**DEPARTMENT OF MATHEMATICS**

**MA 321 (Optimization)**

**Mid-semester Examination**

**Time:** 2 pm - 4 pm  February 25, 2019  **Maximum marks:** 30

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1. Show that exactly one of the following two systems has a solution.

   $A\mathbf{z} = \mathbf{b}$  (1)

   $A^T\mathbf{d} = \mathbf{0}, \quad \mathbf{d}^T\mathbf{b} < 0$  (2) (you can use related results done in class) **[4]**

   **Solution:** $A\mathbf{z} = \mathbf{b} \quad \Leftrightarrow A(\mathbf{z}_1 - \mathbf{z}_2) = \mathbf{b},\ \mathbf{z}_1 \geq \mathbf{0}, \mathbf{z}_2 \geq \mathbf{0},$

   where $\mathbf{z} = \mathbf{z}_1 - \mathbf{z}_2$, such that $z_i = z_{1i}$ if $z_i \geq 0$ and $z_i = -z_{2i}$ if $z_i \leq 0$.

   Hence $A\mathbf{z} = \mathbf{b} \quad \Leftrightarrow \quad [A : -A]\begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = \mathbf{b}, \qquad \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} \geq \mathbf{0}.$  (1)

   Consider the system $\mathbf{d}^T[A : -A] \geq [\mathbf{0} : \mathbf{0}], \quad \mathbf{d}^T\mathbf{b} < 0.$  (2')

   Then from Farka's lemma exactly one of the systems (1) and (2) has a solution.

   But, $\mathbf{d}^T[A : -A] \geq [\mathbf{0} : \mathbf{0}], \quad \mathbf{d}^T\mathbf{b} < 0$  (2') has a solution if and only if $\mathbf{d}^T A = \mathbf{0}$ (or $A^T\mathbf{d} = \mathbf{0}$), $\quad \mathbf{d}^T\mathbf{b} < 0$  (2) has a solution.

2. Consider a LPP (P) with feasible region of the form $S = \{\mathbf{x} \in \mathbb{R}^2 : A_{3\times 2}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ with **only** three extreme points $[0,1]^T$,$[2,0]^T$ and $[5,2]^T$, and extreme directions, $[0,1]^T$ and $[2,2]^T$ such that $\mathbf{x} = [3,0]^T$ satisfies $\mathbf{a}_1^T\mathbf{x} = b_1$ ($\mathbf{a}_1^T$ is the first row of $A$).

   (a) Give the simplex table ( except the $c_j - z_j$ values) for the BFS corresponding to $[2,0]^T$.

   **Solution:**

   The feasible region is given by:

   $Fea(P) = \{[x_1, x_2]^T : 2x_1 - 3x_2 \leq 4, x_1 - x_2 \leq 3, -x_1 - 2x_2 \leq -2, x_1 \geq 0, x_2 \geq 0\}$ **[2]**

   The three possible tables are:

   | | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $B^{-1}\mathbf{b}$ |
   |---|---|---|---|---|---|---|
   | $x_1$ | 1 | 0 | 0 | $-\frac{3}{7}$ | $\frac{2}{7}$ | 2 |
   | $x_2$ | 0 | 1 | 0 | $-\frac{2}{7}$ | $-\frac{1}{7}$ | 0 |
   | $s_1$ | 0 | 0 | 1 | $\frac{1}{7}$ | $-\frac{3}{7}$ | 1 |

   | | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $B^{-1}\mathbf{b}$ |
   |---|---|---|---|---|---|---|
   | $x_1$ | 1 | $-\frac{3}{2}$ | 0 | 0 | $\frac{1}{2}$ | 2 |
   | $s_2$ | 0 | $-\frac{7}{2}$ | 0 | 1 | $\frac{1}{2}$ | 0 |
   | $s_1$ | 0 | $\frac{1}{2}$ | 1 | 0 | $-\frac{1}{2}$ | 1 |

   | | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $B^{-1}\mathbf{b}$ |
   |---|---|---|---|---|---|---|
   | $x_1$ | 1 | 2 | 0 | $-1$ | 0 | 2 |
   | $s_3$ | 0 | $-7$ | 0 | 2 | 1 | 0 |
   | $s_1$ | 0 | $-3$ | 1 | 1 | 0 | 1 |

   **[3]**

   $1\frac{1}{2}$ marks deducted for not showing the required calculations for the table.

