Optimization

Jan-May, 2020

Quiz-1

Date: 6th February | Time: 6:00pm -7:30pm | Maximum marks: 15

Notation: I have used (a, b) some times to denote a column vector (in question 1).
$[a, b]$ is used many a times to denote a column vector (in question 2(a), I was lazy to put the transposes on many occasions) and sometimes a row vectorin question 2(b), I hope you will understand from the context.

1. Consider the following problem (LPP):

$$\min\ 3x_1 - x_2$$

$$\text{subject to} \quad -x_1 + 2x_2 \leq 4$$
$$-2x_1 + x_2 \leq 1$$
$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0, x_2 \geq 0.$$

(a) Write all the extreme points and the extreme directions of the above feasible region. (you can write them directly from the picture, you do not have to give any justification).
Soln: The extreme points are given by: $(1, 0), (0, 1), (\frac{2}{3}, \frac{7}{3})$.
A set of extreme directions are given by: $\mathbf{d}_1 = (1, 0), \mathbf{d}_2 = (2, 1)$.

(b) Check whether the above problem has an optimal solution. If yes, then give an optimal solution. If not, then justify.
Soln: Since $\mathbf{c}^T\mathbf{d}_1 = 3$ and $\mathbf{c}^T\mathbf{d}_2 = 5$ are both positive, and the given LPP is a minimization problem hence the problem has an optimal solution.

(c) If the above feasible region is written as $A_{3\times 2}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, then by changing **exactly** one entry in A, give an A' such that for all $\mathbf{c} \in \mathbb{R}^2$ the LPP,
$\min \mathbf{c}^T\mathbf{x}$, subject to $A'_{3\times 2}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$
has optimal solution ($\mathbf{b}$ remains unchanged). Justify.
Soln: If we just change a_{11} from -1 to $+1$, then the feasible region becomes nonempty (since $(1, 0) \in S$) and bounded hence for all $\mathbf{c} \in \mathbb{R}^2$ the LPP,
$\min \mathbf{c}^T\mathbf{x}$, subject to $A'_{3\times 2}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$
has optimal solution.
Changing a_{21} from -2 to $+1$ ($(1, 0) \in S$) will work.
Changing a_{31} from -1 to $+1$ ($(0, 1) \in S$) will also work.
(There are other choices for changes).

[2+2+1]

2. (a) Give a feasible region $S = \{\mathbf{x} \in \mathbb{R}^2 : A_{4\times 2}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ (the number of variables is 2), such that S has 6_{C_2} (=15) extreme points and two distinct extreme directions (distinct as directions). Briefly justify (you do not have to show linear independence of vectors).

