Matrices

DEFINITION

A rectangular arrangement of elements in rows and columns, is called a matrix. Such a rectangular arrangement of numbers is enclosed by small ( ) or big [ ] brackets. Generally a matrix is represented by a capital latter A, B, C......... etc. and its element are represented by small letters a, b, c, x, y etc.

Following are some examples of a matrix :

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{bmatrix} 1 & 5 & 3 \\ 4 & 0 & 2 \end{bmatrix} \qquad C = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \qquad D = [1, 5, 6], \qquad E = [5]$$

ORDER OF MATRIX

A matrix which has m rows and n columns is called a matrix of order m × n, and its represented by

$A_{m\times n}$ or $A = [a_{ij}]_{m\times n}$

It is obvious to note that a matrix of order m × n contains mn elements. Every row of such a matrix contains n elements and every column contains m elements.

TYPES OF MATRICES

Row matrix

If in a matrix, there is only one row, then it is called a Row Matrix.

Thus $A = [a_{ij}]_{m\times n}$ is a row matrix if m = 1

Column Matrix

If in a matrix, there is only one column, then it is called a column matrix.

Thus $A = [a_{ij}]_{m\times n}$ is a column matrix if n = 1.

Square matrix

If number of rows and number of columns in a matrix are equal, then it is called a square matrix.

Thus $A = [a_{ij}]_{m\times n}$ is a square matrix if m = n.

Note : (a) If $m \neq n$ then matrix is called a rectangular matrix.

(b) The elements of a square matrix A for which i = j i.e., a_{11}, a_{22}, a_{33},......a_{nn} are called principal diagonal elements and the line joining these elements is called the principal diagonal or leading diagonal of matrix A.

(c) Trace of a matrix : The sum of principal diagonal elements of a square matrix A is called the trace of matrix A which is denoted by trace A. Trace A = $a_{11} + a_{22} + ...a_{nn}$

Singleton matrix

If in a matrix there is only one element then it is called singleton matrix.

Thus $A = [a_{ij}]_{m\times n}$ is a singleton matrix if m = n = 1.

Null or zero matrix

If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by O.

Thus $A = [a_{ij}]_{m\times n}$ is a zero matrix if $a_{ij} = 0$ for all i and j.

Diagonal matrix

If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix.

Thus a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

Note : (a) No element of principal diagonal in diagonal matrix is zero.

(b) Number of zero is a diagonal matrix is given by $n^2 - n$ where n is a order of the matrix.

Scalar Matrix

If all the elements of the diagonal in a diagonal matrix are equal, it is called a scalar matrix.

Thus a square matrix A $[a_{ij}]$ is a scalar matrix is

$$a_{ij} = \begin{cases} 0 & i \neq j \\ k & i = j \end{cases} \text{ where k is a constant.}$$

Unit matrix

If all elements of principal diagonal in a diagonal matrix are 1, then it is called unit matrix. A unit matrix of order n is denoted by I_n.

Thus a square matrix

$$A = [a_{ij}] \text{ is a unit matrix if } \quad a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Note : Every unit matrix is a scalar matrix.

Triangular matrix

A square matrix $[a_{ij}]$ is said to be triangular if each element above or below the principal diagonal is zero. It is of two types -

(a) Upper triangular matrix : A square matrix $[a_{ij}]$ is called the upper triangular matrix, if $a_{ij} = 0$ when $i > j$.

(b) Lower triangular matrix : A square matrix $[a_{ij}]$ is called the lower triangular matrix, if $a_{ij} = 0$ when $i < j$

Note : Minimum number of zero in a triangular matrix is given by $\frac{n(n-1)}{2}$ where n is order of matrix.

Equal matrix

Two matrices A and B are said to be equal if they are of same order and their corresponding elements are equal.

Singular matrix

Matrix A is said to be singular matrix if its determinant $|A| = 0$, otherwise non-singular matrix i.e.,

If det $|A| = 0$ $\Rightarrow$ singular and det $|A| \neq 0$ $\Rightarrow$ non-singular

ADDITION AND SUBTRACTION OF MATRICES

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of the same order then their sum A + B is a matrix whose each element is the sum of corresponding elements.

i.e., $A + B = [a_{ij} + b_{ij}]_{m\times n}$

A – B is defined as $A - B = [a_{ij} - b_{ij}]_{m\times n}$

Note : Matrix addition and subtraction can be possible only when matrices are of same order.

