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Unit – 3 ⇝ Basic Probability

Introduction

→ Probability theory is the branch of mathematics that is concerned with random (or chance) phenomena. It has attracted people to its study both because of its intrinsic interest and its successful applications to many areas within the physical, biological, social sciences, in engineering and in the business world.

→ The words PROBABLE and POSSIBLE CHANCES are quite familiar to us. We use these words when we are sure of the result of certain events. These words convey the sense of uncertainty of occurrence of events.

→ Probability is the word we use to calculate the degree of the certainty of events.

→ There are two types of approaches in the theory of probability:

- Classical Approach – By Blaise Pascal
- Axiomatic Approach – By A. Kolmogorov

Method – 1 ⇝ Counting

Factorial Notation

→ The notation n! represents the product of first n natural numbers, i.e.,

$$\mathbf{n! = 1 \times 2 \times 3 \times \ldots \ldots \times (n-1) \times n}$$

→ For Example:

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

→ **Some Important Results:**

(1) $n! = n \times (n-1)!$

(2) $0! = 1$

Permutation (Arrangement)

→ A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

→ Suppose that we are given ‘n’ distinct objects and wish to arrange ‘r’ of these objects in a line without repeating.

→ Then number of such different arrangements is given as below. It is denoted by $^{n}P_{r}$.

$$\mathbf{^{n}P_{r} = \frac{n!}{(n-r)!}} \ ; \ 0 < r \le n.$$

→ For Example:

A number of 3 lettered words which can be formed by without repeating letters of the word “NUMBER” is,

$$^{6}P_{3} = \frac{6\,!}{3\,!} = 120.$$

→ **Some Important Results:**

(1) $^{n}P_{n} = n!$

(2) $^{n}P_{0} = 1$

(3) When all the objects are **distinct**, the number of permutations **with repeating** the object is given by $\mathbf{n^{r}}$.

- For Example:

  A number of 3 lettered words which can be formed by repeating letters of the word "NUMBER" is

  $6^3 = 216$.

(4) When all the objects are **not distinct**, the number of permutations of n object in which n_1 are of one kind, n_2 are of second kind, …, and n_k are of k^{th} kind is given by,

$$\frac{\mathbf{n!}}{\mathbf{n_1!\ n_2!\ \dots\ n_k!}}.$$

Note that, $n = n_1 + n_2 + \cdots + n_k$.

- For Example:

  A number of different permutations of letters of the word MISSISSIPPI is

  Here, total number of letters are 11 among which M repeats 1 time, I & S repeats 4 times each and P repeats 2 times.

  $$\frac{11!}{1!\ 4!\ 4!\ 2!} = 34650$$

Combination (Selection)

→ A combination is selection of a number of objects from given set of objects.

→ We denote the number of unique r selections or combinations out of a group of n objects by $\mathbf{{}^nC_r}$ and defined as below.

$$\mathbf{{}^nC_r = C(n, r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}}\ ;\ 0 < r \le n.$$

→ For Example:

- The number of ways in which 3 card can be chosen from 8 cards is

  $$\binom{8}{3} = \frac{8!}{3!\,(8-3)!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

- A club has 10 male and 8 female members. A committee composed of 3 men and 4 women is formed. In how many ways this can be done?

  $$\binom{10}{3}\binom{8}{4} = 120 \times 70 = 8400$$

- Out of 6 boys and 4 girls, in how many ways a committee of five members can be formed in which there are at most 2 girls are included?

$$\binom{4}{2}\binom{6}{3}+\binom{4}{1}\binom{6}{4}+\binom{4}{0}\binom{6}{5}=120+60+6=186$$

→ **Some Important Results:**

(1) $^{n}C_0 = {}^{n}C_n = 1$

(2) $^{n}C_1 = n$

(3) $^{n}C_k = {}^{n}C_{n-k}$

Example of Method-1: Counting

C	1	How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if **(1)** 4 – letters are used at a time, **(2)** All letters are used at a time, **(3)** All letters are used but the first letter is vowel? **Answer: (1) 360, (2) 720, (3) 240**
C	2	How many different poker hands consists of 5 cards being either 2 or 7? **Answer: 56**
C	3	A bag contains 5 black ball and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected. **Answer: 200**
C	4	What is the number of ways of choosing 4 cards of below choices from pack of 52 cards? **(1)** In how many of these four cards are of the same suit. **(2)** Four cards belong to four different suits. **(3)** Two are red card and two are black card. **Answer: (1) 2860, (2) 28561, (3) 105625**

Method – 2 ⇝ Basic Terminology and Definition of Probability

2.1 Sample Space and Event

Random Experiment

→ An experiment conducted under identical conditions is known as random experiment if it satisfies the following conditions:

(1) It has more than one possible outcome.

(2) It is not possible to predict the outcome in advance.

→ For Examples:

- Tossing a coin.
- Throwing/rolling a die.
- Selecting a card from a pack of playing cards.

→ The result of a random experiment is known as **outcome**.

Sample Space

→ The set of all possible outcome is known as sample space of an experiment.

→ Sample space is denoted by the symbol **S**.

→ Elements of a sample space is known as **sample points**.

→ The sample space of an experiment may consist of a finite or an infinite number of possible outcomes.

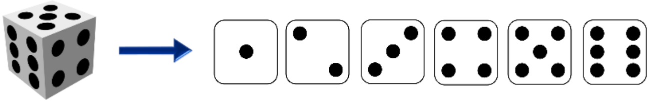
→ For Examples:

- Consider an experiment of tossing a coin. The outcomes of this experiment are head (H) or tail (T).

  Therefore, sample space of this experiment is written as follow:

  $S = \{ H, \ T \}$

- Consider an experiment of rolling a die. The outcomes of this experiment are 1, 2, 3, 4, 5 or 6.

