Practise problems 4

Note that for all the problems $\mathbf{a}_k$ denotes the k-th column of A and $\tilde{\mathbf{a}}_k$ denotes the k-th row of A.

1. **(Repeated from practise 2)**Let $\mathbf{d}$ be an extreme direction of $Fea(LPP)$ (that is $\mathbf{d}$ is a direction and is orthogonal to the normals of $(n-1)$ LI hyperplanes defining the feasible region $Fea(LPP)$) then does there always exist an extreme point $\mathbf{x}$ of $Fea(LPP)$ such that $\mathbf{x}$ lies on $(n-1)$ LI hyperplanes, to which $\mathbf{d}$ is orthogonal?

2. Consider the following linear programming problem (P) given below:

$$\text{Maximize } -x_1 + 2x_2$$

$$\begin{aligned} \text{subject to} \quad & -x_1 + x_2 \le 1 \\ & x_1 + x_2 \le 7 \\ & x_1 + 3x_2 \le 15 \\ & x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

   (a) Write the basic variables and the basis matrix corresponding to the extreme point $[7, 0]^T$ and obtain the optimal solution starting from this table by using simplex algorithm.

   (b) Write the basic variables and the basis matrix of the extreme point of the dual which corresponds to the extreme point $[7, 0]^T$ and using the dual simplex algorithm starting from this table (that is the table of the dual extreme point, its corresponding BFS) obtain the optimal solution.

   (c) At each step compare the $c_j - z_j$ values and the values of the basic variables in the simplex table with the corresponding values in the respective dual simplex table, and also compare the corresponding pivot elements.

   (d) Repeat the above three steps by starting from the extreme point $[0, 1]^T$ of the problem (P).

   (e) If R denotes the the row corresponding to $c_j - z_j$ values of the optimal table of problem (P) and R_i gives the ith row of the simplex table (leaving out the RHS entries) then for each i if possible choose an $\alpha_i < 0$ such that $R + \alpha_i R_i$ again provides a feasible solution of the dual. Verify that it is indeed a feasible solution of the dual.
   (Note that this is a maximization problem, for a minimization problem the question should have been to choose $\alpha_i > 0$, also note that R_i is of the form $[e_i^T : (B^{-1}).iN]$, where $(B^{-1}).i$ is the ith row of B^{-1}).

   (f) By changing the RHS of the last constraint from 15 to 3, check whether the dual of the new (P) has alternate optimal solutions, by looking at the corresponding table for the BFS of the dual.

   (g) Change the vector RHS vector $b = [1, 7, 15]^T$ and the cost vector $c = [-1, 2]^T$ such that $[0, 1]^T$ is optimal, lies at the point of intersection of three lines but both the primal(P) and its dual has a unique optimal solution.

(h) Change the column entries corresponding to the variable x_1 from $[-1,1,1]^T$ to $[-1,2,3]^T$ and find the new optimal solution.

(i) Change the RHS of the second constraint from 7 to 10, from 7 to 6 and guess the optimal solution, without calculating. Similarly guess the optimal solution when RHS of the first constraint is changed from 1 to 3 and from 1 to 0, and indicate whenever the corresponding dual has more than one optimal solution.

3. For a linear programming problem (P) of the form,

$$\begin{aligned}&\text{Minimize } \mathbf{c}^T\mathbf{x}\\ &\text{subject to } \mathbf{A}_{2\times 3}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\end{aligned}$$

where $\tilde{\mathbf{a}}_1 = [1,1]^T, \tilde{\mathbf{a}}_2 = [1,2]^T, \tilde{\mathbf{a}}_3 = [0,1]^T$, $\mathbf{b} = [2,3]^T$ and $\mathbf{c} = [2,1,1)^T$.

(a) Construct the simplex table corresponding to the basis matrix $B = [\mathbf{a}_1, \mathbf{a}_3]$ and find the corresponding basic solution and the basis matrix $B^{'}$ of the dual of (P).

(b) In general for an A of order $m \times n$, where $A = [B : N]$ and B is the basis matrix corresponding to some BFS of (P), find the basic variables and the basis matrix B' for the corresponding basic solution of the dual.

