MANIPAL SCHOOL OF INFORMATION SCIENCES
MANIPAL
(A constituent unit of MAHE, Manipal)

AML5103 | Applied Probability and Statistics | Problem Set-2 Solutions

1. A student taking a test has to select 7 out of 10 questions. How many different choices does she have if:

   (a) there are no other restrictions?

   Solution: $10C_7$ ways

   (b) she has to answer exactly 2 of the last 4?

   Solution: Select 2 out of the last 4 and 5 from the first 6 resulting in $4C_2 \times 6C_5$ ways.

   (c) she has to answer exactly 2 of the first 6?

   Solution: This is not possible as she cannot select the remaining 5 questions from the last 4.

   (d) she has to answer at least 3 of the first 5?

   Solution: Answer 3 out of first 5 and 4 out of next 5 or answer 4 out of first 5 and 3 out of next 5 or answer 5 out of first 5 and 2 out of next 5 resulting in $5C_3 \times 5C_4 + 5C_4 \times 5C_3 + 5C_5 \times 5C_2$ ways.

2. If eight identical blackboards are to be divided among 4 schools, how many divisions are possible? What if each school must receive at least one blackboard?

   Solution: $r = 8$ identical blackboards can be divided among $n = 4$ schools in $(n+r-1)C_r = (4+8-1)C_8 = 11C_8$ ways. If each school must receive at least one blackboard, then we pre-allocate 1 blackboard into each school leaving us with $r = 4$ identical blackboards to be distributed among $n = 4$ schools which can be done in $(n+r-1)C_r = (4+4-1)C_4 = 7C_4$ ways.

3. 9 computers are brought in for servicing (and machines are serviced one at a time). Of the 9 computers, 3 are PCs, 4 are Macs, and 2 are Linux machines. Assume that all computers of the same type are indistinguishable (i.e., all the PCs are indistinguishable, all the Macs are indistinguishable, etc.).

(a) In how many distinguishable ways can the computers be ordered for servicing?

> **Solution:** There are 9 servicing slots. We select without replacement and order does not matter 3 out of 9 slots for the PCS and 4 out of the remaining 6 for the Macs and 2 out of the remaining 2 for the Linux machines resulting in $9C_3 \times 6C_4 \times 2C_2$ ways.

(b) In how many distinguishable ways can the computers be ordered if the first 5 machines serviced must include all 4 Macs?

> **Solution:** We select from the first 5 slots without replacement and order does not matter 4 slots for putting the Macs and
>
> put a PC in the remaining slot and select 2 slots without replacement and order does not matter from the remaining 4 slots to put the 2 PCs and select 2 slots without replacement and order does not matter from the remaining 2 slots to put the 2 Linux machines
>
> or
>
> put a Linux machine in the remaining slot and select 3 slots without replacement and order does not matter from the remaining 4 slots to put the 3 PCs and select 1 slot without replacement and order does not matter from the remaining 1 slot to put the last Linux machine.
>
> which corresponds to $5C_4 \times (4C_2 \times 2C_2 + 4C_3 \times 1C_1)$ ways.

(c) In how many distinguishable ways can the computers be ordered if 2 PCs must be in the first three and 1 PC must be in the last three computers serviced?

> **Solution:** We select from the first 3 slots without replacement and order does not matter 2 slots for putting 2 PCs and select from the last 3 slots without replacement and order does not matter 1 slot for putting the remaining PC and select from the remaining 6 slots without replacement and order does not matter 4 slots for putting the 4 Macs and select from the remaining 2 slots without replacement and order does not matter 2 slots for putting the 2 Linux machines which corresponds to $3C_2 \times 3C_1 \times 6C_4 \times 2C_2$ ways.

4. A 3-member project team for performing literature survey, coding, and documentation for a project has to be selected from a class of 60 students where in each person takes up one role only. How many different choices of teams are possible if:

(a) There are no restrictions.

Solution: This corresponds to sampling without replacement (one role per student) and order matters (specific roles) 3 students from 60 students which can be done in $60P_3$ ways.

(b) Two of the students will not work together.

