11.16.3.8.3

EE24BTECH11047 - Niketh Prakash Achanta

1 Problem Statement

Question:
Three coins are tossed once. Find the probability of getting at least 2 heads.

2 Solution

2.1 Random Variables

Let X_1, X_2, X_3 be three independent random variables representing the outcomes of the three coin tosses:

$$X_i = \begin{cases} 1, & \text{if Head appears,} \\ 0, & \text{if Tail appears.} \end{cases} \tag{0.1}$$

Each X_i follows a Bernoulli distribution with:

$$P_X(1) = p = \frac{1}{2}, \quad P_X(0) = q = \frac{1}{2}. \tag{0.2}$$

2.2 Probability Mass Function (PMF) Using Z-Transform

The PMF of X_i is:

$$P_X(n) = p^n q^{1-n}. \tag{0.3}$$

The z-transform of this PMF is:

$$P_{X_i}(z) = \mathbb{E}[z^{X_i}] = P_X(0) + P_X(1)z^{-1} = q + \frac{p}{z}. \tag{0.4}$$

For each X_i:

$$P_{X_i}(z) = \frac{1}{2} + \frac{1}{2}z^{-1}. \tag{0.5}$$

2.3 Distribution of $Y = X_1 + X_2 + X_3$

Using the independence property, the z-transform of Y is:

$$P_Y(z) = P_{X_1}(z) \cdot P_{X_2}(z) \cdot P_{X_3}(z). \tag{0.6}$$

$$P_Y(z) = \left(\frac{1}{2} + \frac{1}{2z}\right)^3. \tag{0.7}$$

Expanding, we get:

$$P_Y(z) = \frac{1}{8}\left(1 + \frac{3}{z} + \frac{3}{z^2} + \frac{1}{z^3}\right). \tag{0.8}$$

Thus, the PMF of Y is:

$$P_Y(k) = \begin{cases} \frac{1}{8}, & k = 0, \\ \frac{3}{8}, & k = 1, \\ \frac{3}{8}, & k = 2, \\ \frac{1}{8}, & k = 3. \end{cases} \tag{0.9}$$

We can also express this PMF in a summation notation:

$$P_Y(k) = \sum_{k=0}^{3} P_Y(k) \quad \text{where} \quad P_Y(k) \in \left\{\frac{1}{8}, \frac{3}{8}\right\}, \quad k = 0, 1, 2, 3. \tag{0.10}$$

2.4 Cumulative Distribution Function (CDF)

The CDF of Y is:

$$F_Y(k) = \begin{cases} 0, & k < 0, \\ \frac{1}{8}, & 0 \le k < 1, \\ \frac{4}{8}, & 1 \le k < 2, \\ \frac{7}{8}, & 2 \le k < 3, \\ 1, & k \ge 3. \end{cases} \tag{0.11}$$

2.5 Probability of At Least Two Heads

To find $P(Y \ge 2)$, we use:

$$P(Y \ge 2) = 1 - P(Y < 2). \tag{0.12}$$

$$P(Y \ge 2) = 1 - (P(Y = 0) + P(Y = 1)). \tag{0.13}$$

$$P(Y \ge 2) = 1 - \left(\frac{1}{8} + \frac{3}{8}\right) = \frac{4}{8} = \frac{1}{2}. \tag{0.14}$$

3 Conclusion

The probability that at least two of the three coin tosses result in heads is:

$$P(Y \ge 2) = \frac{1}{2}. \tag{0.15}$$

We get the following plot.

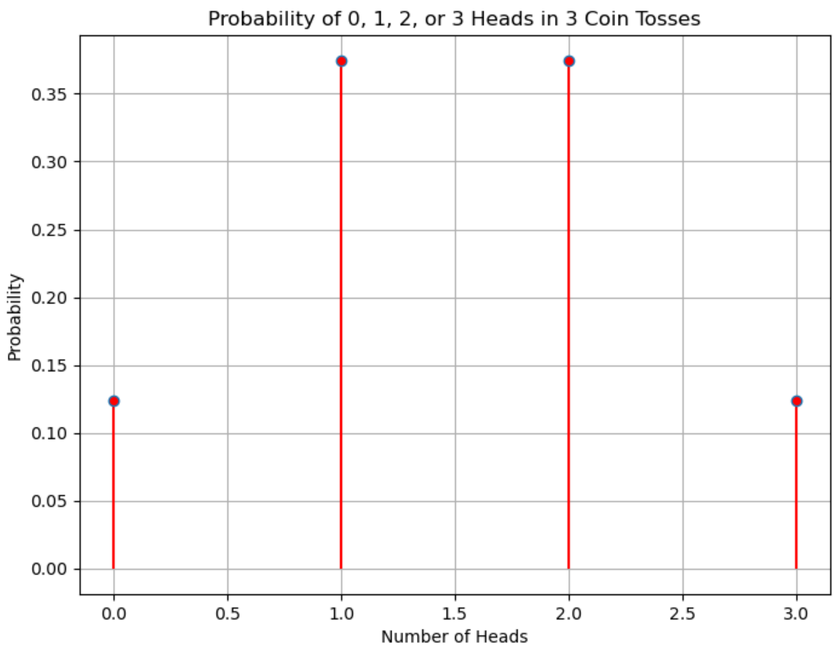
Probability of 0, 1, 2, or 3 Heads in 3 Coin Tosses
Probability
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Number of Heads

Fig. 0.1