is said to be a *geometric* random variable with parameter p. Such a random variable represents the trial number of the first success when each trial is independently a success with probability p. Its mean and variance are given by

$$E[X] = \frac{1}{p} \quad \text{Var}(X) = \frac{1-p}{p^2}$$

The random variable X whose probability mass function is given by

$$p(i) = \binom{i-1}{r-1} p^r (1-p)^{i-r} \quad i \geq r$$

is said to be a *negative binomial* random variable with parameters r and p. Such a random variable represents the trial number of the rth success when each trial is independently a success with probability p. Its mean and variance are given by

$$E[X] = \frac{r}{p} \quad \text{Var}(X) = \frac{r(1-p)}{p^2}$$

A *hypergeometric* random variable X with parameters n, N, and m represents the number of white balls selected when n balls are randomly chosen from an urn that contains N balls of which m are white. The probability mass function of this random variable is given by

$$p(i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}} \quad i = 0, \ldots, m$$

With $p = m/N$, its mean and variance are

$$E[X] = np \quad \text{Var}(X) = \frac{N-n}{N-1} np(1-p)$$

An important property of the expected value is that the expected value of a sum of random variables is equal to the sum of their expected values. That is,

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Problems

4.1. Sophie is choosing two coins randomly from a box containing four \$2, five \$1, eight 50¢, and three 20¢ coins. Let X denote Sophie's income. What are the possible values of X, and what are the probabilities associated with each value?

4.2. Two fair dice are rolled. Let X equal the ratio of the value on the first die to that on the second die. Find the probabilities attached to the possible values that X can take on.

4.3. Three fair dice are rolled. Assume that all $6^3 = 216$ possible outcomes are equally likely. Let X equal the product of the 3 dice. Find the probabilities attached to the possible values that X can take on.

4.4. Six men and 4 women are ranked according to the time they took to complete a 5-mile trail run. Assume that no two individuals took the same time and that all 10! possible rankings are equally likely. What is the probability that at least one out of the three highest ranking individuals is a woman?

4.5. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X?

4.6. In Problem 4.5, for $n = 3$, if the coin is assumed fair, what are the probabilities associated with the values that X can take on?

4.7. Suppose that a die is rolled twice. What are the possible values that the following random variables can take on:

(a) the maximum value to appear in the two rolls;
(b) the minimum value to appear in the two rolls;
(c) the sum of the two rolls;
(d) the value of the first roll minus the value of the second roll?

4.8. If the die in Problem 4.7 is assumed fair, calculate the probabilities associated with the random variables in parts (a) through (d).

4.9. Repeat Example 1c when the balls are selected with replacement.

4.10. Let X be the winnings of a gambler. Let $p(i) = P(X = i)$ and suppose that

$$p(0) = 1/3; p(1) = p(-1) = 13/55;$$
$$p(2) = p(-2) = 1/11; p(3) = p(-3) = 1/165$$

Compute the conditional probability that the gambler wins i, $i = 1, 2, 3$, given that he wins a positive amount.

4.11. The random variable X is said to follow the distribution of Benford's Law if

$$P(X = i) = \log_{10}\left(\frac{i+1}{i}\right), \quad i = 1, 2, 3, \ldots, 9$$

It has been shown to be a good fit for the distribution of the first digit of many real life data values.

(a) Verify that the preceding is a probability mass function by showing that $\sum_{i=1}^{9} P(X = i) = 1$.
(b) Find $P(X \leq j)$.

4.12. In the game of Two-Finger Morra, 2 players show 1 or 2 fingers and simultaneously guess the number of fingers their opponent will show. If only one of the players guesses correctly, he wins an amount (in dollars) equal to the sum of the fingers shown by him and his opponent. If both players guess correctly or if neither guesses correctly, then no money is exchanged. Consider a specified player, and denote by X the amount of money he wins in a single game of Two-Finger Morra.

