Notations:
$\mathbf{x}, \mathbf{d}, \mathbf{b}$, etc, that is characters in boldface represent (column) vectors.
$\tilde{\mathbf{a}}_i$, the ith column of A.

Given a linear programming problem
Max $\mathbf{c}^T\mathbf{x}$ (1)
subject to $A_{m\times n}\mathbf{x} \leq \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$.

Let us denote the set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ by $Fea(P)$.

The **Dual** of this problem is given by
Min $\mathbf{b}^T\mathbf{y}$ (2)
subject to $A^T_{n\times m}\mathbf{y} \geq \mathbf{c}_{n\times 1}$, $\mathbf{y} \geq \mathbf{0}$.

Let us denote the set $\{\mathbf{y} \in \mathbb{R}^m : A^T\mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq \mathbf{0}\}$ by $Fea(D)$.

The linear programming problem (1) is called the **Primal** problem.

Theorem 1: If $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$, then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.

Proof: Since $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$,
$y_i(Ax)_i \leq y_ib_i$ for all $i = 1, 2, ..., m$.
This implies, $\mathbf{y}^TA\mathbf{x} = \sum_{i=1}^m y_i(A\mathbf{x})_i \leq \sum_{i=1}^m y_ib_i = \mathbf{b}^T\mathbf{y}$. (i)
Also since $A^T\mathbf{y} \geq \mathbf{c}$ and $\mathbf{x} \geq \mathbf{0}$, we can similarly get
$\mathbf{x}^TA^T\mathbf{y} \geq \mathbf{x}^T\mathbf{c} = \mathbf{c}^T\mathbf{x}$. (ii)
(i) and (ii) together $\Rightarrow \mathbf{b}^T\mathbf{y} \geq \mathbf{y}^TA\mathbf{x} = \mathbf{x}^TA^T\mathbf{y} \geq \mathbf{c}^T\mathbf{x}$.

Corollary 1: If both the Primal and the Dual have feasible solutions then both have optimal solutions.

Proof: Let $\mathbf{x}_0$ is a feasible solution of the primal (P) and $\mathbf{y}_0$ be a feasible solution of the dual. Since $\mathbf{y}_0$ be a feasible solution of the dual, then for all $\mathbf{x} \in Fea(P)$
$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}_0$,
but this implies that there cannot exist a direction $\mathbf{d}$ of Fea(P) such that $\mathbf{c}^T\mathbf{d} > 0$ (we have understood this before). Since (P) is a maximization problem, this implies that (P) has an optimal solution.
Similarly since $\mathbf{x}_0$ is a feasible solution of the primal (P), then for all $\mathbf{y} \in Fea(D)$,
$\mathbf{b}^T\mathbf{y} \geq \mathbf{c}^T\mathbf{x}_0$,
hence there cannot exist a direction $\mathbf{d}'$ of $Fea(D)$ such that $\mathbf{b}^T\mathbf{d}' < 0$, which implies that (D) has an optimal solution.

Corollary 2: Let $\mathbf{x}_0 \in Fea(P)$ and $\mathbf{y}_0 \in Fea(D)$, be such that $\mathbf{c}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{y}_0$, then $\mathbf{x}_0$ and $\mathbf{y}_0$ are optimal for the Primal and the Dual, respectively.

Proof: If $\mathbf{x}$ is any feasible solution of the Primal, then since $\mathbf{y}_0 \in Fea(D)$, from Theorem 1 we get
$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}_0 = \mathbf{c}^T\mathbf{x}_0$.
Hence the result.

