Using $\int_0^{\pi/2} \frac{dx}{a^2\sin^2x+b^2\cos^2x} = \frac{\pi}{2ab}$, prove that

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\sin^2x+b^2\cos^2x)^2} = \frac{\pi(a^2+b^2)}{4a^3b^3}$$

~~Ans~~ Solution:

Given,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\sin^2x+b^2\cos^2x} = \frac{\pi}{2ab} \quad \text{--- (1)}$$

Diff. w.r.t. a, we get,

$$-\int_0^{\pi/2} \frac{2a\sin^2x}{(a^2\sin^2x+b^2\cos^2x)^2}dx = \frac{\pi}{2b}\left(\frac{-1}{a^2}\right)$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sin^2x}{(a^2\sin^2x+b^2\cos^2x)^2}dx = \frac{\pi}{4a^3b} \quad \text{--- (A)}$$

Similarly,

$$\int_0^{\pi/2} \frac{\cos^2x}{(a^2\sin^2x+b^2\cos^2x)^2}dx = \frac{\pi}{4ab^3} \quad \text{--- (B)}$$

Adding (A) and (B)

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2x+\cos^2x}{(a^2\sin^2x+b^2\cos^2x)^2}dx = \frac{\pi}{4ab}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\sin^2x+b^2\cos^2x)^2} = \frac{\pi(a^2+b^2)}{4a^3b^3}$$

Proved

Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\sin^2x+b^2\cos^2x}$$ and prove that

$$\int_0^{\pi/2} \frac{dx}{(a^2\sin^2x+b^2\cos^2x)^2} = \frac{\pi(a^2+b^2)}{4a^3b^3}$$

Ans Solution:

$$\int_0^{\pi/2} \frac{dx}{a^2\sin^2x+b^2\cos^2x}$$

$$= \int_0^{\pi/2} \frac{\sec^2x}{a^2\tan^2x+b^2}\,dx$$

$$= \int_0^{\infty} \frac{du}{(au)^2+(b)^2} \quad \text{where } u=\tan x$$

$$= \frac{1}{ab}\left[\tan^{-1}\left(\frac{au}{b}\right)\right]_0^{\infty}$$

$$= \frac{1}{ab}\times\frac{\pi}{2}$$

$$= \frac{\pi}{2ab}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{a^2\sin^2x+b^2\cos^2x}\,dx = \frac{\pi}{2ab} \quad \text{——— (1)}$$

Diff. w.r.t a, we get,

$$-\int_0^{\pi/2} \frac{2a\sin^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{-\pi}{2a^2 b}$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sin^2 x}{(a^2\sin^2 x+b^2\cos^2 x)^2}\,dx = \frac{\pi}{4a^3 b} \quad \text{——— } (A)$$

$\Rightarrow$ Similarly,

$$\int_0^{\pi/2} \frac{\cos^2 x}{(a^2\sin^2 x+b^2\cos^2 x)^2}\,dx = \frac{\pi}{4ab^3} \quad \text{——— } (B)$$

From (A) and (B)

$$\int_0^{\pi/2} \frac{\sin^2 x+\cos^2 x}{(a^2\sin^2 x+b^2\cos^2 x)^2}\,dx = \frac{\pi}{4ab}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

$$\therefore \int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x+b^2\cos^2 x)^2} = \frac{\pi(a^2+b^2)}{4a^3b^3}$$

Find the value of $\int_0^{\pi} \frac{dx}{a+b\cos x}$ and deduce

that $\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$

~~Ans~~ Solution:

let $I = \int_0^{\pi} \frac{dx}{a+b\cos x}$

$$= \int_0^{\pi} \frac{dx}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \frac{1}{a-b}\int_0^{\pi} \frac{\sec^2 x/2 \, dx}{\left(\frac{a+b}{a-b}\right)+\tan^2\frac{x}{2}}$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\left[\tan^{-1}\left\{\tan\frac{x}{2}\sqrt{\frac{a+b}{a-b}}\right\}\right]_0^{\pi}$$

$$= \frac{2}{\sqrt{a^2-b^2}}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{2}{\sqrt{a^2-b^2}} \cdot \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}$$

Also,
Diff. w.r.t. a, we get,

$$\int_0^{\pi} (-1)(a+b\cos x)^{-2}\,dx = \frac{1}{2}\,\frac{2\pi a}{(a^2-b^2)^{3/2}}$$

$$\therefore \int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{-\pi a}{(a^2-b^2)^{3/2}}$$

Evaluate:

$$\int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$$

Ans Solution:

$$\text{Let } I = \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$$

$$\Rightarrow \frac{dI}{da} = \int_0^{\infty} \frac{1}{(1+x^2)}\cdot\frac{1}{1+a^2x^2}\,dx$$

$$= \frac{1}{1-a^2}\int_0^{\infty}\left(\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right)dx$$

$$= \frac{1}{1-a^2}\left[\tan^{-1}x - a\tan^{-1}(ax)\right]_0^{\infty}$$

$$= \frac{1}{1-a^2}\left[\frac{\pi}{2} - \frac{a\pi}{2}\right] = \frac{\pi}{2}\,\frac{1}{1+a}$$

$$\therefore \frac{dI}{da} = \frac{\pi}{2}\cdot\frac{1}{1+a}$$

$$\Rightarrow I = \int \frac{\pi}{2} \frac{1}{1+a}\, da$$

$$= \frac{\pi}{2} \ln(a+1) + c$$

As ~~a~~ $I = 0$ when $a = 0$.

So,

$$0 = \frac{\pi}{2} \ln(1) + c$$

$$\therefore c = 0$$

$$\therefore \int_0^\infty \frac{\tan^{-1}(ax)}{1+x^2} \quad \frac{\pi}{2} \ln(a+1)$$

Evaluate:

$$\int_0^\infty \frac{e^{-x} \sin bx}{x}\, dx$$

~~As~~ Solution:

let $I = \int_0^\infty \frac{e^{-x} \sin bx}{x}\, dx$

$$\Rightarrow \frac{dI}{db} = \int_0^\infty \frac{e^{-x}\, x \cos bx}{x}\, dx$$

$$= \int_0^\infty e^{-x} \cos bx\, dx$$

$$= \left[\frac{e^{-x}}{(-1)^2 + b^2} (-1 \cos bx + b \sin bx) \right]_0^\infty$$

$$= \frac{1}{1+b^2}$$

$$\Rightarrow I = \int \frac{1}{1+b^2}\, db$$

$$= \tan^{-1}(b) + c$$

Also,

when $b=0$, $I=0$.

So,

$$0 = \tan^{-1}(0) + c$$

$$\therefore c = 0$$

$$\therefore \int_0^\infty \frac{e^{-x}\sin bx}{x}\, dx = \tan^{-1}(b)$$

Area (Quadrature)
(For Cartesian curves)

Formula 1. (For single curve)
The area of the region bounded by $y=f(x)$ about x-axis is

$$A = \int y\,dx$$

Note: The area of the region bounded by $x=f(y)$ about y axis is

$$A = \int x\,dy$$

Formula 2. (For double curve)
The area of the region bounded by two curves $y_1=f(x)$ and $y_2=g(x)$ about x-axis is

$$A = \left| \int (y_1-y_2)\,dx \right|$$

Note: The area of the region bounded by two the curves ~~y_1~~ $x_1=f(y)$ and $x_2=g(y)$ about Y-axis is

$$A = \left| \int (x_1-x_2)\,dy \right|$$

* Symmetry:
  - → if a function contains even power in y it is symmetrical about x-axis
  - → if a function contains even power in x it is symmetrical about Y-axis.

CamScanner

Exercise-15

1. Find the area bounded by the curve $y^2 = x^3$ and the line $y = 2x$.

Ans Solution:

The given curves are

$$y^2 = x^3 \quad \text{---(i)}$$
$$y = 2x \quad \text{---(ii)}$$

Solving (i) and (ii), we get

$$(2x)^2 = x^3$$
$$\Rightarrow x^3 - 4x^2 = 0$$
$$\Rightarrow x^2(x-4) = 0$$
$$\Rightarrow x = 0, 4$$

$\therefore$ The curves intersect at $x=0$ and $x=4$.

The area bounded by the curve is given by

$$A = \left| \int_0^4 (x^{3/2} - 2x)\,dx \right|$$

$$= \left| \left[\frac{2x^{5/2}}{5} - x^2 \right]_0^4 \right|$$

$$= \left| \frac{2}{5} 4^{5/2} - 4^2 \right|$$

$$= \left| \frac{2}{5} \times 32 - 16 \right|$$

$$= \left| \frac{64-80}{5} \right| = \frac{16}{5} \text{ sq. units}$$

$\therefore$ The required area is $\frac{16}{5}$ sq. units.

② Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

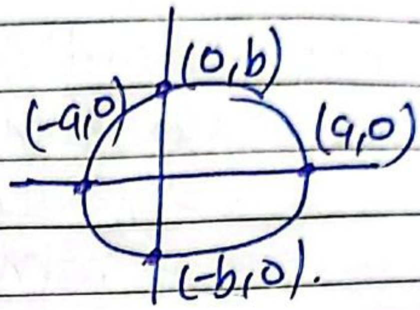
Ans Solution:

The given equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Here,

The area of the ellipse is given by



$$A=4\int_0^a y\,dx$$

$$=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=4\cdot\frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$=4\cdot\frac{b}{a}\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right]$$

$=\pi ab$ sq.units.

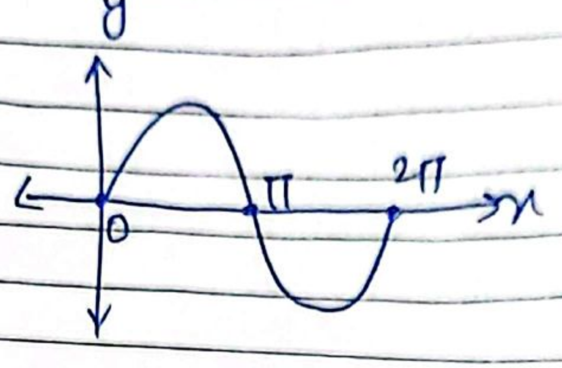
$\therefore$ The area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab. sq.units.

---

③ Find the area of the region bounded by the curve $y=\sin x$ and x-axis between $x=0$ and $x=2\pi$.

Ans Solution:

Given, $y=\sin x$.



$y = \sin(x)$

Here,
The area bounded by $y = \sin x$ and x-axis from $x = 0$ to $x = 2\pi$ is given by

$$A = 2\int_0^{\pi} \sin x \, dx$$

$$= 2\left[-\cos x\right]_0^{\pi}$$
$$= 2[+1 - (-1)]$$
$$= 4 \text{ sq. units.}$$

Hence, the area of the region bounded by the curve $y = \sin x$ and x-axis between $x = 0$ and $x = 2\pi$ is 4 sq. units.

4. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Ans Solution:

Given,

$$x^2 = 4y \quad \text{---(i)}$$
$$x = 4y - 2 \quad \text{---(ii)}$$

solving (i) and (ii), we get,

$$(4y-2)^2 = 4y$$
$$\Rightarrow 16y^2 - 16y + 4 = 4y$$
$$\Rightarrow 16y^2 - 20y + 4 = 0$$
$$\Rightarrow 16y^2 - 16y - 4y + 4 = 0$$

$\Rightarrow 16y(y-1)-4(y-1)=0$

$\Rightarrow (16y-4)(y-1)=0$

$\Rightarrow y=\frac{1}{4}, 1$

When $y=\frac{1}{4}$; $x=-1$

and when $y=1$; $x=2$.

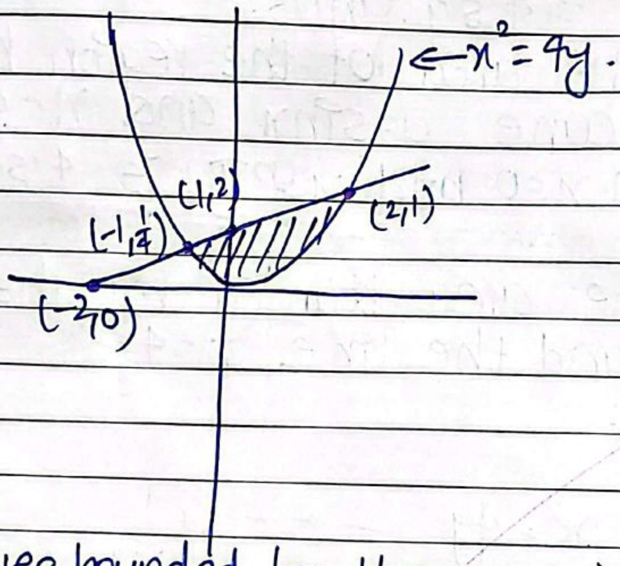
Also, for the line (ii),

$x=4y-2$

When $x=0$, $y=\frac{1}{2}$

and when $y=0$, $x=-2$.

So, the graph becomes.



The area bounded by the curve is given by

$$A=\int_{-1}^{2}\left(\frac{x+2}{4}-\frac{x^2}{4}\right)dx$$

$$=\frac{1}{4}\frac{(x+2)^2}{2}-\frac{1}{4}\cdot\frac{x^3}{3}$$

$$= \left[\frac{1}{4}\frac{(4)^2}{2} - \frac{1}{4}\cdot\frac{8}{3} - \left(\frac{1}{4}\frac{(1)^2}{2} - \frac{1}{4}\frac{(-1)}{3}\right)\right]$$

$$= \left[\frac{1}{4}\cdot\frac{16}{2} - \frac{2}{3} - \left(\frac{1}{8} + \frac{1}{12}\right)\right]$$

$$= \left[\frac{4}{2} - \frac{2}{3} - \left(\frac{5}{24}\right)\right]$$

$$= \left[\frac{4}{2} - \frac{2}{3} - \frac{5}{24}\right]$$

$$= \left[\frac{4\times12 - 2\times8 - 5}{24}\right]$$

$$= \left[\frac{48-16-5}{24}\right]$$

$$= \left[\frac{48-21}{24}\right]$$

$$= \frac{27}{24} \text{ sq.units.} = \frac{9}{8} \text{ sq.units}$$

$\therefore$ The area bounded by the curve $x^2 = 4y$ and the line $x = 4y-2$ is $\frac{27}{24}$ sq.units.

i.e. $\frac{9}{8}$ sq.units.

Find the area between each of the following curve and its asymptotes.

i) $a^2x^2 = y^2(a^2-x^2)$.

Ans Solution:

The given equation of the curve is

$$a^2x^2 = y^2(a^2-x^2) \quad ---(i)$$

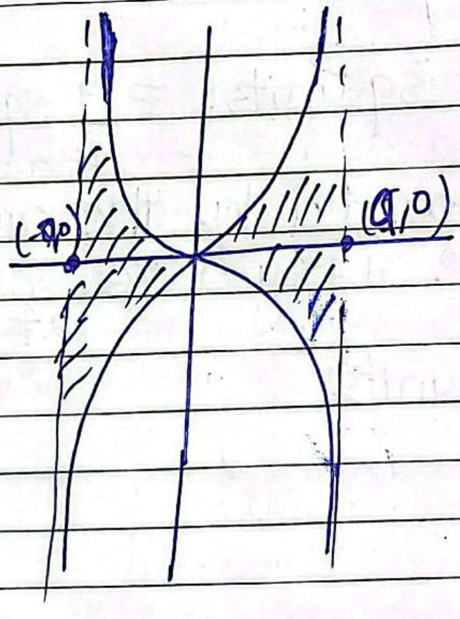
Here,

the equation contains even power in both x and y. So, the graph is symmetrical about both axes.

Also, from eqn (i), it is clear that

as $x \to \pm a$, $y \to \pm\infty$

So, $x = \pm a$ are the asymptotes of the curve. So, the graph of curve looks like



Clearly,

The area between the curve and its asymptotes is given by

$$A = 4\int_0^a \frac{ax}{\sqrt{a^2-x^2}}\,dx$$

$$= 4a \cdot \frac{-1}{2} \, 2\left[\sqrt{a^2-x^2}\right]_0^a$$

$$= -4a[0-a]$$

$$= 4a^2 \text{ sq. units}$$

Hence, the area between the given curve and its asymptotes is $4a^2$. sq. units

ii. $x^2y^2 + a^2b^2 = a^2y^2$

~~Ans~~ Solution:

The given equation of the curve is

$$x^2y^2 + a^2b^2 = a^2y^2 \quad \text{---(i)}$$

Here, the ~~c~~ curve ~~is~~ contains even powers in both x and y. So, the ~~giv~~ curve is symmetrical about both axes.

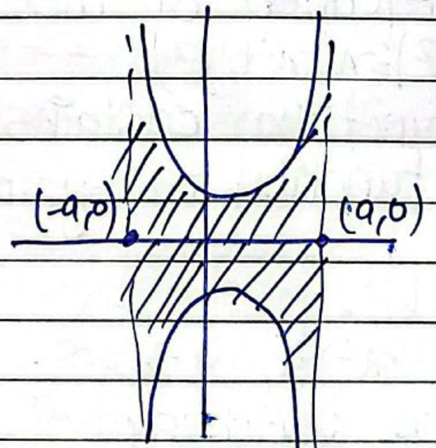
Also, from (i)

$$\frac{-a^2b^2}{(x^2-a^2)} = y^2$$

It is clear that as $x \to \pm a$, $y \to \pm\infty$

So, $x = \pm a$ are the asymptotes of the curve.

So, the graph looks like.



So, the area between the curve and its asymptotes is given by $A = 4\int_0^a \frac{ab}{\sqrt{(x^2-a^2)}}\,dx$

$$= 4ab\int_0^a \frac{1}{\sqrt{(x^2-a^2)}}\,dx \qquad = 4ab\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$= 4ab\left[\cosh^{-1}\left(\frac{x}{a}\right)\right]_0^a \qquad = 4ab\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4ab\left[\cosh^{-1}(1) - \cosh^{-1}(0)\right] \qquad = 4ab\left[\frac{\pi}{2}\right]$$

$$= 4ab\left[\frac{e^1+e^{-1}}{2} - \frac{e^0+e^0}{2}\right] \qquad = 2\pi ab \text{ sq. units}$$

$$= \frac{4ab}{2}\left[e+\frac{1}{e}-2\right]$$

$$= 2ab\left[e+\frac{1}{e}-2\right]$$

$\therefore$ The area between the curve and the asymptotes is ~~$2ab\left[e+\frac{1}{e}-2\right]$~~ $2\pi ab$ sq. units.

(iii) $a(y^2-x^2) = x(x^2+y^2)$

Ans Solution:

The given equation of the curve is

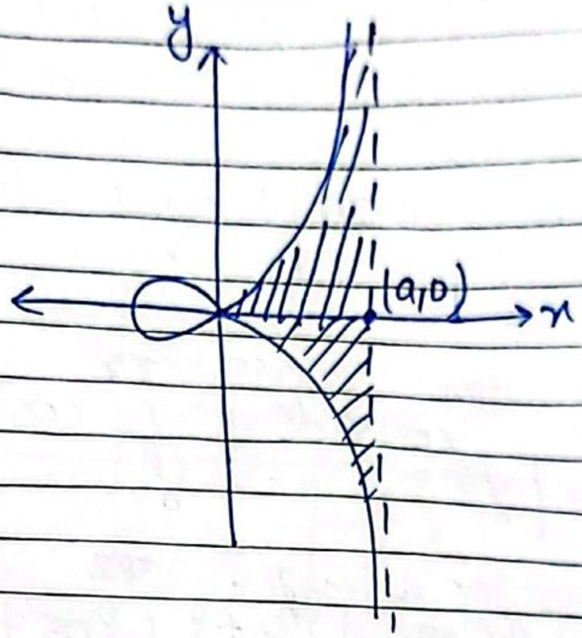
$$a(y^2-x^2) = x(x^2+y^2) \quad \text{---(i)}$$

Since, the function contains even power of y, the function is symmetrical about x-axis.

Clearly, from (i),

When $x \to a$, $y \to \pm\infty$.

So, $x = \pm a$ are the asymptotes of the curve. So, the graph of the curve looks like,



Clearly,
The area between the curve and its asymptotes is given by

$$A = 2\int_0^a \sqrt{\frac{x^3+ax^2}{a-x}}\,dx$$

$$= 2\int_0^a x\sqrt{\frac{a+x}{a-x}}\,dx$$

Let $x = a\cos 2\theta$. When $x \to 0, \theta \to \pi/4$
$\Rightarrow dx = -2a\sin 2\theta\, d\theta$. When $x \to a, \theta \to 0$

So,

$$A = 2\int_{\pi/4}^{0} a\cos 2\theta \cdot \sqrt{\frac{a+a\cos 2\theta}{a-a\cos 2\theta}}\,(-2a)\sin 2\theta\, d\theta$$

$$= 2.2a^2 \int_0^{\pi/4} \cos 2\theta \cdot \frac{\cos\theta}{\sin\theta} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 4a^2 \int_0^{\pi/4} \cos 2\theta \cdot \cos\theta \cdot 2\cos\theta \, d\theta$$

$$= 4a^2 \times 2 \int_0^{\pi/4} \cos 2\theta \left(\frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= \frac{4a^2}{\cancel{2}} \left[\int_0^{\pi/4} \cos 2\theta \, d\theta + \int_0^{\pi/4} \cos^2 2\theta \, d\theta \right]$$

$$= 2a^2 \cancel{\times 2} \left[\left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/2} \cos^2 t \, dt \right]$$

$$= 2a^2 \cancel{\times 2} \left[\frac{1}{2} + \frac{1}{2} \left[\frac{\Gamma(1/2)\Gamma(3/2)}{2 \times \Gamma(2)} \right] \right]$$

$$= 2a^2 \cancel{\times 2} \left[\frac{1}{2} + \frac{1}{2} \times \frac{\sqrt{\pi}}{2} \times \frac{1}{2} \times \sqrt{\pi} \right]$$

$$= 2a^2 \cancel{\times 2} \left[\frac{1}{2} + \frac{1}{8} \pi \right]$$

$$= 2a^2 \left[1 + \frac{\pi}{4} \right] \text{ sq. units}$$

$\therefore$ The area between the curve and its asymptotes is $2a^2 \left[1 + \frac{\pi}{4} \right]$ sq. units

Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum.

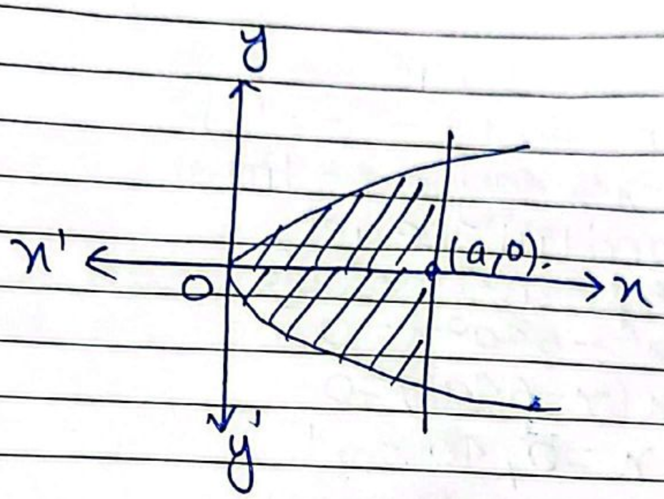
Ans Solution:

The given equation of the parabola

is $y^2 = 4ax$.

So, the equation of the latus rectum is

$x = a$.



Clearly,

The area between the parabola and its latus rectum is given by

$$A = 2\int_0^a \sqrt{4ax}\,dx$$

$$= 2 \times 2\sqrt{a} \left[\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}.\sqrt{a}$$

$$= \frac{8}{3}a^2 \text{ sq. units}$$

Hence, the area of the parabola $y^2 = 4ax$ bounded by its latus rectum is $\frac{8}{3}a^2$ sq. units.