(a) If each player acts independently of the other, and if each player makes his choice of the number of fingers he will hold up and the number he will guess that his opponent will hold up in such a way that each of the 4 possibilities is equally likely, what are the possible values of X and what are their associated probabilities?

(b) Suppose that each player acts independently of the other. If each player decides to hold up the same number of fingers that he guesses his opponent will hold up, and if each player is equally likely to hold up 1 or 2 fingers, what are the possible values of X and their associated probabilities?

4.13. A man wants to buy tablets for his two daughters as Christmas gifts. He goes to an electronics shop that has the two latest models. The probability that Sabrina, the older daughter, will accept the gift is .9, whereas the probability that Samantha, the younger daughter, will accept the gift is .7. These two probabilities are independent. There is a .8 probability that Sabrina will choose the first model, which costs \$600, and a .2 probability that she chooses the second model, which costs \$450. Samantha is equally likely to opt for either model. Determine the expected total cost that will be incurred by the man.

4.14. Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find $P\{X = i\}, i = 0, 1, 2, 3, 4$.

4.15. A state wants to select 10 players along with a goalkeeper from 9 football teams who will represent their state in the national league. An urn consisting of 45 balls is used for the selection of the players. Each of the balls is inscribed with the name of a team: 9 balls have the name of the best-performing team, 8 balls have the name of the second best-performing team, and so on (with 1 ball for the worst-performing team). A ball is chosen at random, and the team whose name is on the ball is instructed to pick a player who will join the state's team. Another ball is then chosen at random and the team named on the ball is asked to pick a player. A third ball is randomly chosen and the team named on the ball (provided that not all 3 chosen balls are of the same team) is asked to choose the third player. If 3 balls are chosen from the same team, the third ball is replaced and another one is chosen. This continues until a ball from another team is chosen. The 7 remaining players are then picked in a way from the teams that were not picked from the urn such that all 9 teams are represented at least once. If all 3 chosen balls are of a different team, then 2 out of the 7 remaining players are selected out of the best-performing team which was not chosen from the urn. What is the probability that the third best-performing team in the competition will have two representative players?

4.16. A deck of n cards numbered 1 through n are to be turned over one a time. Before each card is shown you are to guess which card it will be. After making your guess, you are told whether or not your guess is correct but not which card was turned over. It turns out that the strategy that maximizes the expected number of correct guesses fixes a permutation of the n cards, say $1, 2, \ldots, n$, and then continually guesses 1 until it is correct, then continually guesses 2 until either it is correct or all cards have been turned over, and then continually guesses 3, and so on. Let G denote the number of correct guesses yielded by this strategy. Determine $P(G = k)$.

Hint: In order for G to be at least k what must be the order of cards $1, \ldots, k$.

4.17. Suppose that the distribution function of the random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{2} & 0 \leq x < 1 \\ \dfrac{x+1}{4} & 1 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

(a) Find $P\{X < 1\}$.
(b) Find $P\{X > 2\}$.
(c) Find $P\left\{\frac{1}{3} < X < \frac{5}{3}\right\}$.

4.18. During a tournament, a football team plays a match against 3 different teams. The probabilities that this team wins against the first, second, and third teams are .8, .65, and .3, respectively, and are independent. Let X denote the number of wins obtained. Calculate the probability mass function of X.

4.19. If the distribution function of the random variable X is given by

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{4} & 1 \le x < 3 \\ \frac{5}{8} & 3 \le x < 4 \\ \frac{3}{4} & 4 \le x < 6 \\ \frac{7}{8} & 6 \le x < 7 \\ 1 & x \ge 7 \end{cases}$$

calculate the probability mass function of X.

4.20. A gambling book recommends the following "winning strategy" for the game of roulette: Bet \$1 on red. If red appears (which has probability $\frac{18}{38}$), then take the \$1 profit and quit. If red does not appear and you lose this bet (which has probability $\frac{20}{38}$ of occurring), make additional \$1 bets on red on each of the next two spins of the roulette wheel and then quit. Let X denote your winnings when you quit.

