The axis of the parabola $y^2 = 4ax$ bounded by its latus rectum revolved about its axis. Find the surface area generated.

Ans Solution:

Given

$$y^2 = 4ax$$

When the parabola is rotated about its axis, the surface area of thus generated solid:

$$S = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot 2\pi y \, dx$$

$$= 2\pi \int_0^a y \sqrt{1 + \left(\frac{2a}{y}\right)^2} \, dx$$

$$= 2\pi \int_0^a \sqrt{y^2 + 4a^2} \, dx$$

$$= 2\pi \int_0^a \sqrt{4a} \sqrt{x + a} \, dx$$

$$= 2\pi \times 2\sqrt{a} \cdot \left[\frac{(x+a)^{\frac{3}{2}}}{3/2}\right]_0^a$$

$$= 4\pi\sqrt{a} \left[\frac{(a+a)^{3/2}}{3/2} - \frac{a^{3/2}}{3/2}\right]$$

$$= 4\pi \times \frac{2}{3} \sqrt{a} \left[2^{3/2} a^{3/2} - a^{3/2}\right]$$

$$= \frac{8}{3}\pi a^{1/2 + 3/2} [2^{3/2} - 1] = \frac{8}{3}\pi a^2 [2\sqrt{2} - 1]$$

$\therefore$ Surface Area $= \frac{8}{3}\pi a^2 [2\sqrt{2} - 1]$

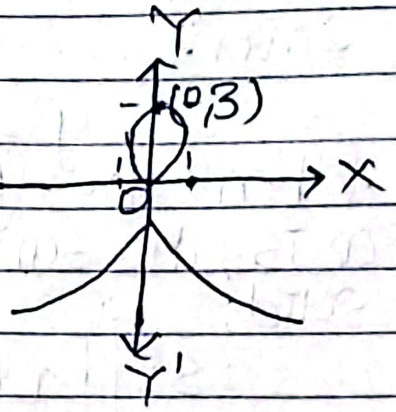
The loop of the curve $9x^2 = y(3-y)^2$ is rotated about x-axis. Find the surface area generated.

Ans Solution:

Given,

$$9x^2 = y(3-y)^2$$

The surface area generated when the curve is rotated about x-axis is given by



$$S = \int_0^3 2\pi y \cdot \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$\cancel{18x = 9y}$

$9x^2 = 9y - 6y^2 + y^3$

$\Rightarrow 18x x_1 = 9 - 12y + 3y^2$

$\Rightarrow x_1 = \frac{9 - 12y + 3y^2}{18x}$

$$= 2\pi \int_0^3 y \cdot \sqrt{1 + \left(\frac{9 - 12y + 3y^2}{18x}\right)^2}\, dy$$

$$= 2\pi \int_0^3 y \cdot \sqrt{\frac{36x^2 + (3 - 4y + y^2)^2}{36x^2}}\, dy$$

$$= 2\pi \int_0^3 y \cdot \sqrt{\frac{4 \times y(3-y)^2 + (y-1)^2(y-3)^2}{\cancel{36xy}\ 4 \times y \times (3-y)^2}}\, dy$$

$$= 2\pi \int_0^3 y \sqrt{\frac{4y + (y-1)^2}{4y}}\, dy$$

$$= 2\pi \int_0^3 y \sqrt{\frac{4y + y^2 - 2y + 1}{4y}}\, dy$$

$$= 2\pi \int_0^3 y \cdot \frac{(y+2)}{\sqrt{y}}\, dy$$

$$= 2\pi \int_0^3 \sqrt{y}\,(y+2)\, dy$$

$$= 2\pi \left[\int_0^3 y^{3/2} dy + 2 \int_0^3 y^{1/2} dy \right]$$

$$= 2\pi \left[\left[\frac{y^{5/2}}{5/2} \right]_0^3 + 2 \left[\frac{y^{3/2}}{3/2} \right]_0^3 \right]$$

$$= 2\pi \left[\frac{2}{5} \times 3^{5/2} + 2 \times \frac{2}{3}\, 3^{3/2} \right]$$

$$= 4\pi \left[\frac{3^{5/2}}{5} + \frac{2}{3}\, 3^{3/2} \right]$$

$$= \frac{76}{5}\sqrt{3}\,\pi$$

$\therefore$ Surface Area $= \frac{76}{5}\sqrt{3}\,\pi$

Show that the area of curved surface generated when loop of the curve $x^2(a^2-x^2) = 8a^2y^2$ is revolved about x-axis is $\frac{\pi a^2}{4}$.

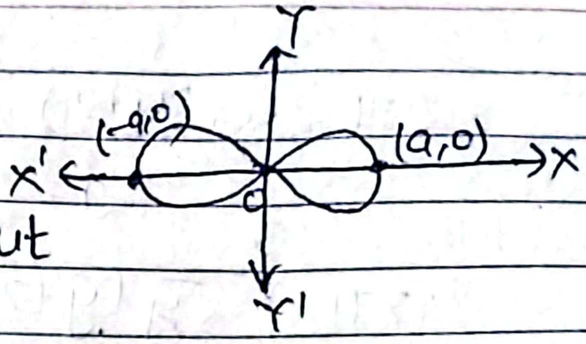
Ans Solution:

Given,

$$x^2(a^2-x^2) = 8a^2y^2$$

Here, the curve looks like :

Now,
Surface area of the solid when the curve is revolved about x-axis is given by



$$\text{Surface area} = 2\int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\int_0^a 2\pi y\sqrt{1+\left(\frac{a^2 2x - 4x^3}{8a^2 . 2y}\right)^2}\,dx$$

$$= 2\int_0^a 2\pi y\sqrt{1+\left(\frac{a^2x - 2x^3}{8a^2 y}\right)^2}\,dx$$

$$= 2\int_0^a 2\pi y\sqrt{\frac{64a^4y^2 + x^2(a^2-2x^2)^2}{64a^4y^2}}\,dx$$

$$= 2\times 2\pi \times \frac{1}{8a^2}\int_0^a \sqrt{64a^4y^2 + x^2(a^2-2x^2)^2}\,dx$$

$$= \frac{\pi}{2a^2}\int_0^a \sqrt{x^4(a^2-x^2)^2 + x^2(a^4 - 4a^2x^2 + 4x^4)}\,dx$$

$$= \frac{\pi}{2a^2}\int_0^a \sqrt{x^4(a^4 - 2a^2x^2 + x^4) + x^2(a^4 - 4a^2x^2 + 4x^4)}\,dx$$

$$= \frac{\pi}{2a^2}\int_0^a \sqrt{x^4a^4 - 2a^2x^6 + x^8 + a^4x^2 - 4a^2x^4 + 4x^6}$$

Find the surface area generated by rotating the portion of the curve $y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$ between $x=0$ and $x=3$ about y-axis.

Ans Solution:

The given curve is

$$y = \frac{1}{3}(x^2+2)^{3/2}$$

$$\Rightarrow y' = \frac{1}{3} \times \frac{3}{2}(x^2+2)^{1/2} \cdot 2x$$

$$\Rightarrow y' = x(x^2+2)^{1/2}$$

Now,

$$\text{Surface Area generated} = \int_0^3 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^3 x\sqrt{1+x^2(x^2+2)}\, dx$$

$$= 2\pi \int_0^3 x\sqrt{1+x^4+2x^2}\, dx$$

$$= 2\pi \int_0^3 x.(x^2+1)\, dx$$

$$= 2\pi \int_0^3 (x^3+x)\, dx$$

$$= 2\pi \left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^3$$

$$= 2\pi \left[\frac{3^4}{4}+\frac{3^2}{2}\right] = \frac{99\pi}{2}$$

$\therefore$ Surface Area $= \dfrac{99\pi}{2}$

Find the area of the surface generated by rotating the curve $y = \dfrac{x^2}{2} + \dfrac{1}{2}$ about y-axis

for $0 \leq x \leq 1$.

Ans Solution:

The given equation of the curve is:

$$y = \frac{x^2+1}{2}$$

$$\Rightarrow y' = x$$

Now,

Surface area generated $= \displaystyle\int_0^1 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$= 2\pi \int_0^1 x \cdot \sqrt{1+x^2}\, dx$$

$$\cancel{= 2\pi \left[\sqrt{1+x^2} \cdot x^2 - \int \frac{1 \cdot 2x}{2\sqrt{1+x^2}} \cdot x^2\, dx \right]_0^1}$$

$$\cancel{= 2\pi \left[x^2 \sqrt{x^2+1} - \int \frac{x^3}{\sqrt{1+x^2}}\, dx \right]_0^1}$$

Let $x = \tan\theta$ $\quad$ As $x \to 0, \theta \to 0$

$\Rightarrow dx = \sec^2\theta\, d\theta$ $\quad$ As $x \to 1, \theta \to \dfrac{\pi}{4}$

$$= 2\pi \int_0^{\frac{\pi}{4}} \tan\theta \cdot \sec\theta \cdot \sec^2\theta\, d\theta$$

Let $\sec\theta = y$.