   (b) By deciding on the suitable entering and leaving variable for the table in part(a), obtain the table corresponding to $[5,2]^T$.

   **Solution:**

   | | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $B^{-1}\mathbf{b}$ |
   |---|---|---|---|---|---|---|
   | $x_1$ | 1 | 0 | 3 | 0 | $-1$ | 5 |
   | $x_2$ | 0 | 1 | 2 | 0 | $-1$ | 2 |
   | $s_2$ | 0 | 0 | 7 | 1 | $-3$ | 7 |

   **[4]**

(c) If the value of $\mathbf{c}^T\mathbf{x}$ at $[5,2]^T$ is 4 less than that at $[2,0]^T$, then what was the value of $c_j - z_j$ for the entering variable in part(b)?

**Solution:** $2(c_j - z_j) = -4$ gives $c_j - z_j = -2$, if $x_2$ is entering (table 2).
$7(c_j - z_j) = -4$ gives $c_j - z_j = -\frac{4}{7}$, if $s_2$ is entering (table 1). **[2]**

(d) If $[5,2]^T$ is optimal for (P), then an optimal solution of the dual of (P) definitely lies on which defining hyperplanes of Fea(D)? Justify. Does the dual of (P) have a unique optimal solution?

**Solution:** Since $x_1 > 0, x_2 > 0$ and $s_2 > 0$, hence $y_1 - y_2 + 2y_3 = c_1$,
$-y_1 - 2y_2 - 3y_3 = c_2, \qquad y_2 = 0.$ **[2]**
Since the three hyperplanes are LI, hence the above system has a unique solution, hence the dual has a unique solution. **[2]**
**[5 + 4 + 2 + 4]**

3. For a linear programming problem (P) of the form,

$$\text{Minimize } \mathbf{c}^T\mathbf{x}$$
$$\text{subject to } A_{3\times5}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}, \quad (rank(A) = 3)$$

let $\mathbf{x}_0$ and $\mathbf{y}_0$ be optimal solutions of (P) and the dual of (P), respectively.

(a) If $[1,-1,0]^T$ is the column corresponding to $x_3$ in the simplex table for a BFS with basic variables $x_1 = 5, x_2 = 6, x_4 = 5$ (written in this order) then if possible give $\mathbf{x}' = [0, x_2, x_3, x_4, 0]^T$ such that $\mathbf{x}' \in Fea(P)$? If possible give $\mathbf{x}'' = [x_1, 0, x_3, x_4, 0]^T$ such that $A\mathbf{x}'' = \mathbf{b}$, but $\mathbf{x}''$ is not in $Fea(P)$. If possible give $\mathbf{x}^*, \mathbf{x}^{**} \in Fea(P)$ $(\mathbf{x}^* \neq \mathbf{x}^{**})$ both satisfying, $x_5 = 0, x_4 = 5$.

**Solution:** The BFS is $\begin{bmatrix} 5 \\ 6 \\ 0 \\ 5 \\ 0 \end{bmatrix}$ and note that $A \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0.$

So $\mathbf{x}' = \begin{bmatrix} 5 \\ 6 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_3' \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \\ 5 \\ 5 \\ 0 \end{bmatrix}$, where $x_3' = 5$, **[1.5]**

and $\mathbf{x}'' = \begin{bmatrix} 5 \\ 6 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_2' \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ -6 \\ 5 \\ 0 \end{bmatrix}$, where $x_2' = -6$. **[1.5]**

Take $x_3' = 4, 5$, then $\mathbf{x}^* = \begin{bmatrix} 5 \\ 6 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_3' \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 4 \\ 5 \\ 0 \end{bmatrix}$ and $\mathbf{x}^{**} = \mathbf{x}'$. **[1 + 1]**

(b) If after adding $[-2,1,1,2,3]$ to the first row of $A$ in (P), $\mathbf{x}_0$ remains optimal for the new problem $(P')$ then can $\mathbf{y}_0$ be optimal for the dual of $(P')$? Justify.

**Solution:** Note that the dual of (P) is given by

Max $\mathbf{c}^T\mathbf{x}$, subject to $A^T\mathbf{y} \leq \mathbf{c}$.