(a) Find $P\{X > 0\}$.

(b) Are you convinced that the strategy is indeed a "winning" strategy? Explain your answer!

(c) Find $E[X]$.

4.21. Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students who were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.

(a) Which of $E[X]$ or $E[Y]$ do you think is larger? Why?

(b) Compute $E[X]$ and $E[Y]$.

4.22. Suppose that two teams play a series of games that ends when one of them has won i games. Suppose that each game played is, independently, won by team A with probability p. Find the expected number of games that are played when (a) $i = 2$ and (b) $i = 3$. Also, show in both cases that this number is maximized when $p = \frac{1}{2}$.

4.23. You have \$1000, and a certain commodity presently sells for \$2 per ounce. Suppose that after one week the commodity will sell for either \$1 or \$4 an ounce, with these two possibilities being equally likely.

(a) If your objective is to maximize the expected amount of money that you possess at the end of the week, what strategy should you employ?

(b) If your objective is to maximize the expected amount of the commodity that you possess at the end of the week, what strategy should you employ?

4.24. A and B play the following game: A writes down either number 1 or number 2, and B must guess which one. If the number that A has written down is i and B has guessed correctly, B receives i units from A. If B makes a wrong guess, B pays $\frac{3}{4}$ unit to A. If B randomizes his decision by guessing 1 with probability p and 2 with probability $1 - p$, determine his expected gain if (a) A has written down number 1 and (b) A has written down number 2.

What value of p maximizes the minimum possible value of B's expected gain, and what is this maximin value? (Note that B's expected gain depends not only on p, but also on what A does.)

Consider now player A. Suppose that she also randomizes her decision, writing down number 1 with probability q. What is A's expected loss if (c) B chooses number 1 and (d) B chooses number 2?

What value of q minimizes A's maximum expected loss? Show that the minimum of A's maximum expected loss is equal to the maximum of B's minimum expected gain. This result, known as the *minimax theorem*, was first established in generality by the mathematician John von Neumann and is the fundamental result in the mathematical discipline known as the theory of games. The common value is called the value of the game to player B.

4.25. Four coins are flipped. The first two coins are fair, whereas the third and fourth coins are biased. The latter coins land on heads with probabilities .7 and .4, respectively. Assume that the outcomes of the flips are independent. Find the probability that

(a) exactly one head appears;

(b) two heads appear.

4.26. One of the numbers 1 through 10 is randomly chosen. You are to try to guess the number chosen by asking questions with "yes–no" answers. Compute the expected number of questions you will need to ask in each of the following two cases:

(a) Your ith question is to be "Is it i?" $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

(b) With each question you try to eliminate one-half of the remaining numbers, as nearly as possible.

4.27. An insurance company writes a policy to the effect that an amount of money A must be paid if some event E occurs within a year. If the company estimates that E will occur within a year with probability p, what should it charge the customer in order that its expected profit will be 10 percent of A?

4.28. A teacher selects a group of 5 students at random from her class consisting of 11 female students and 10 male students. Find the expected number of female students in the group.

4.29. There are two possible causes for a breakdown of a machine. To check the first possibility would cost C_1 dollars, and, if that were the cause of the breakdown, the trouble could be repaired at a cost of R_1 dollars. Similarly, there are costs C_2 and R_2 associated with the second possibility. Let p and $1 - p$ denote, respectively, the probabilities that the breakdown is caused by the first and second possibilities. Under what conditions on $p, C_i, R_i, i = 1, 2$, should we check the first possible cause of breakdown and then the second, as opposed to reversing the checking order, so as to minimize the expected cost involved in returning the machine to working order?
Note: If the first check is negative, we must still check the other possibility.

4.30. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the nth flip, the person wins 2^n dollars. Let X denote the player's winnings. Show that $E[X] = +\infty$. This problem is known as the St. Petersburg paradox.

(a) Would you be willing to pay \$1 million to play this game once?

