9.7.2.1

EE24BTECH11047 - Niketh Prakash Achanta

Question:
Solve the differential equation:

$$xy'' + 2y' - xy + x^2 - 2 = 0 \tag{1}$$

Solution:

Theoretical solution:
The given differential equation is a second-order nonlinear ordinary differential equation and cannot be theoretically solved using known methods.
Computational Solution: Euler's method
By the first principle of derivative,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} \tag{2}$$

$$y(x+h) = y(x) + h\left(y'(x)\right), h \to 0 \tag{3}$$

For a m^{th} order differential equation,

Let

$$y_1 = y \ , \ y_2 = y' \ , \ y_3 = y'' \ , \ \dots \ , \ y_m = y^{m-1} \tag{4}$$

then we obtain the system

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{m-1}' \\ y_m' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \\ f(x, y_1, y_2, \dots, y_m) \end{pmatrix} \tag{5}$$

Here, f is described by the given differential equation. The initial conditions $y_1(x_0) = K_1$, $y_2(x_0) = K_2$, $\dots$, $y_m(x_0) = K_m$.

Representing the system in Euler's form (using first principle of derivative),

$$\begin{pmatrix} y_1(x+h) \\ y_2(x+h) \\ \vdots \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) + hy_2(x) \\ y_2(x) + hy_3(x) \\ \vdots \\ y_m(x) + hf(x, y_1, y_2 \ldots y_m) \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} y_1(x+h) \\ \vdots \\ y_{m-1}(x+h) \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) \\ \vdots \\ y_{m-1}(x) \\ y_m(x) \end{pmatrix} + h \begin{pmatrix} y_2(x) \\ \vdots \\ y_m(x) \\ f(x, y_1, y_2, \ldots, y_m) \end{pmatrix} \tag{7}$$

$$\mathbf{y}(x+h) = \mathbf{y}(x) + h \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & \frac{f(x, y_1, y_2, \ldots, y_m)}{y_m} \end{pmatrix} \mathbf{y}(x) \tag{8}$$

$$\mathbf{y}(x+h) = \begin{pmatrix} 1 & h & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & h & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & h & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & h \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 + \frac{f(x, y_1, y_2, \ldots, y_m)}{y_m} \end{pmatrix} \mathbf{y}(x) \tag{9}$$

Generalizing the system into an iterative format for plotting $y(x)$,

$$\begin{pmatrix} y_{1,n+1} \\ y_{2,n+1} \\ \vdots \\ y_{m,n+1} \end{pmatrix} = \begin{pmatrix} y_{1,n} \\ y_{2,n} \\ \vdots \\ y_{m,n} \end{pmatrix} + h \begin{pmatrix} y_{2,n} \\ y_{3,n} \\ \vdots \\ f(x_n, y_{1,n}, y_{2,n}, \ldots, y_{m,n}) \end{pmatrix} \tag{10}$$

$$\mathbf{y_{n+1}} = \mathbf{y_n} + h \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & \frac{f(x_n, y_{1,n}, y_{2,n}, \ldots, y_{m,n})}{y_{m,n}} \end{pmatrix} \mathbf{y_n}, \text{ where } \mathbf{y_n} = \begin{pmatrix} y_{1,n}(x_n) \\ y_{2,n}(x_n) \\ \vdots \\ y_{m,n}(x_n) \end{pmatrix} \tag{11}$$

$$\mathbf{y_{n+1}} = \begin{pmatrix} 1 & h & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & h & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & h & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & h \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 + \frac{f(x_n, y_{1,n}, y_{2,n}, \ldots, y_{m,n})}{y_{m,n}} \end{pmatrix} \mathbf{y_n} \tag{12}$$

$$x_{n+1} = x_n + h \tag{13}$$

Here, the vector $\mathbf{y_n}$ is not to be confused with y_k which is the $(k-1)^{\text{th}}$ derivative of $y(x)$. The given differential equation can be represented as,

$$xy'' + 2y' + x^2 - xy - 2 = 0 \tag{14}$$

$$y'' = \frac{xy - x^2 + 2 - 2y'}{x} \tag{15}$$

Here $m = 2$, thus,

$$y_3 = y'' = \frac{xy - x^2 + 2 - 2y'}{x} = \frac{xy_1 - x^2 + 2 - 2y_2}{x} \tag{16}$$

$$\tag{17}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ \frac{xy - x^2 + 2 - 2y'}{x} \end{pmatrix} \tag{18}$$

$$\begin{pmatrix} y_{1,n+1} \\ y_{2,n+1} \end{pmatrix} = \begin{pmatrix} y_{1,n} \\ y_{2,n} \end{pmatrix} + h \begin{pmatrix} y_{2,n} \\ \frac{xy_{1,n} - x^2 + 2 - 2y_{2,n}}{x} \end{pmatrix} \tag{19}$$

$$\mathbf{y_{n+1}} = \begin{pmatrix} 1 & h \\ 0 & 1 + \frac{xy_{1,n} - x^2 + 2 - 2y_{2,n}}{x} \end{pmatrix} \mathbf{y_n} \tag{20}$$

Iteratively plotting the above system taking initial conditions as

$$x_0 = 0.1 \tag{21}$$

$$y_{1,0} = 1 \tag{22}$$

$$y_{2,0} = 0 \tag{23}$$

We get the following plot.

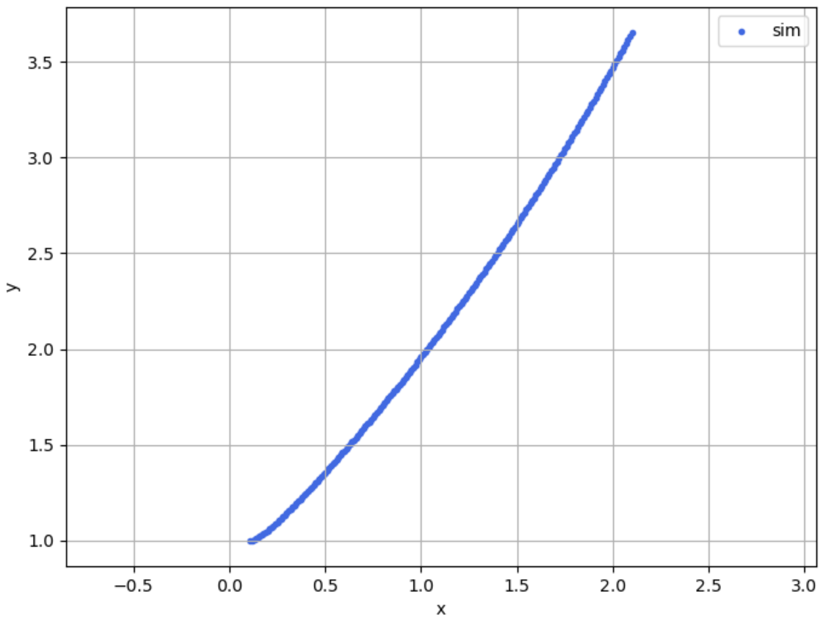

Fig. 0