Since $\mathbf{x}_0$ and $\mathbf{y}_0$ are optimal solutions of (P) and the dual of (P) respectively, they must satisfy the complementary slackness conditions.

Also since $\mathbf{x}_0$ is also feasible for the new problem hence $[-2, 1, 1, 2, 3]\mathbf{x}_0 = 0$.

**Case 1: $\mathbf{b} \neq \mathbf{0}$**:

Since $[-2, 1, 1, 2, 3]\mathbf{x}_0 = 0$, hence $x_1 > 0$ and atleast one of $x_2, x_3, x_4, x_5$ must be positive.

But since $\mathbf{y}_0$ is optimal for the dual of $(P)$ hence the first constraint of the dual must be satisfied as an equality, that $\tilde{\mathbf{a}}_1^T\mathbf{y}_0 = c_1$ and $\tilde{\mathbf{a}}_i^T\mathbf{y}_0 = c_i$ for atleast one of $i = 2, 3, 4, 5$, where $\tilde{\mathbf{a}}_i$ is the $i$ th column of $A$. WLOG let us assume $\tilde{\mathbf{a}}_2^T\mathbf{y}_0 = c_2$.

But then the first and the second constraints of the new problem

$\tilde{\mathbf{a}}_1^T\mathbf{y}_0 - 2(\mathbf{y}_0)_1 \leq c_1$, $\tilde{\mathbf{a}}_2^T\mathbf{y}_0 + (\mathbf{y}_0)_1 \leq c_2$ cannot both be satisfied unless $(\mathbf{y}_0)_1 = 0$. [**4**]

**Case 2:** If $\mathbf{b} = \mathbf{0}$ then $\mathbf{x}_0 = \mathbf{0}$ and $(A^T\mathbf{y}_0)_i \leq c_i$ for all $i = 1, 2, 3, 4, 5$.

If $\tilde{\mathbf{a}}_1^T\mathbf{y}_0 = c_1$ and for atleast one $i \in \{2, \ldots, n\}$, $\tilde{\mathbf{a}}_i^T\mathbf{y}_0 = c_i$, then again $(y_o)_1$ has to be equal to 0 for $\mathbf{y}_0$ to be feasible hence optimal for the new problem.

For all other cases $y_0$ may be feasible for the new problem even if $(y_o)_1 \neq 0$.

(c) If $[1, 3, 5, 0, 0]^T \in Fea(P)$ but is not a BFS then will (P) have a degenerate BFS?

**Solution:** Since $\mathbf{x} \in Fea(P)$ if and only if it is of the form:

$\mathbf{x} = \sum_{i=1}^{r} \lambda_i\mathbf{x}_i + \sum_{j=1}^{l} \mu_i\mathbf{d}_j$, for some $\lambda_i$'s and $\mu_j$'s, such that

$0 \leq \lambda_i \leq 1$, for all $i = 1, \ldots, r$, $\sum_{i=1}^{r} \lambda_i = 1$ and $\mu_j \geq 0$ for all $j = 1, \ldots, l$,

where $\mathbf{x}_i$, $i = 1, \ldots, r$ are the extreme points of Fea(P) and $\mathbf{d}_j$, $j = 1, \ldots, l$ are the distinct extreme directions (which might be empty set) of Fea(P).

If atleast two of the $\lambda_i$'s are strictly greater than 0, then clearly $\mathbf{x}$ will have more than three positive components, if all the extreme points are nondegenerate.

If exactly one $\lambda_i > 0$, that is equal to 1, WLOG let us assume $\lambda_1 = 1$, then clearly $\mathbf{x}_1$ if it is nondegenerate must have exactly three positive components ( which should match with those of $\mathbf{x}$ ) which correspond to three LI columns of $A$, which contradicts that $\mathbf{x}$ is not a BFS.

Hence in any case there must be atleast one degenerate BFS.

**Aliter:** Consider $\mathbf{x}_0 = [1, 3, 5, 0, 0]^T + [-1, 1, 1, 0, 0]^T = [0, 4, 6, 0, 0]^T$, which is a BFS since $\mathbf{x}' = [0, 11, 5, 5, 0]^T$ is BFS which implies columns corresponding to $x_2, x_3, x_4$ is LI, and $\mathbf{x}_0$ is degenerate.

[**4**]

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