(b) Would you be willing to pay \$1 million for each game if you could play for as long as you liked and only had to settle up when you stopped playing?

4.31. Each night different meteorologists give us the probability that it will rain the next day. To judge how well these people predict, we will score each of them as follows: If a meteorologist says that it will rain with probability p, then he or she will receive a score of

$$\begin{array}{ll} 1 - (1 - p)^2 & \text{if it does rain} \\ 1 - p^2 & \text{if it does not rain} \end{array}$$

We will then keep track of scores over a certain time span and conclude that the meteorologist with the highest average score is the best predictor of weather. Suppose now that a given meteorologist is aware of our scoring mechanism and wants to maximize his or her expected score. If this person truly believes that it will rain tomorrow with probability p^*, what value of p should he or she assert so as to maximize the expected score?

4.32. To determine whether they have a certain disease, 100 people are to have their blood tested. However, rather than testing each individual separately, it has been decided first to place the people into groups of 10. The blood samples of the 10 people in each group will be pooled and analyzed together. If the test is negative, one test will suffice for the 10 people, whereas if the test is positive, each of the 10 people will also be individually tested and, in all, 11 tests will be made on this group. Assume that the probability that a person has the disease is .1 for all people, independently of one another, and compute the expected number of tests necessary for each group. (Note that we are assuming that the pooled test will be positive if at least one person in the pool has the disease.)

4.33. A newsboy purchases papers at 10 cents and sells them at 15 cents. However, he is not allowed to return unsold papers. If his daily demand is a binomial random variable with $n = 10, p = \frac{1}{3}$, approximately how many papers should he purchase so as to maximize his expected profit?

4.34. In Example 4b, suppose that the department store incurs an additional cost of c for each unit of unmet demand. (This type of cost is often referred to as a goodwill cost because the store loses the goodwill of those customers whose demands it cannot meet.) Compute the expected profit when the store stocks s units, and determine the value of s that maximizes the expected profit.

4.35. Two cards are drawn at random from an ordinary deck of 52 playing cards. If the two cards display the same suit, you win \$2. If they are of the same color only, you win \$1. Otherwise, you lose 50¢. Calculate

(a) the expected value of the amount you win;

(b) the variance of the amount you win.

4.36. Consider the friendship network described by Figure 4.5. Let X be a randomly chosen person and let Z be a randomly chosen friend of X. With $f(i)$ equal to the number of friends of person i, show that $E[f(Z)] \geq E[f(X)]$.

4.37. Consider Problem 4.22 with $i = 2$. Find the variance of the number of games played, and show that this number is maximized when $p = \frac{1}{2}$.

4.38. Find $\text{Var}(X)$ and $\text{Var}(Y)$ for X and Y as given in Problem 4.21.

4.39. If $E[X] = 3$ and $Var(X) = 1$, find

(a) $E[(4X - 1)^2]$;

(b) $Var(5 - 2X)$.

4.40. A card is drawn at random from an ordinary deck of 52 playing cards. After the card is drawn, it is replaced. The deck is reshuffled and another card is drawn at random. This process goes on indefinitely. What is the probability that exactly 3 out of the first 5 cards that have been drawn are red?

4.41. On a multiple-choice test with 3 possible answers for each of the 8 questions, what is the probability that a student who has not studied for the test will guess at least 6 correct answers?

4.42. A fair die is tossed 10 times consecutively. What is the probability that at most 6 out of the 10 tosses result in an even number?

4.43. A and B will take the same 10-question examination. Each question will be answered correctly by A with probability .7, independently of her results on other questions. Each question will be answered correctly by B with probability .4, independently both of her results on the other questions and on the performance of A.

(a) Find the expected number of questions that are answered correctly by both A and B.

(b) Find the variance of the number of questions that are answered correctly by either A or B.

4.44. A communications channel transmits the digits 0 and 1. However, due to static, the digit transmitted is incorrectly received with probability .2. Suppose that we want to transmit an important message consisting of one binary digit. To reduce the chance of error, we transmit 00000 instead of 0 and 11111 instead of 1. If the receiver of the message uses "majority" decoding, what is the probability that the message will be wrong when decoded? What independence assumptions are you making?

