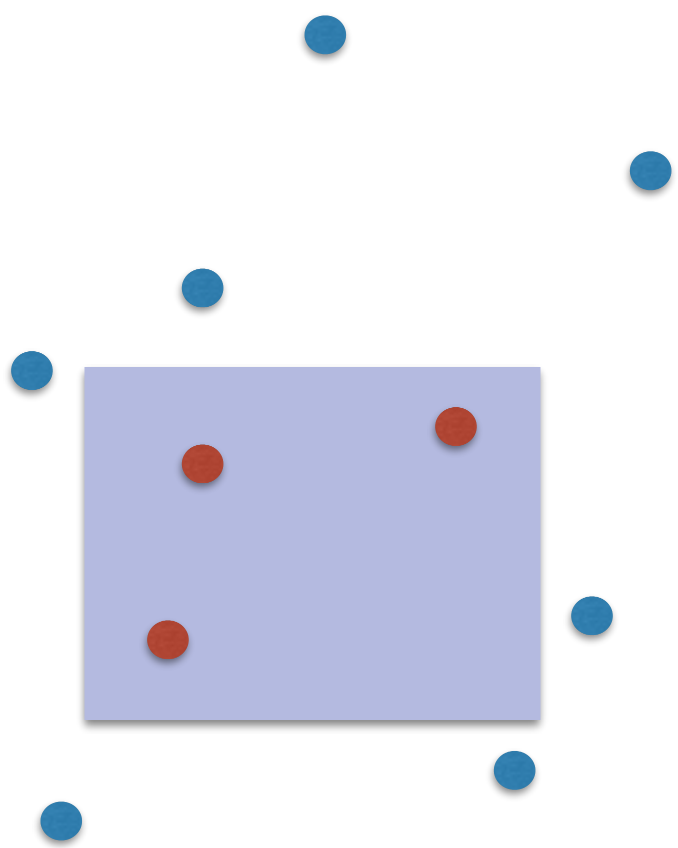
Range Trees

The problem

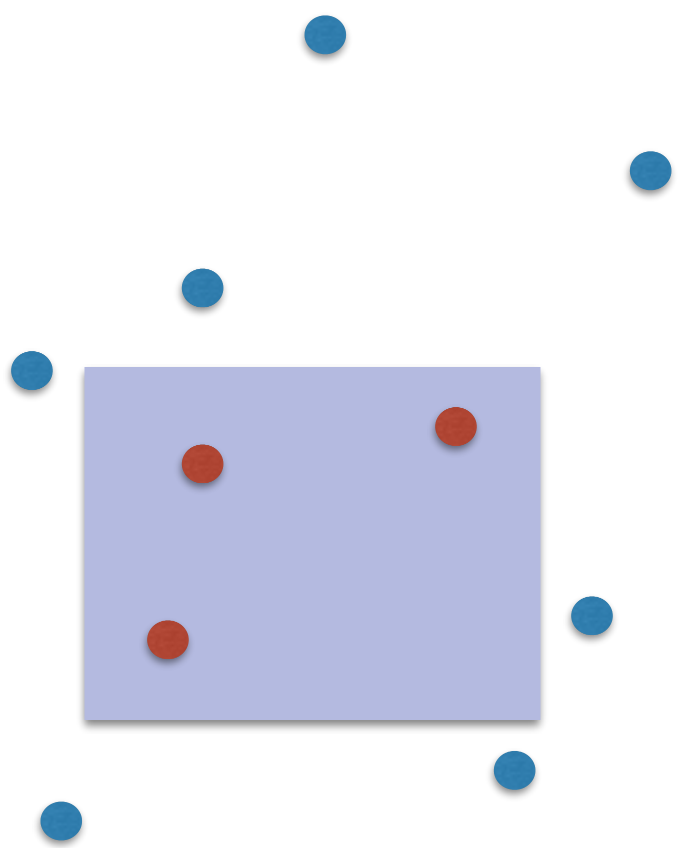
Given a set of n points in 2D, preprocess into a data structure to support fast range queries.



The problem

- n: size of the input (number of points)
- k: size of output (number of points in range)

Given a set of n points in 2D, preprocess into a data structure to support fast range queries.



- No data structure: traverse and check in O(n)
- Goal: static data structure (points are known ahead)
- In 1D: BBST can answer 1D range queries in O(lg n+k) and it's also dynamic (supports inserts and deletes)

Data structures for 2D range queries

- 2D kd-trees
  - Build: $O(n \lg n)$
  - Space: $O(n)$
  - Range queries: $O(n^{1/2} + k)$

Different trade-offs!

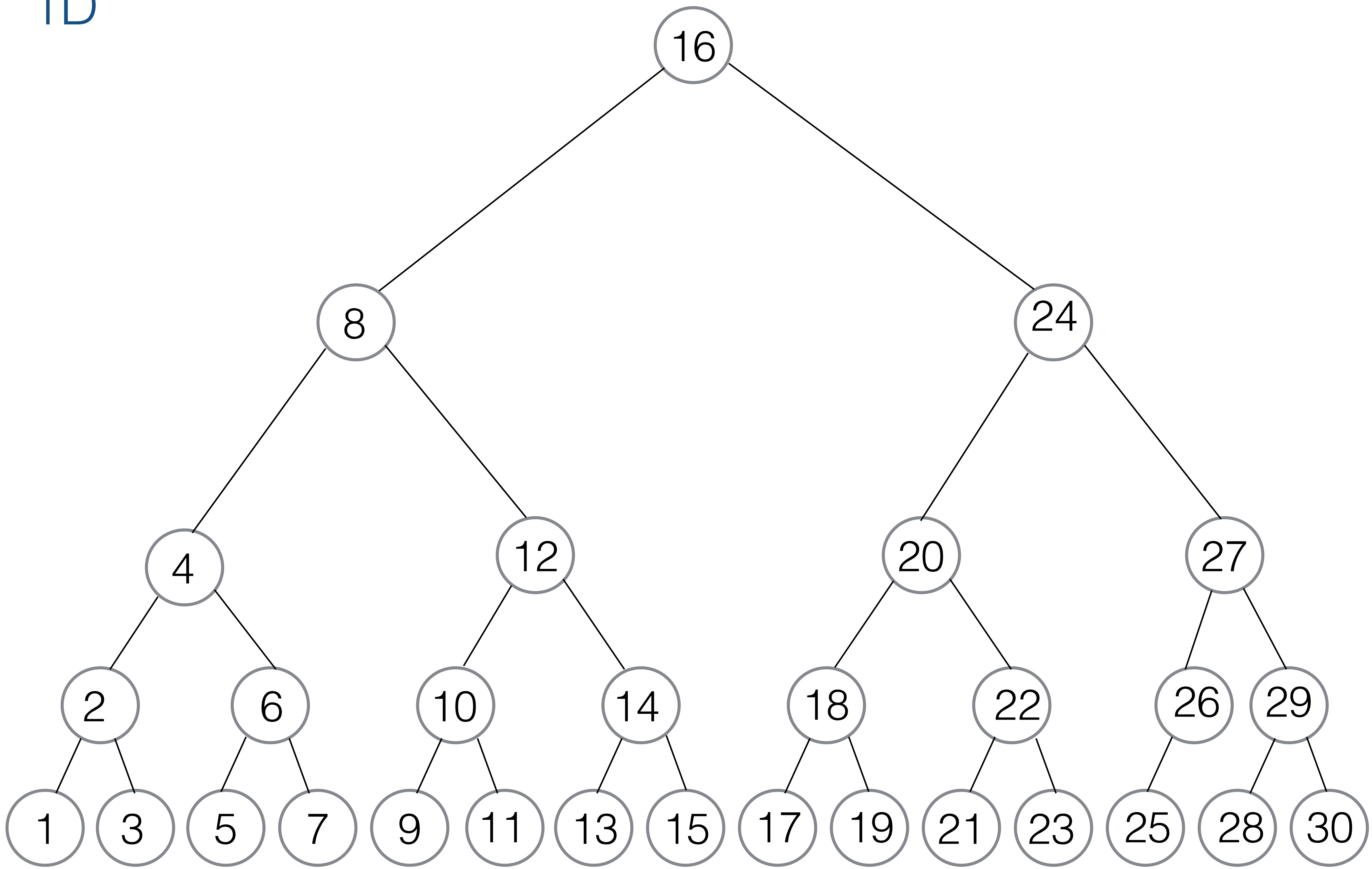
- 2D Range trees
  - Build: $O(n \lg n)$
  - Space: $O(n \lg n)$
  - Range queries: $O(\lg^2 n + k)$

1D

- P is a set of points on the real line
- A range query is an interval

- Store the points as a BBST

1D



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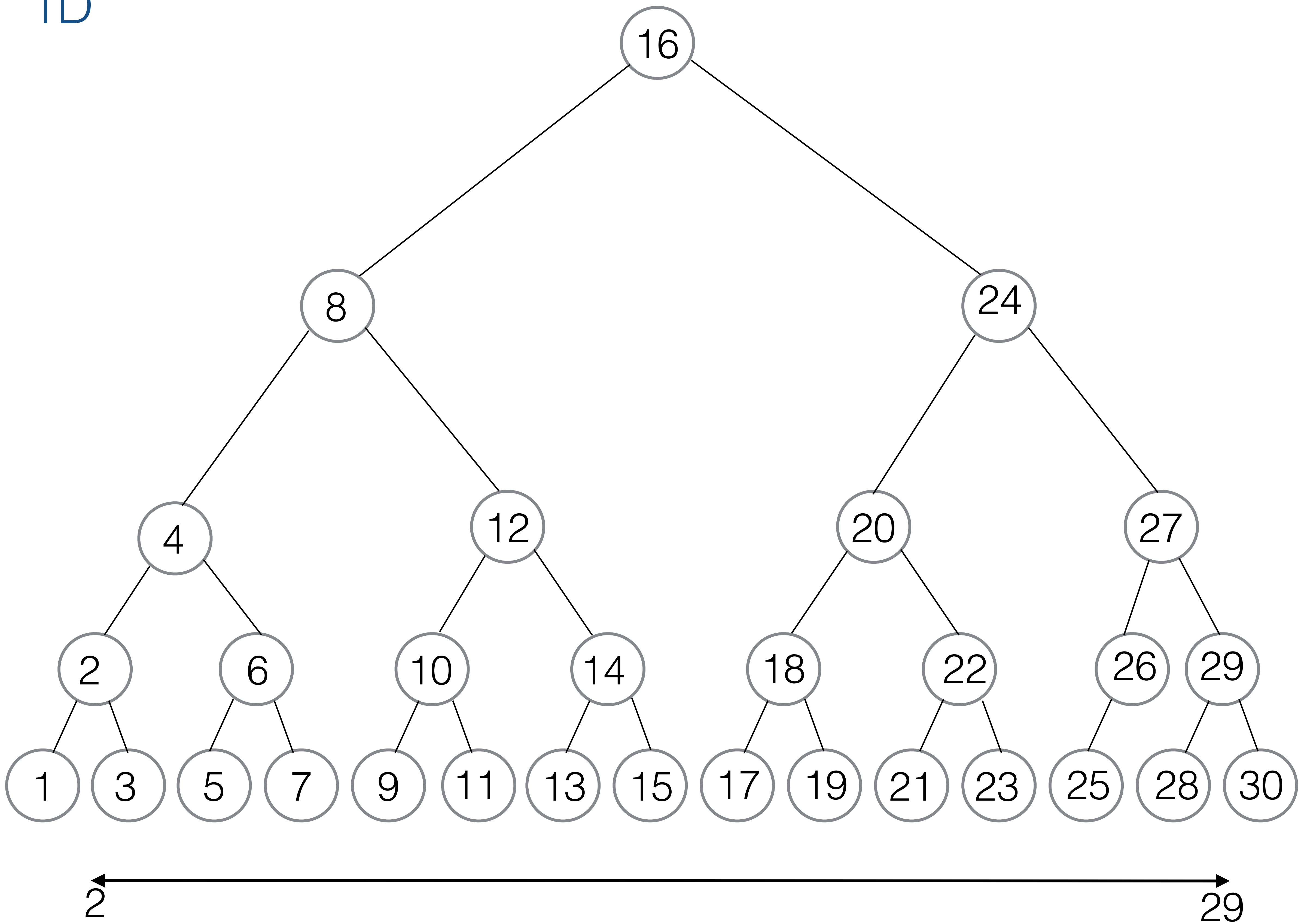
1D



