Practise problems 3

1. Write the dual of the following problems:

   (a)
   $$\begin{aligned} &\text{Maximize } x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 + x_3 \geq 1 \\ & x_1 + 2x_2 + 3x_3 = 2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

   (b)
   $$\begin{aligned} &\text{Maximize } x_1 + x_2 + x_3 \\ \text{subject to} \quad & -x_1 + x_2 + x_3 \leq 2 \\ & x_1 + 2x_2 + 3x_3 \geq 2 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

   (c)
   $$\begin{aligned} &\text{Maximize } x_1 - x_2 + x_3 \\ \text{subject to} \quad & -x_1 + x_2 + x_3 \leq 2 \\ & x_1 + 2x_2 + 3x_3 = 2. \end{aligned}$$

   Hint: Write x_i as $x_i = z_i - w_i$, where $z_i, w_i \geq 0$.

   Check whether the above problems have optimal solutions by using the dual(you need not find the optimal solutions).

2. Find an infeasible (that has no feasible solution) primal problem which also has an infeasible dual.

3. (a) Using similar results done in class show that exactly one of the following two systems has a solution.
   $\mathbf{x} \geq \mathbf{0}$, $A\mathbf{x} > \mathbf{0}$ (all components of $A\mathbf{x}$ is positive). (1)
   $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, $A^T\mathbf{y} \leq \mathbf{0}$. (2)
   Hence can the feasible region of both a primal problem (in standard form) and its dual be bounded?
   Hint: Answer is No.

   (b) Can both the feasible regions be unbounded?
   Hint: Answer is Yes.

4. Fill up the blank with a restatement of the complementary slackness theorem:
   Consider a LPP (P) of the form given below (primal):

   $$\text{Maximize } \mathbf{c}^T\mathbf{x}$$
   $$\text{subject to } A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}.$$

Let $\mathbf{x}_0 \in Fea(P)$ and $\mathbf{y}_0 \in Fea(D)$, then $\mathbf{x}_0$ and $\mathbf{y}_0$ are optimal for (P) and (D) respectively, if and only if

....

5. Consider a LPP of the type:

$$\text{Maximize } \mathbf{c}^T\mathbf{x}$$

$$\text{subject to } A_{m\times n}\mathbf{x} \leq \mathbf{b}.$$

Write the dual of the above problem such that it (the dual) has only n constraints.

6. Given a LPP of the type:

$$\text{Minimize } \mathbf{c}^T\mathbf{x}$$

$$\text{subject to } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}.$$

(a) Write the dual of the above problem.

(b) Obtain the complementary slackness conditions of the above problem.

7. Using the complementary slackness conditions obtain the optimal solution of the following problem and show that it is unique.

$$\begin{aligned} \text{Minimize } & x_1 + 3x_2 + x_3 \\ \text{subject to } & 3x_1 + x_2 \geq 6 \\ & x_1 + x_2 - x_3 \geq 2 \\ & x_1 + x_3 \geq 2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

Does the dual have a unique solution? Obtain feasible solutions $\mathbf{x}$ and $\mathbf{y}$ of the primal(P) and the dual(D) respectively, which does **not** mutually satisfy the complementary slackness condition.

8. Consider the problem

$$\text{Minimize } \mathbf{c}^T\mathbf{x}$$

$$\text{subject to } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$$

where A has m rows and n columns. Suppose an optimal solution exists such that only $x_1, x_2, ..., x_k$ are positive and $x_j = 0$ for $j = k+1, k+2, ..., n$. Now change the vector $\mathbf{b}$ to $\mathbf{b}'$. Prove that if there exists a $\mathbf{z} \geq \mathbf{0}$ such that $A\mathbf{z} = \mathbf{b}'$ and $z_j = 0$ for $j = k+1, k+2, ..., n$, then this $\mathbf{z}$ is automatically optimal for the changed problem.

9. Consider the following problem (P) given below:

$$\begin{aligned} \text{Minimize } & x_1 + 3x_2 + x_3 \\ \text{subject to } & x_1 - x_2 = 6 \\ & x_1 - x_2 - 2x_3 = 2 \\ & x_3 = 2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

(a) Check that (P) has a feasible solution.