4.45. A satellite system consists of n components and functions on any given day if at least k of the n components function on that day. On a rainy day, each of the components independently functions with probability p_1, whereas on a dry day, each independently functions with probability p_2. If the probability of rain tomorrow is α, what is the probability that the satellite system will function?

4.46. A student is getting ready to take an important oral examination and is concerned about the possibility of having an "on" day or an "off" day. He figures that if he has an on day, then each of his examiners will pass him, independently of one another, with probability .8, whereas if he has an off day, this probability will be reduced to .4. Suppose that the student will pass the examination if a majority of the examiners pass him. If the student believes that he is twice as likely to have an off day as he is to have an on day, should he request an examination with 3 examiners or with 5 examiners?

4.47. Suppose that it takes at least 9 votes from a 12-member jury to convict a defendant. Suppose also that the probability that a juror votes a guilty person innocent is .2, whereas the probability that the juror votes an innocent person guilty is .1. If each juror acts independently and if 65 percent of the defendants are guilty, find the probability that the jury renders a correct decision. What percentage of defendants is convicted?

4.48. In some military courts, 9 judges are appointed. However, both the prosecution and the defense attorneys are entitled to a peremptory challenge of any judge, in which case that judge is removed from the case and is not replaced. A defendant is declared guilty if the majority of judges cast votes of guilty, and he or she is declared innocent otherwise. Suppose that when the defendant is, in fact, guilty, each judge will (independently) vote guilty with probability .7, whereas when the defendant is, in fact, innocent, this probability drops to .3.

(a) What is the probability that a guilty defendant is declared guilty when there are (i) 9, (ii) 8, and (iii) 7 judges?

(b) Repeat part (a) for an innocent defendant.

(c) If the prosecuting attorney does not exercise the right to a peremptory challenge of a judge, and if the defense is limited to at most two such challenges, how many challenges should the defense attorney make if he or she is 60 percent certain that the client is guilty?

4.49. A company sells LED bulbs in packages of 20 for \$25. From past records, it knows that a bulb will be defective with probability .01. The company agrees to pay a full refund if a customer finds more than 1 defective bulb in a pack. If the company sells 100 packs, how much should it expect to refund?

4.50. When coin 1 is flipped, it lands on heads with probability .4; when coin 2 is flipped, it lands on heads with probability .7. One of these coins is randomly chosen and flipped 10 times.

(a) What is the probability that the coin lands on heads on exactly 7 of the 10 flips?

(b) Given that the first of these 10 flips lands heads, what is the conditional probability that exactly 7 of the 10 flips land on heads?

4.51. Each member of a population of size n is, independently, female with probability p or male with probability $1 - p$. Let X be the number of the other $n - 1$ members of the population that are the same sex as is person 1. (So $X = n - 1$ if all n people are of the same sex.)

(a) Find $P(X = i)$, $i = 0, \ldots, n - 1$.

Now suppose that two people of the same sex will, independently of other pairs, be friends with probability α; whereas two persons of opposite sexes will be friends with probability β. Find the probability mass function of the number of friends of person 1.

4.52. In a tournament involving players 1, 2, 3, 4, players 1 and 2 play a game, with the loser departing and the winner then playing against player 3, with the loser of that game departing and the winner then playing player 4. The winner of the game involving player 4 is the tournament winner. Suppose that a game between players i and j is won by player i with probability $\frac{i}{i+j}$.

(a) Find the expected number of games played by player 1.

(b) Find the expected number of games played by player 3.

4.53. Suppose that Harry plays 10 rounds of tennis against Smith and wins with probability p during each round. Given that Harry has won a total of 7 rounds, find the conditional probability that his outcomes in the first 3 rounds are

(a) *win, win, lose*;

(b) *lose, win, lose*.