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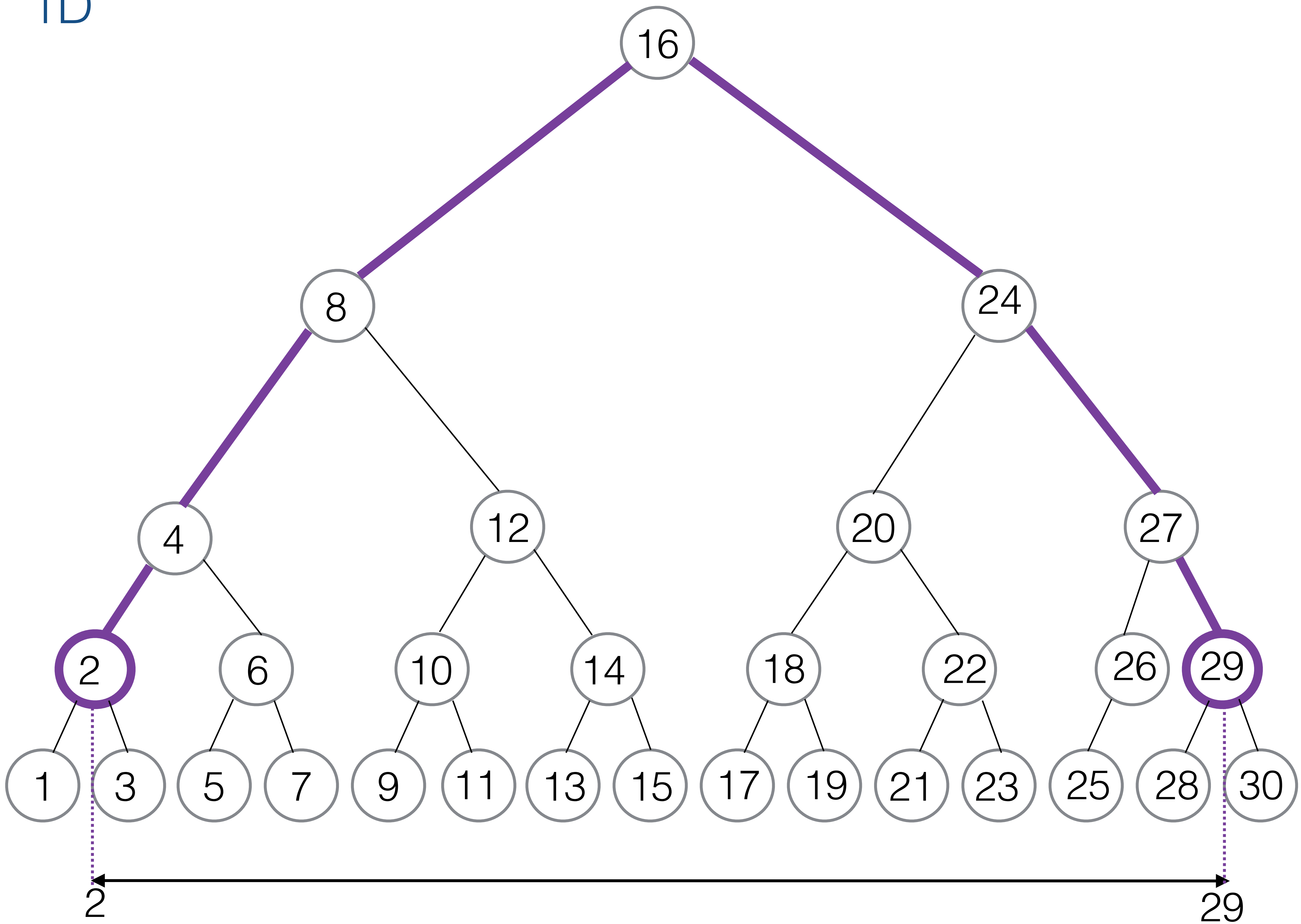

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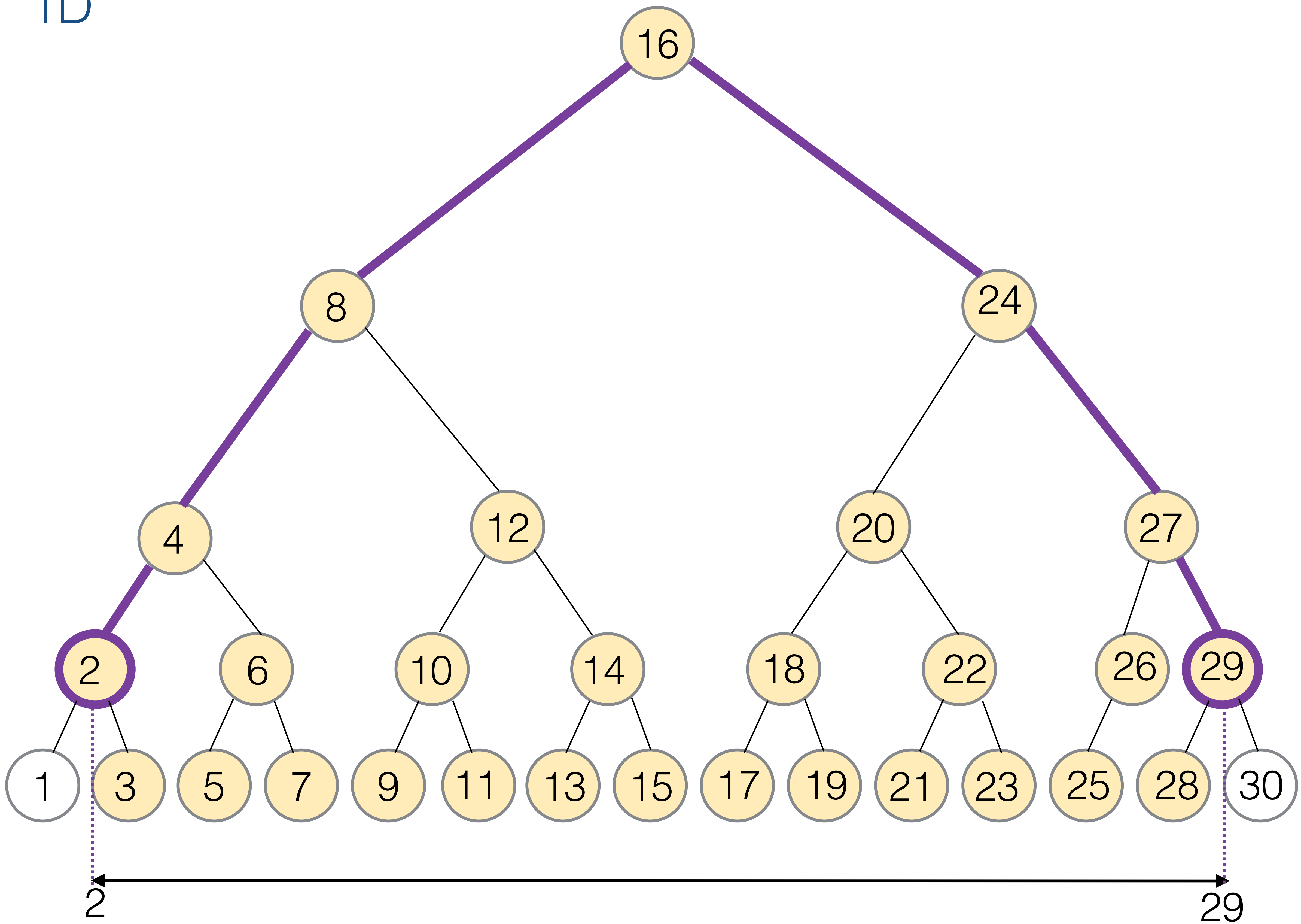

