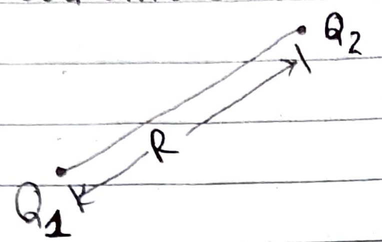
Chapter 2: Electric field (20-25 marks)

Coulomb's law:



Coulomb's law states:

"The force of interaction between two charges separated by a distance R which is greater than the dimension of either charges, is directly proportional to the product of charges and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the two charges."

$$F \propto Q_1 Q_2$$

$$F \propto \frac{1}{R^2}$$

On combining,

$$F \propto \frac{Q_1 Q_2}{R^2}$$

$$F = k\frac{Q_1 Q_2}{R^2}$$

where, $k = \frac{1}{4\pi\epsilon_0}$ is a proportionality constant.

where, ϵ_0 = permittivity of free space

$= 8.85 \times 10^{-12}$ F/m

$$\frac{1}{4\pi\epsilon_0} \approx 9 \times 10^9 \ Nm^2/c^2$$

In vector form,

Force exerted by Q_1 on Q_2,

$$\vec{F}_{Q_1Q_2} = \frac{Q_1 Q_2}{4\pi\epsilon_0 |R_{12}|^2} \cdot \hat{a}_{12}$$

Force exerted by Q_2 on Q_1,

$$\vec{F}_{Q_2Q_1} = \frac{Q_1 Q_2}{4\pi\epsilon_0 |R_{21}|^2} \cdot \hat{a}_{R21}$$

$$\boxed{\therefore \vec{F}_{Q_1Q_2} = -\vec{F}_{Q_2Q_1}}$$

Electric Field Intensity ($\vec{E}$):

→ measurement of the electric field at a point around a charge.

$$\vec{E} = \frac{Q_1}{4\pi\epsilon_0 |R_{1t}|^2} \cdot \hat{a}_{R1t}$$

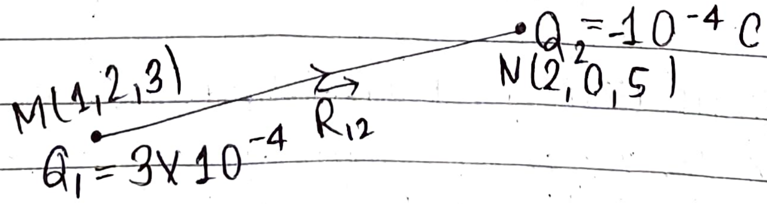
$Q_1 \xrightarrow{R_{1t}} Q_t$

[Fixed charge or source charge], [unit positive test charge]

Example 1:

If a charge of $Q_1 = 3 \times 10^{-4}$ C is at point M(1,2,3) and another charge $Q_2 = -10^{-4}$ C is at point N(2,0,5), find the force exerted on Q_2 by Q_1.

→ Sol^n:



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Here,

$$\vec{R_{12}} = (2-1)\hat{a}_x + (0-2)\hat{a}_y + (5-3)\hat{a}_z$$
$$= \hat{a}_x - 2\hat{a}_y + 2\hat{a}_z$$

$$\hat{a}_{12} = \frac{\vec{R_{12}}}{|\vec{R_{12}}|}$$

$$|\vec{R_{12}}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$$

$$\therefore \hat{a}_{12} = \frac{\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z}{3}$$

Now, force exerted on Q_2 by Q_1,

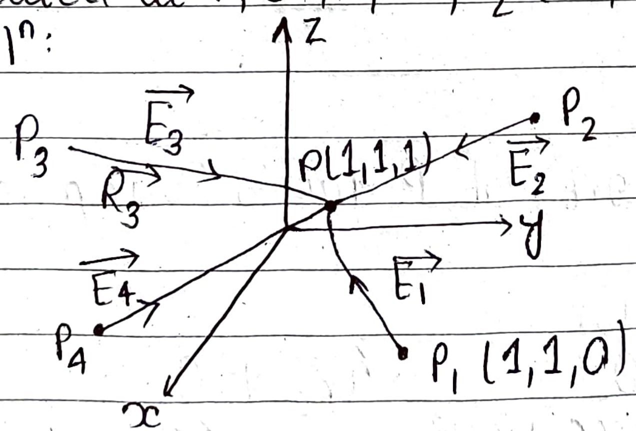
$$\vec{F_{Q_1Q_2}} = \frac{Q_1Q_2}{4\pi\epsilon_0|R_{12}|^2}\cdot\hat{a}_{12}$$

$$= \frac{-3\times10^{-4}\times10^{-4}}{4\pi\epsilon_0(3)^2}\times\frac{(\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z)}{3}$$

$$= -10\hat{a}_x + 20\hat{a}_y - 20\hat{a}_z \text{ N}$$

2) Find $\vec{E}$ at $P(1,1,1)$ caused by four identical 3nC charges located at $P_1(1,1,0)$, $P_2(-1,1,0)$, $P_3(-1,-1,0)$ and $P_4(1,-1,0)$

→ Soln:



Here, $\vec{R_1} = (1-1)\hat{a}_x + (1-1)\hat{a}_y + (1-0)\hat{a}_z$
$$= \hat{a}_z$$

$$\vec{R_2} = (1+1)\hat{a}_x + (1-1)\hat{a}_y + (1-0)\hat{a}_z$$
$$= 2\hat{a}_x + \hat{a}_z$$
$$\vec{R_3} = (1+1)\hat{a}_x + (1+1)\hat{a}_y + (1-0)\hat{a}_z$$
$$= 2\hat{a}_x + 2\hat{a}_y + \hat{a}_z$$
$$\vec{R_4} = (1-1)\hat{a}_x + (1+1)\hat{a}_y + (1-0)\hat{a}_z$$
$$= 2\hat{a}_y + \hat{a}_z$$

$|\vec{R_1}| = 1, \quad \hat{a}_{R_1} = \hat{a}_z$

$|\vec{R_2}| = \sqrt{2^2+1} = \sqrt{5}$

$$\hat{a}_{R_2} = \frac{2\hat{a}_x + \hat{a}_z}{\sqrt{5}}$$

$|\vec{R_3}| = \sqrt{2^2+2^2+1^2} = 3$

$$\hat{a}_{R_3} = \frac{2\hat{a}_x + 2\hat{a}_y + \hat{a}_z}{3}$$

$|\vec{R_4}| = \sqrt{2^2+1^2} = \sqrt{5}$

$$\therefore \hat{a}_{R_4} = \frac{2\hat{a}_y + \hat{a}_z}{\sqrt{5}}$$

The electric field intensity at P due to four point charges are :-

$$\vec{E} = \vec{E_1} + \vec{E_2} + \vec{E_3} + \vec{E_4}$$
$$= \frac{Q}{4\pi\epsilon_0 |R_1|^2} \cdot \hat{a}_{R_1} + \frac{Q}{4\pi\epsilon_0 |R_2|^2} \cdot \hat{a}_{R_2} + \frac{Q}{4\pi\epsilon_0 |R_3|^2} \cdot \hat{a}_{R_3}$$
$$+ \frac{Q}{4\pi\epsilon_0 |R_4|^2} \cdot \hat{a}_{R_4}$$

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$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{1}\hat{a_z} + \frac{1}{(\sqrt{5})^2}\cdot\frac{2\hat{a_x}+\hat{a_z}}{\sqrt{5}} + \frac{1}{3^2}\cdot\frac{2\hat{a_x}+2\hat{a_y}+\hat{a_z}}{3} + \frac{1}{(\sqrt{5})^2}\cdot\frac{2\hat{a_y}+\hat{a_z}}{\sqrt{5}}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{\hat{a_z}}{5} + \frac{2\hat{a_x}+\hat{a_z}}{5\sqrt{5}} + \frac{2\hat{a_x}+2\hat{a_y}+\hat{a_z}}{27} + \frac{2\hat{a_y}+\hat{a_z}}{5\sqrt{5}}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{27\hat{a_z}+10\hat{a_x}+10\hat{a_y}+5\hat{a_z}}{27\times5} + \frac{2\hat{a_x}+2\hat{a_y}+2\hat{a_z}}{5\sqrt{5}}\right]$$

$$= 3\times10^{-9}\times9\times10^{9}\left[\frac{301.86\,\hat{a_z}+111.80\,\hat{a_x}+111.80\,\hat{a_y}+55.90\,\hat{a_z}+270\,\hat{a_x}+270\,\hat{a_y}+270\,\hat{a_z}}{1509.34}\right]$$

$$= 3\times10^{-9}\times9\times10^{9}\left[\frac{627.76\,\hat{a_z}+381.8\,\hat{a_x}+381.8\,\hat{a_y}}{1509.34}\right]$$

$$= 0.017\left[627.76\,\hat{a_z}+381.8\,\hat{a_x}+381.8\,\hat{a_y}\right]$$

$$= 6.4906\,\hat{a_x} + 10.67\,\hat{a_z} + 6.4906\,\hat{a_y}\ \text{N/m}$$

Note:

<u>Differential Parameters</u>

i) Cartesian coordinates:

$$\vec{dl} = dx\,\hat{a_x} + dy\,\hat{a_y} + dz\,\hat{a_z}$$

$$\vec{ds} = dx\,dy\,\hat{a_z} = dy\,dz\,\hat{a_x} = dz\,dx\,\hat{a_y}$$

$$dV = dx\,dy\,dz$$

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ii) Cylindrical coordinates:

$$\vec{dl} = d\rho\,\hat{a}_\rho + \rho\,d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$$
$$\vec{dS} = \rho\,d\rho\,d\phi\,\hat{a}_z = \rho\,d\phi\,dz\,\hat{a}_\rho = d\rho\,dz\,\hat{a}_\phi$$
$$dV = \rho\,d\rho\,d\phi\,dz$$

iii) Spherical coordinates:

$$\vec{dl} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$$
$$\vec{dS} = r\,dr\,d\theta\,\hat{a}_\phi = r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r$$
$$= r\sin\theta\,dr\,d\phi\,\hat{a}_\theta$$
$$dv = r^2\sin\theta\,dr\,d\theta\,d\phi$$

* Charge densities:

a) Volume charge density (ρ_v)
$$\rho_v = \frac{dQ}{dv}$$
$$\therefore Q = \int_v \rho_v\,dv$$

b) Surface charge density (ρ_s)
$$\rho_s = \frac{dQ}{dS}$$
$$\therefore Q = \int_S \rho_s\,dS$$

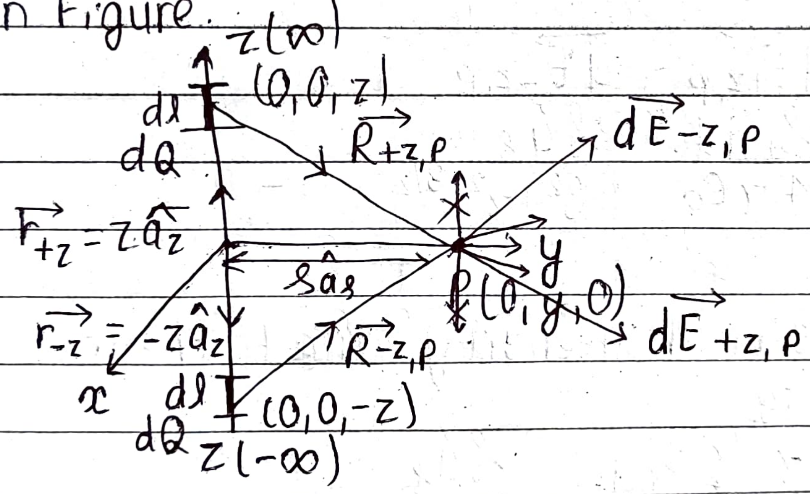
c) Line charge density (ρ_L):

$$\rho_L = \frac{dQ}{dl}$$

$$\therefore Q = \int \rho_L \, dl$$

Electric Field Intensity due to Continuous / Infinite Line Charge:

Let us assume a straight line charge extending along z-axis in the cylindrical co-ordinate system from $-\infty$ to ∞, as shown in figure.



We desire electric field intensity at any point P resulting from a uniform line charge of density ρ_L.

Then,
$$\vec{dE}_{+z,P} = \frac{dQ}{4\pi\epsilon_0 |R_{+z,P}|^2} \cdot \hat{a}_{+z,P}$$

$$= \frac{\rho_L \, dl}{4\pi\epsilon_0 |\vec{R}_{+z,P}|^2} \cdot \hat{a}_{+z,P}$$

Here,
$$\hat{a}_{+z,P} = \frac{\vec{R}_{+z,P}}{|\vec{R}_{+z,P}|}$$

$$= \frac{\vec{r}_P - \vec{r}_z}{|\vec{R}_{+z,P}|}$$

$$= \frac{\rho \hat{a}_\rho - z\hat{a}_z}{\sqrt{\rho^2 + z^2}}$$

$$\therefore \vec{dE}_{+z,p} = \frac{\rho_L dl}{4\pi\epsilon_0 (\sqrt{\rho^2+z^2})^2} \cdot \frac{\rho\hat{a}_\rho - z\hat{a}_z}{\sqrt{\rho^2+z^2}}$$

$$= \frac{\rho_L dl}{4\pi\epsilon_0} \cdot \frac{\rho\hat{a}_\rho - z\hat{a}_z}{(\rho^2+z^2)^{3/2}}$$

Similarly, $\quad \vec{R}_{-z,p} = \rho \vec{a}_\rho + z\hat{a}_z$

$$\vec{dE}_{-z,p} = \frac{\rho_L dl}{4\pi\epsilon_0} \cdot \frac{\rho\hat{a}_\rho + z\hat{a}_z}{(\rho^2+z^2)^{3/2}}$$

Now,

$$\vec{dE} = \vec{dE}_{+z,p} + \vec{dE}_{-z,p}$$

$$= \frac{2\rho_L dl}{4\pi\epsilon_0} \cdot \frac{\rho\hat{a}_\rho}{(\rho^2+z^2)^{3/2}}$$

And electric field intensity due to the infinite line charge is,

$$\vec{E} = \int_0^\infty \frac{2\rho_L dL}{4\pi\epsilon_0} \frac{\rho\hat{a}_\rho}{(\rho^2+z^2)^{3/2}}$$

$$= \frac{\rho_L \hat{a}_\rho}{2\pi\epsilon_0} \int_0^\infty \frac{\rho\, dl}{(\rho^2+z^2)^{3/2}}$$

Here, $dl = dz$,

so, $$\vec{E} = \frac{\rho_L \hat{a}_\rho}{2\pi\epsilon_0} \int_0^\infty \frac{\rho\, dz}{(\rho^2+z^2)^{3/2}}$$

Let ~~$\rho = z\tan\theta$~~ $z = \rho \tan\theta$

$dz = \rho \sec^2\theta \, d\theta$

when $z \to 0$, $\theta \to 0$

$z \to \infty$, $\theta \to \frac{\pi}{2}$

Now,

$$\vec{E} = \frac{\rho_L \hat{a}_\rho}{2\pi\epsilon_0} \int_0^{\pi/2} \frac{\rho \cdot \rho \sec^2\theta \, d\theta}{(\rho^2 \sec^2\theta)^{3/2}}$$

$$= \frac{\rho_L \hat{a}_\rho}{2\pi\epsilon_0} \int_0^{\pi/2} \frac{\rho^2 \sec^2\theta \, d\theta}{\rho^3 \cdot \sec^3\theta}$$

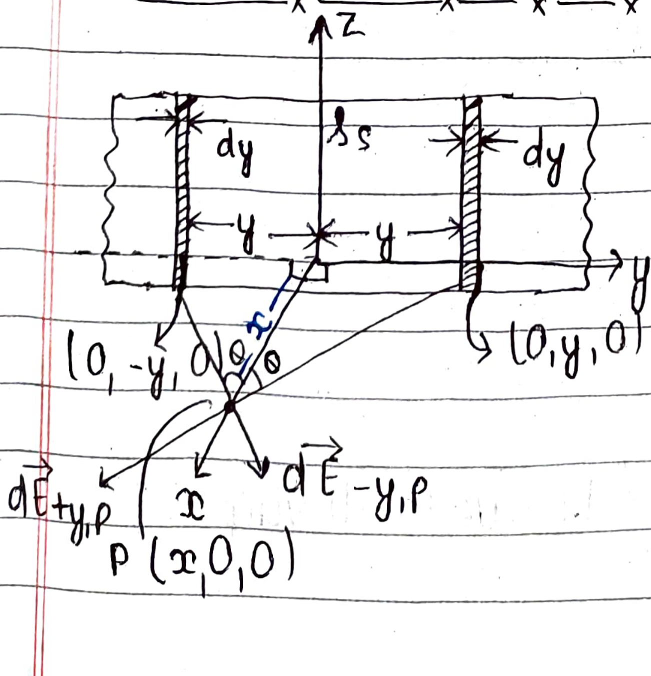
$$= \frac{\rho_L \hat{a}_\rho}{2\pi\epsilon_0} \int_0^{\pi/2} \frac{1}{\rho \sec\theta} \, d\theta$$

$$= \frac{\rho_L \hat{a}_\rho}{2\pi\epsilon_0 \rho} \cdot \int_0^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{\rho_L \hat{a}_\rho}{2\pi\epsilon_0 \rho} \left[\overset{\sin}{\cancel{\cos}} \frac{\pi}{2} - \overset{\sin}{\cancel{\cos}}\, 0 \right]$$

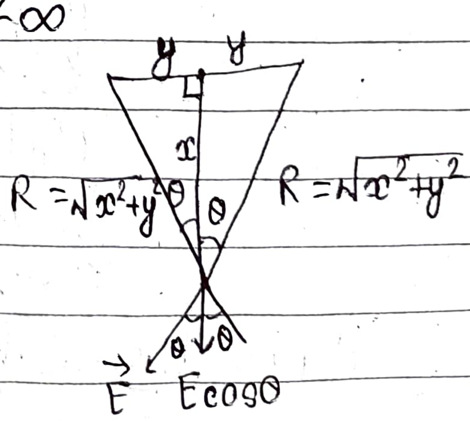
$$\boxed{\vec{E} = \frac{\rho_L \cancel{\hat{a}_\rho}}{2\pi\epsilon_0 \rho} \hat{a}_\rho}$$

Electric Field due to Continuous / Infinite Sheet of Charge:



$-\infty < y < \infty$

$-\infty < z < \infty$



Let us consider an infinite sheet of charge having surface charge density ρ_s in the yz-plane.