Soln: When you are trying to construct an S of this type (6_{C_2} extreme points of a feasible region having 6 defining hyperplanes) then to start with, the defining hyperplanes should be such that any two defining hyperplanes (straight lines in this case) is LI (that is they should intersect).
Also it should have two distinct extreme directions.
So in general if S is defined by the closed half spaces given below :
$a_{11}x_1 + a_{12}x_2 \leq b_1$, $a_{21}x_1 + a_{22}x_2 \leq b_2$, $a_{31}x_1 + a_{32}x_2 \leq b_3$, $a_{41}x_1 + a_{42}x_2 \leq b_4$, $x_1 \geq 0$, $x_2 \geq 0$,
a_{ij}'s should be chosen such that then any two of the corresponding hyperplanes are LI, and the feasible region S is unbounded.
(For example a choice of such constraints could be:
$-x_1 - x_2 \leq b_1$, $-x_1 - 2x_2 \leq b_2$, $-x_1 - 3x_2 \leq b_3$, $-x_1 - 4x_2 \leq b_4$, $x_1 \geq 0$, $x_2 \geq 0$,
the corresponding hyperplanes being:
(i)$-x_1 - x_2 = b_1$,
(ii)$-x_1 - 2x_2 = b_2$,
(iii)$-x_1 - 3x_2 = b_3$,
(iv)$-x_1 - 4x_2 = b_4$
(v) $x_1 = 0$
(vi) $x_2 = 0$)
For all these points of intersections (between pairs of hyperplanes) to be extreme points they should also be in S.
So consider hyperplane H_i, $i = 1, \ldots, 4$ then H_i intersects $x_1 = 0$ and $x_2 = 0$. Let H_i intersect $x_1 = 0$ at some point other that $[0, 0]$ say $[0, b]$, $b > 0$,
($b < 0$ throws the point of intersection outside the feasible region) and intersects $\mathbf{x}_2 = 0$ at $[a, 0]$.
If $a > 0$, and $[0, 0] \in S$ then S is bounded, since for all $\mathbf{x} \in S$, $0 \leq x_1 \leq a$ and $0 \leq x_2 \leq b$.
If $a < 0$ then the point $[a, 0]$ is outside S , which is not allowed (all points of intersection should be inside the feasible region).
Hence in order that $[0, 0]$ and all points of intersection is inside the feasible region and S is unbounded H_i can intersect $x_2 = 0$ only at the origin and similarly H_i can intersect $x_1 = 0$ only at the origin.
Since all the hyperplanes pass through $[0, 0]$ all the points of intersection of any pair of hyperplanes H_i, H_j, $i, j \in \{1, 2, \ldots .6\}$ is also $[0, 0]$ and this is the only extreme point possible (with all the (6_{C_2} points of intersection equal), with two distinct extreme directions (since the resultant feasible region S in this case is a cone which is not a straight line (no two hyperplanes are parallel).
This is just my solution, your solution need not match with mine, but full credit is given for any solution as long as it is correct.
Aliter: Consider hyperplane H_i, $i \in \{1, \ldots, 4\}$ (not $x_1 = 0$ or $x_2 = 0$) then H_i should intersect $x_1 = 0$ and $x_2 = 0$ for S to have 6_{C_2} distinct extreme points. If H_i intersects at $[0, 0]$, then the number of distinct extreme points of S will be

strictly less than 6_{C_2}.
So let H_i intersect $x_1 = 0$ at $[0, b]$, $b > 0$,
($b < 0$ throws the point of intersection outside the feasible region and then again the number of extreme points of S will be strictly less than 6_{C_2})
and intersects $\mathbf{x}_2 = 0$ at $[a, 0]$.
If $a > 0$, and $[0, 0] \in S$ then S is bounded, since for all $\mathbf{x} \in S$, $0 \leq x_1 \leq a$ and $0 \leq \mathbf{x}_2 \leq b$, which gives a contradiction since S is unbounded.
If $a < 0$ then $[a, 0]$ is outside S , which is not allowed
(all points of intersection should be inside the feasible region otherwise the number of extreme points of S will be strictly less than 6_{C_2}).
Hence in any case there is no S which is unbounded and has 6_{C_2} distinct extreme points.
Aliter: Since H_1, H_2 has to intersect $x_1 = 0$, let these points be $[0, a]$ and $[0, b]$ $(a, b > 0)$, and let $[0, a], [0, b] \in S$.
WLOG let $a < b$, then $[0, a] = (1 - \frac{a}{b})[0, 0] + \frac{a}{b}[0, b]$, hence $[0, a]$ is not an extreme point of S (since it is written as a strict convex combination of two distinct points of S), hence S cannot have 6_{C_2} distinct extreme points.

