

## 5.8 Lagrange Multipliers

### Objectives

- ☐ I know the method of Lagrange multipliers.
- ☐ I can use it to maximize a multivariate function subject to a constraint.
- ☐ I understand how the method of Lagrange multipliers can be used to find absolute maximums and absolute minimums of a function over a closed region.
- ☐ I am able to use the method when I have two constraints for a function of three variables.

In the last section, we had to find maximum or minimum values for functions along a curve, namely the boundary of the domain. It was awful. Luckily, the method of **Lagrange multipliers** provides another way to find these extreme values. It's more commonly used in the STEM fields (Science, Technology, Engineering, and Math).

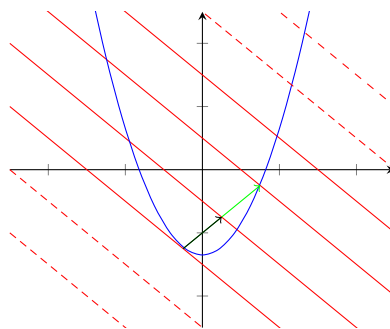
It arises from the notion that extreme points happen when the level curve of a surface ( $f(x, y)$ ) is tangent to a curve (the boundary of  $D$ ). If they are tangent, then their gradients are parallel. Recall that if two vectors are parallel, they differ only by a constant. That is,  $\langle 1, 2, 3 \rangle$  is parallel to  $\langle 2, 4, 6 \rangle$  since

$$\langle 2, 4, 6 \rangle = 2 \langle 1, 2, 3 \rangle$$

For Lagrange multipliers, we will call this constant  $\lambda$  (lambda). We will equate the gradient of our surface  $\nabla f$  with the gradient of a curve  $\nabla g$ :

$$\nabla f = \lambda \nabla g$$

This will force us to find a point where the level curves of  $f$  (in red) are tangent to the curve  $g$  (in blue).



In the picture above, we see a point where the two curves have normal vectors which share the same direction. In the language of Lagrange multipliers, we call  $g(x, y) = k$  the **constraint** on a function  $z = f(x, y)$ . That is, we want to maximize  $z = f(x, y)$  over all points  $(x, y)$  that satisfy  $g(x, y) = k$ . In the last section,  $g(x, y) = k$  was the boundary and  $f(x, y)$  was the function being maximized.

The following steps constitutes the method of Lagrange multipliers:

1. Find  $\nabla f$  and  $\nabla g$  in terms of  $x$  and  $y$ , and set up the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$g(x, y) = k$$

This will give you a system of equations based on the components of the gradients.

2. Solve this system of equations to get  $x$ ,  $y$ , and  $\lambda$ . (There may be multiple solutions.)
3. For each of these solutions, find  $f(x, y)$  and compare the values you get. Every solution that gives a maximum value is a maximum point, and every solution that gives a minimum value gives a minimum point.

### 5.8.1 Examples

**Example 5.8.1.1** Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint  $x^2 + y^2 = 10$ .

$$f(x, y) = 3x + y$$

For this problem,  $f(x, y) = 3x + y$  and  $g(x, y) = x^2 + y^2 = 10$ .

Let's go through the steps:

- $\nabla f = \langle 3, 1 \rangle$
- $\nabla g = \langle 2x, 2y \rangle$

This gives us the following equation

$$\langle 3, 1 \rangle = \lambda \langle 2x, 2y \rangle$$

We break up the above equation and consider the following system of 3 equations with 3 unknowns  $(x, y, \lambda)$

1.  $3 = 2\lambda x$
2.  $1 = 2\lambda y$
3.  $x^2 + y^2 = 10$

Now we want to solve for each variable. At this point, you should take a moment and try to cleverly think of a way to solve for one of the three. Let's plug in equations (1) and (2) into (3). This allows us to solve for  $\lambda$ .

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 10 \implies \lambda = \pm \frac{1}{2}$$

Now, we plug  $\lambda$  back into our original equations and get  $x = \pm 3$  and  $y = \pm 1$ . We get the following extreme points

$$(3, 1), (-3, -1)$$

We can classify them by simply finding their values when plugging into  $f(x, y)$ .

- $f(3, 1) = 9 + 1 = 10$
- $f(-3, -1) = -9 - 1 = -10$

So the maximum happens at  $\boxed{(3, 1)}$  and the minimum happens at  $\boxed{(-3, -1)}$ .

