AML5103: Applied Probability & Statistics – Formula Sheet – August 2, 2024

1. **Counting formulas**:

Setup	Formula
# of ways to select (*without replacement*) and arrange (*order matters*) n distinct objects # of ways to distribute n distinct objects into n distinct bins with one object per bin	$n!$
# of ways to select (*without replacement, order doesn't matter*) r objects out of n distinct objects # of ways to arrange n objects when one object repeats r times and the other $n-r$ times # of ways to divide n distinct objects into two unlabeled groups of unequal sizes r and $n-r$ # of ways to divide n distinct objects into two labeled groups of specific sizes r and $n-r$	$\binom{n}{r} = nC_r$
# of ways to select (*without replacement*) and arrange (*order matters*) r objects out of n distinct objects # of ways to distribute r distinct objects into n distinct bins with at most one object per bin	$r!\binom{n}{r} = nP_r$
# of ways to select (*with replacement*) and arrange (*order matters*) r objects out of n distinct objects # of ways to distribute r distinct objects into n distinct bins	n^r
# of ways to arrange n objects with $r_1, r_2, \ldots, r_k$ repetitions, where $n = r_1 + r_2 + \cdots + r_k$ # of ways to divide n distinct objects into k unlabeled groups of unequal sizes $r_1, r_2, \ldots, r_k$, where $n = r_1 + r_2 + \cdots + r_k$ # of ways to divide n distinct objects into k labeled groups of specific sizes $r_1, r_2, \ldots, r_k$, where $n = r_1 + r_2 + \cdots + r_k$	$\dfrac{n!}{r_1!r_2!\cdots r_k!}$
# of ways to divide n distinct objects into k unlabeled groups with some of equal sizes, say $\underbrace{r_1 = r_2}_{\text{2 groups}}$, $\underbrace{r_3 = r_4 = r_5}_{\text{3 groups}}$, and $r_6, \ldots, r_k$ are different such that $n = r_1 + r_2 + \cdots + r_k$	$\left(\dfrac{1}{2!3!}\right)\left(\dfrac{n!}{r_1!r_2!\cdots r_k!}\right)$
# of ways to select r objects (*with replacement, order doesn't matter*) out of n distinct objects # of ways to distribute r identical objects into n distinct bins # of non-negative integer solutions to the equation $x_1 + x_2 + \cdots + x_n = r$	$\binom{n+r-1}{r}$
# of ways to distribute r identical objects into n distinct bins such that no bin is empty # of positive integer solutions to the equation $x_1 + x_2 + \cdots + x_n = r$	$\binom{r-1}{n-1}$

2. **Inclusion–exclusion principle for 4 events**:

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup A_4) = {} & P(A_1) + P(A_2) + P(A_3) + P(A_4) \\ & - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) - P(A_2 \cap A_3) - P(A_2 \cap A_4) - P(A_3 \cap A_4) \\ & + P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_4) \\ & - P(A_1 \cap A_2 \cap A_3 \cap A_4). \end{aligned}$$

3. **<u>Conditional probability</u>**:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \Leftrightarrow P(A \cap B) = P(A \mid B) \times P(B).$$

4. **<u>Law of total probability</u>**:

For event A and the mutually exclusive & collectively exhaustive set of events $\{B_1, B_2, \ldots, B_k\}$ for the same sample space,

$$\begin{aligned} P\Bigg(\underbrace{A}_{\text{e.g., has Cancer}}\Bigg) &= P\Bigg(\underbrace{A \cap B_1}_{\text{has Cancer AND from State-1}}\Bigg) + P\Bigg(\underbrace{A \cap B_2}_{\text{has Cancer AND from State-2}}\Bigg) + \cdots + P\Bigg(\underbrace{A \cap B_k}_{\text{has Cancer AND from State-k}}\Bigg) \\ &= P(A \mid B_1) \times P(B_1) + P(A \mid B_2) \times P(B_2) + \cdots + P(A \mid B_k) \times P(B_k). \end{aligned}$$

Additional conditioning form:

$$\begin{aligned} P\Bigg(\underbrace{A}_{\text{e.g., has Cancer}} \mid \text{male}\Bigg) &= P\Bigg(\underbrace{A \cap B_1}_{\text{has Cancer AND from State-1}} \mid \text{male}\Bigg) + P\Bigg(\underbrace{A \cap B_2}_{\text{has Cancer AND from State-2}} \mid \text{male}\Bigg) + \cdots + P\Bigg(\underbrace{A \cap B_k}_{\text{has Cancer AND from State-k}} \mid \text{male}\Bigg) \\ &= P(A \mid B_1, \text{male}) \times P(B_1 \mid \text{male}) + P(A \mid B_2, \text{male}) \times P(B_2 \mid \text{male}) + \cdots + P(A \mid B_k, \text{male}) \times P(B_k \mid \text{male}). \end{aligned}$$

5. **Bayes' theorem**:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \cap A)}{P(B)} = \frac{P(B \mid A) \times P(A)}{P(B)} = \frac{P(B \mid A) \times P(A)}{P(B \mid A) \times P(A) + P(B \mid A^c) \times P(A^c)}$$

Additional conditioning form:

$$P(A \mid B, C) = \frac{P(B \mid A, C) \times P(A \mid C)}{P(B \mid C)} = \frac{P(B \mid A, C) \times P(A \mid C)}{P(B \mid A, C) \times P(A \mid C) + P(B \mid A^c, C) \times P(A^c \mid C)}.$$