Let us again consider the problem:

Example 1: Max $5x + 2y$
subject to
$3x + 2y \leq 6$
$x + 2y \leq 4$
$x \geq 0, y \geq 0.$

If we can get a feasible solution $\mathbf{w}$ of the dual of this problem such that $\mathbf{c}^T\mathbf{z} = \mathbf{b}^T\mathbf{w}$, where $\mathbf{c} = [5, 2]^T$, $\mathbf{b} = [6, 4]^T$ and $\mathbf{z} = [2, 0]^T$ (already obtained before), then using the above **corollary**, we will again be able to conclude that $\mathbf{z}$ is indeed optimal for this LPP (we had already seen this before). Note that $\mathbf{c}^T\mathbf{z} = 5 \times 2 + 2 \times 0 = 10$.
The dual of this problem is given by,

Min $6y_1 + 4y_2$
subject to
$3y_1 + y_2 \geq 5$
$2y_1 + 2y_2 \geq 2,$
$y_1 \geq 0, y_2 \geq 0.$

Check that $\mathbf{w} = [1, 1]^T$ is a feasible solution of the dual and $\mathbf{b}^T\mathbf{w} = 6 \times 1 + 4 \times 1 = 10$. Hence $\mathbf{z} = [2, 0]^T$, is optimal for the primal and $\mathbf{w} = [1, 1]^T$ is optimal for the dual.

Theorem 2: (Fundamental Theorem of Duality) If both the Primal (P) and the Dual (D) of the Primal (P) have feasible solutions then both have optimal solutions and the optimal value is equal (that is Min $\mathbf{b}^T\mathbf{y}$ = Max $\mathbf{c}^T\mathbf{x}$). If one of them (the Primal or the Dual) does not have a feasible solution then the other does not have an optimal solution.

Proof: Note that the first part of the theorem is already seen by **Corollary 1.**
We need to show that when both (P) and (D) has feasible solutions then their optimal values are equal.
Also we need to show that if one of (P) or (D) does not have a feasible solution then the other does not have an optimal solution.
We will do these remaining parts of the proof later.

Theorem 3: The Dual of the Dual (D) (of the Primal (P)) is the primal (P).
Proof: Note that Dual (D) given by
Min $\mathbf{b}^T\mathbf{y}$ (2)
subject to $A^T_{n\times m}\mathbf{y} \geq \mathbf{c}_{n\times 1}, \quad \mathbf{y} \geq \mathbf{0}$
is same as the problem (3) given below
Max $-\mathbf{b}^T\mathbf{y}$ (3)
subject to $-A^T_{n\times m}\mathbf{y} \leq -\mathbf{c}_{n\times 1}, \quad \mathbf{y} \geq \mathbf{0}$.
The above problem is of the same form as (P) hence the dual of the above problem is given by
Min $-\mathbf{c}^T\mathbf{z}$ (4)
subject to $(-A^T)^T\mathbf{z} \geq -\mathbf{b}, \quad \mathbf{z} \geq \mathbf{0}$.
But the above problem is same as the problem (5) given below
Max $\mathbf{c}^T\mathbf{z}$ (2)
subject to $A\mathbf{z} \leq \mathbf{b}, \quad \mathbf{z} \geq \mathbf{0}$,
which is same as the primal problem (P).

Hence if we can show that if the Primal (P) is infeasible then the Dual (D) does not have an optimal solution, then by the previous theorem we can see that if the Dual (D) is infeasible then the primal (P) (which is the Dual of (D)) does not have an optimal solution.
But before we can prove this let us first check the validity of the above results through some examples.

Example 2:
Min $-x + 2y$
subject to
$x + 2y \geq 1$
$-x + y \leq 1,$
$x \geq 0, y \geq 0.$
We had already seen that this problem does not have an optimal solution although it has a feasible solution.
The dual of this problem is given by
Max $u - v$
subject to
$u + v \leq -1$
$2u - v \leq 2,$
$u \geq 0, v \geq 0.$
Clearly this problem does not have any feasible solution, since the first constraint cannot be satisfied by any nonnegative u and v.

Example 3:
Max $-x + 2y$
subject to
$x + 2y \leq 1$
$-x + y \geq 1,$
$x, y \geq 0.$
We have seen before that this problem does not have a feasible solution, hence does not have an optimal solution.

The dual of the above problem is given by
Min $u - v$
subject to
$u + v \geq -1$
$2u - v \geq 2$
$u \geq 0, v \geq 0.$
Check that the above problem has a feasible solution but not an optimal solution.