Let us divide the infinite sheet of charge into numerous differential strips of width dy ($dy \to 0$). Each strips behaves as a line charge having line charge density.

$$\rho_L = \rho_s \cdot dy$$

Two such differential strips are shown in the figure. Due to symmetry, only the field along the x-axis exist at the point P where we desire to calculate the electric field intensity.

The contribution to E_x at point P from the differential width strips at $(0, y, 0)$ and $(0, -y, 0)$ is

$$dE_x = dE\cos\theta + dE\cos\theta$$

$$= 2\,dE\cos\theta$$

$$= 2 \cdot \frac{\rho_L}{2\pi\epsilon_0 R} \frac{x}{R}$$

$$dE_x = \frac{\rho_L}{\pi\epsilon_0 R^2} \cdot x$$

$$= \frac{\rho_L \cdot x}{\pi\epsilon_0 (x^2+y^2)}$$

$$= \frac{\rho_s\, x\, dy}{\pi\epsilon_0 (x^2+y^2)}$$

Hence, the total electric field due to all strips is,

$$E_x = \int_0^\infty dE_x$$

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$$= \frac{\rho_s}{\pi \epsilon_0} \int_0^\infty \frac{x}{x^2+y^2} \, dy$$

$$= \frac{\rho_s}{\pi \epsilon_0} \left[\tan^{-1} (y/x) \right]_0^\infty$$

$$= \frac{\rho_s}{\pi \epsilon_0} \left[\pi/2 - 0 \right]$$

$$\therefore E_x = \frac{\rho_s}{2 \epsilon_0}$$

In vector form,

$$\vec{E} = E_x \hat{a}_x$$

$$= \frac{\rho_s}{2 \epsilon_0} \hat{a}_x$$

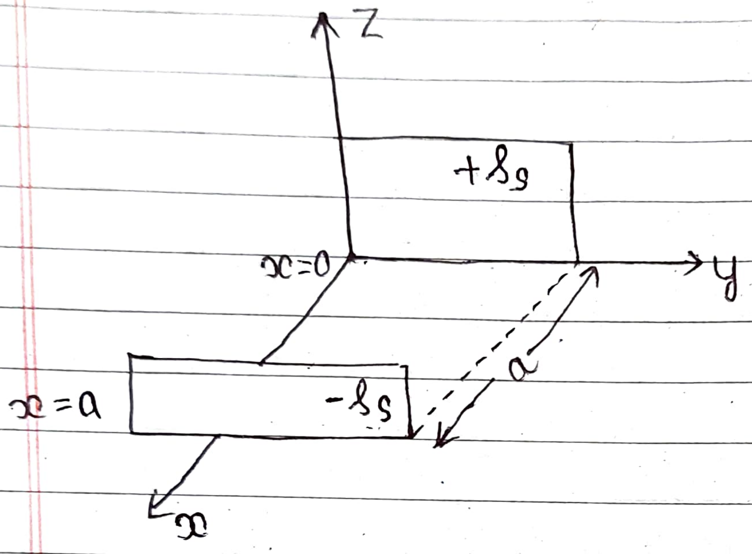
If a point in negative x-axis, then

$$\vec{E} = \frac{\rho_s}{2 \epsilon_0} (-\hat{a}_x) = -\frac{\rho_s}{2 \epsilon_0} \hat{a}_x$$

In general,

$$\vec{E} = \frac{\rho_s}{2 \epsilon_0} \hat{a}_N$$ [$\hat{a}_N$ = unit vector perpendicular to the plane of surface charge]

Field Due to a parallel plate capacitor :



Let us consider a uniform infinite sheet of charge having charge density ρ_s at $x=0$ and another infinite sheet of charge having surface charge density $-\rho_s$ at $x=a$ forming a parallel plate capacitor.

In the region $x>a$,

$$\vec{E}_{+\rho_s} = \frac{\rho_s}{2\epsilon_0}\hat{a}_x$$

$$\vec{E}_{-\rho_s} = -\frac{\rho_s}{2\epsilon_0}\hat{a}_x$$

$$\therefore \vec{E}_{x>a} = \vec{E}_{+\rho_s} + \vec{E}_{-\rho_s}$$
$$= 0$$

In the region $x<0$,

$$\vec{E}_{+\rho_s} = \frac{\rho_s}{2\epsilon_0}(-\hat{a}_x)$$
$$= -\frac{\rho_s}{2\epsilon_0}\hat{a}_x$$

$$\vec{E}_{-\rho_s} = \frac{-\rho_s}{2\epsilon_0}(-\hat{a_x})$$
$$= \frac{\rho_s}{2\epsilon_0}\hat{a_x}$$

$$\therefore \vec{E}_{x<0} = \vec{E}_{+\rho_s} + \vec{E}_{-\rho_s}$$
$$= 0$$

Hence no field exists outside the region of capacitor.

In the region: $0<x<a$

$$\vec{E}_{+\rho_s} = \frac{\rho_s}{2\epsilon_0}\hat{a_x}$$

$$\vec{E}_{-\rho_s} = \frac{-\rho_s}{2\epsilon_0}(-\hat{a_x}) = \frac{\rho_s}{2\epsilon_0}\hat{a_x}$$

$$\therefore \vec{E}_{0<x<a} = \vec{E}_{+\rho_s} + \vec{E}_{-\rho_s}$$

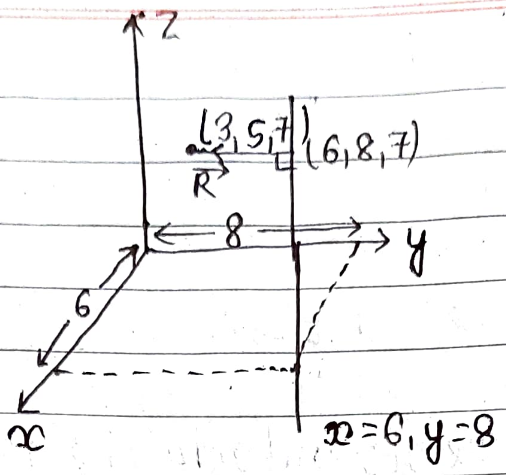
$$= \frac{2\rho_s}{2\epsilon_0}\hat{a_x}$$

$$= \frac{\rho_s}{\epsilon_0}\hat{a_x}$$

Numericals

1) Consider an infinite line charge having uniform charge density $\rho_L = -5\mu C/m$ placed at $x=6$ and $y=8$. Find the electric field intensity at the point $(3,5,7)$

→ Sol^n:



The electric field intensity due to a line charge is,

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\,\hat{a}_\rho$$

where, ρ = distance from the line to the point $= |\vec{R}|$

$$\vec{R} = (3-6)\hat{a}_x + (5-8)\hat{a}_y + (7-7)\hat{a}_z$$
$$= -3\hat{a}_x - 3\hat{a}_y$$

$$\hat{a}_\rho = \frac{\vec{R}}{|\vec{R}|} = \frac{-3\hat{a}_x - 3\hat{a}_y}{3\sqrt{2}} = -\frac{(\hat{a}_x + \hat{a}_y)}{\sqrt{2}}$$

$$\vec{E} = \frac{-5\times10^{-6}}{2\pi\times8.85\times10^{-12}\times3\sqrt{2}} \cdot -\left(\frac{\hat{a}_x + \hat{a}_y}{\sqrt{2}}\right)$$
$$= 14986.34\,(\hat{a}_x + \hat{a}_y)\ \text{V/m}$$

2) Infinite uniform line charges of 5nC/m lie along x-axis and y-axis. Find $\vec{E}$ at:

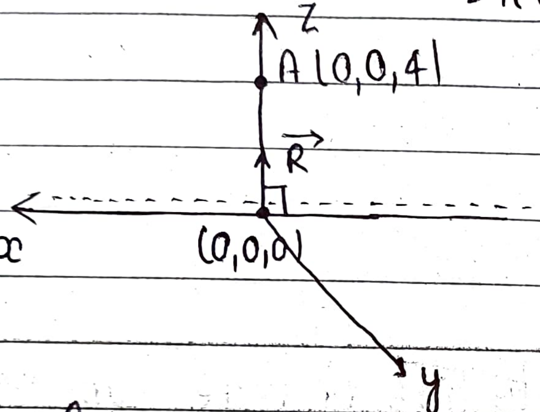
a) A(0,0,4) b) (0,3,4)

→ Sol^n:

a) A(0,0,4)

Electric field intensity due to line charge along X-axis at point A,

$$\vec{E}_{x\text{-axis}} = \frac{\rho_L}{2\pi\epsilon_0 \rho} \cdot \hat{a}_\rho$$



where,

$\rho = |\vec{R}| = \perp r$ distance bet^n line and point

$\vec{R} = 4\hat{a}_z \quad |\vec{R}| = 4$

$\therefore \rho = 4$

$$\hat{a}_\rho = \frac{\vec{R}}{|\vec{R}|} = \hat{a}_z$$

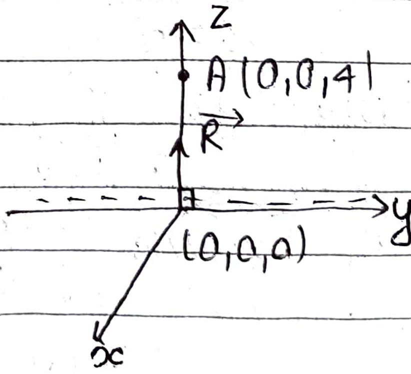
So, $\vec{E}_{x\text{-axis}} = \dfrac{5\times10^{-9}}{2\pi\epsilon_0 \cdot 4} \times \hat{a}_z$

$= \dfrac{5\times10^{-9}\times9\times10^{9}}{2}\hat{a}_z$

$= 22.5\,\hat{a}_z$ V/m

$\vec{E}_{y\text{-axis}} = ?$

Electric field intensity due to line charge along y-axis at point A,



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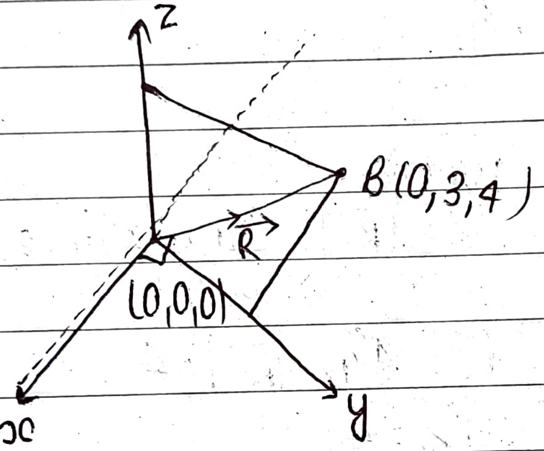
$\vec{E}_{y\text{-axis}} = \dfrac{5\times10^{-9}}{2\pi\epsilon_0 \cdot 4} \times \hat{a}_z$

$= 22.5\,\hat{a}_z$ V/m

$\therefore \vec{E_A} = \overrightarrow{E_{x\text{-axis}}} + \overrightarrow{E_{y\text{-axis}}}$
$= 45\,\hat{a}_z$ V/m

b) B (0,3,4)



Electric field intensity due to line charge along x-axis at point B,

$$\overrightarrow{E_{x\text{-axis}}} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\,\hat{a}_\rho$$

where, $\rho = |\vec{R}| = \perp r$ distance betⁿ line & point

$\vec{R} = 3\hat{a}_y + 4\hat{a}_z$

$|\vec{R}| = \sqrt{3^2+4^2} = 5$

$\rho = 5$

Now, $\hat{a}_\rho = \dfrac{\vec{R}}{|\vec{R}|} = \dfrac{3\hat{a}_y + 4\hat{a}_z}{5}$

So, $\overrightarrow{E_{x\text{-axis}}} = \dfrac{5\times10^{-9}}{2\pi\epsilon_0 \cdot 5} \cdot \dfrac{(3\hat{a}_y + 4\hat{a}_z)}{5}$

$= 10.8\,\hat{a}_y + 14.4\,\hat{a}_z$ V/m

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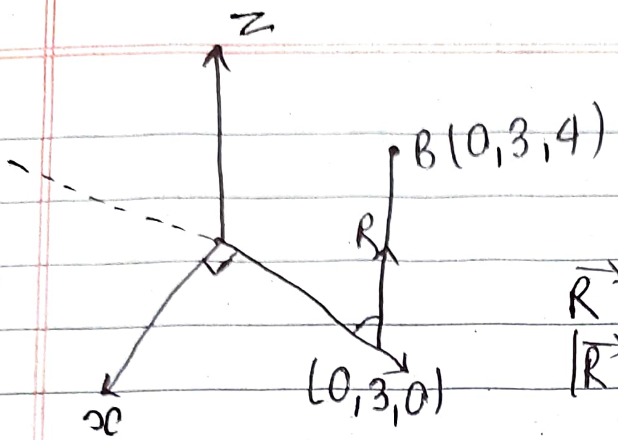


$\vec{R} = 4\hat{a}_z$

$|\vec{R}| = 4 = \rho$

$\hat{a}_\rho = \hat{a}_z$

$\vec{E}$ due to line charge along y-axis is,

$$\vec{E}_{y\text{-axis}} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\hat{a}_\rho$$

$$= \frac{5\times10^{-9}}{2\pi\epsilon_0 \times 4}\cdot\hat{a}_z$$

$$= 22.5\hat{a}_z \text{ V/m}$$

$$\vec{E_B} = \vec{E}_{y\text{-axis}} + \vec{E}_{x\text{-axis}}$$

$$= 10.8\hat{a}_y + 36.9\hat{a}_z \text{ V/m}$$

3) Three uniform sheet of charges are located as follows:

$3n\,C/m^2$ at $z=-4$

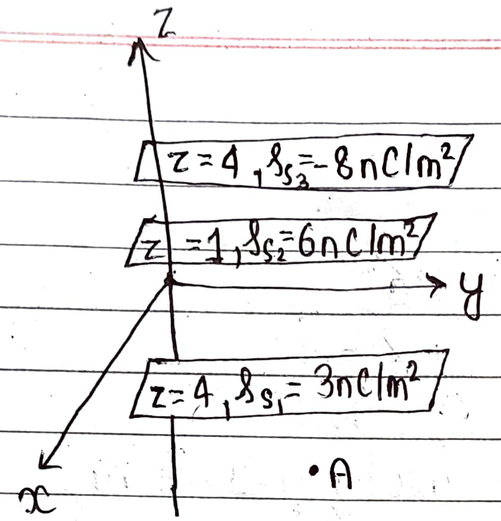
$6n\,C/m^2$ at $z=1$

$-8n\,C/m^2$ at $z=4$

Find $\vec{E}$ at points:

a) A (2,5,-5) b) B(4,2,-3)

c) C(-1,-5,2) d) D(-2,4,5)



a) A(2,5,-5)

$$\vec{E_A} = \frac{\rho_{s_1}}{2\epsilon_0}(-\hat{a}_z) + \frac{\rho_{s_2}}{2\epsilon_0}(-\hat{a}_z) + \frac{\rho_{s_3}}{2\epsilon_0}(-\hat{a}_z)$$

$$= \frac{-\hat{a}_z}{2\epsilon_0}\left[\cancel{8\times 1}\; 3\times 10^{-9} + 6\times 10^{-9} - 8\times 10^{-9}\right]$$

$$= \frac{-\hat{a}_z \times 10^{-9}}{2\times 8.85\times 10^{-12}}$$

$$= -56.49\,\hat{a}_z$$

b) B(4,2,-3)

$$\vec{E_B} = \frac{\rho_{s_1}}{2\epsilon_0}\hat{a}_z + \frac{\rho_{s_2}}{2\epsilon_0}(-\hat{a}_z) + \frac{\rho_{s_3}}{2\epsilon_0}(-\hat{a}_z)$$

$$= \frac{\hat{a}_z}{2\epsilon_0}\left[3\times 10^{-9} - 6\times 10^{-9} + 8\times 10^{-9}\right]$$

$$= \frac{5\times 10^{-9}\,\hat{a}_z}{2\times 8.85\times 10^{-12}}$$

$$= 282.48\,\hat{a}_z$$

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c) (-1, -5, 2)

4) Find electric field intensity at point P(1, 5, 2). if a point charge of $6\mu C$ is located at Q(0, 0, 1), a uniform line charge of 180 nC/m lies along x-axis and a uniform sheet of charge equal to $25\,nC/m^2$ lies in the plane $y=-2$.