4.54. The expected number of dancers falling on stage during a contest is .3. What is the probability that during the next contest, (a) no dancer falls on stage and (b) 3 or more dancers fall on stage? Explain your reasoning.

4.55. The monthly worldwide average number of airplane crashes of commercial airlines is 3.5. What is the probability that there will be

(a) at least 2 such accidents in the next month;

(b) at most 1 accident in the next month?

Explain your reasoning!

4.56. Approximately 80,000 marriages took place in the state of New York last year. Estimate the probability that for at least one of these couples,

(a) both partners were born on April 30;

(b) both partners celebrated their birthday on the same day of the year.

State your assumptions.

4.57 Suppose that the average number of follower requests that an advertising page receives weekly is 50. Approximate the probability that the page will receive

(a) exactly 35 follower requests in the next week;

(b) at least 40 follower requests in the next week.

4.58. An examination board appoints two vetters. The average number of errors per exam paper found by the first vetter is 4, and the average number of errors per exam paper found by the second vetter is 5. If an examiner's paper is equally likely to be vetted by either vetter, approximate the probability that it will have no errors.

4.59. How many people are needed so that the probability that at least one of them has the same first and last name initials as you is at least $\frac{3}{4}$?

4.60 Suppose that the number of weekly traffic accidents occurring in a small town is a Poisson random variable with parameter $\lambda = 7$.

(a) What is the probability that at least 4 accidents occur (until) this week?

(b) What is the probability that at most 5 accidents occur (until) this week given that at least 1 accident will occur today?

4.61. Compare the Poisson approximation with the correct binomial probability for the following cases:

(a) $P\{X = 2\}$ when $n = 8, p = .1$;

(b) $P\{X = 9\}$ when $n = 10, p = .95$;

(c) $P\{X = 0\}$ when $n = 10, p = .1$;

(d) $P\{X = 4\}$ when $n = 9, p = .2$.

4.62. If you buy a lottery ticket in 50 lotteries, in each of which your chance of winning a prize is $\frac{1}{100}$, what is the (approximate) probability that you will win a prize

(a) at least once?

(b) exactly once?

(c) at least twice?

4.63 The number of times that a person contracts a cold in a given year is a Poisson random variable with parameter $\lambda = 5$. Suppose that a new wonder drug (based on large quantities of vitamin C) has just been marketed that reduces the Poisson parameter to $\lambda = 3$ for 75 percent of the population. For the other 25 percent of the population, the drug has no appreciable effect on colds. If an individual tries the drug for a year and has 2 colds in that time, how likely is it that the drug is beneficial for him or her?

4.64 While driving along a long route that has 5,000 intersections, the probability of encountering a red light at any intersection is .001. Find an approximation for the probability that a driver will encounter at least 2 red lights.

4.65. Consider n independent trials, each of which results in one of the outcomes $1, \ldots, k$ with respective probabilities $p_1, \ldots, p_k$, $\sum_{i=1}^{k} p_i = 1$. Show that if all the p_i are small, then the probability that no trial outcome occurs more than once is approximately equal to $\exp(-n(n-1)\sum_i p_i^2/2)$.

4.66. Customers enter a supermarket located on a busy street at a rate of 2 every 3 minutes.

(a) What is the probability that no customer enters the supermarket between 07:00 and 07:06?

(b) What is the probability that at least 5 customers enter during this time?

4.67. In a certain country, babies are born at an approximate rate of 6.94 births per 1,000 inhabitants per year. Assume that the total population is 40,000.

(a) What is the probability that there will be more than 60 births in this country during a 3-month period?

(b) What is the probability that there will be more than 60 births in at least 2 phases of 3 months during the next year?

(c) If the present season (a 3-month period) is identified as Season 1, what is the probability that the first season to have more than 60 births will be Season i $(i = 1, 2, 3, 4)$?

4.68. Each of 500 soldiers in an army company independently has a certain disease with probability $1/10^3$. This disease will show up in a blood test, and to facilitate matters, blood samples from all 500 soldiers are pooled and tested.