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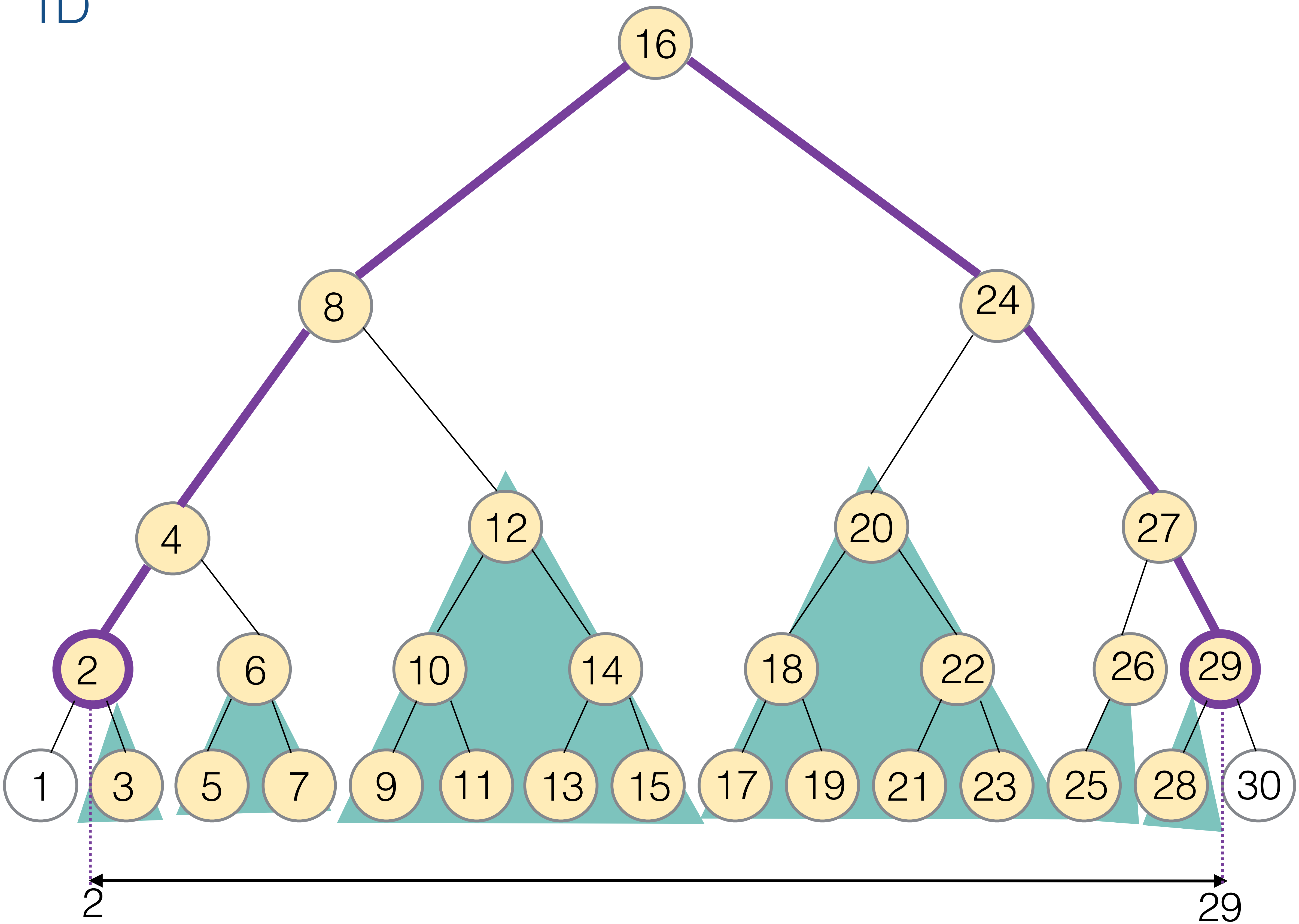
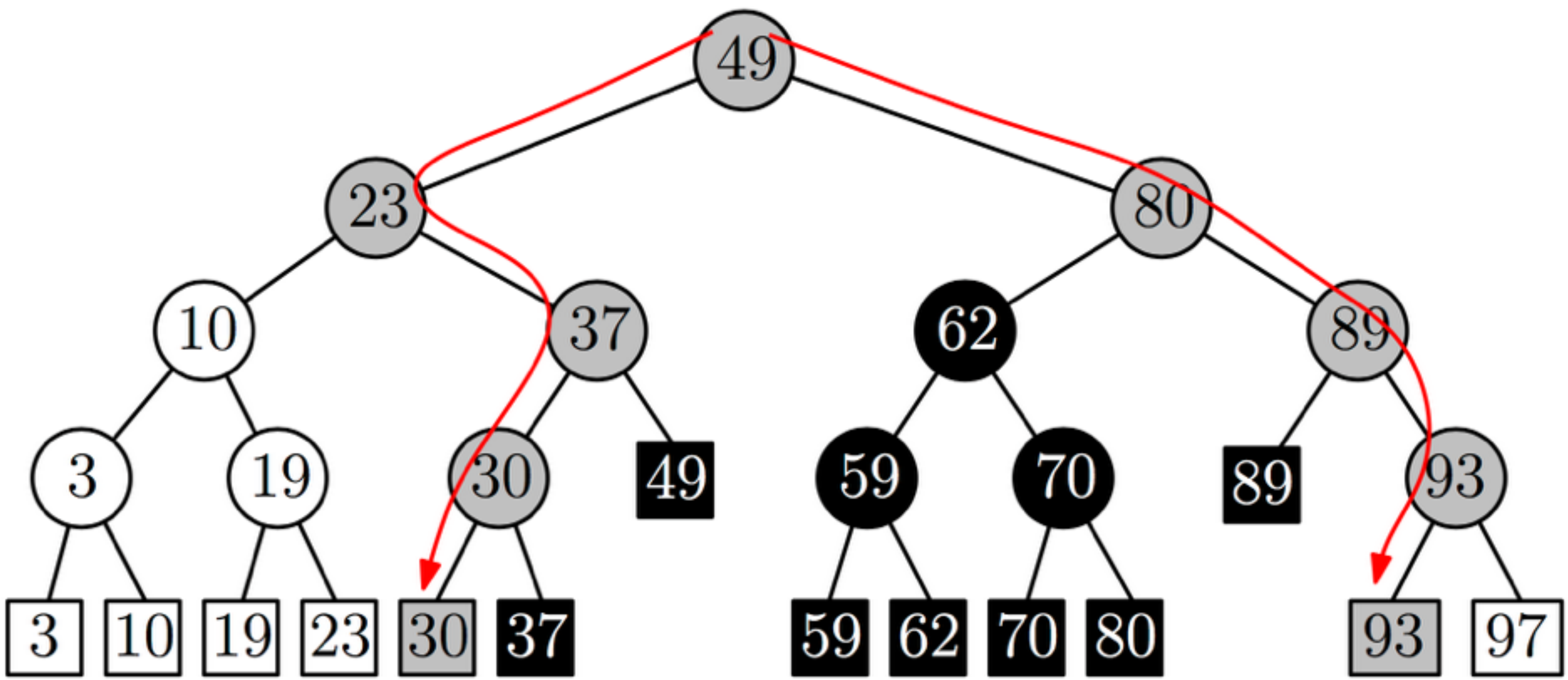
1D



The k points in the range are in O(lg n) subtrees

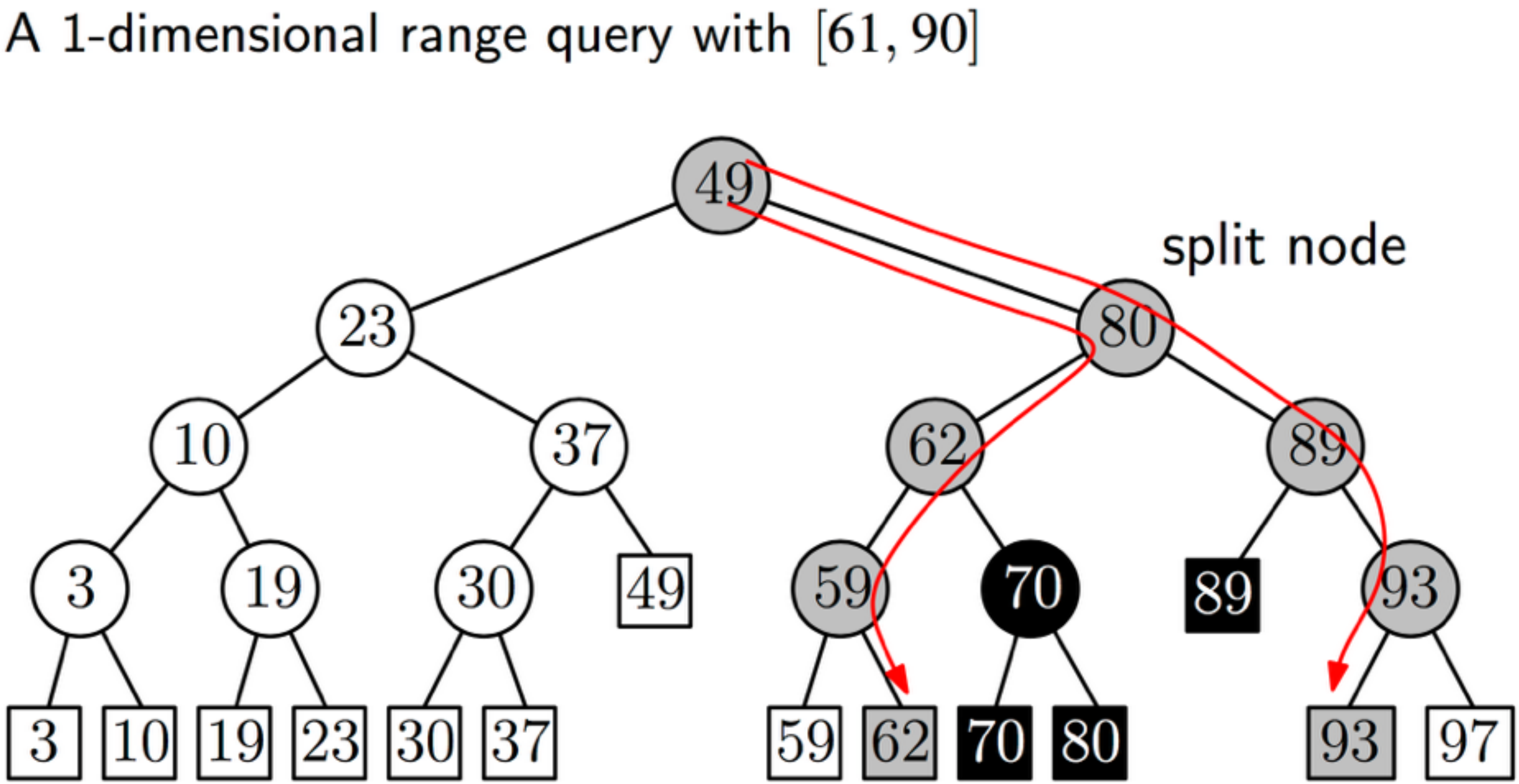
1D: different visual

A 1-dimensional range query with $[25, 90]$



screen shot from Mark van Kreveld slides, http://www.cs.uu.nl/docs/vakken/ga/slides5b.pdf)

1D: different visual



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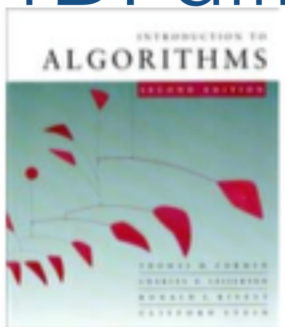
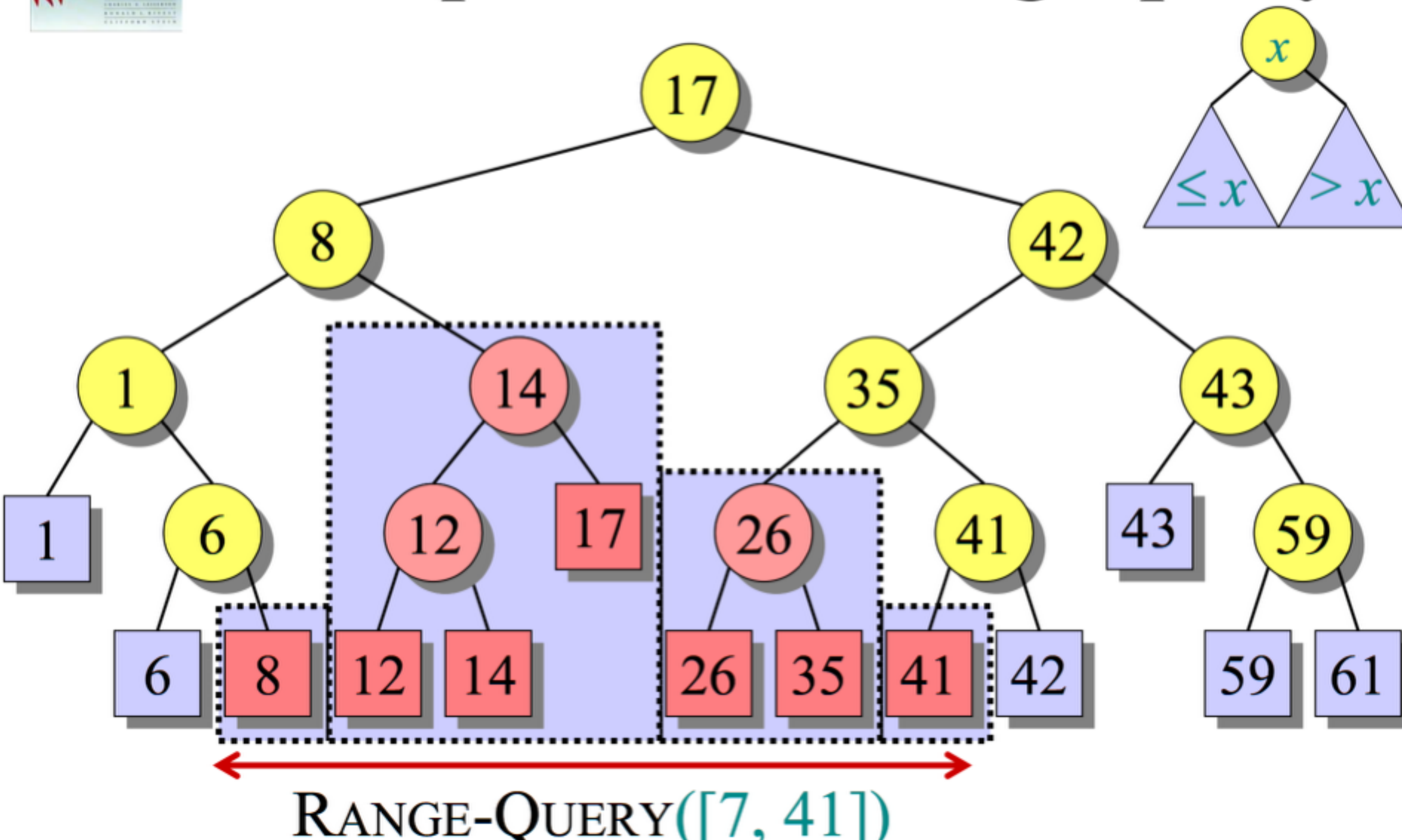
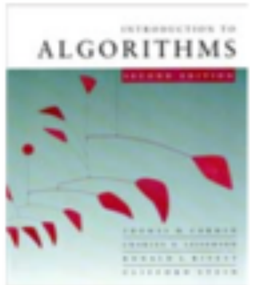
1D: different visual



