11. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x$

Ans solution:

Given,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x$$

$$\Rightarrow (D^2 - 2D + 4)y = e^x \cos x$$

Its A.E. is

$$m^2 - 2m + 4 = 0$$

$$\Rightarrow m = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 4}}{2 \times 1}$$

$$\therefore m = 1 \pm \sqrt{3} i$$

$$\therefore C.F. = e^x [A\cos(\sqrt{3}x) + B\cos(\sqrt{3}x)]$$

Now,

$$P.I. = \frac{1}{D^2 - 2D + 4} e^x \cos x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D + 4} \cos x$$

$$= e^x \frac{1}{D^2 + 5} \cos x$$

$$= e^x \frac{1}{-1 + 5} \cos x$$

$$= -\frac{1}{6} e^x \cos x$$

The general solution is $y = C.F. + P.I.$

i.e. $y = e^x [A\cos(\sqrt{3}x) + B\sin(\sqrt{3}x)] - \frac{1}{6} e^x \cos x.$

12. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$

Ans Solution:

Given,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$$

$$\Rightarrow (D^2 + D - 2)y = x + \sin x$$

Its A.E. is

$m^2+m-2=0$

$\Rightarrow (m+2)(m-1)=0$

$\Rightarrow m=-2,1$

$\therefore$ C.F. $= c_1e^{-2x}+c_2e^x$

Now,

$$P.I. = \frac{1}{D^2+D-2}(x+\sin x)$$

$$= \frac{1}{(D^2+D-2)}x + \frac{1}{D^2+D-2}\sin x$$

$$= \frac{1}{-2\left(1-\frac{D^2+D}{2}\right)}x + \frac{1}{-1+D-2}\sin x$$

$$= -\frac{1}{2}\left(1-\frac{D^2+D}{2}\right)^{-1}x + \frac{1}{D-3}\sin x$$

$$= -\frac{1}{2}\left[1+\frac{D^2+D}{2}+\frac{(D^2+D)^2}{4}+\cdots\right]x + \frac{D+3}{D^2-9}\sin x$$

$$= -\frac{1}{2}\left[x+\frac{0+1}{2}+0+0\right] + \frac{D+3}{-1-9}\sin x$$

$$= -\frac{1}{2}x - \frac{1}{10}(\cos x+3\sin x)$$

The general solution is $y=$ C.F. + P.I.

i.e. $y=c_1e^{-2x}+c_2e^x - \frac{1}{2}x-\frac{1}{10}(\cos x+3\sin x)$

13. $(D^2-4D+4)y=x^2+e^{2x}$

Ans Solution:

Given,

$(D^2-4D+4)y=x^2+e^{2x}$

Its A.E. is:

$m^2-4m+4=0$

$\Rightarrow m=2,2$

$\therefore$ C.F. $=(c_1+c_2x)e^{2x}$

Now,

$$P.I. = \frac{1}{D^2-4D+4} x^2 + e^{2x}$$

$$= \frac{1}{(D-2)^2} x^2 + \frac{1}{(D-2)^2} e^x$$

$$= \frac{-1}{2}\left(1-\frac{D}{2}\right)^{-2} x^2 + e^x$$

$$= -\frac{1}{2}\left[1 - 2\frac{D}{2} + 3\frac{D^2}{4} + \dots\right] x^2 + e^x$$

$$= -\frac{1}{2}\left[x^2 - 2x + \frac{3}{4}.2 + 0\right] + e^x$$

$$= -\frac{1}{2}\left[x^2 - 2x + \frac{3}{2}\right] + e^x$$

The general solution is $y = C.F. + P.I.$

i.e. $y = (c_1 + c_2 x)e^{2x} - \frac{1}{2}\left[x^2 - 2x + \frac{3}{2}\right] + e^x$

---

14. $(D-2)^2 y = x^2 e^{2x}$

Ans Solution:

Given,

$$(D-2)^2 y = x^2 e^{2x}$$

$\Rightarrow$ Its A.E. is:

$$(m-2)^2 = 0$$

$$\Rightarrow m = 2, 2$$

$\therefore C.F. = (c_1 + c_2 x)e^{2x}$

Now,

$$P.I. = \frac{1}{(D-2)^2} x^2 e^{2x}$$

$$= e^{2x}\left(\frac{1}{(D+2-2)^2} x^2\right)$$

$$= e^{2x}\left(\frac{1}{D^2} x^2\right)$$

$$= e^{2x} \int \frac{x^3}{3} dx$$

$$= \frac{e^{2x}}{12} x^4$$

The general solution is $y = CF + PI$

i.e. $y = (C_1 + C_2 x)e^{2x} + \frac{e^{2x}}{12} x^4$.

(15) $(D^2 - 3D + 2)y = \cosh x$

Ans Solution:

Given,

$$(D^2 - 3D + 2)y = \cosh x$$

Its A.E. is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow m = \cancel{1, 2}\ 2, 1\ (1, 2)$$

$\therefore$ C.F. $= c_1 e^{x} + c_2 e^{2x}$

Now,

$$P.I. = \frac{1}{D^2 - 3D + 2} \cosh x$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 3D + 2} e^{x} + \frac{1}{D^2 - 3D + 2} e^{-x}\right]$$

$$= \frac{1}{2}\left[e^{x} \cdot \frac{x}{-1} + e^{-x} \cdot \frac{x}{-5}\right]$$

$$= \frac{-1}{2}\left[xe^{x} + \frac{xe^{-x}}{5}\right]$$

The general solution is $y = C.F. + P.I.$

i.e. $y = c_1 e^{x} + c_2 e^{2x} - \frac{1}{2}\left[xe^{x} + \frac{xe^{-x}}{5}\right]$

(16) $(D^2 - 1)y = \sinh x$

Ans Solution:

Given, $(D^2 - 1)y = \sinh x$

Its A.E. is

$$m^2 - 1 = 0$$

$$\Rightarrow m = -1, 1$$

$\therefore$ C.F. $= c_1 e^{-x} + c_2 e^{x}$

Now,

$$P.I. = \frac{1}{D^2-1}\sinh x$$
$$= \frac{1}{2}\left[\frac{1}{D^2-1}e^x - \frac{1}{D^2-1}e^{-x}\right]$$
$$= \frac{1}{2}\left[\frac{x}{2}e^x + \frac{x}{2}e^{-x}\right]$$
$$= \frac{x}{4}[e^x + e^{-x}]$$
$$= \frac{x}{2}\cosh x$$

The general solution is $y = C.F. + P.I.$

i.e. $y = c_1e^{-x} + c_2e^x + \frac{x}{2}\cosh x$.

---

17. $(D^2+4)y = x\sin^2 x$

Ans Solution:

Given,

$$(D^2+4)y = x\sin^2 x$$

Its A.E is

$$m^2+4=0$$
$$\Rightarrow m = \pm 2i$$

$\therefore C.F. = A\cos(2x) + B\sin(2x)$

Now,

$$P.I. = \frac{1}{D^2+4}x\sin^2 x$$

$= $ Imaginary Part (IP) of $\left[\frac{\cancel{1}}{D^2+4}\cancel{x}\, e\right.$

$$= \frac{1}{D^2+4}x\left(\frac{1-\cos 2x}{2}\right)$$
$$= \frac{1}{2}\left[\frac{1}{D^2+4}x - \frac{1}{D^2+4}x\cos 2x\right]$$
$$= \frac{1}{2}\left[\frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x - \frac{1}{D^2+9}x\cos 2x\right]$$
$$= \frac{1}{2}\left[\frac{1}{4}\left(1-\frac{2D^2}{4}+\dots\right)x - \frac{1}{D^2+4}x\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}x - \frac{1}{D^2+4}\, x\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}x - x\cdot\frac{1}{D^2+4}\cos 2x - \frac{2D}{(D^2+4)^2}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}x - x\cdot x\cdot\frac{1}{2D}\cos 2x - \frac{2D}{2(D^2+4)(2D)}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}x - \frac{x^2}{2}\frac{\sin 2x}{2} - \frac{x}{2(D^2+4)}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}x - \frac{x^2}{4}\sin 2x - \frac{x}{2}\frac{1}{2D}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}x - \frac{x^2}{4}\sin 2x - \frac{x}{4}\frac{\sin 2x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}x + \frac{x^2\sin 2x}{4} - \frac{x}{8}\sin 2x\right]$$

$$= \frac{1}{8}x + \cancel{\frac{x^4\sin 2x}{16}} + \frac{x^2}{8}\sin 2x - \frac{x}{16}\sin 2x$$

The general solution is $y = CF + PI$

i.e. $y = A\cos(2x) + B\sin(2x) + \frac{1}{8}x - \cancel{\frac{x^4\sin 2x}{16}} + \frac{x^2}{8}\sin 2x - \frac{x}{16}\sin 2x$

---

(18) $(D^2-4)y = x\sinh x$

Ans Solution:

Given,

$(D^2-4)y = x\sinh x$

Its A.E. is

$m^2 - 4 = 0$

$\Rightarrow m = -2, 2$

$\therefore$ C.F. $= c_1 e^{-2x} + c_2 e^{2x}$

Now,

$$P.I. = \frac{1}{D^2-4}(x\sinh x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}e^{x}x - \frac{1}{D^2-4}e^{-x}x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{D^2+2D+1-4}x - e^{-x}\frac{1}{D^2-2D+1-4}x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{D^2+2D-3}x - e^{-x}\frac{1}{D^2-2D-3}x\right]$$

$$= \frac{1}{2}\left[\frac{e^{x}}{-3}\left[\left(1-\frac{D^2+2D}{+3}\right)^{-1}x\right] + \frac{e^{-x}}{3}\left[\left(1-\frac{D^2-2D}{3}\right)^{-1}x\right]\right]$$

$$= \frac{1}{2}\left[\frac{e^{x}}{-3}\left[\left[1+\frac{D^2+2D}{3}+\cdots\right]x\right] + \frac{e^{-x}}{3}\left[1+\frac{D^2-2D}{3}\right]x\right]$$

$$= \frac{1}{2}\left[\frac{e^{x}}{-3}\left[x+0+\frac{2}{3}\right] + \frac{e^{-x}}{3}\left[x+0-\frac{2}{3}\right]\right]$$

$$= \frac{1}{2}\left[-\frac{e^{x}}{3}\left[x+\frac{2}{3}\right) + \frac{e^{-x}}{3}\left[x-\frac{2}{3}\right]\right]$$

The general solution is $y = CF + PI$

i.e. $y = c_1e^{-2x} + c_2e^{2x} + \frac{1}{2}\left[\frac{e^{-x}}{3}\left(x+\frac{2}{3}\right) + \frac{e^{-x}}{3}\left(x-\frac{2}{3}\right)\right]$

---

(19) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^{x}\sin x$

Ans solution:

Given,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^{x}\sin x$$

$\Rightarrow (D^2-2D+1)y = xe^{x}\sin x$

Its A.E. is

$m^2-2m+1=0$

$\Rightarrow m = 1, 1$

$\therefore$ C.F. $= (c_1+c_2x)e^{x}$

Now,

$$P.I. = \frac{1}{D^2-2D+1}xe^{x}\sin x$$

$$= e^{x}\frac{1}{D^2+2D+1-2D+1}x\sin x$$

$$= e^{x}\frac{1}{D^2+2}x\sin x$$

$$= e^{x}\left[x\frac{1}{D^2+2}\sin x - \frac{2D}{(D^2+2)^2}\sin x\right]$$

$$= e^x \left[x \sin x - 2 \cos x \right]$$

The general solution is $y = CF + PI$

i.e. $y = e^x (c_1 + c_2 x) + e^x [x \sin x - 2\cos x]$

(20) $(D^2 + 2D + 1) y = x \cos x$

Ans Solution:

Given,

$(D^2 + 2D + 1) y = x \cos x$

Its A.E. is

$m^2 + 2m + 1 = 0$

$\Rightarrow m = -1, -1$

$\therefore C.F. = (c_1 + c_2 x) e^{-x}$

Now,

$$P.I. = \frac{1}{D^2 + 2D + 1} x \cos x$$

$$= \left[x \frac{1}{D^2 + 2D + 1} \cos x - \frac{2D + 2}{(D^2 + 2D + 1)^2} \cos x \right]$$

$$= \left[x \frac{1}{2D} \cos x - \frac{2D + 2}{4D^2} \cos x \right]$$

$$= \left[\frac{x}{2} \sin x - \frac{(2D + 2)}{-4} \cos x \right]$$

$$= \left[\frac{x}{2} \sin x + \frac{1}{2} (-\sin x + \cos x) \right]$$

The general solution is $y = CF + PI$

i.e. $y = (c_1 + c_2 x) e^{-x} + \left[\frac{x}{2} \sin x + \frac{1}{2} (\cos x - \sin x) \right]$

EXERCISE-31

Solve the following differential equations.

1. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \frac{1}{x}$

Ans Solution:

Given,

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \frac{1}{x}$$

Let $x = e^z$. Then,

$$x^2\frac{d^2y}{dx^2} = \delta^2 - \delta \text{ and } x\frac{dy}{dx} = \delta$$

So,

$$(\delta^2 - \delta - 2\delta + 2)y = \frac{1}{e^z}$$

$$\Rightarrow (\delta^2 - 3\delta + 2)y = \frac{1}{e^z}$$

Its A.E. is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-2)(m-1) = 0$$

$$\text{i.e. } m = 2, 1$$

$$\therefore \text{C.F.} = c_1 e^z + c_2 e^{2z}$$

Also,

$$\text{P.I.} = \frac{1}{\delta^2 - 3\delta + 2} e^{-z} = \frac{1}{1+3+2} e^{-z} = \frac{1}{6} e^{-z}$$

The general solution is $y = \text{C.F.} + \text{P.I.}$

$$\text{i.e. } y = c_1 e^z + c_2 e^{2z} + \frac{1}{6} e^{-z}$$

$$\therefore y = c_1 x + c_2 x^2 + \frac{1}{6x}$$

② $(x^2D^2+xD-1)y=x^2$

Ans Solution:

Given,

$(x^2D^2+xD-1)y=x^2$

Let $x=e^z$.

Then,

$(\delta^2-\delta+\delta-1)y=e^{2z}$

$\Rightarrow(\delta^2-1)y=e^{2z}$

Its. A.E. is

$m^2-1=0$

$\Rightarrow m=-1,1$

$\therefore$ C.F. $=c_1e^{-z}+c_2e^{z}$

Now,

$$P.I. = \frac{1}{\delta^2-1}e^{2z}$$

$$= \frac{1}{2^2-1}e^{2z}$$

$$= \frac{1}{3}e^{2z}$$

The general solution is ~~P.I~~

$y=PI+CF$

$\Rightarrow y=\frac{1}{3}e^{2z}+c_1e^{-z}+c_2e^{z}$

i.e. $y=c_1x+\frac{c_2}{x}+\frac{1}{3}x^2$

③ $(x^2D^2+xD+1)y=\sin(\log(x^2))$

Ans Solution:

The given differential equation is

$(x^2D^2+xD+1)y=\sin[\log(x^2)]$

which is second order homogeneous equation. So,

let $x=e^z \Rightarrow z=\ln(x)$

Also,

$x^2D^2=\delta^2-\delta$, where, $\delta=\frac{d}{dz}$ and $\delta^2=\frac{d^2}{dz^2}$

$xD=\delta$

So,

$$(\delta^2-\delta+\delta+1)y = \sin[2z]$$
$$\Rightarrow (\delta^2+1)y = \sin(2z)$$

Which is second order linear differential equation with constant coefficients.

So, its A.E is

$$m^2+1=0$$
$$\Rightarrow m = \pm i$$

So,

$$C.F. = A\cos(z) + B\sin(z)$$

Also,

$$P.I. = \frac{1}{\delta^2+1}\sin(2z)$$
$$= \frac{1}{-4+1}\sin(2z)$$
$$= -\frac{1}{3}\sin(2z)$$

So,

The general solution is

$$y = CF + PI$$
$$= A\cos(z) + B\sin(z) - \frac{1}{3}\sin(2z)$$
$$= A\cos(\ln(x)) + B\sin(\ln x) - \frac{1}{3}\sin[2\ln x]$$

$\therefore$ The general solution is $y = A\cos(\ln x) + B\sin(\ln x) - \frac{1}{3}\sin(2\ln x)$

4. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^6$

Ans Solution:

Given,

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^6$$
$$\Rightarrow (x^2D^2 - 3xD + 4)y = x^6$$

Let $x = e^z$.

$\Rightarrow z = \ln(x)$.

Also,

$x^2D^2 = \delta^2 - \delta$ where $\delta = \frac{d}{dz}$ and $\delta^2 = \frac{d^2}{dz^2}$

and $xD = \delta$

So,

$(\delta^2 - \delta - 3\delta + 4)y = e^{6z}$

$\Rightarrow (\delta^2 - 4\delta + 4)y = e^{6z}$

Which is second order linear differential equation with constant coefficients.

So, its A.E. is

$m^2 - 4m + 4 = 0$

$\Rightarrow m = 2, 2$

$\therefore$ C.F. $= \cancel{c_1e^{2z} + c}\ (c_1 + c_2z)\,e^{2z}$.

Also,

$$P.I. = \frac{1}{\delta^2 - 4\delta + 4} e^{2z}$$

$$= z \cdot \frac{1}{2\delta - 4} e^{2z}$$

$$= z^2 \cdot \frac{1}{2} e^{2z}$$

$$= \frac{z^2 e^{2z}}{2}$$

The general solution is $y = C.F. + P.I.$

$$\Rightarrow y = (c_1 + c_2z)e^{2z} + \frac{z^2e^{2z}}{2}$$

$$\text{i.e. } y = (c_1 + c_2\ln x)x^2 + \frac{[\ln(x)]^2 x^2}{2}$$

5.4. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4x^3$

Ans Solution:

Given,

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4x^3$$

$\Rightarrow (x^2D^2 - 2xD + 2)y = 4x^3$

Where, $D = \frac{d}{dx}$ and $D^2 = \frac{d^2}{dx^2}$

Let $x = e^z$

$\Rightarrow z = \ln(x)$

Here, $x^2D^2 = \delta^2 - \delta$ where $\delta = \frac{d}{dz}$ and $\delta^2 = \frac{d^2}{dz^2}$

and $xD = \delta$

So,

$(\delta^2 - 3\delta + 2)y = 4e^{3z}$

which is ~~a~~ second order line differential equation with constant coefficients.

Its A.E. is

$m^2 - 3m + 2 = 0$

$\Rightarrow m = 1, 2$

$\therefore$ C.F. $= (c_1e^z + c_2e^{2z})$

Now,

P.I. $= \frac{1}{\delta^2 - 3\delta + 2} 4e^{3z}$

$= \frac{4e^{3z}}{9 - 9 + 2}$

$= 2e^{3z}$

The general solution is $y = $ C.F. + P.I.

$= \cancel{c_1 + c_2 x}\ c_1x + c_2x^2 + 2x^3$

$\therefore$ The general solution is $y = c_1x + c_2x^2 + 2x^3$

---

6. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$

~~Ans~~ Solution:

Given,

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$

$\Rightarrow (x^2D^2 + xD - 4)y = x^2$

Let $x = e^z$. $\Rightarrow x^2D^2 = \delta^2 - \delta$ and $xD = \delta$

where, $\delta = \frac{d}{dz}$ and $\delta^2 = \frac{d^2}{dz^2}$

So,
$(\delta^2-4)y = e^{2z}$
Which is second order linear differential equations with constant coefficients.
Its A.E. is
$$m^2-4=0$$
$$\Rightarrow m=2,-2$$
$\therefore$ C.F. $= c_1e^{2z}+c_2e^{-2z}$
Also,
$$P.I. = \frac{1}{\delta^2-4}e^{2z}$$
$$= z\frac{1}{2\delta}e^{2z}$$
$$= \frac{1}{2}e^{2z}$$
$\therefore$ The general solution is $y = c_1x^2+\frac{c_2}{x^2}+\frac{1}{2}x^2$

7. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log(x)$

Ans Solution:
Given,
$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log(x)$$
Let $x=e^z$
So, $x^2D^2 = \delta^2-\delta$ and $xD=\delta$
Where, $\delta = \frac{d}{dz}$ and $\delta^2 = \frac{d^2}{dz^2}$
So,
$(\delta^2-2\delta+1)y = z$
Its A.E. is
$$m^2-2m+1=0$$
$$\Rightarrow m=1,1$$
$\therefore$ C.F. $= (c_1+c_2z)e^z$
Also,

$$P.I. = \frac{1}{\delta^2-2\delta+1} z$$

$$= (1-\delta)^{-2} z$$

$$= (1+2\delta) z$$

$$= z+2$$

$\therefore$ The general solution is $y = (c_1 + c_2 \ln x)x + \ln(x) + 2$

8. $(x^2D^2-2)y = x^2 + \frac{1}{x}$

Ans Solution:

Given,

$$(x^2D^2-2)y = x^2 + \frac{1}{x}$$

let $x = e^z$

$$\Rightarrow x^2D^2 = \delta^2 - \delta$$

Where, $\delta^2 = \frac{d^2}{dz^2}$ and $\delta = \frac{d}{dz}$

So,

$$(\delta^2-\delta-2)y = e^{2z} + e^{-z}$$

Its A.E. is

$$m^2 - m - 2 = 0$$

$$\Rightarrow m = -1, +2$$

$\therefore$ C.F. $= c_1 e^z + c_2 e^{-2z}$

Also,

$$P.I. = \frac{1}{\delta^2-\delta-2}(e^{2z} + e^{-z})$$

$$= z \cdot \frac{1}{2\delta-1} e^{2z} + z \cdot \frac{1}{2\delta-1} e^{-z}$$

$$= z\left(\frac{1}{2\times2-1} e^{2z} + \frac{1}{2\times(-2)-1} e^{-z}\right)$$

$$= z\left(\frac{e^{2z}}{3} - \frac{1}{5} e^{-z}\right)$$

$\therefore$ The general solution is $y = c_1 x + \frac{c_2}{x^2} + \ln(x)\left(\frac{x^2}{3} - \frac{1}{5x}\right)$

9. $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x$

Ans. Solution:

Given,

$$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x$$

Let $x=e^z$.