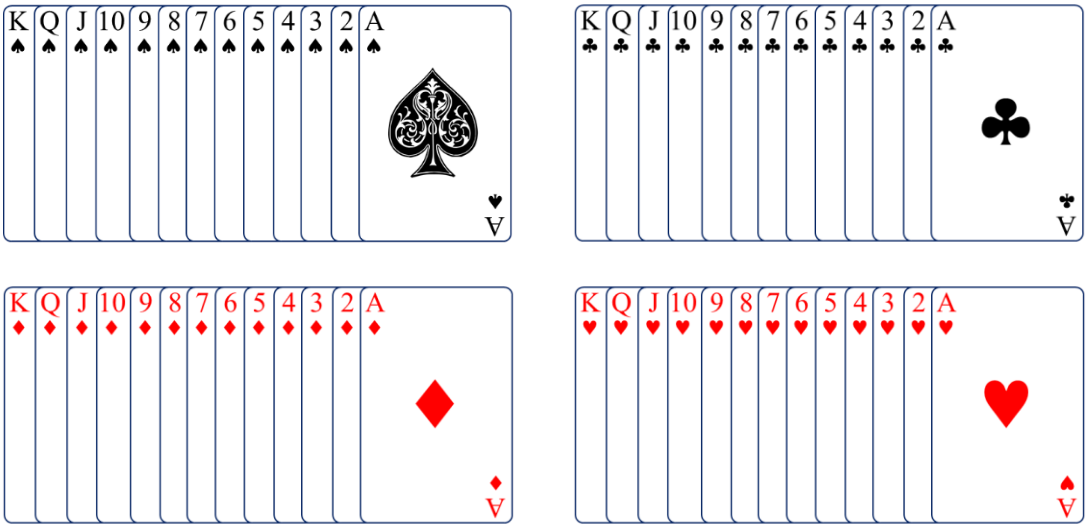


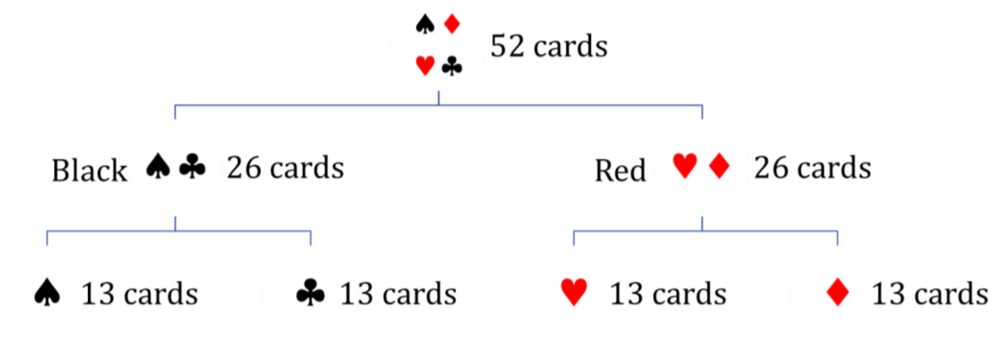
  Therefore, sample space of this experiment is written as follow:

  $S = \{ 1, \ 2, \ 3, \ 4, \ 5, \ 6 \}$

→ Deck of 52 playing cards which has four suits as follow:

SPADES ♠, CLUB ♣, DIAMOND ♦, HEART ♥





Ace → A Picture cards = 12

Experiment **with** Replacement

→ You select something from sample, **put it back** and then select again.

→ Each selection is independent and the size of the sample remains constant throughout the experiment.

Experiment **without** Replacement

→ You select something from sample, **don't put it back** and then select again.

→ Each selection affects the subsequent selections, they are dependent on previous ones and size of a sample space is reduced after each selection.

Event

→ Any subset of a sample space is known as an event.

→ For Example:

- In experiment of Rolling a six-sided die and observing the number (or dots) that appear on top.

  A sample space of an experiment is

  $S = \{1, 2, 3, 4, 5, 6\}$.

  - Let A be an event that **even** number appears on top.

    $\therefore\ A = \{2, 4, 6\}$

  - Let B be an event that number **4** appears on top.

    $\therefore\ B = \{4\}$

  - Let C be an event that number **7** appears on top.

    $\therefore\ C = \phi$

  - Let D be an event that number **less than 7** appears on top.

    $\therefore\ D = \{1, 2, 3, 4, 5, 6\} = S$

Impossible Event

→ The event E is known as impossible event if $\mathbf{E = \phi}$.

→ In above example, event $C = \phi$ therefore, it is impossible event.

Sure Event

→ The event E is known as sure event if $\mathbf{E = S}$.

→ In above example, event $D = S$ therefore, it is sure event.

Simple or Elementary Event

→ If an event E has **only one** sample point of a sample space, then it is known as a simple (or elementary) event.

→ In above example, event B has only one sample point therefore, it is simple event.

Compound Event

→ If an event E has **more than one** sample point of a sample space, then it is known as a compound event.

→ In above example, event A and D are compound events.

Complementary Event

→ For every event E, there corresponds another event $\mathbf{E'}$ $($ or $\overline{\mathbf{E}}$ $)$ known as the complementary event to E.

→ Which is defined as $\mathbf{E' = S - E}$.

→ It is also known as the event "**not E**".

→ For Example:

- For experiment of tossing a coin thrice, a sample space is

  $S = \{\text{HHH}, \ \text{HHT}, \ \text{HTH}, \ \text{HTT}, \ \text{THH}, \ \text{THT}, \ \text{TTH}, \ \text{TTT}\}$

  Consider the following events:

  - Event A: Exactly one head appeared

    $\therefore \ A = \{\text{HTT}, \ \text{THT}, \ \text{TTH}\}$

  The complementary event corresponds to event A is as follow:

  $\therefore \ A' = \{\text{HHH}, \ \text{HHT}, \ \text{HTH}, \ \text{THH}, \ \text{TTT}\}$

Mutually Exclusive Event

→ Events A and B are known as mutually exclusive even if $\mathbf{A \cap B = \phi}$.