$\Rightarrow$ As $\theta \to 0, y \to 1$

As $\theta \to \frac{\pi}{4}, y \to \sqrt{2}$

$$= 2\pi \int_{1}^{\sqrt{2}} y^2 \, dy$$

$$= 2\pi \left[\frac{y^3}{3}\right]_{1}^{\sqrt{2}}$$

$$= 2\pi \left(\frac{2\sqrt{2}}{3} - \frac{1}{3}\right)$$

$$= \frac{4\sqrt{2}}{3}\pi - \frac{2\pi}{3} = \frac{2\pi}{3}(2\sqrt{2}-1)$$

$\therefore$ Surface Area generated $= \frac{4\sqrt{2}}{3}\pi - \frac{2\pi}{3}$

$$= \frac{2\pi}{3}(2\sqrt{2}-1)$$

If $a > 0$, find the surface area generated by rotating the loop of the curve $3ay^2 = x(a-x)^2$ about Y-axis.

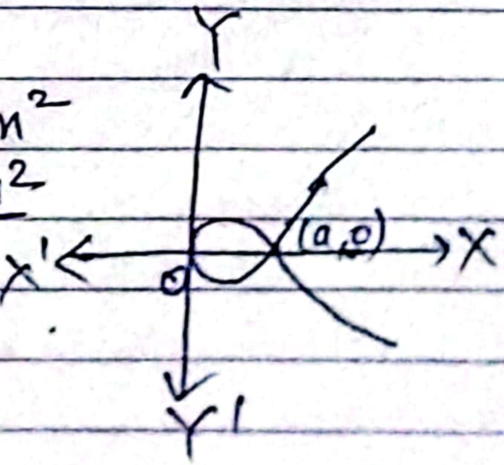
Ans Solution:

Given,

$$3ay^2 = x(a-x)^2$$

$$\Rightarrow 6ayy_1 = a^2 - 4ax + 3x^2$$

$$\Rightarrow y_1 = \frac{a^2 - 4ax + 3x^2}{6ay}$$



Now,

Surface Area Generated $= 2\int_0^a 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$= 2\pi \times 2\int_0^a x \sqrt{1+\left(\frac{a^2-4ax+3x^2}{6ay}\right)^2}\, dx$$

$$= 2\pi \times 2\int_0^a x \sqrt{1+\frac{(3x-a)^2(x-a)^2}{12a\cdot x(a-x)^2}}\, dx$$

$$= 2\pi \times 2\int_0^a x \sqrt{1+\frac{(3x-a)^2}{12ax}}\, dx$$

$$= 2\times 2\pi \int_0^a \frac{x}{\sqrt{12a}\sqrt{x}} \sqrt{12ax+9x^2-6ax+a^2}\, dx$$

$$= \frac{2\times 2\pi}{2\sqrt{3}} \int_0^a \frac{x^{1/2}}{\sqrt{a}} (3x+a)\, dx$$

$$= \frac{2\times \pi}{\sqrt{3a}} \left[\int_0^a (3x^{3/2} + ax^{1/2})\, dx\right]$$

$$= \frac{2\times \pi}{\sqrt{3a}} \left[3\cdot\frac{a^{5/2}}{5/2} + a\cdot\frac{a^{3/2}}{3/2}\right]$$

$$= \frac{56}{15\sqrt{3}}\pi a^2$$

$\therefore$ Surface Area Generated $= \frac{56}{15\sqrt{3}}\pi a^2.$

The curve $y=\sqrt{4-x^2}$, $-1\le x\le 1$ is an arc of the circle $x^2+y^2=4$ Find the area of the surface obtained by rotating this arc about x-axis.

Ans Solution:

Given,

$$y=\sqrt{4-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{4-x^2}}(-2x) = -\frac{x}{y}$$

Now,

$$\text{Area of surface} = \int_0^2 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^2 2\pi y\,\frac{\sqrt{x^2+y^2}}{y}\,dx$$

$$= 2\pi\int_0^2 2\,dx$$

$$= 4\pi\int_0^2 dx$$

$$= 8\pi$$

$\therefore$ Area of the surface obtained $= 8\pi$.

Find the surface area of the solid obtained by rotating the region under curve $y=\sqrt{x}$ from 0 to 1 about x-axis.

Ans Solution:

Given,

$$y=\sqrt{x} \quad \Rightarrow y' = \frac{1}{2y}$$

Area of Surface = $\int_0^1 2\pi y\sqrt{1+(y')^2}\,dx$

$= 2\pi \int_0^1 \cancel{y}\dfrac{\sqrt{4y^2+1}}{\cancel{y}}\,dx$

$= 2\pi \int_0^1 \sqrt{4x+1}\,dx$

$= \dfrac{2\pi}{4}\left[\dfrac{(4x+1)^{3/2}}{3/2}\right]_0^1$

$= \dfrac{\pi}{6}(5\sqrt{5}-1)$

$\therefore$ Area of surface generated $= \dfrac{\pi}{6}(5\sqrt{5}-1)$

Find the surface area obtained by rotating the curve $y=\cos x,\ 0\le x\le \dfrac{\pi}{3}$ about x-axis.

Ans Solution:

Given,

$y=\cos x \quad \Rightarrow y' = -\sin x$

Now

Area of surface generated $= \int_0^{\pi/3} 2\pi y\sqrt{1+(y')^2}\,dx$

$= 2\pi \int_0^{\pi/3} y\sqrt{1+\sin^2(x)}\,dx$

$= 2\pi \int_0^{1} \cos x\sqrt{1+\sin^2(x)}\,dx$

$= 2\pi \int_0^{\sqrt{3}/2} \sqrt{1+y^2}\,dx$

$$= \frac{\sqrt{21}}{4}\pi + \pi \ln\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$$

$\therefore$ Area of surface generated $= \frac{\sqrt{21}}{4}\pi + \pi \ln\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$

Find the volume of the ellipsoid formed by the revolution of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the X-axis.

Ans Solution:

Given,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Now,

$$\text{Volume} = 2\int_0^a \pi\left(1-\frac{x^2}{a^2}\right).\, b^2\, dx$$

$$= \frac{4\pi b^2}{2a^2}\int_0^a a^2-x^2\, dx$$

$$= \frac{4\pi b^2}{2a^2}\left(a^3-\frac{a^3}{3}\right)$$

$$= \frac{4\pi b^2}{2a^2}\,\frac{2a^3}{3}$$

$$= \frac{4}{3}\pi ab^2$$

∴ The volume of the ellipsoid formed by the revolution of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the X-axis is $\frac{4}{3}\pi ab^2$.

Prove that the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$.

Ans Solution:

A sphere is formed by the revolution of a circle.

We have,

for circle; $x^2+y^2=a^2$.

Now,

Volume of sphere $= 2\int_0^a \pi y^2 dx$

$$= 2\int_0^a \pi(a^2-x^2)dx$$

$$= 2\pi \; a^3 - \frac{a^3}{3}$$

$$= \frac{4\pi a^3}{3}$$

$\therefore$ Volume of a sphere of radius a is $\frac{4}{3}\pi a^3$.

An arc of a parabola is bounded at both ends by the latus rectum of length 4a. Find the volume generated when the arc is rotated about the latus rectum.

Ans Solution:

For parabola:

$$y^2=4ax$$

Now,

Volume $= \int_0^a \pi y^2 dx$

$= \pi \int_0^a 4ax\,dx$

$= \pi \cdot 4a \cdot \frac{a^2}{2}$

$= 2\pi a^3$

$\therefore$ Volume $= 2\pi a^3$.

Find the volume of the solid generated by revolving the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ about the axis of x.

Ans Solution:

Given,

$$x^{2/3}+y^{2/3}=a^{2/3}$$

$$\Rightarrow y^{2/3} = a^{2/3}-x^{2/3}$$

$$\Rightarrow y^2 = (a^{2/3}-x^{2/3})^3$$

Now,

Volume of the solid $= 2\int_0^a \pi y^2 dx$

$= 2\int_0^a \pi (a^{2/3}-x^{2/3})^3 dx$

$= 2\pi \int_0^a a^2-x^2-3a^{2/3}x^{2/3}(a^{2/3}-x^{2/3})dx$

$$= 2\pi\left[a^3 - \frac{a^3}{3} - 3a^{2/3}.a^{2/3}.\frac{a^{4/3+1}}{2/3+1} + 3a^{2/3}\frac{a^{4/3+1}}{4/3+1}\right]$$

$$= 2\pi\left[\frac{3a^3 - a^3}{3} - \frac{9a^3}{5} + \frac{9a^3}{7}\right]$$

$$= 2\pi\left[\frac{7(2a^3) + 27a^3}{21} - \frac{9a^3}{5}\right]$$

$$= 2\pi\left[\frac{16}{105}a^3\right]$$

$$= \frac{32}{105}\pi a^3$$

$\therefore$ Volume of the solid $= \frac{32}{105}\pi a^3$.

The part of the parabola $y^2 = 4ax$ bounded by the latus rectum revolves about the tangent at the vertex. Find the volume of the solid thus generated.

