positive s and t,

$$P\{X > s + t | X > t\} = P\{X > s\}$$

If X represents the life of an item, then the memoryless property states that for any t, the remaining life of a t-year-old item has the same probability distribution as the life of a new item. Thus, one need not remember the age of an item to know its distribution of remaining life.

Let X be a nonnegative continuous random variable with distribution function F and density function f. The function

$$\lambda(t) = \frac{f(t)}{1 - F(t)} \qquad t \geq 0$$

is called the *hazard rate*, or *failure rate*, function of F. If we interpret X as being the life of an item, then for small values of dt, $\lambda(t)\,dt$ is approximately the probability that a t-unit-old item will fail within an additional time dt. If F is the exponential distribution with parameter λ, then

$$\lambda(t) = \lambda \quad t \geq 0$$

In addition, the exponential is the unique distribution having a constant failure rate.

A random variable is said to have a *gamma* distribution with parameters α and λ if its probability density function is equal to

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \qquad x \geq 0$$

and is 0 otherwise. The quantity $\Gamma(\alpha)$ is called the gamma function and is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}\,dx$$

The expected value and variance of a gamma random variable are, respectively,

$$E[X] = \frac{\alpha}{\lambda} \qquad \text{Var}(X) = \frac{\alpha}{\lambda^2}$$

A random variable is said to have a *beta* distribution with parameters (a, b) if its probability density function is equal to

$$f(x) = \frac{1}{B(a,b)}x^{a-1}(1 - x)^{b-1} \qquad 0 \leq x \leq 1$$

and is equal to 0 otherwise. The constant $B(a, b)$ is given by

$$B(a,b) = \int_0^1 x^{a-1}(1 - x)^{b-1}\,dx$$

The mean and variance of such a random variable are, respectively,

$$E[X] = \frac{a}{a + b} \qquad \text{Var}(X) = \frac{ab}{(a + b)^2(a + b + 1)}$$

Problems

5.1. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c\left(x - \frac{3}{x^2}\right) & 2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?

(b) What is the cumulative distribution function of X?

5.2. A group of construction workers take time X (in hours) to finish a task. The density function of time X is

$$f(x) = \begin{cases} cxe^{-\sqrt{x}} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that the workers will take more than 10 hours to complete the task?

5.3. For positive c, could the function

$$f(x) = \begin{cases} c(6x^2 - 5x) & 0 < x < \frac{2}{3} \\ 0 & \text{otherwise} \end{cases}$$

be a probability density function? If yes, determine the value of c. Repeat for

$$f(x) = \begin{cases} c(6x^2 - 5x) & 1 < x < \frac{4}{3} \\ 0 & \text{otherwise} \end{cases}$$

5.4. The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \dfrac{10}{x^2} & x > 10 \\ 0 & x \leq 10 \end{cases}$$

(a) Find $P\{X > 20\}$.

(b) What is the cumulative distribution function of X?

(c) What is the probability that of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?

5.5. A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function

$$f(x) = \begin{cases} 5(1 - x)^4 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

what must the capacity of the tank be so that the probability of the supply being exhausted in a given week is .01?

5.6. Compute $E[X]$ if X has a density function given by

(a) $f(x) = \begin{cases} \frac{1}{4}xe^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$;

(b) $f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$;

(c) $f(x) = \begin{cases} \frac{5}{x^2} & x > 5 \\ 0 & x \le 5 \end{cases}$.

5.7. The density function of X is given by

$$f(x) = \begin{cases} a + bx^3 & 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

If $E(X) = 5$, find a and b.

5.8. The wind speed, measured in miles per hour, experienced at a particular site is a random variable having a probability density function given by

$$f(x) = 3x^2e^{-x^3} \qquad x > 0$$

What is the expected wind velocity?

5.9. Consider Example 4b of Chapter 4, but now suppose that the seasonal demand is a continuous random variable having probability density function f. Show that the optimal amount to stock is the value s^* that satisfies

$$F(s^*) = \frac{b}{b + \ell}$$

where b is net profit per unit sale, ℓ is the net loss per unit unsold, and F is the cumulative distribution function of the seasonal demand.

5.10. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.

(a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?

(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 A.M.?

5.11. A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

5.12. A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over $(0, 100)$. There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A. Do you agree? Why?

5.13. A shuttle train completes the journey from an airport to a nearby city and back every 15 minutes.

(a) If the waiting time has a uniform distribution, what is the probability that a passenger has to wait more than 6 minutes for a shuttle train?

(b) Given that a passenger has already waited for 8 minutes, what is the probability that he or she has to wait an additional 2 minutes or more for a shuttle train?

5.14. Let X be a uniform $(0, 1)$ random variable. Compute $E[X^n]$ by using Proposition 2.1, and then check the result by using the definition of expectation.

5.15. The height X, in centimeters, of adult women is normally distributed with mean 165 centimeters and standard deviation 6.5 centimeters. Compute

(a) $P\{X > 160\}$;
(b) $P\{163 < X < 167\}$;
(c) $P\{X < 164\}$;
(d) $P\{X > 171\}$;
(e) $P\{X < 168\}$.