(b) If the feasible region region of (P) is written as $S = \{\mathbf{x} : A_{3\times 3}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ then check that $rank(A) = 2$. Hence by deleting a constraint write the feasible region S as $S = \{\mathbf{x} : A'_{2\times 3}\mathbf{x} = \mathbf{b}', \mathbf{x} \geq \mathbf{0}\}$, where A' is the matrix obtained from A by deleting a row of A and $\mathbf{b}'$ is obtained from $\mathbf{b}$ by deleting the corresponding component of $\mathbf{b}$.

(c) Obtain a basic feasible solution of (P) and write a corresponding basis matrix. If possible obtain both a degenerate basic feasible solution and a non degenerate basic feasible solution of (P).

(d) By changing the R.H.S of the third equation and and then by deleting the third constraint show that in general $S = \{\mathbf{x} : A_{3\times 3}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \neq S' = \{\mathbf{x} : A_{2\times 3}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. So can you give a general condition on S such that $S = S'$?

(e) Give the set of all extreme directions of S.

(f) Check whether this problem has an optimal solution.

10. Consider the problem (P) given below:

$$\begin{aligned} &\text{Maximize } x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 + x_3 \leq 1 \\ & x_1 + 2x_2 + 3x_3 = 2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

Consider the extreme point $\mathbf{x}^0$ which lies in the hyperplanes corresponding to the first constraint, second constraint and the constraint $x_1 \geq 0$. Write the corresponding feasible solution $[\mathbf{x}^0, s_1]^T$ of the system,

$$-x_1 + x_2 + x_3 + s_1 = 1$$

$$x_1 + 2x_2 + 3x_3 = 2$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0.$$

Verify that $[\mathbf{x}^0, s_1]^T$ is a basic feasible solution of the above system.
Verify that $[0, 0, \frac{2}{3}, \frac{1}{3}]^T$ is a basic feasible solution of the above system. Verify that $[0, 0, \frac{2}{3}]^T$ is an extreme point of $Fea(P)$.

11. Use simplex algorithm to solve the following problem:
Min $4x_1 + 4x_2 - 2x_3$
Subject to
$2x_1 + 3x_2 - 2x_3 \leq 10$
$2x_1 - x_2 + 3x_3 \leq 4$
$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

12. For a linear programming problem (P) of the form,

$$\begin{aligned}&\text{Minimize } \mathbf{c}^T\mathbf{x}\\&\text{subject to } A_{m\times n}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\end{aligned}$$

check the correctness of the following statements with proper justification.

(a) If $c_1 - z_1 > 0$ in some iteration (table) and if $\tilde{\mathbf{a}_1}$ is made to enter the basis, then the new solution obtained will not be feasible.

(b) If for some basic feasible solution $\overline{\mathbf{x}}$, $\mathbf{c}^T\overline{\mathbf{x}} = 10$ and for no feasible $\mathbf{x}$, $\mathbf{c}^T\mathbf{x} = 5$, then (P) has an optimal solution.

(c) For $m = 2$ and $n = 4$ in the linear programming problem (P), if $[1, 2, 3, 0]^T$ is an optimal solution, then (P) has infinitely many solutions.

(d) For $m = 2$ and $n = 4$ in the linear programming problem (P), if in some iteration (table) of the simplex method, $\tilde{\mathbf{a}_2}$ enters the basis, then $\tilde{\mathbf{a}_2}$ can leave the basis in the next iteration (table).

(e) For $m = 2$ and $n = 4$ in the linear programming problem (P), if in some iteration (table) of the simplex method, $\tilde{\mathbf{a}_2}$ leaves the basis, then $\tilde{\mathbf{a}_2}$ can again enter the basis in the next iteration (table).

(f) If $\mathbf{B}$ is the basis matrix corresponding to the optimal table (of the simplex method) where $\mathbf{c}_1 - \mathbf{z}_1 = 0$ and $\mathbf{B}^{-1}\tilde{\mathbf{a}_1} \leq \mathbf{0}$, then the optimal solutions of (P) is an unbounded set.

(g) If $\mathbf{c}^T\mathbf{x}^0 = \mathbf{b}^T\mathbf{y}^0$ for some $\mathbf{x}_0$ feasible but not optimal for (P), then $\mathbf{y}^0$ is not feasible for the dual.

(h) For $m = 2$ and $n = 4$ in (P) if $x_1 = 1$ and $x_2 = 3$ are the basic variables of a basic feasible solution of (P), then there exists a feasible solution of (P) with $x_1 > 0, x_2 > 0, x_3 > 0$.