→ Here, Electric field intensity at point P(1, 5, 2)

For point charge,

$$\vec{E_p} = \frac{Q}{4\pi\epsilon_0 R^2}\hat{a}_R = \frac{Q}{4\pi\epsilon_0 R^3}\vec{R}$$

$$\therefore \vec{R} = R\,\hat{a}_R$$

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Here, $\vec{R} = (1,5,2) - (0,0,1)$
$= (1,5,1)$

$|\vec{R}| = \sqrt{1^2+5^2+1^2} = \sqrt{27} = 3\sqrt{3}$

$\hat{a}_R = \dfrac{\hat{a}_x + 5\hat{a}_y + \hat{a}_z}{3\sqrt{3}}$

$\vec{E_p} = \dfrac{6\times10^{-9}\times9\times10^{9}}{(3\sqrt{3})^2} \times \left(\dfrac{\hat{a}_x + 5\hat{a}_y + \hat{a}_z}{3\sqrt{3}}\right)$

$= \dfrac{54}{(3\sqrt{3})^3} (\hat{a}_x + 5\hat{a}_y + \hat{a}_z)$

$= 0.385(\hat{a}_x + 5\hat{a}_y + \hat{a}_z)$

$= 0.385\hat{a}_x + 1.92\hat{a}_y + 0.385\hat{a}_z$

At x axis,
For line charge,
$\vec{R} = (1,5,2) - (1,0,0)$
$= (0,5,2)$

$|\vec{R}| = \sqrt{0^2+25+4} = \sqrt{29} = \rho$

$\vec{E_L} = \dfrac{\rho_L}{2\pi\epsilon_0\rho}\hat{a}_s$

$= \dfrac{2\times180\times10^{-9}\times9\times10^{9}}{(\sqrt{29})^2}(5\hat{a}_y + 2\hat{a}_z)$

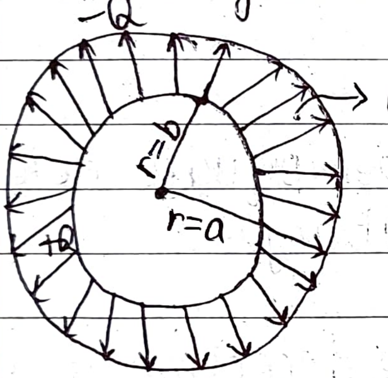
$= 558.62\hat{a}_y + 223.44\hat{a}_z$

For surface charge at $y = -2$,
We have,
$\vec{E_s} = \dfrac{\rho_s}{2\epsilon_0}\hat{a}_N = \dfrac{25\times10^{-9}}{2\times8.85\times10^{-12}}(\hat{a}_y)$

$= 1412.42\hat{a}_y$

$$\vec{E} = \vec{E_P} + \vec{E_L} + \vec{E_S}$$
$$= 0.385\hat{a_x} + 1.92\hat{a_y} + 0.385\hat{a_z} + 558.62\hat{a_y} + 223.44\hat{a_z} + 1412.42\hat{a_y}$$
$$= 0.385\hat{a_x} + 1972.96\hat{a_y} + 223.825\hat{a_z}$$

Concept of Electric flux:
(Finding of Faraday's Experiment)



-charge on outer sphere has equal magnitude and opposite sign as compared to charge at inner sphere regardless of the dielectric material separating the sphere

→ Faraday concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium and we now refer to this flux as displacement flux or simply electric flux.

→ If electric flux is denoted by Ψ and the total charge on the inner sphere by Q, then according to Faraday's experiment,

$$\boxed{\Psi = Q}$$

The unit of electric flux is Coulomb's.

→ Faraday's experiment also showed that a larger positive charges on the inner sphere induced a larger negative charge on the outersphere leading to direct proportionality between the electric flux and the charge on inner Sphere.

→ The constant of proportionality in SI unit is 1.

So, $\psi = Q$

Electric Flux Density ($\vec{D}$):

→ Sometime described as the "lines per square meter" For each line is due to 1 coulomb charge.

→ the direction of $\vec{D}$ at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

$$\vec{D} = \frac{\text{Flux passing through the surface area at that point}}{\text{area of the surface}}$$

$$\vec{D} = \frac{\psi}{S} \hat{a}_r$$

$$\vec{D} = \frac{Q}{S} \hat{a}_r$$

Flux density at a distance r from Q coulomb charge,

$$\vec{D} = \frac{Q}{4\pi r^2} \hat{a}_r \text{, where } S = 4\pi r^2 \text{ for sphere}$$

We know,
$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r$$

So, $\vec{E} = \frac{\vec{D}}{\epsilon_0}$

$\therefore \boxed{\vec{D} = \epsilon_0 \vec{E}}$ for free space

For any material media,
$$\vec{D} = \epsilon \vec{E}$$
$$= \epsilon_0 \epsilon_r \vec{E}$$
where, ϵ = permittivity of the medium
$\epsilon_r = \frac{\epsilon}{\epsilon_0}$ = relative permittivity of the medium.

The generalisation of Faraday's experiment led to the Gauss's law.

$\times \hat{a}_r$

Gauss's law (in integral form)

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.
i.e. $\Psi = \oint \vec{D} \cdot \vec{dS} = Q_{enclosed}$

(Criteria for Gaussian Surface)

Applications of Gauss's law

→ We may use Gauss's law to determine $\vec{D}$ or $\vec{E}$ if the charge distribution is known.

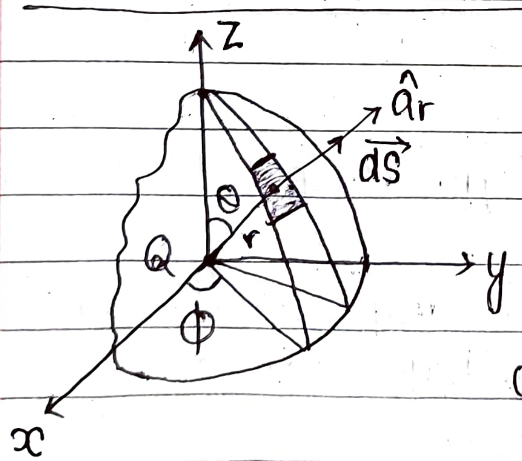
→ The solution is easy if we are able to chose a closed surface which satisfies two conditions:

i) $\vec{D_s}$ is everywhere either normal or tangential to the closed surface, so that $\vec{D_s} \cdot \vec{dS}$ becomes either $D_s dS$ or zero respectively.

$$\vec{D_s} \cdot \vec{dS} = D_s \, dS \cos\theta$$

ii) On that portion of that closed surface for which $\vec{D_s} \cdot \vec{dS}$ is not zero, D_s is constant.

1) Electric field Due to a Point Charge:



Let us consider a point charge Q at the origin.

We take spherical Gaussian surface centered at the origin and the radius of sphere is r.

Then, we have from Gauss's law,

$$Q = \oint \vec{D} \cdot \vec{dS} = \oint D \, dS \cos 0$$

$$= \oint D \, dS$$

$$= D \oint dS$$

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$$= D \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D r^2 \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sin\theta \, d\theta \, d\phi$$

$$= D r^2 \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= D r^2 \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= -D r^2 \left[-1-1\right] \left[2\pi - 0\right]$$

$$= D \, 4\pi r^2$$

$$\boxed{\therefore Q = 4\pi r^2 D}$$

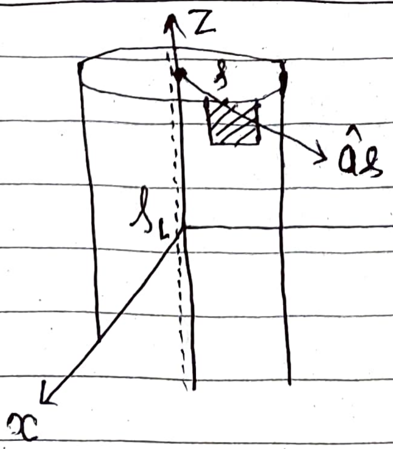
$$\therefore D = \frac{Q}{4\pi r^2}$$

In vector form,

$$\vec{D} = \frac{Q}{4\pi r^2} \hat{a}_r$$

and $\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{Q}{4\pi \epsilon_0 r^2} \hat{a}_r$

2) Electric Field Due to a Line Charge:



Let us consider a uniform line charge, ρ_L lying along the z-axis.
Here only the radial component of $\vec{D}$ is present

i.e. $\vec{D} = D_\rho \hat{a}_\rho$

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→ Let us consider a closed cylinder of radius ρ extending from $z=0$ to $z=L$.

Applying Gauss's law

$$Q = \oint_{cylinder} \vec{D}\cdot\vec{dS} = \int_{sides} \vec{D}\cdot\vec{dS} + \int_{top} \cancel{\vec{D}\cdot\vec{dS}}^{\,0} + \int_{bottom} \cancel{\vec{D}\cdot\vec{dS}}^{\,0}$$

$$= \int_{sides} \vec{D}\cdot\vec{dS}$$

$$= \int_{sides} D\, dS \cos 0^\circ$$

$$= D \int_{\phi=0}^{2\pi}\int_{z=0}^{L} \rho\, d\phi\, dz$$

$$= D\rho \int_0^{2\pi} d\phi \cdot \int_0^{L} dz$$

$$= D\rho\,(2\pi - 0)\cdot(L-0)$$

$$\therefore Q = D\, 2\pi\rho L$$

So, $D = D_\rho = \dfrac{Q}{2\pi\rho L}$

For a line charge,

$$Q = \rho_L \cdot L$$

$$\boxed{\therefore D_\rho = \frac{\rho_L L}{2\pi\rho L} = \frac{\rho_L}{2\pi\rho}}$$

In vector form,

$$\boxed{\vec{D} = D_\rho \hat{a}_\rho = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho}$$

$$\boxed{\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{\rho_L}{2\pi\epsilon_0\rho}\hat{a}_\rho}$$

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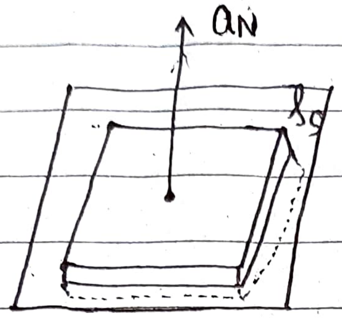
3) Electric Field Due to Sheet Charge:



Consider a plane sheet of charge with charge density ρ_s.
Consider the Gaussian surface as shown in the figure.
From Gauss's ~~surface~~ law,

$$Q = \oint_s \vec{D} \cdot \vec{ds} = \oint_{top} \vec{D} \cdot \vec{ds} + \oint_{bottom} \vec{D} \cdot \vec{ds} + \cancel{\oint_{sides} \vec{D} \cdot \vec{ds}}^{\,0}$$

$$= \int_{top} D\,ds \cos 0^\circ + \int_{bottom} D\,ds \cos 0^\circ$$

$$= D \int_{top} ds + D \int_{bottom} ds$$

$$= DS + DS$$

$\therefore Q = 2DS$ where $S \rightarrow$ area of the flat surface or sheet.

$$D = \frac{Q}{2S}$$

For a surface charge,

$$Q = \rho_s \cdot S$$

So, $D = \frac{\rho_s S}{2S}$

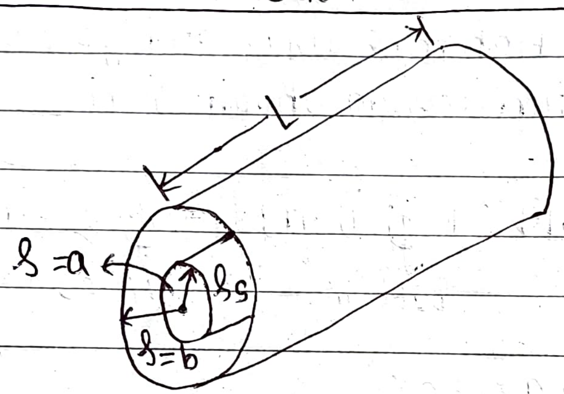
$$\therefore D = \frac{\rho_s}{2}$$

In vector form,

$$\boxed{\vec{D} = \frac{\rho_s}{2}\hat{a}_N}$$

and $$\boxed{\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{\rho_s}{2\epsilon_0}\hat{a}_N}$$

4) Electric Field Due to Coaxial Cable:



→ ρ_s charge on the outer surface of inner conductor.

→ Let us consider a Gaussian cylinder of length L and radius ρ, where $a<\rho<b$ is chosen. Then we have the total flux passing through the Gaussian surface.

$$\Psi = Q = D2\pi\rho L \quad \text{---(i)}$$

and the total charge on the inner conductor of length L

$$Q = \int_{z=0}^{L}\int_{\phi=0}^{2\pi} \rho_s \cdot a \, d\phi \, dz$$

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$Q = \rho_s \cdot$ Area of inner conductor

$$= \int_s \rho_s \cdot ds$$

$$= \int_{\phi=0}^{2\pi} \int_{z=0}^{L} \rho_s \, a \, d\phi \, dz$$

$$Q = \rho_s \, 2\pi a L \quad \text{---(ii)}$$

From eq^n (i) and (ii),

$$D \, 2\pi \rho L = \rho_s \, 2\pi a L$$

$$D = \frac{\rho_s}{\rho} \cdot a$$

In vector form,

$$\boxed{\vec{D} = \frac{\rho_s a}{\rho} \cdot \hat{a}_\rho}$$

and $\boxed{\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{\rho_s \cdot a}{\epsilon_0 \rho} \hat{a}_\rho}$

[$\because$ Since only radial component of field is present $\vec{D} = D_\rho \hat{a}_\rho$ and $\vec{E} = E_\rho \hat{a}_\rho$]

Note: i) For $\rho < a$, $\vec{D} = 0$ and $\vec{E} = 0$ — Since Gaussian surface doesnot enclose any charge

ii) For $\rho < b$, $\vec{D} = 0$ and $\vec{E} = 0$

Due to induction phenomenon, equal amount of charge but of opposite nature is developed on the outer conductor. So the net charge is 0 and the net charge enclosed by Gaussian surface is 0.

Q1 Find the total charge inside the volume indicated:

$\rho_v = 4xyz^2, \; 0 \le \rho \le 2, \; 0 \le \phi \le \frac{\pi}{2}, \; 0 \le z \le 3$

→ Solⁿ:

$$\rho_v = 4xyz^2$$
$$= 4(\rho\cos\phi)(\rho\sin\phi)z^2$$
$$= 4\rho^2\cos\phi\sin\phi z^2$$
$$= 2\rho^2\sin 2\phi\, z^2$$

Total charge inside the given volume,

$$Q = \int_v \rho_v dv$$
$$= \int_{\rho=0}^{2}\int_{\phi=0}^{\pi/2}\int_{z=0}^{3} (2\rho^2 \sin 2\phi z^2)(\rho d\rho d\phi dz)$$
$$= \int_{\rho=0}^{2}\int_{\phi=0}^{\pi/2}\int_{z=0}^{3} 2\rho^3 \sin 2\phi z^2 d\rho\, d\phi dz$$

$$Q = 2\int_{\rho=0}^{2}\rho^3 d\rho \int_{\phi=0}^{\pi/2}\sin 2\phi d\phi \int_{z=0}^{3} z^2 dz$$
$$= 2\left[\frac{\rho^4}{4}\right]_0^2 \left[-\frac{\cos 2\phi}{2}\right]_0^{\pi/2} \left[\frac{z^3}{3}\right]_0^3$$
$$= -1\left[\frac{2^4}{4} - 0\right]\left[\cos\pi - \cos 0\right]\left[\frac{3^3}{3} - 0\right]$$
$$= 72$$

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2.5 DIVERGENCE

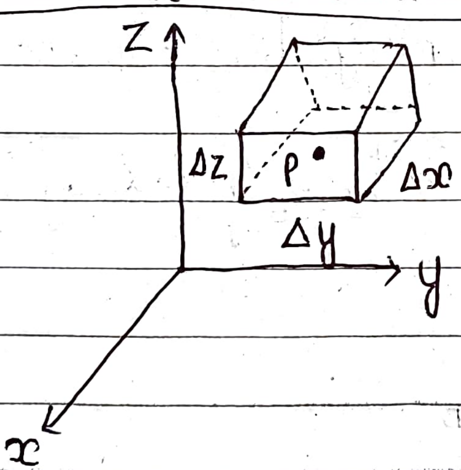
Divergence Concept:

Divergence is an operation which is performed on a vector but the result is scalar. It is the measurement of how much flux is leaving from a small volume on a per unit volume basis and no direction is associated with it.