Ans Solution:

Given,

$$y^2 = 4ax$$

$$\Rightarrow \frac{y^4}{16a^2} = x^2$$

Now,

volume of solid $= 2\int_0^{2a} \pi \frac{y^4}{16a^2}\, dy$

$$= \frac{1}{8a^2} \frac{\pi(2a)^5}{5}$$

$$= \frac{4}{5}\pi a^3$$

$\therefore$ Volume of the solid $= \frac{4}{5}\pi a^3$.

Find the volume of the solid formed by the revolution of the cardioid $r = a(1+\cos\theta)$ about the initial line.

Ans Solution:

Given,

$r = a(1+\cos\theta)$

$\Rightarrow y = a\sin\theta + a\sin\theta\cos\theta$

$\Rightarrow dy = -a\cos\theta \mp a.\frac{1}{2}.2\cos2\theta$

$\Rightarrow dy = (-a\cos\theta \mp a\cos2\theta)$

When $y = -2a, \theta = 0$ and when $y = 0, \theta = \pi$

Now,

Volume of solid $= \int_0^{\pi} \pi (a\sin\theta + a\sin2\theta)^2 \cdot (a\cos2\theta - a\cos\theta)\, d\theta$

$$= \pi \int_0^{\pi} a^2(\sin^2\theta + 2\sin\theta\sin2\theta + \sin^2 2\theta)\, a(\cos2\theta - \cos\theta)\, d\theta$$

$$= -\pi \int_0^{\pi} a^2(1+\cos\theta)^2.\sin^2\theta.a\sin\theta\,(1+2\cos\theta)\, d\theta$$

$$= -\pi \int_0^{\pi} a^2.2^2\cos^4\frac{\theta}{2}.2^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}.a.2.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,(1+2\cos\theta)\, d\theta$$

$$=\pi a^3 \int_0^{\pi} 2\cos^7\frac{\theta}{2} \cdot 8\sin^3\frac{\theta}{2}\left(2+2\cos\theta-1\right)d\theta$$

$$=\pi a^3 \int_0^{\pi} 16\cos^7\frac{\theta}{2}\,\sin^3\frac{\theta}{2}\left(4\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$=\pi a^3 \cdot \frac{40}{15}$$ [From β-F elimination function]

$$=\frac{8}{3}\pi a^3$$

$\therefore$ Volume of solid $=\frac{8}{3}\pi a^3$

Q.

Find the volume of the solids formed by the revolution of the curve $y^2=x^2(a-x)$ about the X-axis.

Ans Solution:

Given, Revolution by loop only.

$$y^2=x^2(a-x)$$

Now,

Volume of the solid $=\int_0^a \pi y^2\,dx$

$$=\pi\int_0^a x^2(a-x)\,dx$$

$$=\pi\left[\frac{a a^3}{3}-\frac{a^4}{4}\right]$$

$$= \pi \left[\frac{a^4}{3} - \frac{a^4}{4} \right]$$

$$= \frac{\pi a^4}{12}$$

$\therefore$ Volume of the solid formed (loop part only) is $\frac{\pi a^4}{12}$.

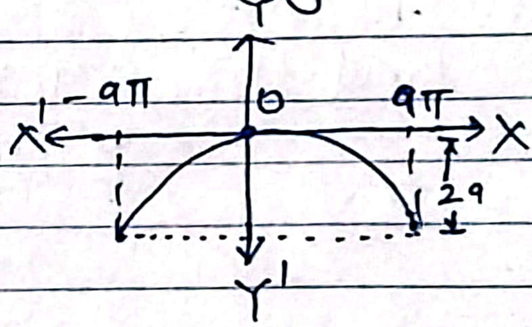
Find the volume of the solid formed by revolving the cycloid $x = a(\theta + \sin\theta), y = a(1+\cos\theta)$ about its base.

Ans Solution:

The tangent at the vertex of the cycloid is the x-axis.

So,

Volume of solid



$$= 2 \int_0^{a\pi} y^2 dx$$

$$= 2\pi \int_0^{a\pi} a^2(1+\cos\theta)^2 . a(1-\cos\theta) d\theta \text{ (Change to } \theta)$$

$$= 2\pi a^3 \int_0^{a\pi} 4\cos^4\frac{\theta}{2} . 2 . \sin^2\frac{\theta}{2} d\theta .$$

let $\frac{\theta}{2} = t \Rightarrow d\theta = 2t$

When $\theta \to 0, t \to 0$ and when $\theta \to a\pi, t \to \frac{\pi}{2}$. So,

$$\text{volume} = 16\pi a^3 \, 2\int_0^{\pi/2} \cos^4 t \sin^2 t \, dt$$

$$= 32\pi a^3 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2 \cdot \Gamma(4)}$$

$$= \pi^2 a^3$$

$\therefore$ Volume of the solid generated $= \pi^2 a^3$ ~~so~~ cubic units.

Model SET-2

GROUP-A.

1. Find the pedal equation of the reciprocal spiral

$$r = \frac{a}{\theta}$$

Ans

Solution:

Given reciprocal spiral is

$$r = \frac{a}{\theta}$$

$$\Rightarrow \frac{dr}{d\theta} = a\left(\frac{-1}{\theta^2}\right)$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{-\theta^2}{a}$$

$$\Rightarrow r\frac{d\theta}{dr} = r\cdot\left(\frac{-\theta^2}{a}\right) = -\theta$$

Also,

We have,

$$\cancel{p = to} \quad r\frac{d\theta}{dr} = \tan\phi \quad \text{— (A)}$$

$$r = \frac{a}{\theta}$$

$$p = r\sin\phi \quad \text{— (C)}$$

So, from (A)

$$-\theta = \tan\phi$$

$$\cancel{\Rightarrow \phi = \tan^{-1}(}$$

$$\Rightarrow \theta^2 = \tan^2\phi$$

$$\Rightarrow \sin^2\phi = \frac{p^2}{r^2} = \frac{\theta^2}{\theta^2+1}$$

So, From ©

$$p^2 = r^2 \sin^2\phi$$

$$\Rightarrow p^2 = r^2 \cdot \frac{\theta^2}{\theta^2+1}$$

$$\Rightarrow p^2 = r^2 \cdot \frac{\frac{a^2}{r^2}}{\frac{a^2}{r^2}+1}$$

$$\Rightarrow p^2 = \frac{a^2}{a^2+r^2}$$

$\therefore$ $p^2(a^2+r^2) - a^2 = 0$ is the required pedal equation.

2. Find the asymptotes of the curve
$x^2y^2 - 4(x^2-y^2) + 2y - 3 = 0$

Ans Solution:

Given,

$x^2y^2 - 4(x^2-y^2) + 2y - 3 = 0$

The given curve is polynomial of degree 4 and doesn't contain the term having x^4 and y^4. So, horizontal and vertical asymptotes are present.

For horizontal asymptote,

$\cancel{y^2 + 4y^2 = 0}$

$\cancel{\Rightarrow y^2\ 8y^2}$

$y^2 - 4 = 0$

$\Rightarrow y = \pm 2$

Also,

For vertical asymptotes,

$$x^2+4=0$$

(Not possible)

Hence, the required asymptotes are $y = \pm 2$.

3. Show that the radius of curvature at a point on $r = e^{\theta \cot\alpha}$ is $r \operatorname{cosec}\alpha$.

~~Ans~~ Solution:

Given,

$$r = e^{\theta\cot\alpha}$$

So,

$$r_1 = \frac{dr}{d\theta} = \cot\alpha \, e^{\theta\cot\alpha} = \cot\alpha . r$$

Also,

$$r_2 = \frac{d^2r}{d\theta^2} = \cancel{r.t}\cot\alpha\frac{dr}{d\theta} = \cot^2\alpha . r$$

Now,

$$\text{Radius of curvature } (\rho) = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+r\cot\alpha)^{3/2}}{\cot^2\alpha . r}$$

$$= \frac{(\operatorname{cosec}^2\alpha - \cot^2\alpha + r\cot\alpha)^{3/2}}{r.\cot^2\alpha}$$

Now,

$$\text{Radius of curvature } (\rho) = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+2r^2\cot^2\alpha - r.r.\cot^2\alpha}$$

$$= \frac{r^3\operatorname{cosec}^3\alpha}{r^2(1+\cot^2\alpha)}$$

$$= r\operatorname{cosec}\alpha$$

$\therefore$ Radius of curvature of the given curve is $r\operatorname{cosec}\alpha$.