→ For Example:

- For experiment of throwing a die, a sample space is

  $S = \{1, \ 2, \ 3, \ 4, \ 5, \ 6\}$

  Consider the following events:

  - Event A: an **even** number appeared

    $\therefore \ A = \{2, \ 4, \ 6\}$

  - Event B: an **odd** number appeared

    $\therefore \ B = \{1, \ 3, \ 5\}$

  - Event C: a **prime** number appeared

    $\therefore \ C = \{2, \ 3, \ 5\}$

  For events A and B

  $A \cap B = \phi$

  Hence, Events A and B are mutually exclusive events.

For events B and C

$B \cap C = \{ 3, \ 5 \} \neq \phi$

Hence, Events B and C are not mutually exclusive events.

Exhaustive Events

→ Events A and B of a sample space S are known as exhaustive events, if $\mathbf{A \cup B = S}$.

→ For Example:

- In experiment of rolling a die, $S = \{ 1, \ 2, \ 3, \ 4, \ 5, \ 6 \}$.

  Let us define the following events:

  - Event A: A number less than 4 appears

    $\therefore \ A = \{ 1, \ 2, \ 3 \}$

  - Event B: A number greater than 2 but less than 5 appears

    $\therefore \ B = \{ 3, \ 4 \}$

  - Event C: A number greater than 4 appears

    $\therefore \ C = \{ 5, \ 6 \}$

  Now,

  $$\begin{aligned} A \cup B \cup C &= \{ 1, \ 2, \ 3 \} \cup \{ 3, \ 4 \} \cup \{ 5, \ 6 \} \\ &= \{ 1, \ 2, \ 3, \ 4, \ 5, \ 6 \} \\ &= S \end{aligned}$$

  Hence, events A, B and C are exhaustive events.

→ Furthermore, if

$\mathbf{A \cap B = \phi}$ and $\mathbf{A \cup B = S}$,

then events A and B are known as mutually exclusive and exhaustive events.

Example of Method-2.1: Sample Space and Event

C	1	A coin is tossed twice, and their up faces are recorded. What is the sample space for this experiment? **Answer:** $\mathbf{S = \{\, HH,\ \ HT,\ \ TH,\ \ TT\,\}}$
C	2	Find the sample space associated with the experiment of rolling a pair of dice once. Also, find the number of elements of this sample space. **Answer:** $\mathbf{S = \left\{\begin{matrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{matrix}\right\}}$ $\mathbf{Number\ of\ elements\ in\ sample\ space = 6 \times 6 = 6^2 = 36}$
C	3	A coin is tossed. If it shows head, we draw a ball from a bag which contains 3 blue and 4 white balls; if it shows tail, we throw a die. Describe the sample space of this experiment. **Answer:** $\mathbf{S = \left\{\begin{matrix} HB_1,\ HB_2,\ HB_3,\ HW_1,\ HW_2,\ HW_3,\ HW_4, \\ T1,\ T2,\ T3,\ T4,\ T5,\ T6 \end{matrix}\right\}}$

C	4	The letters A, B, C and D are written separately on four slips of paper. The slips are put in a box and mixed thoroughly. A person draws two slips from the box, one after the other. Describe the sample space for the experiment in following cases: **(1)** With replacement **(2)** Without replacement **Answer**: (1) $\mathbf{S} = \left\{ \begin{matrix} \mathbf{AA,} & \mathbf{AB,} & \mathbf{AC,} & \mathbf{AD,} & \mathbf{BA,} & \mathbf{BB,} & \mathbf{BC,} & \mathbf{BD,} \\ \mathbf{CA,} & \mathbf{CB,} & \mathbf{CC,} & \mathbf{CD,} & \mathbf{DA,} & \mathbf{DB,} & \mathbf{DC,} & \mathbf{DD,} \end{matrix} \right\}$ (2) $\mathbf{S} = \left\{ \begin{matrix} \mathbf{AB,} & \mathbf{AC,} & \mathbf{AD,} & \mathbf{BA,} & \mathbf{BC,} & \mathbf{BD,} \\ \mathbf{CA,} & \mathbf{CB,} & \mathbf{CD,} & \mathbf{DA,} & \mathbf{DB,} & \mathbf{DC} \end{matrix} \right\}$
C	5	Two unbiased dice are thrown. Write down the following events: Event A: Both the dice show the same number. Event B: The total of the numbers on the dice is 8. Event C: The total of the numbers on the dice is 13. Event D: The total of the number on the dice is any number from 2 to 12. **Answer**: $\mathbf{A} = \{ (\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3}), (\mathbf{4}, \mathbf{4}), (\mathbf{5}, \mathbf{5}), (\mathbf{6}, \mathbf{6}) \}$ $\mathbf{B} = \{ (\mathbf{2}, \mathbf{6}), (\mathbf{3}, \mathbf{5}), (\mathbf{4}, \mathbf{4}), (\mathbf{5}, \mathbf{3}), (\mathbf{6}, \mathbf{2}) \}$ $\mathbf{C} = \emptyset$ $\mathbf{D} = \{ (\mathbf{1}, \mathbf{1}), \ldots, (\mathbf{1}, \mathbf{6}), \ldots, (\mathbf{6}, \mathbf{1}), \ldots, (\mathbf{6}, \mathbf{6}) \}$

2.2 Probability of an Event

Probability of an Event

→ The probability of an event E is denoted as **P(E)** and read as “probability of event E” and defined as follow:

$$P(E) = \frac{\text{number of elements in E}}{\text{number of elements in S}} = \frac{n(E)}{n(S)}$$

or

$$P(E) = \frac{\text{number of outcomes favorable to E}}{\text{total number of all possible outcomes of the experiment}}$$

→ For Example:

- One card is drawn from a well-shuffled deck of 52 cards.

  $$\text{The probability of getting a queen } = \frac{1}{52}$$

- A box contains 3 blue, 2 white, and 4 red marbles. A marble is drawn at random from the box, what is the probability that it will be white?

  Solution:

  $S = \{ B_1, B_2, B_3, W_1, W_2, R_1, R_2, R_3, R_4 \}$

  Number of possible outcomes $= 3 + 2 + 4 = 9$

  i.e., $n(S) = 9$

  Let W be the event that marble is white.