* Show that the ~~lines~~ area bounded by the curves $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3} a^2$.

Ans Solution:

The given curves are:

$$y^2 = 4ax \quad \text{---(i)}$$

and $$x^2 = 4ay \quad \text{---(ii)}$$

Solving (i) and (ii), we get,

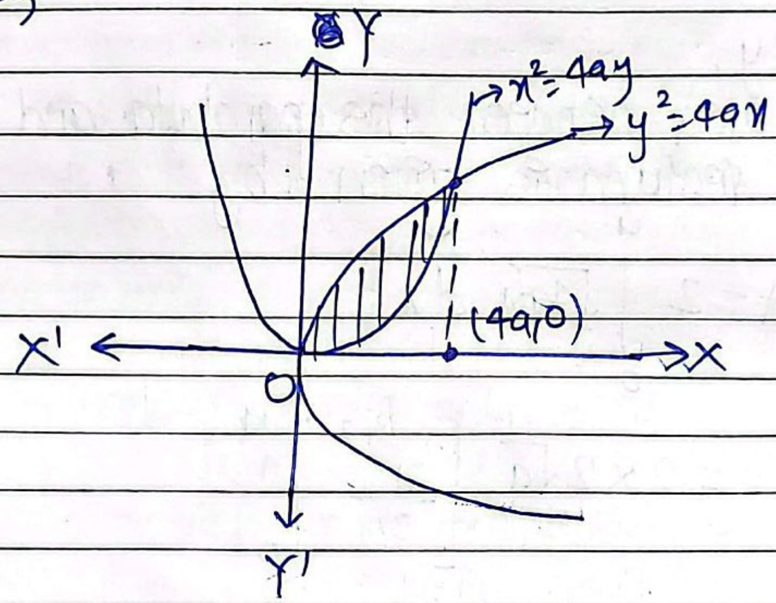
$$x^4 = (4a)^2 4a.x$$

$$\Rightarrow x^4 - 64a^3 x = 0$$

$$\Rightarrow x(x - 64a^3) = 0$$

$$\Rightarrow x = 0, 4a.$$

So,



Clearly,

The area between the curves is given by

$$A = \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a} \right) dx$$

$$= \left[2\sqrt{a} \cdot \frac{x^{3/2}}{3/2} - \frac{x^3}{4a \cdot 3} \right]_0^{4a}$$

$$= \left[\frac{2\sqrt{a}\,(4a)^{3/2}}{3/2} - \frac{(4a)^3}{3 \times 4a} \right]$$

$$= \frac{4a^2 \times 8}{3} - \frac{(4a)^2}{3}$$

$$= \frac{32a^2 - 16a^2}{3}$$

$$= \frac{16a^2}{3} \text{ sq. units.}$$

$\therefore$ The area bounded by the curves $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$. sq. units.

proved

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* Find the area enclosed between the line $y = x$ and the parabola $y^2 = 16x$.

Ans Solution:

The given curves are:

$$y = x \quad --- (i)$$

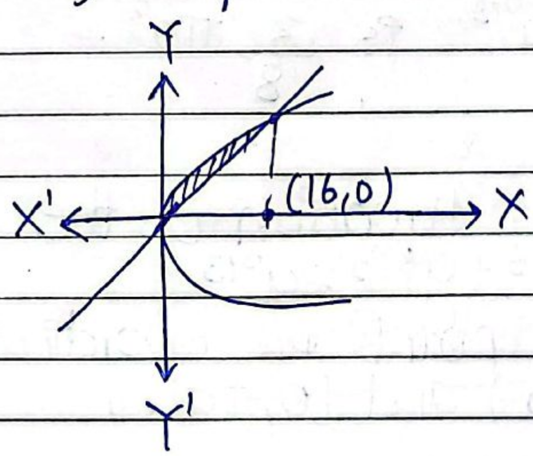
$$y^2 = 16x \quad --- (ii)$$

Solving (i) and (ii), we get,

$$x^2 = 16x$$

$$\Rightarrow x = 0, 16$$

So,



Clearly,
The enclosed area is given by
$$A = \int_0^{16} (\sqrt{16x} - x)\,dx$$

$$= \left[4\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^{16}$$

$$= \left[4 \times 64 \times \frac{2}{3} - \frac{256}{2}\right]$$

$$= \left[\frac{256 \times 2}{3} - \frac{256}{2}\right]$$

$$= \left[\frac{256 \times 2 \times 2 - 256 \times 3}{6}\right]$$

$$= \left[\frac{256}{6}\right]$$

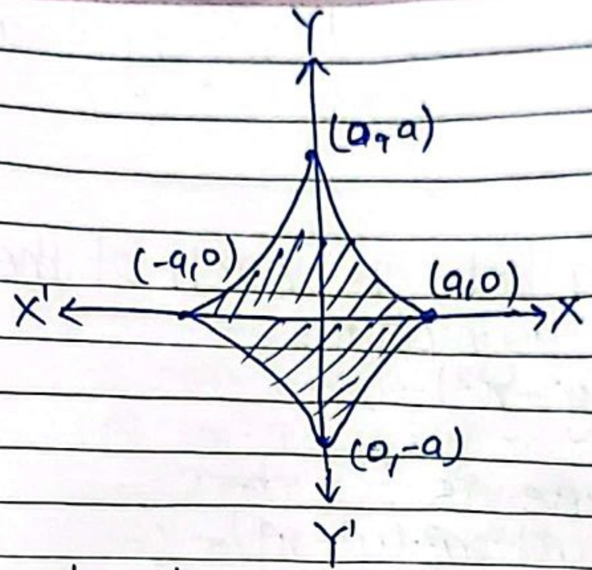
$$= \frac{128}{3} \text{ sq. units.}$$

$\therefore$ The area enclosed between the line $y = x$ and parabola $y^2 = 16x$ is $\frac{128}{3}$ sq. units.

---

* Show that the area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{3}{8}\pi a^2$.

Ans Solution:
The given equation of the astroid is
$$x^{2/3} + y^{2/3} = a^{2/3}$$
Clearly, the points on coordinate axes are $(\pm a, 0)$ and $(0, \pm a)$.
So,



Clearly, the area bounded by the curve is given by $A = 4\int_0^a y\,dx$

$$= 4\int_0^a (-x^{2/3} + a^{2/3})^{3/2}\,dx$$

Let $x = a\sin^3\theta$ When $x \to 0, \theta \to 0$

$\Rightarrow dx = 3a\sin^2\theta\cos\theta$ When $x \to a, \theta \to \frac{\pi}{2}$

$$\text{So, } A = 4\int_0^{\pi/2} 3a\sin^2\theta\cos\theta \cdot (a^{2/3} - a^{2/3}\sin^2\theta)^{3/2}\,d\theta$$

$$= 4 \cdot 3 \cdot a \cdot a \cdot \int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$$

$$= 12a^2 \cdot \frac{\Gamma(\frac{3}{2})\,\Gamma(3/2)}{2 \times \Gamma(6/2)}$$

$$= 6a^2 \cdot \frac{\frac{1}{2} \times \sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{2 \times 2} = \frac{3}{8}\pi a^2$$

$\therefore$ The area of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ is $\frac{3}{8}\pi a^2$.

* Find the area between each of the following curve and its asymptotes.

i. $x^2(x^2+y^2)=a^2(y^2-x^2)$

Ans Solution:

The given curve is:

$$x^2(x^2+y^2)=a^2(y^2-x^2)$$

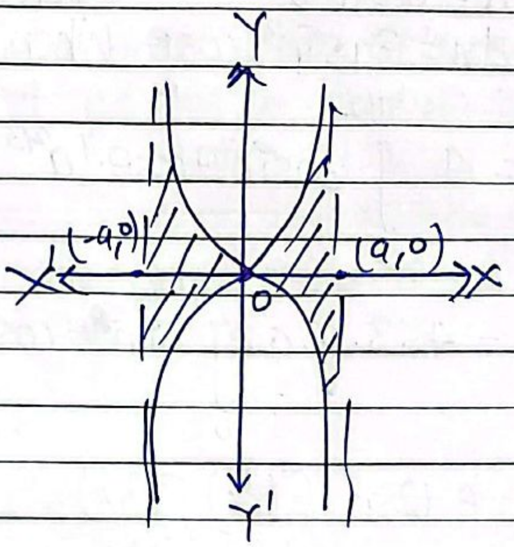
$$\Rightarrow x^4+x^2y^2=a^2y^2-a^2x^2$$

$$\Rightarrow y^2=\frac{x^4+a^2x^2}{a^2-x^2}$$

Clearly, when $x=\pm a$, $y\to\pm\infty$.

Since, the function contains even power is x and y. Then, it is symmetrical about both axes.

So,



Clearly,

The area bounded by the curve and its asymptotes is given by $A=4\int_0 y\,dx$

$$= 4 \int_0^a \sqrt{\frac{x^4 + a^2x^2}{a^2 - x^2}}\, dx$$

$$= 4 \int_0^a x \sqrt{\frac{a^2 + x^2}{a^2 - x^2}}\, dx$$

let $x^2 = a^2 \cos 2\theta$

$\Rightarrow 2x\,dx = a^2 \cdot (-2\sin 2\theta)\, d\theta$

$\Rightarrow x\,dx = -a^2 \sin 2\theta\, d\theta$

When $x \to 0,\ \theta \to \frac{\pi}{4}$ and when $x \to a,\ \theta \to 0$

So,

$$A = 4 \int_{\pi/4}^{0} \sqrt{\frac{a^2 + a^2\cos 2\theta}{a^2 - a^2 \cos 2\theta}}\; -a^2 \sin 2\theta\, d\theta$$

$$= 4a^2 \int_0^{\pi/4} \cot\theta \cdot \sin 2\theta\, d\theta$$

$$= 4a^2 \int_0^{\pi/4} \frac{\cos\theta \cdot 2\sin\theta\cos\theta}{\sin\theta}\, d\theta$$

$$= 4a^2 \int_0^{\pi/4} 2\cos^2\theta\, d\theta$$

$$= 4a^2 \int_0^{\pi/4} (1 + \cos 2\theta)\, d\theta$$

$$= 4a^2 \left[\frac{\pi}{4} + \int_0^{\pi/4} \cos 2\theta\, d\theta \right]$$

$$= 4a^2\left[\frac{\pi}{4} + \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}\right]$$

$$= 4a^2\left[\frac{\pi}{4} + \frac{1}{2}\left(\frac{\pi}{2}\right)\right]$$

$$= 2\pi a^2$$

$\therefore$ The area between the curve and its asymptotes is $2\pi a^2$.

ii. $y^2(a-x) = x^3$

Ans: Solution:

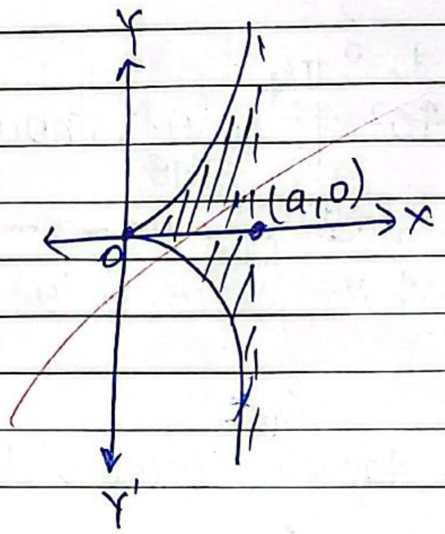
The given equation of the curve is

$$y^2(a-x) = x^3$$

Clearly, when $x \to a$, $y = \pm\infty$.

Also, the function contains even power in y. So, the curve is symmetrical about x-axis.

So,



Clearly,

The enclosed area is given by

$$A = 2\int_0^a y\,dx = 2\int_0^a \sqrt{\frac{x^3}{a-x}}\,dx = 2\int_0^a \frac{x^{3/2}}{\sqrt{a-x}}\,dx$$

Let $x = a\sin^2\theta$

$\Rightarrow dx = a.2\sin\theta\cos\theta\, d\theta$

When $x \to 0, \theta \to 0$; When $x \to a, \theta \to \pi/2$.

So,

$$A = 2\int_0^{\pi/2} \frac{a^{3/2}\sin^3\theta \cdot a.2\sin\theta\cos\theta\, d\theta}{a^{1/2}\cdot\cos\theta}$$

$$= 2\times 2.a\int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= 4a.\frac{\Gamma(5/2)\Gamma(1/2)}{2\times\Gamma(6/2)}$$

$$= 4a\times \frac{3/2\times\frac{1}{2}\times\sqrt{\pi}\times\sqrt{\pi}}{2\times 2}$$

$$= \frac{3}{4}\pi a^2$$

$\therefore$ The area between the curve and its asymptotes is $\frac{3}{4}\pi a^2$.

Find the area of the loop of the curve $y^2 = x^2(x+a)$.

Ans Solution:

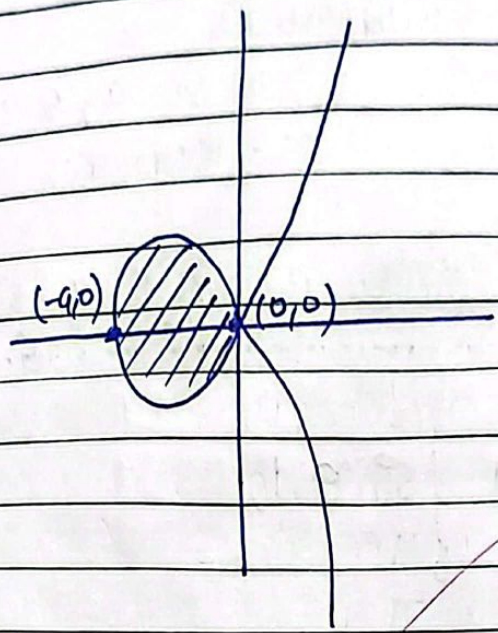
The given curve is

$$y^2 = x^2(x+a)$$

Here, $y=0$, when $x = 0, -a$.

The function contains even power in y. So, the function is symmetrical about x-axis.

So,



Clearly,

The area of the loop $= 2\int_{-a}^{0} y\,dx$

$$= 2\times\int_{-a}^{0} x\sqrt{a+x}\,dx$$

$$= 2\times\int_{-a}^{0} x\sqrt{x+a}\,dx = \cancel{\left[x\frac{(a+x)^{3/2}}{3/2} - \frac{(a+x)^{3/2}}{3/2}\right]_{-a}^{0}}$$

$$= 2\times\left[x\frac{(x+a)^{3/2}}{2} - \int\frac{(x+a)^{3/2}}{3/2}\,dx\right]_{-a}^{0} = \cancel{\left[\frac{x(a-a)^{3/2}}{3/2} - (a-\right.} = \left[-\frac{a^{3/2}}{3/2}\right] = \cancel{\frac{2}{3}a^{3/2}}$$

$$= 2\times\left[\frac{x(x+a)^{3/2}}{2} - \frac{2}{3}\,\frac{(x+a)^{5/2}}{5/2}\right]_{-a}^{0}$$

$$= 2\times -\frac{2}{3}\times\frac{a^{5/2}}{5/2} = -\frac{4}{3}\times\frac{2}{5}a^{5/2} = \frac{-8}{15}a^{5/2}$$

Ignoring -ve sign, we get

$$A = \frac{8}{15} a^{5/2}$$

Hence, the area of the loop of the curve $y^2 = x^2(x+a)$ is $\frac{8}{15} a^{5/2}$.

* Find the area of the curve $y^2 = x(x-1)^2$

Ans Solution:

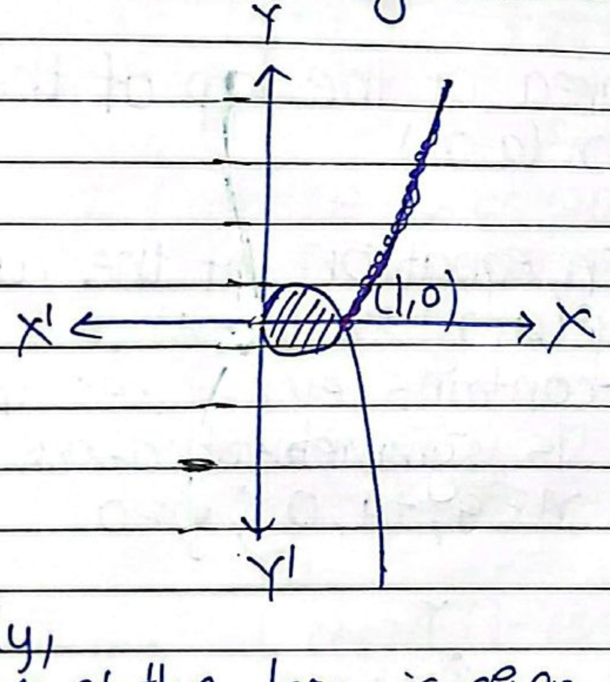
The given curve is

$$y^2 = x(x-1)^2$$

Here, $y=0$ when $x=0,1$

and this eqn of curve contains even power in y. The curve is symmetrical about x-axis.

Also,



Clearly,

The area of the loop is given by

$$A = 2\int_0^1 y\,dx = 2\int_0^1 \sqrt{x}(x-1)\,dx =$$

$$= 2\left[(x-1)\frac{x^{3/2}}{3/2} - \int \frac{x^{3/2}}{3/2}\,dx\right]_0^1$$

$$=2\left[(x-1)\frac{x^{3/2}}{3/2}-\frac{2}{3}\frac{x^{5/2}}{5/2}\right]_0^1$$

$$=2\left[-\frac{2}{3}\cdot\frac{2}{5}\right]$$

$$=2\times-\frac{4}{15}$$

$$=-\frac{8}{15}\text{ sq.unit}$$

Ignoring the -ve sign, we get.

$$A=\frac{8}{15}\text{ sq.unit}$$

Hence, the area of the curve $y^2=x(x-1)^2$ is $\frac{8}{15}$ sq.unit.

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* Find the area of the loop of the curve $y^2(a+x)=x^2(a-x)$

Ans Solution:

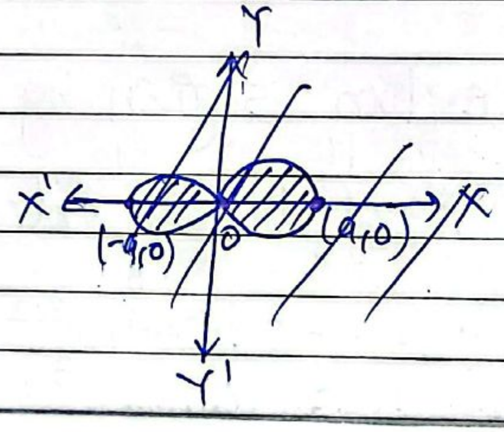
The given equation of the curve is

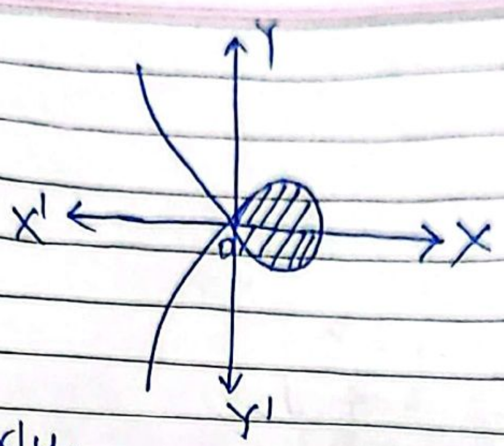
$$y^2(a+x)=x^2(a-x)$$

which contains even power in Y. So, the curve is symmetrical about X-axis.

Also, for $x=a, +a, 0$; $y=0$

So,





Clearly,

The area of the loop is given by

$$A = 2\int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx$$

let $x = a\cos 2\theta$

$\Rightarrow dx = -2a\sin 2\theta$

When $x \to 0$, $\theta \to \frac{\pi}{4}$; $x \to a$, $\theta \to 0$

So,

$$A = 2\int_{\pi/4}^{0} a\cos 2\theta \cdot \sqrt{\frac{a-a\cos 2\theta}{a+a\cos 2\theta}} \cdot -2a\sin 2\theta\, d\theta$$

$$= 2a^2\int_0^{\pi/4} \cos 2\theta \cdot \frac{\sin\theta}{\cos\theta} \cdot 2 \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 4a^2\int_0^{\pi/4} \cos 2\theta \cdot (1-\cos 2\theta)\, d\theta$$

$$= 4a^2\left[\int_0^{\pi/4} \cos 2\theta - \int_0^{\pi/4} \cos^2 2\theta \right\} d\theta\Big]$$

$$= 4a^2\left[\left[\frac{\sin 2\theta}{\theta}\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/2} \cos^2 t\, dt\right]$$

$$= 4a^2\left[\frac{1}{2} - \frac{1}{2}\frac{\Gamma(1/2)\Gamma(3/2)}{2\times\Gamma(2)}\right]$$

$$= 4a^2\left[\frac{1}{2} - \frac{1}{2}\times\frac{\sqrt{\pi}\times 1/2\sqrt{\pi}}{2\times 1}\right]$$

$$= 4a^2\left[\frac{1}{2} - \frac{1}{2}\frac{\pi}{4}\right]$$

$$= 2a^2\left[1 - \frac{\pi}{4}\right]$$

$$= a^2\left[2 - \frac{\pi}{2}\right]$$

$$= a^2\left[2 - \frac{\pi}{2}\right] \text{ sq. units}$$

$\therefore$ The area of the loop of the curve $y^2(a+x) = x^2(a-x)$ is $a^2\left[2 - \frac{\pi}{2}\right]$ sq. units.

Chapter : 19
Transformation of Axes

Formula 1 (Translation Formulae)

To find the coordinates of a point $P(x,y)$ when origin $(0,0)$ is changed to another point $O'(h,k)$ without changing direction of axes:

$x = x' + h$
$y = y' + k$

where (x', y') be the coordinates of the same point w.r.t. the new axes.

Formula of 2 (Rotation Formula)

To find the coordinates of a point $P(x,y)$ when the direction of axes is changed without changing origin:

$x = x'\cos\theta - y'\sin\theta$
$y = x'\sin\theta + y'\cos\theta$

where (x', y') be the coordinates of the same point P w.r.t. new axes.

Exercise-32

1. Transform the equation $x^2-3y^2+4x+6y=0$ by transferring the origin to the point $(-2,1)$ and the coordinate axes remaining parallel.

Solution:

Given:

The origin equation of the curve is:

$$x^2-3y^2+4x+6y=0$$

Since, the origin is transferred to $(-2,1)$ and the coordinate axes are remained parallel, the new equation of the same curve becomes

$$(x-2)^2-3(y+1)^2+4(x-2)+6(y+1)=0$$

$$\cancel{\Rightarrow x^2-2x+4-3y^2-6y-3+4x-8-6y+6=0}$$

$$\cancel{\Rightarrow x^2-3y^2+2x-1=0}$$

~~which is the required equation of the transformed curve.~~

$$\Rightarrow x^2-4x+4-3y^2-6y-3+4x-8+6y+6=0$$

$$\Rightarrow x^2-3y^2-1=0$$

which is the required equation of the transformed curve.

2. Transform the equation $x^2+y^2-5xy+4=0$ to parallel axes through point $(1,3)$.

Solution:

The given equation of the curve is

$$x^2+y^2-5xy+4=0$$

Since, the origin is transferred to $(1,3)$ and the coordinate axes are remained parallel, the new equation of the same curve becomes,

$$(x+1)^2+(y+3)^2-5(x+1)(y+3)+4=0$$
$$\Rightarrow x^2+2x+1+y^2+6y+9-5(xy+3x+y+3)+4=0$$
$$\Rightarrow x^2+2x+1+y^2+6y+9-5xy-15x-5y-15+4=0$$
$$\Rightarrow x^2+y^2-5xy-13x+y-1=0$$

which is the required equation of the transformed axes.