4. Find the area of the loop of the curve $\cancel{y^2+r}$ $y^2 = x(x-1)^2$

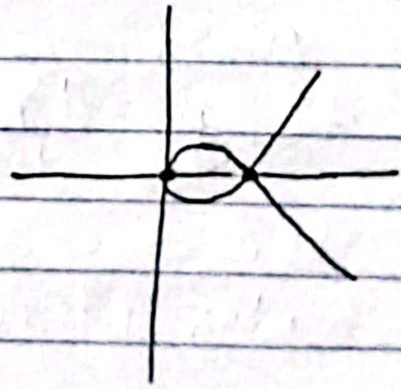
~~Ans~~ Solution:

Given,

$$y^2 = x(x-1)^2$$

For $y=0$; $x=0, 1$

So,



$$\text{Area of loop} = 2\int_0^1 y\,dx$$

$$= 2\int_0^1 \sqrt{x}\,(x-1)\,dx$$

$$= 2\left[\frac{x^{3/2+1}}{3/2+1} - \frac{x^{1/2+1}}{1/2+1}\right]_0^1$$

$$= 2\left[\frac{2}{5} - \frac{2}{3}\right]$$

$$= 2\left[\frac{6-10}{15}\right]$$

$$= \frac{2\times(-4)}{15}$$

$$= \frac{8}{15}$$ (Taking +ve sign only)

$\therefore$ Area of loop $= \frac{8}{15}$ sq. units.

5. Find the moment of inertia of semi-circular disc $x^2+y^2=a^2$, given that its density is ρ.

6. Find the arc length of catenary $y = a\cosh\left(\frac{x}{a}\right)$ from the vertex $(0, a)$ to any point (x, y)

Ans Solution:

Given,

$$y = a\cosh\left(\frac{x}{a}\right)$$

$$\Rightarrow \frac{dy}{dx} = a \times \frac{1}{a}\sinh\left(\frac{x}{a}\right) = \sinh\left(\frac{x}{a}\right)$$

Now,

~~A~~ Req. arc length $= \displaystyle\int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_0^x \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)}\, dx$$

$$= \int_0^x \cosh\left(\frac{x}{a}\right) dx$$

$$= \left[\sinh\left(\frac{x}{a}\right)\right]_0^x$$

$$= \sinh\left(\frac{x}{a}\right)$$

$\therefore$ Req. arc length $= \sinh\left(\frac{x}{a}\right)$

7. Solve the diff^r eq^n $p^2-(e^x+e^{-x})p+1=0$ when $p=\frac{dy}{dx}$

Ans Sol^n:

Given

$$p^2-(e^x+e^{-x})p+1=0$$

$$\Rightarrow \cancel{p^2-e^x-e^{-x}p+1=0}$$

$$\Rightarrow \cancel{0p^2-e^{-x}p-e^x+1=0}$$

$$\Rightarrow$$

$$p^2-e^xp-e^{-x}p+1=0$$

$$\Rightarrow p(p-e^x)-e^{-x}(p-e^x)=0$$

$$\Rightarrow (p-e^{-x})(p-e^x)=0$$

$\Rightarrow$ Either $p-e^{-x}=0$ OR $p-e^x=0$

$\Rightarrow p=+e^{-x}$ $\qquad \Rightarrow p=e^x$

$\Rightarrow y=\frac{e^{-x}}{-1}+c$ $\qquad \Rightarrow y=e^x+c$

$\therefore$ $(y+e^{-x}-c)(y-e^x-c)=0$ is the required general solution.

8. Solve the initial value problem

$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=0$ given that $y(0)=0$ and $\left(\frac{dy}{dx}\right)_{x=0}=1$

Ans Sol^n:

Given,

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=0$$

Its A·E is

$$m^2 + 6m + 9 = 0$$

$$\Rightarrow m = +3, +3$$

$\therefore y = (c_1 + c_2 x)e^{3x}$

Also,

$$y(0) = 0$$

$$\Rightarrow 0 = c_1$$

$$\therefore c_1 = 0$$

and

$$(y_1)_0 = 1$$

$$\Rightarrow \left(e^{3x}(c_2) + (c_1 + c_2 x)e^{3x}\right)_{x=0} = 1$$

$$\Rightarrow c_2 + c_1 = 1$$

$$\therefore c_2 = 1$$

$\therefore y = xe^{3x}$ is the reqn particular solution.

9. Reduce Cauchy's eqn $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log x$ into a linear diff eqn with constant coefficient.

Ans Solution:

Given,

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log(x)$$

Let $x = e^z \Rightarrow z = \log(x)$

Also,

Let ~~$\frac{dz}{dx}$~~ $\frac{dy}{dz} = \delta y$

Then,

$$x^2\frac{d^2y}{dx^2} = \delta(\delta-1)y$$

and $x\frac{dy}{dx} = \delta y$

So,

$$(\delta^2-\delta-(\delta-1)-3)y = e^{2z}.z$$

$$\Rightarrow (\delta^2-2\delta+1-3)y = e^{2z}.z$$

$$\Rightarrow (\delta^2-2\delta-2)y = e^{2z}.z$$

which is required linear diff eqn with constant coefficient.

10. Identify the polar conic $r = \frac{12}{3+2\cos\theta}$

Ans Solution:

Given,

$$r = \frac{12}{3+2\cos\theta}$$

$$\Rightarrow r = \frac{4}{1+\frac{2}{3}\cos\theta}$$

comp. with $r = \frac{de}{1+e\cos\theta}$, we get, $e = \frac{2}{3} < 1$

∴ The given conic is ellipse.

11. Reduce $x+y+z=0=4x+y-2z$ in symmetrical form

Ans Solution:

Given planes are

$$x+y+z=0$$
$$4x+y-2z=0$$

Let l, m, n be the d.r of given line. Then,

$$l+m+n=0$$
$$4l+m-2n=0$$

By the rule of cross multiplication:

$$\frac{l}{-2-1} = \frac{m}{-(-2-4)} = \frac{n}{1-4}$$

$$\Rightarrow \frac{l}{-3} = \frac{m}{+6} = \frac{n}{-3}$$

$$\Rightarrow \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

So, for point $(x, y, 0)$ on line

$$x+y=0$$
$$4x+y=0$$
$$\Rightarrow x=0$$
$$\text{and } y=0$$

∴ line passes through $(0,0,0)$

∴ line in symmetrical form is

$$\frac{x-0}{1} = \frac{y-0}{-2} = \frac{z-0}{1} = k.$$

12. Find the equation of the right circular cone whose vertex is at origin, the axis along x-axis and semi-vertical angle is α.

Ans Soln:

Let pt. on cone be (x,y,z).

Then,

~~$\cos\alpha =$~~

D.r. of line joining vertex and pt $= x, y, z$

D.r. of axis $= 1, 0, 0$.

So,

$$\cos\alpha = \frac{x+0+0}{\sqrt{1}\cdot\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \cos^2\alpha = \frac{x^2}{x^2+y^2+z^2}$$

$$\Rightarrow \cos^2\alpha\,(y^2+z^2) = x^2\sin^2\alpha$$

i.e. $y^2+z^2 = x^2\tan^2\alpha$ is the reqd. eqn of cone.

GROUP-B

13. A

Ans Solution:

Given,

$$f(x) = \ln(\sec x)$$

$$\Rightarrow f'(x) = \tan(x)$$

$$\Rightarrow f''(x) = \sec^2(x)$$

$$\Rightarrow f'''(x) = 2\sec^2 x\tan x$$

$$\Rightarrow f^{iv}(x) = 2\sec^4 x + 2\tan x \cdot 2\sec^2 x\tan x$$

$$= 2\sec^4 x + 4\sec^2 x\tan^2 x$$

$$= 2\sec^2 x(\sec^2 x + 2\tan^2 x)$$

$$= 2\sec^2 x(\sec^2 x + 2\sec^2 x - 2)$$
$$= 2\sec^2 x(3\sec^2 x - 2)$$
$$= 6\sec^4 x - 4\sec^2 x$$
$$\Rightarrow f^{v}(x) = 24\sec^4 x \tan x - 8\sec^2 x \tan(x)$$
$$\Rightarrow f^{vi}(x) = 24\sec^6 x + 24\tan x . 4\sec^4 x . \tan x - 8\sec^4 x - 8\tan x . 2\sec^2 x \tan x$$

and so on.

So,

$f(0) = 0, f'(0) = 0, f''(0) = 1, f'''(0) = 0, f^{iv}(0) = 2, f^{v}(0) = 0, f^{vi}(0) = 16$

and so on.

So,
By Maclaurin series expansion of $f(x)$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$\therefore \ln(\sec x) = \frac{x^2}{2!} + \frac{x^4}{4!} . (2) + \frac{x^6}{6!} (16) + \cdots$$

14.