Solution: Total number of selections =

$$\begin{cases} \text{selections in which the two students are together} \\ \qquad\qquad\qquad + \\ \underbrace{\text{selections in which the two students are not together}}_{\text{need this}} \end{cases}.$$

The number of selections in which the two students are together is equal to the number of ways to select 1 out of the remaining 58 students (2 are already there) <u>and</u> arrange the 3 selected students for the 3 roles which can be done in $58C_1 \times 3!$ ways. This results in $60P3 - (58C_1 \times 3!)$ ways.

(c) Two of the students will work together or not at all.

Solution: When the two students work together, the number of selections is equal to the number of ways to select 1 out of the remaining 58 students <u>and</u> arrange the 3 selected students for the 3 roles which can be done in $58C_1 \times 3!$ ways. When both of the students don't work together, the number of selections is equal to the number of ways to select 3 out of the remaining 58 students <u>and</u> arrange the 3 selected students for the 3 roles which can be done in $58C_3 \times 3!$ ways. This results in $(58C_1 \times 3!) + (58C_3 \times 3!)$ ways.

(d) One of the students must be in the team.

Solution: When that particular student is in the team, the total number of selections is equal to the number of ways to select 2 out of the remaining 59 students <u>and</u> arrange the 3 selected students for the 3 roles which can be done in $59C_2 \times 3!$ ways.

(e) One of the students can only do coding.

Solution: When that particular student is in the team to do coding, the total number of selections is equal to the number of ways to select 2 out of the remaining 59 students <u>and</u> arrange the 2 selected students for the 2 roles (survey and documentation) which can be done in $59C_2 \times 2!$ ways or when that particular student is not in the team, the total number of selections is equal to the number of ways to select 3 out of the remaining 59 students <u>and</u> arrange the 3 selected students for the 3 roles which can be done in $59C_3 \times 3!$ ways. This results in a total of $(59C_2 \times 2!) + (59C_3 \times 3!)$ ways.

5. 100 units of stabilizing weights are to be placed into 5 vehicles. Because of different vehicle characteristics, vehicle 1 needs at least 10 units, vehicles 2 and 3 at least 12 each, vehicles 4 and 5 travel in a convoy and they need at least 4 combined. How many distributions of these weight units are feasible?

Solution: The first step is to put 10 units in vehicle-1, 12 units in vehicles-2 and 3 each. Now we have 100 - 10 - 12 - 12 = 66 weights left, and we can ignore the constraints about vehicles 1,2, and 3.

Ignoring the last constraint (vehicles-4 and 5 need at least 4 combined), for the total number of ways, we are looking at distributing $r = 66$ weights into 5 distinct vehicles which can be done in $(5 + 66 - 1)C_{66}$ ways.

Now we subtract the number of ways where vehicles-4 and 5 has less than 4 units combined:

- vehicles-4 and 5 have 3 units combined. There are 4 ways: $3+0, 2+1, 1+2$, and $0+3$. For each one of the 4 ways, we are looking at distributing $r = 63$ weights into 3 distinct vehicles which can be done in $(3 + 63 - 1)C_{63}$ ways resulting in a total of $4 \times (3 + 63 - 1)C_{63}$ ways.
- vehicles-4 and 5 have 2 units combined. There are 3 ways: $2 + 0, 1 + 1$, and $0+2$. For each one of the 3 ways, we are looking at distributing $r = 64$ weights into 3 distinct vehicles which can be done in $(3 + 64 - 1)C_{64}$ ways resulting in a total of $3 \times (3 + 64 - 1)C_{64}$ ways.
- vehicles-4 and 5 have 1 unit combined. There are 2 ways: $1 + 0$, and $0 + 1$. For each one of the 2 ways, we are looking at distributing $r = 65$ weights into 3 distinct vehicles which can be done in $(3 + 65 - 1)C_{65}$ ways resulting in a total of $2 \times (3 + 65 - 1)C_{65}$ ways.
- vehicles-4 and 5 have no units. Then, we are looking at distributing $r = 66$ weights into 3 distinct vehicles which can be done in $(3 + 66 - 1)C_{66}$ ways.