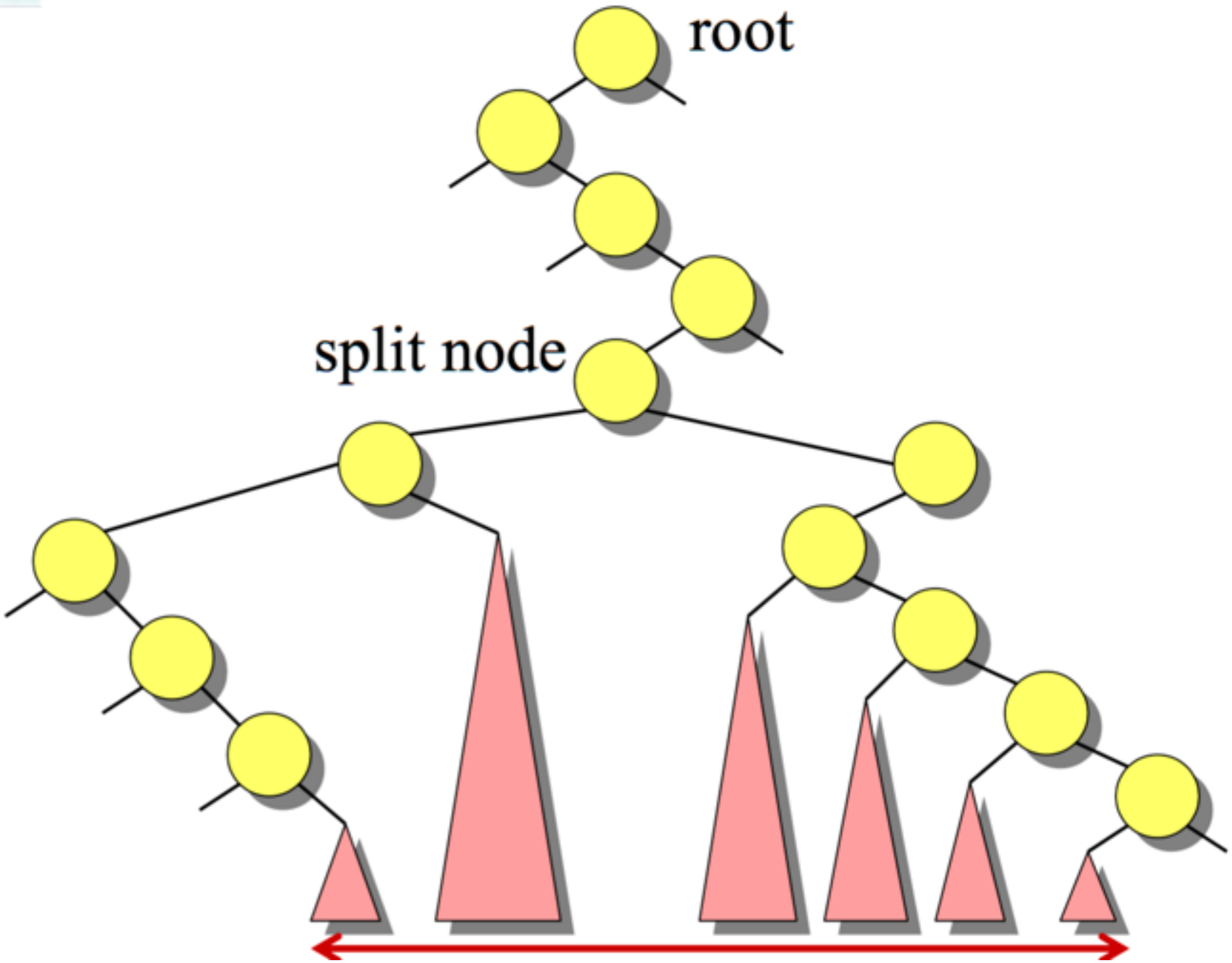
Example of a 1D range query



1D: different visual



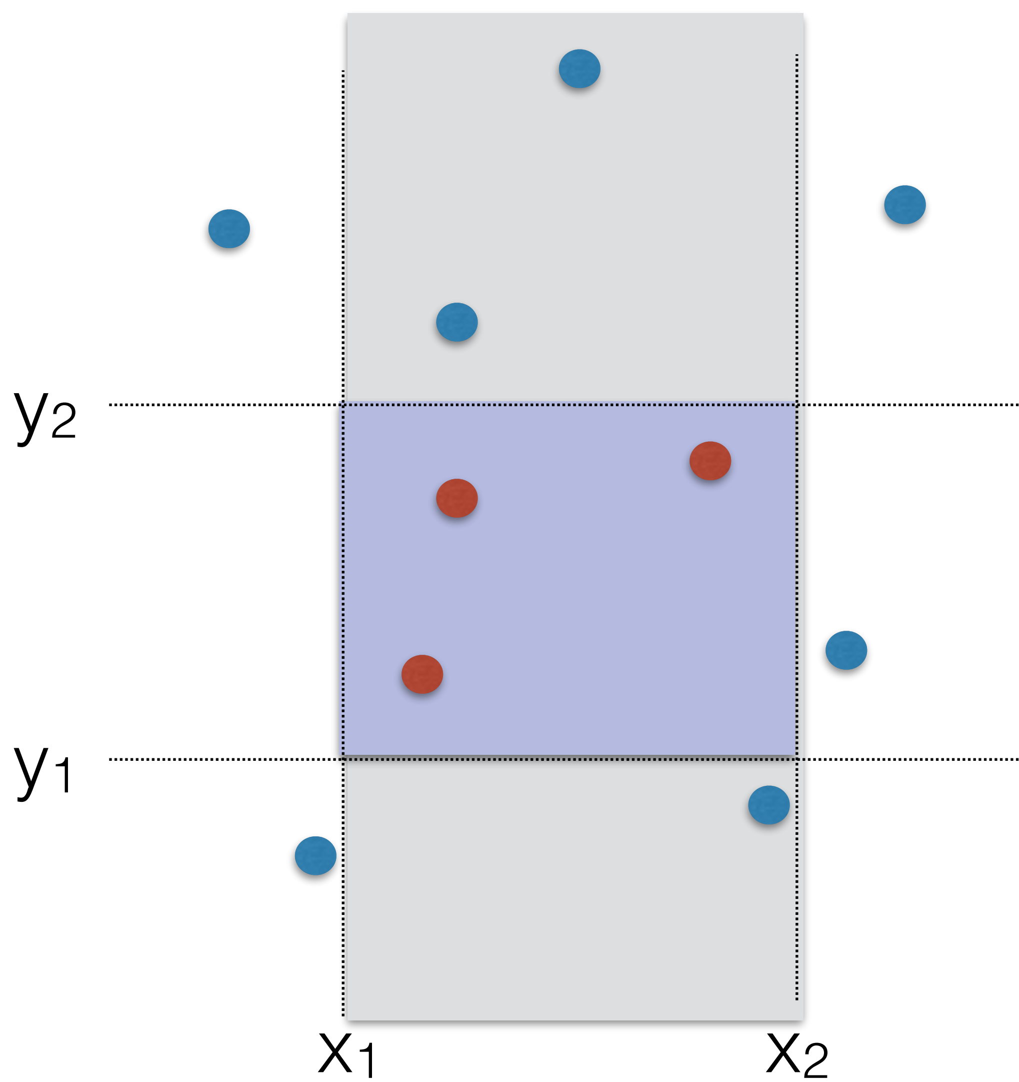
General 1D range query



Towards 2D Range Trees

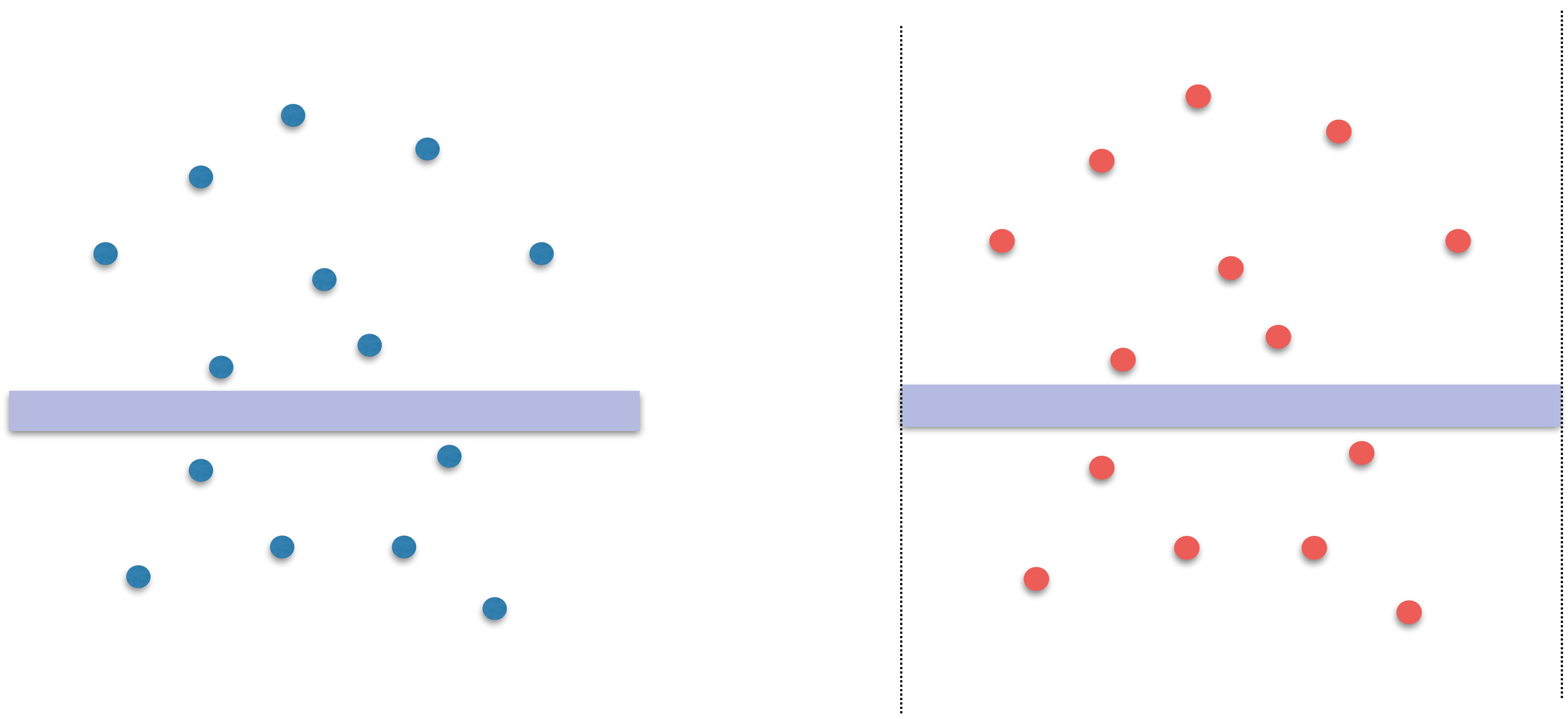
Idea

- Denote query $[x_1, x_2] \times [y_1, y_2]$
- Find all points with the x-coordinates in the correct range $[x_1, x_2]$
- Out of these points, find all points with the y-coord in the correct range $[y_1, y_2]$



