1. For a linear programming problem (P) of the form,

$$\text{Maximize } \mathbf{c}^T\mathbf{x}$$

$$\text{subject to } \mathbf{A}_{2\times 4}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},$$

where $\tilde{\mathbf{a}_1} = [3,5]^T, \tilde{\mathbf{a}_2} = [1,1]^T, \tilde{\mathbf{a}_3} = [4,6]^T,\ \tilde{\mathbf{a}_4} = [-2,7]^T,\ \mathbf{b} = [3,4]^T$ and $\mathbf{c} = [3,2,5,8]^T$.

(a) Construct the simplex table corresponding to the basis matrix $B = [\tilde{\mathbf{a}_3}, \tilde{\mathbf{a}_2}]$ and check whether this basic feasible solution is optimal.

(b) Does this problem (P) have a degenerate basic feasible solution? [4+2]

Soln:

(a)

$c_j - z_j$	0	0	0	$\frac{51}{2}$	
	$B^{-1}\tilde{\mathbf{a}_1}$	$B^{-1}\tilde{\mathbf{a}_2}$	$B^{-1}\tilde{\mathbf{a}_3}$	$B^{-1}\tilde{\mathbf{a}_4}$	$B^{-1}\mathbf{b}$
$\tilde{a_3}$	1	0	1	$\frac{9}{2}$	$\frac{1}{2}$
$\tilde{a_2}$	-1	1	0	-20	1

The above basic feasible solution is not optimal because $c_4 - z_4 > 0$, and since the BFS is nondegenerate, so by entering $\tilde{\mathbf{a}_4}$ in the basis in place of $\tilde{\mathbf{a}_3}$ clearly gives a BFS with higher value of the objective function, $\mathbf{c}^T\mathbf{x}$.

(b) It does not have a degenerate feasible solution, since $\mathbf{b}$ cannot be written as a scalar multiple of any column of A.

2. Using the result (**given below**) that exactly one of the following two systems has a solution:

$$A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0} \qquad (1)$$

$$\mathbf{y}^T A \geq \mathbf{0}, \quad \mathbf{y}^T\mathbf{b} < 0, \qquad \mathbf{y} \geq \mathbf{0} \qquad (2),$$

prove that if both the Primal (P) and its Dual (D) have optimal solutions then the optimal values for the two problems (P) and (D) are equal. [5]

3. Consider the linear programming problem (P):
Min $\mathbf{c}^T\mathbf{x}$, $\quad (\mathbf{c} \neq \mathbf{0})$
subject to $A_{3\times 4}\mathbf{x} = \mathbf{b}_{3\times 1},\ \mathbf{x} \geq \mathbf{0}$, where $rank(A) = 3$.

(a) If $\mathbf{x}_0 = [4,7,0,1]^T$ is an optimal solution of (P) and $\mathbf{b}$ is changed to $2\mathbf{b} + 3\tilde{\mathbf{a}_1} - 2\tilde{\mathbf{a}_2}$, (everything else remaining same in (P)) then if possible find an optimal solution of the new problem.
Soln: Since $A[4,7,0,1]^T = 4\tilde{\mathbf{a}_1} + 7\tilde{\mathbf{a}_2} + \tilde{\mathbf{a}_3} = \mathbf{b}$,
$2\mathbf{b} + 3\tilde{\mathbf{a}_1} - 2\tilde{\mathbf{a}_2} = 2A[4,7,0,1]^T + A[3,0,0,0]^T + A[0,0,0,-2]^T = A[11,12,0,2]^T$.
Hence $[11,12,0,2]^T$ is feasible for the new problem.
Let $\mathbf{y}_0$ be the optimal solution of (P), then since $\mathbf{x}_0$ and $\mathbf{y}_0$ satisfy must satisfy complementary slackness property, so $(A^T\mathbf{y}_0)_1 = (A^T\mathbf{y}_0)_2 = (A^T\mathbf{y}_0)_4 = 0$.
Note that $\mathbf{y}_0$ is again feasible for the Dual of the changed problem. Since $[11,12,0,2]^T$

satisfies complementary slackness with $\mathbf{y}_0$ which is again feasible for the Dual of the new problem, so $[11, 12, 0, 2]^T$ is optimal for the new problem.