Properties of matrices addition

If A, B and C are matrices of same order, then-

(i) A + B = B + A (Commutative Law)

(ii) (A + B) + C = A + (B + C) (Associative law)

(iii) A + O = O + A = A, where O is zero matrix which is additive identity of the matrix.

(iv) A + (–A) = 0 = (–A) + A where (–A) is obtained by changing the sign of every element of A which is additive inverse of the matrix

(v) $\left.\begin{array}{l} A+B=A+C \\ B+A=C+A \end{array}\right\} \Rightarrow$ B = C (Cancellation law)

(vi) Trace (A ± B) = trace (A) ± trace (B)

SCALAR MULTIPLICATION OF MATRICES

Let $A = [a_{ij}]_{m\times n}$ be a matrix and k be a number then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it denoted by kA.

Thus $A = [a_{ij}]_{m\times n} \Rightarrow kA = [ka_{ij}]_{m\times n}$

Properties of scalar multiplication

If A, B are matrices of the same order and m, n are any numbers, then the following results can be easily established.

(i) m(A + B) = mA + mB (ii) (m + n)A = mA + nA (iii) m(nA) = (mn)A = n(mA)

MULTIPLICATION OF MATRICES

If A and B be any two matrices, then their product AB will be defined only when number of column in A is equal to the number of rows in B. If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ then their product $AB = C = [c_{ij}]$, will be matrix of order m × p, where

$$(AB)_{ij} = C_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

Properties of matrix multiplication

If A, B and C are three matrices such that their product is defined, then

(i) $AB \neq BA$ (Generally not commutative)

(ii) (AB) C = A (BC) (Associative Law)

(iii) IA = A = AI I is identity matrix for matrix multiplication

(iv) A (B + C) = AB + AC (Distributive law)

(v) If $AB = AC \not\Rightarrow B = C$ (cancellation Law is not applicable)

(vi) If AB = 0 It does not mean that A = 0 or B = 0, again product of two non-zero matrix may be zero matrix.

(vii) trance (AB) = trance (BA)

Note : (i) The multiplication of two diagonal matrices is again a diagonal matrix.

(ii) The multiplication of two triangular matrices is again a triangular matrix.

(iii) The multiplication of two scalar matrices is also a scalar matrix.

(iv) If A and B are two matrices of the same order, then

(a) $(A + B)^2 = A^2 + B^2 + AB + BA$

(b) $(A - B)^2 = A^2 + B^2 - AB - BA$

(c) $(A - B)(A + B) = A^2 - B^2 + AB - BA$

(d) $(A + B)(A - B) = A^2 - B^2 - AB + BA$

(e) A (–B) = (–A) B = –(AB)

Positive Integral powers of a matrix

The positive integral powers of a matrix A are defined only when A is a square matrix.

Also then $A^2 = A.A$ $A^3 = A.A.A = A^2A$

Also for any positive integers m, n

(i) $A^m A^n = A^{m+n}$ (ii) $(A^m)^n = A^{mn} = (A^n)^m$

(iii) $I^n = I$, $I^m = I$ (iv) $A^\circ = I_n$ where A is a square matrices of order n.

TRANSPOSE OF MATRIX

If we interchange the rows to columns and columns to rows of a matrix A, then the matrix so obtained is called the transpose of A and it is denoted by

$$A^T \text{ or } A^t \text{ or } A'$$

From this definition it is obvious to note that

(i) Order of A is $m \times n \Rightarrow$ order of A^T is $n \times m$

(ii) $(A^T)_{ij} = (A)_{ji}$, " i, j)

Properties of Transpose

If A, B are matrices of suitable order then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(A - B)^T = A^T - B^T$

(iv) $(kA)^T = kA^T$

(v) $(AB)^T = B^T A^T$

(vi) $(A_1 A_2 \ldots\ldots A_n)^T = A_n^T \ldots\ldots A_2^T A_1^T$

(vii) $(A^n)^T = (A^T)^n$, $n \in N$

SYMMETRIC AND SKEW-SYMMETRIC MATRIX

(a) Symmetric matrix : A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all $\hat{i} = \hat{j}$ or $A^T = A$.