So the divergence of the vector flux density $\vec{D}$ is the outflow of electric flux per unit volume as the volume shrinks to zero.

i.e. Divergence of $\vec{D}$ = Div $\vec{D}$ = $\lim_{\Delta V \to 0} \frac{\oint \vec{D} \cdot d\vec{S}}{\Delta V}$

→ Mathematical Formation of Divergence:



$P(x_0, y_0, z_0)$

Let us consider a point $P(x_0, y_0, z_0)$ within the volume ΔV as shown in figure.

We take electric flux density as a reference vector

$$\vec{D} = D_x \hat{a}_x + D_y \hat{a}_y + D_z \hat{a}_z$$

The flux passing through the front surface is,

$$\psi_{front} = [D_x]_{x = x_0 + \Delta x} \Delta y \, \Delta z$$

The flux passing through the back surface is,

$$\Psi_{back} = -[D_x]_{x=x_0 + \cancel{\Delta x}} \Delta y \, \Delta z$$

∴ Net flux passing through front and back surface

$$\Psi_{fb} = \Psi_{front} + \Psi_{back}$$

$$= \left\{ [D_x]_{x=x_0+\Delta x} - [D_x]_{x=x_0} \right\} \Delta y \, \Delta z$$

Similarly, net flux flowing through the left and right surface:

$$\Psi_{r_l} = \left\{ [D_y]_{y=y_0+\Delta y} - [D_y]_{y=y_0} \right\} \Delta x \, \Delta z$$

Similarly, net flux flowing through the top and bottom surface is,

$$\Psi_{tb} = \left\{ [D_z]_{z=z_0+\Delta z} - [D_z]_{z=z_0} \right\} \Delta x \, \Delta y$$

From Gauss's law,

$$\oint_S \vec{D} \cdot \vec{dS} = \Psi_{fb} + \Psi_{rl} + \Psi_{tb}$$

According to the definition of divergence,

$$\text{Div} \cdot \vec{D} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{D} \cdot \vec{dS}}{\Delta V}$$

$$= \Big[\lim_{\Delta V \to 0} \left\{ [D_x]_{x=x_0+\Delta x} - [D_x]_{x=x_0} \right\} \Delta y \, \Delta z$$

$$+ \lim_{\Delta V \to 0} \left\{ [D_y]_{y=y_0+\Delta y} - [D_y]_{y=y_0} \right\} \Delta y \, \Delta z$$

$$+ \lim_{\Delta V \to 0} \left\{ [D_z]_{z=z_0+\Delta z} - [D_z]_{z=z_0} \right\} \Delta x \, \Delta y \Big] \Big/ \Delta V$$

$$= \frac{\lim\limits_{\Delta x \to 0} \left\{ [D_x]_{x = x_0 + \Delta x} - [D_x]_{x = x_0} \right\} \cancel{\Delta y \Delta z}}{\Delta x \cancel{\Delta y \Delta z}}$$

$$+ \frac{\lim\limits_{\Delta y \to 0} \left\{ [D_y]_{y = y_0 + \Delta y} - [D_y]_{y = y_0} \right\} \cancel{\Delta x \Delta z}}{\cancel{\Delta x} \Delta y \cancel{\Delta z}}$$

$$+ \frac{\lim\limits_{\Delta z \to 0} \left\{ [D_z]_{z = z_0 + \Delta z} - [D_z]_{z = z_0} \right\} \cancel{\Delta x \Delta y}}{\cancel{\Delta x \Delta y} \Delta z}$$

$$= \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

- We know, the vector operator

$$\nabla = \frac{\delta}{\delta x} \hat{a}_x + \frac{\delta}{\delta y} \hat{a}_y + \frac{\delta}{\delta z} \hat{a}_z$$

Then,

$$\nabla \cdot \vec{D} = \left(\frac{\delta}{\delta x} \hat{a}_x + \frac{\delta}{\delta y} \hat{a}_y + \frac{\delta}{\delta z} \hat{a}_z \right) \cdot \left(D_x \hat{a}_x + D_y \hat{a}_y + D_z \hat{a}_z \right)$$

$$= \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

Hence, $\boxed{\mathrm{Div}\, \vec{D} = \nabla \cdot \vec{D}}$

Note*: $\nabla = \frac{\delta}{\delta x} \hat{a}_x + \frac{\delta}{\delta y} \hat{a}_y + \frac{\delta}{\delta z} \hat{a}_z$

$$\nabla \cdot \vec{D} = \frac{\delta}{\delta x}(D_x) + \frac{\delta}{\delta y}(D_y) + \frac{\delta}{\delta z}(D_z)$$

$$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z,$$

$$\nabla \cdot \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(D_\phi) + \frac{\partial}{\partial z}(D_z)$$

$$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi$$

$$\nabla \cdot \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(D_\phi)$$

Significance of Divergence of a Vector Field:

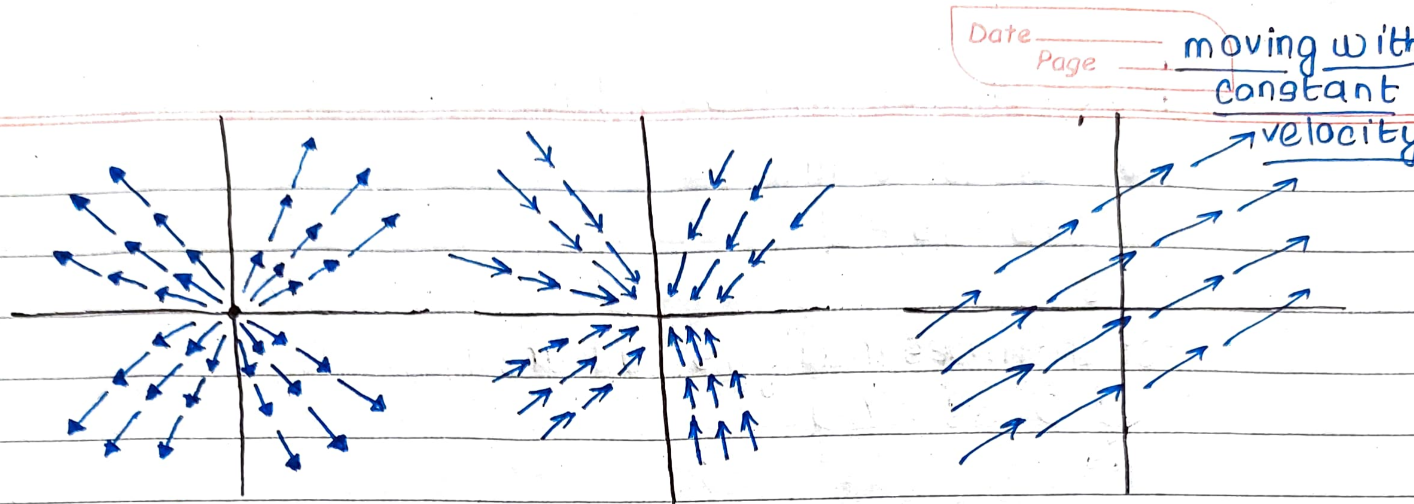
⇒ The Divergence actually measures how much the vector function is spreading.

⇒ Divergence of a Vector Field is a measure of how much a vector field converges to or diverges from a given point. In simple terms, it is a measure of the outgoingness of a vector field.

⇒ Divergence of a vector field is positive, if the vector diverges or spreads out from a given point called source.

⇒ Divergence of a vector field is negative, if the vector field converges at that point called sink.

⇒ If just as much of the vector field points in as an out, the divergence will be approximately zero.



Source — Sink — neither source or sink

VVI

Divergence Theorem:

The integral of dot product of flux density and surface element over a closed surface is equal to the integral of divergence of flux density throughout the volume enclosed by the surface.

i.e., $\oint_s \vec{D}\cdot d\vec{s} = \int_{vol} (\nabla\cdot\vec{D})\, dV$

Proof:

From Maxwell's 1^{st} eq^n in point form,

$\nabla\cdot\vec{D} = \rho_v$

From Gauss's law,

$\oint_s \vec{D}\cdot d\vec{s} = Q = \int_{vol} \rho_v\, dV = \int_{vol} (\nabla\cdot\vec{D})\, dv$

Now, $\oint_s \vec{D}\cdot d\vec{s} = \int_{vol} (\nabla\cdot\vec{D})\, dv$

Point form of Gauss's Law:

From Gauss's law,

$$\oint_S \vec{D} \cdot \vec{ds} = Q$$

For infinitesimally small area,

$$\oint_{\Delta S} \vec{D} \cdot \vec{ds} = \Delta Q$$

where ΔQ is the charge enclosed by the ΔS surface.
Dividing both sides by infinitesimally small volume ΔV and assuming the volume tends to zero.

$$\lim_{\Delta V \to 0} \frac{\oint_{\Delta S} \vec{D} \cdot \vec{ds}}{\Delta V} = \cancel{\Delta V} \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V}$$

$$\boxed{\nabla \cdot \vec{D} = \rho_v}$$

This is Maxwell's eq^n of 1^{st} form.

Gauss's law relates the flux leaving any closed surface to the charge enclosed and Maxwell's 1^{st} eq^n makes an identical statement on a per volume basis for a vanishingly small volume or at a point.

Numericals

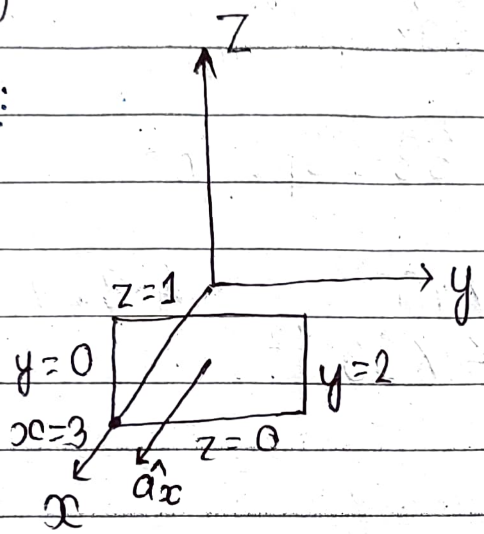
1) Let $\vec{D} = y^2z^3 \hat{a}_x + 2xyz^3 \hat{a}_y + 3xy^2z^2 \hat{a}_z$ pC/m^2 in free space.

a) Find the total electric flux passing through the surface $x=3$, $0 \leq y \leq 2$, $0 \leq z \leq 1$ in the direction away from the origin.

b) Find $|\vec{E}|$ at P(3,2,1)

c) Find the total charge contained in an incremental sphere having radius of 2 μm centered at P(3,2,1).

→ **Solution:**



From Gauss's law,

$$\psi = \oint_S \vec{D} \cdot \vec{dS} = Q_{enclosed}$$

Here, $\vec{D} = y^2 z^3 \hat{a}_x + 2xyz^3 \hat{a}_y + 3xy^2 z^2 \hat{a}_z \ pC/m^2$

i) For flux passing through the surface $x=3$, $0 \le y \le 2$, $0 \le z \le 1$,

$$\psi = \int_S \vec{D} \cdot \vec{dS} = \int_{y=0}^{2} \int_{z=0}^{1} (y^2 z^3 \hat{a}_x + 2xyz^3 \hat{a}_y + 3xy^2 z^2 \hat{a}_z) \cdot (dy\,dz\,\hat{a}_x)$$

$$\psi = \int_{y=0}^{2} \int_{z=0}^{1} y^2 z^3 \, dy\, dz$$

$$= \int_{y=0}^{2} y^2 dy \int_{z=0}^{1} z^3 dz$$

$$= \left[\frac{y^3}{3}\right]_0^2 \left[\frac{z^4}{4}\right]_0^1$$

$$= \frac{2^3}{3} \times \frac{1}{4}$$

$$= 0.67 \ pC$$

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ii) For $|\vec{E}|$ at P(3,2,1)
At P(3,2,1),

$$\vec{D_P} = y^2z^3\hat{a}_x + 2xyz^3\hat{a}_y + 3xy^2z^2\hat{a}_z \Big|_{x=3,y=2,z=1}$$

$$= 4\hat{a}_x + 12\hat{a}_y + 36\hat{a}_z \ pC/m^2$$

$$|\vec{D_P}| = \sqrt{4^2+12^2+36^2} = 38.16 pC/m^2$$

$$\therefore |\vec{E_P}| = \frac{|\vec{D_P}|}{\epsilon_0} = \frac{38.16\times10^{-12}}{8.854\times10^{-12}} = 4.31 V/m$$

iii) We know from Maxwell's 1st eqn,

$$\nabla\cdot\vec{D} = \rho_v$$

$$\text{or, } \rho_v = \nabla\cdot\vec{D}$$

$$= \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

$$= \frac{\delta(y^2z^3)}{\delta x} + \frac{\delta}{\delta y}(2xyz^3) + \frac{\delta}{\delta z}(3xy^2z^2)$$

$$= 2xz^3 + 6xy^2z \ pC/m^3$$

$$\rho_v \text{ at } P(3,2,1) = 2\times3\times1^3 + 6\times3\times2^2$$

$$= 78 \ pC/m^3$$

Total charge in the incremental sphere of $2\mu m$ radius centered at P(3,2,1) is

$$Q = \rho_v \times \text{volume of sphere}$$

$$= 78\times10^{-12}\times\frac{4}{3}\times\pi(2\times10^{-6})^3$$

$$= 2.61\times10^{-27} C$$

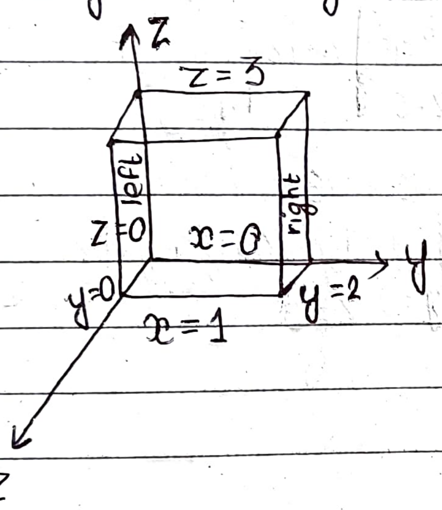
2) Evaluate both sides of divergence theorem for the field $\vec{D} = 2xy\,\hat{a}_x + x^2\hat{a}_y$ C/m² for volume formed by the planes $x = 0$ and 1, $y = 0$ and 2 and $z = 0$ and 3

→ Solution:

We have from Divergence theorem,

$$\oint_S \vec{D}\cdot d\vec{S} = \int_V \rho_v\, dv = \int_V (\nabla\cdot\vec{D})\, dv$$

& $\vec{D} = 2xy\,\hat{a}_x + x^2\hat{a}_y$ C/m²



Then evaluating the surface integral,

$$\oint_S \vec{D}\cdot d\vec{S} = \underbrace{\cancel{\int \vec{D}\cdot d\vec{S}}}_{\text{top}}^{0} + \underbrace{\cancel{\int \vec{D}\cdot d\vec{S}}}_{\text{bottom}}^{0} + \underbrace{\int \vec{D}\cdot d\vec{S}}_{\text{right}} + \underbrace{\int \vec{D}\cdot d\vec{S}}_{\text{left}}$$

$$+ \underbrace{\int \vec{D}\cdot d\vec{S}}_{\text{front}} + \underbrace{\int \vec{D}\cdot d\vec{S}}_{\text{back}}$$

right surface, $d\vec{S} = dx\, dz\, \hat{a}_y$
left surface, $d\vec{S} = dx\, dz\, (-\hat{a}_y)$
front surface, $d\vec{S} = dy\, dz\, \hat{a}_x$
back surface, $d\vec{S} = dy\, dz\, (-\hat{a}_x)$

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$$\oint_S \vec{D}\cdot\vec{dS} = \int_{x=0}^{1}\int_{z=0}^{3} (2xy\hat{a}_x + x^2\hat{a}_y)\cdot(dx\,dz\,\hat{a}_y)$$

$$+ \int_{x=0}^{1}\int_{z=0}^{3} (2xy\hat{a}_x + x^2\hat{a}_y)\cdot(dx\,dz(-\hat{a}_y)) +$$

$$\int_{y=0}^{2}\int_{z=0}^{3} (2xy\hat{a}_x + x^2\hat{a}_y)\cdot(dy\,dz\,\hat{a}_x) +$$

$$\int_{y=0}^{2}\int_{z=0}^{3} (2xy\hat{a}_x + x^2\hat{a}_y)\cdot\{dy\,dz(-\hat{a}_x)\}$$

$$= \cancel{\int_{x=0}^{1}\int_{z=0}^{3} x^2\,dx\,dz\Big|_{y=2} - \int_{x=0}^{1}\int_{z=0}^{3} x^2\,dx\,dz\Big|_{y=0}} +$$

$$\int_{y=0}^{2}\int_{z=0}^{3} 2xy\,dy\,dz\Big|_{x=1} - \int_{y=0}^{2}\int_{z=0}^{3} 2xy\,dy\,dz\Big|_{x=0}$$

$$= \int_{y=0}^{2}\int_{z=0}^{3} 2y\,dy\,dz$$

$$= \int_{y=0}^{2} 2y\,dy \int_{z=0}^{3} dz$$

$$= 2\left[\frac{y^2}{2}\right]_0^2 [z]_0^3$$

$$= 4\times3$$

$$= 12 \text{ C}$$

Again for volume integral, i.e. $\oint_v (\nabla\cdot\vec{D})\,dv$

$$\nabla\cdot\vec{D} = \rho_v = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

$$= \frac{\delta}{\delta x}(2xy) + \frac{\delta}{\delta y}(x^2)$$

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$= 2y$

$= 2y \ C/m^3$

Now, $\oint_V [\nabla \cdot \vec{D}] dv = \int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} 2y \, dx \, dy \, dz$

$= \int_{y=0}^{2} 2y \, dy \int_{x=0}^{1} dx \int_{z=0}^{3} dz$

$= 2 \left[\frac{y^2}{2} \right]_0^2 \cdot 1 \cdot 3$

$= 12 C$

$= \oint_S \vec{D} \cdot d\vec{S}$

Hence, Divergence theorem is proved for given field and volume.

