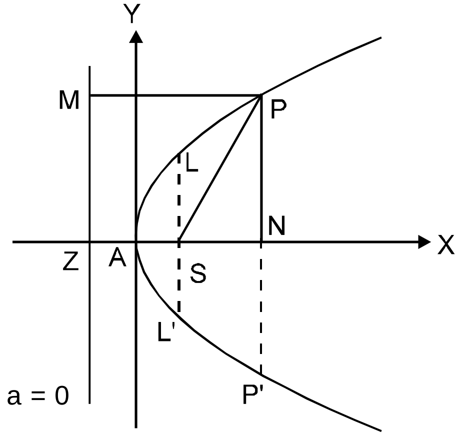
PARABOLA

Definition and terminology



$y^2 = 4ax$

A parabola is the locus of a point, whose distance from a fixed point (focus) is equal to perpendicular distance from a fixed straight line (directrix).

Four standard forms of the parabola are

$y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$

For parabola $y^2 = 4ax$:

(i) Vertex is (0, 0) (ii) focus is (a, 0)

(iii) Axis is $y = 0$ (iv) Directrix is $x + a = 0$

Focal Distance: The distance of a point on the parabola from the focus.

Focal Chord : A chord of the parabola, which passes through the focus.

Double Ordinate: A chord of the parabola perpendicular to the axis of the symmetry.

Latus Rectum: A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the Latus Rectum (L.R.).

For $y^2 = 4ax$. $\Rightarrow$ Length of the latus rectum = 4a.

$\Rightarrow$ ends of the latus rectum are L(a, 2a) & L' (a, – 2a).

NOTE :

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Two parabolas are said to be equal if they have the same latus rectum.

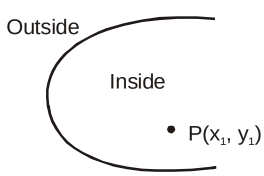
Parametric representation:

The simplest & the best form of representing the co–ordinates of a point on the parabola is $(at^2, 2at)$ i.e. the equations $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

Parametric form for :	$y^2 = -4ax$	$(-at^2, 2at)$
	$x^2 = 4ay$	$(2at, at^2)$
	$x^2 = -4ay$	$(2at, -at^2)$

Position of a point relative to a parabola:

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.



$S_1 : y_1^2 - 4ax_1$

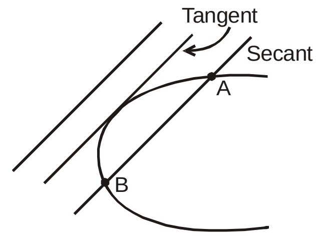
$S_1 < 0 \rightarrow$ Inside

$S_1 > 0 \rightarrow$ Outside

Line & a parabola:

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a \gtreqless cm \Rightarrow$ condition of tangency is, $c = a/m$.

Length of the chord intercepted by the parabola on the line $y = mx + c$ is :

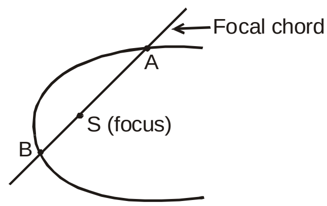


$$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}\,.$$

NOTE :

1. The equation of a chord joining t_1 & t_2 is $2x - (t_1 + t_2)\,y + 2\,at_1t_2 = 0$.

2. If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$. Hence the co–ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$



3. Length of the focal chord making an angle α with the x– axis is $4a\operatorname{cosec}^2\alpha$.

Tangents to the parabola $y^2 = 4ax$:

Equation of tangent at a point on the parabola can be obtained by replacement method or using derivatives.
In replacement method, following changes are made to the second degree equation to obtain T.
$x^2 \to x\,x_1$, $y^2 \to y\,y_1$, $2xy \to xy_1 + x_1y$, $2x \to x + x_1$, $2y \to y + y_1$
So, it follows that the targents are :

(i) $y\,y_1 = 2a\,(x + x_1)$ at the point (x_1, y_1) ;

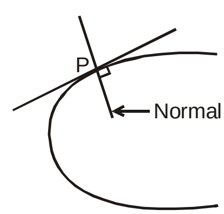
(ii) $y = mx + \frac{a}{m}$ $(m \neq 0)$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(iii) $t\,y = x + a\,t^2$ at $(at^2, 2at)$.

(iv) Point of intersection of the tangents at the point t_1 & t_2 is $\{\, at_1t_2,\ a(t_1 + t_2)\,\}$.

Normals to the parabola $y^2 = 4ax$:

Normal is obtained using the slope of tangent.



Slope of tangent at $(x_1, y_1) = \frac{2a}{y_1}$

$\Rightarrow$ Slope of normal $= -\frac{y_1}{2a}$

(i) $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ at (x_1, y_1) ;

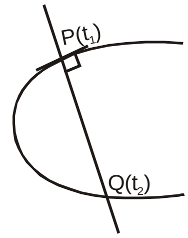
(ii) $y = mx - 2am - am^3$ at $(am^2, -2am)$

(iii) $y + tx = 2at + at^3$ at $(at^2, 2at)$.

NOTE :

(i) Point of intersection of normals at t_1 & t_2 is $(a(t_1^2 + t_2^2 + t_1t_2 + 2), -a\,t_1 t_2 (t_1 + t_2))$.

(ii) If the normals to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.



(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1 t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.