Note : (i) Every unit matrix and square zero matrix are symmetric matrices.

(ii) Maximum number of different element in a symmetric matrix is $\frac{n(n+1)}{2}$

(b) Skew-symmetric matrix : A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix if

$a_{ij} = -a_{ji}$ for all i, j or $A^T = -A$

Note : (i) All principal diagonal elements of a skew-symmetric matrix are always zero because for any diagonal element - $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

(ii) Trace of a skew symmetric matrix is always 0

Properties of symmetric and skew-symmetric matrices

(i) If A is a square matrix, then $A + A^T$, AA^T, A^TA are symmetric matrices while $A - A^T$ is skew-symmetric matrices.

(ii) If A, B are two symmetric matrices, then-

(a) $A \pm B$, $AB + BA$ are also symmetric matrices.

(b) $AB - BA$ is a skew-symmetric matrix.

(c) AB is a symmetric matrix when AB = BA

(iii) If A, B are two skew-symmetric matrices, then-

(a) $A \pm B$, $AB - BA$ are skew-symmetric matrices.

(b) $AB + BA$ is a symmetric matrix.

(iv) If A is a skew-symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

(v) Every square matrix A can be uniquely be expressed as sum of a symmetric and skew symmetric matrix i.e.,

$$A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$$

DETERMINANT OF A MATRIX

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix, then its determinant, denoted by $|A|$ or det. (A) is

defined as $|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Properties of the determinant of a matrix

(i) $|A|$ exist $\Leftrightarrow$ A is a square matrix

(ii) $|AB| = |A|\ |B|$

(iii) $|A^T| = |A|$

(iv) $|kA| = k^n |A|$, if A is a square matrix of order n.

(v) If A and B are square matrices of same order then $|AB| = |BA|$

(vi) If A is skew symmetric matrix of odd order then $|A| = 0$

(vii) If $A = \text{diag}(a_1, a_2, ...a_n)$ then $|A| = a_1 a_2 ....a_n$

(viii) $|A|^n = |A^n|$, $n \in N$

ADJOINT OF A MATRIX

If every element of a square matrix A be replaced by its cofactor in $|A|$, then the transpose of the matrix so obtained is called the adjoint of A and it is denoted by adj A

Thus if $A = [a_{ij}]$ be a square matrix and C_{ij} be the cofactor of a_{ij} in $|A|$, then

$\text{adj } A = [C_{ij}]^T$

$\Rightarrow$ $(\text{adj } A)_{ij} = C_{ij}$

Hence if $A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}$, then $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & .... & C_{1n} \\ C_{21} & C_{22} & .... & C_{2n} \\ .... & .... & .... & .... \\ C_{n1} & C_{n2} & .... & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & .... & C_{n1} \\ C_{12} & C_{22} & .... & C_{n2} \\ .... & .... & .... & .... \\ C_{1n} & C_{2n} & .... & C_{nn} \end{bmatrix}$

Properties of Adjoint Matrix

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A\ (\text{adj } A) = |A|\ I_n = (\text{adj } A)\ A$

(Thus A (adj A) is always a scalar matrix)

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj}\ (\text{adj } A) = |A|^{n-2} A$

(iv) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

(v) $\text{adj}(A^T) = (\text{adj } A)^T$

(vi) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

(vii) $\text{adj}(A^m) = (\text{adj } A)^m$, $m \in N$

(viii) $\text{adj}(kA) = k^{n-1}(\text{adj } A)$, $k \in R$

(ix) $\text{adj}(I_n) = I_n$

(x) $\text{adj } 0 = 0$

(xi) A is symmetric $\Rightarrow$ adj A is also symmetric.

(xii) A is diagonal $\Rightarrow$ adj A is also diagonal.

(xiii) A is triangular $\Rightarrow$ adj A is also triangular.