(b) Does there exist any feasible region $S \subset \mathbb{R}^2$ (of an LPP in **two** variables), with three distinct extreme directions (distinct as directions)?
(Just writing yes or no will not fetch any marks).
Soln: No there cannot exist three extreme directions.
Let $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ be three directions of $S \subset \mathbb{R}^2$, then since any three vectors in $\mathbb{R}^2$ are LD, hence one of $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ can be written as a linear combination of the other two vectors.
WLOG let us assume $\mathbf{d}_3 = \alpha_1 \mathbf{d}_1 + \alpha_2 \mathbf{d}_2$, then both α_1, α_2 cannot be non positive, since $\mathbf{d}_3$ is a direction.
If both α_1, α_2 are nonnegative and only one is strictly positive, then $\mathbf{d}_3$ is same as direction to either $\mathbf{d}_1$ or $\mathbf{d}_2$.
If both α_1, α_2 are strictly positive, then $\mathbf{d}_3$ cannot be an extreme direction if $\mathbf{d}_1$ and $\mathbf{d}_2$ are distinct as directions.
If one of α_1, α_2 is positive and the other is negative, WLOG let $\alpha_1 > 0$ and $\alpha_2 < 0$, then $\mathbf{d}_1$ can be written as a positive linear combination of $\mathbf{d}_2, \mathbf{d}_3$, hence $\mathbf{d}_1$ cannot be an extreme direction if $\mathbf{d}_2$ and $\mathbf{d}_3$ are distinct as directions.
Hence in any case S cannot have three distinct extreme directions.
Aliter: The distinct extreme directions of S are extreme points of the set
$D' = \{\mathbf{d} \in \mathbb{R}^2 : \mathbf{d} \geq \mathbf{0}, A\mathbf{d} \leq \mathbf{0}, [1, 1]\mathbf{d} = 1\}$.
Since $D' \subset \{\mathbf{d} \in \mathbb{R}^2 : \mathbf{d} \geq \mathbf{0}, [1, 1]\mathbf{d} = 1\}$, is a bounded polyhedral set hence if D' is nonempty then D' is either a single point or a straight line segment of length $\leq \sqrt{2}$, hence it has atmost 2 extreme points. [2.5+ 2.5]

3. Suppose if **none** of the two problems P^1, P^2
P^1 : min $3x_1 - 5x_2$ subject to $A_{m\times 2}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$
P^2 : max $3x_1 - 5x_2$ subject to $A_{m\times 2}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$
have optimal solutions (the feasible region is same for both the problems),
then does there exist a $\mathbf{c} \neq \mathbf{0}$ such that **both** the problems P^3, P^4

P^3 : $\max \mathbf{c}^T\mathbf{x}$, subject to $A_{m\times 2}\mathbf{x} \le \mathbf{b}$ $\mathbf{x} \ge \mathbf{0}$
P^4 : $\min \mathbf{c}^T\mathbf{x}$, subject to $A_{m\times 2}\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge \mathbf{0}$
have optimal solutions (feasible region is same for all the above problems)?
(Just writing yes or no will not fetch any marks).
Soln: Since P_1 does not have a solution, hence either the feasible region is empty or there exists an extreme direction $\mathbf{d}_1$ such that $[3, -5]\mathbf{d}_1 < 0$.
Since P_2 does not have a solution, hence either the feasible region is empty or there exists an extreme direction $\mathbf{d}_2$ such that $[3, -5]\mathbf{d}_2 > 0$, hence $\mathbf{d}_2$ is distinct from $\mathbf{d}_1$ as **direction** which implies $\{\mathbf{d}_1, \mathbf{d}_2\}$ is LI.
If P^3 has a solution for some $\mathbf{c} \in \mathbb{R}^2$, then $\mathbf{c}^T\mathbf{d}_1 \le 0$, $\mathbf{c}^T\mathbf{d}_2 \le 0$.
If P^4 has a solution for the same $\mathbf{c}$ then $\mathbf{c}^T\mathbf{d}_1 \ge 0, \mathbf{c}^T\mathbf{d}_2 \ge 0$.
That is if both P^3, P^4 for some $\mathbf{c} \in \mathbb{R}^2$ then $\mathbf{c}^T\mathbf{d}_1 = \mathbf{c}^T\mathbf{d}_2 = 0$ which is not possible since $\{\mathbf{d}_1, \mathbf{d}_2\}$ is LI and $\mathbf{c} \ne \mathbf{0}$. [3]