  $W = \{ W_1, W_2 \}$

  i.e., $n(W) = 2$

  $$\text{Therefore,} \quad P(W) = \frac{n(W)}{n(S)}$$

  $$= \frac{2}{9}$$

→ **Some Important Results:**

(1) For every event A, $0 \leq P(A) \leq 1$.

(2) $\mathbf{P(A) = 0}$ if and only if event A is **impossible** event.

(3) $\mathbf{P(A) = 1}$ if and only if event A is **certain** event.

(4) $P(A') = 1 - P(A)$ **or** $P(A) = 1 - P(A')$

(5) If A and B are mutually exclusive events, , $P(A \cap B) = 0$.

(6) If A and B are mutually exhaustive events, $P(A \cup B) = 1$

(7) The probability that **at least one** out of the events A and B will occur is,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(8) The probability that **only event A** out of the events A and B will occur is,

$P(A \cap B') = P(A) - P(A \cap B)$

(9) The probability that **only event B** out of the events A and B will occur is,

$P(A' \cap B) = P(B) - P(A \cap B)$

(10) The probability that **none** of the event A and B occur is,

$P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$ (De Morgan's Rule)

(11) The probability that events A and B **not occur together** is,

$P(A' \cup B') = P(A \cap B)' = 1 - P(A \cap B)$ (De Morgan's Rule)

(12) The probability that **at least one** of the events A, B and C will occur is,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

(13) The probability that **at least two** of the three events occur is,

$$P[(A \cap B) \cup (B \cap C) \cup (C \cap A)] = P(A \cap B) + P(B \cap C) + P(A \cap C) - 2P(A \cap B \cap C)$$

(14) The probability that **exactly two** of the three events occur is,

$$P\,[(A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A' \cap B \cap C)]$$

$$= P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C)$$

(15) The probability that **exactly one** of the three events occur is,

$$P[(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)]$$

$$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(A \cap C) + 3P(A \cap B \cap C)$$

(16) The probability that **none** of the event A, B and C occur is,

$P(A' \cap B' \cap C') = P(A \cup B \cup C)' = 1 - P(A \cup B \cup C)$ (De Morgan's Rule)

(17) The probability that events A, B and C **not occur together** is,

$P(A' \cup B' \cup C') = P(A \cap B \cap C)' = 1 - P(A \cap B \cap C)$ (De Morgan's Rule)

Example of Method-2.2: Probability of an Event

C	1	If A and B are two mutually exclusive events with $P(A) = 0.30, P(B) = 0.45$. Find the probability of A', $A \cap B$, $A \cup B$, $A' \cap B'$. **Answer**: $\mathbf{P(A') = 0.7}$, $\mathbf{P(A \cap B) = 0}$, $\mathbf{P(A \cup B) = 0.75}$, $\mathbf{P(A' \cap B') = 0.25}$
C	2	Two unbiased dice are thrown. Find the probability that: **(1)** The total of the numbers on the dice is greater than 8. **(2)** The total of the numbers on the dice is 13. **(3)** Total of numbers on the dice is any number from 2 to 12, both inclusive. **Answer**: (1) $\mathbf{\frac{5}{18}}$, (2) $\mathbf{0}$, (3) $\mathbf{1}$
C	3	If 5 cards are drawn from a pack of 52 well-shuffled cards, find the probability of **(1)** 4 ace **(2)** 4 aces and 1 is a king **(3)** 3 are tens and 2 are jacks **(4)** a nine, ten, jack, queen, king is obtained in any order **(5)** 3 are of any one suit and 2 are of another **(6)** at least one ace is obtained **Answer**: (1) $\mathbf{\frac{1}{54145}}$, (2) $\mathbf{\frac{1}{649740}}$, (3) $\mathbf{\frac{1}{108290}}$ (4) $\mathbf{\frac{64}{162435}}$, (5) $\mathbf{\frac{429}{4165}}$, (6) $\mathbf{\frac{18472}{54145}}$
C	4	An urn contains 6 green, 4 red and 9 black balls. If 3 balls are drawn at random, find the probability that at least one is green. **Answer**: $\mathbf{\frac{683}{969}}$

C	5	Consider a poker hand of five cards. Find the probability of getting four of a kind (i.e., four cards of the same face value) assuming the five cards are chosen at random. **Answer**: $\frac{1}{4165}$
C	6	An integer is chosen at random from the first 200 positive integers. What is the probability that the integer is divisible by 6 or 8? **Answer**: 0.25

Method – 3 ⇝ Conditional Probability and Independent Events

Conditional Probability

→ The probability of an event occurring given that another event has already occurred is known as conditional probability.

→ Let A and B be any two events in same sample space S.

→ The probability of the occurrence of event A when it is given that B has already occurred is known as conditional probability.

→ It is denoted as **P(A | B)** and read as "conditional probability of A given B" and defined as follow:

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)},\quad P(B)\neq 0$$

→ Similarly, "conditional probability of B given A" is defined as follow:

$$P(B\mid A)=\frac{P(A\cap B)}{P(A)},\quad P(A)\neq 0$$

→ For Example:

Consider the experiment of tossing three fair coins. To find probability of "getting at least two heads given that first coin shows tail" is conditional probability.

The sample space of the experiment is

$$S=\{\,HHH,\ HHT,\ HTH,\ THH,\ HTT,\ THT,\ TTH,\ TTT\,\}.$$

Let E and F denote the following events:

E: at least two head appear

$$E=\{\,HHH,\ HHT,\ HTH,\ THH\,\}$$

F: first coin shows tail

$$F=\{\,THH,\ THT,\ TTH,\ TTT\,\}$$

Here, event F has occurred before event E.

$$\text{Now, to find } P(E\mid F)=\frac{P(E\cap F)}{P(F)}$$

$$E\cap F=\{\,THH\,\}$$

$$\text{Thus, } P(F)=\frac{4}{8}=\frac{1}{2}\ \text{and}\ P(E\cap F)=\frac{1}{8}$$

Therefore, $P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{\left({}^{1}/_{8}\right)}{\left({}^{1}/_{2}\right)} = \frac{1}{4}$

→ **Properties of Conditional Probability**

- Let A_1, A_2 and B be any three events of a sample space S, then

  $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B);\ P(B) > 0.$

- Let A and B be any two events of a sample space S, then

  $P(A' \mid B) = 1 - P(A \mid B);\ P(B) > 0.$

Multiplicative Law of Probability

→ **Statement:**

Let A and B be any two events in the sample space S, then

$$P(A \cap B) = P(A) \cdot P(B \mid A);\ P(A) \neq 0$$

$$= P(B) \cdot P(A \mid B);\ P(B) \neq 0$$

Let A, B and C be any three events in the sample space S, then

$$P(A \cap B \cap C) = P(A) \cdot P(B \mid A) \cdot (C \mid A \cap B)$$

Independent Events

→ Two events are known as independent events if the probability of occurrence of one of them is not affected by occurrence of the other.