(xiv) A is singular $\Rightarrow$ $|\text{adj } A| = 0$

INVERSE MATRIX

If A and B are two matrices such that

$$AB = I = BA$$

then B is called the inverse of A and it is denoted by A^{-1}. Thus

$$A^{-1} = B \Leftrightarrow AB = I = BA$$

Further we may note from above property (i) of adjoint matrix that if $|A| \neq 0$, then

$$A\frac{\text{adj}(A)}{|A|} = I = \frac{(\text{adj } A)}{|A|}A \qquad \Rightarrow \qquad A^{-1} = \frac{1}{|A|}\text{adj } A$$

Thus A^{-1} exists $\Leftrightarrow$ $|A| \neq 0$.

Note :

(i) Matrix A is called invertible if A^{-1} exists.

(ii) Inverse of a matrix is unique.

Properties of Inverse Matrix

(i) $(A^{-1})^{-1} = A$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(A^n)^{-1} = (A^{-1})^n$, $n \in N$

(v) $\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$

(vi) $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$

(vii) $A = \text{diag}(a_1, a_2, ..., a_n) \Rightarrow A^{-1} = \text{diag}(a_1^{-1}, a_2^{-1}, ..., a_n^{-1})$

(viii) A is symmetric $\Rightarrow$ A^{-1} is also symmetric.

(ix) A is diagonal $|A| \neq 0$ $\Rightarrow$ A^{-1} is also diagonal.

(x) A is scalar matrix $\Rightarrow$ A^{-1} is also scalar matrix.

(xi) A is triangular $|A| \neq 0$ $\Rightarrow$ A^{-1} is also triangular.

SOME IMPORTANT CASES OF MATRICES

Orthogonal Matrix

A square matrix A is called orthogonal if

$AA^T = I = A^TA$; i.e., if $A^{-1} = A^T$

Idempotent matrix

A square matrix A is called an idempotent matrix if $A^2 = A$

Involutory Matrix

A square matrix A is called an involutory matrix if $A^2 = I$ or $A^{-1} = A$

Nilpotent matrix

A square matrix A is called a nilpotent matrix if there exist a p $\in$ N such that $A^P = 0$

Hermition matrix

A square matrix A is skew-Hermition matrix if $A^q = A$; i.e., $a_{ij} = -\bar{a}_{ji}$ " i, j

Skew hermitian matrix

A square matrix A is skew-hermition is $A = -A^q$ i.e., $aij = -\bar{a}_{ji}$ " i, j

Period of a matrix

If for any matrix A $A^{k+1} = A$

then k is called period of matrix (where k is a least positive integer)

Differentiation of matrix

If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & l(x) \end{bmatrix}$

then $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & l'(x) \end{bmatrix}$ is a differentiation of matrix A

Submatrix

Let A be m × n matrix, then a matrix obtained by leaving some rows or columns or both of a is called a sub matrix of A

Rank of a matrix

A number r is said to be the rank of a m × n matrix A if

(a) every square sub matrix of order (r + 1) or more is singular and

(b) there exists at least one square submatrix of order r which is non-singular.

Thus, the rank of matrix is the order of the highest order non-singular sub matrix.

We have |A| = 0 therefore r (A) is less then 3, we observe that $\begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$ is a non-singular square sub matrix of order 2 hence r (A) 2.

Note :

(i) The rank of the null matrix is zero.

(ii) The rank of matrix is same as the rank of its transpose i.e., $r(A) = r(A^T)$

(iii) Elementary transformation of not alter the rank of matrix.

DETERMINANTS

DEFINITION

When an algebraic or numerical expression is expressed in a square form containing some rows and columns, this square form is named as a determinant of that expression. For example when expression $a_1b_2 - a_2 b_1$ is expressed in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

then it is called a determinant of order 2, Clearly a determinant of order 2 contains 2 rows and 2 columns. Similarly

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ is a determinant of order 3.

Obviously in every determinant, the number of rows and columns are equal and this number is called the order of that determinant.

REPRESENTATION OF A DETERMINANT

Generally we use Δ or $|A|$ symbols to express a determinant and a determinant of order 3 is represented by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

It should be noted that the (i, j)th element (i.e., the element of the i^{th} row and j^{th} column) of the determinant has been expressed by a_{ij} , i = 1 , 2 , 3 ; j = 1,2,3 . The elements for which i = j are called diagonal elements and the diagonal containing them is called principal diagonal or simply diagonal of the determinant. For the above determinant a_{11} , a_{22} , a_{33} are diagonal elements.