<u>Ans</u> Solution:

$$\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$$

At $x = 0$, limit takes the form 1^∞.

let

$$y = \left(\frac{\tan x}{x}\right)^{1/x^2}$$

$$\Rightarrow \ln(y) = \frac{1}{x^2}\ln\left(\frac{\tan x}{x}\right)$$

Taking $\lim\limits_{x\to 0}$ on both sides, we get,

$$\lim_{x\to 0} \ln(y) = \lim_{x\to 0} \frac{1}{x^2}\ln\left(\frac{\tan x}{x}\right) \quad \left[\frac{0}{0} \text{ form}\right]$$

$$\Rightarrow \lim_{x\to 0} \ln(y) = \lim_{x\to 0} \frac{\ln\left(\frac{\tan x}{x}\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{x}{\tan x} \times \frac{x\sec^2 x - \tan x}{x^2}}{2x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} 1 \times \frac{x\sec^2 x - \tan x}{2x^3} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{x(2\sec^2 x \tan x) + \sec^2 x - \sec^2 x}{6x^2}$$

$$= \lim_{x\to 0} \frac{2x\sec^2 x \tan x}{6x^2}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\cancel{2}\sec^2 x \sec^2 x + \cancel{2}\tan x \cdot \cancel{2}\sec^2 x \tan x}{3x^2}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x}{6}$$

$$= \frac{1}{3}$$

$$\therefore \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{1/3}$$

Solution:

Given,

$$\int_0^\infty \frac{\tan^{-1}(\alpha x)}{x(1+x^2)}\,dx$$

let $I = \displaystyle\int_0^\infty \frac{\tan^{-1}(\alpha x)}{x(1+x^2)}\,dx$

$$\Rightarrow \frac{dI}{d\alpha} = \int_0^\infty \frac{1}{x(1+x^2)} \cdot \frac{1}{1+(\alpha x)^2} \cdot x \, dx$$

$$\Rightarrow \frac{dI}{d\alpha} = \int_0^\infty \frac{1}{(1+x^2)(1+\alpha^2 x^2)}\,dx$$

$$\Rightarrow \frac{dI}{d\alpha} = \frac{1}{1-\alpha^2}\int_0^\infty \frac{1}{1+x^2} - \frac{\alpha^2}{1+\alpha^2 x^2}\,dx$$

$$\Rightarrow \frac{dI}{d\alpha} = \frac{1}{1-\alpha^2}\int_0^\infty \frac{1}{1+x^2} - \frac{\alpha^2}{1+\alpha^2 x^2}\,dx$$

$$\Rightarrow \frac{dI}{d\alpha} = \frac{1}{1-\alpha^2}\left[\tan^{-1}(x) - \frac{\alpha^2}{\alpha}\cdot\tan^{-1}(\alpha x)\right]_0^\infty$$

$$\Rightarrow \frac{dI}{d\alpha} = \frac{1}{1-\alpha^2}\left[\frac{\pi}{2} - 0 - \alpha\cdot\frac{\pi}{2}\right]$$

$$\Rightarrow \frac{dI}{d\alpha} = \frac{1}{1-\alpha^2}\,\frac{\pi}{2}\,[1-\alpha]$$

$$\Rightarrow \frac{dI}{d\alpha} = \frac{1}{1+\alpha}\,\frac{\pi}{2}$$

$$\frac{\cancel{\alpha^2 x^2}+1-\alpha^2-\cancel{\alpha^2x^2}}{1-\alpha^2} \qquad \frac{1-\alpha^2}{1-\alpha^2}$$

$\Rightarrow I = \int \frac{1}{1+\alpha} \frac{\pi}{2} d$

$\Rightarrow I = \frac{\pi}{2} \ln(1+\alpha) + c$

We have,

when $\alpha = 0, I = 0$

So,

$0 = \frac{\pi}{2} \ln(1) + c$

$\therefore c = 0$

$\therefore I = \frac{\pi}{2} \ln(1+\alpha)$

i.e. $\int_0^\infty \frac{\tan^{-1}(\alpha x)}{x(1+x^2)} dx = \frac{\pi}{2} \ln(1+\alpha)$

16.

Ans Solution:

$$\int_0^{\pi/6} \cos^2 6\theta \sin^4 3\theta \, d\theta$$

$$= \int_0^{\pi/6} (2\cos^2 3\theta - 1)^2 \sin^4 3\theta \, d\theta$$

$$= \int_0^{\pi/6} (4\cos^4 3\theta - 4\cos^2 3\theta + 1) \sin^4 3\theta \, d\theta$$

$$= \int_0^{\pi/6} (4\cos^4 3\theta \sin^4 3\theta - 4\cos^2 3\theta \sin^4 3\theta + \sin^4 3\theta)\, d\theta$$

let $3\theta = t$

$\Rightarrow 3\,d\theta = dt$

and when $\theta \to 0, t \to 0$

and when $\theta \to \frac{\pi}{6}, t \to \frac{\pi}{2}$

$$= \frac{1}{3}\int_0^{\pi/2} (4\cos^4 t \sin^4 t - 4\cos^2 t \sin^4 t + \sin^4 t)\, dt$$

$$= \frac{1}{3}\left[4\frac{\Gamma(\frac{5}{2})\Gamma(5/2)}{2\times\Gamma(10/2)} - 4\times\frac{\Gamma(3/2)\Gamma(5/2)}{2\times\Gamma(8/2)} + \frac{\Gamma(5/2)\Gamma(1/2)}{2\times\Gamma(6/2)}\right]$$

$$= \frac{1}{3}\left[\frac{4\times(3/2\times 1/2\times\sqrt{\pi})^2}{2\times 4!} - \frac{4\times(1/2\sqrt{\pi})\times 3/2\times 1/2\times\sqrt{\pi}}{2\times 3!} + \frac{3/2\times 1/2\sqrt{\pi}\times\sqrt{\pi}}{2\times 2!}\right]$$

$$= \frac{1}{3}\left[4\times\frac{9/4\times 1/4\times\pi\times 1}{2\times 24} - 4\times\frac{1/4\times 3/2\times\pi}{2\times 6} + \frac{3/2\times 1/2\,\pi}{2\times 2}\right]$$

$= \frac{1}{3}$

$$\therefore \int_0^{\pi/6} \cos^2 6\theta \sin^4 3\theta\, d\theta = \frac{7}{192}\pi$$

$$\int_{2}^{\infty} \frac{dx}{x\sqrt{x^2-4}}$$

At $x=2$, $f(x) = \frac{1}{x\sqrt{x^2-4}} = \infty$

So,

$$\int_{2}^{\infty} \frac{dx}{x\sqrt{x^2-4}} = \lim_{a\to 2^+} \int_{a}^{\infty} \frac{dx}{x\sqrt{x^2-4}}\, dx$$

$$= \lim_{a\to 2^+} \int_{a}^{\infty} \frac{\sqrt{x^2-4}}{x(x^2-4)}\, dx$$

$$= \lim_{a\to 2^+} \int_{a}^{\infty} \frac{x}{x^2\sqrt{x^2-4}}\, dx$$

let $x^2 = t$

$\Rightarrow 2x\,dx = dt$

When $x \to a$, $t \to \sqrt{a}$ and

when $x \to \infty$, $t \to \infty$

$$= \lim_{a\to 2^+} \frac{1}{2} \int_{\sqrt{a}}^{\infty} \frac{dt}{t\sqrt{t-4}}$$

$$= \frac{1}{2} \lim_{a\to 2^+} \int_{\sqrt{a}}^{\infty}$$

$$= \lim_{a \to 2^+} \int_a^{\infty} \frac{dx}{x\sqrt{x^2-4}}$$

~~let $x = 2\sin\theta$~~

~~$\Rightarrow dx = 2\cos\theta \, d\theta$~~

~~When $x \to a$, $\theta \to \sin$~~

let $x = 2\sec\theta$

$\Rightarrow dx = 2\sec\theta \tan\theta \, d\theta$

When $x \to a$, $\theta \to \sec^{-1}\left(\frac{a}{2}\right)$

$x \to \infty$, $\theta \to \sec^{-1}(\infty) = \frac{\pi}{2}$

$$= \lim_{a \to 2^+} \int_{\sec^{-1}\left(\frac{a}{2}\right)}^{\pi/2} \frac{2\sec\theta \tan\theta \, d\theta}{2\sec\theta \cdot 2 \cdot \tan\theta}$$

$$= \lim_{a \to 2^+} \frac{1}{2} \int_{\sec^{-1}(a/2)}^{\pi/2} d\theta$$

$$= \frac{1}{2} \lim_{a \to 2^+} \frac{\pi}{2} - \sec^{-1}\left(\frac{a}{2}\right)$$

$$= \frac{1}{2} \lim_{a \to 2^+} \operatorname{cosec}^{-1}\left(\frac{a}{2}\right)$$

$$= \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

$$\therefore \int_2^{\infty} \frac{dx}{x\sqrt{x^2-4}} = \frac{\pi}{4}$$

17. Solution:

Given,

$$(1+y^2)\,dx + (x - e^{\tan^{-1}(x)})\,dy = 0$$

$$\Rightarrow (1+y^2)\,dx = (e^{\tan^{-1}(x)} - x)\,dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+y^2}{e^{\tan^{-1}(x)} - x}$$

$$\Rightarrow \int \frac{1}{e^{\tan^{-1}(x)} - x}\,dx = \int \frac{1}{1+y^2}\,dy$$

$$\Rightarrow \qquad = \tan^{-1}(y) + c$$

Solution:

$$y = px + p(1-p) \quad \text{—— (A)}$$

$$\Rightarrow \frac{dy}{dx} = x\frac{dp}{dx} + p + (1-2p)\frac{dp}{dx}$$

$$\Rightarrow 0 = x\frac{dp}{dx} + (1-2p)\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}(1-2p+x) = 0$$

Either $\frac{dp}{dx} = 0$ OR $(1-2p+x) = 0$

$\Rightarrow p = c$ —— (1) $\qquad \Rightarrow 1-2p+x = 0$

$\Rightarrow 2p = 1+x$

$\Rightarrow p = \frac{1+x}{2}$ —— (2)

From (A) and (1)

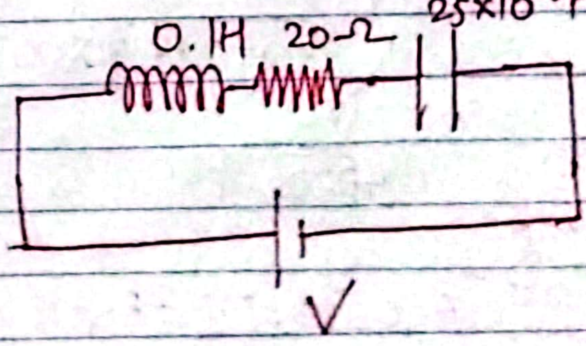
$y = cx + c(1-c)$ (general sol^n) and from (A) and (2)

$$y = \frac{x(x+1)}{2} + \left(\frac{x+1}{2}\right)\left(\frac{1-x}{2}\right)$$

(singular sol^n)

18.

