An arithmetic progression (A.P.)

A.P. is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then A.P. can be written as a, a + d, a + 2 d,....... a + (n – 1) d,........

(i) n^{th} term of an A.P.
Let a be the first term and d be the common difference of an A.P., then

$$t_n = a + (n-1)d \qquad \text{where} \quad d = a_n - a_{n-1}$$

(ii) The sum of first n terms of are A.P.
If a is first term and d is common difference then

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad = \frac{n}{2}[a + l] = nt_{\left(\frac{n+1}{2}\right)},$$

where l is the last term and $t_{\left(\frac{n+1}{2}\right)}$ is the middle term.

(iii) r^{th} term of an A.P. when sum of first r terms is given is $t_r = s_r - S_{r-1}$.

Properties of A.P.

(i) The common difference can be zero, positive or negative.

(ii) If a, b, c are in A.P. $\Rightarrow 2b = a + c$ & if a, b, c, d are in A.P. $\Rightarrow a + d = b + c$.

(iii) Three numbers in A.P. can be taken as a – d, a, a + d; four numbers in A.P. can be taken as a – 3d, a – d, a + d, a + 3d; five numbers in A.P. are a – 2d, a – d, a, a + d, a + 2d & six terms in A.P. are a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d etc.

(iv) The sum of the terms of an A.P. equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) Any term of an A.P. (except the first) is equal to half the sum of terms which are equidistant from it. $a_n = 1/2\,(a_{n-k} + a_{n+k})$, $k < n$. For k = 1, $a_n = (1/2)(a_{n-1} + a_{n+1})$;
For k = 2, $a_n = (1/2)(a_{n-2} + a_{n+2})$ and so on.

(vi) If each term of an A.P. is increased, decreased, multiplied or divided by the sA.M.e non zero number, then the resulting sequence is also an A.P..

Arithmetic Mean (Mean or Average) (A.M.):

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c.

(a) n – Arithmetic Means Between Two Numbers:
If a, b are any two given numbers & a, $A_1, A_2, \ldots, A_n$, b are in A.P. then $A_1, A_2, \ldots A_n$ are the n A.M.'s between a & b.

$$A_1 = a + \frac{b-a}{n+1},\ A_2 = a + \frac{2(b-a)}{n+1}, \ldots\ldots, A_n = a + \frac{n(b-a)}{n+1}$$

NOTE :
Sum of n A.M.'s inserted between a & b is equal to n times the single A.M. between a & b

i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a & b.

Geometric Progression (G.P.)

G.P. is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the proceeding terms multiplied by a constant. Thus in a G.P. the ratio of successive terms is constant. This constant factor is called the common ratio of the series & is obtained by dividing any term by that which immediately proceeds it. Therefore a, ar, ar^2, ar^3, ar^4,...... is a G.P. with a as the first term & r as common ratio.

Example 2, 4, 8, 16 ....

Example $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$ ....

(i) n^{th} term = $a\,r^{n-1}$

(ii) Sum of the first n terms i.e. $S_n = \begin{cases} \dfrac{a(r^n - 1)}{r-1}, & r \neq 1 \\ na, & r = 1 \end{cases}$

(iii) Sum of an infinite G.P. when $|r| < 1$. When $n \to \infty$ $r^n \to 0$ if $|r| < 1$ therefore,

$S_\infty = \frac{a}{1-r} \left(|r| < 1\right)$.

Properties of G.P.

(i) If a, b, c are in G.P. $\Rightarrow b^2 = ac$, in general if $a_1, a_2, a_3, a_4, \ldots\ldots\ldots a_{n-1}, a_n$ are in G.P., then $a_1a_n = a_1a_{n-1} = a_3\,a_{n-2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(ii) Any three consecutive terms of a G.P. can be taken as $\frac{a}{r}$, a , ar, in general we take

$\frac{a}{r^k}, \frac{a}{r^{k-1}}, \frac{a}{r^{k-2}}, \ldots\ldots\ldots a, ar, ar^2, \ldots\ldots\ldots ar^k$ in case we have to take 2k + 1 terms in a G.P.

(iii) Any four consecutive terms of a G.P. can be taken as $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3, in general we take

$\frac{a}{r^{2k-1}}, \frac{a}{r^{2k-3}}, \ldots\ldots\ldots \frac{a}{r}$, ar, ...$ar^{2k-1}$ in case we have to take 2k terms in a G.P.