$\Rightarrow x^2D^2=\delta^2-\delta$ and $xD=\delta$

Where, $\delta^2=\frac{d^2}{dz^2}$ and $\delta=\frac{d}{dz}$

So,

$$(\delta^2-\delta+4\delta+2)y=e^x$$

$$\Rightarrow(\delta^2+3\delta+2)y=e^{e^z}$$

Its A.E. is

$$m^2+3m+2=0$$

$$\Rightarrow m=-1,-2 \qquad \therefore CF=c_1e^{-z}+c_2e^{-2z}$$

Also,

$$P.I.=\frac{1}{\delta^2+3\delta+2}e^{e^z}$$

$$=\left(\frac{1}{\delta+1}-\frac{1}{\delta+2}\right)e^{e^z}$$

$$=\frac{1}{\delta+1}e^{e^z}-\frac{1}{\delta+2}e^{e^z}$$

$$=e^{-z}\int e^ze^{e^z}dz-e^{-2z}\int e^{2z}e^{e^z}dz$$

$$=e^{-z}e^{e^z}-e^{-2z}(e^ze^{e^z}-e^{e^z})$$

$$=e^{-z}e^{e^z}-e^{-z}e^{e^z}+e^{-2z}e^{e^z}$$

$$=e^{-2z}e^{e^z}$$

$\therefore$ The general solution is

$$y=\frac{c_1}{x}+\frac{c_2}{x^2}+\frac{1}{x^2}e^x$$

10. $(x+p)^2\frac{d^2y}{dx^2}-4(x+p)\frac{dy}{dx}+6y=x$

Ans Solution:

Given,

$$(x+p)\frac{d^2y}{dx^2} - 4(x+p)\frac{dy}{dx} + 6y = x$$

let $x+p = X = e^z$

$\Rightarrow (x+p)D^2 = \delta^2 - \delta$

and $(x+p)D = \delta$

Where, $\delta = \frac{d}{dz}$ and $\delta^2 = \frac{d^2}{dz^2}$

So,

$(\delta^2 - \delta - 4\delta + 6)y = e^z - p$

$\Rightarrow (\delta^2 - 5\delta + 6)y = e^z - p$

Its A.E. is

$m^2 - 5m + 6 = 0$

$\Rightarrow m = 2, 3$

$\therefore$ C.F. $= c_1 e^{2z} + c_2 e^{3z}$

Also,

P.I. $= \frac{1}{\delta^2 - 5\delta + 6}(e^z - p)$

$= \frac{1}{\delta^2 - 5\delta + 6} e^z - \frac{1}{\delta^2 - 5\delta + 6} p e^0$

$= \frac{e^z}{1-5+6} - p\frac{1}{0+6}$

$= \frac{1}{2}e^z - \frac{p}{6}$

$\therefore$ The general solution is $y = c_1(x+p)^2 + c_2(x+p)^3 + \frac{1}{2}(x+p) - \frac{p}{6}$

---

11. $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4)$

Ans. Solution:

Given,

$$(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4)$$

let $x+1 = X = e^z$

$\Rightarrow (x+1)^2 D^2 = \delta^2 - \delta$

and $(x+1) D = \delta$

So,

$(\delta^2 - \delta + \delta) y = (2(x+1)+1)(2(x+1)+2)$

$\Rightarrow \delta^2 y = (2e^z+1)(2e^z+2)$

Its A.E. is

$m^2 = 0$

$\Rightarrow m = 0, 0$

$\therefore$ ~~The~~ $\therefore$ C.F. $= c_1 + c_2 z$

Also,

$$P.I. = \frac{1}{\delta^2}(2e^z+1)(2e^z+2)$$

$$= \frac{1}{\delta^2}(4e^{2z} + 6e^z + 2)$$

$$= \frac{4}{4}e^{2z} + \frac{6}{1}e^z + 2 \cdot \frac{1}{\delta^2}e^0$$

$$= e^{2z} + 6e^z + 2z \frac{1}{2\delta}e^0$$

$$= e^{2z} + 6e^z + \frac{2z^2}{2}e^0$$

$$= e^{2z} + 6e^z + z^2$$

$\therefore$ The general solution is

$$y = c_1 + c_2 \ln(x+1) + (x+1)^2 + 6(x+1) + [\ln(x+1)]^2$$

12. $(2x+1)^2 \frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$

Ans Solution:

Given,

$$(2x+1)^2 \frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$$

let $(2x+1) = X = e^z$

$\Rightarrow (2x+1)^2 \frac{d^2y}{dx^2} = 4(\delta^2 - \delta)y$ where $\delta y = \frac{dy}{dz}$

and $(2x+1)\frac{dy}{dx} = 2\delta y$ and $\delta^2 y = \frac{d^2y}{dz^2}$

So,

$(4\delta^2-4\delta-6\times2\delta+16)y=8e^{2z}$

$\Rightarrow (4\delta^2-16\delta+16)y=8e^{2z}$

$\Rightarrow (\delta^2-4\delta+4)y=2e^{2z}$

Its A.E. is

$m^2-4m+4=0$

$\Rightarrow m=2,2$

$\therefore$ C.F. $=(c_1+c_2z)e^{2z}$

Also,

$$P.I.=\frac{1}{\delta^2-4\delta+4}2e^{2z}$$

$$=2.z.\frac{1}{2\delta-4}e^{2z}$$

$$=\frac{2z^2}{2}.e^{2z}$$

$$=z^2e^{2z}$$

$\therefore$ The general solution is

$$y=(c_1+c_2\ln(2x+1))(2x+1)^2+\frac{[\ln(2x+1)]^2}{(2x+1)^2}$$

---

* $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4$

Ans Solution:

Given,

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4$$

Let $x=e^z$.

$\Rightarrow x^2D^2=\delta^2-\delta$ and $xD=\delta$

So, $(\delta^2-\delta-2\delta-4)y=e^{4z}$

$\Rightarrow (\delta^2-3\delta-4)y=e^{4z}$

Its A.E. is

$m^2-3m-4=0$

$\therefore m=-1,4$

$\therefore$ C.F. $=c_1e^{-z}+c_2e^{4z}$

Also,

$$P.I = \frac{1}{\delta^2-3\delta-4} e^{4z}$$

$$= z \cdot \frac{1}{2\delta-3} e^{4z}$$

$$= \frac{z}{5} e^{4z}$$

∴ The general solution is

$$y = \frac{c_1}{x} + c_2 x^4 + \frac{\ln(x)}{5} x^4$$

Assuming the validity of expansion, find the ~~follow~~ expansion of the following functions ~~verifying Mac~~ very Maclaurin's se theorem.

~~Solution~~.

a) $\log(1+x)$

~~Ans~~ Solution:

The given function is

$$f(x) = \log(1+x) \Rightarrow f(0) = 0$$

Here,

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$$

$$f^{iv}(x) = -6\frac{1}{(1+x)^4} \Rightarrow f^{iv}(0) = -6 \text{ and so on.}$$

Applying Maclaurin's theorem, we get,

$$f(x) = \log(1+x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

$$= 1 + \frac{x}{1!}(+1) + \frac{x^2}{2!}\cdot(-1) + \frac{x^3}{3!}(2) - \frac{x^4}{4!}f^{iv}(0) + \cdots$$

$$= 1 + \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\therefore \log(1+x) = 1 + \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

b) $\sinh x$

~~Ans~~ Solution:

The given function is

$$y = \sinh(x).$$

So,

$f(x) = \sinh(x) \quad \Rightarrow f(0) = 0$

$\Rightarrow f'(x) = \cosh(x) \quad \Rightarrow f'(0) = 1$

$\Rightarrow f''(x) = \sinh(x) \quad \Rightarrow f''(0) = 0$

$\Rightarrow f'''(x) = \cosh(x) \quad \Rightarrow f'''(0) = 1$

$\Rightarrow f^{iv}(x) = \sinh(x) \quad \Rightarrow f^{iv}(0) = 0$

Applying Maclaurin's theorem, we get,

$$\sinh x = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\therefore \sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

---

c) $\cosh(x)$

Ans Solution:

Given,

$f(x) = \cosh(x) \quad \Rightarrow f(0) = 1$

$\Rightarrow f'(x) = \sinh(x) \quad \Rightarrow f'(0) = 0$

$\Rightarrow f''(x) = \cosh(x) \quad \Rightarrow f''(0) = 1$

$\Rightarrow f'''(x) = \sinh(x) \quad \Rightarrow f'''(0) = 0$

$\Rightarrow f^{iv}(x) = \cosh(x) \quad \Rightarrow f^{iv}(0) = 1$

Applying Maclaurin's theorem, we get,

$$\cosh(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\therefore \cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

4. $\tanh(x)$.

Ans Solution:

Given,

$f(x) = \tanh(x) \quad \Rightarrow f(0) = 0$

So,

$f'(x) = \operatorname{sech}^2(x)$

$f''(x) = -2\operatorname{sech}^2(x) \cdot \tanh(x)$

$= -2\operatorname{sech}^2(x) \cdot f(x)$

$f'''(x) = -2[f'(x) \cdot 2\operatorname{sech}^2(x) \cdot \tanh(x) + f'(x) \cdot \operatorname{sech}^2 x]$

$= -2[f(x) \cdot f''(x) + f'(x) \cdot \cancel{\text{se}} f'(x)]$

$f^{iv}(x) = -2[f'(x) \cdot f''(x) + f'''(x) f(x) + f'(x) f''(x) + f''(x) f'(x)]$

$f^{v}(x) = -2[f'(x) \cdot f'''(x) + f''(x) f''(x) + f'''(x) f'(x) + f(x) \cdot f^{iv}(x) + f'(x) f'''(x) + f''(x) \cdot f''(x) + f''(x) \cdot f''(x) + f'(x) \cdot f'''(x)]$

So,

$f'(0) = 1$

$f''(0) = 0$

$f'''(0) = -2[0 \times 0 + 1] = -2$

$f^{iv}(0) = -2[1 \times 0 + (-2) \times 0 + 1 \times 0 + 0 \times 1] = 0$

$f^{v}(0) = -2[1 \times (-2) + 0 + (-2) + 0 + 1 \times (-2) + 0 + 0 + (-2)]$

$= -2[-8]$

$= 16$

According to Maclaurian theorem,

$f(x) = \tanh(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \cdots$

$\therefore \tanh(x) = 1 - \frac{2}{3!} x^3 + \frac{16}{5!} x^5 - \cdots$

5) $\tan x$.

Ans Solution:

The given function is

$$f(x) = \tan x$$

Here,

$f'(x) = \sec^2 x$

$f''(x) = 2\sec^2 x \tan x = 2\sec^2 x \cdot f(x)$

$f'''(x) = 2[f(x) \cdot 2\sec^2 x \cdot \tan x + f'(x) \cdot \sec^2 x]$

$= 2[f(x) \cdot f''(x) + f'(x) \cdot f'(x)]$

$f^{iv}(x) = 2[f(x) \cdot f'''(x) + f''(x) \cdot f'(x) + f'(x) \cdot f''(x) + f'(x) f''(x)]$

$f^{v}(x) = 2[f(x) \cdot f^{iv}(x) + f'''(x) \cdot f'(x) + f'(x) f'''(x) + f''(x) \cdot f''(x) + f'''(x) \cdot f'(x) + f''(x) \cdot f''(x) + f'(x) \cdot f'''(x) + f''(x) f''(x)]$

So, and so on:

$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = 2$,

$f^{iv}(0) = 0$, $f^{v}(0) = 2[a + 8] = 16$

Applying maclaurin's theorem,

$$f(x) = \tan x = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \cdots$$

$$= 0 + \frac{x}{1!} + 0 + \frac{x^3}{3!} \cdot 2 + 0 + \frac{x^5}{5!} 2 \times 8 + \cdots$$

or, $\tan(x) = x + \frac{x^3}{3!} + 2 \times 8 \frac{x^5}{5!} + \cdots$

$\therefore \tan(x) = x + \frac{x^3}{3!} + 16 \frac{x^5}{5!} + \cdots$

6) $e^x \sin x$.