(b) If in (P), $\tilde{\mathbf{a}_1} = [r, 2, -1]^T$, $\tilde{\mathbf{a}_2} = [-1, 3, s]^T$, $\tilde{\mathbf{a}_3} = [2, 1, t]^T$, $\tilde{\mathbf{a}_4} = [p, 1, -2]^T$ and $\mathbf{x} = [3, a, 2, e]^T \in Fea(P)$, then does (P) have an optimal solution?
Soln: The Primal (P) has atleast one feasible solution so Fea(P) is nonempty. Also consider the second equation of the above system of equations
$2x_1 + 3x_2 + x_3 + x_4 = b_2$. Since $x_i \geq 0$ for all $i = 1, \ldots, 4$ we get that Fea(P) is bounded, hence by Weierstrass's theorem (P) has an optimal solution.
Aliter: The feasible region of the Dual is given by $A^T\mathbf{y} \leq \mathbf{c}$, ($\mathbf{y}$ unrestricted in sign).
Since the second column of A^T is a nonnegative vector, hence by choosing $\mathbf{y} = [0, -\alpha, 0]^T$, where $\alpha > 0$ is sufficiently large, we can get a feasible solution of the Dual. Since both the Primal and its Dual have feasible solutions, hence both must have optimal solutions.

(c) If possible find a problem of the type (P), such that for all $m \in \mathbb{R}$
$\mathbf{c}^T\mathbf{x} = m$, for some $\mathbf{x} \in Fea(P)$.
Soln: If we can give directions $\mathbf{d}_1$ and $\mathbf{d}_2$ of $Fea(P)$ such that $\mathbf{c}^T\mathbf{d}_1 < 0$ and $\mathbf{c}^T\mathbf{d}_2 > 0$, then for any $n \in \mathbb{R}$ there exists an $\mathbf{x}_n \in Fea(P)$ such that $\mathbf{c}^T\mathbf{x} < n$ and for any $N \in \mathbb{R}$ there exists an $\mathbf{x}_N \in Fea(P)$ such that $\mathbf{c}^T\mathbf{x} > N$.
The problem given below is same as the first problem given in **quiz -1**,
Consider the following problem:

$$\begin{aligned} \text{Min } & -x_1 + 3x_2 \\ \text{subject to } & \quad x_1 + 2x_2 \geq 4 \\ & -x_1 + 2x_2 \leq 4 \\ & -2x_1 + 8x_2 \geq 8 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Add variables s_1, s_2, s_3 to get equations for the above three inequalities.
Then the above problem will be of the form given by (P) with six variables. Check that the above problem satisfies the conditions mentioned above.
But the situation changes if you want to get a similar problem in four variables with $rank(A) = 3$. If for every $m \in \mathbb{R}$, there exists an $\mathbf{x}_m \in Fea(P)$ such that $\mathbf{c}^T\mathbf{x}_m = m$ then neither the problem, Min$\mathbf{c}^T\mathbf{x}$ nor the problem, Max$\mathbf{c}^T\mathbf{x}$ has optimal solution, that means there exists atleast two directions $\mathbf{d}_1$ and $\mathbf{d}_2$ of $Fea(P)$ such that $\mathbf{c}^T\mathbf{d}_1 < 0$ and $\mathbf{c}^T\mathbf{d}_2 > 0$.
But since $rank(A) = 3$ and A has four columns, there exists atmost one LI direction $\mathbf{d}$ such that $A\mathbf{d} = \mathbf{0}$, ($\mathbf{d} \geq \mathbf{0}$). Hence for all $\alpha > 0$, $\mathbf{c}^T(\alpha\mathbf{d})$ is either non negative or non positive, which contradicts the fact that neither the problem, Min$\mathbf{c}^T\mathbf{x}$ nor the problem, Max$\mathbf{c}^T\mathbf{x}$ has optimal solution.
(**All the parts in the above question are independent**) [3+3+4]

4. Consider the LPP, Max $\mathbf{c}^T\mathbf{x}$, subject to $A_{3\times5}\mathbf{x} = \mathbf{b}_{3\times1}$, $\mathbf{x} \geq \mathbf{0}$, with the following table corresponding to a BFS $\mathbf{x}_0$.