(iv) If each term of a G.P. be multiplied or divided or raised to power by the some non–zero quantity, the resulting sequence is also a G.P..

(v) If $a_1, a_2, a_3, \ldots\ldots\ldots$ and $b_1, b_2, b_3, \ldots\ldots\ldots$ are two G.P's with common ratio r_1 and r_2 respectively then the sequence $a_1b_1, a_2b_2, a_3b_3, \ldots\ldots$ is also a G.P. with common ratio $r_1\,r_2$.

(vi) If $a_1, a_2, a_3, \ldots\ldots\ldots\ldots$ are in G.P. where each $a_i > 0$, then $\log a_1, \log a_2, \log a_3, \ldots\ldots\ldots\ldots$ are in A.P. and its converse is also true.

Geometric Means (Mean Proportional) (G.M.):

If a, b, c are in G.P., b is the G.M. between a & c.

$b^2 = ac$, therefore $b = \sqrt{ac}$; a > 0, c > 0.

(a) n–Geometric Means Between a, b:

If a, b are two given numbers & a, $G_1, G_2, \ldots\ldots, G_n$, b are in G.P.. Then $G_1, G_2, G_3, \ldots\ldots, G_n$ are n G.M.s between a & b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$,......, $G_n = a(b/a)^{n/n+1}$

NOTE :

The product of n G.M.s between a & b is equal to the nth power of the single G.M. between a & b

i.e. $\overset{n}{\underset{r=1}{\pi}} G_r = (G)^n$ where G is the single G.M. between a & b.

Relation between means :

(i) If A, G, H are respectively A.M., G.M., H.M. between a & b both being unequal & positive then, $G^2 = AH$ i.e. A, G, H are in G.P.

(ii) A.M. $\geq$ G.M. $\geq$ H.M.

Let $a_1, a_2, a_3, ...a_n$ be n positive real numbers, then we define their

A.M. = $\dfrac{a_1 + a_2 + a_3 + ... + a_n}{n}$, their

G.M. = $(a_1 a_2 a_3 ....a_n)^{1/n}$ and their H.M. = $\dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}$ It can be shown that

A.M. $\geq$ G.M. $\geq$ H.M. and equality holds at either places iff

$a_1 = a_2 = a_3 = ...= a_n$

Arithmetico-Geometric Series:

A series each term of which is formed by multiplying the corresponding term of an A.P. & G.P. is called the AritH.M.etico–Geometric Series. e.g. $1 + 3x + 5x^2 + 7x^3 + ...$

Here 1, 3, 5,.... are in A.P. & 1, x, x^2, x^3.... are in G.P..

Sum of n terms of an Arithmetico–Geometric Series:

Let $S_n = a + (a + d)\, r + (a + 2\,d)\, r^2 + ... + [a + (n - 1)d]\, r^{n-1}$

then $S_n = \dfrac{a}{1-r} + \dfrac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \dfrac{[a+(n-1)d]r^n}{1-r}$, $r \neq 1$.

Sum To Infinity: If $|r| < 1$ & $n \to \infty$ then $\lim\limits_{n\to\infty} r^n = 0 \Rightarrow S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$.

Important Results

(i) $\sum\limits_{r=1}^{n} (a_r \pm b_r) = \sum\limits_{r=1}^{n} a_r \pm \sum\limits_{r=1}^{n} b_r$.

(ii) $\sum\limits_{r=1}^{n} k\, a_r = k \sum\limits_{r=1}^{n} a_r$.

(iii) $\sum\limits_{r=1}^{n} k = k + k + k...............n$ times $= nk$; where k is a constant.

(iv) $\sum\limits_{r=1}^{n} r = 1 + 2 + 3 +..........+ n = \dfrac{n\,(n+1)}{2}$

(v) $\sum\limits_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 +..........+ n^2 = \dfrac{n\,(n+1)\,(2n+1)}{6}$

(vi) $\sum\limits_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 +..........+ n^3 = \dfrac{n^2\,(n+1)^2}{4}$

(vii) $2 \sum\limits_{i<j=1}^{n} a_i\, a_j = (a_1 + a_2 + ...+ a_n)^2 - (a_1^2 + a_2^2 + ... + a_n^2)$