1. Regarding the proof of problem 23(b).

Since $\mathbf{x}_0$ is an extreme point it lies on n LI hyperplanes defining $Fea(P) = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}$ and it lies on atleast n defining hyperplanes of $Fea(P)$.
WLOG let the set of normals to those chosen set of n LI hyperplanes (if $\mathbf{x}_0$ lies on more than n defining hyperplanes of $Fea(P)$) be $\tilde{\mathbf{a}}_j$, $j = 1, 2, \ldots, n$,
hence $\mathbf{x}_0$ satisfies $\tilde{\mathbf{a}}_j^T\mathbf{x}_0 = \tilde{b}_j$, $j = 1, \ldots, n$.
Let $\mathbf{d}_i \in \mathbb{R}^n$ be such that $\tilde{\mathbf{a}}_j^T\mathbf{d}_i = 0$, for all $j = 1, \ldots, i-1, i+1, \ldots, n$, and $\tilde{\mathbf{a}}_i^T\mathbf{d}_i < 0$ (that is $\mathbf{d}_i$ is orthogonal to the normals of all the chosen LI hyperplanes on which $\mathbf{x}_0$ lies except the i th one).
We have to show that all the $\mathbf{d}_i$'s defined above cannot be directions, and hence by moving in the positive direction of one of them, starting from $\mathbf{x}_0$ we hope to find an obstruction, which will give an extreme point adjacent to $\mathbf{x}_0$.
Check that $\{\mathbf{d}_1, \ldots, \mathbf{d}_n\}$ is LI hence forms a basis of $\mathbb{R}^n$.
If $\mathbf{y}_0$ is an extreme point of $Fea(P)$, then $(\mathbf{y}_0 - \mathbf{x}_0) \in \mathbb{R}^n$ can be expressed as a linear combination of $\mathbf{d}_1, \ldots, \mathbf{d}_n$.
Let $\mathbf{y}_0 - \mathbf{x}_0 = \alpha_1\mathbf{d}_1 + \ldots + \alpha_n\mathbf{d}_n$. (**)
Case 1: All the α_i's are nonnegative.
Since $\mathbf{y}_0 - \mathbf{x}_0$ can be expressed as a nonnegative linear combination of $\mathbf{d}_1, \ldots, \mathbf{d}_n$, and $\mathbf{y}_0$ is an extreme point $\mathbf{y}_0 - \mathbf{x}_0$ is not a direction hence at least one of the $\mathbf{d}_i$'s is not a direction. Choose a $\mathbf{d}_i$ such that $\mathbf{x}_0 + \alpha\mathbf{d}_i$ does not belong to $Fea(P)$ for $\alpha > 0$ large but $\mathbf{x}_0 + \alpha\mathbf{d}_i \in Fea(P)$ for $\alpha > 0$ sufficiently small
(will there always exist such a $\mathbf{d}_i$? Consider the exercise given below).
Let $\beta = \max\{\alpha : \mathbf{x}_0 + \alpha\mathbf{d}_i \in Fea(P)\}$, then $\beta > 0$ and check that $\mathbf{x}_0 + \beta\mathbf{d}_i \ne \mathbf{x}_0$ is an extreme point which lies on $(n-1)$ LI hyperlanes in common with $\mathbf{x}_0$, that is an adjacent extreme point of $\mathbf{x}_0$.
Case 2: If not then WLOG let $\alpha_1 < 0$, which implies for all $\gamma > 0$,
$\tilde{\mathbf{a}}_1^T(\mathbf{x}_0 + \gamma(\mathbf{y}_0 - \mathbf{x}_0))$
$= \tilde{\mathbf{a}}_1^T(\mathbf{x}_0 + \gamma(\alpha_1\mathbf{d}_1 + \ldots + \alpha_n\mathbf{d}_n))$
$= \tilde{\mathbf{a}}_1^T\mathbf{x}_0 + (\gamma\alpha_1)\tilde{\mathbf{a}}_1^T\mathbf{d}_1 > \tilde{b}_1$ (***)
But since $\mathbf{x}_0 + \gamma(\mathbf{y}_0 - \mathbf{x}_0) \in Fea(P)$ for all $0 \le \gamma \le 1$,
$\tilde{\mathbf{a}}_j^T(\mathbf{x}_0 + \gamma(\mathbf{y}_0 - \mathbf{x}_0)) \le \tilde{b}_j$ for all $0 \le \gamma \le 1$, for all $j = 1, \ldots, n$, which contradicts (***).
(The above proof was given by a student Debanjan chakrabarty of CSE in the 2016 batch, also almost the same proof was given by a student yesterday, I dont remember his name).