In order to however completely prove the fundamental theorem of duality we need to look at solutions of certain systems of equations and inequalities.

Definition: A nonempty set $T \subseteq \mathbb{R}^n$ is said to be a **cone** if $\mathbf{x} \in T$ implies $\lambda\mathbf{x} \in T$ for all $\lambda \geq 0$. So a cone always contains the origin.
Also a **cone** T is said to be a **convex cone** if it is also a convex subset of $\mathbb{R}^n$.

Exercise: If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are all vectors in $\mathbb{R}^n$ then check that the set of all nonnegative linear

combinations of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ is a convex cone, that is,
$T = \{\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \ldots + \alpha_k\mathbf{a}_k : \alpha_i \geq 0 \text{ for all } i = 1, \ldots, k\}$ is a **convex cone**.
Then $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are called **generators** of the convex cone T.

Exercise: Check that if $S = Fea(LPP)$ is unbounded then $D \cup \{\mathbf{0}\}$ is a convex cone, where D (as defined before) is the set of all directions of S.

Theorem 4: (**Farka's Lemma**) Exactly one of the following two systems has a solution.

$$A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \qquad (1)$$

$$\mathbf{y}^T A \geq \mathbf{0}_{1\times n}, \quad \mathbf{y}^T\mathbf{b} < 0 \qquad (2)$$

System 1 has a solution means that $\mathbf{b}$ belongs to the cone generated by the columns of A.
Whereas system 2 has a solution means there exists a hyperplane (with normal $\mathbf{y}$) which separates the cone generated by the columns of A and the vector $\mathbf{b}$, in the sense that the cone lies in one of the closed half spaces associated with this hyperplane and the vector $\mathbf{b}$ lies in the other open half space.

Proof: It is easy to check that both the two systems cannot have a solution.
If we multiply equation (1) with $\mathbf{y}^T$, the solution of (2) then we get
$0 \leq \mathbf{y}^T A\mathbf{x} = \mathbf{y}^T\mathbf{b}$, but this contradicts that $\mathbf{y}^T\mathbf{b} < 0$.
The proof of the part that if (1) does not have a solution then (2) has a solution, and vice versa, we will postpone it for later consideration.

Corollary 4: Exactly one of the following two systems has a solution.

$$A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0} \qquad (1)$$

$$\mathbf{y}^T A \geq \mathbf{0}, \quad \mathbf{y}^T\mathbf{b} < 0, \quad \mathbf{y} \geq \mathbf{0} \qquad (2)$$

Proof: Note that system (1) has a solution if and only if system $(1)'$ given below, has a solution.
$A_{m\times n}\mathbf{x} + \mathbf{u} = [A : I]\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}, \mathbf{u} \geq \mathbf{0} \qquad (1)'$

But from the previous theorem, exactly one of the following two systems has a solution.

$$[A : I]\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \mathbf{b}, \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \geq \mathbf{0} \qquad (1)'$$

$$\mathbf{y}^T[A : I] \geq [\mathbf{0}_{1\times n}, \mathbf{0}_{1\times m}], \quad \mathbf{y}^T\mathbf{b} < 0 \qquad (2)'$$

But note that system $(2)'$ is same as system (2).
Hence the result.

Fundamental Theorem of Duality (revisited):
Note that since $Fea(D) = \{\mathbf{y} \in \mathbb{R}^m : A^T\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}\}$,
the set of all directions of $Fea(D)$ is given by
$D_D = \{\mathbf{d} \in \mathbb{R}^m : A^T\mathbf{d} \geq \mathbf{0}, \quad \mathbf{d} \geq \mathbf{0}\}$, and the dual (D) has an optimal solution if and only if either $Fea(D)$ is bounded or if not then $\mathbf{b}^T\mathbf{d} \geq 0$ for all $\mathbf{d} \in D_D$.
So from **corollary 4** it is obvious that if the primal does not have a feasible solution, that is system (1) does not have a solution, then since (2) must have a solution so there exists a $\mathbf{y} \in \mathbb{R}^m$ such that $A^T\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{y}^T\mathbf{b} < 0$, that is there exists a direction $\mathbf{d}(= \mathbf{y})$ of $Fea(D)$ such that $\mathbf{d}^T\mathbf{b} < 0$, which implies that the dual does not have an optimal solution (the dual may or may not

have a feasible solution).
Similarly (since the dual can be converted to a problem of the same form as the primal (P)) we can show that if the dual does not have a feasible solution then the primal (dual of the dual) does not have an optimal solution.