3. Transform the equation $x^2+3y^2+3x-40=0$ to parallel axes through $(4,-1)$.

Ans Solution:

The given equation of the curve is:

$$x^2+3y^2+3x-40=0$$

Since, the origin is transferred to $(4,-1)$ and the coordinate axes are remained parallel, the new equation of the same curve becomes,

$$(x+4)^2+3(y-1)^2+3(x+4)-40=0$$
$$\Rightarrow x^2+4x+16+3(y^2-2y+1)^2+3x+12-40=0$$
$$\Rightarrow x^2+3y^2+7x-6y-9=0$$

which is the required equation of the transformed axes.

4. What does the equation $2x+3y=\sqrt{52}$ becomes when the axes are turned through the angle $45°$ to the original axes.

Ans Solution:

The given equation of the curve is:

$2x+3y=\sqrt{2}$

Since, the coordinate axes of the curve is rotated through 45°, the equation of the curve becomes,

$$2\left(\frac{x}{\sqrt{2}}-\frac{y}{\sqrt{2}}\right)+3\left(\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}\right)=\sqrt{2}$$

$$\Rightarrow \frac{2x-2y+3x+3y}{\sqrt{2}}=\sqrt{2}$$

$$\Rightarrow 2x-2y+3x+3y=2$$

$$\Rightarrow 5x+y=2$$

$$\Rightarrow 5x+y-2=0$$

$$\Rightarrow 5x+y=2$$

$$\Rightarrow \frac{5}{2}x+\frac{y}{2}=1$$

$$\Rightarrow \frac{5}{2}x+\frac{y}{2}-1=0$$

$\therefore \frac{5}{2}x+\frac{y}{2}-1=0$ is the required equation of the curve.

5. What does the equation $x^2+2\sqrt{3}xy-y^2=2a^2$ become when the axes are rotated through the angle 30° to the original axes.

Ans Solution:

The given equation of the curve is

$$x^2+2\sqrt{3}xy-y^2=2a^2$$

Since, the curve is rotated through 30°, the new coordinates of reference becomes

$$X=x\cos\theta-y\sin\theta=\frac{x\sqrt{3}}{2}-\frac{y}{2}$$

$$Y=x\sin\theta+y\sin\theta=\frac{x}{2}+\frac{y\sqrt{3}}{2}$$

So, the equation of the rotated curve becomes

$$\left(\frac{\sqrt{3}x-y}{2}\right)^2 + 2\sqrt{3}\left(\frac{x\sqrt{3}-y}{2}\right)\left(\frac{x+\sqrt{3}y}{2}\right) - \left(\frac{x+\sqrt{3}y}{2}\right)^2 = 2a^2$$

$$\Rightarrow \frac{3x^2-2\sqrt{3}xy+y^2}{4} + \frac{2\sqrt{3}(\sqrt{3}x^2+3xy-xy-\sqrt{3}y^2)}{4} - \frac{(x^2+2\sqrt{3}xy+y^2)}{4} = 2a^2$$

$$\Rightarrow 3x^2-2\sqrt{3}xy+y^2+6x^2+6\sqrt{3}xy-2\sqrt{3}xy-6y^2-x^2-2\sqrt{3}xy-3y^2=8a^2$$

$$\Rightarrow 3x^2+6x^2+y^2-6y^2-3y^2-x^2=8a^2$$

$$\Rightarrow 8x^2-8y^2=8a^2$$

$$\Rightarrow x^2-y^2=a^2$$

which is the required equation of the transformed curve.

6. The equation $x^2-y^2=a^2$ is transformed to $xy=c^2$ by change of rectangular axes. Find the inclination of the new axes to the original axes and the value of c^2.

Ans. Solution:

The given equation of the curve is

$$x^2-y^2=a^2$$

Let the original axes be rotated by angle θ such that the equation of the new axes becomes $xy=c^2$.

So,

which implies

$xy = c^2$

$\equiv (x\cos\theta - y\sin\theta)^2 - (x\sin\theta + y\cos\theta)^2 = a^2$

$\equiv x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin^2\theta$
$- x^2\sin^2\theta - 2xy\sin\theta\cos\theta$
$- y^2\cos^2\theta = a^2$

$\equiv (\cos^2\theta - \sin^2\theta)x^2 - 4xy\sin\theta\cos\theta$
$+ y^2(\sin^2\theta - \cos^2\theta) = a^2$

$\Rightarrow \cos^2\theta - \sin^2\theta = 0 \qquad \Rightarrow \cos 2\theta = 0$

and $\underline{\sin^2\theta - \cos^2\theta = 0} \qquad \Rightarrow -\cos 2\theta = 0$

$\Rightarrow \sin^2\theta = \cos^2\theta \qquad \Rightarrow 2\theta = \pi/2$

$\Rightarrow \theta = \pm\frac{\pi}{4} \qquad \Rightarrow \theta = \pm\frac{\pi}{4}$

So,

$xy = c^2$

$\equiv -4xy\sin\theta\cos\theta = a^2$

$\equiv 4xy \frac{+1}{\sqrt{2}} \frac{-1}{\sqrt{2}} = -a^2 \quad (\text{for } \theta = -\frac{\pi}{4})$

$\equiv 4xy\cancel{\sin\theta} = 2a^2$

$\equiv xy = \frac{2a^2}{4} = \frac{a^2}{2} \qquad$ and for $\theta = \frac{\pi}{4}$

$\therefore c^2 = \frac{a^2}{2} \qquad c^2 = -a^2/2$

Hence, the inclination of the new ~~and~~ the axes to the original axes is $\frac{\pi}{4}$ or $-\frac{\pi}{4}$ and the value of c^2 is

$\frac{a^2}{2}$ (for $\theta = \frac{\pi}{4}$) and $-\frac{a^2}{2}$ (for $\theta = -\frac{\pi}{4}$)

7. Transform the equation $12x^2 - 10xy + 2y^2 + 11x$

$-5y+2=0$ by translating the axes into an equation with linear term missing.

Ans. Solution:

Let the original curve be ~~thro~~ translated through the origin by the point (h,k). Then, the equation of the transformed curve becomes:

$$12(x+h)^2-10(x+h)(y+k)+2(y+k)^2+11(x+h)-5(y+k)+2=0$$

$$\Rightarrow 12(x^2+2hx+h^2)-10(xy+kx+hy+hk)+2(y^2+2yk+k^2)+11x+11h-5y-5k+2=0$$

$$\Rightarrow 12x^2+24xh+12h^2-10xy-10kx-10hy-10hk+2y^2+4yk+2k^2+11x+11h-5y-5k+2=0$$

$$\Rightarrow 12x^2+2y^2-10xy+(24h-10k+11)x+(-10h+4k-5)y+12h^2-10hk+2k^2+11h-5k+2=0$$

For the linear terms to be missing,

$$24h-10k+11=0 \quad \text{---(i)}$$

$$-10h+4k-5=0 \quad \text{---(ii)}$$

From (ii)

$$10h=4k-5$$

$$h=\frac{4k-5}{10} \quad \text{---(iii)}$$

From (i) and (iii), we get

$$24\frac{(4k-5)}{10}-10k+11=0$$

$$\Rightarrow \cancel{96-120-100k+110=0}$$

$$\cancel{\Rightarrow 100k=}$$

$$k = \frac{86}{100} = \frac{43}{50}$$

Also,
$$h = \frac{4k-5}{10} = \frac{4 \times \frac{43}{50} - 5}{10} = \frac{172-250}{500}$$
$$= -\frac{78}{500}$$
$$= -\frac{39}{250} = -\frac{39}{250}$$

$$24\left(\frac{4k-5}{10}\right) - 10k + 11 = 0$$
$$\Rightarrow 96k - 120 - 100k + 110 = 0$$
$$\Rightarrow -4k - 10 = 0$$
$$\Rightarrow k = -\frac{5}{2}$$

Also,
$$h = \frac{4 \times (-5/2) - 5}{10} = -\frac{3}{2}$$

So, the required equation becomes,
$$12x^2 + 2y^2 - 10xy + 12 \times \left(-\frac{3}{2}\right)^2 - 10 \times \frac{3}{2} \times \frac{5}{2}$$
$$+ 2 \times \left(-\frac{5}{2}\right)^2 + 11 \times \left(-\frac{3}{2}\right) - 5 \times \left(-\frac{5}{2}\right)$$
$$+ 2 = 0$$

$$\Rightarrow 12x^2 + 2y^2 - 10xy + 27 - \frac{75}{2} + \frac{25}{2} - \frac{33}{2} + \frac{25}{2} = 0$$
$$\Rightarrow 12x^2 + 2y^2 - 10xy + \frac{67}{4} - \frac{67}{4} = 0$$
$$\Rightarrow 48x^2 + 8y^2 - 40xy + 67 - 67 = 0$$
$$\Rightarrow 48x^2 + 8y^2 - 40xy = 0$$

$\Rightarrow \cancel{-6y}\ 6x^2 + y^2 - 5xy = 0$

$\Rightarrow 6x^2 - 5xy + y^2 = 0$

is the required transformed equation.

8. Through what angle must the axes be rotated to remove the term containing xy in $3x^2 + 2xy + 3y^2 - \sqrt{2}x = 0$. Also, find the transformed equation.

Ans Solution:

The given equation of the curve is

$$3x^2 + 2xy + 3y^2 - \sqrt{2}x = 0$$

Let the curve be rotated through an angle θ. Then, the equation of transformed curve be

$$3(x\cos\theta - y\sin\theta)^2 + 2(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + 3(x\sin\theta + y\cos\theta)^2 - \sqrt{2}(x\cos\theta - y\sin\theta) = 0$$

$$\Rightarrow 3(x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin^2\theta) + 2(x^2\sin\theta\cos\theta + xy\cos^2\theta - xy\sin^2\theta - y^2\sin\theta\cos\theta) + 3(x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta) - \sqrt{2}x\cos\theta + \sqrt{2}y\sin\theta = 0$$

$$\Rightarrow 3x^2\cos^2\theta - 6xy\sin\theta\cos\theta + 3y^2\sin^2\theta + 2x^2\sin\theta\cos\theta + 2xy\cos^2\theta - 2xy\sin^2\theta - 2y^2\sin\theta\cos\theta + 3x^2\sin^2\theta + 6xy\sin\theta\cos\theta + 3y^2\cos^2\theta - \sqrt{2}x\cos\theta + \sqrt{2}y\sin\theta = 0$$

$$\Rightarrow (3\cos^2\theta + 2\sin\theta\cos\theta + 3\sin^2\theta)x^2 + (-6\sin\theta\cos\theta + 2\cos^2\theta - 2\sin^2\theta + 6\sin\theta\cos\theta)xy + (3\sin^2\theta - 2\sin\theta\cos\theta + 3\cos^2\theta)y^2 - \sqrt{2}x\cos\theta + \sqrt{2}y\sin\theta = 0$$

Since, xy term must be missing,

$$2\cos^2\theta - 2\sin^2\theta = 0$$

$$\therefore \theta = \frac{\pi}{4}.$$

So the equation becomes

$$\left(3\times\frac{1}{2} + \frac{2}{2} + \frac{3}{2}\right)x^2 + \left(\frac{3}{2} - \frac{2}{2} + \frac{3}{2}\right)y^2 - \sqrt{2}x\cdot\frac{1}{\sqrt{2}} + \sqrt{2}y\cdot\frac{1}{\sqrt{2}} = 0$$

$$\Rightarrow 4x^2 + 2y^2 - x + y = 0$$

$\Rightarrow 4x^2 + 2y^2 - x + y = 0$ is the required transformed equation.

9. Transform the equation $x^2 - 2xy + y^2 + x - 3y = 0$ to axes through the point $(-1, 0)$ parallel to the lines bisecting the angles between original axes.

Ans Solution:

The given equation of the curve is

$$x^2 - 2xy + y^2 + x - 3y = 0$$

Since, the origin of the given curve is shifted to $(-1,0)$. The equation of the new curve becomes,

$$(x-1)^2 - 2(x-1)y + y^2 + (x-1) - 3y = 0$$

$$\Rightarrow x^2 - 2x + 1 - 2xy + 2y + y^2 + x - 1 - 3y = 0$$

$$\Rightarrow x^2 - 2xy + y^2 - x - y = 0$$

Also, coordinate axes are

The curve is parallel to the lines bisecting the angles between the original axes i.e. the curve is rotated through 45°. So,

$$x = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$$

$$y = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$$

So, the new equation of the curve becomes,

$$\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)^2 - 2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) + \left(\frac{y}{\sqrt{2}} + \frac{x}{\sqrt{2}}\right)^2$$

$$-\left(\frac{x-y}{\sqrt{2}}\right) - \left(\frac{x+y}{2}\right) = 0$$

$$\Rightarrow \frac{x^2-2xy+y^2}{2} - \frac{2(x^2-y^2)}{2} + \frac{x^2+2xy+y^2}{2}$$

$$-\left\{\frac{x+y-x-y}{2}\right.$$

$$\cancel{\Rightarrow x^2-2xy+y^2-2x^2+2y^2+x^2+2xy+y^2+x+3y=0}$$

$$\cancel{\Rightarrow 2x^2+2y^2-2x^2+2y^2+x+3y=0}$$

$$\cancel{\Rightarrow 4y^2+x+3y=0}$$

$$\Rightarrow x^2-2xy+y^2-2x^2+2y^2+x^2+2xy+y^2-$$

$$x^2-2xy+y^2-2x^2+2y^2+x^2+2xy+y^2-x+y-x-y = 0$$

$$\Rightarrow 4y^2-2x=0$$

$$\Rightarrow x = 2y^2$$

$\therefore x-2y^2=0$ is the required equation of the curve after successive transformation.

10. Transform the equation $2x^2+4xy+5y^2-4x-22y+7=0$ to parallel axes through $(-2,3)$.

Ans Solution:

The given equation of the curve is:

$$2x^2+4xy+5y^2-4x-22y=0$$

Since, the origin is shifted to $(-2,3)$ and the coordinate axes are remained parallel,

$$2(x-2)^2+4(x-2)(y+3)+5(y+3)^2-4(x-2)-22(y+3)=0$$

$$\Rightarrow 2(x^2-4x+4)+4(xy+3x-2y-6)+5(y^2+6y+9)-4x+8-22y-66=0$$

$$\Rightarrow 2x^2-8x+8+4xy+12x-8y-24+5y^2+30y+45-4x-22y-58=0$$

$\Rightarrow 2x^2+4xy+5y^2+29=0$ is the required equation of the curve.

---

11. Find the angle through which the axes may be turned so that the equation $x+2y+5=0$ may be reduced to the form $x=c$ and determine the value of c.

Ans Solution:

The given equation of the curve is

$$x+2y+5=0 \quad \text{---(i)}$$

Let the axes be rotated through an angle θ. Then,

$$x = x\cos\theta - y\sin\theta$$
$$y = x\sin\theta + y\cos\theta$$

Then, the equation (i) becomes,

$$x\cos\theta - y\sin\theta + 2(x\sin\theta + y\cos\theta) + 5 = 0$$

$$\Rightarrow x\cos\theta - y\sin\theta + 2x\sin\theta + 2y\cos\theta + 5 = 0$$

$\Rightarrow (\cos\theta + 2\sin\theta)x + (2\cos\theta - \sin\theta)y + 5 = 0$

Since,

the reduced should be in the form $x = c$.

$\Rightarrow \cos\theta + 2\sin\theta = 1$ and $2\cos\theta - \sin\theta = 0$

$\Rightarrow \frac{\cos\theta - 1}{2} = -\sin\theta$

$\Rightarrow 2\cos\theta - \left(\frac{\cos\theta - 1}{2}\right) = 0$

$\Rightarrow 4\cos\theta - \cos\theta + 1 = 0$

$\Rightarrow 3\cos\theta = -1$

$\Rightarrow \cos\theta = \frac{-1}{3}$

$\therefore \theta = \cos^{-1}\left(\frac{-1}{3}\right)$.

Also, $c = -5$

Hence, the axes should be rotated through an angle of $\cos^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ and the value of c is -5.

---

12. What does the equation of the straight lines $7x^2 + 4xy + 4y^2 = 0$ become when the axes are the bisectors of the angles between them.

Ans Solution:

The given equation of the straight lines is: $7x^2 + 4xy + 4y^2 = 0$ ---(i)

The equation of bisectors of (i) is

$2(x^2 - y^2) = (7-4)xy$

$\Rightarrow 2x^2 - 2y^2 = 3xy$

$\Rightarrow 2x^2 - 3xy - 2y^2 = 0$

$\Rightarrow 2x^2 - 4xy + xy - 2y^2 = 0$

$$\Rightarrow 2x(x-2y)+y(x-2y)=0$$

$$\Rightarrow (2x+y)(x-2y)=0$$

Either $2x+y=0$ OR $x-2y=0$

$$\Rightarrow x-2y=0$$

$$\Rightarrow x=2y$$

$$\Rightarrow y=\frac{1}{2}x$$

$$\therefore \theta=\tan^{-1}\left(\frac{1}{2}\right)$$

Since, the acute angle made by the bisector is $\tan^{-1}\left(\frac{1}{2}\right)$. The curve should be rotated through $\theta=\tan^{-1}\left(\frac{1}{2}\right)$

$$\Rightarrow \sin\theta=\frac{1}{\sqrt5} \text{ and } \cos\theta=\left(\frac{2}{\sqrt5}\right)$$

We have,

After rotation, the equation of the curve becomes,

$$\cancel{7\left(\frac{x}{\sqrt5} - \frac{2}{\sqrt5}\right)}$$

$$7\left(\frac{2x}{\sqrt5}-\frac{y}{\sqrt5}\right)^2+4\left(\frac{2x}{\sqrt5}-\frac{y}{\sqrt5}\right)\left(\frac{x}{\sqrt5}+\frac{2y}{\sqrt5}\right)+4\left(\frac{x}{\sqrt5}+\frac{2y}{\sqrt5}\right)^2=0$$

$$\Rightarrow 7\frac{(4x^2-4xy+y^2)}{5}+4\frac{(2x^2+4xy-xy-2y^2)}{5}+4\frac{(x^2+4xy+4y^2)}{5}=0$$

$$\Rightarrow 28x^2-28xy+7y^2+8x^2+16xy-4xy-8y^2+4x^2+16xy+16y^2=0$$

$\Rightarrow 40x^2 + 15y^2 = 0$
$\Rightarrow 8x^2 + 3y^2 = 0$
Hence, the equation of the required curve is $8x^2 + 3y^2 = 0$.

Examples:

* If the axes be turned through an angle $\tan\theta = 2$, what does the equation $4xy - 3x^2 = a^2$ become?

Ans Solution:
The given equation of the curve is:
$$4xy - 3x^2 = a^2$$
Also, the curve is rotated through angle θ such that $\tan\theta = 2$.
So,
$$\sin\theta = \frac{2}{\sqrt5} \text{ and } \cos\theta = \frac{1}{\sqrt5}.$$
So, the equation of the curve becomes,
$$4\left(\frac{1}{\sqrt5}x - \frac{2}{\sqrt5}y\right)\left(\frac{2x}{\sqrt5} + \frac{y}{\sqrt5}\right) - 3\left(\frac{x}{\sqrt5} - \frac{2}{\sqrt5}y\right)^2 = a^2$$
$$\Rightarrow \frac{4(x-2y).(2x+y)}{5} - \frac{3(x-2y)^2}{5} = a^2$$
$$\Rightarrow \frac{4(2x^2 + xy - 4xy - 2y^2)}{5} - \frac{3}{5}(x^2 - 4xy + 4y^2) = a^2$$
$$\Rightarrow 4(2x^2 - 3xy - 2y^2) - 3(x^2 - 4xy + 4y^2) = 5a^2$$
$$\Rightarrow 8x^2 - 12xy - 8y^2 - 3x^2 + 12xy - 12y^2 = 5a^2$$
$$\Rightarrow 5x^2 - 20y^2 = 5a^2$$
$\Rightarrow x^2 - 4y^2 = a^2$ is the required equation of the curve.

* What does the equation $3x^2+3y^2+2xy=2$ become when the axes are turned an through an angle $45°$ to the original axes?

Ans Solution:

The given equation of the given curve is:

$$3x^2+3y^2+2xy=2$$

Since, the original axes are rotated through an angle $45°$, we get,

$$3\left(\frac{x}{\sqrt{2}}-\frac{y}{\sqrt{2}}\right)^2+3\left(\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}\right)^2+2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right)=2$$

$$\Rightarrow 3(x^2-2xy+y^2)+3(x^2+y^2+2xy)+2(x^2-y^2)=4$$

$$\Rightarrow 3x^2-6xy+3y^2+3x^2+3y^2+6xy+2x^2-2y^2=4$$

$$\Rightarrow 8x^2+4y^2=4$$

$$\Rightarrow 2x^2+y^2=1$$ is the required equation of the curve.

---

* Through what angle must the axes be rotated to remove the term containing xy in $11x^2+4xy+14y^2=5$.

Ans Solution:

The given equation of the given curve is:

$$11x^2+4xy+14y^2=5$$

Let the curve be rotated through an angle θ such that the term containing xy will be absent. Then,

$$11(x\cos\theta-y\sin\theta)^2+4(x\cos\theta-y\sin\theta)(x\sin\theta+y\cos\theta)+14(x\sin\theta+y\cos\theta)^2=5$$

$$\Rightarrow 11(x^2\cos^2\theta-2xy\sin\theta\cos\theta+y^2\sin^2\theta)+$$

$4(x^2\sin\theta\cos\theta + xy\cos^2\theta - xy\sin^2\theta - y^2\sin\theta\cos\theta)$
$+14(x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta) = 5$

$\Rightarrow (11\cos^2\theta + 4\sin\theta\cos\theta + 14\sin^2\theta)x^2$
$+(-22\sin\theta\cos\theta + 4\cos^2\theta - 4\sin^2\theta +$
$28\sin\theta\cos\theta) + (11\sin^2\theta - 4\sin\theta\cos\theta$
$+ 14\cos^2\theta)y^2 = 5$

Since, xy containing term must be absent,

$6\sin\theta\cos\theta + 4\cos^2\theta - 4\sin\theta = 0$

$\Rightarrow 2\sin^2\theta - 3\sin\theta\cos\theta - 2\cos^2\theta = 0$

$\Rightarrow 2\sin\theta(\sin\theta - 2\cos\theta) + \cos\theta(\sin\theta - 2) = 0$

$\Rightarrow (2\sin\theta + \cos\theta)(\sin\theta - 2) = 0$

Since, $\sin\theta - 2\cos\theta =$ ~~(not possible)~~.

Also, $2\sin\theta = -\cos\theta$ $\quad\Rightarrow 2\cos\theta = \sin\theta$

$\Rightarrow \tan\theta = -\frac{1}{2}$. $\quad\Rightarrow \tan\theta = 2$

Hence, the axes should be rotated through angle $\theta = \tan^{-1}\left(-\frac{1}{2}\right)$ or $\tan^{-1}(2)$ to remove the term containing xy!

Through what angle should the axes be rotated so that the equation $9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ may be changed to $3x^2 + 5y^2 = 5$.

Ans Solution:

The given equation of the curve is:

$9x^2 - 2\sqrt{3}xy + 7y^2 = 10$

let the given curve through an angle θ such that it may be changed to

$3x^2 + 5y^2 = 5$.