Ans Solution:

The given function is

$$f(x) = e^x \sin(x).$$

So,

$$f'(x) = e^x \cos x + e^x \sin(x)$$
$$= e^x \cos x + f(x)$$
$$f''(x) = e^x(-\sin x) + e^x \cos x + f'(x)$$
$$= -e^x \sin x + e^x \cos x + f'(x)$$
$$= -f(x) + f'(x) - f(x) + f'(x)$$
$$= 2[f'(x) - f(x)]$$
$$f'''(x) = 2[f''(x) - f'(x)]$$
$$f^{iv}(x) = 2[f'''(x) - f''(x)]$$ and so on

So,

$f(0) = 0$ $\quad f''(0) = 2$ $\quad f^{iv}(0) = 4$

$f'(0) = 1$ $\quad f'''(0) = 2$

Applying Maclaurin's theorem, we get

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$$= 0 + \frac{x}{1!} + \frac{x^2}{2!} \cdot 2 + \frac{x^3}{3!} \cdot 2 + \frac{x^4}{4!} \cdot 4 + \cdots$$

$$\therefore e^x \sin x = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

---

7) $\ln(1+\sin x)$

Solution:

Let the given function be

$$f(x) = \ln(1+\sin x).$$

So,

$$f'(x) = \frac{\cos x}{1+\sin x}$$

$$f''(x) = \frac{(1+\sin x).(-\sin x) - \cos x . \cos x}{(1+\sin x)^2}$$

$$= -\frac{1}{1+\sin x}$$

$$f'''(x) = \frac{1}{(1+\sin x)^2} . \cos x$$

$$f^{iv}(x) = \frac{(1+\sin x)^2 d(\cos x) - \cos x . 2(1+\sin x) . \cos x}{(1+\sin x)^4}$$

$$= \frac{(1+\sin x)^2 . (-\sin x) - \cos^2 x . 2(1+\sin x)}{(1+\sin x)^4}$$

and so on.

So,

$f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 1$, $f^{iv}(0) = -2$

Applying maclaurin's series, we get,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$$\therefore \ln(1+\sin x) = x + \frac{x^3}{3!} - \frac{x^2}{2!} + \frac{x^4}{4!} . (-2) + \cdots$$

$$\therefore \ln(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$$

8) $\log(\sec x)$

Solution:

Let the given function be

$$f(x) = \log(\sec x).$$

So,

$$f'(x) = \frac{1}{\sec x} . \sec x \tan x = \tan x$$

$$f''(x) = \sec^2 x$$

$$f'''(x) = 2\sec^2 x \tan x = 2 f''(x) . f'(x)$$

$$f^{iv}(x) = 2[f''(x) f''(x) + f'(x) f'''(x)]$$

and so on.

Clearly,

$f(0) = 0$, $f'(0) = 0$, $f''(0) = 1$, $f'''(0) = 0$, $f^{iv}(0) = 2$

Applying Maclaurin's series, we get,

$$f(x) = \ln(\sec x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$$\therefore \ln(\sec x) = \frac{x^2}{2} + \frac{2x^4}{4!} + \frac{16x^6}{6!} + \cdots$$

Previous part

Also, $f^{v}(x) = 2[f''(x).f'''(x) + f''(x).f'''(x) + f'(x).f^{iv}(x) + f'''(x).f''(x)]$

$= 2[2f''(x)f'''(x) + f'(x).f^{iv}(x) + f'''(x)f''(x)]$

$f^{vi}(x) = 2[2(f''(x)f^{iv}(x) + f'''(x).f'''(x)) + f'(x).f^{v}(x) + f^{iv}(x)f''(x) + f'''(x)f'''(x) + f''(x)f^{iv}(x)]$

and so on.

$\Rightarrow f^{v}(0) = 0,\ f^{vi}(x) = 16$

9) $\frac{e^x}{1+e^x}$

~~Ans~~ Solution:

Let the given function be

$$f(x) = \frac{e^x}{1+e^x}$$

So,

$$f'(x) = \frac{(1+e^x).e^x - e^x.e^x}{(1+e^x)^2}$$

$$= \frac{e^x}{(1+e^x)^2}$$

$$f''(x) = \frac{(1+e^x)^2 e^x - e^x.2(1+e^x).e^x}{(1+e^x)^4}$$

$$= \frac{e^x + 2e^{2x} + e^{3x} - 2e^{2x} - 2e^{3x}}{(1+e^x)^4}$$

$$= \frac{e^x - e^{3x}}{(1+e^x)^4} \qquad f''''(x) = \frac{(1+e^x)^4 . (e^x - 3e^{3x}) - (e^x - 3e^{3x}) [4(1+e^x)^3 e^x]}{(1+e^x)^8}$$

and so on.

Also,

$$f(0) = \frac{1}{2}, f'(0) = \frac{1}{4}, f''(0) = 0, f'''(0) = -\frac{1}{8}$$

Applying Maclaurin's series, we get,

$$f(x) = \frac{e^x}{1+e^x} = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\text{or, } \frac{e^x}{1+e^x} = \frac{1}{2} + \frac{1}{4} x + 0 + \frac{x^3}{3!} \times \left(-\frac{1}{8}\right) + \dots$$

$$\therefore \frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{1}{48} x^3 + \dots$$

10) $\log(1+e^x)$

Ans Solution:

The function is $f(x) = \ln(1+e^x)$.

So,

$$f'(x) = \frac{1}{1+e^x} . e^x$$

$$f''(x) = \frac{e^x}{(1+e^x)^2}$$

$$f'''(x) = \frac{e^x - e^{3x}}{(1+e^x)^4}$$

$$f^{iv}(x) = \frac{(1+e^x)^4 . (e^x - 3e^{3x}) - (e^x - 3e^{3x}) . 4(1+e^x)^3 . e^x}{(1+e^x)^8}$$

and so on.

Also,

$$f(0) = \ln(2), f'(0) = \frac{1}{2}, f''(0) = \frac{1}{4}, f'''(0) = 0$$

$$f^{iv}(0) = -\frac{1}{8}$$

Applying Maclaurin's series, we get,

$$f(x) = \ln(1+e^x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(x) + \cdots$$

$$\therefore \ln(1+e^x) = \ln(2) + \frac{1}{2}x + \frac{x^2}{8} - \frac{x^4}{24\times 8} + \cdots$$

xi) ~~e^x~~ $e^{\sin x}$

Solution:

Let the given function be

$$f(x) = e^{\sin x}$$

So,

$$f'(x) = \cos x\, f(x)$$

$$\Rightarrow f''(x) = f'(x)\cos x - \sin x\, f(x)$$

$$\Rightarrow f'''(x) = \cos x\, f''(x) - \sin x\, f''(x) - f(x)\cos x - f'(x)\sin x$$

$$\Rightarrow f^{iv}(x) = \cos x\, f'''(x) - \sin x\, f''(x) - \sin x\, f'''(x) - f''(x)\cos x + f(x)\sin x - \cos x\, f'(x) - f''(x)\sin x - \cos x\, f''(x)$$

and so on.

So,

$f(0) = 1$, $f''(0) = 1$, $f^{iv}(x) = -3$

$f'(0) = 1$, $f'''(0) = 0$,

Applying Maclaurin's series,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\therefore e^{\sin x} = 1 + \frac{x}{1} + \frac{x^2}{2} - 3\frac{x^4}{4!} + \cdots$$

$$\therefore e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

xii. $x\cot x$.

Ans Solution:

Let the function be

$$f(x) = x\cot x.$$
$$= \frac{x\cos x}{\sin x}$$

Let $f(x) = x\cot x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$ — (A)

$\Rightarrow x = \tan x\,(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots)$

$\Rightarrow x = \left(x + \frac{x^3}{3!} + \frac{16x^5}{5!} + \cdots\right)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots)$

$\Rightarrow x = (a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + \cdots)$
$+ \left(\frac{x^3}{3!}a_0 + \frac{x^4}{3!}a_1 + \frac{x^5}{3!}a_2 + \frac{x^6}{3!}a_3 + \cdots\right)$

$\Rightarrow \left(+ \frac{16x^5}{5!}a_0 + \frac{16x^6}{5!}a_1 + \frac{16}{5!}x^7a_2 + \cdots\right)$

$\Rightarrow x = a_0x + a_1x^2 + \left(a_2 + \frac{a_0}{3!}\right)x^3 + \left(a_3 + \frac{a_1}{3!}\right)x^4 + \cdots$

Equating the corresponding coefficients,

$a_0 = 1;\ a_1 = 0$

$a_2 + \frac{1}{3} = 0 \qquad a_3 + \frac{a_1}{3!} = 0;\ a_4 + \frac{a_2}{3!} + \frac{16}{5!} = 0$

$\Rightarrow a_2 = -\frac{1}{3} \qquad \Rightarrow a_3 = 0 \Rightarrow a_4 = -\frac{16}{5!}$

Substituting the values in equation (A),

We get,

$f(x) = x\cot x = 1 - \frac{1}{3}x^2 - \frac{16}{5!}x^4 + \cdots$

$\therefore x\cot(x) = 1 - \frac{1}{3}x^2 - \frac{16}{5!}x^4 + \cdots$

Assuming the validity of expansion, prove the following series using Maclaurin's series.

i. $\sin^{-1}(x) = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \cdots$

and hence $\cos^{-1}(x) = \frac{\pi}{2} - x - \frac{x^3}{3!} - \cdots$

Ans Solution:

The given function is

$y = \sin^{-1}(x) \Rightarrow y_0 = 0$

Also,

$y_1 = \frac{1}{\sqrt{1-x^2}} \Rightarrow (y_1)_0 = 1$

$\Rightarrow (1-x^2) y_1^2 = 1$

Diff. using leibnitz's rule, we get,

$(1-x^2) \cdot 2y_1 \cdot y_2 + (-2x) \cdot y_1^2 = 0$

$\Rightarrow 2(1-x^2) y_1 y_2 - 2xy_1^2 = 0$

$\Rightarrow (1-x^2) y_2 - xy_1 = 0$

$\Rightarrow (y_2)_0 = 0$

Also,

$(1-x^2) \cdot y_3 - (2x) \cdot y_2 - (2x) y_2 - 2y_1 = 0$

$\Rightarrow (1-x^2) y_3 - 2xy_2 - 2xy_2 - 2y_1 = 0$

$\Rightarrow (y_3)_0 = 2$

Differentiating n times using leibnitz rule, we get,

$(1-x^2) 2y_1 \cdot y_{n+1} - (2x) \cdot 2y_1 \cdot y_n - 2 \cdot 2y_1 y_{n-1} = 0$

For $x = 0$,

$2(y_1)_0 (y_{n+1})_0 - 4(y_1)_0 (y_{n-1})_0 = 0$

$\Rightarrow (y_1)_0 (y_{n+1})_0 = 2(y_1)_0 (y_{n-1})_0$

So,

$f''(0) = \frac{2 \times 1 \cdot 0}{1} = 0$; $f'''(0) = \frac{2 \times 1 \times 1}{1 \times}$

ii. $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots$

Ans Solution:

Let $f(x) = \sec x$.

$\Rightarrow y = \sec x$

$\Rightarrow y_0 = 1$

Also,

$y_1 = \sec x \tan x = y_0 \tan x \Rightarrow f'(0) = 0$

$y_2 = y_1 \tan x + y_0 \sec^2 x$

$= y_1 \tan x + y_0 . (y_0)^2$

$= y_1 \tan x + (y_0)^3 \Rightarrow f''(0) = 1$

$y_3 = y_1 \sec^2 x + \tan x y_2 + 3(y_0)^2 y_1$

$= y_1 \sec^2 x + y_2 \tan x + 3(y_0)^2 y_1 \Rightarrow f'''(0) = 0$

$y_4 = y_1 . 2\sec^2 x \tan x + \sec^2 x y_2 + y_2 \sec^2 x + \tan x y_3 + 3.(2y_0 y_1 + (y_0)^2 . y_2)$

$\Rightarrow f^{iv}(0) = 1 + 1 + 3(1)$
$= 5$

According to Maclaurin's expansion,

$y = f(x) = y_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \cdots$

$\therefore \sec x = 1 + \frac{x^2}{2} + \frac{5}{24} x^4 + \cdots$

iii. $\log(1 + x + x^2) = x - \frac{x^2}{2} - \frac{2}{3} x^3 + \frac{x^4}{4} + \cdots$

Ans Solution:

Let $f(x) = \log(1 + x + x^2)$.