→ Let A and B be two events associated with the same random experiment, then A and B are known as independent if and only if

$\mathbf{P(A \cap B) = P(A) \cdot P(B)}$

→ Furthermore, if events A and B are independent events, then

(1) $P(A \mid B) = P(A)$

(2) $P(B \mid A) = P(B)$

Mutually Independent Events

→ Three events A, B and C are known as mutually independent if

$P(A \cap B) = P(A) \cdot P(B)$

$P(B \cap C) = P(B) \cdot P(C)$

$P(A \cap C) = P(A) \cdot P(C)$

$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

Example of Method-3: Conditional Probability and Independent Events

<table>
<tr><td>C</td><td>1</td><td>$P(A) = \frac{1}{3},\ P(B') = \frac{1}{4},\ \ P(A \cap B) = \frac{1}{6}$, then find $P(A \cup B),\ P(A' \cap B')$ and $P(A' \mid B')$.

Answer: $\mathbf{\frac{11}{12}},\quad \mathbf{\frac{1}{12}},\quad \mathbf{\frac{1}{3}}$</td></tr>
<tr><td>C</td><td>2</td><td>If $P(A) = \frac{1}{3},\ \ P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{1}{2}$, then find $P(B \mid A),\ \ \ P(A \mid B')$.

Answer: $\mathbf{\frac{1}{4}},\quad \mathbf{\frac{1}{3}}$</td></tr>
<tr><td>C</td><td>3</td><td>If A and B are independent events, with $P(A) = \frac{3}{8},\ \ P(B) = \frac{7}{8}$.
Find $P(A \cup B),\ \ P(A \mid B)$ and $P(B \mid A)$.

Answer: $\mathbf{\frac{59}{64}},\quad \mathbf{\frac{3}{8}},\quad \mathbf{\frac{7}{8}}$</td></tr>
<tr><td>C</td><td>4</td><td>A person is known to hit the target in 3 out of 4 shots, whereas another person is known to hit the target in 2 out of 3 shots. What is probability that target will be hit?

Answer: $\mathbf{\frac{11}{12}}$</td></tr>
<tr><td>C</td><td>5</td><td>A market survey was conducted in four cities to find out the preference for brand X soap. The responses are shown below:
<table>
<tr><th></th><th>Delhi</th><th>Kolkata</th><th>Chennai</th><th>Mumbai</th></tr>
<tr><td>Yes</td><td>45</td><td>55</td><td>60</td><td>50</td></tr>
<tr><td>No</td><td>35</td><td>45</td><td>35</td><td>45</td></tr>
<tr><td>No opinion</td><td>5</td><td>5</td><td>5</td><td>5</td></tr>
</table>
(1) What is the probability that a consumer preferred brand X, given that he was from Chennai?
(2) Given that a consumer preferred brand X, what is the probability that he was from Mumbai?

Answer: (1) 0.6,   (2) 0.23</td></tr>
</table>

C	6	In a group of 200 students 40 are taking English, 50 are taking Mathematics, 12 are taking both. **(1)** If a student is selected at random, what is the probability that the student is taking English? **(2)** A student is selected at random from those taking Mathematics. What is the probability that the student is taking English? **(3)** A student is selected at random from those taking English, what is the probability that the student is taking Mathematics? **Answer**: $(\mathbf{1})\ \mathbf{0.20},\qquad (\mathbf{2})\ \mathbf{0.24},\qquad (\mathbf{3})\ \mathbf{0.3}$
C	7	A bag contains 6 white, 9 black balls. 4 balls are drawn at a time. Find the probability for first draw to give 4 white & second draw to give 4 black balls in each of following cases: **(1)** The balls are replaced before the second draw. **(2)** The balls are not replaced before the second draw. **Answer**: $(\mathbf{1})\ \dfrac{\mathbf{6}}{\mathbf{5915}},\qquad (\mathbf{2})\ \dfrac{\mathbf{3}}{\mathbf{715}}$

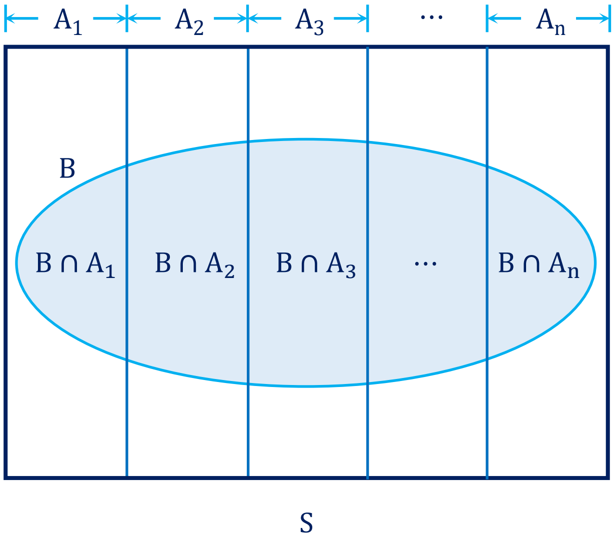
Method – 4 ⇝ Total Probability and Bayes' Theorem

Total Probability

→ Let $A_1, A_2, \dots, A_n$ be mutually exclusive and exhaustive events of the sample space S with $P(A_i) \neq 0$, for $i = 1, 2, \dots, n$. Let B be any event associated with S. Then, the probability of B is

$$\mathbf{P(B) = P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) + \cdots + P(A_n) \cdot P(B \mid A_n)}$$

Explanation:



Using multiplicative law of probability, we get

$$P(B \cap A_i) = P(A_i) \cdot P(B \mid A_i), \ \forall i = 1, 2, \dots, n$$ ('∀' means **for every**)

Thus,

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_i)$$

$$P(B) = P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) + \cdots + P(A_n) \cdot P(B \mid A_n)$$

→ For Example:

Factory A and Factory B, that produce electronic components. Factory A produces 60% of the components and among those, 10% are defective. Factory B produces the remaining 40% of the components and among those, 5% are defective.