Then,

The equation:

$$9(x\cos\theta - y\sin\theta)^2 - 2\sqrt{3}(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + 7(x\sin\theta + y\cos\theta)^2 = 10$$

$$\Rightarrow 9(x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin^2\theta) - 2\sqrt{3}(x^2\sin\theta\cos\theta + xy\cos^2\theta - xy\sin^2\theta - y^2\sin\theta\cos\theta) + 7(x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta) = 10$$

$$\Rightarrow (9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta)x^2 + (-18\sin\theta\cos\theta - 2\sqrt{3}\cos^2\theta + 2\sqrt{3}\sin^2\theta + 14\sin\theta\cos\theta)xy + (9\sin^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 7\cos^2\theta)y^2 = 10$$

So,

$$\frac{9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta}{3} = \frac{10}{5}$$

$$\Rightarrow 9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta = 6 \quad -(A)$$

Also,

$$9\sin^2\theta + 2\sqrt{3}\cos\theta\sin\theta + 7\cos^2\theta = 5 \times \frac{10}{5} = 10 \quad -(B)$$

~~From (1) and (2)~~

Also,

$$(2\sqrt{3}\sin^2\theta - 2\sqrt{3}\cos^2\theta - 4\sin\theta\cos\theta) = 0$$

$$\Rightarrow 2\sqrt{3}\cos 2\theta = -2\sin 2\theta$$

$$\Rightarrow \tan 2\theta = -\sqrt{3}$$

$$\cancel{\Rightarrow 2\theta = \tan^{-1}(-\sqrt{3}) = (-60^\circ)}$$

$$\cancel{\Rightarrow \theta = (-30^\circ)}$$

$$\Rightarrow \tan 2\theta = \tan(120^\circ)$$

$$\Rightarrow 2\theta = 120$$

$$\Rightarrow \theta = 60^\circ$$

Hence, the axes should be rotated through 60°.

* Transform the equation $3x^2-2xy+4y^2+8x-10y+8=0$ by translating the axes into an equation with linear term missing.

~~Ans~~ Solution:

The given equation of the curve is $3x^2-2xy+4y^2+8x-10y+8=0$.

Let the ~~cur~~ coordinate axes of the curve be translated through the point (a,b). Then,

$$3(x+a)^2-2(x+a)(y+b)+4(y+b)^2+8(x+a)-10(y+b)+8=0$$

$$\Rightarrow 3(x^2+2ax+a^2)-2(xy+bx+ay+ab)+4(y^2+2by+b^2)+8x+8a-10y-10b+8=0$$

$$\Rightarrow 3x^2+6ax+3a^2-2xy-2bx-2ay-2ab+4y^2+8by+4b^2+8x+8a-10y-10b+8=0$$

$$\Rightarrow 3x^2-2xy+4y^2+(6a-2b+8)x+(-2a+8b-10)y+3a^2-2ab+4b^2+8a-10b+8=0$$

Since, linear terms must be absent,

$6a-2b+8=0$ and

$-2a+8b-10=0$.

$\Rightarrow a=-1, b=1$

So, the equation of the curve becomes

$$3x^2-2xy+4y^2+3-2\times(-1)\times1+4+(-8)-10+8=0$$

$$\Rightarrow 3x^2-2xy+4y^2+3+2+4-8-10+8=0$$

$$\Rightarrow 3x^2-2xy+4y^2-1=0$$

i.e. $3x^2-2xy+4y^2=1$ is the required equation of the given curve.

* If (x,y) and (x_1,y_1) be the coordinates of the same point referred to two sets of rectangular axes with same origin and if $ux+vy=0$ where u and v are independent x,y become $u_1x_1+v_1y_1=0$ then show that $u^2+v^2=u_1^2+v_1^2$.

Ans Solution:

The equation of the curves are:

$$ux+vy=0 \quad ---(i)$$

$$u_1x_1+v_1y_1=0 \quad ---(ii)$$

Let eqn (i) be rotated through angle θ. Then,

$$u(x'\cos\theta-y'\sin\theta)+v(x'\sin\theta+y'\cos\theta)=0$$

$$\Rightarrow (u\cos\theta+v\sin\theta)x'+(v\cos\theta-u\sin\theta)y'=0$$

Which must be identical to (ii), then,

$$u_1=u\cos\theta+v\sin\theta$$

$$v_1=v\cos\theta-u\sin\theta.$$

$$RHS=u_1^2+v_1^2$$

$$=(u\cos\theta+v\sin\theta)^2+(v\cos\theta-u\sin\theta)^2$$

$$=u^2\cos^2\theta+v^2\sin^2\theta+v^2\cos^2\theta+u^2\sin^2\theta$$

$$=u^2+v^2=RHS$$

proved

Exercise-38

Describe and sketch the graph of the following polar equation of conic section.

1. $r = \frac{2}{1-\cos\theta}$

Solution:

Given,

$r = \frac{2}{1-\cos\theta}$

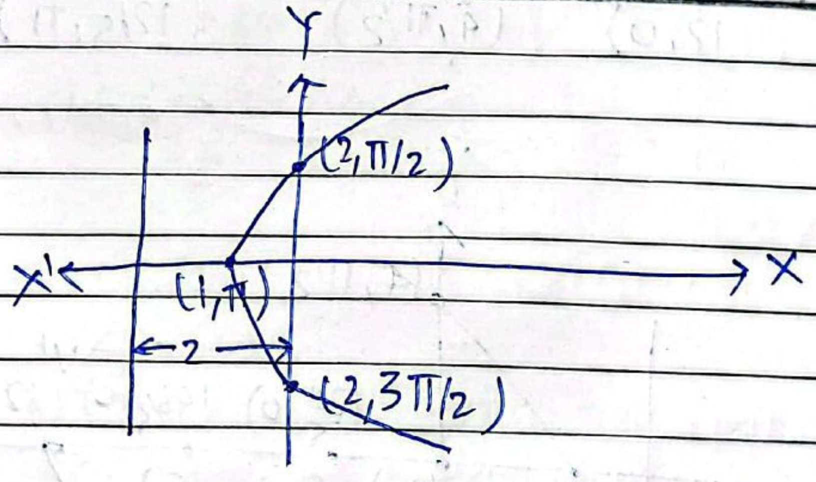
$\Rightarrow e = 1,\quad ed = 2$

$\Rightarrow d = 2$

Since, e = 1. the given conic section is a parabola.

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	∞	2	1	2
(r,θ)	$(\infty, 0)$	$(2, \pi/2)$	$(1, \pi)$	$(2, 3\pi/2)$



Description:

① Equation of Directrix:

$x = r\cos\theta$

$= ~~2\sin\pi~~\ 2\cos\pi$

$= -2$

$\therefore x = -2$ is the equation of

the directrix, vertex $(1, \pi)$ and focus $(0,0)$.

2. $r = \dfrac{12}{3-2\cos\theta}$

~~Ans~~ Solution:

Given,

$$r = \frac{12}{3-2\cos\theta} = \frac{4}{1-\frac{2}{3}\cos\theta}$$

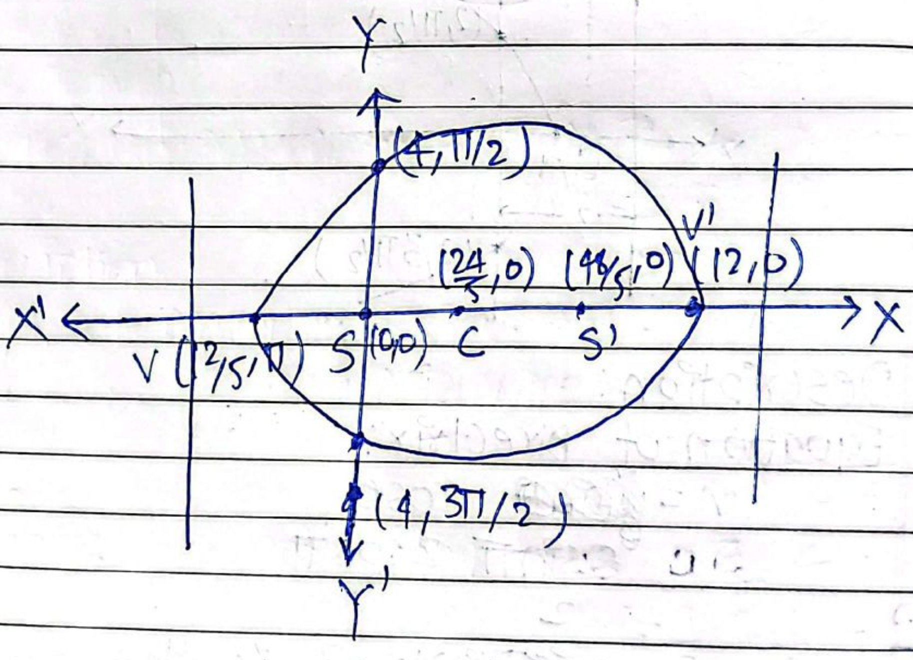
$\Rightarrow e = \dfrac{2}{3}$ and $ed = 4$

$\Rightarrow d = \dfrac{4}{2} \times 3 = 6.$

Since, $e = 2/3\ (<1)$, the given ~~ell~~ conic section is ellipse.

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	12	4	$12/5$	4
r, θ	$(12, 0)$	$(4, \pi/2)$	$(12/5, \pi)$	$(4, 3\pi/2)$



Description:

① length of major axis $= 2a = \frac{12}{5} + 12$

$\Rightarrow a = \frac{36}{5}$

② center $= ae$

$= \frac{36}{5} \times \frac{2}{3} = \frac{24}{5}$

③ Length of minor axis $= 2b$

$= 2 \cdot \sqrt{a^2(1-e^2)}$

$= 2 \times \sqrt{\left(\frac{36}{5}\right)^2 \left(1 - \left(\frac{2}{3}\right)^2\right)}$

$= \frac{24}{\sqrt{5}}$

④ Equation of directrix:

$x = r_1 \cos\theta_1$

$= 6 \cos(\pi)$

$= -6$

and $x = r_2 \cos\theta_2$

$= \left(\frac{48}{5} + 6\right) \cos 0$

$= \frac{78}{5}$

$\therefore x = -6$ and $x = \frac{78}{5}$ are the equations of the directrix.

3. $r = \frac{12}{3 + 8\cos\theta}$

Ans Solution:

Given,

$$r = \frac{12}{3+\cos\theta} = \frac{4}{1+\frac{1}{4}\cos\theta}$$

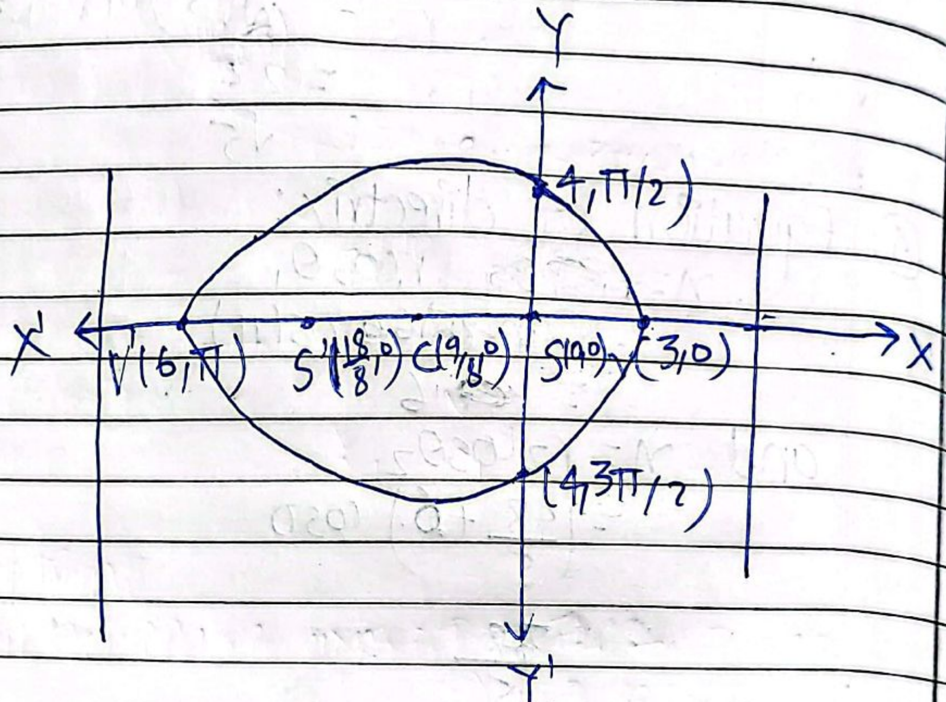
$\Rightarrow e = \frac{1}{4}$ and $ed = 4$

$\Rightarrow d = 16.$

Since, $e = \frac{1}{4} (<1)$, the given conic section is an ellipse.

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	3	4	6	4
(r,θ)	$(3,0)$	$(4,\pi/2)$	$(6,\pi)$	$(4,3\pi/2)$



Description:

① length of major axis $= 2a = 6+3 = 9$

$\therefore a = \frac{9}{2}$

Also,

Distance of centre from focus $= ae$

$= \frac{9}{2} \times \frac{1}{4} = \frac{9}{8}$

② length of Minor axis $= 2b$

$$= 2\sqrt{\left(\frac{9}{2}\right)^2\left(1-\frac{1}{4}^2\right)}$$

$$= \frac{9\sqrt{15}}{4}$$

③ Equation of directrix:

$$x = r_1 \cos\theta_1$$
$$= d\cos 0$$
$$= 16$$

and $x = r_2 \cos\theta_2$

$$= \left(\frac{18}{8} + 16\right)\cos\pi$$

$$= -\frac{73}{4}$$

$\therefore$ $x = 16$ and $x = -\frac{73}{4}$ are the equations of the directrix of the ellipse.

---

4. $r = \dfrac{10}{3+2\cos\theta}$

Ans Solution:

Given,

$$r = \frac{10}{3+2\cos\theta} = \frac{10/3}{1+2/3\cos\theta}$$

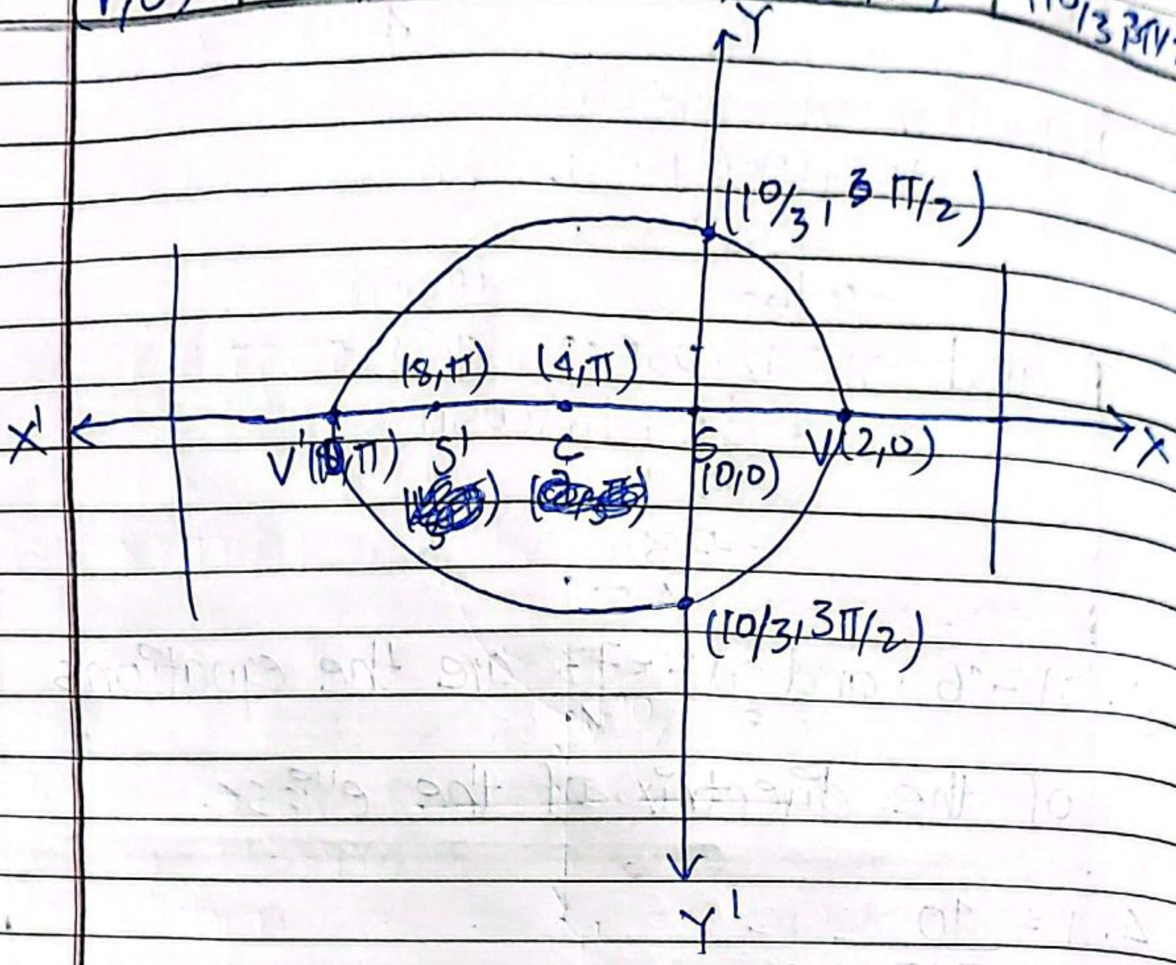
$\Rightarrow e = \frac{2}{3}$ and $ed = \frac{10}{3}$

$$\Rightarrow d = \frac{10}{3} \times \frac{3}{2} = 5$$

Since, $e = \frac{2}{3}$ (<1); the given conic section is ellipse.

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	2	$10/3$	10	$10/3$
(r,θ)	$(2,0)$	$(10/3, \pi/2)$	$(10, \pi)$	$(10/3, 3\pi/2)$



Description:

① length of major axis $= 2a = 12$

$\Rightarrow a = \frac{12}{2} = 6$

Also,
Distance of center from focus $= ae$

$= 6 \times \frac{2}{3}$

$= 4$

~~② length of Major Axis $= 2b = 2\times\frac{7}{2}\sqrt{1-\frac{4}{9}}$~~

~~$= \frac{7\sqrt{5}}{3}$~~

~~③ Equation of the directrix:~~

② length of Minor axis $= 2\times 4\sqrt{1-\frac{4}{9}}$

$= \frac{8\sqrt{5}}{3}$

③ Equation of directrix:

$x = r_1 \cos\theta_1$

$= 5\cos 0$

$= 5$

and $x = r_2 \cos\theta_2$

$= 13\cos\pi$

$= -13$

Hence, $x = 5$ and $x = -13$ are the equations of the directrix of the curve.

---

5. $r = \frac{10}{3+2\sin\theta}$

Ans Solution:

Given,

$$r = \frac{10}{3+2\sin\theta} = \frac{10/3}{1+2/3\sin\theta}$$

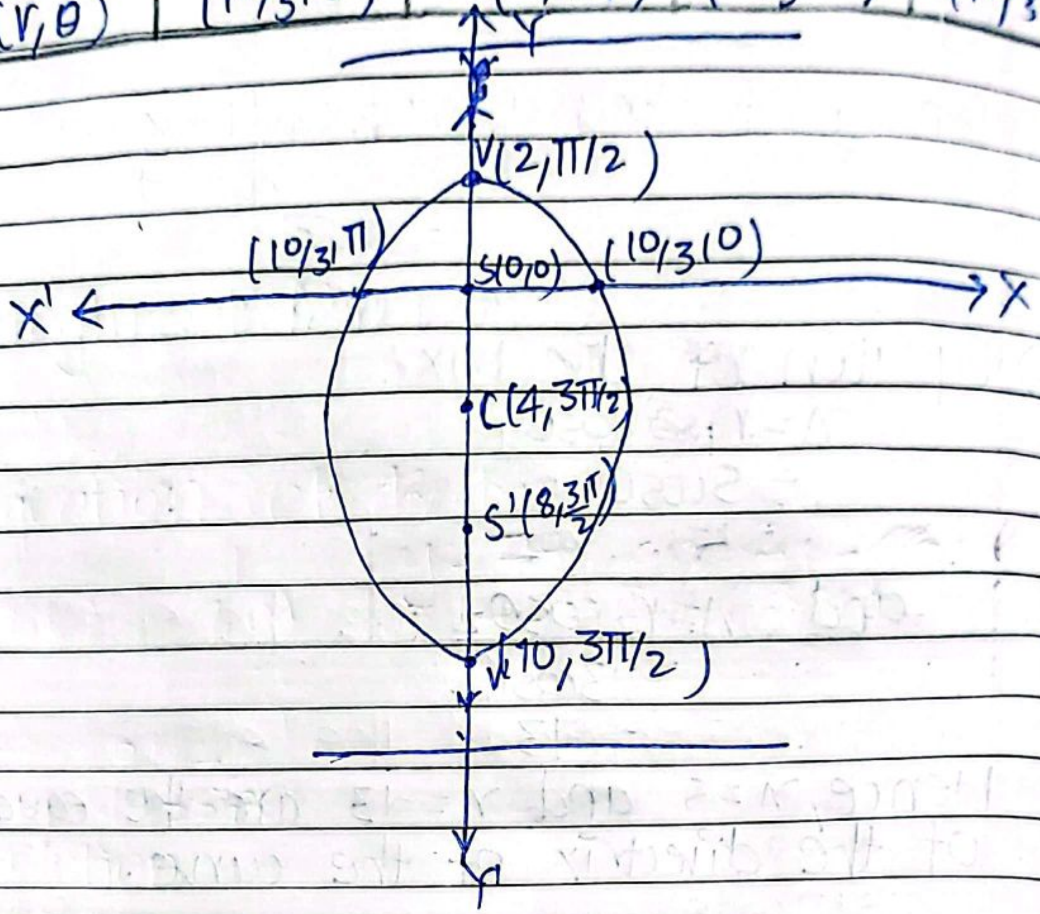
$\Rightarrow e = \frac{2}{3}$ and $ed = \frac{10}{3}$

$\Rightarrow d = \frac{10}{3}\times\frac{3}{2} = 5$

Since; $e = 2/3 (<1)$, the given conic section is ellipse:

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	$10/3$	2	$10/3$	10
(r,θ)	$(10/3, 0)$	$(2, \pi/2)$	$(10/3, \pi)$	$(10, 3\pi/2)$



Description:

① length of Major Axis $= 2a = 10 + 2 = 12$

$\therefore a = 6$

Also,

Distance from center $= ae = 4$

② length of Minor Axis $= 2b$

$= \frac{8\sqrt{5}}{3}$

③ Equation of the directrix

$y = 5$ and $y = -13$

6. $r = \frac{12}{3-2\sin\theta}$

Ans. Solution:

Given,

$$r = \frac{12}{3-2\sin\theta} = \frac{4}{1-2/3\sin\theta}$$

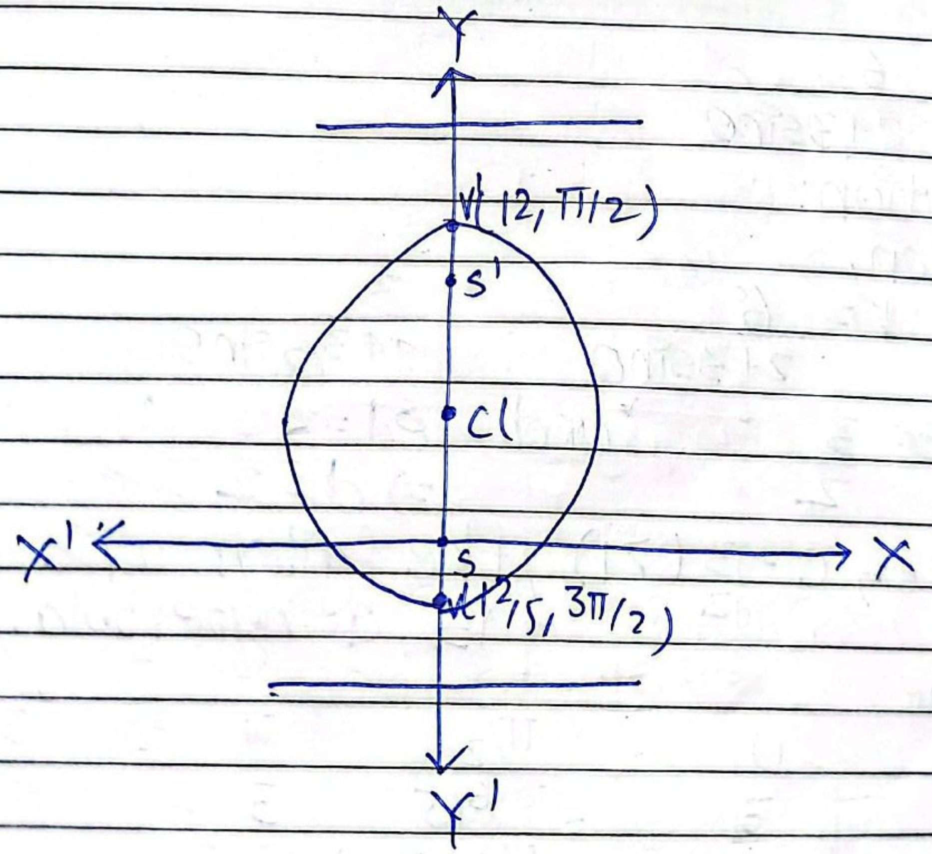
$\Rightarrow e = \frac{2}{3}$ and $ed = 4$

$\Rightarrow d = \frac{4}{2} \times 3 = 6$

Since, $e = \frac{2}{3} (<1)$; the given conic section is an ellipse

Table:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
r	4	12	4	$12/5$
(r,θ)	$(4,0)$	$(12,\pi/2)$	$(4,\pi)$	$(12/5, 3\pi/2)$



Description:

① length of Major Axis $= 2a = 12 + \frac{12}{5} = \frac{72}{5}$

$\therefore a = \frac{36}{5}$

Also,
Distance of centre from focus $= ae$

$= \frac{24}{5}$

② length of Minor Axis $= 2b$

$= \frac{24}{\sqrt{5}}$

③ Equation of directrix:

$y = \frac{\cancel{7}8}{5}$ and $y = -6$

---

7. $r = \dfrac{6}{2+3\sin\theta}$

~~Ans~~ Solution:
Given,

$$r = \frac{6}{2+3\sin\theta} = \frac{3}{1+3/2\sin\theta}$$

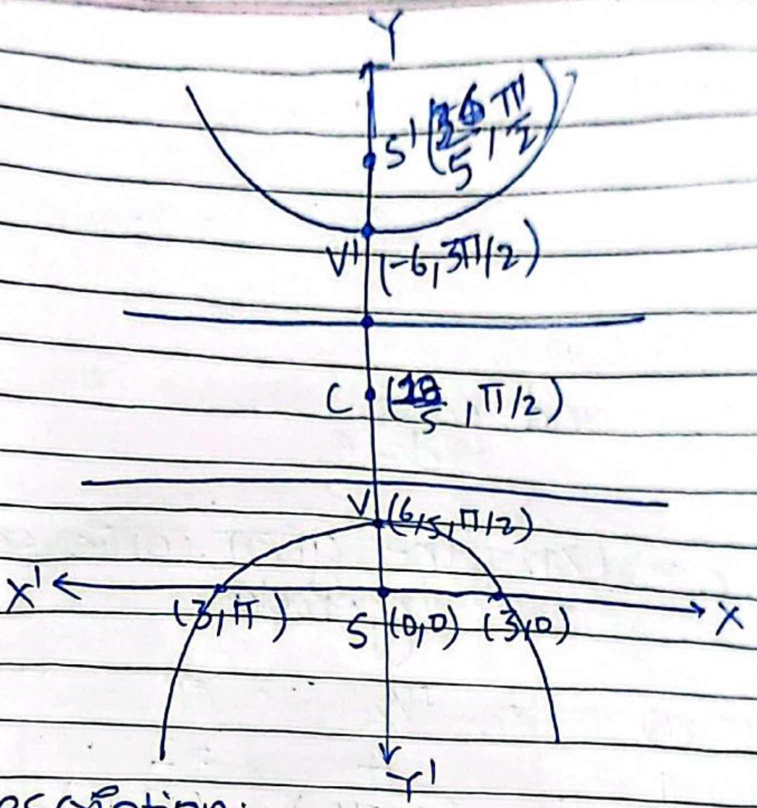
$\Rightarrow e = \frac{3}{2}$ and $ed = 3$

$\Rightarrow d = 2$

Since, $e = \frac{3}{2} (>1)$; the given conic section is a hyperbola.

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	3	$6/5$	3	-6
(r,θ)	$(3,0)$	$(6/5,\pi/2)$	$(3,\pi)$	$(-6,3\pi/2)$



Description:

① length of transverse axis $= 2a = \cancel{+6}\ \frac{6}{5}$

$= \frac{24}{5}$

$\therefore a = \frac{12}{5}$

② Distance of centre from focus $= ae$

$= \frac{12}{5} \times \frac{3}{2} = \frac{18}{5}$

③ length of conjugate axis $= 2 \times \frac{12}{5} \sqrt{\frac{9}{4} - 1}$

$= \frac{12}{\sqrt{5}}$

④ Equation of directrix:

$y = 2$ and $y = \frac{34}{5}$

8. $r = \dfrac{4}{1+3\cos\theta}$

Ans Solution:

Given,

$r = \dfrac{4}{1+3\cos\theta}$

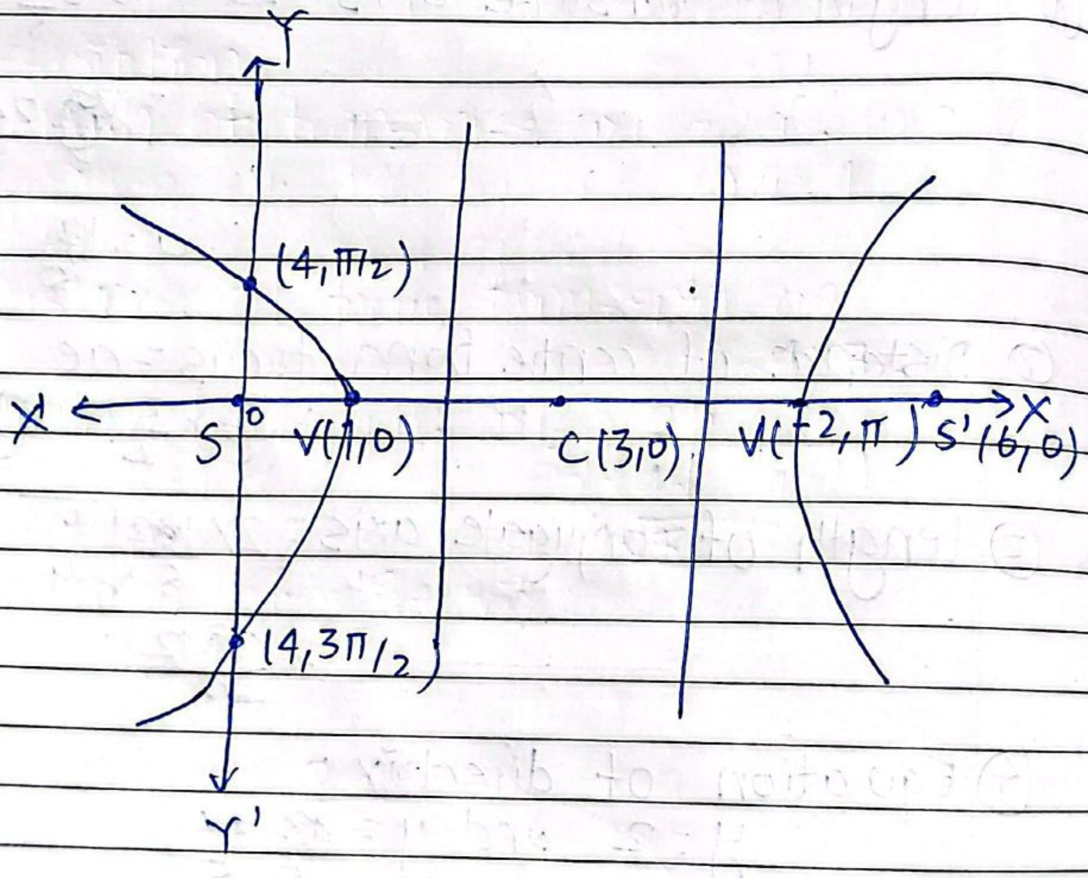
$\Rightarrow e = 3$ and $ed = 4$

$\Rightarrow d = \dfrac{4}{3}$

Since, $e = 3 (>1)$; the given conic section is hyperbola

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	1	4	-2	4
(r,θ)	$(1,0)$	$(4,\pi/2)$	$(-2,\pi)$	$(4,3\pi/2)$



Description:

① Length of transverse axis $= 2a = 2\cdot 1 = 1$

Also,

Distance of centre from focus $= ae$

$= 1 \times 3$

$= 3$

② length of conjugate axis $= 2b$

$= 2 \times 1\sqrt{9-1}$

$= 2\sqrt{8}$

③ Equation of the directrix:

$x = \frac{4}{3}$ and $x = \frac{14}{3}$

9. $r = \frac{10\operatorname{cosec}\theta}{2\operatorname{cosec}\theta + 3}$

Ans. Solution:

Given,

$r = \frac{10\operatorname{cosec}\theta}{2\operatorname{cosec}\theta+3} = \frac{10}{2+3\sin\theta} = \frac{5}{1+3/2\sin\theta}$

$\Rightarrow e = \frac{3}{2}$ and $ed = 5$

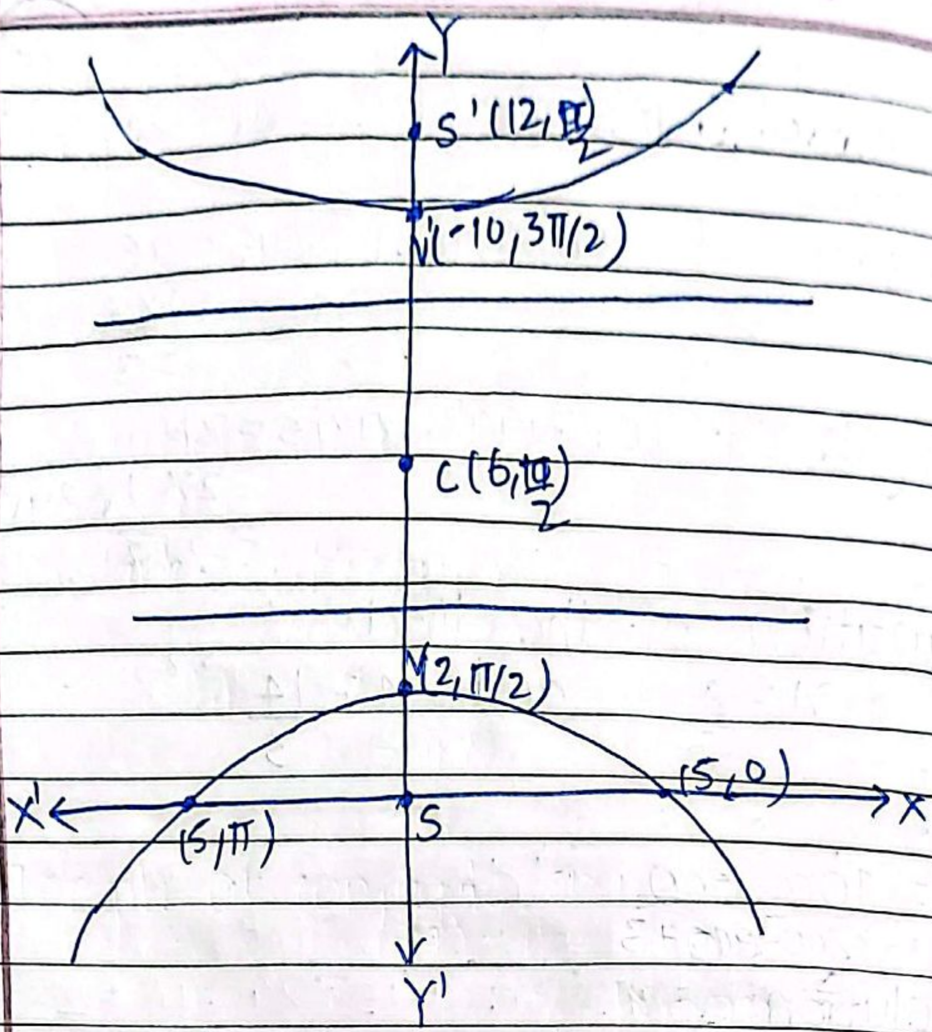
$\Rightarrow d = \frac{5}{3} \times 2 = \frac{10}{3}$

Since, $e = \frac{3}{2} (>1)$; the given conic section is hyperbola.

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	5	2	5	-10
(r,θ)	$(5,0)$	$(2,\pi/2)$	$(5,\pi)$	$(-10,3\pi/2)$

CamScanner



Description:

① Length of transverse axis $= 2a = 10 - 2 = 8$
$\Rightarrow a = 4$

Also,
Distance of focus from center $= ae$
$= 4 \times 3/2 = 6$

② Length of conjugate axis $= 2b$
$= 2 \times 4\sqrt{(9/4 - 1)}$
$= 4\sqrt{5}$

③ Equation of directrix:
$y = \frac{10}{3}$
and $y = \frac{26}{3}$

10. $r = \dfrac{7}{2+5\sin\theta}$

Ans Solution:

Given,

$$r = \frac{7}{2+5\sin\theta} = \frac{7/2}{1+5/2\sin\theta}$$

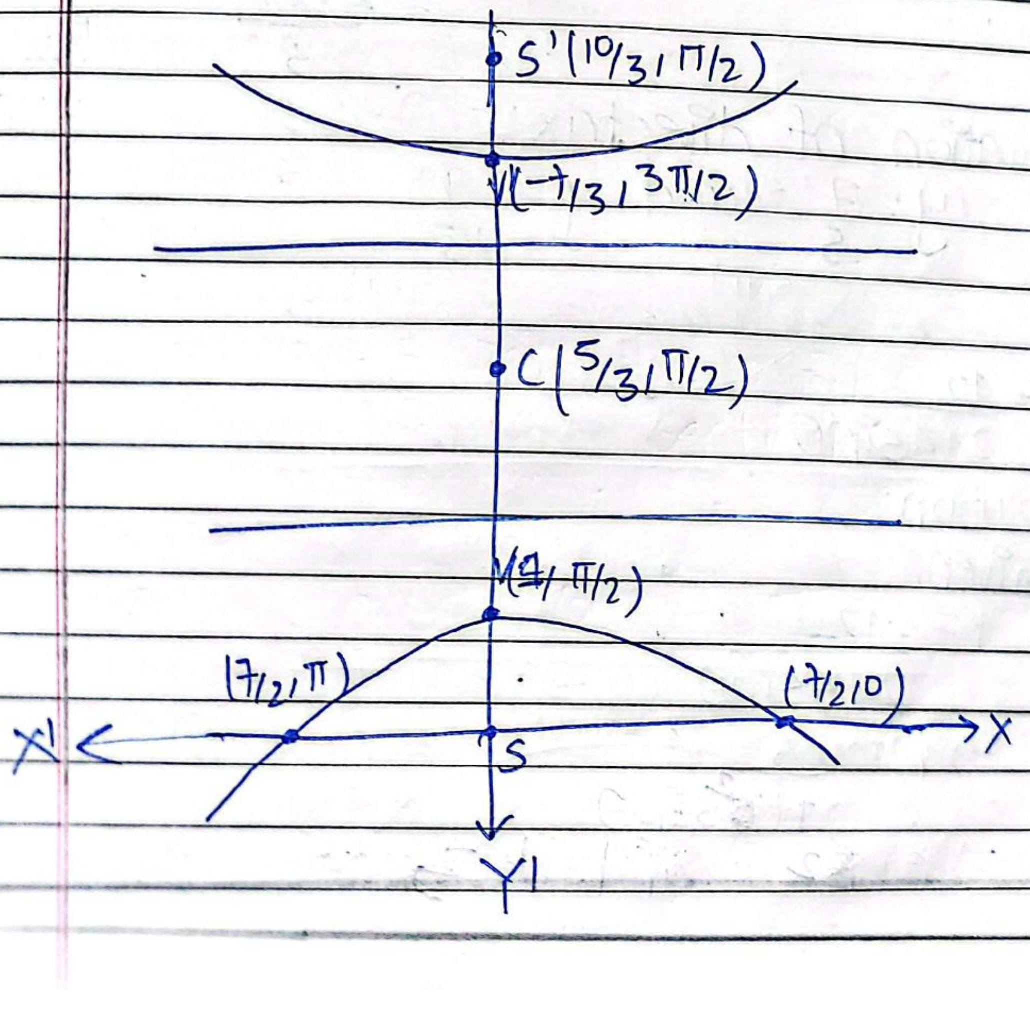
$\Rightarrow e = \dfrac{5}{2}$ and $ed = \dfrac{7}{2}$

$\Rightarrow d = \dfrac{7}{2} \times \dfrac{2}{5} = \dfrac{7}{5}$

Since; $e = \dfrac{5}{2}$ (>1), the given conic section is hyperbola.

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	$7/2$	1	$7/2$	$-7/3$
(r,θ)	$(7/2, 0)$	$(1, \pi/2)$	$(7/2, \pi)$	$(-7/3, 3\pi/2)$



Description:

① length of transverse axis $= 2a$

$= \cancel{2 \times 2a}$

$= \frac{7}{3} - 1 = \frac{4}{3}$

$\therefore a = \frac{2}{3}$

Also,

Distance of center from focus $= ae$

$= \frac{2}{3} \times \frac{5}{2}$

$= \frac{5}{3}$

② length of conjugate axis $= 2b$

$= 2 \times \frac{2}{3}\sqrt{\frac{25}{4} - 1}$

$= \frac{2\sqrt{21}}{3}$

③ Equation of directrix:

$y = \frac{7}{5}$ and $y = \frac{29}{15}$

---

11. $r = \frac{12}{2 + 4\sin\theta}$

~~Ans~~ Solution:

Given,

$r = \frac{12}{2 + 4\sin\theta}$

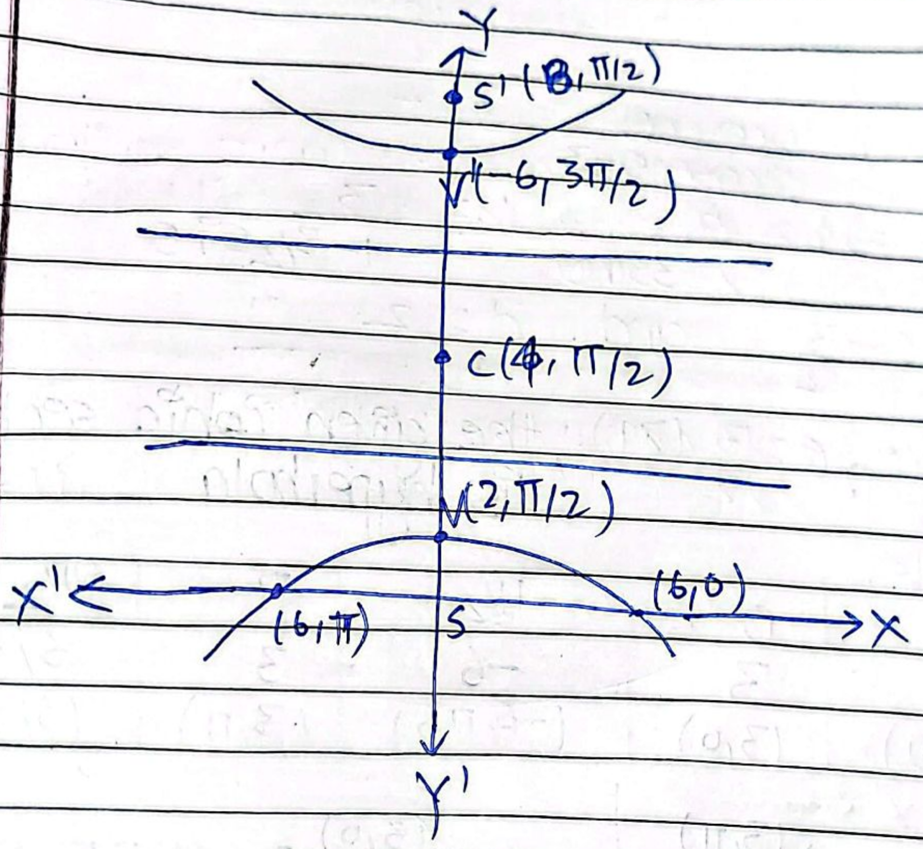
$\Rightarrow r = \frac{6}{1 + 2\sin\theta}$

$\Rightarrow e = 2$ and $d = 2,$

Since, $e = 2 (>1)$; the given conic section is hyperbola.

Table:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
r	6	2	6	-6
(r,θ)	$(6,0)$	$(2,\pi/2)$	$(6,\pi)$	$(-6,3\pi/2)$



Description:

① length of transverse axis $= 2a = \cancel{2\times}$
$= 6-2 = 4$
$\therefore a = 2$

Also,
Distance of center from focus $= ae = 2\times 2 = 4$

② length of conjugate axis $= 2\times 2\sqrt{4-1}$
$= 4\sqrt{3}$
$= 4\sqrt{3}$

③ Equation of directrix
$y = 3$ and $y = 5$

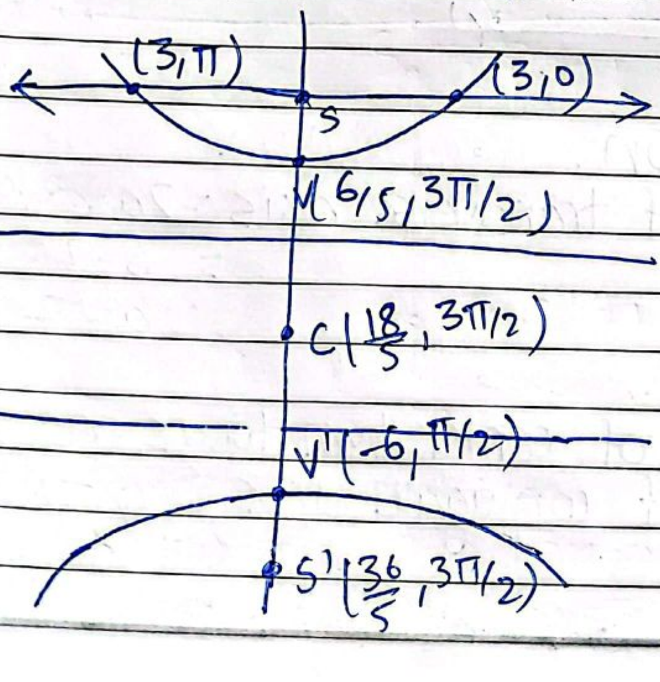
12. $r = \dfrac{6\operatorname{cosec}\theta}{2\operatorname{cosec}\theta - 3}$

Ans Solution:
Given,
$$r = \frac{6\operatorname{cosec}\theta}{2\operatorname{cosec}\theta - 3}$$
$$\Rightarrow r = \frac{6}{2 - 3\sin\theta} = \frac{3}{1 - 3/2 \sin\theta}$$
$$\Rightarrow e = \frac{3}{2} \quad \text{and} \quad d = 2$$

Since, $e = \frac{3}{2}$ (>1); the given conic section is hyperbola.