So the proof of the Fundamental theorem of duality will be complete if we can show that if $Fea(P)$ and $Fea(D)$ are both nonempty, then their optimal values will be same (we have already seen that in this case both will have optimal solutions).
So we need to show that if $Fea(P) \neq \phi$ and $Fea(D) \neq \phi$, then the following system (1) has a solution:

$$\begin{gathered} A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0} \qquad (1) \\ A^T\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0} \\ \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y} \end{gathered}$$

(**Note:** If we can show that system (1) has a solution then we would have actually shown that both the problems (P) and (D) have optimal solutions (which anyway we have already seen previously) and that the optimal value for both (P) and (D) are same.)
The first two inequalities correspond to $Fea(P)$ and $Fea(D)$, respectively. We have already seen that if $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$ then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$, hence system (1) given above has a solution if and only if, system $(1)''$ given below has a solution:

$$\begin{gathered} A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0} \qquad (1)'' \\ A^T\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0} \\ \mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y} \end{gathered}$$

System $(1)''$ can be written as:

$$\begin{bmatrix} A_{m\times n} & \mathbf{0}_{m\times m} \\ \mathbf{0}_{n\times n} & -A^T_{n\times m} \\ -\mathbf{c}^T_{1\times n} & \mathbf{b}^T_{1\times m} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \geq \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \mathbf{0}_{m\times 1} \end{bmatrix} \qquad (1)''.$$

By **corollary 4**, exactly one of the two systems, $(1'')$ and $(2)''$ (given below) has a solution.

$$\begin{bmatrix} \mathbf{z}^T_{1\times m} & \mathbf{w}^T_{1\times n} & a_{1\times 1} \end{bmatrix} \begin{bmatrix} A_{m\times n} & \mathbf{0}_{m\times m} \\ \mathbf{0}_{n\times n} & -A^T_{n\times m} \\ -\mathbf{c}^T_{1\times n} & \mathbf{b}^T_{1\times m} \end{bmatrix} \geq \begin{bmatrix} \mathbf{0}_{1\times n} & \mathbf{0}_{1\times m} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{z}^T_{1\times m} & \mathbf{w}^T_{1\times n} & a_{1\times 1} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \\ 0 \end{bmatrix} < 0 \quad (2)''$$

$$\begin{bmatrix} \mathbf{z}^T_{1\times m} & \mathbf{w}^T_{1\times n} & a_{1\times 1} \end{bmatrix} \geq \mathbf{0}_{m+n+1}$$

If we simplify the inequalities in system $(2)''$, then we get the following:

$\mathbf{z}^TA - a\mathbf{c}^T \geq \mathbf{0}$, $\mathbf{z}^T \geq \mathbf{0}$, $a \geq 0$ or
$A^T\mathbf{z} \geq a\mathbf{c}$, $\mathbf{z} \geq \mathbf{0}$, $a \geq 0$ **(i)**

$-\mathbf{w}^TA^T + a\mathbf{b}^T \geq \mathbf{0}$, $\mathbf{w}^T \geq \mathbf{0}$, $a \geq 0$ or
$A\mathbf{w} \leq a\mathbf{b}$, $\mathbf{w} \geq \mathbf{0}$, $a \geq 0$ **(ii)**

$\mathbf{z}^T\mathbf{b} < \mathbf{w}^T\mathbf{c}$ **(iii)**

Case1: If $a > 0$, then from **(i)** and **(ii)** it follows that
$\frac{\mathbf{z}}{a} \in Fea(D)$ and $\frac{\mathbf{w}}{a} \in Fea(P)$, but this contradicts **(iii)**.