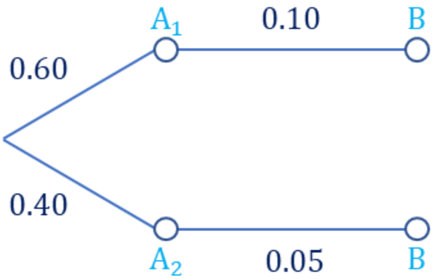
Find the probability that a randomly selected electronic component is defective.

Solution:

Let A_1 : The component is from Factory A.

A_2 : The component is from Factory B.

B : The component is defective.



Therefore,

$P(A_1) = 60\% = 0.60$ $\quad$ $P(B \mid A_1) = 10\% = 0.10$

$P(A_2) = 40\% = 0.40$ $\quad$ $P(B \mid A_2) = 5\% = 0.05$

Thus,

$$\begin{aligned} P(B) &= P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) \\ &= (0.60) \cdot (0.10) + (0.40) \cdot (0.05) \\ &= 0.06 + 0.02 \\ &= 0.08 \end{aligned}$$

Bayes' Theorem

→ **Statement:**

Let $A_1, A_2, A_3, \ldots, A_n$ be mutually exclusive and exhaustive events of the sample space S with $P(A_i) \neq 0$, for $i = 1, 2, 3, \ldots, n$. Let B be any event associated with S with $P(B) \neq 0$. The probability of an event A_i when the event B has occurred is

$$\mathbf{P(A_i \mid B) = \frac{P(A_i) \cdot P(B \mid A_i)}{P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) + \cdots + P(A_n) \cdot P(B \mid A_n)}}$$

Example of Method-4: Total Probability and Bayes' Theorem

C	1	In a certain assembly plant, three machines, B_1, B_2 and B_3, make 30%, 45% and 25%, respectively, of the products. It is known form the past experience that 2%, 3% and 2% of the products made by each machine respectively are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective? **Answer: $\mathbf{0.0245}$**

C	2	An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second urn and then a ball is drawn from the later. What is the probability that it is a white ball? **Answer**: $\mathbf{\frac{59}{130}}$
C	3	Consider two boxes, first with 5 green and 2 pink and second with 4 green and 3 pink balls. Two balls are selected from random box. If both balls are pink, find the probability that they are from second box. **Answer**: $\mathbf{\frac{3}{4}}$
C	4	A company has two plants to manufacture hydraulic machine. Plant I manufacture 70% of the hydraulic machines and plant II manufactures 30%. At plant I, 80% of hydraulic machines are rated standard quality and at plant II, 90% of hydraulic machine are rated standard quality. A machine is picked up at random and is found to be of standard quality. What is the chance that it has come from plant I? **Answer**: $\mathbf{0.6747}$
C	5	If proposed medical screening on a population, the probability that the test correctly identifies someone with illness as positive is 0.99 and the probability that test correctly identifies someone without illness as negative is 0.95. The incident of illness in general population is 0.0001. You take the test the result is positive then what is the probability that you have illness? **Answer**: $\mathbf{0.002}$

Method – 5 ⇝ Random Variable and Probability Function

Random Variable

→ A variable whose values can be obtained from the results of a random experiment is known as random variable.

→ A random variable is a function associated with a sample space of a random experiment.

→ Random variable is often denoted by X, Y.

→ Random variable can be classified as bellow:

(1) Discrete random variable

(2) Continuous random variable

Discrete Random variable:

→ A random variable can take only finite values or countable infinite values is known as discrete random variable.

→ Discrete random variables can be measured exactly.

→ For Example:

If two coins are tossed simultaneously the following sample space is generated:

$S = \{ HH,\ HT,\ TH,\ TT \}$

Now if X denotes the number of heads, then

$X(HH) = 2,\ X(HT) = 1,\ X(TH) = 1,\ X(TT) = 0$

i.e., random variable X can take values 0, 1, 2.

→ Some other examples of discrete random variable.

- Number of children in a family.
- Numbers of stars in the sky.
- Profit made by investor in a day.

Continuous random variable:

→ If a random variable can take all values within an interval, is known as continuous random variable.

→ Continuous random variable cannot be measured exactly.

→ For Examples:

- The age of a person
- Weight and height of a person
- Life of an electric bulb

Probability Distribution of random variable

→ Probability distribution of random variable is the set of its possible values together with their respective probabilities.

i.e.,

X	x_1	x_2	x_3	…	…	x_n
$P(X = x)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	…	…	$p(x_n)$

Where, $p(x_i) \geq 0$ and $\sum_{i=1}^{n} p(x_i) = 1.$

→ For Example:

Two balanced coins are tossed, then $S = \{ HH,\ HT,\ TH,\ TT \}$

We find the probability distribution of head

$$P(X = 0) = P(\text{no head}) = \frac{1}{4} = 0.25$$

$$P(X = 1) = P(\text{one head}) = \frac{2}{4} = \frac{1}{2} = 0.5$$

$$P(X = 2) = P(\text{two heads}) = \frac{1}{4} = 0.25$$

Probability distribution is as follow:

X	0	1	2
$P(X = x)$	0.25	0.5	0.25

Probability Function

→ If for random variable X, the real valued function f(x) is such that $P(X = x) = f(x)$, then f(x) is called probability function of random variable X.