Aliter: let $\mathbf{y}_0$ be another extreme point of $Fea(P)$. Let $\bar{S} = S \cap H$ where H is a halfspace such that $\bar{S}$ is bounded and includes no other extreme point of S except $\mathbf{x}_0$ (check that you can do it, for example consider a positive vector $\mathbf{a}$ such that $\mathbf{a}^T\mathbf{x}_0 \ne \mathbf{a}^T\mathbf{x}_i$ for all other extreme points $\mathbf{x}_i$ of S. Consider $H = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} \le \mathbf{a}^T\mathbf{x}_0 + \epsilon\}$, where $\epsilon > 0$ is sufficiently small).
Then $\bar{S}$ is a bounded polyhedral set and check that a part of the line segment joining $\mathbf{x}_0$ and $\mathbf{y}_0$ must be inside $\bar{S}$, that is $\mathbf{x}_0 + \gamma(\mathbf{y}_0 - \mathbf{x}_0) \in \bar{S}$ for $\gamma > 0$ small, (*)
hence $\bar{S}$ must have atleast two extreme points and the extreme points of $\bar{S}$ are $\mathbf{x}_0$ and points of the form $\mathbf{x}_0 + \alpha_i\mathbf{d}_i$, (where the $\mathbf{d}_i$'s are as defined above) which lie on $(n-1)$ LI hyperplanes in common with $\mathbf{x}_0$ and on the hyperplane associated with H. Also by

(*) and the representation theorem, $\mathbf{x}_0 + \gamma(\mathbf{y}_0 - \mathbf{x}_0) \in \bar{S}$ can be written as a convex combination of the extreme points of $\bar{S}$, which implies that $(\mathbf{y}_0 - \mathbf{x}_0)$ can be written as a nonnegative linear combination of the $\mathbf{d}_i$'s, and since $(\mathbf{y}_0 - \mathbf{x}_0)$ is not a direction (because $\mathbf{y}_0$ is an extreme point), all the $\mathbf{d}_i$'s cannot be directions.
Hence by starting from $\mathbf{x}_0$ and moving along the positive direction of one such $\mathbf{d}_i$ we hope to find an adjacent extreme point of $\mathbf{x}_0$.

Exercise: Check that if $\mathbf{x}_0$ is an extreme point then there can exist $\mathbf{d}_i$ not a direction (where $\mathbf{d}_i$ is as defined in the previous proof) such that none of $\mathbf{x}_0 + \alpha\mathbf{d}_i$, and $\mathbf{x}_0 - \alpha\mathbf{d}_i$ belong to the feasible region for any $\alpha > 0$. So the initial choice of the $\mathbf{d}_i$'s in the above proof has to be done judiciously.
Hint: Think of an extreme point for a feasible region in $\mathbb{R}^2$ lying on three defining hyperplanes of the feasible region.

2. There are also errors (the transpose sign are missing, the word LI missing etc) and gaps in the Hint of 23(a). I am trying to correct it, I will send it to you as soon as possible.

3. Regarding the proof of the existence of a direction if the set is unbounded is not done for a general convex set, but for a polyhedral set the existence of direction if the set is unbounded is sort of included in the proof of **Representation theorem** (go through the proof carefully).

4. Regarding the question of a student that whether the extreme directions are LI. I had given a counterexample for a general polyhedral set (a single half space) which was not like the feasible region of a LPP, a better counterexample is given below:
Consider the feasible region of a LPP in $\mathbb{R}^3$ as:
$-x_1 + x_2 - x_3 \leq 1$
$-x_1 - x_2 + x_3 \leq -2$
$-x_1 - x_2 - x_3 \leq -5$
$x_1 \geq 0$, $x_2 \geq 0$,$x_3 \geq 0$.
Check that the above feasible region has more than three extreme directions (as far as I have calculated), hence the extreme directions are LD.

5. Regarding the **Exercise** given in notes(2) immediately before Example 2 (revisited), there is an issue with the statement as pointed out by the student Manan (maybe that is his name), so I have rephrased the statement of the problem as follows:
Exercise: Check that if a $\mathbf{d} \in D$ lies on $(n-1)$ LI hyperplanes (out of the $(m+n)$ defining hyperplanes of D) given by $\{H_1, \ldots, H_{n-1}\}$, then $\{H, H_1, \ldots, H_{n-1}\}$ is LI where $H = \{\mathbf{d} \in \mathbb{R}^n : [1, 1, \ldots, 1]\mathbf{d} = 1\}$.