4. Consider the following problem:
Min $\mathbf{b}^T\mathbf{y}$
subject to $\mathbf{A}^T_{n\times m}\mathbf{y} \geq \mathbf{c}$.
If every $\mathbf{y}$ feasible for this problem satisfies the condition that at most m components of $\mathbf{c} - \mathbf{A}^T y$ is equal to 0, then the dual of this problem has at most one optimal solution. (Hint: Note that $\mathbf{y}$ is unrestricted in sign. Write $\mathbf{y} = \mathbf{w} - \mathbf{z}$, where $\mathbf{w} \geq \mathbf{0}, \mathbf{z} \geq \mathbf{0}$, then obtain the dual of this problem.)

5. Consider a linear programming problem (P) of the form,

$$\begin{aligned}&\text{Minimize } \mathbf{c}^T\mathbf{x}\\ &\text{subject to } A_{m\times 2}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.\end{aligned}$$

If every extreme point of $Fea(P)$ lies at the point of intersection of exactly 2 LI defining hyperplanes of $Fea(P)$, then what can you say about the number of optimal solutions of the dual of (P)?

6. For a linear programming problem (P) of the form,

$$\begin{aligned}&\text{Minimize } \mathbf{c}^T\mathbf{x}\\ &\text{subject to } A_{m\times n}\mathbf{x} \geq \mathbf{b}(\mathbf{b} \geq \mathbf{0}),\ \mathbf{x} \geq \mathbf{0},\end{aligned}$$

check the correctness of the following statements with proper justification.

(a) For $n = 2$ if the optimal solution of (P) lies at the point of intersection of more than two distinct lines given by the constraints of the feasible region, then the dual of (P) has infinitely many solutions.

(b) If $\mathbf{c}$ is a nonpositive vector (all the components of $\mathbf{c}$ are less than or equal to 0) then the dual simplex method can be used to solve this problem with the initial basis as $-I$.

(c) If the dual simplex method can be used to solve the above problem and if the value of the objective function of the primal (P) corresponding to a table (of the dual-simplex algorithm) is 3, then the value of the objective function of the primal in the next iteration is ≥ 3.

(d) If x_1 is a basic variable in an optimal basic feasible solution of the given problem and if after decreasing the value of c_1 the modified problem has an optimal solution, then there exists an optimal basic feasible solution having x_1 as a basic variable.

(e) If a new variable is added to the problem (P) then the optimal value of the new problem is greater than or equal to the optimal value of the given problem.

(f) If a new constraint is added to the problem (P) then the optimal value of the new problem is greater than or equal to the optimal value of the given problem.

7. If we change problem (P) to a maximization problem and solve it by dual simplex method then give the rule for the entering variable.

8. Solve the following problem (P) by adding artificial variables with cost M, where M is large.
Minimize $3x - 2y$
subject to $-x + y \geq 2$
$x + y \geq 4$
$y \leq 3$
$x \geq 0, y \geq 0$.
(Hint: Convert the above inequality constraints into equalities by subtracting s_1 and s_2 from the first two constraints and adding s_3 to the last. Then add artificial variables in the first and the second constraint **only** to get the initial basic feasible solution.)

(a) Show that if $M \leq 1$ then although (P) has an optimal solution, the optimal solution of $P(M)$ will have at least one artificial variable as nonzero.

(b) What should be the minimum value of M such that the optimal solution of P(M) should have all the artificial variables at zero value?

9. Consider the problem:
Minimize $2x + 3y$
subject to $x + 2y \geq 2$
$3x + 3y \leq 2$
$x \geq 0, y \geq 0$.
By using dual simplex method find the direction along which the objective function of the dual is unbounded.

10. Consider the following linear programming problem and its optimal final tableau shown below.

$$\begin{aligned} &\text{Maximize } 2x_1 + x_2 - x_3 \\ &\text{subject to } x_1 + 2x_2 + x_3 \leq 8, \\ &\qquad\qquad -x_1 + x_2 - 2x_3 \leq 4, \\ &\qquad\qquad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{aligned}$$

	x_1	x_2	x_3	s_1	s_2	b
$z_j - c_j$	0	3	3	2	0	
x_1	1	2	1	1	0	8
s_2	0	3	-1	1	1	12

(a) If the coefficient of x_2 in the objective function is changed from 1 to 6, then find the new optimal solution.

(b) If the coefficient of x_2 in the first constraint is changed from 2 to $\frac{1}{4}$, find the new optimal solution.

(c) if the coefficient of x_1 in the first constraint is changed from 1 to 0, find the new optimal solution.