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	3	-6	3	$6/5$
(r, θ)	$(3, 0)$	$(-6, \pi/2)$	$(3, \pi)$	$(6/5, 3\pi/2)$



Description:

(1) ~~Equ~~ Length of Transverse axis $= 2a$

$$= 6 - \frac{6}{5} = \frac{24}{5}$$

$$\therefore a = \frac{12}{5}$$

Distance of center from focus $= \frac{18}{5}$

(2) length of conjugate axis $= \frac{12}{\sqrt{5}}$

(3) Equation of directrix: $y = -2$ and

$$y = \frac{-34}{5}$$

Examples:

1. $r = \frac{3}{2+2\cos\theta}$

Solution:

Given,

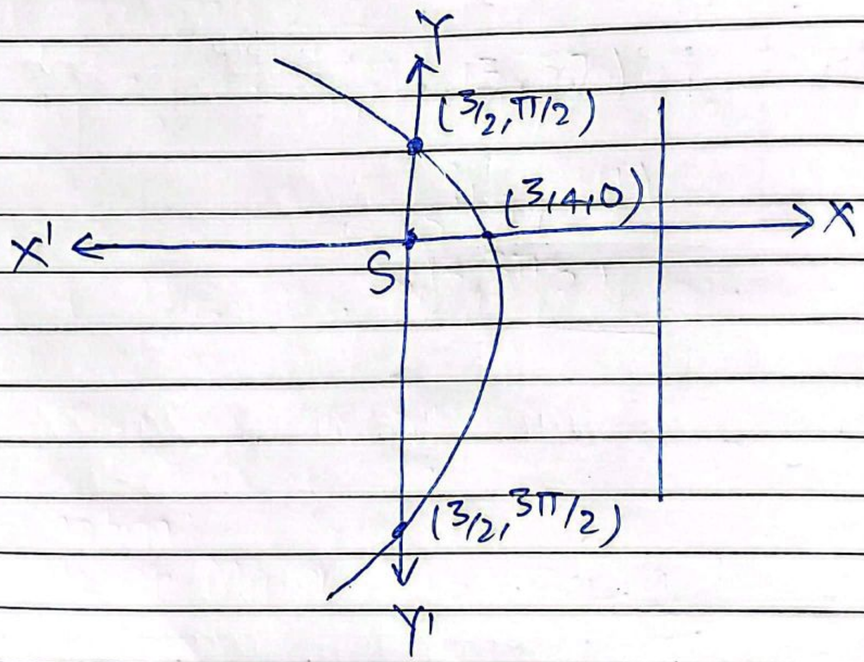
$r = \frac{3}{2+2\cos\theta} = \frac{3/2}{1+\cos\theta}$

$\Rightarrow e = 1$ and $d = \frac{3}{2}$

Since, $e = 1$; the given conic section is parabola.

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	$3/4$	$3/2$	∞	$3/2$
(r,θ)	$(3/4, 0)$	$(3/2, \pi/2)$	(∞, π)	$(3/2, 3\pi/2)$



Description:

① Focus: $S(0,0)$ and vertex $(3/4, 0)$

② Equation of directrix: $x = \frac{3}{2}$

2. $r = \frac{12}{2-6\cos\theta}$

Ans Solution:

Given,

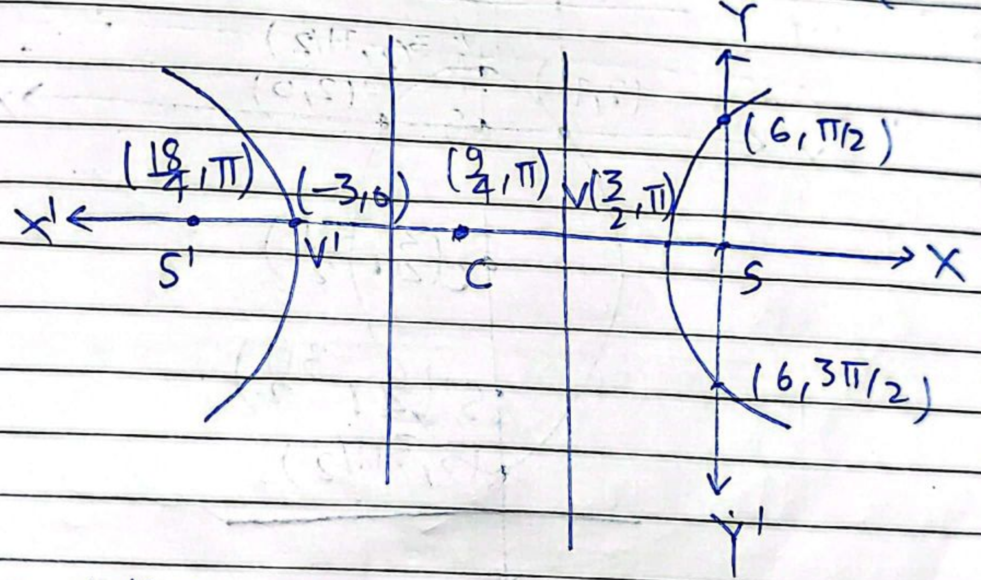
$$r = \frac{12}{2-6\cos\theta} = \frac{6}{1-3\cos\theta}$$

$\Rightarrow e=3$ and $d=2$

Since, $e=3(>1)$; the given conic section is hyperbola

Table

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
r	-3	6	$3/2$	6
(r,θ)	$(-3,0)$	$(6,\pi/2)$	$(3/2,\pi)$	$(6,3\pi/2)$



Description:

① Length of transverse axis $= 2a = \frac{3}{2}$

$\therefore$ center $= \left(\frac{9}{4},\pi\right)$

② Length of conjugate axis $= 2b = 3\sqrt{2}$

③ Equation of directrix: $x = -2$

and $x = -5/2$

3. $r = \frac{12}{6+2\sin\theta}$

Ans Solution:

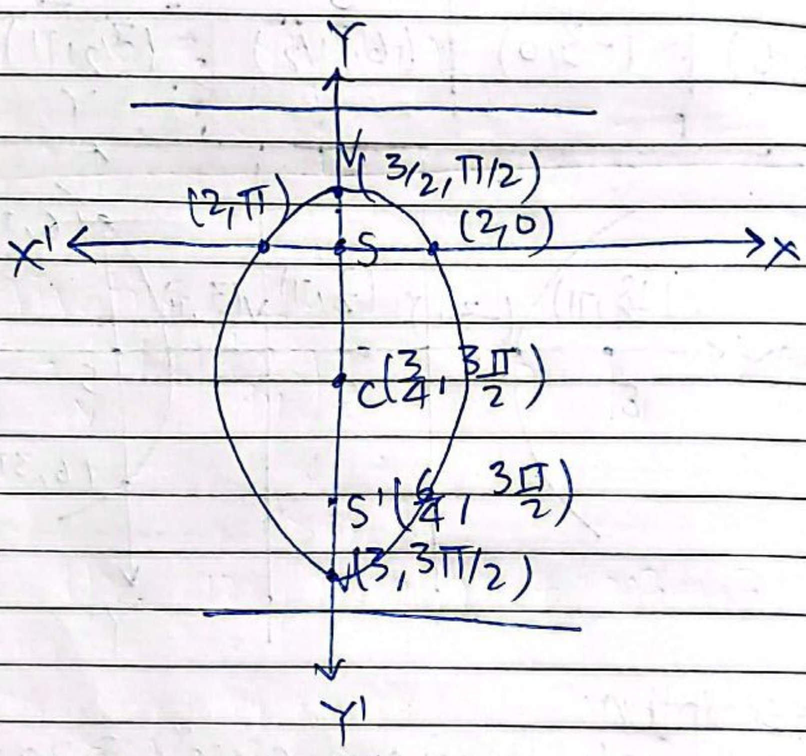
Given,

$$r = \frac{12}{6+2\sin\theta} = \frac{6}{3+\sin\theta}$$

$\Rightarrow e = 1$ and $d = 6$

Table:

θ	0	$\frac{\pi}{2}$	π	$3\pi/2$
r	2	$3/2$	2	3
(r,θ)	$(2,0)$	$(3/2, \pi/2)$	$(2,\pi)$	$(3, 3\pi/2)$.



Description:

① length of Major Axis $= 2a = \frac{9}{2}$

② Also, distance of center from focus $= (3/4)$

② length of Minor Axis $= 2b = 3\sqrt{2}$ and eq^n of directrix $y = 6$ and $y = -\frac{15}{2}$

4. $r = \frac{4\sec\theta}{2\sec\theta - 1}$

Ans Solution:

Given,

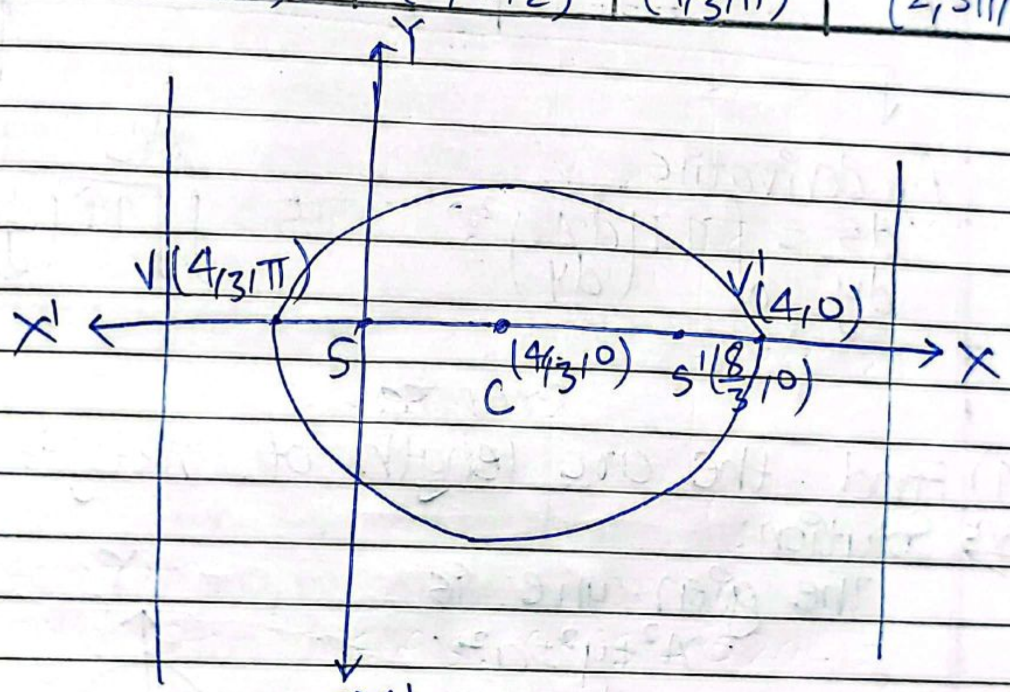
$$r = \frac{4\sec\theta}{2\sec\theta - 1} = \frac{4}{2-\cos\theta} = \frac{2}{1-\frac{1}{2}\cos\theta}$$

$\Rightarrow e = \frac{1}{2}$ and $d = 4$

Since, $e = \frac{1}{2} (<1)$; the given conic section is an ellipse

Table:

θ	0	$\pi/2$	π	$3\pi/2$
r	4	2	$4/3$	2
(r,θ)	$(4,0)$	$(2,\pi/2)$	$(4/3,\pi)$	$(2,3\pi/2)$.



Description:

1. Length of Major Axis $= 2a = 16/3 \Rightarrow a = 8/3$
Also, distance of center from focus $= 4/3$
2. Length of minor axis $= 2b = 8/\sqrt{3}$
3. Equation of directrix: $x = -4$ and $x = 20/3$

Rectification:

(Arc length for cartesian curve)

Formula 1:

For curve $y = f(x)$

Arc derivative

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Arc length

$$S = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Formula (2)

For curve $x = f(y)$

Arc derivative

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

Arc length

$$S = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

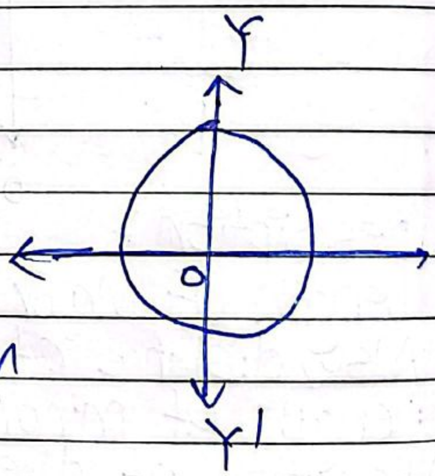
Exercise -

1) Find the arc length of $x^2 + y^2 = a^2$

Ans Solution:

The given curve is

$$x^2 + y^2 = a^2$$



Now,

length of the arc

$$= 4\int_0^a \sqrt{1 + \left(\frac{-x}{y}\right)^2}\, dx$$

$$= 4\int_0^a \sqrt{1+\frac{x^2}{y^2}}\, dx$$

$$= 4\int_0^a \frac{a}{y}\, dx$$

$$= 4a\int_0^a \frac{1}{\sqrt{a^2-x^2}}\, dx$$

$$= 4a\left[\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4a.\frac{\pi}{2}$$

$$= 2\pi a$$

$\therefore$ ~~Leng~~ Arc length of $x^2+y^2=a^2$ is $2\pi a$.

2) Find the arc length of the parabola $y^2=4x$ cut of by the line $y=2x$.

~~Ans~~ Solution:

The given curve is.

$$y^2=4x$$

and the line is $y=2x$.

The point of intersections are:

$$(0,0) \text{ and } (1,2)$$

So, the arc length is

$$l = \int_0^1 \sqrt{1+\left(\frac{2}{y}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1+\frac{4}{4x}}\, dx$$

$m^2 = 4x$

$$= \int_0^1 \sqrt{\frac{1+x}{x}}\,dx$$

Let $x = t^2$. Then,
$dx = 2t\,dt$.

$$= \int_0^1 \frac{\sqrt{1+t^2}}{\cancel{t}} \cdot 2\cancel{t}\,dt$$

$$= 2\int_0^1 \sqrt{1+t^2}\,dt$$

$$= 2\cdot\left[\frac{t\sqrt{1+t^2}}{2} + \frac{1}{2}\,\overset{\ln(t+\sqrt{1+t^2})}{\cancel{\sinh^{-1}}}(t)\right]_0^1$$

$$= 2\left[1\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\sinh^{-1}(1) - 0 - \frac{1}{2}\sinh^{-1}(0)\right]$$

$$\cancel{= 2\Big[\frac{\sqrt{2}}{2} +}$$

$$= 2\left[\frac{1\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2}) - 0 - \frac{1}{2}\ln(0+\sqrt{1+0})\right]$$

$$= 2\,\frac{(\sqrt{2} + \ln(1+\sqrt{2}))}{2}$$

$\therefore$ length of arc $= \sqrt{2} + \ln(1+\sqrt{2})$

Find the length of arc of the parabola $x^2 = 4ay$ from the vertex to an extremity of latus rectum.

Ans Solution:

The given curve is

$$x^2 = 4ay$$
$$\Rightarrow 2x\,dx = 4a\,dy$$
$$\Rightarrow \frac{dx}{dy} = \frac{2a}{x}$$

Now,

$$\text{length of arc} = \int_0^a \sqrt{1+\left(\frac{2a}{x}\right)^2}\,dy$$
$$= \int_0^a \sqrt{1+\frac{4a^2}{4ay}}\,dy$$
$$= \int_0^a \sqrt{1+\frac{a}{y}}\,dy$$
$$= \int_0^a \sqrt{\frac{a+y}{y}}\,dy$$

let $y = t^2$
$$\Rightarrow dy = 2t\,dt$$

$$= 2\int_0^{\sqrt{a}} \sqrt{a+t^2}\,dt$$

$$= 2\left[\frac{t\sqrt{a+t^2}}{2} + \frac{1}{2}\ln\left(t+\sqrt{a+t^2}\right)\right]_0^{\sqrt{a}}$$

$$= \sqrt{a}\sqrt{a+a} + \ln\left(\sqrt{a}+\sqrt{a+a}\right) - a\ln\sqrt{a}$$
$$= \sqrt{a}\sqrt{a+a} + \ln\left(\sqrt{a+a}+\sqrt{a}\right) - a\ln\sqrt{a}$$

$\therefore$ length of arc $= \sqrt{a}\sqrt{a+a} + \ln\left(\sqrt{a+a}+\sqrt{a}\right)$

$\cancel{= 2\left(a^{3/2} + \ln\left(a^{3/2}\right)\right.}$

$\therefore$ length of arc $= \sqrt{2a^2} + a\ln(\sqrt{\ }a + \sqrt{2}a) - a\ln a$

$= a\sqrt{2} + a\ln(1+\sqrt{2})$

$= a(\sqrt{2} + \ln(1+\sqrt{2}))$

$\therefore$ length of arc $= a[\sqrt{2} + \ln(1+\sqrt{2})]$

In case of catenary $y = a\cosh\left(\frac{x}{a}\right)$, show that length of the arc measured from vertex $(0,a)$ to any point (x,y) is $a\sinh\left(\frac{x}{a}\right)$. Also, show that $s^2 = y^2 - a^2$.

Ans Solution:

Given,

$$y = a\cosh\left(\frac{x}{a}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{a}{a}\sinh\left(\frac{x}{a}\right) = \sinh\left(\frac{x}{a}\right)$$

Now,

$$s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^x \sqrt{1+\sinh^2\left(\frac{x}{a}\right)}\, dx$$

$$= \int_0^x \cosh\left(\frac{x}{a}\right) dx$$

$$= a\left[\sinh\left(\frac{x}{a}\right)\right]_0^x$$

$$= a\left[\sinh\left(\frac{x}{a}\right) - \sinh(0)\right]$$

$$= a\sinh\left(\frac{x}{a}\right)$$

$\therefore$ length of arc $= a\sinh\left(\frac{x}{a}\right)$

We have,

$$s = a\sinh\left(\frac{x}{a}\right)$$

$$\Rightarrow s^2 = a^2\sinh^2\left(\frac{x}{a}\right)$$

$$\Rightarrow s^2 = a^2\left(\cosh^2\left(\frac{x}{a}\right) - 1\right)$$

$$\Rightarrow s^2 = y^2 - a^2$$

$$\therefore s^2 = y^2 - a^2$$

Proved

Solid of Revolution

Surface Area | Volume

Formula : (Surface Area)

1. For the revolution/rotation about x-axis

$$S = \int 2\pi y \, ds$$

2. For the revolution/rotation about Y-axis

$$S = \int 2\pi x \, ds$$

Where,

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{OR} \quad \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

* Prove that surface area of a sphere of radius a is $4\pi a^2$.

OR

A circle $x^2+y^2=a^2$ is revolved about x-axis. Find the surface area of solid of revolution.

Solution:

The given curve is

$$x^2+y^2=a^2.$$

Now,

$$\text{Surface Area} = 2\int_0^a 2\pi y \cdot \sqrt{1+\frac{x^2}{y^2}}\, dx$$

$$= 2\int_0^a 2\pi y \cdot \frac{a}{y}\, dx$$

$$= 4\pi a \int_0^a dx$$

$$= 4\pi a^2$$

$\therefore$ Surface area $= 4\pi a^2$.

Find the surface area of solid formed by revolving the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis.

Ans Solution:

The given curve is

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Now,

$$\text{Surface Area} = 2\int_0^a 2\pi y \sqrt{1 + \left(\frac{-x^{-1/3}}{y^{-1/3}}\right)^2}\, dx$$

$$= 2\int_0^a 2\pi y \sqrt{\frac{y^{-2/3} + x^{-2/3}}{y^{-2/3}}}\, dx$$

$$= 2\int_0^a 2\pi y \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}}\, dx$$

$$= 2\int_0^a 2\pi y \, \frac{a^{1/3}}{x^{1/3}}\, dx$$

$$= 2a^{1/3} \cdot 2\pi \int_0^a \frac{(a^{2/3} - x^{2/3})^{3/2}}{x^{1/3}}\, dx$$

$$\cancel{=4\pi a^{1/3}\left[a^{1/3}\left[\frac{x^{-1/3+1}}{-1/3+1}\right]_0^a - \left[\frac{x^{1/3+1}}{1/3+1}\right]_0^a\right]}$$

$$\cancel{=4\pi a^{1/3}\left[a^{1/3}\left[\frac{a^{2/3}}{2/3}\right] - \frac{a^{4/3}}{4/3}\right]}$$

$$\cancel{=4\pi a^{1/3}\left[\frac{3a}{2} - b\frac{3}{4}\right.}$$

$$= 4\pi a^{1/3}\int_0^{\pi/2} \frac{(a^{2/3}-a^{2/3}\sin^3\theta)^{3/2}}{a^{1/3}\sin\theta}\, 3a\sin^2\theta\cos\theta\, d\theta$$

$$=4\pi a^2\times 3\int_0^{\pi/2}\cos^4\theta\sin\theta\, d\theta$$

$$=4\pi a^2\times 3\times\frac{\Gamma(5/2)\,\Gamma(1)}{2\times\Gamma(7/2)}$$

$$=\frac{12}{5}\pi a^2$$

$\therefore$ Surface Area $= \dfrac{12}{5}\pi a^2$.

Find the arc length of the following:

i. $x^{2/3}+y^{2/3}=a^{2/3}$.

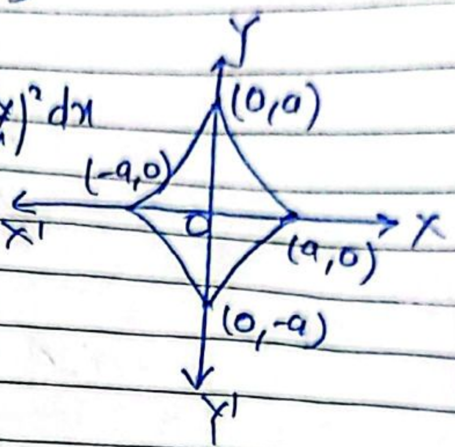
Ans Solution:

The given curve is

$$x^{2/3}+y^{2/3}=a^{2/3}$$

Now,

$$\text{length of arc}=4\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



$$=4\int_0^a\sqrt{1+\frac{x^{-2/3}}{y^{-2/3}}}\,dx$$

$$=4\int_0^a\sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}}\,dx$$

$$=4a^{1/3}\int_0^a x^{-1/3}\,dx$$

$$=4a^{1/3}\left[\frac{x^{2/3}}{2/3}\right]_0^a$$

$$=4\times\frac{3}{2}a$$

$$=6a$$

$\therefore$ length of arc $=6a$.

ii. $\frac{x^{2/3}}{a^{2/3}}+\frac{y^{2/3}}{b^{2/3}}=1$

Ans Solution:

The given curve is: $\frac{x^{2/3}}{a^{2/3}}+\frac{y^{2/3}}{b^{2/3}}=1$

Now,

length of arc $= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$= 4\int_0^a \sqrt{1+\frac{x^{-2/3}b^{4/3}}{a^{4/3}y^{-2/3}}}\, dx$$

$$= 4\int_0^a \sqrt{\frac{a^{4/3}y^{-2/3}+x^{-2/3}b^{4/3}}{a^{4/3}y^{-2/3}}}\, dx$$

$$= 4\int_0^a \sqrt{\frac{a^{4/3}.x^{2/3}+y^{2/3}b^{4/3}}{a^{4/3}.x^{2/3}}}$$

$$= 4\int_0^a \sqrt{a^{6/3}}$$

If s be the length of the an arc of the curve $3ay^2 = x(x-a)^2$ measured from the origin to the point (x, y), show that $3s^2 = 4x^2 + 3y^2$. Show also that the entire length of the loop is $\frac{4}{3}\sqrt{3}\,a$.

