Pricing European style options under the Black-Scholes model

References:

1. F. Black, M. Scholes "**The pricing of options and corporate liabilities",** J. Polit. Econ., 81, pp. 637-654, (1973).
2. P. Wilmott, S. Howison, J. Dewynne "**Option Pricing: Mathematical Models and Computation",** Oxford Financial Press, Oxford (1993).

Black-Scholes model

Under certain assumptions, the value (or price) $V(S_\tau, \tau)$ of a European call option on a stock, whose price at time τ is S_τ, is given by the Black-Scholes model (1973)

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S_\tau^2 \frac{\partial^2 V}{\partial S_\tau^2} + r S_\tau \frac{\partial V}{\partial S_\tau} - rV = 0 \quad \forall\ (S_\tau, \tau) \in (0, \infty) \times [0, T)$$

$$V(S_T, T) = \max\{S_T - K, 0\}$$

whose analytical solution is given by

$$V(S_\tau, \tau) = S_\tau \Phi(d_1) - K e^{-r(T-\tau)} \Phi(d_2),\ 0 \le \tau < T$$

where

$$d_1 = \frac{\ln\left(\frac{S_\tau}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-\tau)}{\sigma\sqrt{T-\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{T-\tau}$$

$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds$ is the standard normal cumulative distribution function

σ is the volatility of the stock

r is the risk-free interest rate

T is the maturity/expiry date

K is the strike price

Transformation to a constant coefficient initial value problem (IVP):

Using the transformation $\tau = T - t, \quad S_\tau = K e^x, \quad \text{and} \quad V(S_\tau, \tau) = V(Ke^x, T-t) := u(x,t),$

the original Black-Scholes model can be transformed to the following IVP:

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial u}{\partial x} + ru = 0 \quad \forall\ (x,t) \in (-\infty,\infty) \times (0,T]$$
$$u(x,0) = K \max\{e^x - 1, 0\}$$

Localization: Imposing artificial boundary conditions

In order to implement implicit finite difference methods to approximate the solution u, the above problem needs to be localized on a bounded domain $(x_L, x_R),\, x_L,\, x_R \in \mathbb{R}$, that is,

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial u}{\partial x} + ru = 0 \quad \forall\ (x,t) \in (x_L, x_R) \times (0,T]$$
$$u(x,0) = K \max\{e^x - 1, 0\}$$
$$u(x_L, t) = 0$$
$$u(x_R, t) = K(e^{x_R} - 1)$$

where, in the spatial direction, the artificial boundary conditions 0 and $K(e^{x_R} - 1)$ are imposed on the left boundary x_L and right boundary x_R, respectively.

Implement the following finite difference method

Consider the following parameter values

$\sigma = 0.2$

$r = 0.06$

$T = 1$

$K = 5$

$x_L = -2$

$x_R = 2$

and a uniform partition

$$\left\{(x_j, t_n) \in [x_L, x_R] \times [0,T] : x_j = x_L + (j-1)h,\, t_n = (n-1)k,\, h = \frac{x_R - x_L}{N_x},\, k = \frac{T}{N_t},\ j = 1, 2, \ldots, N_x + 1,\, n = 1, 2, \ldots, N_t + 1\right\}$$

of the rectangle $[x_L, x_R] \times [0,T]$.

Find $U = (U_j^n) \in \mathbb{R}^{(N_x+1)\times(N_t+1)}, \quad U_j^n \approx u(x_j, t_n), \quad j = 1, 2, \ldots, N_x + 1,\, n = 1, 2, \ldots, N_t + 1$, such that

$$U_j^1 = K \max\{e^{x_j} - 1, 0\}, \quad j = 1, 2, \dots, N_x + 1$$

$$U_1^n = 0, \quad n = 1, 2, \dots, N_t + 1$$

$$U_{N_x+1}^n = K(e^{x_R} - 1), \quad n = 1, 2, \dots, N_t + 1$$

$$\frac{U_j^2 - U_j^1}{k} - \frac{1}{2}\sigma^2 \frac{U_{j+1}^2 - 2U_j^2 + U_{j-1}^2}{h^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{U_{j+1}^2 - U_{j-1}^2}{2h} + rU_j^2 = 0 \quad \forall\ j = 2, 3, \dots, N_x$$

$$\frac{3U_j^{n+1} - 4U_j^n + U_j^{n-1}}{2k} - \frac{1}{2}\sigma^2 \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2h} + rU_j^{n+1} = 0 \quad \forall\ n = 2, 3, \dots, N_t,\ j = 2, 3, \dots, N_x$$