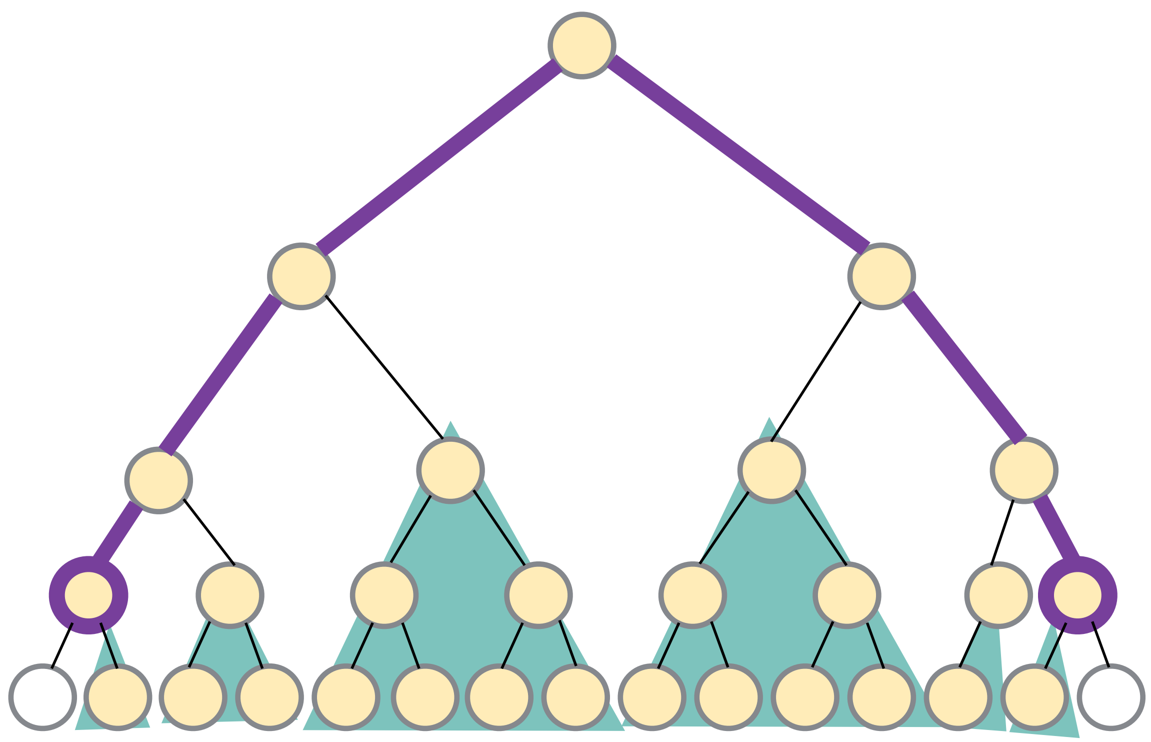
Towards 2D Range Trees

- How?
  - Store points in a BBST by x-coordinate
  - Use it to find all points with the x-coordinates in the correct range $[x_1, x_2]$
  - This alone is not enough..



Towards 2D Range Trees

- How?
  - Store points in a BBST by x-coordinate
  - Find all points with the x-coordinates in the correct range [x_1, x_2]
    - They are sitting in O(lg n) subtrees!



Towards 2D Range Trees

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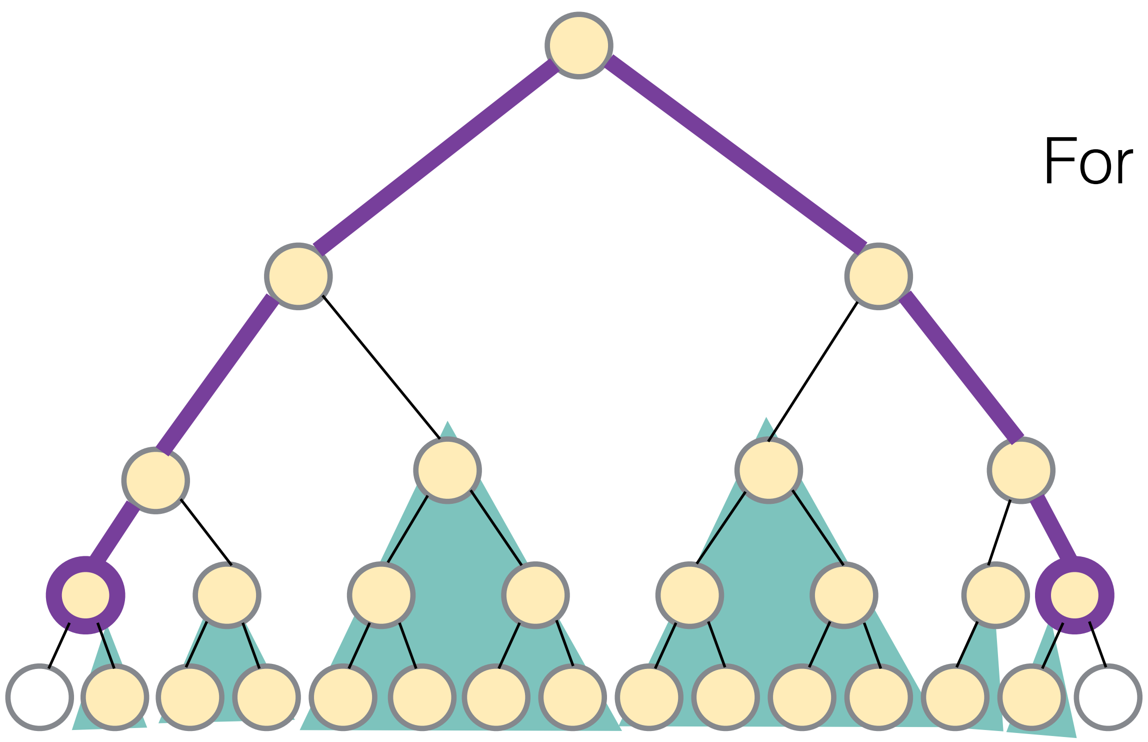

For each subtree we need all points in $[y_1, y_2]$

Towards 2D Range Trees

- How?
  - Store points in a BBST by x-coordinate
  - Find all points with the x-coordinates in the correct range [x_1, x_2]
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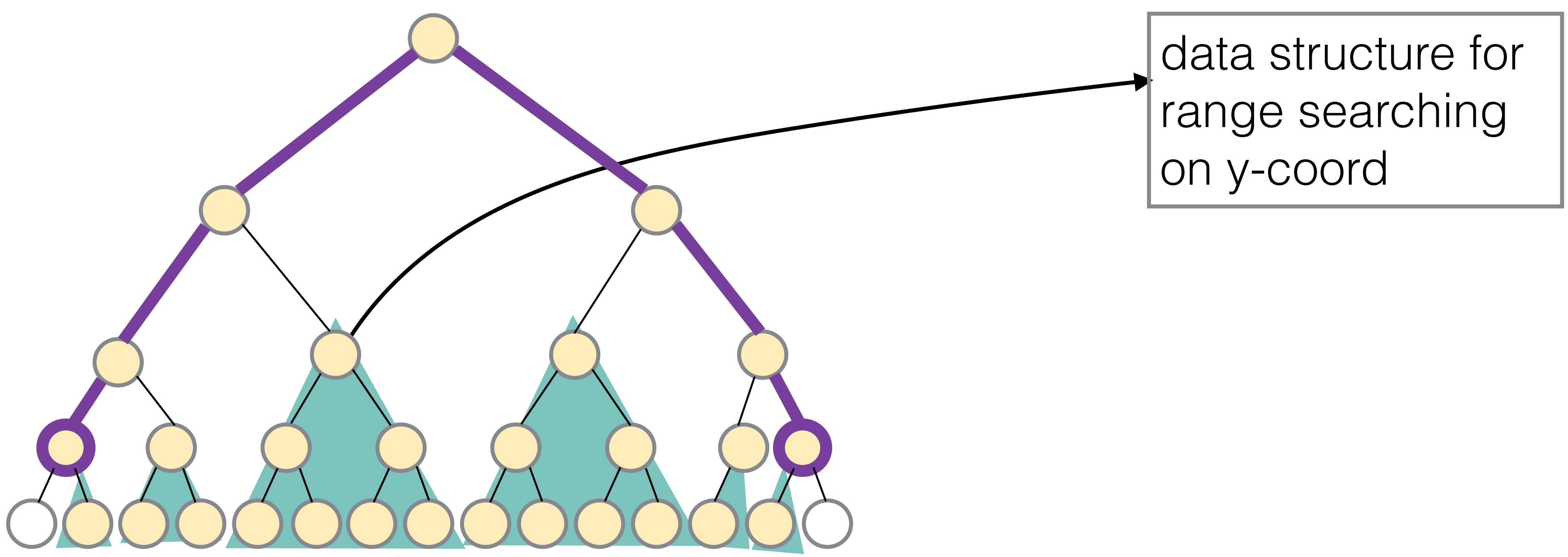


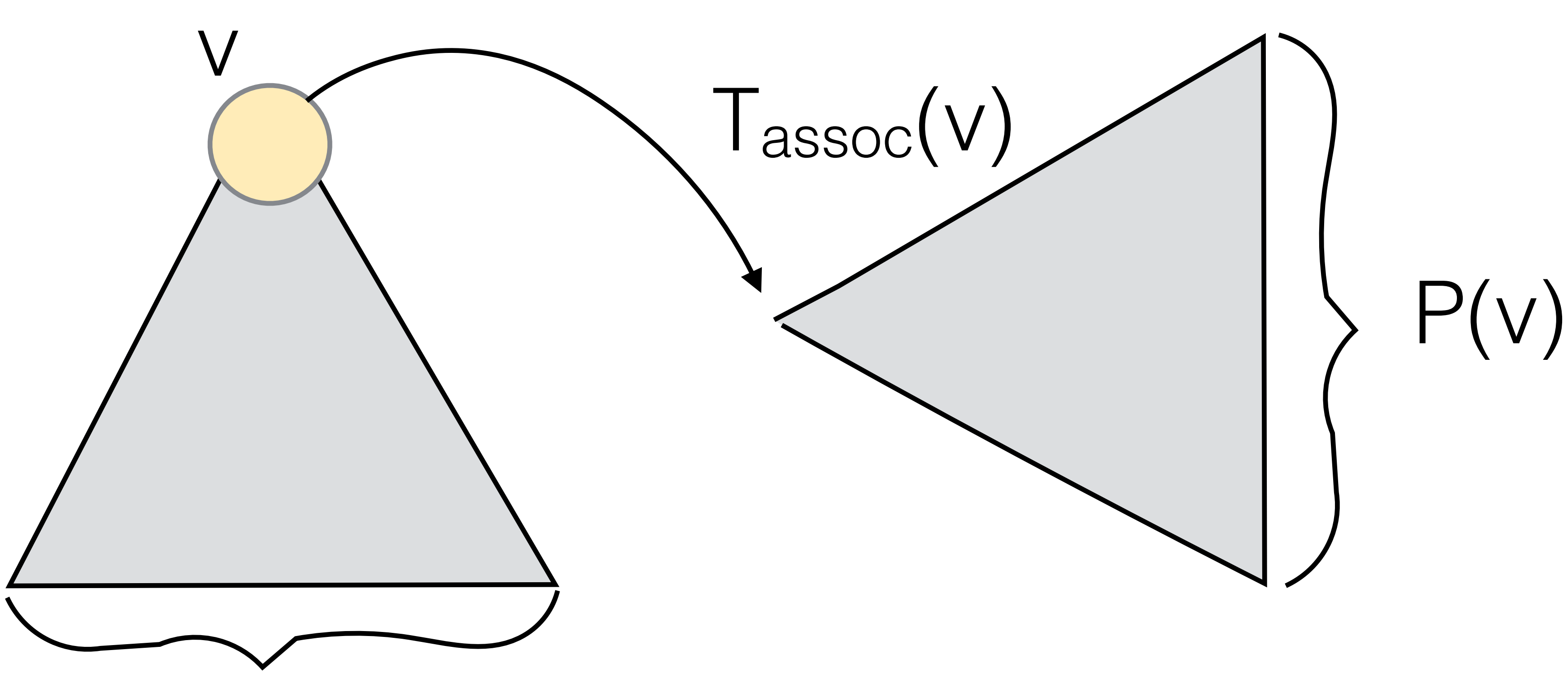
What is a good data structure for range search on y?