Ans Solution:

The given curve is:

$$3ay^2 = x(x-a)^2$$

$$\Rightarrow 3a(2y)\cdot\frac{dy}{dx} = \frac{d}{dx}(x^3 - 2ax^2 + a^2x)$$

$$\Rightarrow 6ay\frac{dy}{dx} = 3x^2 - 4ax + a^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{6ay}(3x^2 - 4ax + a^2)$$

Now,

$$\text{Length of Arc} = \int_0^x \sqrt{1 + \left(\frac{3x^2 - 4ax + a^2}{6ay}\right)^2}\, dx$$

$$= \int_0^x \sqrt{\frac{36a^2y^2 + (3x^2 - 4ax + a^2)^2}{36a^2y^2}}\, dx$$

$$= \int_0^x \sqrt{\frac{36a^2y^2 + (3x^2 - 4ax)^2 + 2a^2(3x^2 - 4ax) + a^4}{36a^2y^2}}\, dx$$

$$= \int_0^x \sqrt{\frac{36a^2y^2 + 9x^4 - 24ax^3 + 16a^2x^2 + 6a^2x^2 - 8a^3x + a^4}{36a^2y^2}}$$

$$= \frac{1}{\sqrt{12a}} \int_0^{x} 3x^{1/2} + ax^{-1/2}\, dx$$

$$= \frac{1}{\sqrt{12a}} \left[3 . \frac{x^{3/2}}{3/2} + a \frac{x^{\frac{1}{2}}}{1/2} \right]$$

$$= \frac{1}{\sqrt{12a}} \left[2x^{3/2} + 2ax^{1/2} \right]$$

$$\Rightarrow s^2 = \frac{1}{12a} \times 4 (x^{3/2} + ax^{1/2})^2$$

$$\Rightarrow 3as^2 = x^3 + 2.ax + a^2x$$

$$\Rightarrow 3as^2 = x(x^2 + 2ax + a^2) = x(x+a)^2$$

$$\Rightarrow 3as^2 = x[(x-a)^2 + 4ax]$$

$$\Rightarrow 3as^2 = 3ay^2 + 4ax^2$$

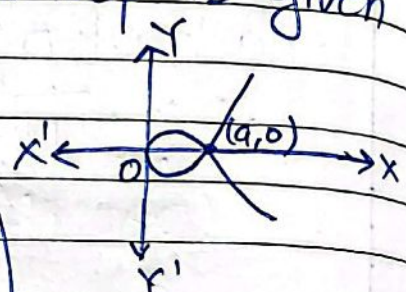
$$\Rightarrow 3s^2 = 3y^2 + 4x^2$$

$$\therefore 3s^2 = 3y^2 + 4x^2$$ proved

Also,

The entire length of the loop is given by

$$L = 2 \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$



$$= 2(s)_{\substack{x=a \\ y=0}} = 2 \left(\sqrt{\frac{(3y^2 + 4x^2)}{3}} \right)_{x=a,\, y=0}$$

$$= 2 . \sqrt{\frac{0 + 4a^2}{3}} = \frac{2.2.a}{\sqrt{3}} = \frac{4a}{\sqrt{3}}$$

$$= \frac{4\sqrt{3}}{3} a$$

$\therefore$ The ~~te~~ entire length of the loop is $\frac{4\sqrt{3}}{3} a$

showed

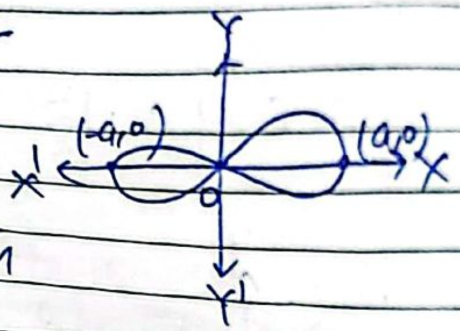
Show that the arc length of the curve $x^2(a^2-x^2)=8a^2y^2$ is $\pi a\sqrt{2}$.

Ans Solution:

The given curve is:

$$x^2(a^2-x^2)=8a^2y^2$$

Now,



$x^2(a^2-x^2)=8a^2y^2$

$\Rightarrow a^2\cdot 2x-4x^3=8a^2\cdot 2y\frac{dy}{dx}$

$\Rightarrow \frac{2a^2x-4x^3}{16a^2y}=\frac{dy}{dx}$

$\Rightarrow \frac{2x(2a^2-3x^2)}{16a^2y}=\frac{dy}{dx}$

Now,

Length of Arc $=4\int_0^a\sqrt{1+\frac{x^2(2a^2-4x^3)^2}{16\times 16a^4y^2}}\,dx$

$$=4\int_0^a\sqrt{1+\frac{x^2\times 4(a^2-2x^2)^2\times 8a^2}{16\times 16\times a^4\cdot x^2(a^2-x^2)}}\,dx$$

$$=4\int_0^a\sqrt{1+\frac{(a^2-2x^2)^2}{8a^2(a^2-x^2)}}\,dx$$

$$=4\int_0^a\frac{3a^2-2x^2}{\sqrt{8}a(a^2-x^2)^{1/2}}\,dx$$

$$=4\int_0^a\frac{2(a^2-x^2)+a^2}{\sqrt{8}a(a^2-x^2)^{1/2}}\,dx=\frac{4}{\sqrt{8}a}\left[\int_0^a(a^2-x^2)^{1/2}\,dx+a^2\int_0^a\frac{dx}{\sqrt{a^2-x^2}}\right]$$

$$=\frac{2\sqrt{2}}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a+\sqrt{2}a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]=\pi a\sqrt{2}$$

∴ Arc length of the curve $x^2(a^2-x^2)=8a^2y^2$ is $\pi a\sqrt{2}$ showed

Prove that the loop of the curve $x = t^2$, $y = t - \frac{t^3}{3}$ is of length $4\sqrt{3}$.

Solution:

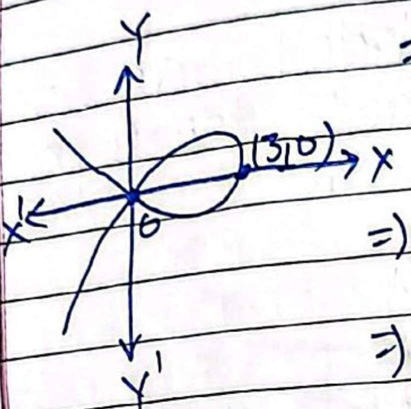
Given,

$x = t^2, y = t - \frac{t^3}{3}$

$\Rightarrow y^2 = t^2\left(1 - \frac{t^2}{3}\right)^2$

$\Rightarrow y^2 = x\left(1 - x/3\right)^2$

$\Rightarrow 2y\frac{dy}{dx} = x \cdot 2\left(1 - \frac{x}{3}\right)\cdot\left(\frac{-1}{3}\right) + \left(1 - x/3\right)^2$



$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}\left(\frac{2x(3-x)(-1) + (3-x)^2}{9}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{18y}\left(2x(x-3) + (x-3)^2\right)$

$\Rightarrow \frac{dy}{dx} = \frac{3(x-3)(x-1)3}{18\sqrt{x}(x-3)}$

$\Rightarrow \frac{dy}{dx} = \frac{x-1}{2\sqrt{x}}$

Now,

length of arc

$= 2\int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\int_0^3 \sqrt{1 + \frac{(x-1)^2}{2x}}\, dx$

$= 2\int_0^3 \sqrt{\frac{4x + x^2 - 2x + 1}{2x}}\, dx = \frac{2}{2}\int_0^3 \sqrt{\frac{x^2 + 1 + 2x}{x}}\, dx$

$= \int_0^3 \frac{(x+1)}{\sqrt{x}}\, dx = \int_0^3 x^{1/2} + x^{-1/2}\, dx$

$= 2\left[3^{1/2} + (3)^{1/2}\right]$

$= 4\sqrt{3}$

$\therefore$ Length of loop $= 4\sqrt{3}$ units.

Proved

Find the perimeter of the cardioid $r=a(1+\cos\theta)$ and show that the arc of upper half is bisected by $\theta=\frac{\pi}{3}$.

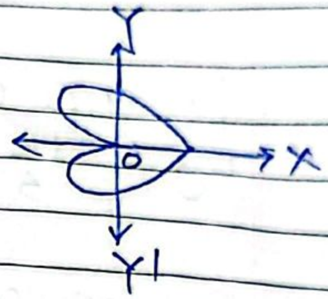
Ans Solution:

The given equation of the cardioid is:

$$r=a(1+\cos\theta)$$

$$\Rightarrow \frac{dr}{d\theta}=-a\sin\theta.$$

Now,



Perimeter of the cardioid

$$=2\int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$=2\int_0^{\pi}\sqrt{r^2+a^2\sin^2\theta}\,d\theta$$

$$=2\int_0^{\pi}\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,d\theta$$

$$=2\int_0^{\pi}\sqrt{a^2+2a^2\cos\theta+a^2\cos^2\theta+a^2\sin^2\theta}\,d\theta$$

$$=2\int_0^{\pi}\sqrt{a^2+a^2+2a^2\cos\theta}\,d\theta=2\int_0^{\pi}\sqrt{2a^2}\sqrt{2\cos^2\frac{\theta}{2}}\,d\theta$$

$$=2\sqrt{2}\cdot a\sqrt{2}\cdot\int_0^{\pi}\cos\theta/2\,d\theta$$

$$=4a\int_0^{\pi}\cos\theta/_2\,d\theta=4a\times 2\left[\sin\frac{\theta}{2}\right]_0^{\pi}$$

$$=\cancel{2\times}4a[1-0]=4a\times 2=8a$$

$\therefore$ The perimeter of cardioid $= 8a$.
(i.e. $4a$ for upper half)

Also,
Arc length from $\theta = 0$ to $\theta = \pi/3$ (for upper half)

$$= 4a \times \left[\sin \theta/2\right]_0^{\pi/3}$$

$$= 4a \times \frac{1}{2} = 2a = \frac{1}{2} \times 4a = \frac{1}{2} \times \text{upper half}$$

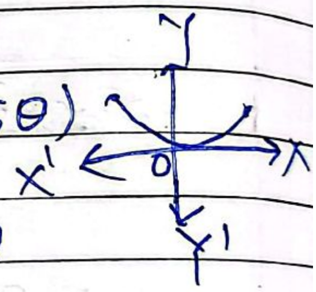
$\therefore$ The arc of the upper half is bisected by $\theta = \frac{\pi}{3}$ (from $\theta = 0$).

In the cycloid, $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. Show that, $s^2 = 8ay$, s being measured from the vertex to any point.

~~Ans~~ Solution:

The given polar form of the cycloid is

$$x = a(\theta + \sin\theta)$$
$$y = a(1 - \cos\theta)$$



Now,

$$\frac{dx}{d\theta} = a + a\cos\theta = a(1 + \cos\theta)$$

$$\frac{dy}{dx} = a(0 + \sin\theta) = a\sin\theta$$

Now,
length of the arc from vertex

$$= \int_0^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\theta} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= \int_0^\theta \sqrt{a^2 + 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}\, d\theta$$

$$= \int_0^\theta \sqrt{2a^2 + 2a^2\cos\theta}\, d\theta$$

$$= \sqrt{2}a \cdot \sqrt{2} \int_0^\theta \cos\theta/2\, d\theta$$

$$= 4a \left[\sin\frac{\theta}{2}\right]_0^\theta$$

$$= 4a\left[\sin\frac{\theta}{2}\right]$$

$$= 4a\sin\frac{\theta}{2}$$

$\Rightarrow S = 4a\sin\theta/2$

Squaring on both sides, we get,

$$s^2 = 16a^2\sin^2\theta/2$$

$$\Rightarrow s^2 = 8a.a.\, 2\sin^2\theta/2$$

$$\Rightarrow s^2 = 8a.\, a(1-\cos\theta)$$

$$\Rightarrow s^2 = 8ay$$

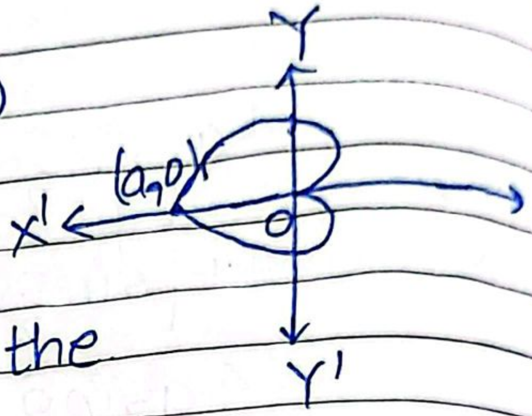
$$\therefore s^2 = 8ay$$

proved

Find the perimeter of the cardioid $r = a(1-\cos\theta)$ and show that the arc of the upper half is bect bisected by $\theta = \frac{2\pi}{3}$

Ans Solution:
The given equation of the cardioid is:

$$r = a(1-\cos\theta)$$
$$\Rightarrow \frac{dr}{d\theta} = a\sin\theta$$



Now,
The perimeter of the cardioid is

$$L = 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{2a^2 - 2a^2\cos\theta}\, d\theta$$

$$= 2\sqrt{2}a \cdot \sqrt{2}\int_0^{\pi} \sin\frac{\theta}{2}\, d\theta$$

$$= 8a\left[-\cos\frac{\theta}{2}\right]_0^{\pi}$$

$= 8a[1]$
$= 8a$

$\therefore$ The total perimeter of the cardioid is $8a$.

Also,

length of upper half $= \frac{1}{2} \times 8a = 4a$

Now,

length of the arc ~~fo~~ from $\theta = 0$ to $\theta = \frac{2\pi}{3}$

$$= \int_0^{2\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 4a\left[-\cos\frac{\theta}{2}\right]_0^{2\pi/3}$$

$$= 4a\left(-\frac{1}{2} + 1\right)$$

$$= 2a$$

$= \frac{1}{2} \times$ length of upper half.

$\therefore$ length of the arc of the upper half is bisected by $\theta = \frac{2\pi}{3}$

showed

Find the area of the surface of the solid generated by the revolution of the cardioid $r=a(1-\cos\theta)$ about the initial line.

Ans. Solution:

The given equation of the cardioid is
$$r=a(1-\cos\theta)$$
$$\Rightarrow \frac{dr}{d\theta}=a\sin\theta$$

The area of the surface generated by revolution of cardioids above initial line is defined as:
$$S=\int_0^{\pi} 2\pi y\frac{ds}{d\theta}.d\theta$$

We have,
$$x=r\cos\theta=a\cos\theta(1-\cos\theta)$$
$$y=r\sin\theta=a\sin\theta(1-\cos\theta)$$

So,
$$S=\int_0^{\pi} 2\pi a\sin\theta(1-\cos\theta)\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$
$$=\int_0^{\pi} 2\pi a\sin\theta(1-\cos\theta).\,4a\sin\frac{\theta}{2}\,d\theta$$
$$=8\pi a^2\times 4\times 2\times 2\int_0^{\pi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\times\sin^2\frac{\theta}{2}\times\sin\frac{\theta}{2}d\theta$$
$$=16\pi a^2\int_0^{\pi}\sin^4\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta$$

let $\frac{\theta}{2}=t \Rightarrow \frac{1}{2}d\theta=dt \Rightarrow$ When $t\to0, \theta\to0$

When $t\to\pi, \theta\to\pi/2$

$$= 16\pi a^2 \times 2 \int_0^{\pi/2} \sin^4 t \cos t \, dt$$

$$= 32\pi a^2 \times \frac{\Gamma(5/2)\,\Gamma(1)}{\Gamma(7/2)}$$

$$= 32/5\, \pi a^2 \text{ sq. units.}$$

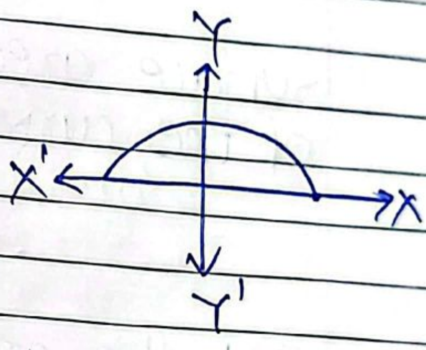
Hence, the area of the surface revolved is $32/5\, \pi a^2$ sq. units.

Find the surface area of solid generated by revolving the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base.

Ans Solution:

The given equation is

$$x = a(\theta + \sin\theta)$$
$$y = a(1 + \cos\theta)$$



Now,

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$\frac{dy}{d\theta} = -a\sin\theta$$

Now,

$$\text{Surface area} = \int_0^{\pi} 2\pi a(1+\cos\theta)\sqrt{a^2(1+\cos\theta) + a^2\sin^2\theta}\; d\theta$$

$$= \int_0^{\pi} 2\pi a \cdot 2\cos^2\frac{\theta}{2} \cdot \sqrt{2} \cdot \sqrt{2}\, a\cos\frac{\theta}{2}\, d\theta$$

$$= 8\pi a^2 \int_0^{\pi} \cos^3\frac{\theta}{2}$$

$$= 8\pi a^2 \, 2\int_0^{\pi/2} \cos^3 t \, dt$$

$$= 16\pi a^2 \frac{\Gamma(4/2)\,\Gamma(1/2)}{2\times\Gamma(5/2)}$$

$$= 16\pi a^2 \times \frac{1\times\sqrt{\pi}}{2\times 3/2\times \cancel{\tfrac{1}{2}}\times\sqrt{\pi}}$$

$$= \frac{64}{3\times 2}\pi a^2$$

$$= \frac{32}{3}\pi a^2$$

$\therefore$ Surface area generated by the revolution of the curve is

$$\frac{32}{3}\pi a^2.$$

Find the area of the curved surface generated when the part of the parabola $y^2 = 4ax$ bounded by the latus rectum revolves about the tangent at the vertex.

Ans Solution:

The given equation of the curve is

$$y^2 = 4ax.$$

Now,

$$\text{Surface Area} = 2\int_0^{2a} 2\pi x \sqrt{1+\left(\frac{y}{2a}\right)^2}\, dy$$

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$$= 4\pi \int_0^{2a} x \sqrt{1 + \frac{4ax}{4a^2}}\, dy$$

$$= 4\pi \int_0^{2a} x \sqrt{1 + \frac{x}{a}}\, dy$$

$$= 4\pi \int_0^{2a} x \sqrt{\frac{a+x}{a}}\, dy$$

$$= 4\pi \int_0^{2a} \frac{y^2}{4a} \sqrt{\frac{a + y^2/4a}{a}}\, dy$$

$$= 4\pi \int_0^{2a} \frac{y^2}{4a} \sqrt{\frac{4a^2+y^2}{4a^2}}\, dy$$

$$= 4\pi \int_0^{2a} \frac{y^2 \times 2a}{4a \times 4a \cdot 2a} \sqrt{4a^2+y^2}\, dy$$

$$= \frac{\cancel{4}\pi}{8a^2} \int_0^{2a} y^2 \sqrt{y^2+4a^2}\, dy$$

$$= \frac{\pi}{2a^2} \int_0^{2a} y^2 \sqrt{y^2+4a^2}\, dy$$

Let $y = 2a\tan\theta$

$\Rightarrow dy = 2a\sec^2\theta\, d\theta$

Also, when $y \to 0$, $\theta \to 0$

when $y \to 2a$, $\theta \to \frac{\pi}{4}$

$$= \frac{\pi}{2a^2} \int_0^{\pi/4} 4a^2\tan^2\theta \sqrt{4a^2+4a^2\tan^2\theta} \cdot 2a \sec^2\theta\, d\theta$$

$$= \frac{\pi}{2a^2} \int_0^{\pi/4} 4a^2 . 2a . \sec^3\theta \tan^2\theta \, d\theta$$

$$= \frac{\pi}{2a^2} \times 8a^3 \int_0^{\pi/4} \sec^3\theta \tan^2\theta \, d\theta$$

$$= \frac{4\pi a}{\cancel{4}} \int_0^{\pi/4} \sec^3\theta \tan^2\theta \, d\theta$$

let $y = \sec\theta$

$\Rightarrow dy = \sec\theta \tan\theta \, d\theta$

When $\theta \to 0, y \to 1, \theta \to \pi/4, y = \sqrt{2}$

$$= 4\pi a \int_0^{\sqrt{2}} \cancel{y^3 . (1-y^2)} . \frac{\sec^3\theta \tan^2\theta . \, dy}{\sec\theta \tan\theta}$$

$$= 4\pi a \int_0^{\sqrt{2}} \sec^2\theta \tan\theta \, dy$$

$$= 4\pi a \int_0^{\sqrt{2}} (y^2) \sqrt{1-y^2} \, dy$$

$$= 4\pi a \int_0^{\sqrt{2}} y^2 \sqrt{1-y^2} \, dy$$

let $t^2 = 1 - y^2$

$\Rightarrow 2t = -2y \frac{dy}{dt}$ When $y \to 0, t \to 1$
When $y \to \sqrt{2}, t \to 1$

$\Rightarrow \frac{-t}{y} = \frac{dy}{dt}$

$$= 4\pi a \int_1^{-1} (1-t^2) . t . \frac{(-t)}{y} . \, dt$$

$$= 4\pi a \int_{-1}^{1} \frac{(1-t^2) \cdot t^2}{4} \, dt.$$

Exercise - 37

1. Show that the conic $9x^2-24xy+16y^2-50x-100y+225=0$ represents a parabola. Find the latus rectum, vertex, focus and directrix of the parabola.

Ans Solution:

The given equation of the conic is

$$9x^2-24xy+16y^2-50x-100y+225=0$$

Here,

$a=9,\ h=-12,\ b=16,\ g=-25,\ f=-50,\ c=225$

Now,

$\Delta = abc+2fgh-af^2-bg^2-ch^2$

$= 9\times16\times225+2\times(-50)\times(-25)\times(-12) - 9\times(-50)^2-16\times(-25)^2-225\times(-12)^2$

$\therefore \Delta = -62500 \neq 0$

Also,

$h^2=ab$

i.e. $144 = 9\times16$ (True)

$\therefore$ The conic $9x^2-24xy+16y^2-50x-100y+225=0$ represents a parabola.

We have,

$9x^2-24xy+16y^2-50x-100y+225=0$

$\Rightarrow (3x-4y)^2 = 50x+100y-225$

$\Rightarrow (3x-4y+\lambda)^2 = 50x+100y-225+6\lambda x-8\lambda y+\lambda^2$

$\Rightarrow (3x-4y+\lambda)^2 = (50+6\lambda)x+(100-8\lambda)y-225+\lambda^2$

We have,

The equation of axis:

$3x-4y+\lambda=0$ —— (i)

The equation of tangent to vertex

$(50+6\lambda)x+(100-8\lambda)y-225-\lambda^2=0$ —— (ii)

Since (i) and (ii) are perpendicular.

$$\frac{-3}{-4} \times -\left(\frac{50+6\lambda}{100-8\lambda}\right) = -1$$

$$\Rightarrow 150+18\lambda = 400-32\lambda$$

$$\Rightarrow 50\lambda = 250$$

$$\therefore \lambda = 5$$

So, the equation of the parabola is

$$(3x-4y+5)^2 = 80x+60y-200$$

$$\Rightarrow \left(\frac{3x-4y+5}{5}\right)^2 \times 25 = \frac{80x+60y-200}{100} \times 100$$

$$\Rightarrow \left(\frac{3x-4y+5}{5}\right)^2 = 4\left(\frac{80x+60y-200}{100}\right)$$

which is of the form $Y^2 = 4AX$.

Where, $Y = \frac{3x-4y+5}{5}$

$A = 1$; $X = \frac{80x+60y-200}{100}$

Now,

The equation of latus rectum is

$$X = A$$

$$\Rightarrow \frac{80x+60y-200}{100} = 1$$

$$\Rightarrow 80x+60y-300 = 0$$

$$\Rightarrow 4x+3y-15 = 0$$

and length of latus rectum $= 4A = 4$.

Also,

Vertex is the point of intersection of lines $X=0$ and $Y=0$

i.e. $3x-4y+5=0$

and $80x+60y-200=0$

$$\therefore (x,y) = (1,2)$$

Hence, the vertex is (1,2).