The final number of ways is

$$\binom{5+66-1}{66} - 4\binom{3+63-1}{63} - 3 \times \binom{3+64-1}{64} - 2 \times \binom{3+65-1}{65} - \binom{3+66-1}{66} = 895440.$$

6. Eleven soccer players are to be divided into 4 functional groups: 3 forwards, 3 midfields, 4 defenses, and 1 goalie. There are only 2 people who can play goalie. Both of these two players can play any other position. Of the remaining 9, 4 can play only forward or midfield; the other 5 can play only defense or midfield. We want to calculate the number of possible ways to divide the team into the 4 functional groups. Follow the hint below and get to the answer:

$$\text{Select 1 goalie out of 2 in } \binom{?}{?} \text{ ways } \textbf{AND} \begin{cases} \text{Remaining goalie plays defense} \\ \textbf{OR} \\ \text{Remaining goalie plays midfield} \\ \textbf{OR} \\ \text{Remaining goalie plays forward} \end{cases}$$

Solution:

$$\binom{2}{1} \times \Bigg[\underbrace{\binom{5}{3}}_{\text{choose remaining defense players}} \times \underbrace{\binom{4}{3}}_{\text{choose forward players}} \times \underbrace{\binom{3}{3}}_{\text{choose midfield players}}$$

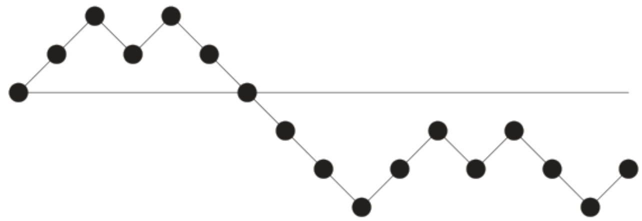
$$+ \underbrace{\binom{5}{4}}_{\text{choose defense players}} \times \underbrace{\binom{4}{3}}_{\text{choose forward players}} \times \underbrace{\binom{2}{2}}_{\text{choose remaining midfield players}}$$

$$+ \underbrace{\binom{5}{4}}_{\text{choose defense players}} \times \underbrace{\binom{4}{2}}_{\text{choose remaining forward players}} \times \underbrace{\binom{3}{3}}_{\text{choose midfield players}} \Bigg] = 180.$$

7. In how many ways can r identical server requests be distributed among n servers so that the ith server receives *at least* m_i requests, for each $i = 1, 2, \ldots, n$? You can assume that $r \geq (m_1 + m_2 + \cdots + m_n)$.

Solution: We will first pre-assign the minimum number of requests for each server: m_1 for server-1, m_2 for server-2, and so on.This will leave is with distributing the remaining $r - (m_1 + m_2 + \cdots + m_n)$ identical requests to the n distinct servers which can be done in $\left(n + \underbrace{r - (m_1 + m_2 + \cdots + m_n)}_{\text{new } r} - 1 \right) C_{r-(m_1+m_2+\cdots+m_n)}$ ways.

8. Suppose a particle starting from the origin can move *only* up or down; the *binomial option pricing model* addresses stock price movements using such an idea.



Show that the number of ways the particle can move from the origin to position k in n steps is $\binom{n}{\frac{n+k}{2}}$. Assume that $n+k$ is even.

Solution: For the particle to reach position k in n steps, it has to make a certain number of upward movements n_U and a certain number of downward movements n_D such that $n_U + n_D = n$ and $n_U - n_D = k$. You can check this for $k = 0$ in $n = 10$ steps which can happen, for example in one possible way, as $\underbrace{UUUUU}_{n_U=5}\underbrace{DDDDD}_{n_D=5}$. Solving the two equations involving n_U and n_D, we get $n_U = (n+k)/2$ and $n_D = (n-k)/2$. Thus, we have to select from n moves without replacement and order does not matter $n_U - \frac{n+k}{2}$ moves to put the Us. This can happen in $nC_{\frac{n+k}{2}}$ ways.