$\Rightarrow f(0) = 0$

Also,

$f'(x) = \frac{1}{1 + x + x^2} . (2x + 1) = \frac{2x + 1}{1 + x + x^2}$

$\Rightarrow f'(0) = 1$

Also,

$$y = \ln(1+x+x^2)$$

$$\Rightarrow y_1 = \frac{1+2x}{1+x+x^2}$$

$$\Rightarrow y_1(1+x+x^2) = 1+2x \Rightarrow y_1 = 1$$

Differentiating 'n' times using leibnitz rule, we get,

$$(1+x+x^2)y_{n+1} + n(1+2x)\cdot y_n + 2\cdot\frac{n(n-1)}{2} y_{n-1} + 0 = \frac{d^n}{dx^n}(1+2x)$$

$\Rightarrow$ For $x=0$

$$(y_{n+1})_0 + n(y_n)_0 + n(n-1)(y_{n-1})_0 = 0$$

$$\Rightarrow (y_{n+1})_0 = -n(y_n)_0 - n(n-1)(y_{n-1})_0$$

So,

$$(y_2)_0 = -1y_1 - 0$$
$$= -1\times 1$$
$$= -1$$

$$(y_3)_0 = -2(y_2)_0 - 2(y_1)_0$$
$$= -2\times(-1) - 2\times 1$$
$$= 2-2$$
$$= 0$$

$$(y_4)_0 = 3y_3 - 3\times 2 y_2$$
$$= 3\times 0 - 3\times 2\times(-1)$$
$$= 6$$

According to Maclaurin's series,

$$f(x) = y(0) + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0$$

$$\therefore \ln(1+x+x^2) = \frac{x}{1!} - \frac{x^2}{2!} + \frac{6}{4!}x^4$$

iv. $x\operatorname{cosec}x = 1 + \frac{1}{6}x^6 + \frac{7}{360}x^4 + \cdots$

~~Ans~~ Solution:

let $f(x) = x\operatorname{cosec}x$

$\cancel{\Rightarrow f(0)}$

$\Rightarrow f(x) = \frac{x}{\sin x}$

$\Rightarrow f(x) = \frac{x}{\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]}$

$\Rightarrow f(x) = x\left[1 - \left(1 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots\right)\right]^{-1}$

$\Rightarrow f(x) = x\left[1 + \left(1 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots\right)\right.$
$\left. + \left(1 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots\right)^2\right.$

Solution:

let $f(x) = x\operatorname{cosec}(x)$ $[0.\infty$ form$]$

So,

Let $f(x) = x\operatorname{cosec}(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

$\Rightarrow x = \sin x\,(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)$

$\Rightarrow x = \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots\right)$

$\Rightarrow x = (a_0x + a_1x^2 + a_2x^3 + \cdots) - \left(a_0\frac{x^3}{3!} + a_1\frac{x^4}{3!} + a_2\frac{x^5}{3!} + \frac{a_3x^6}{3!} + \cdots\right)$
$+ \left(a_0\frac{x^5}{5!} + a_1\frac{x^6}{5!} + \frac{a_2}{5!}x^7 + \cdots\right)$

$\Rightarrow x = a_0x + a_1x^2 + \left(a_2 - \frac{a_0}{3!}\right)x^3 + \left(a_3 - \frac{a_1}{3!}\right)x^4 +$
$\left(a_4 - \frac{a_2}{3!} + \frac{a_0}{5!}\right)x^5 + \cdots$

Equating the corresponding terms, we get,

$$a_0 = 1,\ a_1 = 0,\ a_2 - \frac{a_0}{3!} = 0,$$

$$\Rightarrow a_2 = \frac{1}{3!},$$

$$a_3 - \frac{a_1}{3!} = 0 \Rightarrow a_3 = 0,\ a_4 - \frac{a_2}{3!} + \frac{a_0}{5!} = 0$$

$$\Rightarrow a_4 = \frac{1}{3! \times 3!} - \frac{1}{5!}$$

$$\Rightarrow a_4 = \frac{1}{36} - \frac{1}{120}$$

$$\Rightarrow a_4 = \frac{120-36}{120 \times 36} = \frac{84}{36 \times 120}$$

$$\Rightarrow a_4 = \frac{21}{9 \times 120} = \frac{21}{1080} = \frac{7}{36}$$

So, the series becomes,

$$x \operatorname{cosec} x = 1 + \frac{1}{3!} x^2 + \frac{7}{360} x^4 + \cdots$$

V. $\frac{1}{\sqrt{x^2+1}} = 1 - \frac{x^2}{2!} + \frac{9}{4!}x^4 - \cdots$

Ans Solution:

Let $f(x) = \frac{1}{\sqrt{x^2+1}}$

$\Rightarrow f(0) = 1$

$\Rightarrow y = \frac{1}{\sqrt{x^2+1}}$

$\Rightarrow (x^2+1)y^2 = 1 \quad --- (i)$

Differenting 'n' times by Leibnitz rule, we get,

$(x^2+1).2y\,y_{n+1} + n(2x).2y\,y_n + \frac{n(n-1)}{2}.2y.2y_{n-1} = 0$

For $x=0$,

$2(y)_0(y_{n+1})_0 + 2\,n(n-1)\,y_0\,(y_{n+3})_0 = 0$

$\Rightarrow y_0(y_{n+1})_0 + n(n-1)y_0(y_{n+3})_0 = 0$

So,

$f'(0) = 0$

$f''(0) = -\frac{3(3-1)\,y_0.(y_0)}{y_0}$

$= -3\times2\times1 = -6$

$f'''(0)$

$f^{iv}(0) = -\frac{5(5-1)\,y_0\times1\times(y_2)_0}{y_0}$

$\Rightarrow (y_{n+1})_0 + n^2 y_{n+3} - n\,y_{n+3}$

Differentiating (i) n times using Leibnitz rule, we get,

$2y(x^2+1)y_n + n.(2x).2y.y_{n-1} + \frac{n(n-1)}{2}.2.2y.y_{n-2} = 0$

$\Rightarrow (x^2+1)y_n + 2nx\,y_{n-1} + n^2 y_{n-2} - n\,y_{n-2} = 0$

For $x=0$, we get.

$y_n + n^2 y_{n-2} - n\,y_{n-2} = 0$

So,

$y_1 = 0$

Also,

$y_2 + 4y_0 - 2y_0 = 0$

$\Rightarrow y_2 = -2y_0 = -2$

$(x^2+1).2y.y_n + (2x).n\,2y\,y_{n-1}$
$+ 2.\frac{n(n-1)}{2}.2y\,y_{n-2} = 0$

For $x=0$

$(y)_0 (y_n)_0 + n(n-1)(y)_0 (y_{n-2})_0 = 0$ — ①

So, from ①, it is clear that

$y_1 = 0$

$y_2 = -2(2-1).y_0 = -2\times 1\times 1 = -2$

${}^{n}C_r = \frac{n!}{(n-r)!\,r!}$

${}^{n+1}C_r = \frac{(n+1)!}{}$

If $y = e^{a\sin^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$, then show that

i) $(1-x^2)y_2 - xy_1 - a^2y = 0$

ii) $(n+1)(n+2)a_{n+2} - (n^2+a^2)a_n = 0$

and hence obtain the expansion of $e^{a\sin^{-1}x}$.

Ans Solution:

The given function is

$$y = e^{a\sin^{-1}x} \quad \Rightarrow f(0) = 1$$

Diff. w.r.t. x, we get,

$$y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}} = \frac{ya}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)y_1^2 = a^2y^2$$

Diff. using leibnitz rule, we get,

$$(1-x^2) \cdot 2y_1 \cdot y_2 + (-2x) \cdot y_1^2 = a^2 \cdot 2yy_1$$

$$\Rightarrow (1-x^2)2y_1y_2 - 2xy_1^2 = a^2 2yy_1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0$$

showed

Diff. using leibnitz rule, we get,

$$\cancel{(1-x^2)y_{n+2} - n(2x)y_{n+1} - 2y_n\left(\frac{n(n-1)}{2}\right) - nxy_{n+1} - ny_n - a^2y_n = 0}$$

$$\cancel{\Rightarrow (1-x^2)y_{n+2} - 2xy_{n+1} - 2y_n - y_n - xy_{n+1} - a^2y_n = 0}$$

Differentiang n times using leibnitz rule, we get,

$$(1-x^2) \cdot y_{n+2} - (2x) \cdot y_{n+1} \cdot n - 2 \cdot \frac{n(n-1)}{2} \cdot y_n - xy_{n+1} - 1y_n \cdot n - a^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - a^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2nx+x)y_{n+1} - (n(n-1)+n)y_n = a^2y_n$$

Putting $x=0$, we have
$(y_{n+2})_0 - (n(n-1)+n)(y_n)_0 = a^2(y_n)_0$. ──→ (A)

We have,
$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$
$\Rightarrow (y_1)_0 = a_1 = 1!\, a_1$

$\Rightarrow (y_2)_0 = 2a_2 \Rightarrow 2!\, a_2$
$\Rightarrow (y_3)_0 = 6a_3 \Rightarrow 3! \cdot a_3$

$\cdots\cdots$

$(y_{n+2})_0 = (n+2)!\, a_{n+2}$

$(y_n)_0 = n!\, a_n$

So,
$(n+2)!\, a_{n+2} - (n(n-1)+n)\, n!\, a_n = a^2(n!\, a_n)$

$\Rightarrow (n+2)!\, a_{n+2} - (n^2 - n + n)\, n!\, a_n = a^2 \cdot n!\, a_n$
$\Rightarrow (n+2)!\, a_{n+2} - n^2 n!\, a_n = a^2 \cdot n!\, a_n$
$\Rightarrow (n+2)!\, a_{n+2} - n!\, a_n(n^2 + a^2) = 0$
$\Rightarrow (n+2)(n+1)\, a_{n+2} - a_n(n^2 + a^2) = 0$
$\therefore (n+1)(n+2)\, a_{n+2} - a_n(a^2 + n^2) = 0$

showed

Also, from (A)
$f(0) = 1$
$f'(0) = a$
$f''(0) = a^2$
$f'''(0) = (a^2+1) \cdot a$
$f^{iv}(0) = a^2(a^2 + 2^2)$
and so on.