Electric Potential:

The work required to move a charge through a finite distance in an electric field is:

$$W = -Q \int_{initial}^{final} \vec{E} \cdot d\vec{l}$$

Note: $\vec{F} = Q\vec{E} \longrightarrow$ Force due to electric field

work done $= \vec{F} \cdot d\vec{l}$

$= -Q \vec{E} \cdot d\vec{l}$

$W = -Q \int \vec{E} \cdot d\vec{l}$

↳ Work done around a closed path in an electric field is zero i.e. $\oint \vec{E} \cdot \vec{dl} = 0$

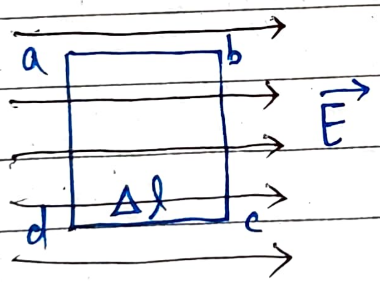
↳ Potential difference, V is the work done (by an external source) in moving a unit positive charge from one point to another in an electric field, i.e.

Potential difference, $V = \frac{W}{Q} = -\int_{initial}^{final} \vec{E} \cdot \vec{dl}$

$$V_{AB} = -\int_{B}^{A} \vec{E} \cdot \vec{dl} = \int_{A}^{B} \vec{E} \cdot \vec{dl}$$

V_{AB} signifies the potential difference points A and B and is the work done in moving unit positive charge from B to A. Thus, in determining V_{AB}, B is the initial point and A is the final point.

Conservative Field:



When a charge is moved along a closed path in an electric field, $\vec{E}$ such that the initial point and the final point coincide, then

$$W = -Q \int_{a}^{a} \vec{E} \cdot \vec{dl} = 0$$

i.e. $\boxed{\vec{E} \cdot \vec{dl} = 0}$

Any field which satisfies the above eqn is said to be conservative field.

Electric field intensity ($\vec{E}$) as the negative gradient of potential (i.e. $-\nabla V = \vec{E}$)

We have,

$$V = -\int \vec{E} \cdot \vec{dl}$$

$$dV = -\vec{E} \cdot \vec{dl} = -E\,dl \cos\theta$$

where θ is the angle between $\vec{E}$ and $\vec{dl}$.

For a small potential difference, dV in space can be represented as,

$$dV = \frac{\delta V}{\delta x} dx + \frac{\delta V}{\delta y} dy + \frac{\delta V}{\delta z} dz \quad \text{---(i)}$$

Also,

$$dV = -\vec{E} \cdot \vec{dl}$$

$$= -(E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z) \cdot (dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z)$$

$$= -[E_x dx + E_y dy + E_z dz] \quad \text{---(ii)}$$

On comparing eq^n (i) and (ii),

$$E_x = -\frac{\delta V}{\delta x}, \quad E_y = -\frac{\delta V}{\delta y}, \quad E_z = -\frac{\delta V}{\delta z}$$

Now,

$$\vec{E} = E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z$$

$$= -\frac{\delta V}{\delta x} \hat{a}_x - \frac{\delta V}{\delta y} \hat{a}_y - \frac{\delta V}{\delta z} \hat{a}_z$$

$$= -\left[\frac{\delta}{\delta x} \hat{a}_x + \frac{\delta}{\delta y} \hat{a}_y + \frac{\delta}{\delta z} \hat{a}_z\right] V$$

$$\therefore \vec{E} = -\nabla V$$

$$\therefore \vec{E} = -\text{gradient of } V = -\nabla V$$

Gradient:

a) $\nabla V = \frac{\delta V}{\delta x}\hat{a}_x + \frac{\delta V}{\delta y}\hat{a}_y + \frac{\delta V}{\delta z}\hat{a}_z \rightarrow$ cartesian

b) $\nabla V = \frac{\delta V}{\delta \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\delta V}{\delta \phi}\hat{a}_\phi + \frac{\delta V}{\delta z}\hat{a}_z \rightarrow$ cylindrical

c) $\nabla V = \frac{\delta V}{\delta r}\hat{a}_r + \frac{1}{r}\frac{\delta V}{\delta \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\delta V}{\delta \phi}\hat{a}_\phi \rightarrow$ spherical

Note:

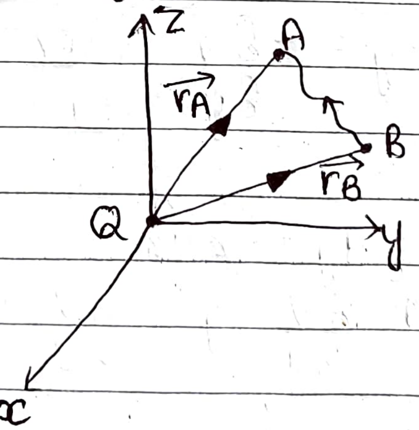
$\vec{dl} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z \rightarrow$ cartesian

$\vec{dl} = d\rho\,\hat{a}_\rho + \rho\, d\phi\,\hat{a}_\phi + dz\,\hat{a}_z \rightarrow$ cylindrical

$\vec{dl} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\, d\phi\,\hat{a}_\phi \rightarrow$ spherical

Potential Difference in the Field of a Point charge:

Let us consider a point charge at the origin. We need to find the potential difference from B to A as shown in Figure.



We know, the electric field intensity due to point charge,

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r = E_r\,\hat{a}_r$$

and $\vec{dl} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$

Then, potential difference between points A and B is,

$$V_{AB} = -\int_{r_B}^{r_A} \vec{E}\cdot\vec{dl}$$

$$= -\int_{r_B}^{r_A} E_r\,dr$$

$$= -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2}\,dr$$

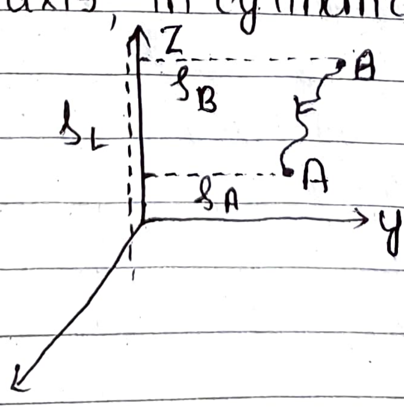
$$= -\frac{Q}{4\pi\epsilon_0}\int_{r_B}^{r_A} \frac{1}{r^2}\,dr$$

$$= -\frac{Q}{4\pi\epsilon_0}\left[\frac{r^{-1}}{-1}\right]_{r_B}^{r_A}$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

Potential Difference in the Field of a Line Charge:

Let us consider a line charge with line charge density ρ_L along the z-axis, in cylindrical co-ordinate system



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The potential difference between points A and B along the path B to A as shown in the figure is to be determined.
We know, for a line charge,

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\hat{a}_\rho = E_\rho\hat{a}_\rho$$

and

$$\vec{dl} = d\rho\hat{a}_\rho + \rho d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$$

Then, potential difference between points A and B is,

$$V_{AB} = -\int_{\rho_B}^{\rho_A} \vec{E}\cdot\vec{dl}$$

$$= -\int_{\rho_B}^{\rho_A} E_\rho d\rho$$

$$= -\int_{\rho_B}^{\rho_A} \frac{\rho_L}{2\pi\epsilon_0\rho} d\rho$$

$$= \frac{-\rho_L}{2\pi\epsilon_0}\int_{\rho_B}^{\rho_A} \frac{1}{\rho} d\rho$$

$$= \frac{-\rho_L}{2\pi\epsilon_0}[\ln\rho]_{\rho_B}^{\rho_A}$$

$$\therefore V_{AB} = \frac{-\rho_L}{2\pi\epsilon_0}[\ln\rho_A - \ln\rho_B]$$

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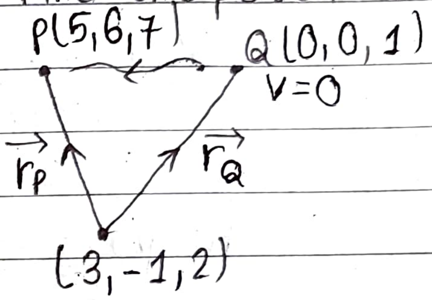
Numericals:

1) Given a point charge of $200\pi\epsilon_0$ C at $(3,-1,2)$, a line charge of $40\pi\epsilon_0$ C/m on the x-axis and a surface charge of $8\epsilon_0$ C/m^2 on the plane $x=-3$ all in the free space, find the potential at $P(5,6,7)$ if $V=0$ at $Q(0,0,1)$

→ Soln:



* For potential due to point charge,

$$V_{PQ} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_P} - \frac{1}{r_Q}\right]$$

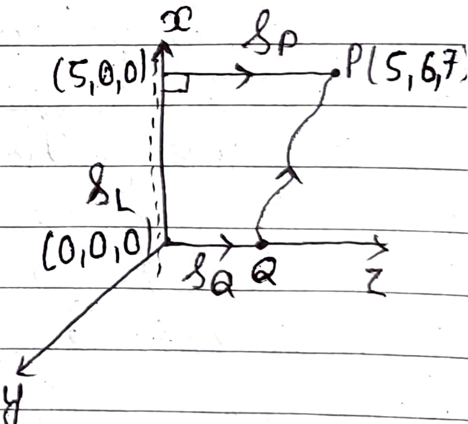
$$r_P = \sqrt{(5-3)^2+(6+1)^2+(7-2)^2} = \sqrt{4+49+25} = \sqrt{78}$$

$$r_Q = \sqrt{(0-3)^2+(0+1)^2+(1-2)^2} = \sqrt{9+1+1} = \sqrt{11}$$

where r_P and r_Q are the distances from the point charge to point P & Q. so

So, $$V_{PQ} = \frac{200\pi\epsilon_0}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{78}} - \frac{1}{\sqrt{11}}\right]$$

$$= -9.414 \text{ Volts}$$



* For potential due to line charge:

$$V_{PQ} = \frac{-\rho_L}{2\pi\epsilon_0}\left[\ln\rho_P - \ln\rho_Q\right]$$

where, ρ_P and ρ_Q are the ~~tr~~ distance between line charge and points P and Q respectively

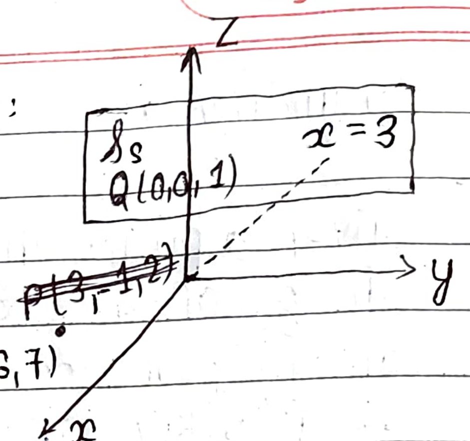
$$\rho_P = \sqrt{(5-5)^2+(6-0)^2+(7-0)^2} = \sqrt{36+49} = \sqrt{85}$$

$$\rho_Q = \sqrt{0^2+0^2+1^2} = 1$$

$$V_{PQ} = \frac{-40\pi\epsilon_0}{2\pi\epsilon_0}\left[\ln\sqrt{85} - \ln 1\right]$$

$$= -44.43 \text{ volts}$$

For potential due to surface charge:



Electric field due to surface charge is, $\vec{E} = \dfrac{\rho_s}{2\epsilon_0}\hat{a}_N$

At point P,
$$\vec{E_P} = \frac{\rho_s}{2\epsilon_0}\hat{a}_x$$
$[\hat{a}_N = \hat{a}_x]$

$$\vec{E} = \frac{\rho_s}{2\epsilon_0}\hat{a}_x$$

At point Q,
$$\vec{E_Q} = \frac{\rho_s}{2\epsilon_0}\hat{a}_x$$
$[\hat{a}_N = \hat{a}_x]$

Now,
$$V_{PQ} = -\int_Q^P \vec{E}\cdot\vec{dl}$$

and $\vec{dl} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$

So, $\vec{E}\cdot\vec{dl} = \dfrac{\rho_s}{2\epsilon_0}dx$

Then, $V_{PQ} = -\displaystyle\int_Q^P \frac{\rho_s}{2\epsilon_0}dx = -\int_{x=0}^{5}\frac{\rho_s}{2\epsilon_0}dx$

$$V_{PQ} = -\frac{8\epsilon_0}{2\epsilon_0}[x]_0^5$$
$$= -20 \text{ volts}$$

Total $V_{PQ} = -9.414 - 44.43 - 20$
$= -73.844$ volts

Also, $V_{PQ} = V_P - V_Q$

so, $V_P = V_{PQ} + V_Q$

$\therefore V_P = -73.844$ volts

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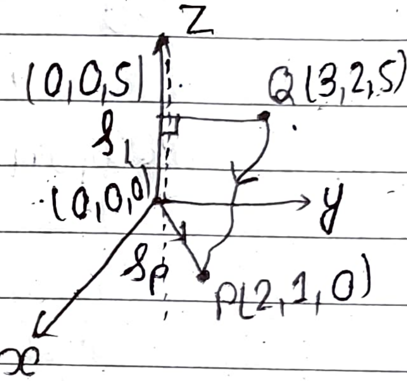
2) In free space, a line charge $\rho_L = 80\,nC/m$ lies along the entire Z-axis, while point charge of 100nC each located at (1,0,0) and (0,1,0). Find the potential difference V_{PQ} given P(2,1,0) and Q(3,2,5).

→ Soln:



For the potential due to line charge,

$$V_{PQ} = \frac{-\rho_L}{2\pi\epsilon_0}\left[\ln(\rho_P) - \ln(\rho_Q)\right]$$

where ρ_P and ρ_Q are perpendicular distances betn the line charge and points P and Q respectively.