9. Imagine a criminal appeals court consisting of five judges; let's call them A, B, C, D, and E. The judges meet regularly to vote (independently, of course) on the fate of prisoners who have petitioned for a review of their convictions. The result of each of the court's deliberations is determined by a simple majority; for a petitioner to be granted or denied a new trial requires three or more votes. Based on long-term record keeping, it is known that A votes correctly 95% of the time; i.e., when A votes to either uphold or to reverse the original conviction, he is wrong only 5% of the time. Similarly, B, C, D, and E vote correctly $95\%, 90\%, 90\%$, and 80% of the time. (There are, of course, two different ways a judge can make a mistake. The judge may uphold a conviction, with new evidence later showing that the petitioner was in fact innocent. Or the judge may vote to reverse a conviction when in fact the petitioner is actually guilty, as determined by the result of a second conviction at the new trial.) Suppose we want to calculate the probability that the court, as an entity, makes an incorrect decision.

(a) Write the sample space showing any two outcomes in it clearly. Explain what the outcomes mean.

Solution: Let 0 encode an incorrect judgment and 1 encode a correct judgment by a judge. Then the sample space appears like:

$$S = \{(1,1,1,0,0), (1,1,0,0,0), \ldots\}.$$

The first outcome $(1, 1, 1, 0, 0)$ means that judges A, B, C passed a correct judgment and judges D, E passed an incorrect judgment.

(b) How many outcomes n are there in the sample space?

Solution: To build a specific outcome of the sample space, say $(1, 1, 1, 0, 0)$, we sample 5 objects from the sampling space $\{0, 1\}$ with replacement and order matters which can be done in 2^5 ways.

(c) How many outcomes $n(E)$ are there in the event of interest?

Solution: An outcome in the event E appears like $(1, 1, 0, 0, 0)$ which means that judges A, B passed a correct judgment and judges C, D, E passed an incorrect judgment in majority resulting in an incorrect judgment from the court. The number of such outcomes in the event is the number of ways to select at least 3 slots out of 5 without replacement and order does not matter to put the zeros representing the incorrect judgment. This can be done in $5C_3 + 5C_4 + 5C_5 = 16$ ways.

(d) Explain briefly why or why not the probability of the event of interest can be calculated as $n(E)/n$.

Solution: The probability of the event of interest cannot be calculated as $n(E)/n$ because the outcomes in the sample space are not equally likely. For example, the probability of the outcome $(1, 1, 1, 0, 0)$ assuming independence is:

$$\begin{aligned}
&P\left(A_{\text{correct}}\ \underline{\text{AND}}\ B_{\text{correct}}\ \underline{\text{AND}}\ C_{\text{correct}}\ \underline{\text{AND}}\ D_{\text{incorrect}}\ \underline{\text{AND}}\ E_{\text{incorrect}}\right)\\
&= P\left(A_{\text{correct}}\right) \times P\left(B_{\text{correct}}\right) \times P\left(C_{\text{correct}}\right) \times P\left(D_{\text{incorrect}}\right) \times P\left(E_{\text{incorrect}}\right)\\
&= 0.95 \times 0.95 \times 0.9 \times 0.1 \times 0.2,
\end{aligned}$$

which is not equal to the probability of the outcome, say, $(1, 1, 1, 1, 0)$ which is equal to $0.95 \times 0.95 \times 0.9 \times 0.9 \times 0.2$.

10. Mr. Brown needs to take 1 tablet of type A and 1 tablet of type B together on a regular basis. One tablet of type A corresponds to a 1 mg dosage, and so does 1 tablet of type B. He keeps these two types of tablets in two separately labeled bottles as they cannot be differentiated easily. One day, on a business trip, Mr. Brown brought 10 tablets of type A and 10 tablets of type B.. Unfortunately, he drops the bottles and breaks them. He does not have the time to go to a pharmacy to buy a new set of tablets but he needs to take his required dosage of both tablets A and B. The safe dosage that he needs for both tablets A and B is given by

$$0.9\,\text{mg} \leq \text{safe dosage} \leq 1.1\,\text{mg}.$$

Taking either an excess or a shortage of the required intake will result in serious health issues.

(a) Suppose that after investigating the broken bottles, Mr. Brown finds 2 tablets that are still intact in the bottle for tablet A. The other 18 tablets are found to be mixed in a pile. Is it better for him to take one known tablet from the bottle and one from the pile, or take two tablets from the pile? Answer this by calculating the respective probabilities that he will not have any serious health issues for both options.