The 2D Range Tree

P = set of points

RangeTree(P) is

- A BBST T of P on x-coord
- Any node v in T stores a BBST T_{assoc} of P(v), by y-coord



P(v): all points in subtree rooted at v

The 2D Range Tree

- To simplify, we'll use BBSTs that store all data in leaves

The 2D Range Tree

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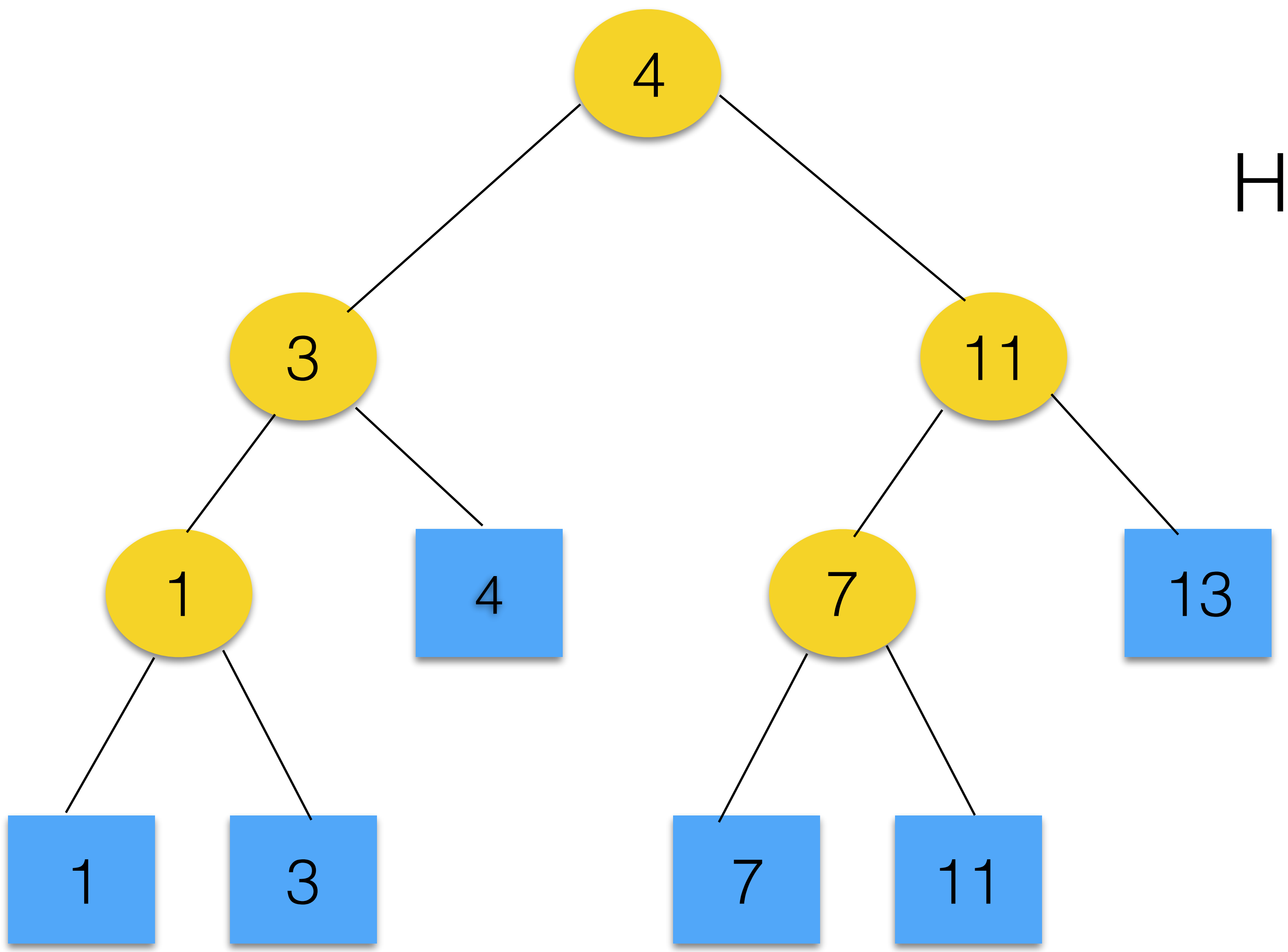
Example: P = (1,3,4,7,11,13}



How do you build it?

Class work

- Show the BBST with all dat in leaves for P = {1,2,3,4,5,6,7,8,9,10}

- Write pseudocode for the algorithm to build on

  BuildBBST(P)

The 2D Range Tree

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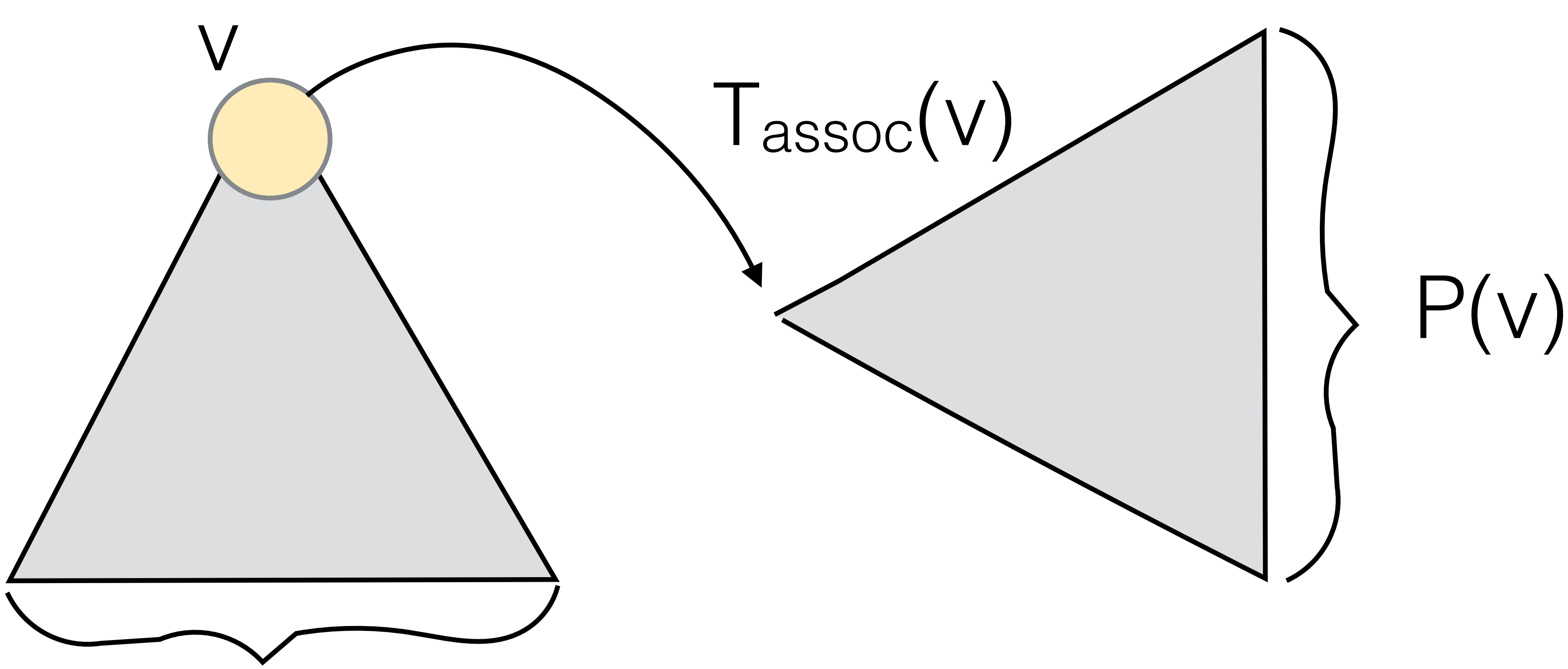
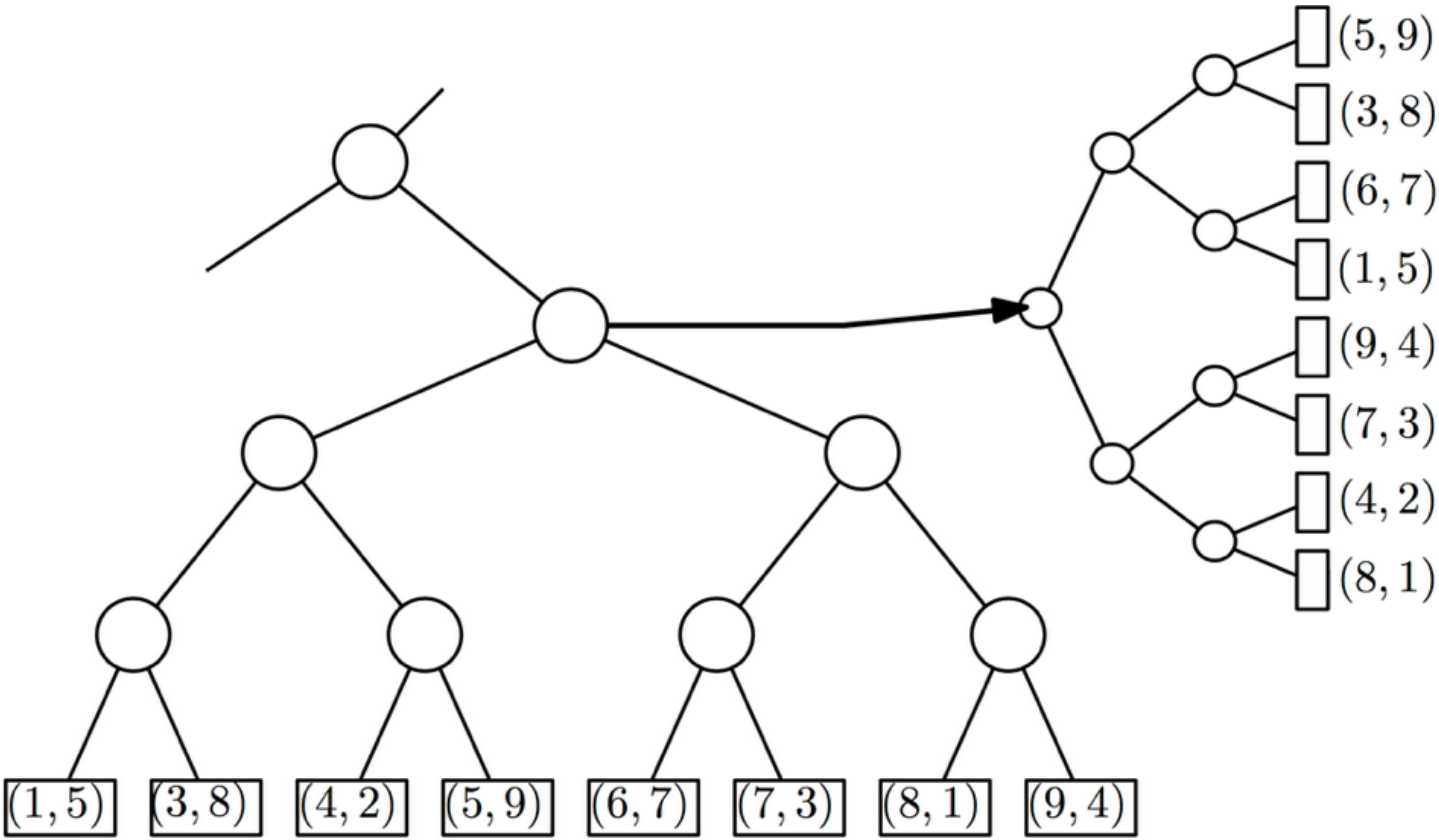


