Optimization

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Unanswered questions and their solutions:

To show that if the feasible region of a linear programming problem is unbounded then it should have atleast one direction.

Proof: Let us assume representation theorem to be true for all nonempty bounded feasible regions.

Let S have extreme points $\mathbf{x}_1, \ldots, \mathbf{x}_r$ and consider $S \cap H$, where $H = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i \leq M\}$ and $M > 0$ is such that $\mathbf{x}_1, \ldots, \mathbf{x}_r \in S \cap H$ but none of $\mathbf{x}_1, \ldots, \mathbf{x}_r$ lie on the hyperplane corresponding to H.

Let $\mathbf{x}_0 \in S$ be such that $M \geq \sum_{i=1}^n \mathbf{x}_{0i} > \max\{\sum_{i=1}^n x_{ki} : k = 1, \ldots, r\}$ (there exists such an $\mathbf{x}_0$, why?).

Check that this $\mathbf{x}_0$ **cannot** be written as a convex combination of the extreme points of S (check this), but it can be written as a convex combination of the extreme points of $S \cap H$, at least one of which (call that extreme point $\mathbf{u}$) must lie on the hyperplane corresponding to H.

Then by repeating the proof of representation theorem (for unbounded feasible region) conclude that $\mathbf{u}$ lies on an extreme direction of S, that extreme direction is orthogonal to the normals of all the hyperplanes on which $\mathbf{u}$ lies except the hyperplane corresponding to H.

Problem 23(a): Let $\mathbf{d}$ be an extreme direction of $Fea(LPP)$ (that is $\mathbf{d}$ is a direction and is orthogonal to the normals of a set of $(n-1)$ LI hyperplanes defining the feasible region $Fea(LPP)$) then does there always exist an extreme point $\mathbf{x}$ of $Fea(LPP)$ such that $\mathbf{x}$ lies on $(n-1)$ LI hyperplanes, the normals to which are orthogonal to $\mathbf{d}$?

Proof: Suppose there exists no extreme point which satisfies $\tilde{\mathbf{a}_i}^T\mathbf{x} = \tilde{b}_i$ for $i = 1, 2, \ldots, (n-1)$, where $\tilde{\mathbf{a}_1}, \ldots, \tilde{\mathbf{a}_{n-1}}$ is such that $\tilde{\mathbf{a}_i}^T\mathbf{d} = 0$ for all $i = 1, 2, \ldots, n-1$.

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ be the extreme points of S.

Now consider $S \cap H_1$, where $H_1 = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i \leq M_1\}$ is such that all the extreme points of S are in $S \cap H_1$ but none of $\mathbf{x}_1, \ldots, \mathbf{x}_r$ satisfy the equation $\sum_{i=1}^n x_i = M_1$.

Then all $\mathbf{x} \in S \cap H_1$ can be written as a convex combination of the extreme points of $S \cap H_1$.

If $\mathbf{x}_0 + \alpha\mathbf{d}$ is an extreme point of $S \cap H_1$ for some $\mathbf{x}_0 \in S$ and some $\alpha > 0$, then $\mathbf{x}_0 + \alpha\mathbf{d}$ satisfies the equation $\sum_{i=1}^n x_i = M_1$, (since $\mathbf{x}_0 \in S$ and $\mathbf{d}$ is a direction) hence either $\mathbf{x}_0$ is itself an extreme point of S which satisfies $\tilde{\mathbf{a}_i}^T\mathbf{x} = \tilde{b}_i$ for $i = 1, 2, \ldots, (n-1)$, or by starting from $\mathbf{x}_0 + \alpha\mathbf{d}$ (or from $\mathbf{x}_0$ if it is not an extreme point of S) and moving towards $-\mathbf{d}$, we would have been able to get an extreme point of S, which satisfies $\tilde{\mathbf{a}_i}^T\mathbf{x} = \tilde{b}_i$ for $i = 1, 2, \ldots, (n-1)$, which is in contradiction to our assumption.

If $\mathbf{d}$ is the only distinct extreme direction of S then since by the previous argument $S \cap H_1$ has atleast one extreme point which is not an extreme point of S, that extreme point will be of the form $\mathbf{x} + \alpha\mathbf{d}$ for some $\mathbf{x} \in S$, $\alpha > 0$ (recall proof of representation theorem). Starting from $\mathbf{x} + \alpha\mathbf{d}$ and moving towards $-\mathbf{d}$ will give us an extreme point of S satisfying $\tilde{\mathbf{a}_i}^T\mathbf{x} = \tilde{b}_i$ for $i = 1, 2, \ldots, (n-1)$, which is a contradiction to our assumption.