Solution:

- Let's label the two tablets of type A in the bottle as A_1, A_2. Let's label the remaining 8 tablets of type A in the pile as $A_3, A_4, \ldots, A_{10}$ and the 10 tablets of type B as $B_1, B_2, \ldots, B_{10}$.
- If Mr. Brown chooses a tablet from the bottle and another one from the pile, the sample space is $S = \{(A_1, A_3), (A_2, A_3), (A_1, B_1), (A_2, B_1), \ldots\}$.
- The outcomes in the sample space are (intuitively) equally likely because any one of tablet of type A from the bottle is equally likely to be selected and any one tablet from the pile is equally likely to be selected.
- A particular outcome in the sample space, say (A_1, A_3) is built as follows: select one object from the sampling space $\{A_1, A_2\}$ and one object from the sampling space $\{A_3, A_4, \ldots, A_{10}, B_1, B_2, \ldots, B_{10}\}$. Thus, the number of outcomes in the sample space is $\binom{2}{1} \times \binom{18}{1}$.
- The event E (corresponding to Mr. Brown not developing a serious health issue) comprise outcomes in the set $\{(A_1, B_1), (A_2, B_2), \ldots\}$.
- A particular outcome in the event E, say (A_1, B_1), is built as follows: select one object from $\{A_1, A_2\}$ and one object from $\{B_1, B_2, \ldots, B_{10}\}$, which can be done in $\binom{2}{1} \times \binom{10}{1}$ ways.
- Therefore the probability of Mr. Brown not developing a serious health issue when he picks a tablet of type A from the bottle and a random tablet from the pile is $\frac{\binom{2}{1}\times\binom{10}{1}}{\binom{2}{1}\times\binom{18}{1}} = 0.56$.
- If Mr. Brown chooses both tablets from the pile, the sample space is $S = \{(A_3, A_4), (A_3, B_1), (A_3, B_2), \ldots\}$.
- The outcomes in the sample space are (intuitively) equally likely because selecting any two tablets from the pile is equally likely.
- A particular outcome in the sample space, say (A_3, A_4), is built as follows: select two objects from the sampling space $\{A_3, A_4, \ldots, A_{10}, B_1, B_2, \ldots, B_{10}\}$ without replacement and order does not matter, which can be done in $\binom{18}{2}$ ways.

- The event E (corresponding to Mr. Brown not developing a serious health issue) comprise outcomes in the set $\{(A_3, B_1), (A_3, B_2), \ldots\}$.
- A particular outcome in the event E, say (A_3, B_1), is built as follows: select one object from $\{A_3, A_4, \ldots, A_{10}\}$ and one object from $\{B_1, B_2, \ldots, B_{10}\}$, which can be done in $\binom{8}{1} \times \binom{10}{1}$ ways.
- Therefore the probability of Mr. Brown not developing a serious health issue when he picks a tablet of type A from the bottle and random tablet from the pile is $\frac{\binom{8}{1}\times\binom{10}{1}}{\binom{18}{2}} = 0.52$.
- Therefore, it is better for Mr. Brown to pick a tablet from the bottle and another one from the pile.

(b) Suppose that after investigating the broken bottles, Mr. Brown finds that the tablets are all mixed up. What is the probability that he will not have any serious health issues if he randomly picks 2 tablets?

Solution:

- If Mr. Brown chooses a tablet from the pile, the sample space is

$$S = \{(A_1, A_2), (A_1, B_1), (A_1, B_2), \ldots, (A_2, A_3), (A_2, B_1), (A_2, B_2), \ldots\}.$$

- The outcomes in the sample space are (intuitively) equally likely because selecting any two tablets from the pile is equally likely.
- A particular outcome in the sample space, say (A_1, A_2), is built as follows: select two objects from the sampling space $\{A_1, A_2, \ldots, A_{10}, B_1, B_2, \ldots, B_{10}\}$ without replacement and order does not matter, which can be done in $\binom{20}{2}$ ways.
- The event E (corresponding to Mr. Brown not developing a serious health issue) comprise outcomes in the set

$$\{(A_1, B_1), (A_1, B_2), \ldots, (A_2, B_1), (A_2, B_2), \ldots\}.$$

- A particular outcome in the event E, say (A_1, B_1), is built as follows: select one object from $\{A_1, A_2, \ldots, A_{10}\}$ and one object from $\{B_1, B_2, \ldots, B_{10}\}$, which can be done in $\binom{10}{1} \times \binom{10}{1}$ ways.
- Therefore the probability of Mr. Brown not developing a serious health issue when he picks two tablets from the pile is $\frac{\binom{10}{1}\times\binom{10}{1}}{\binom{20}{2}} = 0.53$.