(d) If the constraint $x_2 + x_3 = 3$ is added to the problem then find the new optimal solution.

(e) If you were to choose between increasing the right hand side of the first and second constraints, which one would you choose? Why? What will be the effect of this change in the value of the objective function.

(f) Suppose a new variable x_4 is added with $c_4 = 4$ and $a_4 = (1, 2)^T$, find the new optimal solution.

11. For a balanced transportation problem of the form given below,

$$\text{Minimize } \sum_i \sum_j c_{ij} x_{ij}$$

$$\text{subject to } \sum_i x_{ij} = b_j, \quad j = 1, 2, ..., n,$$

$$\text{subject to } \sum_j x_{ij} = a_i, \quad i = 1, 2, ..., m,$$

$$\mathbf{x} \geq \mathbf{0},$$

with $\sum_i a_i = \sum_j b_j$, check the correctness of the following statements with proper justification.

(a) If $\mathbf{B}$ is a basis matrix of the above problem then the entries of $\mathbf{B^{-1}}$ are either $1, -1$ or 0.

(b) If corresponding to a optimal basic feasible solution $\mathbf{x}^*$ the corresponding optimal solution of the dual is $\mathbf{y}^*$ taking $\mathbf{v}_n = 0$ then there exists an optimal solution $\mathbf{y}'$ of the dual with $\mathbf{v}_n = 5$.

(c) If the transportation problem (P) given above is feasible then it has an optimal solution.

(d) If $m = 4$ and $n = 4$, then there exists a $\theta-$loop with exactly $m + n - 1$ cells.

(e) In every row and column of the transportation array there must be at least one basic cell.

(f) If all the a_i and b_j are positive integers and x^* is an optimal solution satisfying the condition that $\Delta = \{(i, j) : i, j, \text{ such that } x^*_{ij} \text{ is not an integer } \} \neq \phi$, then the cells corresponding to nonzero components of x^* contains a θ-loop.

(g) If $\mathfrak{B}$ is a collection of $m+n-1$ basic cells and if the $\theta-$loop in $\mathfrak{B}\cup\{(p,q)\}$ is considered, then $\sum\{c_{ij} : (i,j)$ gets the allocation $+\theta\}-\sum\{c_{ij} : (i,j)$ gets the allocation $-\theta\} = c_{pq} - z_{pq}$.

(h) If in the 1st row of the array the cost c_{1j} is changed to $c_{1j}+5$, for all $j = 1, 2, .., n$, then can we say that the optimal solution for the new problem is always equal to the optimal solution of the old problem?

(i) If all the a_i's and b_j's of the above transportation problem are integers then any basic feasible solution of the above problem will also take integer values.

(j) If $\bar{a_1}, \bar{a_2}, \bar{a_3}$ (where $\bar{a_i} = B^{-1}a_i$, a_i is the i th column of A), are three columns in any iteration (table) of the simplex algorithm applied to this problem, then $\{\bar{a_1}, \bar{a_2}, \bar{a_3}\}$ is linearly independent and the sum of the elements of the vector $u = \bar{a_1} + \bar{a_2} + \bar{a_3}$, is equal to 3.

(k) If $\mathcal{B}$ is a collection of basic cells of a transportation array and if $(p,q) \in \mathcal{B}$, then there exists at least one more cell from $\mathcal{B}$ in either the p-th row or q-th column of the array.

12. Consider the following transportation problem with c_{ij}'s, a_i's and b_j's as given below.

							a_i
	2	0	3	5	6	3	10
	1	3	4	0	9	7	24
	3	7	9	1	8	1	9
	4	8	7	3	6	8	36
b_j	24	4	1	19	23	8	

(a) Construct a basic feasible solution such that $\{(1,1),(1,3),(3,4)\} \subseteq \mathcal{B}$.

(b) Construct a $\theta-$loop which includes the cells $(1,1),(1,4),(3,4)$ and $(3,6)$.

(c) Solve the following transportation problem with, $\{(1,2),(1,5),(2,1),(2,4),(3,2),(3,3),(3,6),(4,4),(4,5)\}$ as the initial set of basic cells and by taking

i. $u_3 = 0$ in every stage of solving (that is the third supply is removed).

ii. $u_3 = 1$ initially (that is the third supply constraint is removed) and then in the next iteration (if any) by taking $v_3 = 0$ (that is the third destination constraint is removed).