5.16. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that starting with this year, it will take more than 10 years before a year occurs having a rainfall of more than 50 inches? What assumptions are you making?

5.17. The salaries of physicians in a certain speciality are approximately normally distributed. If 25 percent of these physicians earn less than \$180,000 and 25 percent earn more than \$320,000, approximately what fraction earn

(a) less than \$200,000?
(b) between \$280,000 and \$320,000?

5.18. Suppose that X is a normally distributed random variable with mean μ and variance σ^2. If $P\{X < 10\} = .67$ and $P\{X < 20\} = .975$, approximate μ and σ^2.

5.19. Let X be an exponentially distributed random variable with mean 1/2. Find the value of c for which $P\{X > c\} = .25$.

5.20. In a city, 55 percent of the population is in favor of constructing a new shopping center. If a random sample of 400 people is selected, find the probability that

(a) at least 200 support the construction of the new shopping center;

(b) people between 250 and 350 support the construction of the new shopping center;

(c) at most 225 support the construction of the new shopping center.

5.21. The weight of a group of people is independently and normally distributed, with mean 70 kg and standard deviation 4 kg. What percentage of individuals from this group weigh less than 75 kg? What percentage of individuals from this group weigh more than 67 kg?

5.22. Tom is throwing darts at a dartboard repeatedly. Each of his throws, independently of all previous throws, has a success probability of .05 of hitting the bullseye. What is the approximate probability that Tom takes more than 50 throws to hit the bullseye?

5.23. A card is picked at random from a shuffled card deck for 500 consecutive times.

(a) What is the approximate probability that a red card will be picked between 250 and 300 times inclusively?

(b) What is the approximate probability that an even-numbered card will be picked more than 200 times?

5.24. The lifetimes of interactive computer chips produced by a certain semiconductor manufacturer are normally distributed with parameters $\mu = 1.4 \times 10^6$ hours and $\sigma = 3 \times 10^5$ hours. What is the approximate probability that a batch of 100 chips will contain at least 20 whose lifetimes are less than 1.8×10^6?

5.25. A die is biased in such a way that even numbers are three times as likely to be rolled as odd numbers. Approximate the probability that the number 5 will appear at most 15 times in 100 throws.

5.26. Two types of coins are produced at a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test: We shall toss the coin 1000 times. If the coin lands on heads 525 or more times, then we shall conclude that it is a biased coin, whereas if it lands on heads fewer than 525 times, then we shall conclude that it is a fair coin. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

5.27. In 10,000 independent tosses of a coin, the coin landed on heads 5800 times. Is it reasonable to assume that the coin is not fair? Explain.

5.28. About 17 percent of the world's population has blue eyes. What is the approximate probability of spotting at least 40 blue-eyed individuals in a crowd of 300 people? State your assumptions.

5.29. A model for the movement of a stock supposes that if the present price of the stock is s, then after one period, it will be either us with probability p or ds with probability $1 - p$. Assuming that successive movements are independent, approximate the probability that the stock's price will be up at least 30 percent after the next 1000 periods if $u = 1.012, d = .990$, and $p = .52$.

5.30. An image is partitioned into two regions, one white and the other black. A reading taken from a randomly chosen point in the white section will be normally distributed with $\mu = 4$ and $\sigma^2 = 4$, whereas one taken from a randomly chosen point in the black region will have a normally distributed reading with parameters $(6, 9)$. A point is randomly chosen on the image and has a reading of 5. If the fraction of the image that is black is α, for what value of α would the probability of making an error be the same, regardless of whether one concluded that the point was in the black region or in the white region?

5.31. (a) A fire station is to be located along a road of length $A, A < \infty$. If fires occur at points uniformly chosen on $(0, A)$, where should the station be located so as to minimize the expected distance from the fire? That is, choose a so as to

$$\text{minimize } E[|X - a|]$$

when X is uniformly distributed over $(0, A)$.

(b) Now suppose that the road is of infinite length—stretching from point 0 outward to ∞. If the distance of a fire from point 0 is exponentially distributed with rate λ, where should the fire station now be located? That is, we want to minimize $E[|X - a|]$, where X is now exponential with rate λ.

5.32. The time X (in minutes) between customer arrivals at a bank is exponentially distributed with mean 1.5 minutes.

(a) If a customer has just arrived, what is the probability that no customer will arrive in the next 2 minutes?

(b) What is the probability that no customer will arrive within the next minute, given that no customer had arrived in the past minute?

5.33. Suppose that U is a uniform random variable on $(0, 1)$. What is the distribution of $V = aU^{-\frac{1}{\lambda}}$ for $a, \lambda > 0$?

5.34. Jones figures that the total number of thousands of miles that a racing auto can be driven before it would need to be junked is an exponential random variable with parameter $\frac{1}{20}$. Smith has a used car that he claims has been