Question 1

Question 2

Question 3

A. The semi-implicit Backward Euler discretization is:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

$$u^2(1-u) \approx (u_i^n)^2(1-u_i^n)$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = (u_i^n)^2(1-u_i^n) + D\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}$$

B. For a domain with 5 nodes (i = 0,1,2,3,4), with

$$u_0^{n+1} = 1$$

and

$$u_4^{n+1} = 0$$

as boundary conditions, we get these 3 interior equations (i = 1,2,3):

For i = 1:

$$\frac{u_1^{n+1} - u_1^n}{\Delta t} = D\frac{u_2^{n+1} - 2u_1^{n+1} + 1}{(\Delta x)^2} + (u_1^n)^2(1-u_1^n)$$

For i = 2:

$$\frac{u_2^{n+1} - u_2^n}{\Delta t} = D\frac{u_3^{n+1} - 2u_2^{n+1} + u_1^{n+1}}{(\Delta x)^2} + (u_2^n)^2(1-u_2^n)$$

For i = 3:

$$\frac{u_3^{n+1} - u_3^n}{\Delta t} = D\frac{0 - 2u_3^{n+1} + u_2^{n+1}}{(\Delta x)^2} + (u_3^n)^2(1-u_3^n)$$

C. Let's rearrange these equations to standard form and write them as a matrix equation. Let's define:

$$r = \frac{D\Delta t}{(\Delta x)^2}$$

Then our system becomes:

$$\begin{bmatrix} (1+2r) & -r & 0 \\ -r & (1+2r) & -r \\ 0 & -r & (1+2r) \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \end{bmatrix} = \begin{bmatrix} u_1^n + \Delta t(u_1^n)^2(1-u_1^n) + r \\ u_2^n + \Delta t(u_2^n)^2(1-u_2^n) \\ u_3^n + \Delta t(u_3^n)^2(1-u_3^n) \end{bmatrix}$$