A determinant is called a triangular determinant if its every element above or below the diagonal is zero. For example

$$\begin{vmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{vmatrix}$$

is a triangular determinant. In particular when all the elements except diagonal elements are zero, then it is called a diagonal determinant. For example

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

is a diagonal determinant.

We generally use R_1, R_2, R_3, ... to denote first, second, third .... row and C_1, C_2 , C_3 .... to denote first, second, third ..... column of a determinant.

VALUE OF A DETERMINANT

The expression which has been expressed in a determinant form is called the value of that determinant.

To find the value of a third order determinant

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

be a third order determinant. To find its value we expand it by any row or column as the sum of three determinants of order 2. If we expand it by first row then

$$\Delta = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

MINOR AND COFACTOR OF AN ELEMENT

MINOR OF AN ELEMENT

Minor of an element of the determinants is obtain by leaving the row and column containing that element and retaining rest of elements.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$. Similarly $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Using this concept the value of Determinant can be

$\Delta = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$

or $\Delta = -a_{21} M_{21} + a_{22} M_{22} - a_{23} M_{23}$

or $\Delta = a_{31} M_{31} - a_{32} M_{32} + a_{33} M_{33}$

COFACTOR OF AN ELEMENT

The cofactor of an element a_{ij} is denoted by C_{ij} and is equal to $(-1)^{i+j} M_{ij}$ where M_{ij} is a minor of element a_{ij}

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

then $C_{11} = (-1)^{1+1} M_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Note :- (i) The sum of products of the element of any row with their corresponding cofactor is equal to the value of determinant i.e. $\Delta = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$

(ii) The sum of the product of element of any row with corresponding cofactor of another row is equal to zero i.e. $a_{11} C_{21} + a_{12} C_{22} + a_{13} C_{23} = 0$

(iii) If order of a determinant (Δ) is 'n' then the value of the determinant formed by replacing every element by its cofactor is Δ^{n-1}

PROPERTIES OF DETERMINANTS

If the elements of a determinant are complicated expressions or numbers, then it is very difficult to find its value by expansion method. In such cases we reduce the determinant into a simple one using the following properties.

P–1 The value of a determinant is unchanged if its rows and columns are interchanged. For example

$$\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & p & u \\ b & q & v \\ c & r & w \end{vmatrix}$$

P–2 The interchange of any two consecutive rows or columns will simply change the sign of the value of the determinant. For example

$$\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = -\begin{vmatrix} b & a & c \\ q & p & r \\ v & u & w \end{vmatrix} = -\begin{vmatrix} p & q & r \\ a & b & c \\ u & v & w \end{vmatrix}$$

P–3 If any two rows or columns of a determinant are identical then its value is zero. For example

$$\begin{vmatrix} a & b & c \\ a & b & c \\ u & v & w \end{vmatrix} = 0 = \begin{vmatrix} a & a & b \\ p & p & q \\ u & u & v \end{vmatrix}$$

P–4 If each element of a row or column of a determinant be multiplied by a number, then its value is also multiplied by that number. For example

$$\begin{vmatrix} ka & kb & kc \\ p & q & r \\ u & v & w \end{vmatrix} = k\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} ka & b & c \\ kp & q & r \\ ku & v & w \end{vmatrix}$$

P–5 If each entry in a row or column of a determinant is the sum of two numbers, then the determinant can be written as the sum of two determinants. For example

$$\begin{vmatrix} a+\alpha & b+\beta & c+\gamma \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} \alpha & \beta & \gamma \\ p & q & r \\ u & v & w \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a+\alpha & b & c \\ p+\beta & q & r \\ u+\gamma & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} \alpha & b & c \\ \beta & q & r \\ \gamma & v & w \end{vmatrix}$$

P–6 The value of a determinant does not change if the elements of a row (column) are added to or subtracted from the corresponding elements of another row (column). For example

$$\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a+\alpha b+\beta c & b & c \\ p+\alpha q+\beta r & q & r \\ u+\alpha v+\beta w & v & w \end{vmatrix}$$

P–7 If $\Delta = f(x)$ and $f(a) = 0$, then $(x - a)$ is a factor of Δ. For example in the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \text{ if we replace a by b then } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ b & b & c \\ b^2 & b^2 & c^2 \end{vmatrix} = 0$$

$\Rightarrow$ $(a - b)$ is a factor of Δ.