11. A total of 28% of American males smoke cigarettes, 7% smoke cigars, and 5% smoke both cigars and cigarettes. Let A and B represent the events that a randomly chosen person is a cigarette smoker and a cigar smoker, respectively. Explain in plain English what the following compound events represent and calculate their probabilities: (1) $(A \cup B)^c$ (2) $B \cap A^c$.

Solution: $(A \cup B)^c$ represents the event that the person neither smokes cigarettes nor cigars. The corresponding probability is

$$\begin{aligned} P\left(A \cup B\right)^c &= 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - [0.28 + 0.07 - 0.05] = 0.7. \end{aligned}$$

$B \cap A^c$ represents the event that the person smokes cigar and not cigarettes. The corresponding probability can be calculated as follows:

$$\begin{aligned} P(B) &= P(B \cap A \ \underline{\text{or}}\ B \cap A^c) = P(B \cap A) + P(B \cap A^c) \\ \Rightarrow P(B \cap A^c) &= P(B) - P(B \cap A) \\ &= 0.07 - 0.05 = 0.02. \end{aligned}$$

12. What is more likely? Provide quantitative support.

(a) Obtaining at least one 6 in 4 rolls of a single die.

Solution: The random experiment is rolling a single die. This corresponds to the sampling space

$$S = \{(1111), (1234), \ldots, (6666)\}.$$

The outcomes are intuitively equally likely, so we assign a likelihood to each outcome that is equal to $1/n(S)$, where $n(S)$ is the number of outcomes in the sample space. To calculate $n(S)$, we use the sampling space $s = \{1, 2, 3, 4, 5, 6\}$ to identify that each outcome in the sample space S is built by selecting from the sample space with replacement and order matters 4 objects out of 6 which can be done in 6^4 ways. Therefore, the likelihood for each outcome in the sample space is $1/6^4$.

The event space E corresponds to at least one 6. The complementary event space E^c corresponds to no 6. Let us calculate $P(E^c)$ and use that to calculate $P(E)$ as $1 - P(E^c)$. The complementary event space E^c corresponds to the outcomes

$$\{(1111), (1234), \ldots, (5555)\}.$$

The number of outcomes in E^c, denoted as $n(E^c)$, can be calculated as the number of ways to select from the sampling space with replacement and order matters 4 objects out of 5 which is equal to 5^4. Therefore,

$$P(E^c) = \frac{1}{6^4} \times n(E^c) = \frac{1}{6^4} \times 5^4$$
$$\Rightarrow P(E) = 1 - P(E^c) = 1 - \frac{5^4}{6^4} \approx 0.52.$$

A simpler way to calculate the same probability is as follows using the independence of the rolls:

$$\begin{aligned}
P(\text{at least one 6 in 4 rolls}) &= 1 - P(\text{no 6 in 4 rolls}) \\
&= 1 - P\begin{pmatrix}\text{roll-1 is not 6}\\ \underline{\text{and}}\\ \text{roll-2 is not 6}\\ \underline{\text{and}}\\ \text{roll-3 is not 6}\\ \underline{\text{and}}\\ \text{roll-4 is not 6}\end{pmatrix} \\
&= 1 - \begin{pmatrix}P(\text{roll-1 is not 6})\\ \times\\ P(\text{roll-2 is not 6})\\ \times\\ P(\text{roll-3 is not 6})\\ \times\\ P(\text{roll-4 is not 6})\end{pmatrix} \\
&= 1 - \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right) = 1 - \left(\frac{5}{6}\right)^4.
\end{aligned}$$

(b) Obtaining at least one 12 in 24 rolls of a pair of dice.