Using Maclaurin's series
$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$\therefore e^{a\sin^{-1}(x)} = 1 + ax + \frac{a^2x^2}{2} + \frac{a(a^2+1)x^3}{6} + \cdots$

Find the pedal equation of the following curves:

a) $y^2 = 4a(x+a)$

Ans Solution:

The given equation of the curve is

$$y^2 = 4a(x+a)$$

Let (X, Y) be any point on the tangent at any point $P(x, y)$ to the curve. Then, the equation of the tangent curve will be

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow Y - y = \frac{2a}{y}(X - x)$$

$$\Rightarrow 2aX - Yy - 2ax + y^2 = 0 \quad \text{--- (i)}$$

Let p be the length of perpendicular drawn from origin to the tangent (i). Then,

$$p^2 = \frac{(-2ax+y^2)^2}{(2a)^2 + (-y)^2}$$

$$\Rightarrow p^2 = \frac{(4ax + 4a^2 - 2ax)^2}{4a^2 + 4ax + 4a^2}$$

$$\Rightarrow p^2 = \frac{(2a)^2(x+2a)^2}{4a(x+2a)}$$

$$\Rightarrow p^2 = a(x+2a)$$

$$\Rightarrow p^2 = a\sqrt{x^2 + 4ax + 4a^2}$$

$$\Rightarrow p^2 = a\sqrt{x^2+y^2}$$

$$\Rightarrow p^2 = ar$$

$\therefore p^2 = ar$ is the required pedal equation of the given curve.

i) $y^2 = 4ax$

Ans Solution:

The given equation of the curve is

$$y^2 = 4ax$$

$$\Rightarrow 2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

Let (X,Y) be any point on the tangent line at $P(x,y)$ to the curve. Then, the equation of tangent becomes:

$$Y - y = \frac{2a}{y}(X - x)$$

$$\Rightarrow 2aX - Yy - 2ax + y^2 = 0 \quad \text{---(i)}$$

Let p be the length of perpendicular from the origin to the tangent line (i). Then,

$$p^2 = \frac{(y^2-2ax)^2}{4a^2+y^2}$$

$$\Rightarrow p^2 = \frac{(4ax-2ax)^2}{4a^2+4ax}$$

$$\cancel{\Rightarrow p^2 = \frac{4a^2x^2}{8ax}}$$

$$\cancel{\Rightarrow p^2 = \frac{1}{2}ax}$$

$$\cancel{\Rightarrow p^2 = \frac{1}{2}a\sqrt{x^2}}$$

$$\cancel{\Rightarrow p^2}$$

$$\Rightarrow p^2 = \frac{4a^2x^2}{4a(a+x)}$$

$$\Rightarrow p^2 = \frac{ax^2}{a+x}$$

$$\cancel{\Rightarrow p^2 = \frac{a(a+x)}{a+x} - \frac{a^2}{a+x}}$$

$$\cancel{\Rightarrow p^2 = a - \frac{a^2}{a+x}}$$

$$\cancel{\Rightarrow p^2 = a - \frac{a^2}{\sqrt{a^2+x^2+2ax}}}$$

$$\Rightarrow p^2 = \frac{ax^2}{a+x} \quad \cancel{\Rightarrow p^2 = \frac{ax^2}{\sqrt{a^2+2ax+x^2}}}$$

$$\Rightarrow ap^2 + p^2x = ax^2$$

$$\Rightarrow ax^2 - p^2x - ap^2 = 0 \quad \text{— (1)}$$

Also, $r^2 = x^2 + y^2$

$$\Rightarrow r^2 = x^2 + 4ax \Rightarrow x^2 + 4ax - r^2 = 0 \quad \text{— (ii)}$$

From (1) and (2), we get,

$$\frac{x^2}{p^2r^2+4a^2p^2} = \frac{x}{-(-ar^2+ap^2)} = \frac{1}{4a^2+p^2}$$

$$\Rightarrow x^2 = \frac{p^2(r^2+4a^2)}{4a^2+p^2} \text{ and } x^2 = \frac{(ar^2-ap^2)^2}{(4a^2+p^2)^2}$$

$\therefore (ar^2-ap^2)^2 = p^2(r^2+4a^2)(4a^2+p^2)$ is the required pedal equation of the curve.

iii. $x^{2/3}+y^{2/3}=a^{2/3}$

~~Ans~~ Solution:

The given equation of the curve is:

$$x^{2/3}+y^{2/3}=a^{2/3}$$

Let (X,Y) be any point on the tangent line drawn at $P(x,y)$ on to the curve. Then, the equation of the tangent line becomes:

$$Y-y=-\frac{x^{-1/3}}{y^{-1/3}}(X-x)$$

$$\Rightarrow x^{-1/3}X+Yy^{-1/3}-yy^{-1/3}-xx^{-1/3}=0$$

Let p be the length of the perpendicular from the origin to the tangent line.

Then,

$$p^2=\frac{(yy^{-1/3}+xx^{-1/3})^2}{x^{-2/3}+y^{-2/3}}$$

$$\Rightarrow p^2=\frac{\cancel{y^{4/3}+x^{4/3}+2.y^{2/3}x^{2/3}}}{\cancel{x^{-2/3}+y^{-2/3}}}$$

$$\Rightarrow p^2=\frac{(y^{2/3}+x^{2/3})^2}{x^{-2/3}+y^{-2/3}}=\frac{a^{4/3}}{x^{-2/3}+y^{-2/3}}$$

$$=\frac{a^{4/3}x^{2/3}y^{2/3}}{y^{2/3}+x^{2/3}}$$

$$=a^{2/3}x^{2/3}y^{2/3}$$

Also,

$$r^2+3p^2=x^2+y^2+3a^{2/3}x^{2/3}y^{2/3}$$

$$=(x^{2/3})^3+(y^{2/3})^3+3x^{2/3}y^{2/3}(x^{2/3}+y^{2/3})$$

$$=(x^{2/3}+y^{2/3})^3$$

$$=(a^{2/3})^3$$

$$=a^2$$

$\therefore r^2+3p^2=a^2$ is the required pedal equation of the curve.

Find the pedal equation of the following curves.

i. $r=a(1-\cos\theta)$

Ans. Solution:

The given equation of the curve is

$$r=a(1-\cos\theta)$$

Diff. w.r.t. θ, we get,

$$\frac{dr}{d\theta}=a\sin\theta$$

$$\Rightarrow \frac{d\theta}{dr}=a^{-1}\operatorname{cosec}\theta.$$

From (1), we get,

$$r\frac{d\theta}{dr}=\tan\phi$$

$$\Rightarrow r.\frac{1}{a\sin\theta}=\tan\phi$$

$$\Rightarrow a\frac{(1-\cos\theta)}{\sin\theta}=\tan\phi \quad [\text{From } (3)] \text{ i.e. } (iii)$$

$$\Rightarrow \frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}=\tan\phi$$

$$\Rightarrow \tan\frac{\theta}{2}=\tan\phi \Rightarrow \phi=\frac{\theta}{2} \quad \text{—— (A)}$$

Using relation (A) in equation (1), we get,

$$p=r\sin\left(\frac{\theta}{2}\right)$$

$$=r\sqrt{\frac{1-\cos\theta}{2}}$$

$$=r\sqrt{\frac{r}{2a}}$$

$$\Rightarrow p^2=\frac{r^3}{2a}$$

$\therefore p^2=\frac{r^3}{2a}$ is the required pedal equation of the given curve.

Find the pedal equation of the following curves.

i. $r=a(1-\cos\theta)$

Ans Solution:

The given equation of the curve is

$$r=a(1-\cos\theta)$$

Diff. w.r.t. θ, we get,

$$\frac{dr}{d\theta}=a\sin\theta$$

$$\Rightarrow \frac{d\theta}{dr}=a^{-1}\operatorname{cosec}\theta.$$

We have,

$$p=r\sin\phi \quad \text{——— (i)}$$

$$r\frac{d\theta}{dr}=\tan\phi \quad \text{——— (ii)}$$

$$r=a(1-\cos\theta) \quad \text{——— (iii)}$$

From (ii)

$$ra\operatorname{cosec}\theta=\tan\phi$$

$$a.a(1-\cos\theta)\operatorname{cosec}\theta=\tan\phi$$

$$\Rightarrow a^2-a^2\cot\theta=\tan\phi$$

$$\Rightarrow a^2(1-\cot\theta)=\tan\phi$$

$$\Rightarrow a^2(\operatorname{cosec}\theta-\cot\theta)=\tan\phi$$

$$\Rightarrow a^2\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right)=\tan\phi$$

$$\Rightarrow a^2.\sec\frac{\theta}{2}=\tan\phi$$

Also,

$$p=r\sin\phi$$

$$=r.\frac{a^2\sec\theta/2}{\sqrt{a^2\sec^2\theta/2+1}}$$

ii. $r = a(1+\cos\theta)$

Ans Solution:

The given equation of the curve is:

$$r = a(1+\cos\theta)$$

Differentiating w.r.t. θ, we get,

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\Rightarrow \frac{d\theta}{dr} = -\frac{1}{a(\operatorname{cosec}\theta)^{-1}}$$

We have,

$$p = r\sin\phi \quad \text{——— (A)}$$

$$r.\frac{d\theta}{dr} = \tan\phi \quad \text{——— (T)}$$

$$r = a(1+\cos\theta)$$

~~From~~ From eqn (T), we get,

$$\cancel{a(1+\cos} \; r. -\frac{1}{a\operatorname{cosec}\theta} = \tan\phi$$

$$\Rightarrow \cancel{a}. -\frac{1+\cos\theta}{(\operatorname{cosec}\theta)^{-1}} = \tan\phi$$

$$\Rightarrow -\frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2} = \tan\phi$$

$$\Rightarrow -\cot\frac{\theta}{2} = \tan\phi$$

$$\Rightarrow \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = \tan\phi$$

$$\Rightarrow \phi = \frac{\pi}{2}+\frac{\theta}{2}$$

From equation (A), we get,

$$p = r\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\Rightarrow p = r\cos\frac{\theta}{2}$$

$$\Rightarrow p = r.\sqrt{\frac{1+\cos\theta}{2}}$$

$$\Rightarrow p = r\sqrt{\frac{r}{a.2}}$$

$\therefore \cancel{p^2 = \frac{r^2}{2a}}$

$\Rightarrow p^2 = \frac{r^2 \cdot r}{2a}$

$\Rightarrow p^2 = \frac{r^3}{2a}$ is the required pedal equation of the curve.

iii. $r^2 = a^2 \cos 2\theta$

Ans Solution:

The given equation of the curve is

$$r^2 = a^2 \cos 2\theta$$

Differentiating w.r.t. θ, we get,

$$2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta$$

$$\Rightarrow r\frac{d\theta}{dr} = -\left(\frac{a^2 \sin 2\theta}{r}\right)^{-1}$$

$$\Rightarrow \frac{d\theta}{dr} = -\frac{r}{a^2 \sin 2\theta}$$

We have,

$$p = r\sin\phi \quad \text{--- (T)}$$

$$r\frac{d\theta}{dr} = \tan\phi \quad \text{--- (B)}$$

$$r^2 = a^2 \cos 2\theta \quad \text{--- (A)}$$

From equation (B),

$$-\frac{r^2}{a^2 \sin 2\theta} = \tan\phi$$

$$\Rightarrow -\cot 2\theta = \tan\phi$$

$$\Rightarrow \tan\left(\frac{\pi}{2} + 2\theta\right) = \tan\phi$$

$$\Rightarrow \phi = \frac{\pi}{2} + 2\theta$$

From equation (T), we get,

$$p = r\sin\left(\frac{\pi}{2} + 2\theta\right)$$

$$\Rightarrow p = r\cos 2\theta$$
$$\Rightarrow p = \frac{r \cdot r^2}{a^2}$$
$$\Rightarrow p = \frac{r^3}{a^2}$$
$\therefore p = \frac{r^3}{a^2}$ is the required pedal equation of the given curve.

---

iv. $r^2\cos 2\theta = a^2$

Ans Solution:

The given equation of the curve is
$$r^2\cos 2\theta = a^2$$
Differentiating w.r.t. θ, we get,
$$2r \cdot \frac{dr}{d\theta} = a^2 \cdot 2\sec 2\theta \tan 2\theta$$
$$\Rightarrow r\frac{dr}{d\theta} = a^2\sec 2\theta \tan 2\theta$$
$$\Rightarrow \frac{d\theta}{dr} = \frac{r\cos 2\theta \cdot \cot 2\theta}{a^2}$$
We have,
$$p = r\sin\phi \quad \text{——— (1)}$$
$$r\frac{d\theta}{dr} = \tan\phi \quad \text{——— (2)}$$
$$r^2\cos 2\theta = a^2 \quad \text{——— (3)}$$
From equation (2), we get,
$$\frac{r^2\cos 2\theta \cot 2\theta}{a^2} = \tan\phi$$
$$\Rightarrow \cot 2\theta = \tan\phi \quad \text{[From eqn (3)]}$$
$$\Rightarrow \tan\left(\frac{\pi}{2} - 2\theta\right) = \tan\phi$$
$$\Rightarrow \phi = \frac{\pi}{2} - 2\theta \quad \text{——— (A)}$$
Using relation (A) in eqn (1), we get,
$$p = r\sin\left(\frac{\pi}{2} - 2\theta\right) = r\cos 2\theta$$

$$\Rightarrow p = r \cdot \frac{a^2}{r^2}$$

$$\Rightarrow p = \frac{a^2}{r}$$

$\therefore p = \frac{a^2}{r}$ is the required pedal equation of the curve.