(a) What is the (approximate) probability that the blood test will be positive (that is, at least one person has the disease)?

Suppose now that the blood test yields a positive result.

(b) What is the probability, under this circumstance, that more than one person has the disease?

Now, suppose one of the 500 people is Jones, who knows that he has the disease.

(c) What does Jones think is the probability that more than one person has the disease?

Because the pooled test was positive, the authorities have decided to test each individual separately. The first $i - 1$ of these tests were negative, and the ith one—which was on Jones—was positive.

(d) Given the preceding scenario, what is the probability, as a function of i, that any of the remaining people have the disease?

4.69. A total of $2n$ people, consisting of n married couples, are randomly seated (all possible orderings being equally likely) at a round table. Let C_i denote the event that the members of couple i are seated next to each other, $i = 1, \ldots, n$.

(a) Find $P(C_i)$.

(b) For $j \neq i$, find $P(C_j|C_i)$.

(c) Approximate the probability, for n large, that there are no married couples who are seated next to each other.

4.70. Repeat the preceding problem when the seating is random but subject to the constraint that the men and women alternate.

4.71. In response to an attack of 10 missiles, 500 antiballistic missiles are launched. The missile targets of the antiballistic missiles are independent, and each antiballstic missile is equally likely to go towards any of the target missiles. If each antiballistic missile independently hits its target with probability .1, use the Poisson paradigm to approximate the probability that all missiles are hit.

4.72. A fair coin is flipped 10 times. Find the probability that there is a string of 4 consecutive heads by

(a) using the formula derived in the text;

(b) using the recursive equations derived in the text.

(c) Compare your answer with that given by the Poisson approximation.

4.73. At time 0, a coin that comes up heads with probability p is flipped and falls to the ground. Suppose it lands on heads. At times chosen according to a Poisson process with rate λ, the coin is picked up and flipped. (Between these times, the coin remains on the ground.) What is the probability that the coin is on its head side at time t?

Hint: What would be the conditional probability if there were no additional flips by time t, and what would it be if there were additional flips by time t?

4.74. Consider a roulette wheel consisting of 38 numbers 1 through 36, 0, and double 0. If Smith always bets that the outcome will be one of the numbers 1 through 12, what is the probability that

(a) Smith will lose his first 5 bets;

(b) his first win will occur on his fourth bet?

4.75. Two athletic teams play a series of games; the first team to win 4 games is declared the overall winner. Suppose that one of the teams is stronger than the other and wins each game with probability .6, independently of the outcomes of the other games. Find the probability, for $i = 4, 5, 6, 7$, that the stronger team wins the series in exactly i games. Compare the probability that the stronger team wins with the probability that it would win a 2-out-of-3 series.

4.76. Suppose in Problem 4.75 that the two teams are evenly matched and each has probability $\frac{1}{2}$ of winning each game. Find the expected number of games played.

4.77. An interviewer is given a list of people she can interview. If the interviewer needs to interview 5 people, and if each person (independently) agrees to be interviewed with probability $\frac{2}{3}$, what is the probability that her list of people will enable her to obtain her necessary number of interviews if the list consists of (a) 5 people and (b) 8 people? For part (b), what is the probability that the interviewer will speak to exactly (c) 6 people and (d) 7 people on the list?

4.78. During assembly, a product is equipped with 5 control switches, each of which has probability .04 of being defective. What is the probability that 2 defective switches are encountered before 5 nondefective ones?

4.79. Solve the Banach match problem (Example 8e) when the left-hand matchbox originally contained N_1 matches and the right-hand box contained N_2 matches.

4.80. In the Banach matchbox problem, find the probability that at the moment when the first box is emptied (as opposed to being found empty), the other box contains exactly k matches.

4.81. An urn contains 4 red, 4 green, and 4 blue balls. We randomly choose 6 balls. If exactly two of them are red, we stop. Otherwise, we replace the balls in the urn and randomly choose 6 balls again. What is the probability that we shall stop exactly after n selections?