Case 2: If $a = 0$, then from **(i)** and **(ii)** it follows that
$\mathbf{z}^T A \geq \mathbf{0}$, $\mathbf{z}^T \geq \mathbf{0}$ and $\mathbf{w}^T A^T \leq \mathbf{0}$, $\mathbf{w}^T \geq \mathbf{0}$ **(iv)**
Since $Fea(P) \neq \phi, Fea(D) \neq \phi$, let $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$,
hence $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ and $A^T\mathbf{y} \geq \mathbf{c}$, $\mathbf{y} \geq \mathbf{0}$. **(v)**
Then from **(iv)** and **(v)** after appropriate multiplication we get
$0 \leq \mathbf{z}^T A\mathbf{x} \leq \mathbf{z}^T\mathbf{b}$ and $0 \geq \mathbf{w}^T A^T\mathbf{y} \geq \mathbf{w}^T\mathbf{c}$ which contradicts **(iii)**.

Aliter for Case 2: (**suggested by a student**) Note that since $Fea(P) \neq \phi$ and $Fea(D) \neq \phi$ from **(iv)** we get $\mathbf{z}$ and $\mathbf{w}$ are directions of (D) and (P) respectively. Also since we have already seen that if both Fea(P) and Fea(D) are nonempty then both have optimal solutions hence $\mathbf{c}^T\mathbf{w} \leq 0$ and $\mathbf{b}^T\mathbf{z} \geq 0$ which contradicts **(iii)**.

Hence in both the two cases system $(2)''$ does not have a solution, hence system $(1)''$ has a solution.
Hence the proof of the **Fundamental Theorem of Duality** is complete.

Interpretation of the Dual and the Complementary Slackness Theorem (or Equilibrium Theorem).
Now consider again the **diet problem** which is of the form
Min $\sum_{i=1}^{n} c_i x_i$
subject to
$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$, for all $i = 1, 2, ..., m$,
$x_j \geq 0$ for all $j = 1, 2, \ldots, n$.

The dual of the above problem is given by
Max $\sum_{i=1}^{m} b_i y_i$
subject to
$\sum_{i=1}^{m} a_{ij} y_i \leq c_j$, $j = 1, 2, ..., n$, or $A^T_{n\times m}\mathbf{y}_{m\times 1} \leq \mathbf{c}_{n\times 1}$,
$y_i \geq 0$ for all $i = 1, 2, \ldots, m$, or $\mathbf{y} \geq \mathbf{0}$.
Note that here the y_i's should correspond to some type of cost or value associated with the ith nutrient, since the right hand side of the constraints of the dual give costs and a_{ij}'s give quantity of nutrients in unit amount of F_j. For example if there were tablets or pills of the corresponding nutrients each tablet corresponding to unit amount of the nutrient, then y_i could be thought of as price of one pill of the ith nutrient. So if the jth food product, F_j was to be substituted by pills of nutrients N_i, then in order to get the same value of nutrients as one unit of F_j, one will have to take
(a_{1j} amount of N_1)+(a_{2j} amount of N_2) +... + (a_{mj} amount of N_m),
the total cost of which comes out to be $\sum_{i=1}^{m} a_{ij} y_i$.
So a seller of these nutrient pills can fix the price of these pills so as to maximize his return or profit which is given by $\sum_{i=1}^{m} b_i y_i$ as long as $\sum_{i=1}^{m} a_{ij} y_i$ does not exceed the market cost of a unit amount of F_j given by c_j, for all $j = 1, 2, \ldots, n$.

If the Primal (P) is given by
Max $\mathbf{c}^T\mathbf{x}$
(1) subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.
We have already seen that the Dual (D) of this problem is given by

Min $\mathbf{b}^T\mathbf{y}$
(2) subject to $A^T\mathbf{y} \geq \mathbf{c}$, $\mathbf{y} \geq \mathbf{0}$.