V. $r^m = a^m \cos(m\theta)$

Solution:

The given equation of the curve is

$$r^m = a^m \cos(m\theta)$$

Differentiating w.r.t. θ, we get,

$$m \cdot r^{m-1} \frac{dr}{d\theta} = -m a^m \sin(m\theta)$$

$$\Rightarrow \frac{d\theta}{dr} = -\frac{r^{m-1}}{a^m \sin(m\theta)}$$

We have,

$$p = r\sin\phi \quad \text{--- (1)}$$

$$r\frac{d\theta}{dr} = \tan\phi \quad \text{--- (2)}$$

$$r^m = a^m \cos(m\theta) \quad \text{--- (3)}$$

From equation (2),

$$-\frac{r^m}{a^m \sin(m\theta)} = \tan\phi$$

$$\Rightarrow -\frac{\cos(m\theta)}{\sin(m\theta)} = \tan\phi \; [\text{from (3)}]$$

$$\Rightarrow -\cot(m\theta) = \tan\phi$$

$$\Rightarrow \tan\left(\frac{\pi}{2} + m\theta\right) = \tan\phi$$

$$\Rightarrow \phi = \frac{\pi}{2} + m\theta \quad \text{--- (A)}$$

using relation (A) in (1), we get,

$$p = r\sin\left(\frac{\pi}{2} + m\theta\right) = r\cos(m\theta)$$

$$\Rightarrow p = r \cdot \frac{r^m}{a^m}$$

$\therefore p = \dfrac{r^{m+1}}{a^m}$ is the required pedal equation of the curve.

vi. $r = ae^{\theta\cot\alpha}$

Ans Solution:

The given equation of the curve is

$$r = ae^{\theta\cot\alpha}$$

Differentiating w.r.t. θ, we get,

$$\frac{dr}{d\theta} = a\cot\alpha \cdot e^{\theta\cot\alpha}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{a\cot\alpha e^{\theta\cot\alpha}}$$

we have,

$$p = r\sin\phi \quad \text{———} \quad (1)$$

$$r\frac{d\theta}{dr} = \tan\phi \quad \text{———} \quad (2)$$

$$r = ae^{\theta\cot\alpha} \quad \text{———} \quad (3)$$

From equation (2), we get,

$$\frac{r}{a\cot\alpha\, e^{\theta\cot\alpha}} = \tan\phi \qquad [\text{From } (3)]$$

$$\Rightarrow \tan\alpha = \tan\phi \qquad \text{~~From~~}$$

$$\Rightarrow \phi = \alpha. \quad \text{———} \quad (A)$$

Using relation (A) in (1), we get,

$$p = r\sin\alpha$$

$\Rightarrow p = r\sin\alpha$ is the required pedal equation of the curve.

$p = r\sin\phi.$

vii. $r = \frac{a}{\theta}$

$\frac{1}{p} = \frac{1}{r}\operatorname{cosec}\phi$

$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$

Ans. Solution:

The given equation of the curve is:

$$r = \frac{a}{\theta}$$

Differentiating w.r.t. θ, we get,

$$\frac{dr}{d\theta} = -\frac{a}{\theta^2}$$

$$\Rightarrow \cancel{r\,dr} \Rightarrow \frac{d\theta}{dr} = -\frac{1}{a}\theta^2$$

$$\Rightarrow r\frac{d\theta}{dr} = \frac{r(-\theta^2)}{a} = \frac{a}{\theta}\cdot\left(\frac{-\theta^2}{a}\right)$$

$$\Rightarrow r\frac{d\theta}{dr} = -\theta$$

$$\Rightarrow \tan\phi = -\theta$$

$$\Rightarrow \cancel{\sin\phi = 1+\theta}$$

$$\Rightarrow \tan\phi = -\theta$$

$$\Rightarrow \cot\phi = -\frac{1}{\theta}$$

Also,

$$p = r\sin\phi$$

$$\Rightarrow p^2 = r^2\sin^2\phi$$

$$\Rightarrow p^2 = \frac{r^2}{1+\cot^2\phi}$$

$$\Rightarrow p^2 = \frac{r^2}{1+\left(\frac{1}{\theta}\right)^2}$$

$$\Rightarrow p^2 = \frac{r^2}{\frac{\theta^2+1}{\theta^2}}$$

$$\Rightarrow p^2 = \frac{a^2/\theta^2}{1+\frac{1}{\theta^2}}$$

$$\Rightarrow p^2 = \frac{a^2}{\theta^2+1} = \frac{a^2}{\frac{a^2}{r^2}+1}$$

$$\Rightarrow p^2 = \frac{a^2}{a^2+r^2} \cdot r^2$$

$\therefore p^2(a^2+r^2) - a^2r^2 = 0$ which is the required pedal equation of the given curve.

viii. $r = a\theta$

Solution:

The given equation of the curve is

$$r = a\theta$$

Differentiating w.r.t. θ, we get,

$$\frac{dr}{d\theta} = a$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{a}$$

$$\Rightarrow r\frac{d\theta}{dr} = \frac{r}{a}$$

$$\Rightarrow \tan\phi = \frac{a\theta}{a}$$

$$\Rightarrow \tan\phi = \theta$$

$$\Rightarrow \cot\phi = \frac{1}{\theta}$$

Also,

$$p = r\sin\phi$$

$$\Rightarrow p^2 = \frac{r^2}{1+\cot^2\phi} = \frac{r^2}{1+\frac{1}{\theta^2}}$$

$$\Rightarrow p^2 = \frac{r^2}{1+\frac{a^2}{r^2}}$$

$$\Rightarrow p^2 = \frac{r^4}{r^2+a^2}$$

$\Rightarrow p^2(r^2+a^2) - r^4 = 0$ is the required pedal equation of the curve.

solve the differential equation $x'' + 4x' + 4x = 0$ for the motion of the spring attached to mass.

Ans. Solution:

The given differential equation is:

$$x'' + 4x' + 4x = 0$$

where,

$$x'' = \frac{d^2x}{dt^2} \text{ and } x' = \frac{dx}{dt}$$

Then,

$$(D^2 + 4D + 4)x = 0$$

Its A.E. is:

$$m^2 + 4m + 4 = 0$$

$$\Rightarrow (m+2)(m+2) = 0$$

$$\Rightarrow m = -2, -2$$

So, the general equation of the given differential equation is

$$x = (c_1 + c_2 t)e^{-2t}$$

$\therefore$ The required equation is

$$x = (c_1 + c_2 t)e^{-2t}$$

Solve the differential equation of spring mass system with external force,

$$x'' + \frac{1}{2}x' + 4x = \cos(t).$$

Solution:

Given,

$$x'' + \frac{1}{2}x' + 4x = \cos(t)$$

$$\Rightarrow 2x'' + x' + 8x = 2\cos(t) \quad \text{——} \quad (1)$$

The A.E of (1) is

$\cancel{(2D^2+D+8)}$ $2m^2 + m + 8 = 0$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1 - 4\times2\times8}}{2\times2}$$

$$\therefore m = \frac{-1 \pm \sqrt{63}\,i}{4}$$

$$\therefore \text{C.F.} = e^{-1/4\,t}\left[A\cos\left(\frac{\sqrt{63}}{4}t\right) + B\sin\left(\frac{\sqrt{63}}{4}t\right)\right]$$

Also,

$$\text{P.I.} = \frac{1}{2D^2+D+8}\,2\cos(t)$$

$$= 2\,\frac{1}{-2\times4+D+8}\cos(t)$$

$$= 2\,\frac{1}{D}\cos(t)$$

$$= 2\sin(t)$$

$$\therefore x = e^{-1/4t}\left[A\cos\left(\frac{\sqrt{63}}{4}t\right) + B\sin\left(\frac{\sqrt{63}}{4}t\right)\right] + 2\sin(t)$$

$\therefore$ The required equation is:

$$x = e^{-1/4}\left[A\cos\left(\frac{\sqrt{63}}{4}t\right) + B\sin\left(\frac{\sqrt{63}}{4}t\right)\right] + 2\sin(t)$$

Solve the second order differential equation of the series LCR circuit.

$$L\frac{d^2V_c}{dt^2} + R\frac{dV_c}{dt} + \frac{1}{C}V_c = 0$$

where, $R = 10\Omega$, $L = 1H$ $C = 16\times10^{-4}F$, $V_c(0) = 6V$, $V_c'(0) = 6A$

Solution:

Given,

$$L\frac{d^2V_c}{dt^2} + R\frac{dV_c}{dt} + \frac{1}{C}V_c = 0$$

Where, $R = 10\Omega$, $L = 1H$ and $C = 16\times10^{-4}F$. Then,

$$\frac{d^2V_c}{dt^2} + 10\frac{dV_c}{dt} + \frac{1}{16\times10^{-4}}V_c = 0$$

whose A-E. is

$$m^2 + 10m + \frac{10^4}{16} = 0$$

$$\Rightarrow m = \frac{-10 \pm \sqrt{100 - 4\times1\times\frac{10^4}{16}}}{2\times1}$$

$$\Rightarrow m = -5 \pm 10\sqrt{6}\,i$$

$$\therefore V_c = e^{-5t}[A\cos(10\sqrt{6}t) + B\sin(10\sqrt{6}t)]$$

~~Also~~ Given,

$V_c(0) = 6$

$\Rightarrow 6 = [A]$

$\therefore A = 6$

and

$V_c'(0) = 6$

$$\Rightarrow 6 = e^{-5\times0}[-A\sin(10\sqrt{6}\times0).10\sqrt{6} + 10\sqrt{6}B\cos(0)] + [A\cos 0 + B\sin(0)].(-5)e^{-0}$$

$$\Rightarrow 6 = [10\sqrt{6}B] + A(-5)$$

$$\Rightarrow \frac{6+5A}{10\sqrt{6}} = B$$

$$\Rightarrow B = \frac{36}{10\sqrt{6}} = \frac{18}{5\sqrt{6}}$$

$$\therefore V = e^{-5t}\left[6\cos(10\sqrt{6}t) + \frac{18}{5\sqrt{6}}\sin(10\sqrt{6}t)\right]$$

Find the pedal equation of the curve
$r^m = a^m \cos(m\theta)$

Ans Solution:

The given equation of the curve is:

$$r^m = a^m \cos(m\theta)$$

$$\Rightarrow m r^{m-1} \frac{dr}{d\theta} = -a^m . m \sin(m\theta)$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{m r^{m-1}}{-a^m . m \sin(m\theta)}$$

$$\Rightarrow r\frac{d\theta}{dr} = \frac{r^m}{-a^m \sin(m\theta)} = \frac{a^m \cos(m\theta)}{-a^m \sin(m\theta)}$$

$$\Rightarrow \tan\phi = -\cot(m\theta)$$

$$\Rightarrow \tan\phi = \tan\left(\frac{\pi}{2} + m\theta\right)$$

$$\Rightarrow \phi = \frac{\pi}{2} + m\theta$$

Also,

$$p = r\sin\phi$$

$$\Rightarrow p = r . \sin\left(\frac{\pi}{2} + m\theta\right)$$

$$\Rightarrow p = r\cos m\theta$$

$$\Rightarrow p = r . \frac{r^m}{a^m}$$

$\therefore p = \dfrac{r^{m+1}}{a^m}$ is the required pedal equation of the curve.

