HardHaQ '25 Trapped Ion Problem Set Submission

Team Name: *Hard Nanos*
Members: *Nikhil, Rebanta, Lucas*

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1 Mathematical Framework

1.1 Shorthand useful relations

The Mathieu parameter (radial) is

$$q = \frac{2QV}{mr_0^2\Omega^2}. \tag{1}$$

Here Q is the ion charge (C), m is the ion mass (kg), V is the RF peak amplitude (V), Ω is the RF angular frequency ($\mathrm{rad\,s^{-1}}$), and r_0 is the characteristic trap radius (m). The secular frequency (radial secular motion) in the small-q limit is approximated by

$$\omega_{\text{sec}} \approx \frac{q}{2\sqrt{2}}\Omega = \frac{QV}{\sqrt{2}\,mr_0^2\Omega}. \tag{2}$$

1.2 Pseudopotential depth (U_{depth})

1.2.1 Governing formula

The time-averaged pseudopotential (in 1D radial coordinate r for an ideal quadrupole) is

$$\Psi(r) = \frac{Q^2}{4m\Omega^2}\,|\mathbf{E}_{\text{rf}}(r)|^2.$$

Here $\Psi(\mathbf{r})$ is the time-averaged pseudopotential (J). For the ideal quadrupole field amplitude $|\mathbf{E}_{\text{rf}}| \sim Vr/r_0^2$, this gives a quadratic pseudopotential:

$$\Psi(r) = \kappa\,\frac{Q^2V^2}{4m\Omega^2r_0^4}\,r^2,$$

so a characteristic *trap depth* (energy scale between center and effective boundary $r \sim r_0$) is commonly approximated as

$$U_{\text{depth}} \approx \kappa\,\frac{Q^2V^2}{4m\Omega^2r_0^2}. \tag{3}$$

Here U_{depth} denotes the trap depth (J) and κ is a dimensionless geometry factor (order unity). An alternative useful form in terms of the Mathieu parameter q is

$$U_{\text{depth}} = \kappa \frac{m\Omega^2 r_0^2}{16} q^2. \tag{4}$$

1.2.2 How each variable affects depth

- **RF amplitude V:** $U_{\text{depth}} \propto V^2$. *Increasing V increases depth quadratically.*
- **Drive frequency Ω:** $U_{\text{depth}} \propto 1/\Omega^2$. *Increasing Ω reduces depth (inverse square).*
- **Trap size r_0:** $U_{\text{depth}} \propto 1/r_0^2$. *Increasing r_0 decreases depth (inverse square).*
- **Ion properties Q, m:** $U_{\text{depth}} \propto Q^2/m$. Higher charge increases depth; heavier mass decreases depth.
- **Geometry κ:** improves depth linearly with κ (ideal electrodes have $\kappa \approx 1$).

1.2.3 Practical remarks and trade-offs

- Increasing V increases q. Large q invalidates the pseudopotential approximation and increases micromotion.

1.3 Center offset (x_0) — static displacement and micromotion

1.3.1 Governing formula

If a stray DC electric field E_{dc} exists at the trap center, it exerts a force QE_{dc} on the ion. Balance this against the secular restoring force $m\omega_{\text{sec}}^2 x$ to obtain the static displacement (offset) x_0:

$$x_0 = \frac{QE_{\text{dc}}}{m\omega_{\text{sec}}^2}. \tag{5}$$

where E_{dc} is the stray DC electric field (V/m) producing the static offset. Using the small-q expression for ω_{sec},

$$\omega_{\text{sec}} \approx \frac{QV}{\sqrt{2}\, m r_0^2 \Omega},$$

we can express x_0 in terms of trap parameters:

$$x_0 \approx \frac{QE_{\text{dc}}}{m} \left(\frac{\sqrt{2}\, m r_0^2 \Omega}{QV} \right)^2 = \frac{2 m r_0^4 \Omega^2}{QV^2} E_{\text{dc}}. \tag{6}$$

1.3.2 Excess micromotion amplitude (leading order)

For small q the amplitude of *excess* micromotion driven at RF frequency is approximately proportional to q times the static offset:

$$x_{\mathrm{mm}} \approx \frac{q}{2}\, x_0. \tag{7}$$

Kinetic energy associated with excess micromotion scales as $\sim \frac{1}{2}m(\Omega x_{\mathrm{mm}})^2$.