→ Probability function f(x) gives the measures of probability for different values of X say $x_1,\ x_2,\ \ldots,\ x_n$.

→ Probability functions can be classified as

(1) Probability Mass Function (P. M. F.) (for discrete random variable)

(2) Probability Density Function (P. D. F.) (for continuous random variable)

Probability Mass Function

→ If X is a **discrete** random variable, then its probability function f(x) or $P(X = x_i)$ is known as probability mass function, if it satisfies below conditions:

(1) $p(x_i) \geq 0,$ for all i

(2) $$\sum_{i=1}^{n} p(x_i) = 1$$

→ Note:

If $a < x_1 < x_2 < \ldots < x_k < \ldots < x_{n-1} < x_n < b$, then

$$\mathbf{P(x < b)} = P(X = a) + P(X = x_1) + \cdots + P(X = x_{n-1}) + P(X = x_n)$$

$$\mathbf{P(x_1 \leq x \leq x_k)} = P(X = x_1) + P(X = x_2) + \cdots + P(X = x_k)$$

$$\mathbf{P(x > a)} = P(X = x_1) + P(X = x_2) + \cdots + P(X = b)$$

Probability Density Function

→ If X is a **continuous** random variable, then its probability function f(x) is known as continuous probability function, if it satisfies below conditions:

(1) $f(x_i) \geq 0$, for all i

(2) $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

→ Note:

if … … $a < x_1 < x_2 < \ldots < x_k < \ldots < x_{n-1} < x_n < b$ … …, then

$$\mathbf{P(x \leq b)} = \int_{-\infty}^{b} f(x)\,dx$$

$$\mathbf{P(a < x < b)} = \int_{a}^{b} f(x)\,dx$$

$$\mathbf{P(a < x)} = \int_{a}^{\infty} f(x)\,dx$$

Example of Method-5: Random Variable and Probability Function

<table>
<tr>
<td>C</td>
<td>1</td>
<td>Is $P(X = x) = \left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{1-x}$; $x = 0,\ 1$ probability function?

Answer: Yes</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>A random variable X has the following probability function.
Find the value of k and then evaluate $P(x < 6)$, $P(x \geq 6)$ and $P(0 < x < 5)$.
<table>
<tr><td>X</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td></tr>
<tr><td>$P(X = x)$</td><td>0</td><td>k</td><td>$2k$</td><td>$2k$</td><td>$3k$</td><td>k^2</td><td>$2k^2$</td><td>$7k^2 + k$</td></tr>
</table>
Answer: $k = 0.1$, $P(x < 6) = 0.81$,
$P(x \geq 6) = 0.19$, $P(0 < x < 5) = 0.8$</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>Verify that the following function is P.D.F or not?
$$f(x) = \begin{cases} \frac{2x}{9}\left(2 - \frac{x}{2}\right) & ;\ 0 \leq x \leq 3 \\ 0 & ;\text{elsewhere} \end{cases}.$$
Answer: Yes</td>
</tr>
</table>

Method – 6 ⇝ Various Measures of Statistics

Mathematical Expectation

→ If X is a **discrete** random variable having various possible values $x_1, x_2, \dots, x_n$ and $P(X = x)$ is the probability mass function, the mathematical expectation of X is defined & denoted by

$$\mathbf{E(X) = \sum_{i=1}^{n} x_i \cdot p(x_i).}$$

→ If X is a **continuous** random variable which has probability density function f(x), the mathematical expectation of X is defined as

$$\mathbf{E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.}$$

→ E(X) is also known as the mean value of the probability distribution of x.

Properties of Mathematical Expectation

(1) Expected value of constant term is constant. i.e., $E(c) = c$

(1) If a, b and c are constants, then

$$E\left(\frac{aX \pm b}{c}\right) = \frac{1}{c}[a \cdot E(X) \pm b\,]$$

(2) For Probability mass function,

$$E(X^2) = \sum_{i=1}^{n} x_i^2 \cdot p(x_i)$$

(3) For Probability density function,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

(4) If X and Y are two random variables, then $E(X + Y) = E(X) + E(Y)$

(5) If X and Y are two **independent** random variables, then $E(X \cdot Y) = E(X) \cdot E(Y)$

Variance of Random Variable:

→ Variance is a characteristic of random variable X and it is used to measure dispersion (or variation) of X.

→ If X is a random variable with probability mass function P(X), then expected value of $[\,X - E(X)\,]^2$ is known as the variance of X and it is denoted by $\mathbf{V(X)}$ and defined as $\mathbf{V(X) = E(X^2) - [\,E(X)\,]^2}$

Properties of Variance

(1) $V(c) = 0$, where, c is a constant.

(2) If a and b are constants, then $V(aX + b) = a^2 \cdot V(X)$

(3) If X and Y are the **independent** random variables, then $V(X + Y) = V(X) + V(Y)$

Standard Deviation of Random Variable

→ The positive square root of V(X) (Variance of X) is known as standard deviation of random variable X and is denoted by $\boldsymbol{\sigma}$ and defined as

$$\boldsymbol{\sigma} = \sqrt{\mathbf{V(X)}}$$

Example of Method-6: Various Measures of Statistics

C	1	Probability distribution of a random variable X is given below. Find $E(X),\ V(X),\ \sigma(X),\ E(3X + 2),\ V\,(3X + 2)$. <table><tr><td>X</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>P(X = x)</td><td>0.1</td><td>0.2</td><td>0.5</td><td>0.2</td></tr></table> **Answer**: $\mathbf{E(X) = 2.8,}$ $\mathbf{V(X) = 0.76,}$ $\boldsymbol{\sigma}\mathbf{(X) = 0.8718,}$ $\mathbf{E(3X + 2) = 10.4,}$ $\mathbf{V\,(3X + 2) = 6.84}$
C	2	Let mean and standard deviation of a random variable X be 5 & 5 respectively, find $E(X^2)$ and $E(2X + 5)^2$. **Answer**: $\mathbf{E(X^2) = 50,}$ $\mathbf{E(2X + 5)^2 = 325}$

<table>
<tr><td>C</td><td>3</td><td>The following table gives the probabilities that a certain computer will malfunction 0, 1, 2, 3, 4, 5 or 6 times on any one day.
<table>
<tr><td>Number of malfunctions x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
<tr><td>Probability p(x)</td><td>0.17</td><td>0.29</td><td>0.27</td><td>0.16</td><td>0.07</td><td>0.03</td><td>0.01</td></tr>
</table>
Find the mean and variance of this probability distribution.