(i) If for some basic feasible solution $\mathbf{x}$ and a nonbasic column $\tilde{\mathbf{a}_s}$, $min\{\frac{x_i}{u_{is}} : u_{is} > 0\} = \frac{x_t}{u_{ts}} = \frac{x_r}{u_{rs}} > 0$ for some $t \neq r$, then there exists a basic feasible solution of (P) which corresponds to two different basis.

13. Given a LPP with feasible region F as follows
$S = \{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$.
It is given that the feasible region S has only the three extreme points $[0, 0]^T, [0, 3]^T$ and $[2, 5]^T$.
Convert the feasible region S (by adding variables) to the form $S' = \{[\mathbf{x}, \mathbf{s}]^\mathbf{T} \in \mathbb{R}^4 : A'[\mathbf{x}, \mathbf{s}]^\mathbf{T} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{0}\}$, where $A' = [A : I]$.
The following is the simplex table for the basic feasible solution corresponding to the basis $B = [\mathbf{a}_1, \mathbf{a}_2]$.

	a_1	a_2	s_1	s_2	b
a_1			-2		
a_2			-1		

(a) How many constraints (other than the non negativity constraints) does S have?

(b) If possible give all the missing entries of the above table.

(c) If possible give a direction $[\mathbf{d}, \mathbf{s}]^T = [d_1, d_2, s_1, s_2]^T$ of S' such that the corresponding direction $\mathbf{d} = [d_1, d_2]^T$ of the feasible region S satisfies $\mathbf{a}_1^T \mathbf{d} < 0$ while $\mathbf{a}_2^T \mathbf{d} = 0$.

(d) If possible give a direction $[\mathbf{d}', \mathbf{s}]^T = [d_1', d_2', s_1, s_2]^T$ of S' such that the corresponding direction $\mathbf{d}' = [d_1', d_2']^T$ of the feasible region S satisfies $\mathbf{a}_1^T \mathbf{d}' < 0$ and $\mathbf{a}_2^T \mathbf{d}' < 0$.

(e) Obtain a feasible region S of the above problem.

(f) Let $\mathbf{u}$ and $\mathbf{v}$ be the extreme points of S' corresponding to the extreme points $[0,0]^T$ and $[0,3]^T$, respectively. Give a $[\mathbf{d}, \mathbf{s}]^T$ such that $\mathbf{u} + \alpha[\mathbf{d}, \mathbf{s}]^T = \mathbf{v}$. Give the value of α.

(g) Similarly let $\mathbf{v}$ and $\mathbf{w}$ be the extreme points of S' corresponding to the extreme points $[0,3]^T$ and $[2,5]^T$, respectively. If $[\mathbf{d}, \mathbf{s}]^T$ is such that $\mathbf{v} + \alpha[\mathbf{d}, \mathbf{s}]^T = \mathbf{w}$, $\alpha > 0$ then can $\mathbf{d}$ be a nonnegative vector?

(h) Which of the following can possibly constitute a basis matrix for the above problem:
$B = [\tilde{\mathbf{a}_2}, \mathbf{e}_1], B = [\tilde{\mathbf{a}_1}, \mathbf{e}_1], B = [\mathbf{e}_1, \mathbf{e}_2]$ and $B = [\tilde{\mathbf{a}_2}, \mathbf{e}_2]$, where $\mathbf{e}_i$ denotes the i th column of the identity matrix.

(i) Find a $\mathbf{c}$ such that if the objective function is of the form:
Minimize $\mathbf{c}^T \mathbf{x}$
then the problem does not have an optimal solution.

14. Consider a LPP for which we get the following optimal table.

$c_j - z_j$	0	0	1	0	0	
	$B^{-1}\tilde{\mathbf{a}_1}$	$B^{-1}\tilde{\mathbf{a}_2}$	$B^{-1}\tilde{\mathbf{a}_3}$	$B^{-1}\tilde{\mathbf{a}_4}$	$B^{-1}\tilde{\mathbf{a}_5}$	$B^{-1}\mathbf{b}$
$\tilde{a_1}$		-1				4
$\tilde{a_4}$		-2				4
		-3				3

(a) Can you suggest two different optimal solutions for this problem?

(b) If you change the entry -1 in the above table to +1, can you find two optimal basic feasible solutions, by entering a new variable in the basis?

(c) If there are two optimal basic feasible solutions then there are infinitely many optimal solutions of a LPP. What about the converse?