Assuming the validity of expansion using Maclaurin's theorem, prove the following series:

$$e^x \sec x = 1 + x + x^2 + \frac{2}{3}x^3 + \dots$$

Ans Solution:

The given function is:

$$y = e^x \sec x.$$

$\Rightarrow y' = e^x \sec x \tan x + \sec x e^x$

$\Rightarrow y' = f(x) \tan x + f(x)$

Also,

$y'' = f(x)\sec^2 x + \tan(x) f'(x) + f'(x)$

Also,

$y''' = f(x) \cdot 2\sec x \tan x + \sec^2 x f'(x) + \tan x f''(x) + f'(x)\sec^2(x) + f''(x)$

and so on.

Therefore,

$f(0) = y_0 = 1$

$f'(0) = (y_1)_0 = 1 \times 0 + 1 = 1$

$f''(0) = (y_2)_0 = 1 \times 1 + 0 \times 1 + 1 = 2$

$f'''(0) = (y_3)_0 = 1 \times 2 \times 1 \times 0 + 1 \times 1 + 0 \times 2 + 1 \times 1 + 2$
$= 1 + 1 + 2$
$= 4$

and so on.

By using Maclaurin's series expansion,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

or, $e^x \sec x = 1 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times 4 + \cdots$

$= 1 + x + x^2 + \frac{2x^3}{3} + \cdots$

$\therefore e^x \sec(x) = 1 + x + x^2 + \frac{2}{3} x^3 + \cdots$

Proved

Find the pedal equation of the curve $r = \frac{2a}{1-\cos\theta}$

Ans Solution:

Given, $r = \frac{2a}{1-\cos\theta}$

$\Rightarrow r = \frac{2a}{2\sin^2\theta/2}$

$\Rightarrow r = a\operatorname{cosec}^2\theta/2$

$\Rightarrow \frac{dr}{d\theta} = -a \cdot 2\operatorname{cosec}\theta/2 \cdot \operatorname{cosec}\theta/2 \cdot \cot\theta/2$

$\Rightarrow \frac{d\theta}{dr} = \frac{-1}{2\operatorname{cosec}^2\theta/2 \cot\theta/2 \cdot a}$

$\Rightarrow r\frac{d\theta}{dr} = -\frac{2a\operatorname{cosec}^2\theta/2}{2\operatorname{cosec}^2\theta/2 \cdot \cot\theta/2 \cdot a}$

$\Rightarrow r\frac{d\theta}{dr} = -\tan\frac{\theta}{2}$

$\Rightarrow \tan\phi = \tan\left(\pi - \frac{\theta}{2}\right)$

$\Rightarrow \phi = \pi - \theta/2$

Also,

$p = r\sin\phi$

$\Rightarrow p = r\sin(\pi - \theta/2)$

$\Rightarrow p = r\sin\theta/2$

$\Rightarrow p = r \cdot \sqrt{\frac{1-\cos\theta}{2}}$

$\Rightarrow p^2 = \frac{r^2}{a} \cdot r$

$\Rightarrow ap^2 - r^3 = 0$ is the required pedal equation of the curve.

Evaluate: $\int_3^5 \frac{1}{\sqrt{x-3}}\,dx$

Ans Solution:

Given,

Let $I = \int_3^5 \frac{1}{\sqrt{x-3}} dx$

Here, for $x=3$, $f(x) = \frac{1}{\sqrt{x-3}} = \infty$ (undefined).

So,

$$I = \lim_{a \to 3^+} \int_a^5 \frac{1}{\sqrt{x-3}} dx$$

$$= \lim_{a \to 3^+} \left[\frac{(x-3)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right]_a^5$$

$$= \lim_{a \to 3^+} \left[2 . \sqrt{x-3}) \right]_a^5$$

$$= \lim_{a \to 3^+} 2 [\sqrt{2} - \sqrt{a-3}]$$

$$= 2\sqrt{2} - 2 \lim_{a \to 3^+} \sqrt{a-3}$$

$$\Rightarrow 2\sqrt{2} - 2 . 0 \quad [\text{As } \sqrt{3^+ - 3} = 0]$$

$$= 2\sqrt{2}$$

$\therefore I = \int_3^5 \frac{1}{\sqrt{x-3}} dx = 2\sqrt{2}$ which converges.

Hence, the given integral $\int_3^5 \frac{1}{\sqrt{x-3}} dx$ converges with an integral value of $2\sqrt{2}$.

Show that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the radius of curvature at the extremity of the major axis is equal to half the latus rectum.

Ans Solution:

Given,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

Also,

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{y - x\,dy/dx}{y^2}\right)$$

For the extremity of the major axis $(a,0)$ and $(-a,0)$.

$$\rho_1 = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left(1+\frac{b^4 x^2}{a^4 y^2}\right)^{3/2}}{-\frac{b^2}{a^2}\left(\frac{y + x\frac{b^2 x}{a^2 y}}{y^2}\right)}$$

$$= \frac{(a^4y^2+b^4x^2)^{3/2}}{a^6 y^3} \times \frac{a^2}{-b^2} \times \frac{y^2 \times a^2 y}{a^2y^2+b^2x^2}$$

$$= \frac{(a^4y^2+b^4x^2)^{3/2}}{-a^2b^2(a^2y^2+b^2x^2)}$$

For vertex, put $x=a$ and $y=0$, we get,

$$\rho_1 = \frac{(b^4 x^2)^{3/2}}{-a^2b^2(b^2x^2)} = \frac{b^6 . x^3}{a^2b^2 . x^2 . -b^2}$$

$$= \frac{b^6 a^3}{a^2 b^2 a^2} = -\frac{b^2}{a}$$

Similarly,

$$\rho_2 = \frac{b^2}{a}$$

Since, length of latus rectum $= \frac{2b^2}{a}$

$|\rho_1| = |\rho_2| = \frac{1}{2} \times$ length of latus rectum

Showed

Find the radius of curvature of the curve $y = x^2(x-3)$ at the points

Find the radius of curvature at the origin for the following curves.

a. $4x^2 - 3xy + y^2 - 3y = 0$

Ans Solution:

The given equation of the curve is:

$$4x^2 - 3xy + y^2 - 3y = 0$$

Here,

$$8x - 3\left[x\frac{dy}{dx} + y\right] + 2y\frac{dy}{dx} - 3\frac{dy}{dx} = 0$$

$$\Rightarrow 8x - 3xy_1 + 3y + 2yy_1 - 3y_1 = 0$$

$$\Rightarrow \frac{8x + 3y}{(3x - 2y + 3)} = y_1$$

$$\Rightarrow y_1 = \frac{8x + 3y}{3x - 2y + 3}$$

At origin, $y_1 = 0$.

Also,

$$8 - 3\left[x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx}\right] + 2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] - 3\frac{d^2y}{dx^2} = 0$$

At origin,

$$8 - 3[0] + 2[0] - 3y_2 = 0$$

$$\Rightarrow y_2 = \frac{8}{3}$$

We have,

radius of curvature $(\rho) = \frac{(1+y_1^2)^{3/2}}{y_2}$

$= \frac{(1+0^2)^{3/2}}{8} \times 3$

$= \frac{3}{8}$

$\therefore$ Radius of curvature $= \frac{3}{8}$.

---

b. $x^3+y^3-2x^2+6y=0$.

Ans. Solution:

Given,

$x^3+y^3-2x^2+6y=0$

$\Rightarrow 3x^2+3y^2y_1-4x+6y_1=0$

At origin,

$0+0-0+6y_1=0$

$\therefore y_1=0$

Also,

$6x+3[y^2y_2+y_1.2yy_1]-4+6y_2=0$

At origin

$0+0-4+6y_2=0$

$\therefore y_2=\frac{2}{3}$

We have,

Radius of curvature $(\rho)=\frac{(1+y_1^2)^{3/2}}{y_2}$

$=\frac{(1+0)^{3/2}}{2/3}$

$=\frac{3}{2}$

$\therefore$ Radius of curvature $=\frac{3}{2}$.

Find the radius of curvature at any point

(x, y) of the following curves.

a. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

~~Ans~~ Solution:

Given,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2} y_1 = 0$$

$$\Rightarrow y_1 = -\frac{x\, b^2}{y\, a^2}$$

$$\Rightarrow y_2 = -\frac{b^2}{a^2}\left[\frac{y - x y_1}{y^2}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{y - x \cdot \left(-\frac{x}{y} \cdot \frac{b^2}{a^2}\right)}{y^2}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{y^2 a^2 + x^2 b^2}{a^2 y \cdot y^2}\right]$$

$$= -\frac{b^2}{a^2} \cdot \frac{1}{a^2 y^3}[a\, x^2 b^2 + a^2 y^2]$$

We have,

$$\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2}$$

$$= \left(1 + \frac{x^2 b^4}{y^2 a^4}\right)^{\frac{3}{2}} \times \frac{a^4 y^3}{-b^2[x^2 b^2 + a^2 y^2]}$$

$$= \frac{(x^2 b^4 + y^2 a^4)^{\frac{3}{2}}}{-y^3 a^6} \times \frac{a^4 y^3}{b^2[x^2 b^2 + a^2 y^2]}$$

$$= -\frac{1}{a^2 b^2} \cdot \frac{[x^2 b^4 + y^2 a^4]}{[x^2 b^2 + a^2 y^2]}$$

$$= -\frac{1}{a^2 b^2 \cdot a^2 b^2} \cdot [x^2 b^4 + y^2 a^4]$$

$$= -\frac{(x^2 b^4 + y^2 a^4]^{3/2}}{a^4 b^4}$$

$$\therefore \rho = \left|\frac{(x^2 b^4 + y^2 a^4)^{3/2}}{a^4 b^4}\right|$$

b. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Ans Solution:

Given,

$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$\Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow y_1 = \frac{-x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

$$\Rightarrow y_2 = -\left[\frac{x^{1/3}.\frac{1}{3}y^{-2/3}y_1 - y^{1/3}.\frac{1}{3}.x^{-2/3}}{x^{2/3}}\right]$$

$$= \frac{-1}{3}\left[\frac{x^{1/3}.y^{-2/3}.\left(\frac{-y^{1/3}}{x^{1/3}}\right) - y^{1/3}.x^{-2/3}}{x^{2/3}}\right]$$

$$= \frac{-1}{3}\left[\frac{-y^{-1/3} - y^{1/3}.x^{-2/3}}{x^{2/3}}\right]$$

$$= \frac{1}{3}\left[\frac{y^{-1/3} + y^{1/3}.x^{-2/3}}{x^{2/3}}\right]$$

We have,

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$$

$$= \frac{\left(1 + \frac{y^{2/3}}{x^{2/3}}\right)^{\frac{3}{2}} \times 3 \times x^{2/3}}{[y^{-1/3} + y^{1/3}.x^{-2/3}]}$$

$$= \frac{3(x^{2/3}+y^{2/3})^{\frac{3}{2}}, x^{-1/3}}{y^{-1/3} + y^{1/3}x^{-2/3}}$$

$$= 3\frac{x^{-1/3}}{y^{1/3}} \cdot \frac{a}{x^{-2/3}+y^{-2/3}}$$

$$= 3.\frac{x^{-1/3}}{y^{1/3}}.x^{2/3}.y^{2/3}\ a^{1/3}$$

$$= 3x^{1/3}y^{1/3}a^{1/3}$$

$$\therefore \rho = 3x^{1/3}y^{1/3}a^{1/3}$$

c. $y^2 = 4x$ at the vertex $(0,0)$.

Ans Solution:

Given,

$$y^2 = 4x$$

$$\Rightarrow 2yy_1 = 4$$

$$\Rightarrow y_1 = \frac{2}{y}$$

$$\Rightarrow x_1 = \frac{y}{2}$$

$$\Rightarrow x_2 = \frac{1}{2}$$

Now, At Vertex $(0,0)$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+0)^{3/2}}{1/2} = 2$$

$$\therefore \rho = 2.$$