Answer: $\mathbf{Mean = 1.8,}$ $\mathbf{Variance = 1.8}$</td></tr>
<tr><td>C</td><td>4</td><td>4 raw mangoes are mixed accidentally with the 16 ripe mangoes. Find the probability distribution of the raw mangoes in a draw of 2 mangoes.

Answer: $\mathbf{P(X = 0) = \frac{12}{19}}$, $\mathbf{P(X = 1) = \frac{32}{95}}$, $\mathbf{P(X = 2) = \frac{3}{95}}$</td></tr>
<tr><td>C</td><td>5</td><td>There are 3 red and 2 white balls in a box and 2 balls are taken at random from it. A person gets Rs. 20 for each red ball and Rs. 10 for each white ball. Find his expected gain.

Answer: $\mathbf{32}$</td></tr>
<tr><td>C</td><td>6</td><td>A random variable X has P. D. F $f(x) = \begin{cases} \dfrac{3 + 2x}{18} & ; \ 2 \le x \le 4 \\ 0 & ; \text{otherwise} \end{cases}$.
Find the standard deviation of the distribution.

Answer: $\mathbf{0.5726}$</td></tr>
</table>

Method – 7 ⇝ Cumulative Distribution Function

Cumulative Distribution Function

(1) Discrete Distribution Function

→ The distribution function **F(x)** of the discrete random variable X is defined by

$$F(x) = P(X \le x) = \sum_{i=1}^{x} p(x_i)$$

Where, X be a discrete random variable which takes the values $x_1, x_2, x_3, \ldots\ldots$ such that $x_1 < x_2 < \ldots\ldots$ with probabilities $p(x_1),\ p(x_2),\ p(x_3), \ldots$ & $p(x_i) \ge 0$ for all values of i

and $\sum_{i=1}^{x} p(x_i) = 1$; where x is any integer.

→ The set of pairs $\{x_i, F(x)\}, i = 1, 2, \ldots$ is known as the **cumulative probability distribution**.

x	x_1	x_2	…
$F(x)$	$p(x_1)$	$p(x_1) + p(x_2)$	…

(2) Continuous Distribution Function

→ If X is a continuous random variable having the probability density function f(x), then the function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx \ ; \ -\infty < x < \infty$$

is known as the continuous distribution function.

Properties of Continuous Distribution Function

(1) For $-\infty < x < \infty,\ 0 \le F(x) \le 1$

(2) $F(-\infty) = 0,\quad F(\infty) = 1$

(3) $P(a < X < b) = F(b) - F(a)$

(4) $P(X > x) = 1 - P(X \le x) = 1 - F(x)$

(5) $F'(x) = f(x);\ f(x) \ge 0$

Example of Method-7: Cumulative Distribution Function

<table>
<tr>
<td>C</td>
<td>1</td>
<td>
A random variable X takes the values $-3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3$ such that
$P(X = 0) = P(X > 0) = P(X < 0)$
$P(X = -3) = P(X = -2) = P(X = -1) \ = P(X = 1) = P(X = 2) = P(X = 3)$.
Obtain the probability distribution and the distribution function of X.

Answer:
<table>
<tr><th>X</th><th>-3</th><th>-2</th><th>-1</th><th>0</th><th>1</th><th>2</th><th>3</th></tr>
<tr><td>$P(X)$</td><td>$\frac{1}{9}$</td><td>$\frac{1}{9}$</td><td>$\frac{1}{9}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{9}$</td><td>$\frac{1}{9}$</td><td>$\frac{1}{9}$</td></tr>
<tr><td>$F(x)$</td><td>$\frac{1}{9}$</td><td>$\frac{2}{9}$</td><td>$\frac{1}{3}$</td><td>$\frac{2}{3}$</td><td>$\frac{7}{9}$</td><td>$\frac{8}{9}$</td><td>1</td></tr>
</table>
</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>
A discrete random variable X has the following distribution function:

$$F(x) = \begin{cases} 0 \ ; & x < 1 \\ \frac{1}{3}; & 1 \le x < 4 \\ \frac{1}{2}; & 4 \le x < 6 \\ \frac{5}{6}; & 6 \le x < 10 \\ 1 \ ; & x \ge 10 \end{cases}$$

Find $P(2 < X \le 6), P(X = 5), P(X = 4), P(X \le 6), P(X = 6)$.

Answer: $\mathbf{P(2 < X \le 6) = \frac{1}{2}}$, $\mathbf{P(X = 5) = 0}$, $\mathbf{P(X = 4) = \frac{1}{6}}$,
$\mathbf{P(X \le 6) = \frac{5}{6}}$, $\mathbf{P(X = 6) = \frac{1}{3}}$
</td>
</tr>
</table>

C	3	The probability density function of a continuous random variable X is given by $f(x)=\begin{cases} ax & ;\ 0\le x<1 \\ a & ;\ 1\le x<2 \\ 3a-ax & ;\ 2\le x<3 \\ 0 & ;\ \text{otherwise} \end{cases}.$ Find the value of a and also find C.D.F of X. **Answer:** $\mathbf{a=\frac{1}{2},\quad F(x)=\begin{cases} \frac{ax^2}{2} & ;\ 0\le x<1 \\ ax-\frac{a}{2} & ;\ 1\le x<2 \\ 3ax-\frac{ax^2}{2}-\frac{5a}{2} & ;\ 2\le x<3 \\ 0 & ;\ \text{otherwise} \end{cases}.}$

*** * * * * End of the Unit * * * * ***