8.3.16

EE24BTECH11047 - Niketh Prakash Achanta

Question:
Find the area of the region bounded by $y = x^3$, the x-axis, and the ordinates $x = -2$ and $x = 1$.
Solution:
Theoretical Solution:
Area under the curve is given by:

$$A = \int_{-2}^{1} |x^3| \, dx \tag{0.1}$$

Split the integral at $x = 0$, as $y = x^3$ changes sign:

$$A = \int_{-2}^{0} -x^3 \, dx + \int_{0}^{1} x^3 \, dx \tag{0.2}$$

Compute each integral:

$$\int_{-2}^{0} -x^3 \, dx = -\int_{-2}^{0} x^3 \, dx \tag{0.3}$$

$$= -\frac{x^4}{4}\Big|_{-2}^{0} \tag{0.4}$$

$$= -\left(0 - \frac{(-2)^4}{4}\right) \tag{0.5}$$

$$= -\left(0 - \frac{16}{4}\right) \tag{0.6}$$

$$= 4 \tag{0.7}$$

$$\int_{0}^{1} x^3 \, dx = \frac{x^4}{4}\Big|_{0}^{1} \tag{0.8}$$

$$= \frac{1^4}{4} - \frac{0^4}{4} \tag{0.9}$$

$$= \frac{1}{4} \tag{0.10}$$

Add the results:

$$A = 4 + \frac{1}{4} \tag{0.11}$$

$$= \frac{17}{4} \tag{0.12}$$

Computational Solution: Taking trapezoid-shaped strips of small area and adding them all up. Say we have to find the area of y_x from $x = x_0$ to $x = x_n$, discretize the points on the x axis $x_0, x_1, x_2, \ldots, x_n$ such that they are equally spaced with the step size h.
Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2}h\left(y\left(x_1\right) + y\left(x_0\right)\right) + \frac{1}{2}h\left(y\left(x_2\right) + y\left(x_1\right)\right) + \cdots + \frac{1}{2}h\left(y\left(x_n\right) + y\left(x_{n-1}\right)\right) \tag{0.13}$$

$$= h\left[\frac{1}{2}\left(y\left(x_0\right) + y\left(x_n\right)\right) + y\left(x_1\right) + \cdots + y\left(x_{n-1}\right)\right] \tag{0.14}$$

Let $A\left(x_n\right)$ be the area enclosed by the curve $y\left(x\right)$ from $x = x_0$ to $x = x_n$, $\left(x_0, x_1, \ldots x_n\right)$ be equidistant points with step-size h.

$$A\left(x_n + h\right) = A\left(x_n\right) + \frac{1}{2}h\left(y\left(x_n + h\right) + y\left(x_n\right)\right) \tag{0.15}$$

We can repeat this till we get the required area.
Discretizing the steps, making $A\left(x_n\right) = A_n, y\left(x_n\right) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h\left(y_{n+1} + y_n\right) \tag{0.16}$$

We can write y_{n+1} in terms of y_n using the first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left(\left(y_n + hy'_n\right) + y_n\right) \tag{0.17}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{0.18}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{0.19}$$

$$x_{n+1} = x_n + h \tag{0.20}$$

In the given question, $y_n = x_n^3$ and $y'_n = 3x_n^2$.
The general difference equation will be given by

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{0.21}$$

$$A_{n+1} = A_n + h\left(x_n^3\right) + \frac{1}{2}h^2\left(3x_n^2\right) \tag{0.22}$$

$$x_{n+1} = x_n + h \tag{0.23}$$

Iterating till we reach $x_n = 1$ will return the required area.

Area obtained computationally ≈ 4 sq. units.

Area obtained theoretically: $\frac{17}{4} = 4.25$ sq.units

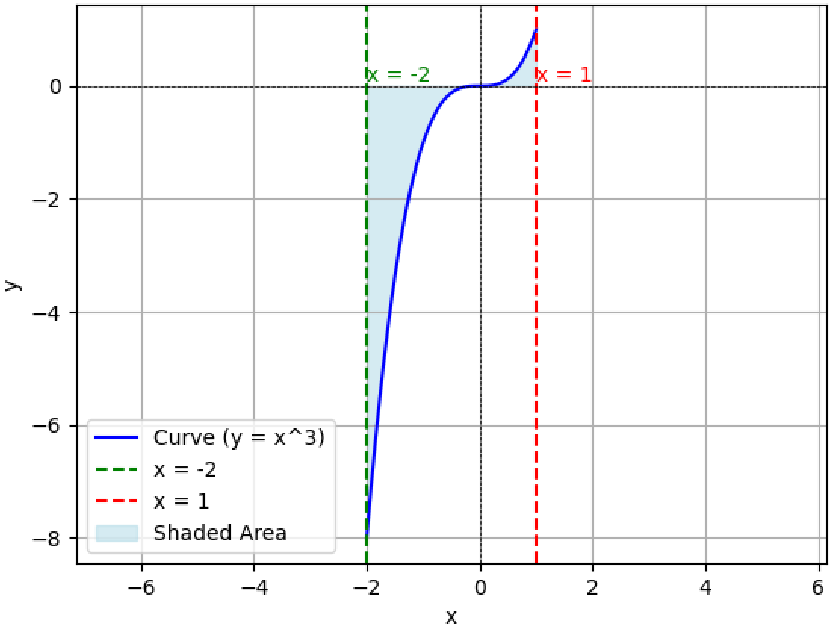

Fig. 0.1