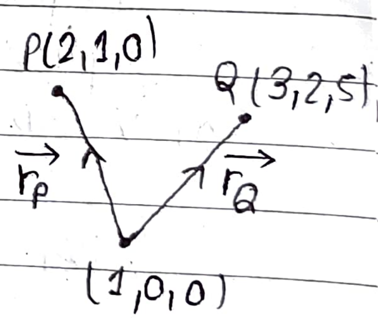
$$\rho_P = \sqrt{2^2+1^2+0^2} = \sqrt{5}$$

$$\rho_Q = \sqrt{(3-0)^2+(2-0)^2+(5-5)^2} = \sqrt{13}$$

So,

$$V_{PQ_{line}} = \frac{-80\times10^{-9}}{2\pi\epsilon_0}\left[\ln(\sqrt{5}) - \ln(\sqrt{13})\right]$$

$$= 687.34 \text{ volts}$$



Potential due to point charge at (1,0,0).

$$V_{PQ} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_P} - \frac{1}{r_Q}\right]$$

$$r_P = \sqrt{(2-1)^2+(1-0)^2+(0-0)^2} = \sqrt{2}$$

$$r_Q = \sqrt{(3-1)^2+(2-0)^2+(5-0)^2} = \sqrt{33}$$

So,

$$V_{PQ_{point1}} = \frac{100\times10^{-9}}{4\pi\times\epsilon_0}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{33}}\right]$$

$$= 479.73 \text{ volts}$$

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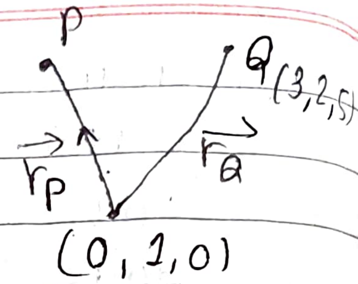
Potential due to point charge at (0, 1, 0)



$\rho_P = \sqrt{(2-0)^2+(1-1)^2+(0-0)^2}$

$= \sqrt{2^2} = 2$

$\rho_Q = \sqrt{(3-0)^2+(2-1)^2+(5-0)^2}$

$= \sqrt{35}$

$V_{PQ \, point\,2} = \dfrac{100 \times 10^{-9}}{4\pi\epsilon_0}\left[\dfrac{1}{2} - \dfrac{1}{\sqrt{35}}\right]$

$= 297.87$ volts

$\therefore$ Total P.d. $= 687.34 + 479.73 + 297.87$

$= 1464.94$ volts

3) For the point $P(3, 60°, 2)$ in cylindrical co-ordinates, the potential is $V = [100(\rho+1)z^2\cos\phi]$ volt in free space. Find

a) V b) $\vec{E}$ c) $\vec{D}$ d) ρ_v e) $\dfrac{dV}{dN}$ f) $\hat{a}_N$

Given, $V = 100(\rho+1)z^2\cos\phi$ volts

a) $V_P = 100(3+1)\cdot 2^2 \cdot \cos(60°)$

$= 800$ volts

b) $\vec{E} = -\nabla V$

$= -\left[\dfrac{\partial V}{\partial \rho}\hat{a}_\rho + \dfrac{1}{\rho}\dfrac{\partial V}{\partial \phi}\hat{a}_\phi + \dfrac{\partial V}{\partial z}\hat{a}_z\right]$

$= -\left[100z^2\cos\phi\,\hat{a}_\rho - \dfrac{1}{\rho}100(\rho+1)z^2\sin\phi\,\hat{a}_\phi + 200(\rho+1)z\cos\phi\,\hat{a}_z\right]$

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$\therefore \vec{E_P} = -200\,\hat{a}_\rho + 461.88\,\hat{a}_\phi + -800\,\hat{a}_z$

c) $\vec{D} = \epsilon_0 \vec{E}$

$= -1.77 \times 10^{-9}\,\hat{a}_\rho + 4.087 \times 10^{-9} - 7.08 \times 10^{-9}\,\hat{a}_z$

d) $\rho_v = \nabla \cdot \vec{D}$

$= \nabla \cdot (\epsilon_0 \vec{E})$

$= \epsilon_0\, \nabla \cdot \vec{E}$

$= \epsilon_0 \left[\frac{1}{\rho}\frac{\delta(\rho E_\rho)}{\delta \rho} + \frac{1}{\rho}\frac{d}{\delta\phi} E_\phi + \frac{\delta}{\delta z} E_z \right]$

$= \epsilon_0 \left[-200 + \frac{1}{3} \times 461.88 + -800 \right] = -7.48 \times 10^{-9}\, C/m^2$

e) $\frac{dV}{dN} = |\vec{E_P}| = \sqrt{(E_\rho)^2 + (E_z)^2} = \sqrt{(-200)^2 + (-800)^2}$

f) $\hat{a}_N = -\frac{\vec{E_P}}{|\vec{E_P}|}$

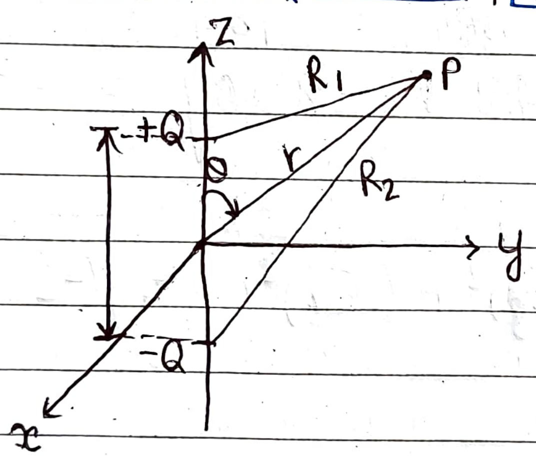
Electric Field due to Electric Dipole:

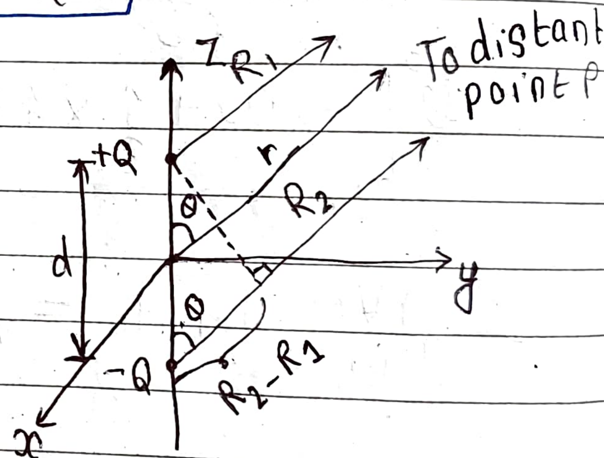
Dipole:

Electric dipole or simply dipole is the separation between a positive charge and negative charge of equal magnitude by a small distance.

If Q is the magnitude of charge and $\vec{d}$, represents the vector from negative charge to positive charge then,

electric dipole moment, $\boxed{\vec{P} = Q\vec{d}}$





For a distant point P, R_1 is parallel to R_2 and we find
$$R_2 - R_1 = d\cos\theta$$

Suppose, an electric dipole with center at origin is located in spherical co-ordinate system as shown in figure. We have to find the field at distant point P due to the dipole.
We know, potential at point P is,

$$V = \frac{Q}{4\pi\epsilon_0 R_1} - \frac{Q}{4\pi\epsilon_0 R_2}$$
$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{R_2 - R_1}{R_1 R_2}\right]$$

We have,

$$R_2 - R_1 = d\cos\theta$$

For a distant point P,

$$R_1 = R_2 = r \ [\text{for } d \ll r]$$

So, $V = \frac{Q}{4\pi\epsilon_0}\left[\frac{d\cos\theta}{r^2}\right] = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}$

We know, $\vec{E} = -\nabla V$

$$= -\left[\frac{\delta V}{\delta r}\hat{a}_r + \frac{1}{r}\frac{\delta V}{\delta \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\delta V}{\delta \phi}\hat{a}_\phi\right]$$

$$= -\left[\frac{\delta}{\delta r}\left(\frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}\right)\hat{a}_r + \frac{1}{r}\frac{\delta}{\delta\theta}\left(\frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}\right)\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\delta}{\delta\phi}\left(\frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}\right)\hat{a}_\phi\right]$$

$$= \cancel{\frac{Qd\cos\theta}{4\pi\epsilon_0}} \ -\frac{Qd}{4\pi\epsilon_0}\left[\cos\theta\frac{\delta}{\delta r}\left(\frac{1}{r^2}\right)\hat{a}_r + \frac{1}{r^3}\frac{\delta}{\delta\theta}\cos\theta\,\hat{a}_\theta + 0\right]$$

$$= -\frac{Qd}{4\pi\epsilon_0}\left[\cos\theta\left(\frac{-2}{r^3}\right)\hat{a}_r - \frac{\sin\theta}{r^3}\hat{a}_\theta\right]$$

$$\boxed{\vec{E} = \frac{Qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right]}$$

For dipole moment,

$$\vec{p} = Q\vec{d}$$

Here, $\vec{d} = d\hat{a}_z$

So, $\vec{p}\cdot\hat{a}_r = Q\vec{d}\cdot\hat{a}_r$

$$= Q\,d\hat{a}_z\cdot\hat{a}_r$$

$$= Qd\cos\theta$$

Also,

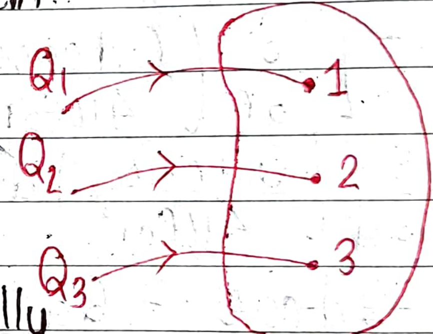
$$V = \frac{\vec{p} \cdot \hat{a}_r}{4\pi\epsilon_0 r^2}$$

(potential difference in terms of dipole moment)

Energy Density in the Electrostatic Field:

Note: To determine the energy present in an assembly of charges, we must first determine the amount of work necessary to assemble them.

Suppose three charges Q_1, Q_2 and Q_3 are located at infinity and consider a region which is charge free, i.e. no field is initially present.



The work required to place the charge Q_1 into position 1 is zero, i.e. $W_1 = 0$.

Then when Q_2 is moved towards the region, work equal to the product of this charge Q_2 and potential due to Q_1 is required,

$$\boxed{\text{i.e. } W_2 = Q_2 V_{2,1}}$$

where $V_{2,1}$ means the potential at position 2 due to charge Q_1 at point 1.

Similarly, work done to position charge Q_3 at 3 is,

$$\boxed{W_3 = Q_3 V_{3,1} + Q_3 V_{3,2}}$$

Total work done for the assembly of 3 charges is,

$$W_E = W_1 + W_2 + W_3$$
$$= 0 + Q_2 V_{2,1} + Q_3 (V_{3,1} + V_{3,2}) \text{ --- (i)}$$

Now, if the charges were brought in reverse order,

$$W_E = W_3 + W_2 + W_1$$
$$= 0 + Q_2 V_{2,3} + Q_1 (V_{1,3} + V_{1,2}) \text{ --- (ii)}$$

Since, $V_{1,2} + V_{1,3} = V_1$

$V_{2,1} + V_{2,3} = V_2$

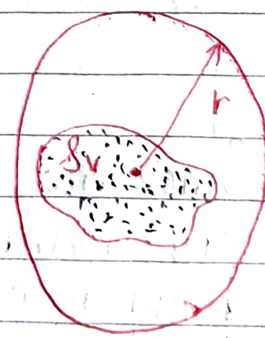
$V_{3,1} + V_{3,2} = V_3$

Here, adding eq^n (i) and eq^n (ii)

$$2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3$$

For n number of charges,

$$W_E = \frac{1}{2} \sum_{m=1}^{n} Q_m V_m$$



For a region with a continuous charge density ρ_v C/m³, the summation becomes an integration,

$$\boxed{W_E = \frac{1}{2} \int_{vol} \rho_v V dv}$$

Fig: charge distribution ρ_v enclosed within a large sphere of a radius r

Using Maxwell's 1st eq^n

$$\rho_v = \nabla \cdot \vec{D}$$

we get,

$$W_E' = \frac{1}{2}\int_{vol} (\nabla \cdot \vec{D}) V \, dV$$

Now, using the vector identity,

$$\nabla \cdot (V\vec{D}) = V(\nabla \cdot \vec{D}) + \vec{D} \cdot (\nabla V)$$

So, $V(\nabla \cdot \vec{D}) = \nabla \cdot (V\vec{D}) - \vec{D} \cdot (\nabla V)$

Then,

$$W_E' = \frac{1}{2}\int_{vol} \{\nabla \cdot (V\vec{D}) - \vec{D} \cdot (\nabla V)\} \, dv$$

$$= \frac{1}{2}\int_{vol} \nabla \cdot (V\vec{D}) \, dV - \frac{1}{2}\int_{vol} \vec{D} \cdot (\nabla V) \, dV$$

Using Divergence Theorem,

$$\int_{vol} \nabla \cdot (V\vec{D}) \, dV = \oint_S (V\vec{D}) \cdot d\vec{S}$$

So, $W_E = \frac{1}{2}\oint_S (V\vec{D}) \cdot d\vec{S} - \frac{1}{2}\int_{vol} \vec{D} \cdot (\nabla V) \, dV$

Since the volume must contain every charge and there can then be no charge outside of the volume, we may therefore consider the volume as infinite in extent.

And

$$\left[V \propto \frac{1}{r}, \; \vec{D} \propto \frac{1}{r^2} \; \& \; d\vec{S} \propto r^2 \right]$$

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For $\lim_{r\to\infty} \oint_S (V\vec{D})\cdot d\vec{S} = 0$

Therefore,

$$W_E = -\frac{1}{2}\int_{vol.} \vec{D}\cdot(\nabla V)\,dV$$

We have, $-\nabla V = \vec{E}$

So, $\nabla V = -\vec{E}$

$$W_E = \frac{1}{2}\int_{vol.} \vec{D}\cdot\vec{E}\,dV$$

$$= \frac{1}{2}\int_{vol.} \epsilon_0 \vec{E}\cdot\vec{E}\,dV$$

$$\boxed{\therefore W_E = \frac{1}{2}\int_{vol.} \epsilon_0 E^2\,dV}$$

In differential form,

$$dW_E = \frac{1}{2}\vec{D}\cdot\vec{E}\,dV$$

$$\boxed{\therefore \frac{dW_E}{dV} = \frac{1}{2}\vec{D}\cdot\vec{E}}$$

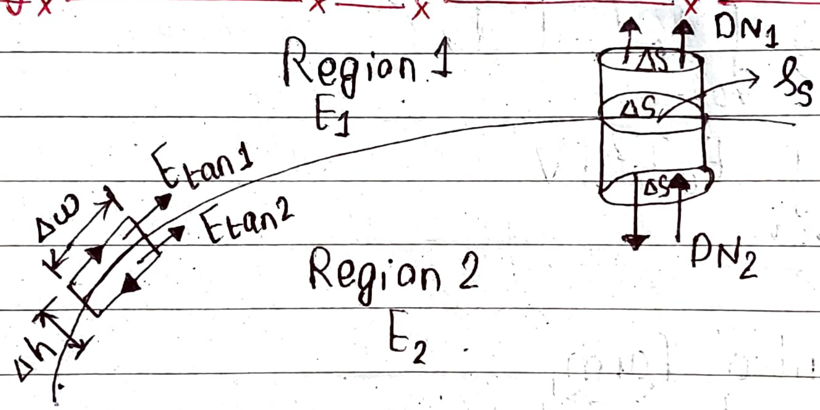
→ electrostatic charge density

$$\boxed{\frac{dW_E}{dV} = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\frac{D^2}{\epsilon_0}}$$

Boundary Conditions:

So far we have considered electric field in a homogeneous medium. If the field exists in the region consisting of two different media, the conditions which the field must satisfy at the interface separating the media are called "boundary conditions".

* Boundary Conditions for Dielectric Materials:



Let us consider the interface betⁿ the two dielectrics having permittivities ϵ_1 and ϵ_2 and occupying region 1 and region 2 as shown in figure.

→ We first examine the tangential component by using

$$\oint \vec{E} \cdot \vec{dl} = 0$$

around the small closed path on the left of the figure, obtaining

$$E_{tan1}\Delta w - E_{tan2}\Delta w = 0 \quad (\text{by considering } \Delta h \to 0)$$

$$\therefore \boxed{E_{tan1} = E_{tan2}}$$

i.e. tangential components of electric field intensity is continuous (remains same) across the boundary

Moreover,

$$E_{tan1} = E_{tan2}$$

$$\frac{D_{tan1}}{E_1} = \frac{D_{tan2}}{E_2}$$

$$\therefore \frac{D_{tan1}}{D_{tan2}} = \frac{E_1}{E_2}$$

i.e. tangential components of electric flux density is not continuous across the boundary.

⇒ For normal components, referring to the figure on the right side. The sides are again very short and the flux leaving the surface (top and bottom) is

$$\oint_S \vec{D} \cdot \vec{dS} = Q$$

$$\Rightarrow D_{N1} \cdot \Delta S - D_{N_2} \Delta S = \Delta Q$$

$$\Rightarrow D_{N_1} \Delta S - D_{N_2} \Delta S = \rho_s \Delta S$$

$$\Rightarrow D_{N_1} - D_{N_2} = \rho_s \quad (\rho_s \rightarrow \text{surface charge density})$$

if $\rho_s = 0$, then

$$\boxed{D_{N_1} = D_{N_2}}$$

i.e. normal components of electric flux density is continuous across boundary.

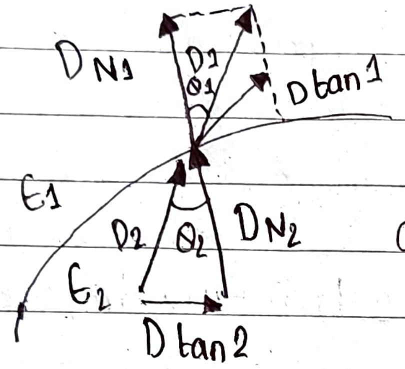
Moreover,

$$\boxed{D_{N_1} = D_{N_2}}$$

$$\epsilon_1 E_{N1} = \epsilon_2 \cancel{N} E_{N_2}$$

$$\therefore \boxed{\frac{E_{N_1}}{E_{N_2}} = \frac{\epsilon_2}{\epsilon_1}}$$

i.e. normal components of electric field intensity is not continuous across the boundary.



These conditions can be combined to show the changes in vectors $\vec{D}$ and $\vec{E}$ at the surface.

Let $\vec{D_i}$ (or $\vec{E_i}$) makes an angle θ_i with the normal to the surface.

