Practise problems 5

Note that for all the problems $\tilde{\mathbf{a}}_k$ denotes the k-th column of A or of $[A : I]$ as the case may be.
$N_{i.}^T$ gives the i th row of N^T.

1. Let (P) be a problem of the form,

$$\begin{aligned}&\text{Maximize } \mathbf{c}^T\mathbf{x} \quad (\mathbf{c} \neq \mathbf{0})\\ &\text{subject to } A_{m\times n}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.\end{aligned}$$

   (a) If every extreme point of Fea(P) lies in exactly n hyperplanes defining Fea(P) then how many optimal solutions can the dual of (P) have?

   (b) If for any $\mathbf{x} \in Opt(P)$ and $\mathbf{y} \in Opt(D)$,
   $x_i = 0$ implies $(A^T\mathbf{y})_i > c_i$ for $i = 1, 2, \ldots, n$, and
   $y_j = 0$ implies $(A\mathbf{x})_j < b_j$ for $j = 1, 2, \ldots, m$,
   then how many optimal solutions will (P) and the dual (D) of (P) have?

   (c) If for some $\mathbf{x} \in Opt(P)$, $\mathbf{y} \in Opt(D)$ and exactly one $k \in \{1, \ldots, n\}$,
   $x_k = 0$ and $(A^T\mathbf{y})_k = c_k$,
   but for any $j = 1, 2, \ldots, m$, $y_j = 0$ implies $(A\mathbf{x})_j < b_j$,
   then what can you say about the number of optimal solutions of (P) and the dual (D) of (P)?

   (d) If for some $\mathbf{x} \in Opt(P)$, $\mathbf{y} \in Opt(D)$ and some $i \in \{1, \ldots, n\}$ and some $j \in \{1, \ldots, m\}$,
   $x_i = 0$ and $(A^T\mathbf{y})_i = c_i$, and $y_j = 0$ and $(A\mathbf{x})_j = b_j$,
   then what can you say about the number of optimal solutions of (P) and the dual (D) of (P)?

   (e) If $n = 3, m = 3$ and $\mathbf{d}_1 = [2, 1, -1, 0, 0, 3]^T$ is of the form $\begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_j \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ where B is the basis matrix corresponding to the BFS $[\mathbf{x}_0^T, \mathbf{s}_0^T]^T$, then can you guess the basic and the nonbasic variables of this BFS and the variable which corresponds to the index j? Also give (indicate) the hyperplane/s which will finally obstruct movement from $[\mathbf{x}_0^T, \mathbf{s}_0^T]^T$ along the positive direction of $\mathbf{d}_1$.

2. Consider a problem (P) of the form:
   Min $\mathbf{c}^T\mathbf{x}$
   subject to $A_{3\times 5}\mathbf{x} = \mathbf{b}_{3\times 1}$, $\mathbf{x} \geq \mathbf{0}$, $rank(A) = 3$.
   Given that $S = Fea(P)$ is nonempty, check the correctness of the following statements.

   (a) If $\mathbf{x}_0 = [1, 0, 3, 0, 0]^T$ and $\mathbf{y}_0 = [3, 0, 1]^T$ are optimal solutions of (P) and (D) respectively and if we change $\mathbf{c}$ to $\mathbf{c}' = \mathbf{c} + [0, 1, 0, 2, 0]^T$, (everything else in (P) remaining same) then $\mathbf{x}_0$ and $\mathbf{y}_0$ will again be optimal solutions for the changed (P) and its dual, respectively.

(b) If $\mathbf{u}$ and $\mathbf{v}$ are two (distinct) adjacent extreme points of S, and if $\mathbf{d}$ be such that $\mathbf{u} = \mathbf{v} + \alpha\mathbf{d}$ for some $\alpha > 0$, then the vector $\mathbf{d}$ has a zero component, a positive component and a negative component.

(c) If every nonzero $\mathbf{d} \in \mathbb{R}^5$ is orthogonal to the normals of at most 4 hyperplanes defining Fea(P) then in any (simplex) table all the columns corresponding to nonbasic variables will have all nonzero entries.

(d) If $\mathbf{x}_0$ is optimal for (P) and $\mathbf{c}$ is changed $\mathbf{c}'$ such that $\mathbf{x}_0$ is no longer optimal for the new problem then $\mathbf{c}^T\mathbf{x}_0$ is greater than equal to the optimal value (if there is one) of the new problem.

(e) If $\mathbf{x}_0$ is optimal for (P) and $\mathbf{b}$ is changed $\mathbf{b}'$ such that $[(B^{-1}\mathbf{b}')^T, 0^T]^T$ is no longer feasible for the new problem then $\mathbf{c}^T\mathbf{x}_0$ is greater than equal to the optimal value (if there is one) of the new problem.

(f) If every $\mathbf{d} \in \mathbb{R}^5$ which is orthogonal to at least 4 LI hyperplanes defining Fea(P) has atleast one negative component and a positive component, then Fea(P) is bounded.

(g) If $\mathbf{x}_0$ is optimal for (P) and $\mathbf{c}$ is changed $\mathbf{c}'$ such that $c_1' - z_1' < 0$ but for all $j = 2, \ldots, 5$ $c_j' - z_j' \geq 0$, then x_1 will be a basic variable for an optimal solution to the new problem (if there is an optimal solution).