1.3.3 How each variable affects center offset

- **Stray field E_{dc}:** $x_0 \propto E_{\mathrm{dc}}$ (linear). More stray field $\Rightarrow$ larger offset.
- **Secular stiffness ω_{sec}:** $x_0 \propto 1/\omega_{\mathrm{sec}}^2$. Stronger confinement (higher ω_{sec}) $\Rightarrow$ smaller x_0.
- **RF amplitude V:** Increasing V increases ω_{sec} (for fixed Ω and r_0) and thus reduces x_0; but increasing V also increases q and so multiplies offset into micromotion amplitude.
- **Drive frequency Ω:** Appears in ω_{sec} inversely; higher Ω (for fixed V) tends to reduce x_0 via ω_{sec} formula in practice.
- **Trap size r_0:** Larger r_0 reduces secular frequency (for fixed V, Ω) so x_0 grows with r_0.

1.3.4 Minimization strategies

- **Increase secular stiffness:** Raise V or reduce r_0 (within other constraints) to reduce x_0.
- **Balance trade-offs:** Increasing V reduces x_0 but increases q (and thus micromotion amplitude $x_{\mathrm{mm}} \propto q x_0$). Optimize to minimize residual micromotion energy.

1.4 RF power and power-per-depth (P/U_{depth})

1.4.1 Model for RF power

The RF power is modelled by the following equation extracted from the COMSOL file:

$$P_{\mathrm{est}} = 10^3 \left(\omega_{\mathrm{sec}}\big(\varepsilon \pi L\big)\frac{V_{\mathrm{rf}}}{\sqrt{2}}\right)^2 \tag{8}$$

Here V_{rf} is the RF peak amplitude. In practice, P_{est} also depends on the effective electrode capacitance and series resistance of the RF drive and electrode structure (but, we assume it is negligible).

1.4.2 Power per depth

Combine (8) with the depth formula (3) which scales as $U_{\text{depth}} \propto V^2$. Eliminating V^2 yields:

$$\frac{P_{\text{est}}}{U_{\text{depth}}} \propto \frac{10^3 \left(\omega_{\text{sec}} \left(\varepsilon \pi L\right) \frac{V_{\text{rf}}}{\sqrt{2}}\right)^2}{\kappa \frac{m\Omega^2 r_0^2}{16} q^2} \Rightarrow \ldots \Rightarrow \frac{P_{\text{est}}}{U_{\text{depth}}} \propto \frac{10^3 (\varepsilon \pi L)^2 V_{\text{rf}}^2}{\kappa m r_0^2}. \tag{9}$$

where the constant prefactor depends on the precise definitions used for U_{depth} and P_{est}; the displayed scaling is the relevant design dependence.

1.4.3 How each variable affects P/U_{depth}

- **Trap size r_0:** $P/U_{\text{depth}} \propto r_0^2$ (scales with trap area/size) — larger traps cost more power per depth.
- **Ion properties:** $P/U_{\text{depth}} \propto m/Q^2$ — heavier ions increase power per depth; higher ionic charge reduces it.
- **RF voltage V_{rf}:** $P/U_{\text{depth}} \propto V_{\text{rf}}^2$; smaller V_{rf} reduces power cost.
- **Electrode length L:** $P/U_{\text{depth}} \propto L^2$; smaller L reduces power cost if the quadrupole field quality is preserved.

1.4.4 Minimization of P/U_{depth}

1. **Choose moderate Ω:** Avoid excessively high drive frequencies unless required for stability/performance.
2. **Reduce r_0 where acceptable:** Smaller traps require less RF voltage for a given depth.