Ans Solution:



Sol: Here,

$$V = iR + L\frac{di}{dt} + \frac{1}{C}q$$

$$\Rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V$$

Its A.E. is

$$Lm^2 + Rm + \frac{1}{C} = 0$$

$$\Rightarrow 0.1m^2 + 20m + \frac{1}{25\times10^{-6}} = 0$$

$$\Rightarrow \cancel{m = -200 \pm \sqrt{\frac{125}{10^8}}}$$

$$\therefore m = -100 + 625i$$

So,

$$C.F. = e^{-100t}[A\cos(625t) + B\sin(625t)]$$

Also,

$$P.I = \frac{1}{0.1D^2 + 20D + \frac{1}{25\times10^{-6}}} V$$

$$= \frac{10}{D^2 + 200D + 400000} V$$

$$= 10V \cdot \frac{1}{0+0+400000}$$

$$= \frac{V}{40000}$$

$$\therefore q = CF + PI$$

$$= e^{-100t}[A\cos(625t) + B\sin(625t)] + \frac{V}{40000}$$

Given,

At $t=0, q=0.05$

So,

$$0.05 = 1[A + B\times 0] + \frac{V}{40000}$$

$$\Rightarrow 0.05 - \frac{V}{40000} = A$$

$$\therefore A = \frac{200-V}{40000}$$

Also,

$$\left(\frac{dq}{dt}\right)_{t=0} = (I)_{t=0} = 0$$

So,

$$0 = 1[A+0]\cdot(-100) + 1[-0 + 625B] + 0$$

$$\Rightarrow 0 = -A\times 100 + 625B$$

$$\Rightarrow B = \frac{100}{625}A = \frac{100}{625}\times\frac{200-V}{40000}$$

$$\therefore B = \frac{200-V}{250000}$$

$$\therefore q = e^{-100t}\left[\frac{200-V}{40000}\cos(625t) + \frac{200-V}{250000}\sin(625t)\right] + \frac{V}{40000}$$

Solution:

$$r = \frac{10}{3+2\cos\theta}$$

$$\Rightarrow r = \frac{10/3}{1+2/3\cos\theta}$$

comp. with $r = \frac{de}{1+e\cos\theta}$

we get $e = \frac{2}{3} < 1$ (ellipse)

$de = 10/3$

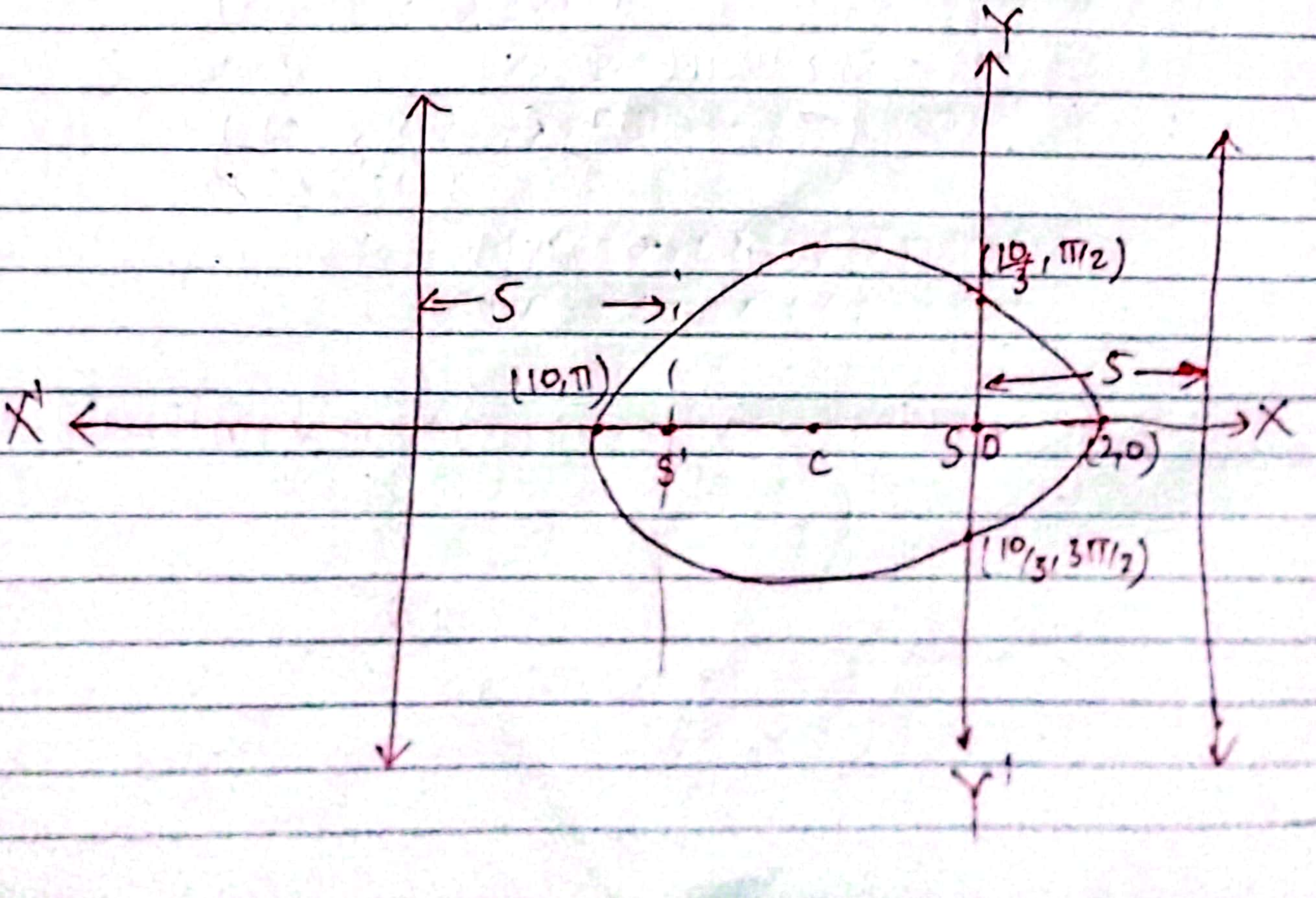
$\Rightarrow d \times 2/3 = 10/3$

$\therefore d = 5$

Now,

Table

θ	0	$\pi/2$	π	$3\pi/2$
r	2	$10/3$	10	$10/3$
(r,θ)	$(2,0)$	$(10/3, \pi/2)$	$(10, \pi)$	$(10/3, 3\pi/2)$



20.