Theorem 5 (Complementary Slackness Theorem): Let $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$, then $\mathbf{x}$ and $\mathbf{y}$ are optimal for the primal and the dual respectively if and only if
$x_j = 0$ whenever $(A^T\mathbf{y})_j > c_j$, $j = 1, 2, .., n$ (1)
and
$y_i = 0$ whenever $(A\mathbf{x})_i < b_i$, $i = 1, 2, ..., m$. (1*)

Proof: If part.
If (1) and (1)* holds, then from (1)
$x_j(A^T\mathbf{y})_j = x_jc_j$ for all $j = 1, 2, ..., n$.
Hence $\sum_{j=1}^n x_j(A^T\mathbf{y})_j = \sum_{j=1}^n x_jc_j$.
Hence $\mathbf{x}^T A^T\mathbf{y} = \sum_{j=1}^n x_j(A^T\mathbf{y})_j = \sum_{j=1}^n x_jc_j = \mathbf{c}^T\mathbf{x}$. (2)
Similarly from (1)* we get
$y_i(A\mathbf{x})_i = y_ib_i$ for all $i = 1, 2, ..., m$.
Hence $\mathbf{y}^T A\mathbf{x} = \sum_{i=1}^m y_i(A\mathbf{x})_i = \sum_{i=1}^m y_ib_i = \mathbf{b}^T\mathbf{y}$. (2*)
Since $\mathbf{x}^T A^T\mathbf{y} = \mathbf{y}^T A\mathbf{x}$,
(2) and (2*) together implies $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$, hence from **corollary 1** we can conclude that $\mathbf{x}$ and $\mathbf{y}$ are optimal for (P) and (D), respectively.
Only If part.
On the other hand if $\mathbf{x}$ and $\mathbf{y}$ are optimal then by the **fundamental theorem of duality** the optimal values should be equal, hence $\mathbf{y}^T\mathbf{b} = \mathbf{x}^T\mathbf{c}$.
Also since $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$, we get the following after appropriate multiplication,
$\mathbf{x}^T\mathbf{c} \leq \mathbf{x}^T A^T\mathbf{y} = \mathbf{y}^T A\mathbf{x} \leq \mathbf{y}^T\mathbf{b} = \mathbf{x}^T\mathbf{c}$.
Hence $\mathbf{x}^T\mathbf{c} = \mathbf{x}^T A^T\mathbf{y}$ or $\mathbf{x}^T(\mathbf{c} - A^T\mathbf{y}) = 0$,
and $\mathbf{y}^T A\mathbf{x} = \mathbf{y}^T\mathbf{b}$ or $\mathbf{y}^T(A\mathbf{x} - \mathbf{b}) = 0$.
Since $\mathbf{x} \geq \mathbf{0}$ and $A^T\mathbf{y} - \mathbf{c} \geq \mathbf{0}$,
$\mathbf{x}^T(A^T\mathbf{y} - \mathbf{c}) = 0 \Rightarrow x_j((A^T\mathbf{y})_j - c_j) = 0$ for all $j = 1, 2, ..., n$.
Similarly since $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b} - A\mathbf{x} \geq \mathbf{0}$,
$\mathbf{y}^T(A\mathbf{x} - \mathbf{b}) = 0 \Rightarrow y_i((A\mathbf{x})_i - b_i) = 0$ for all $i = 1, 2, ..., m$.
Hence we get (1) and (1)*.

If we look at the application of this theorem to the diet problem then condition (1*) tells us that if the cost of the nutrients in a unit amount of F_j taken together, obtained by taking pills of the nutrients, is strictly less than the unit cost of F_j, that is c_j, then $x_j = 0$ or F_j need not be used in the diet. Similarly (1) says that if the amount of N_i obtained from all the food products taken together exceeds the required amount b_i, then no one will buy the pill of nutrient N_i, so it's cost given by y_i would eventually drop to 0.
If $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$ satisfy condition (1) and (1)*, then $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$ is said to satisfy the **complementary slackness property**.

The following example illustrates how complementary slackness theorem can be used to solve certain (not so complicated problems).