4. Let $\mathbf{d} \in \mathbb{R}^2$ be an extreme direction of $S = \{\mathbf{x} \in \mathbb{R}^2 : A_{m\times 2}\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}$ then does there exist an extreme point $\mathbf{x}$ of S (lying) on H, where H is such that the normal(s) of H is orthogonal to $\mathbf{d}$? (Drawing pictures or giving examples will not fetch any marks) If yes, then how many **such** extreme points can S have?
Soln: Let the hyperplane H be given by $\tilde{\mathbf{a}_i}^T\mathbf{x} = \tilde{b_i}$ such that $\tilde{\mathbf{a}_i}^T\mathbf{d} = 0$ and let $\mathbf{x} \in S$ be such that $\tilde{\mathbf{a}_i}^T\mathbf{x} = \tilde{b_i}$.
(If there is no $\mathbf{x} \in S$ which belongs to H then there is no question of finding an extreme point of S in H.)
If the chosen $\mathbf{x} \in S \cap H$ is an extreme point of S then done, if not then consider points of the form $\mathbf{x} - \alpha\mathbf{d}$, $\alpha > 0$. Since $\mathbf{d} \ge \mathbf{0}$, $\mathbf{d} \ne \mathbf{0}$, atleast one component of $\mathbf{x} - \alpha\mathbf{d}$ will for be < 0 for α sufficiently large.
Since $\mathbf{x}$ is not an extreme point, there exists an $\alpha > 0$ such that $\mathbf{x} - \alpha\mathbf{d} \in S$, let $\gamma = \max\{\alpha : (\mathbf{x} - \alpha\mathbf{d}) \in S\}$, then $\gamma > 0$ and there exists a hyperplane H_j with normal $\tilde{\mathbf{a}_j}$ such that
$\tilde{\mathbf{a}_j}^T(\mathbf{x} - \alpha\mathbf{d}) \le \tilde{b_j}$ for all $\alpha \le \gamma$
$\tilde{\mathbf{a}_j}^T(\mathbf{x} - \alpha\mathbf{d}) > \tilde{b_j}$ for all $\alpha > \gamma$
and $\tilde{\mathbf{a}_j}^T(\mathbf{x} - \gamma\mathbf{d}) = \tilde{b_j}$,
hence $\tilde{\mathbf{a}_j}$ clearly cannot be orthogonal to $\mathbf{d}$, which implies $\{\tilde{\mathbf{a}_i}, \tilde{\mathbf{a}_j}\}$ is LI and $\mathbf{x} - \gamma\mathbf{d}$ is an extreme point of S lying on H_i and H_j.
There cannot be more than one extreme point of S on H.
To see this let $\mathbf{x}_1, \mathbf{x}_2 \in S$ be two points lying on H, then $\tilde{\mathbf{a}_i}^T(\mathbf{x}_2 - \mathbf{x}_1) = 0$, since $\tilde{\mathbf{a}_i} \in \mathbb{R}^2$ is a nonzero vector and $\tilde{\mathbf{a}_i}^T\mathbf{d} = 0$, hence $(\mathbf{x}_2 - \mathbf{x}_1) = \alpha\mathbf{d}$ for some $\alpha \in \mathbb{R}$, WLOG let $\alpha > 0$.
Then $\mathbf{x}_2 = (\mathbf{x}_1 + \alpha\mathbf{d}) = \frac{1}{2}(\mathbf{x}_1) + \frac{1}{2}(\mathbf{x}_1 + 2\alpha\mathbf{d})$. Since $\mathbf{x}_1 \in S$, $\mathbf{x}_1 + 2\alpha\mathbf{d} \in S$ (because $\mathbf{d}$ is a direction of S), hence $\mathbf{x}_2$ cannot be extreme points of S (if $\alpha < 0$, then consider $\mathbf{x}_1 - \mathbf{x}_2 = -\alpha\mathbf{d}$, then $\mathbf{x}_1$ will not be an extreme point of S).
Hence there cannot be more than one extreme point of S on H.
Aliter: (for the second part, which is essentially same as the first one) If $\mathbf{x}_1, \mathbf{x}_2$ are two extreme points of S on H then since $\tilde{\mathbf{a}_i} \in \mathbb{R}^2$ is a nonzero vector and $\tilde{\mathbf{a}_i}^T\mathbf{d} = 0$, hence $(\mathbf{x}_2 - \mathbf{x}_1) = \alpha\mathbf{d}$ for some $\alpha \in \mathbb{R}$, WLOG let $\alpha > 0$. Since $\mathbf{x}_2 + \alpha\mathbf{d} \in S$ for all $\alpha > 0$ ($\mathbf{d}$ is a direction), $\mathbf{x}_2 - \alpha\mathbf{d} = \mathbf{x}_1$ does not belong to S for any $\alpha > 0$ (since $\mathbf{x}_2$ is an extreme point) which is a contradiction.

[2+2]