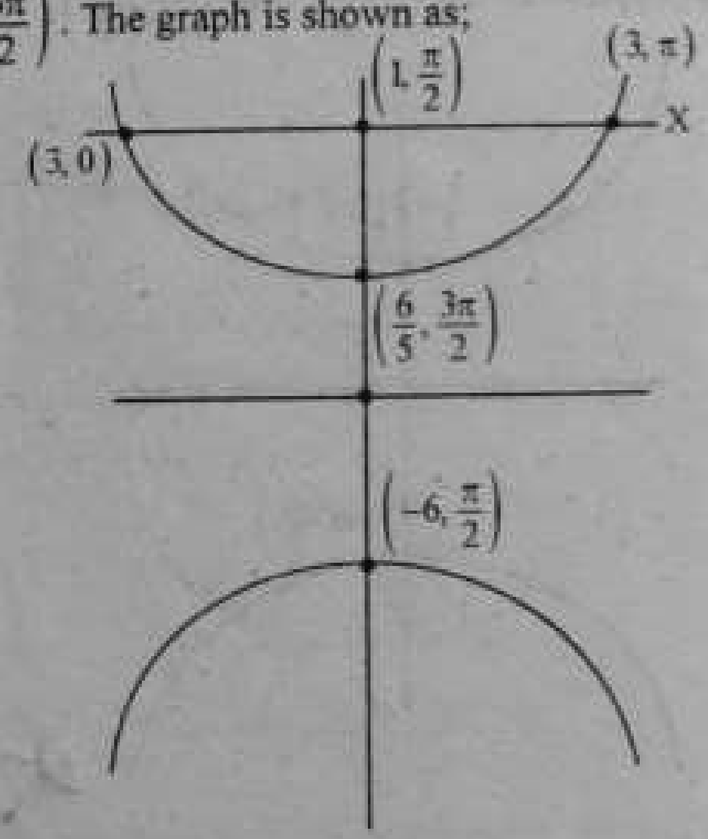
Comparing it with $r = \frac{de}{1-e\sin\theta}$

We get, $d = 6$, $a = \frac{3}{2} > 1$

Therefore, conic represents a hyperbola with focus as pole. The expression $\sin\theta$ tells us that the transverse axis of the hyperbola is perpendicular to polar axis. The co-ordinates p (r, θ) on the curve is shown on the table.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{6\operatorname{cosec}\theta}{2\operatorname{cosec}\theta - 3}$	3	-6	3	$\frac{6}{5}$
(r, θ)	$(3, 0)$	$\left(-6, \frac{\pi}{2}\right)$	$(3, \pi)$	$\left(\frac{6}{5}, \frac{3\pi}{2}\right)$

The points (3, 0) and (3, π) on the graph helps us to sketch the lower branch of hyperbola. The vertex of hyperbola are $v\left(-6, \frac{\pi}{2}\right)$ and $v'\left(\frac{6}{5}, \frac{3\pi}{2}\right)$. The graph is shown as;



Old Question 2056-2068 with Solution

Tribhuvan University
Institute of Engineering
Examination Control Division

2068 Baishakh

Exam	Regular/Back		
Level	BE	Full Marks	80
Programme	All (Except B. Arch.)	Pass Marks	32
Year/ Part	I/I	Time	3hrs.

1. If $y = a\cos(\log x) + b\sin(\log x)$. Prove that $x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1) y_n = 0$

Sol^n. *See page 14, Ex- 2, Q. No. 9 in this book.*

2. State and prove rolle's theorem.

Sol^n. *For proof. See [A text book of engineering mathmatics] (page 23)*

3. Determine the values of a, b, c so that $\lim_{x \to 0} \frac{(a + b\cos x) - c\sin x}{x^5} = 1$

Sol^n. *See page 13, Ex- 2, Q. No. 6 (iii) in this book.*

4. Find the asymptotes of the curve $(x+y)^2(x+2y+2) = x+9y-2$

Sol^n. *See [A set book of engineering mathmatics page 70 (example 5)*

5. If e_1 and e_2 be the radius of curvature at the ends of a focal Chord of the parabola $y^2 = 4ax$

Sol^n. *See A text book of engineering mathmatics page 94 [example 10]*

6. Prove that $\int_0^{\pi} \frac{x\tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$

Sol^n. *See A textbook of engineering mathematics- I page 150.*

7. Apply the method of differentiation under integral sign to prove

$$\int_0^{\frac{\pi}{2}} \frac{1}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^2} dx = \frac{\pi\left(a^2+b^2\right)}{4a^3b^3}$$

Sol^n. *See page 184, Q. No. 11 in this book.*