P(v): all points in subtree rooted at v

screen shot from Mark van Kreveld slides, http://www.cs.uu.nl/docs/vakken/ga/slides5b.pdf)



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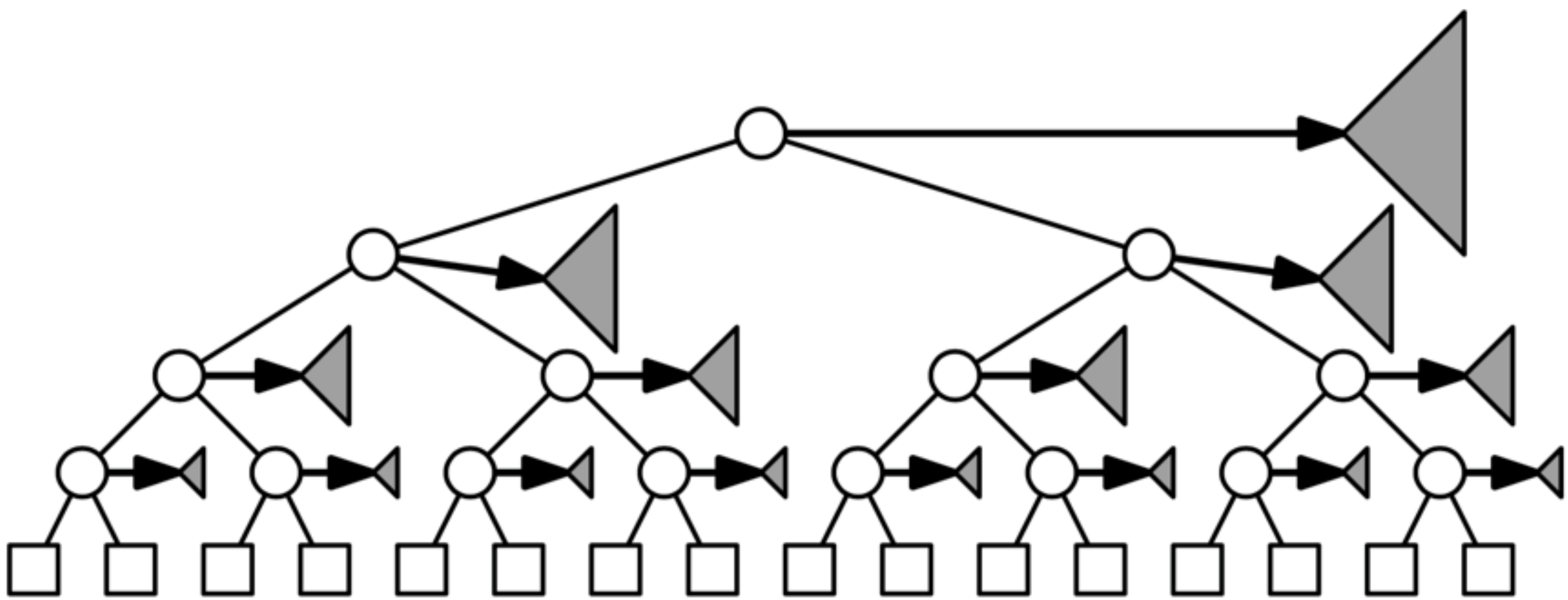
Every internal node stores a whole tree in an *associated structure*, on y-coordinate



Class work

- Let P = {(1,4), (5,8), (4,1), (7,3), (3, 2), (2, 6), (8,7)}.

  Draw the points and show the range tree for it.

The 2D Range Tree

Questions

- How do you build it and how fast?
- How much space does it take?
- How do you answer range queries and how fast?

The 2D Range Tree

- How do you build it and how fast?

Building a 2D Range Tree

Let $P = \{p_1, p_2, \ldots p_n\}$. Assume P sorted by x-coord.

Algorithm Build2DRT(P)

1. *Construct the associated structure: build a BBST T_{assoc} on the set of y-coordinates of P*
2. *if P contains only one point:*

   create a leaf v storing this point, create its T_{assoc} and return v
3. *else*
   1. *partition P into 2 sets wrt the median coordinate x_{middle}:*

      $P_{left} = \{p \text{ in } P.\ p_x <= x_{middle}\}$, $P_{right} = \ldots$ *//keep in x-order*
   2. $v_{left} = Build2DRT(P_{left})$
   3. $v_{right} = Build2DRT(P_{right})$
   4. *create a node v storing x_{middle}, make v_{left} its left child, make v_{right} its right child, make T_{assoc} its associate structure*
   5. *return v*

Building a 2D Range Tree

How fast?

- Constructing a BBST on an unsorted set of keys takes O(n lg n)

- The construction algorithm Build2DRT(P) takes

$$T(n) = 2T(n/2) + O(n \lg n)$$

- This solves to $O(n \lg^2 n)$

Building a 2D Range Tree

- Build2DRT can be improved to O(n lg n) with a common trick:

  pre-sort P on y-coord and pass it along as argument

  //P_{sx} is set of points sorted by x-coord, P_{sy} is set of points sorted by y-coord

  Build2DRT(P_{sx}, P_{sy})

- Maintain the sorted sets through recursion

  P_1 sorted-by-x, P_1 sorted-by-y

  P_2 sorted-by-x, P_2 sorted-by-y

- If we got the keys in order, a BBST can be built in O(n) time and we got

  $T(n) = 2T(n/2) + O(n)$ which solves to $O(n \lg n)$

The 2D Range Tree

- How much space does a range tree use?

The 2D Range Tree

- How much space does a range tree use?

Two arguments can be made:

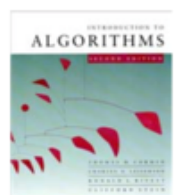
- At each level in the tree, each point is stored exactly once (in the associated structure of precisely one node). So every level stores all points and uses O(n) space => O(n lg n)

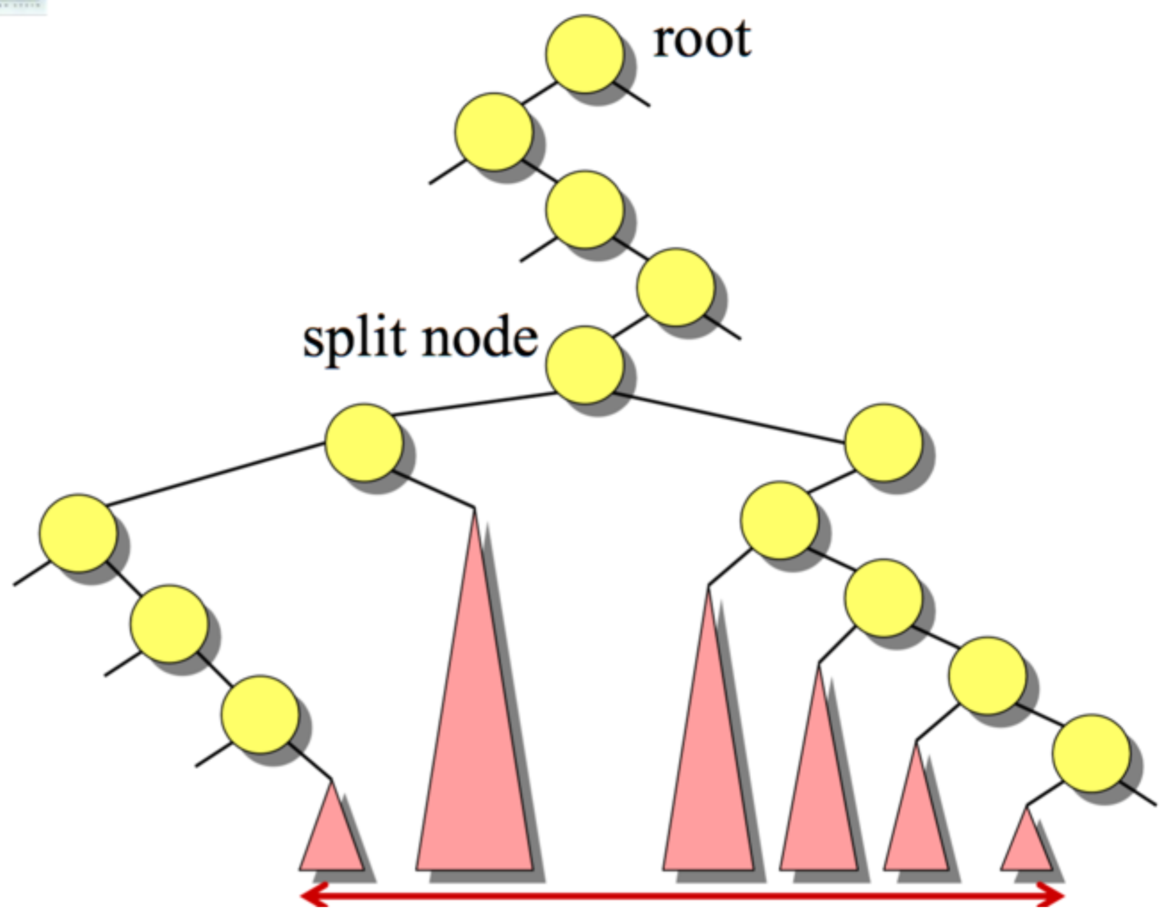
or

- Each point p is stored in the associated structures of all nodes on the path from root to p. So one point is stored O(lg n) times => O(n lg n)

The 2D Range Tree

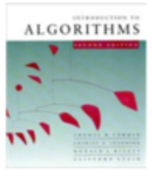
- How do you answer range queries with a range tree, and how fast?