Example 1: Consider the following primal problem (P):
Max $4x_1 + 4x_2 + 2x_3$
subject to

$2x_1 + 3x_2 + 4x_3 \leq 10 \quad (i)$
$2x_1 + x_2 + 3x_3 \leq 4 \quad (ii)$
$x_1, x_2, x_3 \geq 0$

The dual (D) of the above problem is given by:

Min $10y_1 + 4y_2$
subject to
$2y_1 + 2y_2 \geq 4 \quad (i)'$
$3y_1 + y_2 \geq 4 \quad (ii)'$
$4y_1 + 3y_2 \geq 2 \quad (iii)'$
$y_1, y_2 \geq 0$

Let us try to obtain an optimal solution of the primal by starting with simple feasible solutions of the primal (P) and trying to find feasible solutions of the dual satisfying complementary slackness property with the feasible solutions of the primal.
Note that since the primal (P) is a maximization problem, and the increase in the value of the objective function for unit increase in x_1 and x_2 is twice as compared to x_3. Also from the inequalities it can be checked that the maximum value of x_3 is $\frac{4}{3}$, whereas the maximum value of x_1 is 2 and the maximum value of x_2 is $\frac{10}{3}$. Hence if only one of the variables is to remain positive, then it is more sensible to take either x_1 or x_2 to be positive.

So let us start by taking an $\mathbf{x} \in Fea(P)$ such that $x_1 > 0$ and $x_2 = x_3 = 0$, then note that inequality (i) of the primal cannot be satisfied as an equality by $\mathbf{x}$, hence if $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$ satisfy the **complementary slackness property**, then $y_1 = 0$. Also since $x_1 > 0$, if $\mathbf{x}$ and $\mathbf{y}$ satisfy the **complementary slackness property**, then $2y_1 + 2y_2 = 4$, which gives $y_2 = 2$, but then inequality $(ii)'$ of the dual cannot be satisfied by $\mathbf{y}$. Hence there exists no $\mathbf{y} = [y_1, y_2]^T \in Fea(D)$ which satisfies **complementary slackness property** with the above chosen $\mathbf{x} \in Fea(P)$, hence the optimal solution of (P) cannot be of the form $x_1 > 0$ and $x_2 = x_3 = 0$.

Suppose now if we take an $\mathbf{x} \in Fea(P)$ such that $x_2 > 0$ and $x_1 = x_3 = 0$, then note that inequality (ii) of the primal cannot be satisfied as an equality by $\mathbf{x}$, hence if $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$ satisfy the **complementary slackness property**, then, $y_2 = 0$. Also since $x_2 > 0$, if $\mathbf{x}$ and $\mathbf{y}$ satisfy the **complementary slackness property**, then $3y_1 + y_2 = 4$, which gives $y_1 = \frac{4}{3}$, but then such a $\mathbf{y}$ cannot satisfy inequality $(i)'$ of the dual. Hence there exists no $\mathbf{y} = [y_1, y_2]^T \in Fea(D)$ which satisfies **complementary slackness property** with the above chosen $\mathbf{x} \in Fea(P)$, hence the optimal solution of (P) cannot be of the form $x_2 > 0$ and $x_1 = x_3 = 0$.

Now suppose if we take both $x_1, x_2 > 0$ and $x_3 = 0$, then again if $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$ satisfy the **complementary slackness property**, then both inequality $(i)'$ and $(ii)'$ of the dual has to be satisfied as an equality by $\mathbf{y}$. Solving $(i)'$ and $(ii)'$ we get $y_1 = 1, y_2 = 1$.
The value of the objective function of the dual (D) for this $\mathbf{y}$ is equal to 14.
If $\mathbf{x}$ and $\mathbf{y}$ satisfy the **complementary slackness property**, then $\mathbf{x}$ has to satisfy inequality (i) and (ii) of (P) as an equality, hence on solving we get $x_2 = 3$ and $x_1 = \frac{1}{2}$.
The value of the objective function of the primal (P) for this $\mathbf{x}$ is also 14.
Hence either by **complementary slackness theorem** or by **corollary 2** we can conclude that $\mathbf{x}$ and $\mathbf{y}$ are optimal for the primal and the dual respectively.