Ans Soln:

Given lines are.

$$x = \frac{y-2}{2} = \frac{z+3}{3} \quad \text{and} \quad \frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$$

let l, m, n be d.c's of perpendicular line to both. Then,

$$l + 2m + 3n = 0$$

$$2l + 3m + 4n = 0$$

So,

$$\frac{l}{8-9} = \frac{m}{-(4-6)} = \frac{n}{(3-4)}$$

$$\Rightarrow \frac{l}{-1} = \frac{m}{2} = \frac{n}{-1} = \frac{1}{\sqrt{6}}$$

$$\Rightarrow l = -\frac{1}{\sqrt{6}},\ m = \frac{2}{\sqrt{6}},\ n = -\frac{1}{\sqrt{6}}$$

Now,

$$S.D = (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$$

$$= (0-2)\times\left(\frac{-1}{\sqrt{6}}\right) + (2-6)\times\frac{2}{\sqrt{6}} + (-3-3)\times\left(\frac{-1}{\sqrt{6}}\right)$$

$$= \cancel{1 - 4 + 1/3} = 0$$

$$= \frac{2 - 8 + 6}{\sqrt{6}}$$

Solution:

Given sphere is

$$x^2+y^2+z^2-8x+4y+8z-45=0$$

center $(-u,-v,-w) = (4,-2,-4)$

Radius $(r) = \sqrt{16+4+16+45}$

$= \sqrt{36+45}$

$= \sqrt{81}$

$= 9$

Eqn of plane is $x-2y+2z=3$

Line passing through cent. of sphere and $\perp^{r}$ to plane is

$$\frac{x-4}{1} = \frac{y+2}{-2} = \frac{z+4}{2}$$

let cent of circle be at r. Then,

$x = r+4$

$y = -2r-2$

$z = 2r-4$

So,

$r+4-2(-2r-2)+2(2r-4)=3$

$\Rightarrow r+4+4r+4+4r-8=3$

$\Rightarrow 9r=3$

$\therefore r = 1/3$

$\therefore \left(\frac{7}{3}, \frac{-8}{3}, \frac{-10}{3}\right)$ is center

$\frac{2}{3} - 4$

$= \frac{2-12}{3}$

$-\frac{2}{3}-2$

$= \frac{-2-6}{3}$

Also

$\perp^r$ from cent. of sphere to plane

$$= \left|\frac{1\times4+(-2)\times(-2)+2\times(-4)-3}{\sqrt{1^2+4+4}}\right|$$

$$= \left| \frac{4+4-8-3}{3} \right|$$

$$= 1 \text{ unit}$$

Radius of circle $= \sqrt{r_s^2 - 1^2}$

$= \sqrt{81 - 1}$

$= \sqrt{80}$

$= 4\sqrt{5}$ units.

GROUP-B

13

Ans Solution:

Given,

$$x^{2/3}+y^{2/3}=a^{2/3}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

Let (X,Y) be any point on the tangent line at point (x,y) on the given astroid curve. Then,

$$Y-y = \frac{-x^{-1/3}}{y^{-1/3}}(X-x)$$

$$\Rightarrow y^{-1/3}Y + x^{-1/3}X - y^{2/3} - x^{2/3} = 0$$

If 'p' be the ⊥r distance from origin to the tangent line. Then,

$$p^2 = \frac{(x^{2/3}+y^{2/3})^2}{(\sqrt{x^{-2/3}+y^{-2/3}})^2}$$

$$\Rightarrow p^2 = \frac{(a^{2/3})^2}{x^{-2/3}+y^{-2/3}}$$

$$\Rightarrow p^2 = \frac{a^{4/3}\cdot x^{2/3}y^{2/3}}{x^{2/3}+y^{2/3}}$$

$$\Rightarrow p^2 = a^{+2/3}x^{2/3}y^{2/3}$$

Now,

$$r^2 = x^2+y^2$$

So,

$$r^2+3p^2 = (x^2+y^2)+3a^{2/3}x^{2/3}y^{2/3}$$

$$= (x^{2/3})^3+(y^{2/3})^3+3(x^{2/3}+y^{2/3})x^{2/3}y^{2/3}$$

$$= (x^{2/3}+y^{2/3})^3 = (a^{2/3})^3 = a^2$$

$\therefore r^2+3p^2=a^2$ is the reqⁿ pedal eqⁿ of the curve.

14.
Ans Solⁿ:

Given,

$$(x+y)^2(x+2y+2)=x+9y-2$$

$$\Rightarrow (x^2+2xy+y^2)(x+2y+2)=x+9y-2$$

$$\Rightarrow x^3+2x^2y+xy^2+2x^2y+4xy^2+2y^3+2y^2+2x^2+4xy-x-9y-2=0$$

$$\Rightarrow x^3+2y^3+4x^2y+5xy^2+2x^2+2y^2+4xy-x-9y-2=0$$

which is polynomial eqⁿ of degree 3. and since both x^3 and y^3 is present. The curve has oblique asymptotes.

Here,

$$\phi_3(m)=1+2m^3+4m+5m^2$$

$$\phi_2(m)=2+2m^2+4m$$

$$\phi_1(m)=-1-9m$$

Now,

$$\phi_3'(m)=6m^2+10m+4$$

For the value of m,

$$\phi_3(m)=0$$

$$\Rightarrow 2m^3+5m^2+4m+1=0$$

$$\Rightarrow 2m^3+2m^2+3m^2+3m+m+1=0$$

$$\Rightarrow 2m^2(m+1)+3m(m+1)+1(m+1)=0$$

$$\Rightarrow (2m^2+3m+1)(m+1)=0$$

$$\Rightarrow (2m^2+2m+m+1)(m+1)=0$$

$\Rightarrow (2m(m+1)+1(m+1))(m+1)=0$

$\Rightarrow (2m+1)(m+1)(m+1)=0$

$\Rightarrow m=-\frac{1}{2}, -1, -1$ (Repeated).

Now,

For $m=-1/2$

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{2+2m^2+4m}{6m^2+10m+4}$$

$$= -\frac{2+2\times\frac{1}{4}+4\times(-1/2)}{6\times\frac{1}{4}+10\times\left(-\frac{1}{2}\right)+4}$$

$$= -\frac{2+1/2-2}{3/2-5+4}$$

$$= -\frac{1/2}{1/2}$$

$$= -1$$

$\therefore y=-\frac{1}{2}x-1$ is one of the asymptotes.

For repeated case,

$$\frac{c^2}{2!}\phi_3''(m)+\frac{c}{1!}\phi_2'(m)+\phi_1(m)=0$$

$$\Rightarrow \frac{c^2}{2}(12m+10)+\frac{c}{1}(4m+4)+(-1-9m)=0$$

$$\Rightarrow \frac{c^2}{2}(-12+10)+0-1+9=0$$

$$\Rightarrow c^2(-1) = -8$$

$$\therefore c=\pm 2\sqrt{2}$$

$\therefore y=-1x\pm 2\sqrt{2}$ are other two asymptotes.

Solution:

$$\text{let } I = \int_0^{\infty} \frac{e^{-kx}\sin(mx)}{x}\,dx$$

$$\Rightarrow \frac{dI}{dm} = \int_0^{\infty} \frac{\cancel{\sin(mx)}}{\cancel{x}} \cdot \frac{e^{-kx}}{x} \cdot \cos(mx) \cdot x \, dx$$

$$\Rightarrow \frac{dI}{dm} = \int_0^{\infty} e^{-kx}\cos(mx)\,dx$$

$$\Rightarrow \frac{dI}{dm} = \frac{e^{-kx}}{k^2+m^2}\left[\overset{\cos(mx)}{\cancel{a\sin(mx)}} + m\sin(mx)\right]$$

$$\Rightarrow \frac{dI}{dm} = \frac{e^{-kx}}{k^2+m^2}\left[a\cos(mx) + m\sin(mx)\right]$$

$$\Rightarrow I = \int \frac{e^{-kx}}{k^2+m^2}\left[a\cos(mx) + m\sin(mx)\right] dm$$

$$\Rightarrow I = e^{-kx}\left[\frac{a\int\cos(mx)\,dm + m\sin(mx)\,dm}{k^2+m^2}\right]$$

$$\Rightarrow \frac{dI}{dk} = \int_0^{\infty} \frac{\sin(mx)}{x} \cdot -x \cdot e^{-kx}\,dx$$

$$\Rightarrow \frac{dI}{dk} = -\int_0^{\infty} e^{-kx}\sin(mx)\,dx$$

$$\Rightarrow \frac{dI}{dk} = \left[\frac{-e^{-kx}}{k^2+m^2}\left[-k\sin(mx) + k\cos(mx)\right]\right]_0^{\infty}$$

$$\Rightarrow I = -\int \frac{e^{-kx}[k\cos(mx) - \cancel{k}\sin(mx)}{\cancel{k^2+m^2}}\, \cancel{dk}$$

$$\Rightarrow I = -\cos(mx) \cancel{\int \frac{e^{-kx}}{k^2+m^2}\, dk} + \cancel{\sin(mx) \int \frac{e^{-kx}}{k^2+m^2}\, dk}$$

$$\Rightarrow \frac{dI}{dk} = \left[\frac{1}{k^2+m^2} [-0 + k] \right]$$

$$\Rightarrow \frac{dI}{dk} = \frac{k}{k^2+m^2}$$

$$\Rightarrow I = \int \frac{k}{k^2+m^2}\, dk$$

$$\Rightarrow I = \frac{1}{2} \ln|k^2+m^2) + c$$

When $k \to \infty$, $I = 0$

$$\text{let } I = \int_0^\infty \frac{e^{-kx}\sin(mx)}{x}\,dx$$

$$\Rightarrow \frac{dI}{dm} = \int_0^\infty \frac{e^{-kx}}{x}\cdot x\cos(mx)\,dx$$

$$\Rightarrow \frac{dI}{dm} = \int_0^\infty e^{-kx}\cos(mx)\,dx$$

$$\Rightarrow \frac{dI}{dm} = \left[\frac{e^{-kx}}{k^2+m^2}\left[-k\cos(mx) - k\sin(mx)\right]\right]$$

$$\Rightarrow \frac{dI}{dm} = \left[0 - \left(\frac{1}{k^2+m^2}(-k-0)\right]\right.$$

$$\Rightarrow \frac{dI}{dm} = \frac{k}{k^2+m^2}$$

$$\Rightarrow I = k\int \frac{1}{k^2+m^2}\,dm$$

$$\Rightarrow I = \frac{k}{k}\tan^{-1}\left(\frac{m}{k}\right) + c$$

$$\Rightarrow I = \tan^{-1}\left(\frac{m}{k}\right) + c$$

When $m = 0$, $I = 0$

$\therefore c = 0$

$$\therefore I = \tan^{-1}\left(\frac{m}{k}\right)$$

$$\therefore \int_0^\infty e^{-kx} \frac{\sin(mx)}{x}\, dx = \tan^{-1}\left(\frac{m}{k}\right)$$