Now,

$$D_{N_1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N_2} \quad \text{---(a)}$$

Again,

$$\frac{D\tan 1}{D\tan 2} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{E_1}{E_2}$$

$$\Rightarrow \frac{\sin\theta_1}{\sin\theta_2} = \frac{E_1 D_1}{E_2 D_1}$$ $\Rightarrow$ Snell's law for dielectric interface.

$$\Rightarrow E_2 D_1 \sin\theta_1 = E_1 D_2 \sin\theta_2 \quad \text{---(b)}$$

Dividing eq^n (b) by eq^n (a),

$$\frac{E_2 \cancel{D_1} \sin\theta_1}{\cancel{D_1} \cos\theta_1} = \frac{E_1 \cancel{D_2} \sin\theta_2}{\cancel{D_2} \cos\theta_2}$$

$$E_2 \tan\theta_1 = E_1 \tan\theta_2$$

$$\therefore \frac{\epsilon_1}{\epsilon_2} = \frac{\tan\theta_1}{\tan\theta_2}$$

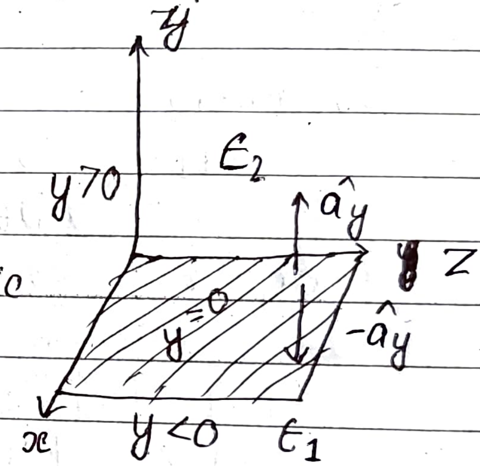
Numerical

1) The region $y<0$ contains a dielectric material for which $\epsilon_{R_1} = 2.5$ while the region $y<0$ is characterised by $\epsilon_{R_2} = 4$. Let,

$$\vec{E_1} = -30\,\hat{a}_x + 50\,\hat{a}_y + 70\,\hat{a}_z \ \text{V/m}$$

Find:

a) $\vec{E_{N_1}}$ b) $\vec{E_{t_1}}$ c) θ_1 d) $\vec{D_{N_2}}$ e) $\vec{D_{t_2}}$ f) $\vec{P_2}$ (polarization vector)
g) θ_2



→ Soln,

$$\vec{E_1} = -30\,\hat{a}_x + 50\,\hat{a}_y + 70\,\hat{a}_z \ \text{V/m}$$

a) The plane $y=0$ separates two electric materials.
To this plane $\pm\hat{a}_y$ are the normal direction.
Hence, $\vec{E_N}$ is that part of $\vec{E_1}$ which contains $\pm\hat{a}_y$.

$$\therefore \vec{E_{N_1}} = 50\,\hat{a}_y \ \text{V/m}$$

b) $\vec{E_1} = \vec{E_{N_1}} + \vec{E_{t_1}}$

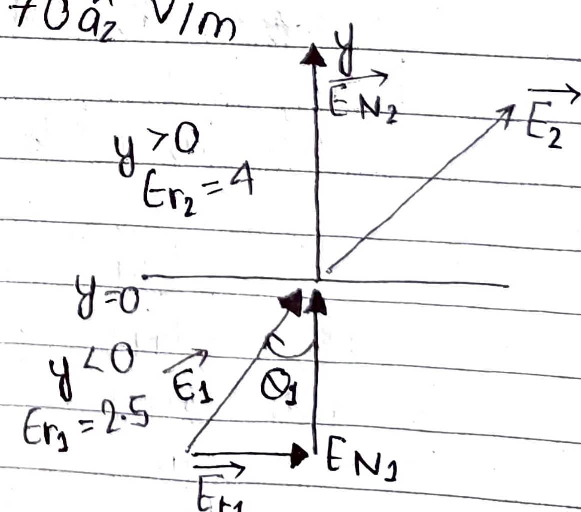
$$\therefore \vec{E_{t_1}} = \vec{E_1} - \vec{E_{N_1}} = -30\,\hat{a}_x + 70\,\hat{a}_z \ \text{V/m}$$

c) $\cos\theta_1 = |\vec{E_{N_1}}|$

$$\cos\theta_1 = \frac{\sqrt{(50)^2}}{\sqrt{(-30)^2+50^2+70^2}}$$

$$\cos\theta_1 = 0.5488$$

$$\therefore \theta_1 = 56.71^\circ$$



[Note: $\epsilon_1 = \epsilon_0 \epsilon_{R1}$
$\epsilon_2 = \epsilon_0 \epsilon_{R2}$]

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d) $\vec{D_{N_2}} = ?$

$$\vec{D_{N_2}} = D_{N1}$$
$$= \epsilon_1 \cdot \vec{E_{N1}}$$
$$= \epsilon_0 \epsilon_{R1} \vec{E_{N1}}$$
$$= \cancel{\epsilon_0}\ 125\,\epsilon_0\,\hat{a}_y\ C/m^2$$

e) $\vec{D_{t_2}} = ?$

We know,

$$\vec{E_{t1}} = \vec{E_{t_2}}$$
$$= \cancel{\epsilon_2}\ \vec{D_{t2}} / \epsilon_2$$

or, $\vec{D_{t_2}} = \epsilon_2 \vec{E_{t_1}}$

$$= \epsilon_{R_2} \epsilon_0 \vec{E_{t_1}}$$
$$= 4\epsilon_0(-30\hat{a}_x + 70\hat{a}_z)$$
$$= -120\epsilon_0 \hat{a}_x + 280\epsilon_0 \hat{a}_z\ C/m^2$$
$$= -1.062 \times 10^{-9} \hat{a}_x + 2.478 \times 10^{-9} \hat{a}_z\ C/m^2$$

f) $\vec{P_2} = ?$

$$\vec{D_2} = \epsilon_0 \vec{E_2} + \vec{P_2}$$
$$\cancel{\vec{D_2}} = \quad \vec{P_2} = \vec{D_2} - \epsilon_0 \vec{E_2}$$

Here,

$$\vec{D_2} = \vec{D_{t_2}} + \vec{D_{N_2}} = \epsilon_0(-120\hat{a}_x + 125\hat{a}_y + 280\hat{a}_z)\ C/m^2$$

$$\vec{E_2} = \vec{E_{t1}} + \vec{E_{t_2}}$$
$$\vec{E_{t_2}} = \vec{E_{t1}} = \cancel{50\hat{a}_y\ V/m}\ -30\hat{a}_x + 70\hat{a}_z\ V/m$$

and $\dfrac{\vec{E_{N1}}}{\vec{E_{N_2}}} = \dfrac{\epsilon_2}{\epsilon_1}$

or, $\vec{E_{N2}} = \dfrac{\epsilon_1}{\epsilon_2} \vec{E_{N1}}$

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$$= \frac{\epsilon_{R1} \cancel{\epsilon_0}}{\epsilon_{R_2} \cancel{\epsilon_0}} \overrightarrow{E_{N1}}$$

$$= \frac{2.5}{4} (50 \hat{a}_y)$$

$$= 31.25 \hat{a}_y \text{ V/m}$$

$$\therefore \overrightarrow{E_2} = \overrightarrow{E_{t_2}} + \overrightarrow{E_{N_2}}$$

$$= -30 \hat{a}_x + 31.25 \hat{a}_y + 70 \hat{a}_z \text{ V/m}$$

So,

$$\overrightarrow{P_2} = \overrightarrow{D_2} - \epsilon_0 \overrightarrow{E_2}$$

$$= \epsilon_0 (-120 \hat{a}_x + 125 \hat{a}_y + 280 \hat{a}_z) - \epsilon_0 (-30 \hat{a}_x + 31.25 \hat{a}_y + 70 \hat{a}_z)$$

$$= \epsilon_0 [-120 \hat{a}_x + 125 \hat{a}_y + 280 \hat{a}_z + 30 \hat{a}_x - 31.25 \hat{a}_y - 70 \hat{a}_x]$$

$$= -0.797 \hat{a}_x + 0.830 \hat{a}_y + 1.859 \hat{a}_z \text{ nC/m}^2$$

g) $\theta_2 = ?$

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_1}{\epsilon_2} = \frac{\epsilon_{R_1} \epsilon_0}{\epsilon_{R_2} \epsilon_0}$$

$$\tan \theta_2 = \frac{\epsilon_{R_2}}{\epsilon_{R_1}} \tan \theta_1$$

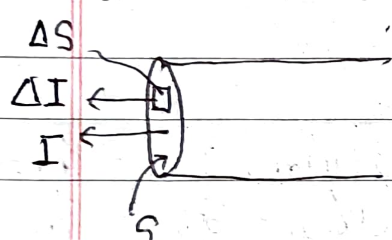
$$= \frac{4}{2.5} \tan (56.71^\circ)$$

$$\therefore \theta_2 = 67.68^\circ$$

Current and Current Density

- Current, $I = \frac{dQ}{dt}$ ⇒ rate of change of charge per unit time.
- Current density, $\vec{J}$ is the current crossing an area perpendicularly ie. $\vec{J} = \frac{I}{S}\hat{a}_N$.

where $\hat{a}_N$ is a unit vector perpendicular to the surface S.



The current passing through a small area Δs is given by;

$$\Delta I = \vec{J} \cdot \vec{\Delta S}$$

and the total current passing through the whole plane of area S is given by;

$$I = \oint_S \vec{J} \cdot \vec{dS}$$

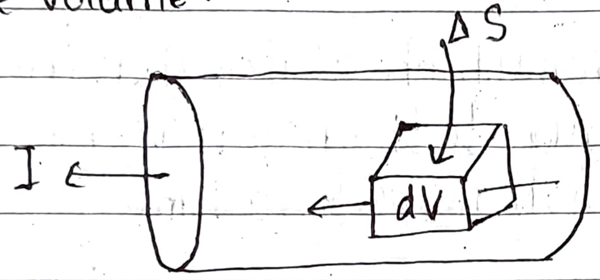
Principal of Conservation of Charges:

"Charges can neither be created nor destroyed, although equal amounts of positive and negative charges may be simultaneously created by separation and destroyed by recombination."

Continuity Eq^n:

The continuity equation defines the basic conservation of charge, relationship between current and charge.

"A net current in or out of a given volume must equal the net increase or decrease in total charge in the volume".



The current flowing out through the surface is,

$$I = \oint_S \vec{J} \cdot d\vec{S}$$

Let us consider a small volume ΔV enclosed by the surface ΔS within the conductor. Then from the principle of conservation of charges the net outflow of charges through the closed surface ΔS implies the decrease of same charges.

$$\therefore \Delta I = \oint_{\Delta S} \vec{J} \cdot d\vec{S} = -\frac{dQ_i}{dt}$$

Using the divergence theorem to change the surface integral to volume integral

$$\oint_{\Delta S} \vec{J} \cdot d\vec{S} = \int_{vol.} (\nabla \cdot \vec{J})\, dV$$

So, $$\int_{vol.} (\nabla \cdot \vec{J})\, dV = -\frac{dQ_i}{dt}$$

If ρ_v is the volume charge density then,

$$Q_i = \int_{vol} \rho_v \, dV$$

Now, $$\int_{vol} (\nabla \cdot \vec{J})\, dV = -\frac{d}{dt} \int_{vol.} \rho_v \, dV$$

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Hence, $\boxed{\nabla \cdot \vec{J} = -\frac{d \rho_v}{dt}}$

This is continuity equation which states that the divergence of current density $\vec{J}$ is equal to the negative rate of change of volume charge density with respect to time.

Point form of Ohm's Law

$\vec{J} = \sigma \vec{E}$ where $\sigma = \frac{L}{SR} \Rightarrow$ conductivity

Relaxation Time Constant:

If some amount of charge is placed inside a volume of conducting material, the Coulomb forces on the individual charges cause them to migrate away from each other. (assuming the charges is all positive or all negative). The end result is a surface charge on the outer surface of the conductor while the inside of the conductor remains charge neutral. The time required for the conductor to reach the charge neutral state is related to a time constant designated as the relaxation time.

" A time that shows how fast the charge decays at a point within the conductor and reappears on the surface is termed as the relaxation time constant for the conductor"

From continuity equation:

$$\Delta \cdot \vec{J} = -\frac{\delta \rho_v}{\delta t}$$

From point form of Ohm's law,

$$\vec{J} = \sigma \vec{E}$$

Then $\nabla \cdot (\sigma \vec{E}) = -\frac{\delta \rho_v}{\delta t}$

$$\nabla \cdot \left(\frac{\sigma \vec{D}}{\epsilon}\right) = -\frac{\delta \rho_v}{\delta t}$$

$$\frac{\sigma}{\epsilon}(\nabla \cdot \vec{D}) = \frac{-\delta \rho_v}{\delta t}$$

$$\frac{\sigma}{\epsilon} \rho_v = -\frac{\delta \rho_v}{\delta t} \quad [\because \nabla \cdot \vec{D} = \rho_v]$$

$$\rho_v + \frac{\epsilon}{\sigma} \frac{\delta \rho_v}{\delta t} = 0$$

The soln of this differential eqn is,

$$\boxed{\rho_v = \rho_o \cdot e^{-(\sigma/\epsilon)t}}$$

$$= \rho_o e^{-t/T}$$

where $T = \frac{\epsilon}{\sigma}$ = Relaxation time constant

If $t = T = \frac{\epsilon}{\sigma}$, then

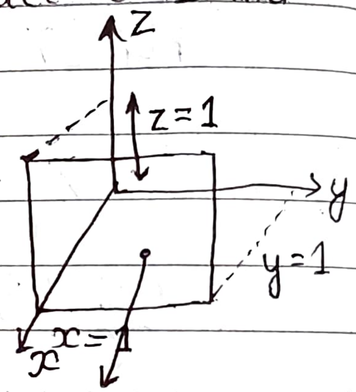
$$\rho_v = 0.37 \rho_o$$

It means that when time 't' equals T, the charge within the conductor decays to 37% of initial charge ρ_o and simultaneously reappear on the conductor surface.

Numericals:

[set calculator in radian]

1) The current density in a certain region is expressed as $\vec{J} = (-10^5 \Delta V)\ A/m^2$ where $V = 10e^{-x}\sin y$ Volts. Find the current in the $\hat{a}_x$ direction crossing the surface $x = 1$ and bounded by $y = 0$ and 1, $z = 0$ and 1.



$\vec{dS} = dy\,dz\,\hat{a}_x$

→ Sol^n:

$\vec{J} = -10^5 \nabla V\ A/m^2,\ V = 10e^{-x}\sin y$ Volts

Now,

$$\Delta V = \frac{\delta V}{\delta x}\hat{a}_x + \frac{\delta V}{\delta y}\hat{a}_y + \frac{\delta V}{\delta z}\hat{a}_z$$

$$= 10\sin y \frac{\delta e^{-x}}{\delta x}\hat{a}_x + 10\,e^{-x}\frac{\delta \sin y}{\delta y}\hat{a}_y + 0$$

$$\nabla V = -10e^{-x}\sin y\,\hat{a}_x + 10e^{-x}\cos y\,\hat{a}_y$$

So, $\vec{J} = -10^5 \nabla V$

$$= -10^5(-10e^{-x}\sin y\,\hat{a}_x + 10e^{-x}\cos y\,\hat{a}_y)$$

$$= 10^6 e^{-x}\sin y\,\hat{a}_x - 10^6 e^{-x}\cos y\,\hat{a}_y\ A/m^2$$

Now,

$$I = \int_S \vec{J}\cdot\vec{dS}$$

$$= \int_S (10^6 e^{-x}\sin y\,\hat{a}_x - 10^6 e^{-x}\cos y\,\hat{a}_y)\cdot(dy\,dz\,\hat{a}_x)$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} 10^6 e^{-x}\sin y\,dy\,dz\,\Big|_{x=1}$$

$$= 10^6 e^{-x}\int_{y=0}^{1}\sin y\,dy\int_{z=0}^{1} dz$$

$$= 10^6 e^{-1}\,[\cos y]_0^1\,[z]_0^1$$

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$= 1.6 \times 10^5$ Amp.

2) If $\vec{J} = \frac{100\cos\theta}{r^2+1}\hat{a_r}$ A/m², find the amount of current passing through the spherical cape $r=3$, $0<\theta<\frac{\pi}{6}$, $0<\phi<2\pi$ and $\frac{\delta \rho_v}{\delta t}$.

→ Soln:

$I = \int \vec{J}\cdot\vec{ds}$

where, $\vec{J} = \frac{100\cos\theta}{r^2+1}\hat{a_r}$

$\vec{ds} = r d\theta . r\sin\theta\, d\phi\, \hat{a_r}$

$= r^2 \sin\theta\, d\theta\, d\phi\, \hat{a_r}$

So, $I = \int_{\theta=0}^{\pi/6}\int_{\phi=0}^{2\pi} \frac{100 r^2 \sin\theta\cos\theta\, d\theta\, d\phi}{r^2+1}$

$= \int_{\theta=0}^{\pi/6}\int_{\phi=0}^{2\pi} \frac{50r^2 \cdot \sin 2\theta\, d\theta\, d\phi}{r^2+1}$

$= \frac{50r^2}{r^2+1} \int_{\theta=0}^{\pi/6} \sin 2\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi \Big|_{r=3}$

$= \left[\frac{50r^2}{r^2+1}\left[-\cos 2\theta\right]_0^{\pi/6} \left[\phi\right]_0^{2\pi}\right]_{r=3}$

$= \frac{45}{2} \times \frac{1}{\cancel{2}} \times 2\pi$

$= 70.65$ Amp.