**Example 5.8.1.2** Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint  $x^4 + y^4 + z^4 = 1$ .

$$f(x, y, z) = x^2 + y^2 + z^2$$

- $\nabla f = \langle 2x, 2y, 2z \rangle$
- $\nabla g = \langle 4x^3, 4y^3, 4z^3 \rangle$

This gives us the following equation

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 4x^3, 4y^3, 4z^3 \rangle$$

Therefore, we have the following equations:

1.  $2x = 4\lambda x^3$
2.  $2y = 4\lambda y^3$
3.  $2z = 4\lambda z^3$

$$4. \ x^4 + y^4 + z^4 = 1$$

If  $x, y, z$  are nonzero, then we can consider  
Therefore, we have the following equations:

$$1. \ 1 = 2\lambda x^2$$

$$2. \ 1 = 2\lambda y^2$$

$$3. \ 1 = 2\lambda z^2$$

$$4. \ x^4 + y^4 + z^4 = 1$$

Remember, we can only make this simplification if all the variables are nonzero! In this form, we can plug in (1), (2), and (3) into (4). This gives us

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1$$

From this, we can solve for  $\lambda$  to get

$$\lambda = \pm \frac{\sqrt{3}}{2}$$

Now, we plug  $\lambda$  back into our original equations and get  $\pm \frac{1}{\sqrt[4]{3}}$  for each variable.

Regardless of the sign, we see that

$$f\left(\pm \frac{1}{\sqrt[4]{3}}, \pm \frac{1}{\sqrt[4]{3}}, \pm \frac{1}{\sqrt[4]{3}}\right) = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \boxed{\sqrt{3}}$$

Now, what if one of the variables is zero? How can we deal with that? We can assume one of the variables is zero and see what happens. That means when we plug into equation (4), we only get two nonzero terms.

If  $x$  is zero, then

$$(0)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1 \implies \lambda = \pm \frac{1}{\sqrt{2}}$$

Notice that if  $y$  or  $z$  were chosen to be zero instead of  $x$ , we'd still conclude that  $\lambda = \pm \frac{1}{\sqrt{2}}$ . That's why we can just consider one of the variables and think of it as considering all three possibilities. In this case, we get one variable to be zero and the remaining nonzero variables as  $\pm \frac{1}{\sqrt[4]{2}}$ . Therefore, we get the critical points

$$\left(0, \pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}\right), \left(\pm \frac{1}{\sqrt[4]{2}}, 0, \pm \frac{1}{\sqrt[4]{2}}\right), \left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}, 0\right)$$

For either of these points, we get

$$f(\text{critical point}) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Still, we haven't considered *all* possible values. What if two variables were zero?

Then, when we plug into equation (4), we get

$$(0)^2 + (0)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1 \implies \lambda = \pm \frac{1}{2}$$

For the variable that is not zero, we'd get the value  $\pm 1$ . Therefore, we have the critical points

$$(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$$

For either of these points, we get

$$f(\text{critical point}) = 1$$

It's not possible that all three variables are zero. Otherwise equation (4) would be false.

Therefore, the maximums are obtained at the points

- $\left(\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$
- $\left(-\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$
- $\left(\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$
- $\left(\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$
- $\left(-\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$
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- $\left(\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$
- $\left(-\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$

The minimums are obtained at the points

- $(1, 0, 0)$
- $(-1, 0, 0)$
- $(0, 1, 0)$
- $(0, -1, 0)$
- $(0, 0, 1)$
- $(0, 0, -1)$

This example shows how complicated these problems can get, especially with an added dimension. We could have easily missed the minimum values if we weren't careful.

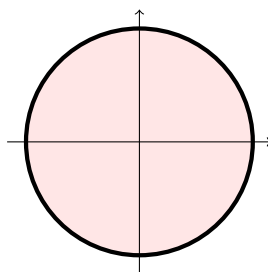
**Example 5.8.1.3** Use Lagrange multipliers to find the absolute maximum and absolute minimum of

$$f(x, y) = xy$$

over the region  $D = \{(x, y) \mid x^2 + y^2 \leq 8\}$ .

As before, we will find the critical points of  $f$  over  $D$ . Then, we'll restrict  $f$  to the boundary of  $D$  and find all extreme values. It is in this second step that we will use Lagrange multipliers.

The region  $D$  is a circle of radius  $2\sqrt{2}$ .



- $f_x(x, y) = y$
- $f_y(x, y) = x$

We therefore have a critical point at  $(0, 0)$  and  $f(0, 0) = 0$ .

Now let us consider the boundary. We will use Lagrange multipliers and let the constraint be  $x^2 + y^2 = 9$ . We begin with  $\nabla f = \lambda \nabla g$ .

$$\langle y, x \rangle = \lambda \langle 2x, 2y \rangle$$

This gives us the following equations:

1.  $y = 2\lambda x$
2.  $x = 2\lambda y$
3.  $x^2 + y^2 = 8$

Notice that if one variable is zero, then the other is as well. This violates equation (3), so we don't need to consider it. Let's substitute (1) into (2).

$$x = 4\lambda^2 x \implies \lambda = \pm \frac{1}{2}$$

Plugging this value into equations (1) and (2) give us the following equation

$$y = \pm x$$

We can then plug this into equation (3). Then  $2x^2 = 8 \implies x = \pm 2$ . We therefore have four critical points  $(2, 2)$ ,  $(2, -2)$ ,  $(-2, 2)$  and  $(-2, -2)$ . The maximums  $f(2, 2) = f(-2, -2) = 4$  and the minimums are  $f(2, -2) = f(-2, 2) = -4$ .