Now;
for $\int_0^\infty \frac{\sin(mx)}{x}\, dx$; $k=0$

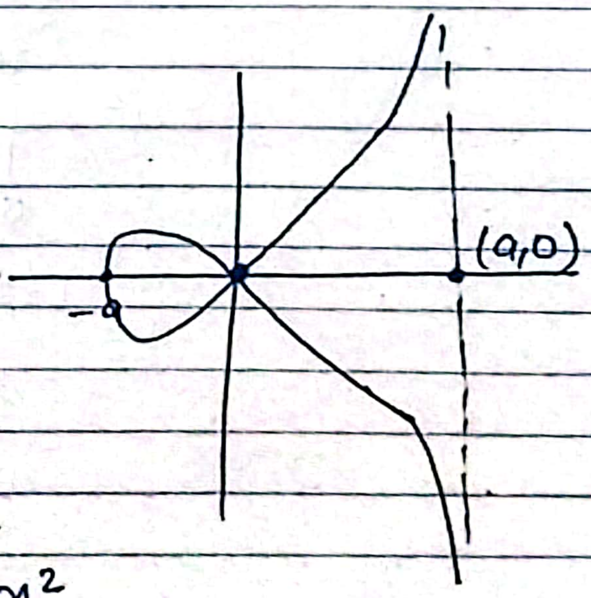
$$= \tan^{-1}\left(\frac{m}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{\sin(mx)}{x}\, dx = \frac{\pi}{2}$$

The given curve is $a(y^2-x^2) = x(x^2+y^2)$.
Given eqn has even power of y. So, the eqn is symmetrical about x-axis only.

So,
for $x=0$
$\therefore ay^2 = 0$
$y = 0$

For $y=0$
$-ax^2 = x^3$
$\Rightarrow -a = x$
$\therefore x = -a$



For $y \to \infty$
$ay^2 - ax^2 = x^3 + xy^2$
$\Rightarrow (a-x)y^2 = x^3 + ax^2$
$\Rightarrow y^2 = \frac{x^3+ax^2}{a-x}$; $x \to a$

Now,

$$\text{Area} = 2\int_0^a \sqrt{\frac{x^3+ax^2}{a-x}}\,dx$$

$$= 2\int_0^a x\sqrt{\frac{a+x}{a-x}}\,dx$$

Let $x = a\cos 2\theta$

$\Rightarrow dx = -2a\sin 2\theta\, d\theta$

When $x \to 0$, $\theta \to \pi/4$ and when $x \to a$, $\theta \to 0$

$$= 2\int_{\pi/4}^{0} a\cos 2\theta \sqrt{\frac{a+a\cos 2\theta}{a-a\cos 2\theta}}\; -2a\sin 2\theta\, d\theta$$

$$= 4a^2 \int_0^{\pi/4} \cos 2\theta \cdot \frac{\cos\theta}{\sin\theta} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 8a^2 \int_0^{\pi/4} (\cos^2\theta - \sin^2\theta)\, \cancel{\sin}\, \cos^2\theta\, d\theta$$

$$= 8a^2 \int_0^{\pi/4} \cos^4\theta - \sin^2\theta\cos^2\theta\, d\theta$$

Let $t = 2\theta$

$= \cancel{8}a^2$

$$= 4a^2 \int_0^{\pi/4} \cos 2\theta \cdot 2\cos^2\theta \, d\theta$$

$$= \frac{4a^2}{\cancel{2}} \int_0^{\pi/4} \cos 2\theta \, (1+\cos 2\theta) \, d\theta$$

let $2\theta = t$

$\Rightarrow 2d\theta = dt$

$$= 2a^2 \int_0^{\pi/2} \cos t \, (1+\cos t) \, dt$$

$$= 2a^2 \left\{ [\sin t]_0^{\pi/2} + \frac{\Gamma(1/2)\Gamma(3/2)}{2 \times \Gamma(4/2)} \right\}$$

$$= 2a^2 \left[1 + \frac{\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{2 \times 1} \right]$$

$$= 2a^2 \left[1 + \frac{\pi}{4} \right]$$

$$\cancel{= \frac{2a^2}{8} [8-\pi] \text{ sq. units}}$$

$$= \frac{2a^2}{4} [\pi + 4]$$

$$= \frac{a^2}{2} (\pi + 4) \text{ sq. units.}$$

Solution:

Given,

$$\frac{dy}{dx} - y\tan x = -y^2\sec x$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx}$$

Solution:

$$4yp^2 - 2px + y = 0$$

$$\Rightarrow 4yp^2 + y = 2px$$

$$\Rightarrow \cancel{x = \frac{4yp^2+y}{2p}}$$

$$\Rightarrow \cancel{4y}\ (4p^2+1)\,y - 2px = 0$$

$$\Rightarrow \cancel{y = \frac{2px}{4p^2+1}}$$

$$\Rightarrow \cancel{\frac{dy}{dx} = \frac{(4p^2+1)\cdot 2(x\,dp/dx + p)}{(4p^2+1)^2}}$$

$$4yp^2 + y = 2px$$

$$\Rightarrow x = \frac{4yp^2+y}{2p}$$

$$\Rightarrow x = 2py + \frac{y}{2p}$$

$$\Rightarrow \frac{dx}{dy} = 2p + y\cdot 2\frac{dp}{dy} + \frac{1}{2}\left(\frac{p + y\,d}{p^2}\right.$$

Solution:

Given,

$$4yp^2 - 2px + y = 0 \; ; \quad \text{— (A)}$$

~~Ans~~

$$\Rightarrow 2px = 4yp^2 + y$$

$$\Rightarrow 2x = 4py + \frac{y}{p}$$

$$\Rightarrow 2\frac{dx}{dy} = 4p + 4y\frac{dp}{dy} + y\left(-\frac{1}{p^2}\right)\frac{dp}{dy} + \frac{1}{p}\cancel{\frac{dy}{dy}}$$

$$\Rightarrow \frac{2}{p} = 4p + 4y\frac{dp}{dy} - \frac{y}{p^2}\frac{dp}{dy} + \frac{1}{p}\cancel{\frac{dp}{dy}}$$

$$\Rightarrow \frac{2}{p} - 4p - \frac{1}{p} = \left(4y - \frac{y}{p^2}\right)\frac{dp}{dy}$$

$$\Rightarrow \cancel{\frac{2-4p^2}{p} - \frac{1}{p}} = \left(\cancel{\frac{4p^2y - y + p}{p^2}}\right)\frac{dp}{dy}$$

$$\Rightarrow \cancel{\frac{2(1-2p^2)}{p}} =$$

$$\Rightarrow \frac{1}{p} - 4p = \left(4y - \frac{y}{p^2}\right)\frac{dp}{dy}$$

$$\Rightarrow \frac{1-4p^2}{p} = y\frac{(4p^2-1)}{p^2}\frac{dp}{dy}$$

$$\Rightarrow -1 = \frac{y}{p}\frac{dp}{dy}$$

$$\Rightarrow -\frac{1}{y}dy = \frac{1}{p}dp$$

$$\Rightarrow py = c$$

$$\Rightarrow p = \frac{c}{y}$$

$\frac{dp}{dx}$

$$\Rightarrow \frac{-1(4p^2-1)}{} = y\frac{(4p^2-1)}{p}\frac{dp}{dy}$$

$$\Rightarrow (4p^2-1)\left(1+\frac{y}{p}\frac{dp}{dy}\right)=0$$

$\Rightarrow$ Either $4p^2-1=0$

$\Rightarrow p = \pm\frac{1}{2}$ — (1) OR $1+\frac{y}{p}\frac{dp}{dy}=0$

$$\Rightarrow py = c$$

$$\Rightarrow p = \frac{c}{y} \quad \text{— (2)}$$

From (1) and eqn (A)

$$4y\frac{c^2}{y^2} - 2\frac{c}{y}\cdot x + y = 0$$

$$\Rightarrow \frac{4c^2}{y} - \frac{2cx}{y} + y = 0$$

$$\Rightarrow 4c^2 - 2cx + y^2 = 0$$

From (2) and eqn (A)

$$4y\left(\frac{1}{4}\right) \pm 2\times\frac{1}{2}x + y = 0$$

$$\Rightarrow 2y \mp x = 0$$