We have, $\nabla\cdot\vec{J} = -\frac{\delta \rho_v}{\delta t}$

$\therefore \frac{\delta \rho_v}{\delta t} = -\nabla\cdot\vec{J}$

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$$\nabla\cdot\vec{J} = \frac{\delta J_r}{\delta r} + \cancelto{0}{\frac{1}{r}\frac{\delta J_\theta}{\delta\theta}} + \cancelto{0}{\frac{1}{r\sin\theta}\frac{\delta J_\phi}{\delta\phi}}$$

$$= \frac{\delta}{\delta r}\left(\frac{100\cos\theta}{r^2+1}\right)$$

$$= 100\cos\theta\,\frac{\delta}{\delta r}\left(\frac{1}{r^2+1}\right)$$

$$= 100\cos\theta\left\{\frac{-2r}{(r^2+1)^2}\right\}$$

$$= \frac{-200r\cos\theta}{(r^2+1)^2}$$

Hence, $\frac{\delta \rho_v}{\delta t} = -\nabla\cdot\vec{J}$

$$= \frac{200r\cos\theta}{(r^2+1)^2}$$

Boundary Value Problems in Electrostatics

Solving boundary value problem ~~re~~means find the potential function which governs the distribution of potential between the boundaries and is also satisfied when the potential at the boundaries are substituted.

Here boundary means two or more conducting objects maintained at the same potential.

Solution can be obtained using:-

i) Iterative methods

ii) Direct solution of Poisson's and Laplace's eq^n

iii) Graphical method

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Poisson's and Laplace's Equations:

From Maxwell's 1^{st} eq^n,

$$\nabla \cdot \vec{D} = \rho_v$$

We know, $\vec{D} = \epsilon \vec{E}$

Also, $\vec{E} = -\nabla V$

So, $\nabla \cdot (\epsilon \vec{E}) = \rho_v$

$$\epsilon \nabla \cdot (-\nabla V) = \rho_v$$

$$\therefore \boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}}$$ $\Rightarrow$ This is called Poisson's eq^n

A special case of this equation occurs when $\rho_v = 0$ (i.e. charge free region) above eq^n becomes

$$\boxed{\nabla^2 V = 0}$$ $\Rightarrow$ This is called Laplace eq^n

Imp

Uniqueness Theorem: (or Poisson)

It states that if Laplace eq^n is solved along with the boundary conditions, we will have only one or unique solution regardless of the method used to solve those equations.

Proof: (by contradiction method)

Let us assume we have two solutions of Laplace eq^n V_1 and V_2. Therefore, both of them should satisfy Laplace eq^n, i.e.

$$\nabla^2 V_1 = 0 \quad \text{--- (i)}$$

$$\nabla^2 V_2 = 0 \quad \text{--- (ii)}$$

From (i) and (ii)

$$\nabla^2 (V_1 - V_2) = 0 \quad \text{--- (iii)}$$

Let us assume that the potential at the boundary be V_0. Then if V_{1b} be the boundary pontential for V_1 and V_{2b} be the boundary potential for V_2.

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In order to satisfy the uniqueness theorem, the value of V_1 on the boundary V_{1b} and the value of V_2 on the boundary V_{2b} must both be identical.

$$V_{1b} = V_{2b} = V_b$$

$$\text{or, } V_{1b} - V_{2b} = 0$$

Using vector identity,

$$\nabla \cdot (V\vec{D}) = V(\nabla \cdot \vec{D}) + \vec{D} \cdot (\nabla V)$$

let us apply the above identity for scalar $(V_1 - V_2)$ and vector $\nabla(V_1 - V_2)$.

$$\nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)] = (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)] + \nabla(V_1 - V_2) \cdot [\nabla(V_1 - V_2)]$$

$$\nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)] = (V_1 - V_2)\cancel{\nabla^2(V_1 - V_2)}^{\to 0} + [\nabla(V_1 - V_2)]^2$$

[From eqn (iii)]

$$\therefore \nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)] = [\nabla(V_1 - V_2)]^2$$

Integrating throughout the volume enclosed by the boundary surface,

$$\int_{vol} \nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)]\,dv = \int_{vol} [\nabla(V_1 - V_2)]^2 \quad \text{--- (iv)}$$

By using Divergence theorem,

$$\int_{vol} \nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)]\,dv = \oint_S [(V_1 - V_2)\nabla(V_1 - V_2)] \cdot d\vec{S} \quad \text{--- (v)}$$

As this surface consists of the boundary already specified on which $V_{1b} = V_{2b}$

$$\therefore \oint_S [(V_{1b} - V_{2b})\nabla(V_{1b} - V_{2b})] \cdot d\vec{S} = 0 \quad \text{--- (vi)}$$

Hence, from eqn (iv), (v) and (vi)

$$\int_{vol} [\nabla(V_1 - V_2)]^2\,dv = 0$$

$$\text{or, } [\nabla(V_1 - V_2)]^2\,dv = 0$$

$$\nabla(V_1 - V_2) = 0$$

$\therefore V_1 - V_2 = \text{constant}$

If we can show that the constant is 0, then the uniqueness theorem is proved.

The constant is evaluated by considering a point on the boundary. Here, $V_1 - V_2 = V_{1b} - V_{2b} = 0$

And we find that the constant is indeed zero

$$V_1 - V_2 = 0$$

$$\therefore V_1 = V_2$$

giving two identical solution.

Note: $\nabla^2 V = \frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V}{\delta z^2} \rightarrow$ cartesian

$$\nabla^2 V = \frac{1}{\rho}\frac{\delta}{\delta \rho}\left(\rho \frac{\delta V}{\delta \rho}\right) + \frac{1}{\rho^2}\frac{\delta^2 V}{\delta \phi^2} + \frac{\delta^2 V}{\delta z^2}$$

$\hookrightarrow$ cylindrical

$$\nabla^2 V = \frac{1}{r^2}\left(\frac{\delta}{\delta r} r^2 \frac{\delta V}{\delta r}\right) + \frac{1}{r^2 \sin\theta}\frac{\delta}{\delta \theta}\left(\sin\theta \frac{\delta V}{\delta \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\delta^2 V}{\delta \phi^2}$$

$\Rightarrow$ spherical

Some typical Capacitance Calculation by One Dimensional Laplace Equation:

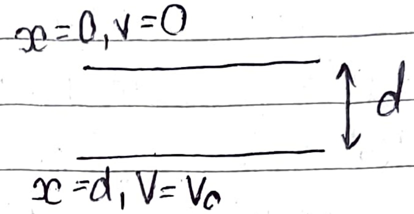
Solution: * Steps to be followed

1) Given V, use $\vec{E} = -\nabla V$ to find $\vec{E}$.
2) Use $\vec{D} = \epsilon \vec{E}$ to find $\vec{D}$
3) Evaluate $\vec{D}$ on any one of the capacitor surface D_s.
4) Use $\rho_s = D_s$

5| Find Q by $Q = \int_S \rho_s \, dS$

6| Find C by $C = \dfrac{Q}{V_o}$

a| Parallel Plate Capacitor:



Let us consider two parallel plates at $x=0,\ V=0$ and at $x=d$, $V=V_o$

Now using one dimensional Laplace eqn,

$$\frac{\delta^2 V}{\delta x^2} = 0$$

Integrating, $\dfrac{\delta V}{\delta x} = A$

Again integrating, $V = Ax + B$

Using boundary conditions, at $x=0,\ v=0$

$$0 = A \cdot 0 + B$$
$$\therefore B = 0$$

At $x=d,\ V=V_o$

$$V_o = Ad + B$$
$$V_o = Ad$$
$$\therefore A = \frac{V_o}{d}$$

So, eqn of V is $V = \dfrac{V_o x}{d}$

Now, $\vec{E} = -\nabla V$

$$= -\frac{\delta}{\delta x} V \hat{a}_x$$

$$= -\frac{\delta}{\delta x}\left(\frac{V_o}{d} x\right) \hat{a}_x$$

$$= -\frac{V_o}{d} \hat{a}_x$$

$$\vec{D} = \epsilon \vec{E}$$

$$= -\frac{\epsilon V_o}{d} \hat{a}_x$$

$$D_S = \vec{D}\,|_{x=x_0} = -\frac{\epsilon V_o}{d}$$

$$\therefore \rho_s = D_S = -\frac{\epsilon V_o}{d}$$

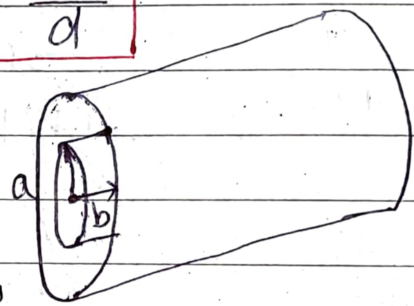
$$Q = \int_S \rho_s \, dS$$

$$= -\int_S \frac{\epsilon V_0}{d} \, dS$$

$$= -\frac{\epsilon V_0}{d} S \quad [S \rightarrow \text{area of plate}]$$

$$C = \frac{|\vec{Q}|}{V_0} = \frac{\frac{\epsilon V_0 S}{d}}{V_0} = \frac{\epsilon S}{d} \qquad \boxed{\therefore C = \frac{\epsilon S}{d}}$$

b) Cylindrical Capacitor:



Let us consider a cylindrical capacitor with radii 'a' and 'b' for inner and outer cylinder respectively.

At $\rho = a$, $V = V_0$

At $\rho = b$, $V = 0$

By Laplace eq^n,

$$\nabla^2 V = \frac{1}{\rho}\frac{\delta}{\delta\rho}\left(\rho \frac{\delta V}{\delta \rho}\right) = 0$$

$$\frac{\delta}{\delta\rho}\left(\rho \frac{\delta V}{\delta \rho}\right) = 0$$

Integrating, $\rho \frac{dV}{\delta\rho} = A$

$$\frac{\delta V}{\delta \rho} = \frac{1}{\rho} A$$

Again integrating,

$$V = A \ln \rho + B \quad \text{--- (i)}$$

Now using boundary condition,

At $\rho = a$, $V = V_0$, $V_0 = A \ln a + B$ --- (ii)

At $\rho = b$, $V = 0$

$$0 = A \ln b + B \quad \text{--- (iii)}$$

Then, $B = -A \ln B$

and $V_0 = A \ln a - A \ln b$

$$V_0 = A(\ln a - \ln b)$$

$$A = \frac{V_0}{\ln(a/b)}$$

So, $B = -A \ln b$

$$= \frac{-V_0}{\ln(a/b)} \times \ln b$$

Substituting in (i)

$$V = \frac{V_0 \ln \rho}{\ln(a/b)} - \frac{V_0 \ln b}{\ln(a/b)}$$

$$\therefore V = V_0 \frac{\ln(\rho/b)}{\ln(a/b)}$$

$\vec{E} = -\nabla V$

$= -\frac{\delta V}{\delta \rho}\hat{a}_\rho$

$= -\frac{\delta}{\delta\rho}\left[V_0 \frac{\ln(\rho/b)}{\ln(a/b)}\right]\hat{a}_\rho$

$= -\frac{V_0}{\ln(a/b)} \frac{1}{\rho}\hat{a}_\rho$

$\therefore \vec{E} = -\frac{V_0}{\rho\ln(a/b)}\hat{a}_\rho$

$\vec{D} = \epsilon\vec{E}$

$= -\frac{\epsilon V_0}{\rho \ln(a/b)}\hat{a}_\rho$

$D_s = \vec{D}\big|_{\rho=a} = \frac{-\epsilon V_0}{a\ln(a/b)}\hat{a}_\rho$

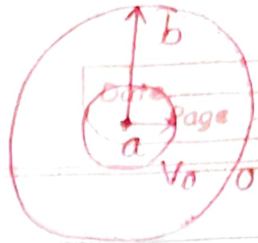
$\rho_s = D_s = \frac{-\epsilon V_0}{a\ln(a/b)}$

$Q = \int \rho_s \, dS = \int \frac{-\epsilon V_0}{a\ln(a/b)} dS$

$= \frac{-\epsilon V_0}{a\ln(a/b)} \times 2\pi a L$

$\therefore Q = \frac{2\pi\epsilon L V_0}{\ln(a/b)}$

$C = \frac{Q}{V_0} = \frac{-2\pi\epsilon L}{\ln(a/b)} = \frac{2\pi\epsilon L}{\ln(b/a)}$



Spherical Capacitor

Consider two concentric spheres with $V=0$ at $r=b$ and $V=V_0$ at $r=a$

Using Laplace eqn,

$\nabla^2 V = 0$

$\Rightarrow \frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2 \frac{\delta V}{\delta r}\right) = 0$

$\frac{\delta}{\delta r}\left(r^2 \frac{\delta V}{\delta r}\right) = 0$

Integrating, $r^2 \frac{\delta V}{\delta r} = A$

$\frac{\delta V}{\delta r} = \frac{A}{r^2}$

Integrating,

$V = -\frac{A}{r} + B$ --- (i)

Using boundary conditions,

at $r=a$, $V=V_0$

$V_0 = -\frac{A}{a} + B$ --- (ii)

at $r=b$, $V=0$,

$0 = -\frac{A}{b} + B$

$\therefore B = \frac{A}{b}$ --- (iii)

From (ii) and (iii),

$V_0 = -\frac{A}{a} + \frac{A}{b}$

$V_0 = A\left(\frac{1}{b} - \frac{1}{a}\right)$

$\therefore A = \frac{V_0}{(1/b - 1/a)}$

So, $B = \frac{V_0}{b(1/b - 1/a)}$

Eqn (i) becomes

$V = \frac{-V_0}{r(1/b - 1/a)} + \frac{V_0}{b(1/b - 1/a)}$

$= \frac{-V_0}{(1/b - 1/a)}\left(\frac{1}{r} - \frac{1}{b}\right)$

$= \frac{V_0}{(1/a - 1/b)}\left(\frac{1}{r} - \frac{1}{b}\right)$

$V = V_0 \frac{(1/r - 1/b)}{(1/a - 1/b)}$

Then, $\vec{E} = -\nabla V = -\frac{\delta V}{\delta r}\hat{a}_r$

$= -\frac{\delta}{\delta r}\left[\frac{V_0(1/r - 1/b)}{(1/a - 1/b)}\right]\hat{a}_r$

$\therefore \vec{E} = \frac{V_0}{r^2(1/a - 1/b)}\hat{a}_r$

$\vec{D} = \epsilon\vec{E} = \frac{\epsilon V_0}{r^2(1/a - 1/b)}\hat{a}_r$

$D_s = \vec{D}\big|_{r=a} = \frac{\epsilon V_0}{a^2(1/a - 1/b)}$

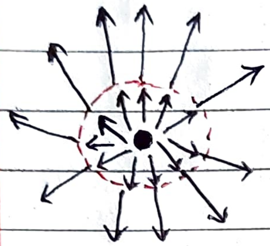
$\rho_s = D_s = \frac{\epsilon V_0}{a^2(1/a - 1/b)}$

$Q = \int \rho_s \, dS$

$= \frac{\epsilon V_0}{a^2(1/a - 1/b)} \times 4\pi a^2$

$= \frac{4\pi\epsilon V_0}{(1/a - 1/b)} \Rightarrow C = \frac{4\pi\epsilon_0}{(1/a - 1/b)}$

Streamlines:



Q_1 Given, $V = 50(x^2 - y^2)$ and a point $P(2, -1, 3)$. Determine the equation of equipotential surface and the eq^n of streamline passing through the point P.

→ Sol^n:

Given potential $V = 50(x^2 - y^2)$

Point $P(2, -1, 3)$

- To find equation of equipotential surface passing through $P(2, -1, 3)$

At point P,

$$V_P = 50(2^2 - (-1)^2)$$
$$= 150$$

So, the eq^n of equipotential surface,

$$V_P = 50(x^2 - y^2)$$
$$150 = 50(x^2 - y^2)$$
$$\therefore (x^2 - y^2) = 3$$

- To find eq^n of streamline.

$$\vec{E} = -\nabla V$$
$$= -\left[\frac{\delta V}{\delta x}\hat{a}_x + \frac{\delta V}{\delta y}\hat{a}_y + \frac{\delta V}{\delta z}\hat{a}_z\right]$$
$$= -\left[\frac{\delta}{\delta x}[50(x^2 - y^2)]\hat{a}_x + \frac{\delta}{\delta y}[50(x^2 - y^2)]\hat{a}_y + 0\right]$$
$$= -100x\hat{a}_x + 100y\hat{a}_y$$

$$E_x = -100x, \quad E_y = 100y$$

$$\frac{E_y}{dy} = \frac{E_x}{dx}$$

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$$\frac{100y}{dy} = \frac{-100x}{dx}$$

or, $\frac{y}{dy} = \frac{-x}{dx}$

or, $\frac{dy}{y} = \frac{-dx}{x}$

Integrating both sides,

$\ln y = -\ln x + c$

$\ln y + \ln x = c$

$\ln (xy) = c$

taking antiderivative on both sides,

$xy = e^c$

$\therefore xy = C_1$

At point $P(2,-1,3)$,

$2(-1) = C_1$

$\therefore C_1 = -2$

$\therefore$ The eq^n of streamline is, $\boxed{xy+2=0}$

Q, Find the eq^n of streamline for $\vec{E} = 2(y-1)\hat{a}_x + 2x\hat{a}_y$ at the point $P(-2,7,10)$.