$c_j - z_j$		-1	1			
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\tilde{\mathbf{a}}_4$	$B^{-1}\tilde{\mathbf{a}}_5$	$B^{-1}\mathbf{b}$
$\tilde{a}_1$		-1	$-\frac{1}{2}$			2
$\tilde{a}_4$		$+2$				1
		-3				0

(a) Give the BFS $\mathbf{x}_0$, and give the number of defining hyperplanes on which it lies.
Soln: $\mathbf{x}_0 = [2, 0, 0, 1, 0]^T$. It lies on six defining hyperplanes, three of which are given by $A_{3\times5}\mathbf{x} = \mathbf{b}_{3\times1}$ and the rest are $x_2 = x_3 = x_5 = 0$.

(b) If possible give a $\mathbf{d} \neq \mathbf{0}$ such that $A\mathbf{d} = \mathbf{0}$. If yes, then what is the value of $\mathbf{c}^T\mathbf{d}$?
Soln: $\mathbf{d} = [1, 1, 0, -2, 3]^T$ satisfies $A\mathbf{d} = \mathbf{0}$, $\mathbf{c}^T\mathbf{d} = -1$.

(c) If possible give another BFS which also lies on two of the defining hyperplanes on which $\mathbf{x}_0$ lies.
Soln: $\mathbf{x}_1 = [\frac{5}{2}, \frac{1}{2}, 0, 0, \frac{3}{2}]^T$, which lies on 5 hyperplanes, 4 of which also passes through $\mathbf{x}_0$.

(d) If possible change the value of **exactly one** c_i such that the above table becomes optimal.
Soln: Change c_3 to $c_3 - k$, where $k \geq 1$.

(e) If possible give a feasible $\mathbf{x}$ such that $x_1 > 0$, $x_2 > 0$, $x_4 > 0$ and $x_5 > 0$.
Soln: Consider $\mathbf{x} = \mathbf{x}_0 + \alpha\mathbf{d}$, where $0 < \alpha < \frac{1}{2}$ and $\mathbf{d}$ is as in part(b).
Or take $\mathbf{x} = \frac{1}{2}\mathbf{x}_0 + \frac{1}{2}\mathbf{x}_1$.

(f) Does there exist a feasible $\mathbf{x}$ such that $x_i > 0$ for all $i = 1, 2, 3, 4, 5$?
Soln: Yes.
Take $\mathbf{x} = \mathbf{x}_0 + \alpha\mathbf{d} + \beta\mathbf{d}_0 = [2, 0, 0, 1, 0]^T + \alpha[1, 1, 0, -2, 3]^T + \beta[\frac{1}{2}, 0, 1, p, q]^T$
where $\alpha < \frac{1}{2}$, $\mathbf{d}_0 = [\frac{1}{2}, 0, 1, p, q]^T$ is obtained from the column corresponding to $\tilde{\mathbf{a}}_3$ in the simplex table and p, q denotes the unknown entries.
Hence $\beta > 0$, can be chosen sufficiently small such that $\mathbf{x}$ has all the components strictly positive and also since $A\mathbf{d} = A\mathbf{d}_0 = 0$ and $A\mathbf{x}_0 = \mathbf{b}$ so $A\mathbf{x} = \mathbf{b}$.

(g) If possible, without changing any existing entry, fill up **exactly one** missing entry in the above table (**there may be various ways of filling it**), such that $\mathbf{c}^T\mathbf{x}_0$ is the optimal value for (P).
Soln: Take $(B^{-1}\tilde{\mathbf{a}}_3)_3 = 6$. If you pivot on this entry check that the BFS does not change hence the value of the objective function $\mathbf{c}^T\mathbf{x}$ remains same, although we get a different basis $\{\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_4\tilde{\mathbf{a}}_3\}$.
Also verify that all the $c_j - z_j$ values are nonpositive in the next table.
(**All the parts in the above question are independent**) [2+2+2+2+2+2+3]