The absolute minimums are

$$(-2, 2) \text{ and } (2, -2)$$

The absolute maximums are

$$(2, 2) \text{ and } (-2, -2)$$

For functions of three variables  $f(x, y, z)$  (this does not work with functions of only two variables), we can also solve for extreme values given *two* constraints on  $(x, y, z)$ . For two constraints  $g(x, y, z) = k$  and  $h(x, y, z) = c$ , we can consider

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

where  $\lambda$  and  $\mu$  are constants.

The following algorithm is used to find  $(x, y, z)$  giving maximal or minimal values for  $f(x, y, z)$  so that the constraints,  $g(x, y, z) = k$  and  $h(x, y, z) = c$ , are true.

1. Find  $\nabla f$ ,  $\nabla g$ , and  $\nabla h$  and set up the equations

$$\nabla f(x, y, z) = \nabla g(x, y, z) + \nabla h(x, y, z)$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

Note that now we have five unknowns:  $x, y, z, \lambda$ , and  $\mu$ , but the vector equation at the top can be taken apart component by component and viewed as three equations of numbers, so we have a total of five equations.

2. Solve this system of equations to get (possibly multiple) solutions for  $x, y, z, \lambda$ , and  $\mu$ .
  3. For each solution  $(x, y, z, \lambda, \mu)$ , find  $f(x, y, z)$  and compare the values you get. The largest value corresponds to maximums, the smallest value corresponds to minimums.
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### 5.8.2 Examples

**Example 5.8.2.1** Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraints  $x + y - z = 0$  and  $x^2 + 2z^2 = 1$ .

$$f(x, y, z) = 3x - y - 3z$$

As you'll see, the technique is basically the same. It only requires that we look at more equations.

- $\nabla f = \langle 3, -1, -3 \rangle$
- $\nabla g = \langle 1, 1, -1 \rangle$
- $\nabla h = \langle 2x, 0, 4z \rangle$

These combine to

$$\langle 3, -1, -3 \rangle = \lambda \langle 1, 1, -1 \rangle + \mu \langle 2x, 0, 4z \rangle$$

Therefore, we have the following equations:

1.  $3 = \lambda + 2\mu x$
2.  $-1 = \lambda$
3.  $-3 = -\lambda + 4\mu z$
4.  $x + y - z = 0$
5.  $x^2 + 2z^2 = 1$

Already, we that  $\lambda = -1$  and

$$\begin{aligned} -3 &= 1 + 4\mu z \implies -\frac{1}{\mu} = z \\ 3 &= -1 + 2\mu x \implies \frac{2}{\mu} = x \end{aligned}$$



We can combine these two facts to get  $x = -2z$ . Let's use our fifth equation to solve.

$$1 = 4z^2 + 2z^2 \implies 6z^2 = 1 \implies z = \pm \frac{1}{\sqrt{6}}, \quad x = \mp \frac{2}{\sqrt{6}}$$

Plugging in either  $x$  or  $z$  to solve for  $\mu$  will give you  $\mu = \mp\sqrt{6}$ . That means we know

$$\lambda = -1, \quad \mu = \mp\sqrt{6}, \quad x = \mp \frac{2}{\sqrt{6}} \quad \text{and} \quad z = \pm \frac{1}{\sqrt{6}}$$

We now use equation 4 to find  $y$ .

$$\mp \frac{2}{\sqrt{6}} + y \mp \frac{1}{\sqrt{6}} = 0 \implies y = \pm \frac{3}{\sqrt{6}}$$

Therefore, my two points are

$$\left( \frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right) \quad \text{and} \quad \left( -\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$$

- $f\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = \frac{6}{\sqrt{6}} + \frac{3}{\sqrt{6}} + \frac{3}{\sqrt{6}} = 2\sqrt{6}$  (This is a max)
- $f\left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -\frac{6}{\sqrt{6}} - \frac{3}{\sqrt{6}} - \frac{3}{\sqrt{6}} = -2\sqrt{6}$  (This is a min)

For all these problems, it's important to take your time when simplifying. When we do algebra, we often want to just "plug and chug" through the problem. After all, that has worked pretty well in the past. Here, however, you'll quickly find that you will go in circles and never solve for anything. Just take your time and think about each step you take. It often helps to write out all the equations before trying to plug in anything.

**Summary of Ideas: Lagrange Multipliers**

- Lagrange multipliers allow you to maximize a function  $f(x, y)$  subject to a constraint  $g(x, y) = k$ .
- It is a method that can be used to find the extreme points of a function on the boundary of a closed region.
- The method asks you to solve for  $x, y$ , and  $\lambda$  given the following expressions

$$\nabla f = \lambda \nabla g \text{ and } g(x, y) = k$$

- We plug in to determine what points are maximums and what points are minimums.
- We can also restrict to more than one constraint,  $g(x, y, z) = k$  and  $h(x, y, z) = c$  when we have three or more variables. In this case, we solve for  $x, y, z, \lambda$  and  $\mu$  given the following expressions

$$\nabla f = \lambda \nabla g + \mu \nabla h, \quad g(x, y) = k \text{ and } h(x, y, z) = c$$