P–8 If each entry in any row (or column) of a determinant is zero, then the value of determinant is equal to zero.

MULTIPLICATION OF TWO DETERMINANTS

Multipication of two second order determinants is defined as follows

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$$

Multiplication of two third order determinants is defined as follows

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 + c_1 l_3 & a_1 m_1 + b_1 m_2 + c_1 m_3 & a_1 n_1 + b_1 n_2 + c_1 n_3 \\ a_2 l_1 + b_2 l_2 + c_2 l_3 & a_2 m_1 + b_2 m_2 + c_2 m_3 & a_2 n_1 + b_2 n_2 + c_3 n_3 \\ a_3 l_1 + b_3 l_2 + c_3 l_3 & a_3 m_1 + b_3 m_2 + c_3 m_3 & a_3 n_1 + b_3 n_2 + c_3 n_3 \end{vmatrix}$$

Note : In above case the order of Determinant is same, if the order is different then for their multiplication first of all they should be expressed in the same order

SYMMETRIC & SKEW SYMMETRIC DETERMINANT

Symmetric determinant

A determinant is called symmetric Determinant if for its every element.

$a_{ij} = a_{ji} \ \forall \ i, j$

Skew Symmetric determinant

A determinant is called skew Symmetric determinant if for its every element

$a_{ij} = -a_{ji} \ \forall \ i, j$;

Note : (i) Every diagonal element of a skew symmetric determinant is always zero

(ii) The value of a skew symmetric determinant of even order is always a perfect square and that of odd order is always zero.

i.e. (order = 2) i.e. (order = 3)

APPLICATIONS OF DETERMINANT

CRAMMER'S RULE :

Let the system of equations be

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$

and $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

then (i) If $\Delta \neq 0$, then given system of equations is consistent i.e. has unique solution and its solution is

$$\boxed{x = \frac{\Delta_1}{\Delta}, \ y = \frac{\Delta_2}{\Delta}, \ z = \frac{\Delta_3}{\Delta}}$$

This is known as Crammer's rule

(ii) If $\Delta = 0$ and atleast one of $\Delta_1, \Delta_2, \Delta_3$ is not zero, then the system of equations is inconsistent i.e. it has no solution.

(iii) If $\Delta = 0$ and $\Delta_1 = 0 = \Delta_2 = \Delta_3$, then the system has infinite solutions.

(iv) If $\Delta = 0$ and $d_1 = 0 = d_2 = d_3$, then the system of equations has infinite solutions (non-zero solution) i.e. non-trivial solutions

(v) If $\Delta \neq 0$ and $d_1 = 0 = d_2 = d_3$, then the system of equations has a unique solution x = 0 , y = 0 , z = 0 i.e., zero solution or trivial solution.

DIFFERENTIATION OF A DETERMINANT :

Let $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix}$,

then $\Delta'(x) = \begin{vmatrix} f_1'(x) & g'_1(x) & h'_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f'_2(x) & g'_2(x) & h'_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f'_3(x) & g'_3(x) & h'_3(x) \end{vmatrix}$

INTEGRATION OF DETERMINATION

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ p & q & r \\ l & m & n \end{vmatrix}$, where p, q, r, l, m and n are constants.

Then $\int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)dx \\ p & q & r \\ l & m & n \end{vmatrix}$

USE OF SUMMATION

If $f(r) = \begin{vmatrix} r & r^2 & r^3 \\ p & q & t \\ 1 & 2 & 3 \end{vmatrix}$, where p, q, t are constants, then $\sum_{r=1}^{n} f(r) = \begin{vmatrix} \sum_{r=1}^{n} r & \sum_{r=1}^{n} r^2 & \sum_{r=1}^{n} r^3 \\ p & q & t \\ 1 & 2 & 3 \end{vmatrix}$