Hence S has more than one distinct extreme direction and let $\mathbf{d}, \mathbf{d}_1, \ldots, \mathbf{d}_s$ be a set of all distinct extreme directions of S(you can take them to be elements of D') Hence all the extreme points of $S \cap H_1$ are either extreme points of S or elements of the form $\mathbf{x}_i + \alpha_i\mathbf{d}_j$, for some $i \in \{1, \ldots, r\}$, some $j \in \{1, 2, \ldots, s\}$ (where $\mathbf{d}_j$'s are extreme directions each of which

is orthogonal to normals of $(n-1)$ LI defining hyperplanes, out of n LI hyperplanes on which the corresponding $\mathbf{x}_i$ lies, there can be more than one such set of n LI hyperplanes).
Consider $\mathbf{x}_1+\alpha\mathbf{d}$, $\alpha>0$ such that $\mathbf{x}_1+\alpha\mathbf{d}$ lies on the hyperplane corresponding to H_1 (there must exist one such $\alpha>0$, note here $\mathbf{x}_1$ is one of the extreme points of S as listed before).
Then $\mathbf{x}_1+\alpha\mathbf{d}$ can be expressed as a convex combination of the extreme points of $S\cap H_1$ (but not as a convex combination of the extreme points of S), or in other words $\mathbf{x}_1+\alpha\mathbf{d}$ can be written as a convex combination of the extreme points of S plus some direction $\bar{\mathbf{d}}$, where $\bar{\mathbf{d}}$ is a nonnegative linear combination of the directions $\mathbf{d}_1,\ldots,\mathbf{d}_s$ of S (excluding $\mathbf{d}$).
That is $\mathbf{x}_1+\alpha\mathbf{d}=\bar{\mathbf{x_1}}+\bar{\mathbf{d}_1}=\bar{\mathbf{x_1}}+\Sigma_{j=1}^{s}\mu_{j1}\mathbf{d}_j$, where $\bar{\mathbf{x_1}}$ is a convex combination of extreme points $\mathbf{x}_1,\ldots,\mathbf{x}_r$ of S and $\mu_{j1}\geq 0$ for all $j=1,\ldots,s$.
We now denote $\alpha\mathbf{d}$ by $\mathbf{d}$ itself (since anyway as direction it is same as $\mathbf{d}$), hence we get $\mathbf{x}_1+\mathbf{d}=\bar{\mathbf{x_1}}+\Sigma_{j=1}^{s}\mu_{j1}\mathbf{d}_j$.
Hence by similar argument for all $k\in\mathbf{N}$, there exists M_k and corresponding $H_k=\{\mathbf{x}\in\mathbb{R}^n:\sum_{i=1}^{n}x_i\leq M_k\}$, where $M_k>M_{k-1}>\ldots>M_1$ such that $\mathbf{x}_1+k\mathbf{d}$ satisfies the equation $\sum_{i=1}^{n}x_i=M_k$(obviously then $\mathbf{x}_1+k\mathbf{d}\in S\cap H_k$).
Hence $\mathbf{x}_1+k\mathbf{d}=\bar{\mathbf{x}_k}+\bar{\mathbf{d}_k}=\bar{\mathbf{x}_k}+\Sigma_{j=1}^{s}\mu_{jk}\mathbf{d}_j$, where $\bar{\mathbf{x}_k}$ is a convex combination of the extreme points $\mathbf{x}_1,\ldots,\mathbf{x}_r$ of S and $\mu_{jk}\geq 0$ for all $j=1,\ldots,s$, which implies
$\|\mathbf{x}_1-\bar{\mathbf{x}_k}\|=k\|\mathbf{d}-\frac{1}{k}\Sigma_{j=1}^{s}\mu_{jk}\mathbf{d}_j\|$. (here $\|\cdot\|$ denotes Euclidean length) (**)
Since $\mathbf{x}_1$ and $\bar{\mathbf{x}_k}$, for all $k\in\mathbf{N}$ belong to the same bounded polyhedral set (obtained by taking all possible convex combinations of extreme points of S) there exists an $M>0$ such that $\|\mathbf{x}_1-\bar{\mathbf{x}_k}\|<M$ for all $k\in\mathbf{N}$.
Also since for all k, $\frac{1}{k}\Sigma_{j=1}^{s}\mu_{jk}\mathbf{d}_j$ belongs to the **closed** convex cone generated by $\mathbf{d}_1,\ldots,\mathbf{d}_s$, and $\mathbf{d}$ does not belong to this cone (since $\mathbf{d}$ is an extreme direction), there exists a $\gamma>0$ such that for all k, $\|\mathbf{d}-\frac{1}{k}\Sigma_{j=1}^{s}\mu_{jk}\mathbf{d}_j\|>\gamma$ (that is there is positive distance between a closed set and an element not in that closed set).
Hence the RHS of (**) is strictly bigger than $k\gamma$, whereas the LHS of (**) $<M$ (for all k) hence by choosing k large we arrive at a contradiction.
Hence there must exist atleast one extreme point which satisfies $\tilde{\mathbf{a}_i}^T\mathbf{x}=\tilde{b}_i$ for $i=1,2,\ldots,(n-1)$, where $\tilde{\mathbf{a}_1},\ldots,\tilde{\mathbf{a}_{n-1}}$, are normals to defining hyperplanes of S such that $\tilde{\mathbf{a}_i}^T\mathbf{d}=0$ for all $i=1,2,\ldots,n-1$.