Solution: The random experiment is rolling a pair of dice 24 times. This corresponds to the sampling space

$$S = \{((11), (12), \ldots, (46)), \ldots, ((61), (62), \ldots, (36))\}.$$

The outcomes are intuitively equally likely, so we assign a likelihood to each outcome that is equal to $1/n(S)$, where $n(S)$ is the number of outcomes in the sample space. To calculate $n(S)$, we use the sampling space $s = \{1, 2, 3, 4, 5, 6\}$ to identify that each outcome in the sample space S is built by selecting from the sample space with replacement and order matters 2 objects out of 6 and repeat that 24 times which can be done in $6^2 \times \cdots \times 6^2 = 36^{24}$ times. Therefore, the likelihood for each outcome in the sample space is $1/36^{24}$.

The event space E corresponds to at least one 12. The complementary event space E^c corresponds to no 12. Let us calculate $P(E^c)$ and use that to calculate $P(E)$ as $1 - P(E^c)$. The complementary event space E^c corresponds to the outcomes that do not have (66) in them. This corresponds to

$$\left(\underbrace{5C_1 \times 5C_1}_{\text{except (66)}} + \underbrace{1C_1 \times 5C_1}_{\text{(61),(62), etc.}} + \underbrace{5C_1 \times 1C_1}_{\text{(16),(26), etc.}}\right)^{24} = 35^{24}$$

ways to select the outcomes in E^c from the sampling space $s = \{1, 2, 3, 4, 5, 6\}$. Therefore,

$$P(E^c) = \frac{1}{36^{24}} \times n(E^c) = \frac{1}{36^{24}} \times 35^{24}$$
$$\Rightarrow P(E) = 1 - P(E^c) = 1 - \left(\frac{35}{36}\right)^{24} \approx 0.49,$$

which is smaller than the probability of getting at least one 6 in 4 rolls of a single die. A simpler way to calculate the same probability is as follows using the independence of the rolls:

$$\begin{aligned}
P(\text{at least one 12 in 24 rolls of a pair of dice}) &= 1 - P(\text{no 12 in 24 rolls of a pair of dice}) \\
&= 1 - P\begin{pmatrix} \text{roll-1 is not 12} \\ \underline{\text{and}} \\ \text{roll-2 is not 12} \\ \underline{\text{and}} \\ \vdots \\ \underline{\text{and}} \\ \text{roll-24 is not 12} \end{pmatrix} \\
&= 1 - \begin{pmatrix} P(\text{roll-1 is not 12}) \\ \times \\ P(\text{roll-2 is not 6}) \\ \times \\ \vdots \\ \times \\ P(\text{roll-24 is not 12}) \end{pmatrix} \\
&= 1 - \left(\frac{35}{36} \times \frac{35}{36} \times \cdot \times \frac{35}{36}\right) = 1 - \left(\frac{35}{36}\right)^{24}.
\end{aligned}$$

13. Data was collected from the residents of a town and displayed as follows:

		Income <\$25k	\$25k – \$70k	> 70k
Age (years)	< 25	952	1,050	53
	25 – 45	456	2,055	1,570
	> 45	54	952	1,008

Answer the following:

(a) What fraction of people are less than 25 years old?

Solution: $\frac{952+1050+53}{952+1050+53+456+2055+1570+54+952+1008} \approx 0.25.$

(b) What is the probability that a randomly chosen person is more than 25 years old?

Solution: $1 - 0.25 = 0.75$

(c) What fraction of people earn less than \$70,000?

Solution: $\frac{952+1050+456+2055+54+952}{952+1050+53+456+2055+1570+54+952+1008} \approx 0.68.$

(d) What is the probability that a randomly chosen person is less than 25 years old and earns more than \$70,000?

Solution: $\frac{53}{952+1050+53+456+2055+1570+54+952+1008} \approx 0.007.$

(e) What fraction of people among those who earn less than \$25,000 are between 25-45 years old?

Solution: $\frac{456}{952+456+54} \approx 0.31.$

(f) If the next random person you see happens to be more than 45 years old, what is the probability that the person earns less than \$70,000?

Solution: $\frac{54+952}{54+952+1008} \approx 0.50.$