4.82. Suppose that a class of 50 students has appeared for a test. Forty-one students have passed this test while the remaining 9 students have failed. Find the probability that in a group of 10 students selected at random

(a) none have failed the test;

(b) at least 3 students have failed the test.

4.83. A game popular in Nevada gambling casinos is Keno, which is played as follows: Twenty numbers are selected at random by the casino from the set of numbers 1 through 80. A player can select from 1 to 15 numbers; a win occurs if some fraction of the player's chosen subset matches any of the 20 numbers drawn by the house. The payoff is a function of the number of elements in the player's selection and the number of matches. For instance, if the player selects only 1 number, then he or she wins if this number is among the set of 20, and the payoff is \$2.20 won for every dollar bet. (As the player's probability of winning in this case is $\frac{1}{4}$, it is clear that the "fair" payoff should be \$3 won for every \$1 bet.) When the player selects 2 numbers, a payoff (of odds) of \$12 won for every \$1 bet is made when both numbers are among the 20.

(a) What would be the fair payoff in this case?

Let $P_{n,\,k}$ denote the probability that exactly k of the n numbers chosen by the player are among the 20 selected by the house.

(b) Compute $P_{n,\,k}$

(c) The most typical wager at Keno consists of selecting 10 numbers. For such a bet, the casino pays off as shown in the following table. Compute the expected payoff:

Keno Payoffs in 10 Number Bets

Number of matches	Dollars won for each \$1 bet
0–4	–1
5	1
6	17
7	179
8	1,299
9	2,599
10	24,999

4.84. In Example 8i, what percentage of i defective lots does the purchaser reject? Find it for $i = 1, 4$. Given that a lot is rejected, what is the conditional probability that it contained 4 defective components?

4.85. An automotive manufacturing company produces brake pads in lots of 100. This company inspects 15 brake pads from each lot and accepts the whole lot only if all 15 brake pads pass the inspection test. Each brake pad is, independently of the others, faulty with probability .09. What proportion of the lots does the company reject?

4.86. A neighborhood consists of five streets. Assume that the numbers of daily traffic accidents that occur on these streets are Poisson random variables with respective parameters .45, .2, .4, .5, and .35. What is the expected number of traffic accidents that will occur in this neighborhood next Monday?

4.87. Suppose that a group of 15 female students is selecting one shop out of the 6 available shops nearby to buy their prom dress. Each student, independently of the others, selects shop i with probability p_i, where $\sum_{i=1}^{6} p_i = 1$.

(a) What is the expected number of shops that will not be visited by any student from this group?

(b) What is the expected number of shops that will be visited by exactly 3 students from this group?

4.88. Martha makes a necklace by randomly selecting n beads from a large jar containing beads of k different colors. Independently of the selection of the previous bead, Martha selects a bead of color i with probability p_i, where $\sum_{i=1}^{k} p_i = 1$. What is the expected number of different colored beads in the necklace?

4.89. An urn contains 10 red, 8 black, and 7 green balls. One of the colors is chosen at random (meaning that the chosen color is equally likely to be any of the 3 colors), and then 4 balls are randomly chosen from the urn. Let X be the number of these balls that are of the chosen color.

(a) Find $P(X = 0)$.

(b) Let X_i equal 1 if the i^{th} ball selected is of the chosen color, and let it equal 0 otherwise. Find $P(X_i = 1)$, $i = 1, 2, 3, 4$.

(c) Find $E[X]$.

Hint: Express X in terms of X_1, X_2, X_3, X_4.

Theoretical Exercises

4.1. There are N distinct types of coupons, and each time one is obtained it will, independently of past choices, be of type i with probability $P_i, i = 1, \ldots, N$. Let T denote the number one need select to obtain at least one of each type. Compute $P\{T = n\}$.

Hint: Use an argument similar to the one used in Example 1e.

4.2. If X has distribution function F, what is the distribution function of e^X?