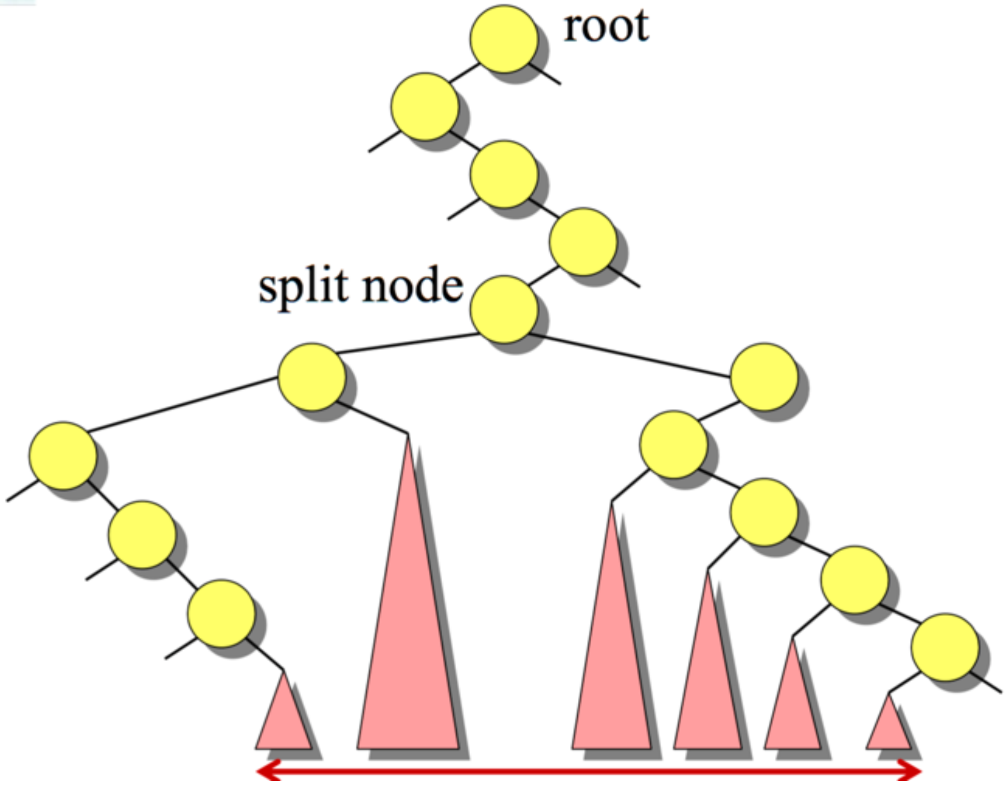




Range queries with the 2D Range Tree

- Find the split node x_{split} where the search paths for x_1 and x_2 split
- Follow path root to x_1: for each node v to the **right** of the path, query its associated structure $T_{assoc}(v)$ with $[y_1,y_2]$
- Follow path root to x_2: for each node v to the **left** of the path, query its associated structure $T_{assoc}(v)$ with $[y_1,y_2]$





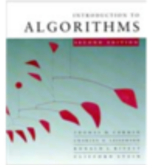
How long does this take?

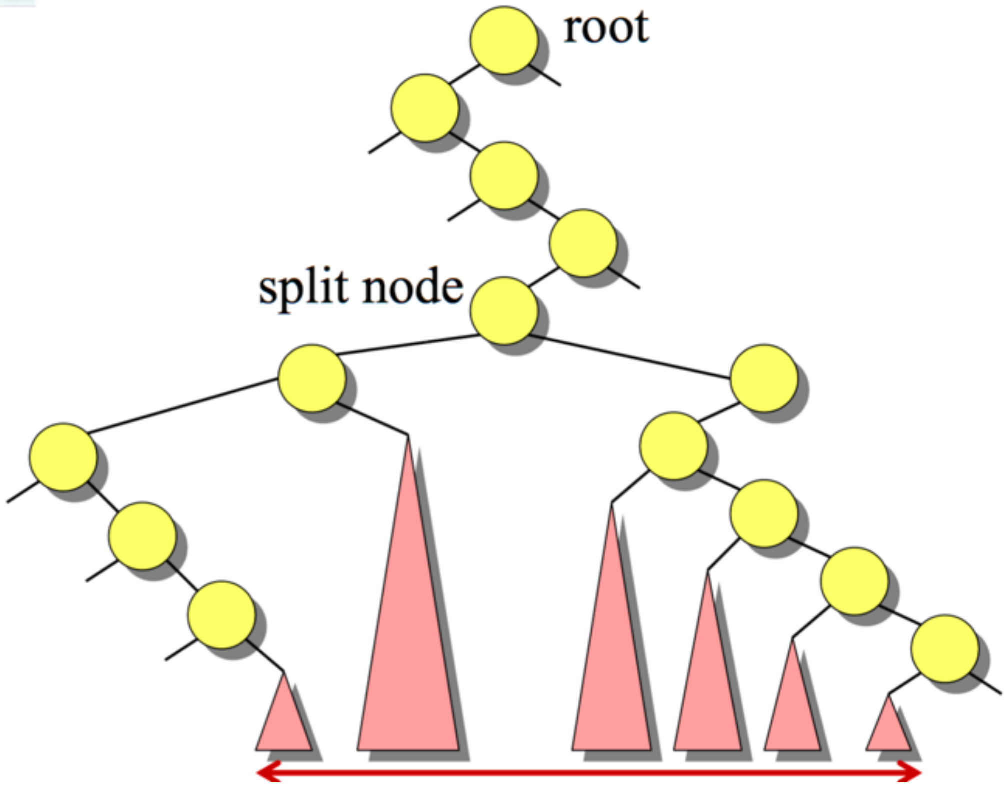
Range queries with the 2D Range Tree

How long does a range query take?

- There are O(lg n) subtrees in between the paths
- We query each one of them using its associated structure
- Querying it's T_{assoc} takes O(lg n_v + k')
- Overall it takes

  SUM O(lg n_v + k') = O($\lg^2 n$+k)





Comparison Range Tree and kd Tree

2D

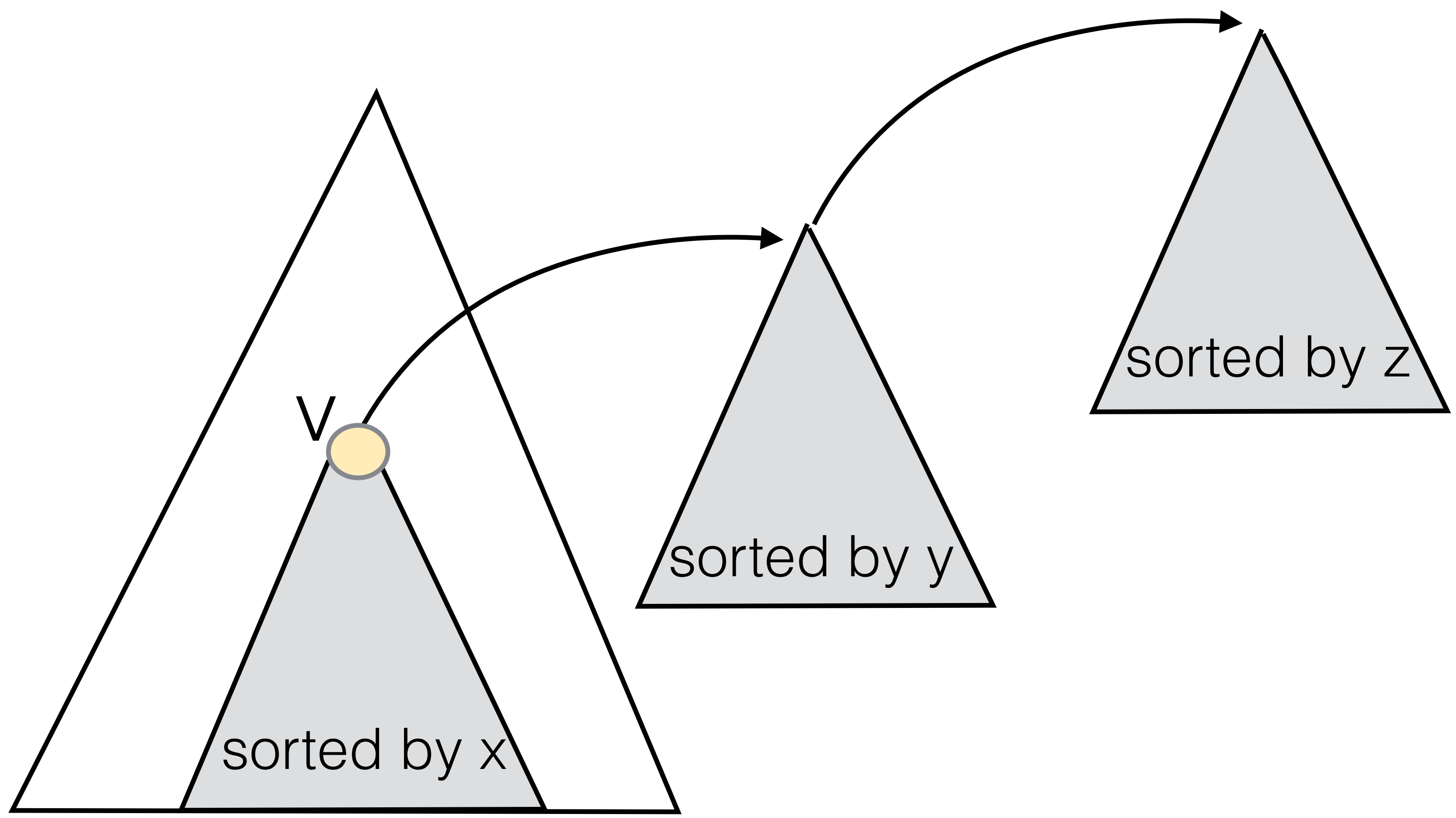
n	$\log n$	$\log^2 n$	$\sqrt{n}$
16	4	16	4
64	6	36	8
256	8	64	16
1024	10	100	32
4096	12	144	64
16384	14	196	128
65536	16	256	256
1M	20	400	1K
16M	24	576	4K

screen shot from Mark van Kreveld slides, http://www.cs.uu.nl/docs/vakken/ga/slides5b.pdf)

3D Range Trees

P = set of points in 3D

- 3DRangeTree(P)
  - Construct a BBST on x-coord
  - Each node v will have an associated structure that's a 2D range tree for P(v) on the remaining coords



3D Range Trees

3D Range Trees

Size:

3D Range Trees

Size:

- An associated structure for n points uses O(n lg n) space. Each point is stored in all associated structures of all its ancestors => $O(n \lg^2 n)$

3D Range Trees

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Let's try this recursively

3D Range Trees

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- An associated structure for n points uses O(n lg n) space. Each point is stored in all associated structures of all its ancestors => O (n $\lg^2$ n)

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- Let $S_3(n)$ be the size of a 3D Range Tree of n points

3D Range Trees

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- Let $S_3(n)$ be the size of a 3D Range Tree of n points
- Find a recurrence for $S_3(n)$

3D Range Trees

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Let's try this recursively

- Let $S_3(n)$ be the size of a 3D Range Tree of n points
- Find a recurrence for $S_3(n)$
    - Think about how you build it : you build an associated structure for P that's a 2D range tree; then you build recursively a 3D range tree for the left and right half of the points

3D Range Trees

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  - $S_3(n) = 2S_3(n/2) + S_2(n)$

3D Range Trees

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- An associated structure for n points uses O(n lg n) space. Each point is stored in all associated structures of all its ancestors => $O(n \lg^2 n)$

Let's try this recursively

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  - $S_3(n) = 2S_3(n/2) + S_2(n)$
  - This solves to $O(n \lg^2 n)$

3D Range Trees

Build time:

- Think recursively
- Let $B_3(n)$ be the time to build a 3D Range Tree of n points
- Find a recurrence for $B_3(n)$
  - Think about how you build it : you build an associated structure for P that's a 2D range tree; then you build recursively a 3D range tree for the left and right half of the points
  - $B_3(n) = 2B_3(n/2) + B_2(n)$
  - This solves to $O(n \lg^2 n)$

3D Range Trees

Query:

- Query BBST on x-coord to find O(lg n) nodes
- Then perform a 2D range query in each node …

Time?

- Let $Q_3(n)$ be the time to answer a 3D range query
- Find a recurrence for $Q_3(n)$
  - $Q_3(n) = O(\lg n) + O(\lg n) \times Q_2(n)$
  - This solves to $O(\lg^3 n + k)$

Comparison Range Tree and kd Tree

3D

n	$\log n$	$\log^4 n$	$n^{3/4}$
1024	10	10,000	181
65,536	16	65,536	4096
1M	20	160,000	32,768
1G	30	810,000	5,931,641
1T	40	2,560,000	1G

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