Also,

Focus is the point of intersection of $X=A$ and $Y=0$

i.e. $4x+3y-15=0$

$3x-4y+5=0$

$\therefore$ Focus $= \left(\frac{9}{5}, \frac{13}{5}\right)$

Also,

Equation of directrix is

$X=-A$

$\Rightarrow \frac{80x+60y-200}{100} = -1$

i.e. $4x+3y-5=0$ is the equation of the directrix.

2 Show that the conic $x^2-2xy+y^2-2x-2y+3=0$ represents a parabola. Find latus rectum, vertex, focus and directrix of the parabola.

Ans Solution:

Given,

$x^2-2xy+y^2-2x-2y+3=0$

Here,

$a=1, b=1, h=-1, g=-1, f=-1, c=3$

Here, for parabola

$h^2=ab$

$\Rightarrow 1=1$ (True)

Hence, the conic $x^2-2xy+y^2-2x-2y+3=0$ represents a parabola.

Now,

$(x^2-2xy+y^2-2x-2y+3=0$

$\Rightarrow (x-y)^2 = 2x+2y-3$

A changing in the λ-formation form:

$(x-y+\lambda)^2 = 2x+2y-3+2\lambda x - 2\lambda y + \lambda^2$

$\Rightarrow (x-y+\lambda)^2 = (2+2\lambda)x + (2-2\lambda)y + \lambda^2 - 3.$

Here, the equation of axis is:

$x-y+\lambda = 0$ —— ①

and the equation of tangent to vertex is

$(2+2\lambda)x + (2-2\lambda)y + \lambda^2 - 3 = 0$ —— ⑪

Since ① and ⑪ are perpendicular,

$-\frac{1}{-1} \times -\left(\frac{2+2\lambda}{2-2\lambda}\right) = -1$

$\Rightarrow 2+2\lambda = 2-2\lambda$

$\Rightarrow \lambda = 0.$

So,

$(x-y)^2 = 2x+2y-3$

$\Rightarrow \left(\frac{x-y}{\sqrt{2}}\right)^2 \times 2 = \frac{2x+2y-3}{2\sqrt{2}} \times 2\sqrt{2}$

$\Rightarrow \left(\frac{x-y}{\sqrt{2}}\right)^2 = \sqrt{2}\left(\frac{2x+2y-3}{2\sqrt{2}}\right)$

Which is of the form ~~$y^2=4ax$~~ $Y^2 = 4AX$

Where, $Y = \frac{x-y}{\sqrt{2}}$; $A = \frac{\sqrt{2}}{4}$ and $X = \frac{2x+2y-3}{2\sqrt{2}}$

Now,

Equation of latus rectum is given by:

~~$Y = A$~~ $\quad X = A$

~~$\Rightarrow \frac{x-y}{\sqrt{2}} = \frac{\sqrt{2}}{4}$~~ $\quad$ i.e. $\frac{2x+2y-3}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

~~$\Rightarrow x-y = \frac{1}{4}$~~ $\quad$ i.e. $2x+2y-4=0$

$\Rightarrow x+y-2=0$

~~i.e. $4(x-y)=1$~~

Length of latus rectum $= 4A = \sqrt{2}$

Also,
Vertex is the point of intersection of axis and tangent to the parabola at vertex. (i.e. $X=0$ and $Y=0$)

i.e. $x-y=0$

$2x+2y-3=0$

$\therefore (x,y) = \left(\frac{3}{4}, \frac{3}{4}\right)$

Also,
Focus is the point of intersection of latus rectum and axis i.e. $X=A$ and $Y=0$

i.e. $\cancel{4(x+y)\sqrt{2}}$ $x+y-2=0$

$x-y=0$

$\therefore$ Focus $=(1,1)$.

Also,
Equation of the directrix is

$X=-A$

i.e. $2x+2y-2=0$

$\Rightarrow x+y-1=0.$

3. Find the positions and lengths of the axes of the conic

$$x^2-4xy-2y^2+10x+4y=0.$$

Ans Solution:

The given equation of the conic is

$$x^2-4xy-2y^2+10x+4y=0$$

Here,

$a=1, b=-2, c=0, g=5, f=2$ and $h=-2$

Also,

$h^2 - ab = 4 - 1 \times (-2) = 4 + 2 = 6 > 0$

$\therefore$ The given conic section is hyperbola.

~~The equation of the~~

The center of Hyperbola:

$x^2 - 4xy - 2y^2 + 10x + 4y = 0$

$\Rightarrow 2x - 4y + 10 = 0$ —— (I)

and $-4x - 4y + 4 = 0$ —— (II)

From (I) and (II), we get,

$x = -1$ and $y = 2$.

Let $x = X - 1$ and $y = Y + 2$.

The equation of the hyperbola referred to center as origin is

$AX^2 + 2HXY + BY^2 = 0$

Where,

$A = \frac{a(h^2 - ab)}{\Delta} = \frac{1 \times 6}{6} = 1.$

Where,

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 1 \times (-2) \times 0 + 2 \times 2 \times 5 \times (-2) - 1 \times (2)^2 + 2 \times (5)^2 - 0 \times (-2)^2$

$= -40 - 4 + 50 = 6.$

$B = \frac{b(h^2 - ab)}{\Delta} = -2$

and $H = \frac{h(h^2 - ab)}{\Delta} = -2$

The length of the axes is given as.

$\frac{1}{r^4} - (A+B) \cdot \frac{1}{r^2} + (AB - H^2) = 0$

$\Rightarrow x^2 - (-1) \cdot x + (-2 - 4) = 0$

$\Rightarrow x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0$

$\Rightarrow x = -3$ and $x = 2$

$\Rightarrow r_2^2 = -\frac{1}{3}$ and $r_1^2 = \frac{1}{2}$

The equation of transverse axes:

$$\left(A - \frac{1}{r_1^2}\right)X + HY = 0$$

$$\Rightarrow (A-2)X + HY = 0$$

$$\Rightarrow -X - 2Y = 0$$

$$\Rightarrow X + 2Y = 0$$

$$\Rightarrow x + 1 + 2(y-2) = 0$$

$$\Rightarrow x + 2y - 3 = 0$$

and its length $= \sqrt{2}$

Also,

The equation of conjugate axes:

$$\left(A - \frac{1}{r_2^2}\right)X + HY = 0$$

$$\Rightarrow (A-3)X + HY = 0$$

$$\Rightarrow -2X - 2Y = 0$$

$$\Rightarrow X + Y = 0$$

$$\Rightarrow x + 1 + y - 2 = 0$$

$$\Rightarrow x + y - 1 = 0$$

and its length $= \frac{2}{\sqrt{3}}$

---

4. Find the product of semi-axis of the conic $x^2 - 4xy + 5y^2 = 2$

Ans Solution:

The given conic is:

$$x^2 - 4xy + 5y^2 = 2$$

$$\Rightarrow x^2 - 4xy + 5y^2 - 2 = 0$$

Here, $a = 1$, $b = 5$, $h = -2$, $c = -2$

Also,

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= -10 + 2 \times 0 \times 0 \times (-2) - 0 - 0 + 2 \times 14)$

$= -10 + 8$

$= -2 \neq 0$

$\therefore$ The given equation is conic.

Also,

$h^2 - ab = 4 - 1 \times 5 = -1 < 0$

$\therefore$ The given conic is ellipse.

Equation of Ellipse where center is Referred to origin is

$$AX^2 + 2HXY + BY^2 = 0$$

where, $A = \frac{a(h^2 - ab)}{\Delta} = \frac{1}{2}$

$B = \frac{b(h^2 - ab)}{\Delta} = \frac{5}{2}$

$H = \frac{h(h^2 - ab)}{\Delta} = \frac{-2}{2} = -1.$

Now,

Length of Axes.

$\frac{1}{r^4} - \frac{1}{r^2}(A+B) - (H^2 - AB) = 0$

$\Rightarrow x^2 - 3x - (1 - 5/4) = 0$

$\Rightarrow x^2 - 3x + \frac{1}{4} = 0$

$\Rightarrow 4x^2 - 12x + 1 = 0 \quad \Rightarrow 4x^2 - 12x + 1 = 0$

Here, Product of roots $= x_1 x_2 = \frac{1}{4}$

$\Rightarrow \frac{1}{r_1^2} \cdot \frac{1}{r_2^2} = \frac{1}{4}$

$\Rightarrow r_1 r_2 = \sqrt{4} = 2$

Hence, the product of semi-major axis of the conic $x^2-4xy+5y^2=2$ is 2.

5. Find the center, lengths of the axes and eccentricity of the conics

i. $9x^2+4xy+6y^2-22x-16y+9=0$.

Ans Solution:

The given equation of the conic is

$$9x^2+4xy+6y^2-22x-16y+9=0$$

Here,

$a=9, h=2, b=6, g=-11, f=-8, c=9$.

Now,

$\Delta = abc+2fgh-af^2-bg^2-ch^2 = -500$

Also,

$h^2-ab = 4-9\times 6 = -50\ (<0)$

$\therefore$ The given conic is ellipse.

Now,

The equation of the ellipse (center referred to origin) is:

$$AX^2+2HXY+BY^2=0$$

Where,

$A = \dfrac{a(h^2-ab)}{\Delta} = \dfrac{9}{10}$

$B = \dfrac{b(h^2-ab)}{\Delta} = \dfrac{6}{10}$

$H = \dfrac{h(h^2-ab)}{\Delta} = \dfrac{2}{10}$

For center.

$18x+4y-22=0$

and $4x+12y-16=0$

$\therefore (\alpha,\beta)=(1,1)$

For the length of the axes:

$$\frac{1}{r^4} - \frac{1}{r^2}(A+B) - (H^2-AB) = 0$$

$$\Rightarrow x^2 - \left(\frac{9}{10} + \frac{6}{10}\right)x - \left(\left(\frac{2}{10}\right)^2 - \frac{9}{10} \times \frac{6}{10}\right) = 0$$

$$\Rightarrow x^2 - \frac{15}{10}x - \left(\frac{4}{100} - \frac{54}{100}\right) = 0$$

$$\Rightarrow 100x^2 - 150x + 50 = 0$$

$$\Rightarrow 20x^2 - 30x + 10 = 0$$

$$\Rightarrow 2x^2 - 3x + 1 = 0$$

$$\Rightarrow 2x^2 - 2x - x + 1 = 0$$

$$\Rightarrow 2x(x-1) - 1(x-1) = 0$$

$$\Rightarrow (2x-1)(x-1) = 0$$

$$\Rightarrow x = \frac{1}{2} \quad \text{and} \quad x = 1$$

$$\Rightarrow r_1^2 = 2 \quad \text{and} \quad r_2^2 = 1.$$

Also, $\therefore L_1 = 2\sqrt{2}$ and $L_2 = 2$

Eccentricity $(e) = \sqrt{1 - \frac{r_2^2}{r_1^2}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$

ii. $2x^2 + 3y^2 - 4x - 12y + 13 = 0$

Ans. Solution:

The given equation of the curve is:

$$2x^2 + 3y^2 - 4x - 12y + 13 = 0$$

Here, $a = 2$, $b = 3$, $h = 0$, $g = -2$, $f = -6$, $c = 13$.

Also,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= 2 \times 3 \times 13 + 0 - 2 \times 36 - 3 \times 4 - 13 \times 0$$

$$= 78 - 72 - 12$$

$$= 78 - 84$$

$$= -6$$

and $h^2 - ab = 0 - 2 \times 3 = -6 \; (<0)$

$\therefore$ The given conic section is ellipse.

Now,

The equation of the ellipse center referred to the origin is

$$AX^2+2HXY+BY^2=0$$

Where,

$$A=\frac{a(h^2-ab)}{\Delta}=2$$

$$B=\frac{b(h^2-ab)}{\Delta}=3$$

$$H=\frac{h(h^2-ab)}{\Delta}=0.$$

For the length of the axes.

$$\frac{1}{r^4}-\frac{1}{r^2}(A+B)-(H^2-AB)=0$$

$$\Rightarrow x^2-5x+(6)=0$$

$$\Rightarrow (x-2)(x-3)=0$$

$$\Rightarrow x=2. \quad \text{and } x=3$$

$$\Rightarrow r_1=\frac{1}{\sqrt{2}} \quad \text{and } r_2=\frac{1}{\sqrt{3}}$$

$$\therefore l_1=\sqrt{2} \quad \text{and } l_2=\frac{2}{\sqrt{3}}$$

Also,

$$\text{Eccentricity }(e)=\sqrt{1-\frac{r_2^2}{r_1^2}}$$

$$=\sqrt{1-\frac{1}{3}\times 2}$$

$$=\frac{1}{\sqrt{3}}$$

For center.

$$4x-4=0$$

$$6y-12=0$$

$$\therefore (\alpha,\beta)=(1,2)$$

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6. Find the asymptotes of the hyperbola $2x^2+5xy+2y^2+4x+5y=0$

Ans: Solution:

The given equation of the hyperbola is

$$2x^2+5xy+2y^2+4x+5y=0$$

Here,

$a=2,\ h=\frac{5}{2},\ b=2,\ g=2,\ f=\frac{5}{2},\ c=0.$

We have,

$$\Delta = abc+2fgh-af^2-bg^2-ch^2$$

$$= 2\times\frac{5}{2}\times2\times\frac{5}{2} - 2\times\left(\frac{5}{2}\right)^2 - 2\times(4)$$

$$= 25-\frac{25}{2}-8$$

$$= \frac{25}{2}-8$$

$$= 9/2.$$

$$h^2-ab = \frac{25}{4}-2\times2 = \frac{25}{4}-4 = \frac{9}{4}$$

We have,

The equation of the asymptote of a hyperbola is given by:

$$2x^2+5xy+2y^2+4x+5y+\frac{\Delta}{h^2-ab}=0$$

$$\Rightarrow 2x^2+5xy+2y^2+4x+5y+\frac{9/2}{9/4}=0$$

i.e. $2x^2+5xy+2y^2+4x+5y+2=0$ is the required equation of the asymptotes.

7. Find the foci and eccentricity of the conics.

i. $x^2+4xy+y^2-2x+2y-6=0$

Ans. Solution:

Given,

$x^2+4xy+y^2-2x+2y-6=0$

Here, $a=1$, $b=1$, $h=2$, $g=-1$, $f=-1$, $c=-6$

Now,

$\Delta = abc+2fgh-af^2-bg^2-ch^2$

$= -6+2\times(-1)\times(-1)\times 2 - 1\times 1 - 1\times 1 + 6\times 4$

$= -6+4-2+24$

$= 28-8$

$= 20$

Also,

$h^2-ab = 4-1\times 1 = 4 > 0$

$\therefore$ The given conic is hyperbola.

Now,

For center

$2x+4y-2=0$

and $4x+2y+2=0$

$\therefore (\alpha, \beta) = (-1, 1)$

Also,

Equation of the hyperbola center referred as origin is

$AX^2+2HXY+BY^2$ where;

$A = \frac{a(h^2-ab)}{\Delta} = \frac{1}{5}$

$H = \frac{h(h^2-ab)}{\Delta} = \frac{2}{5}$

$B = \frac{b(h^2-ab)}{\Delta} = \frac{1}{5}$

For the length of axes:

$\frac{1}{r^4} - \frac{1}{r^2}(A+B) - (H^2-AB) = 0$

$$\Rightarrow x^2 - \frac{2}{5}x - \left(\frac{4}{25} - \frac{1}{25}\right) = 0$$

$$\Rightarrow 25x^2 - 10x - 3 = 0$$

$$\Rightarrow 25x^2 - 15x + 5x - 3 = 0$$

$$\Rightarrow 5x(5x-3) + 1(5x-3) = 0$$

$$\Rightarrow (5x+1)(5x-3) = 0$$

$$\Rightarrow x = -\frac{1}{5} \quad \text{and} \quad x = \frac{3}{5}$$

$$\Rightarrow r_2^2 = -5 \quad \text{and} \quad r_1^2 = \frac{5}{3}.$$

The equation of the transverse axis:

$$\left(A - \frac{1}{r_1^2}\right)X + HY = 0$$

$$\Rightarrow \left(\frac{1}{5} - \frac{3}{5}\right)X + \frac{2}{5}Y = 0$$

$$\Rightarrow -2X + 2Y = 0$$

$$\Rightarrow X - Y = 0$$

$$\Rightarrow x + 1 - (y-1) = 0$$

$$\Rightarrow x + y + 2 = 0$$

$$e = \sqrt{1 + \frac{r_2^2}{r_1^2}} = \sqrt{1 + \frac{5 \times 3}{5}} = \sqrt{4} = 2$$

Slope $= \tan\theta = -1$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos\theta = \frac{-1}{\sqrt{2}}$$

Also,

$$\text{Foci} = (\alpha \pm e r_1 \cos\theta, \beta \pm e r_1 \sin\theta)$$

$$= \left(-1 \pm 2 \times \sqrt{\frac{5}{3}} \times \frac{-1}{\sqrt{2}}, 1 \pm 2 \times \sqrt{\frac{5}{3}} \times \frac{1}{\sqrt{2}}\right)$$

$$\therefore \text{Foci} = \left(-1 \pm \frac{2\sqrt{5}}{\sqrt{6}}, 1 \pm \frac{2\sqrt{5}}{\sqrt{6}}\right)$$

ii. $x^2 + xy + y^2 + x + y = 1$

Ans Solution:

Given,

$x^2+xy+y^2+x+y=1.$

$\Rightarrow x^2+xy+y^2+x+y-1=0$

Here, $a=1, b=1, c=-1, f=\frac{1}{2}, g=\frac{1}{2}, h=\frac{1}{2}$

Now,

$\Delta = abc+2fgh-ch^2-af^2-bg^2$

$= -1+2\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+1\times\frac{1}{4}-1\times\frac{1}{4}-1\times\frac{1}{4}$

$= -1+\frac{1}{4}+\frac{1}{4}-\frac{1}{4}-\frac{1}{4}$

$= -1$

Also,

$h^2-ab = \frac{1}{4}-1\times1 = -\frac{3}{4} (<0)$

$\therefore$ The given conic section is ellipse.

For the center,

$2x+y+1=0$

and $x+2y+1=0$

$\therefore (\alpha, \beta) = \left(\frac{-1}{3}, \frac{-1}{3}\right)$

Also,

Equation of the ellipse center referred to origin is given by:

$AX^2+2HXY+BY^2=0$

Where;

$A = \frac{a(h^2-ab)}{\Delta} = \frac{3}{4}$

$B = \frac{b(h^2-ab)}{\Delta} = \frac{3}{4}$

$$H = \frac{h(h^2-ab)}{\Delta} = \frac{3}{8}$$

For the length of the axes:

$$\frac{1}{r^4} - \frac{1}{r^2}(A+B) - (H^2 - AB) = 0$$

$$\Rightarrow x^2 - \frac{6}{4}x - \left(\frac{9}{64} - \frac{9}{16}\right) = 0$$

$$\Rightarrow 64x^2 - 96x - (9-36) = 0$$

$$\Rightarrow 64x^2 - 96x + 27 = 0$$

$$\Rightarrow x = \frac{3}{8} \text{ and } x = \frac{9}{8}$$

$$\Rightarrow r_1 = \frac{2\sqrt{2}}{\sqrt{3}} \text{ and } r_2 = \frac{2\sqrt{2}}{3}$$

Also,

Eccentricity$(e) = \sqrt{1 - \frac{r_2^2}{r_1^2}} = \sqrt{1 - \frac{1}{9} \times 3} = \frac{\sqrt{2}}{3}$

And.

Foci $= (\alpha \pm r_1 e \sin\theta, \beta \pm r_1 e \sin\theta)$

We have,

Equation of Axes (Major)

$$\left(A - \frac{1}{r_1^2}\right)X + HY = 0$$

$$\Rightarrow \left(\frac{3}{4} - \frac{3}{8}\right)X + \frac{3}{8}Y = 0$$

$$\Rightarrow 3X + 3Y = 0 \Rightarrow \tan\theta = -1$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}, \cos\theta = \frac{-1}{\sqrt{2}}$$

$$\therefore \text{Foci} = \left(-\frac{1}{3} \pm \frac{\sqrt{2}}{3\sqrt{3}}, -\frac{1}{3} \pm \frac{\sqrt{2}}{3\sqrt{3}}\right)$$

$y = A\sin(\omega t)$

$\Rightarrow \frac{dy}{dx} = \cancel{A}\, \cancel{\omega}\, A\cos(\omega t).\frac{d\omega t}{dx}$

$\Rightarrow \frac{dy}{dt} = A\cos(\omega t)$

$\Rightarrow \frac{d^2y}{dx^2} =$

$\Rightarrow \frac{d^2y}{dt^2} = -\omega^2 A$

Similarly, taking the curl of equation (iv) and proceeding in similar way,

$$\nabla^2 \cdot \vec{B} = \mu_0 \varepsilon_0 \frac{d^2\vec{B}}{dt^2}$$

$v^2 = \mu_0 \varepsilon_0$

$\cancel{v = \sqrt{\mu_0 \varepsilon_0} = c}$

$v^2 = \frac{1}{\mu_0 \varepsilon_0}$

$\Rightarrow v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$

$$\nabla^2 y = \frac{1}{v^2} \frac{d^2 y}{dt^2}$$

Electromagnetic Wave Equation in Free Space.

In free space, the charge density ρ and current density $\vec{J}$ are zero. So, the Maxwell's equations are as given:

① $\nabla \cdot \vec{E} = 0$

② $\nabla \cdot \vec{B} = 0$

③ $\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$

④ $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt}$

We have,

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\Rightarrow \nabla \times \nabla \times \vec{E} = -\frac{d}{dt}(\nabla \times \vec{B})$$

$$\Rightarrow \nabla \cdot (\nabla \cdot \vec{E}) - \vec{E}(\nabla \cdot \nabla) = -\frac{d}{dt}\left(\mu_0 \varepsilon_0 \cdot \frac{d\vec{E}}{dt}\right)$$

$$\Rightarrow -\vec{E} \cdot \nabla^2 = -\mu_0\varepsilon_0 \frac{d^2 \vec{E}}{dt^2}$$

$$\therefore \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{d^2 \vec{E}}{dt^2}$$

From Faraday's law of electromagnetic Induction

$$\oint E \cdot d\vec{l} = -\frac{d\phi_B}{dt}$$

$$\Rightarrow \oint \vec{E} \cdot d\vec{l} = -\frac{d \oint \vec{B} \cdot d\vec{A}}{dt}$$

$\Rightarrow$ Using curl Stoke's theorem, we get,

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

which is Maxwell's third equation in differential form.

Maxwell's Fourth Equation:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I + \varepsilon_0 \frac{d\phi_E}{dt} \right)$$

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 \left(\oint \vec{J} \cdot d\vec{A} + \varepsilon_0 \frac{d \oint \vec{E} \cdot d\vec{A}}{dt} \right)$$

Applying curl Stoke's theorem, we get,

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \frac{d\vec{E}}{dt})$$

which is required fourth Maxwell law in differential form.

From Guass law of electrostatics.

$$\oint \vec{E} \cdot \vec{dA} = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \oint \vec{E} \cdot \vec{dA} = \frac{1}{\varepsilon_0} \oint \rho dV$$

Using Guass-Divergence Theorem,

$$\oint (\nabla \cdot \vec{E}) dV = \frac{1}{\varepsilon_0} \oint \rho dV$$

$$\Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}.$$

Which is required maxwell's first equation in differential form.

From Guass law of magnetism, we have,

$$\oint \vec{B} \cdot \vec{dA} = 0$$

Using Guass divergence Theorem,

$$\oint (\nabla \cdot \vec{B}) dV = \oint 0$$

$$\nabla \cdot \vec{B} = 0.$$

Which is required Maxwell's second equation is differential form.