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Unit – 2 ⇝ Numerical Linear Algebra

Method 1 ⇝ Eigen Values and Eigen Vectors

Eigen Value

→ Let A be square matrix.

→ If there exists a non-zero vector X such that $\mathbf{AX = \lambda X}$, then the scalar $\boldsymbol{\lambda}$ is known as **eigen value** of A.

→ The set of all eigenvalues of matrix A is known as **Spectrum** of A.

→ Eigen value is also known as characteristic root/latent value/proper roots.

Eigen Vector

→ Let A be square matrix.

→ If there exists a non-zero vector X such that $\mathbf{AX = \lambda X}$, then the **non-zero** vector **X** is known as **eigen vector** of A corresponding to eigen value λ.

→ Eigen vector is also known as characteristic vector/latent vector/proper vector corresponding to the eigen value λ of the matrix.

Eigen Space

→ The **solution space** of the system $\mathbf{(A - \lambda I) \cdot X = O}$ is known as **eigen space** of matrix A corresponding to eigen value λ.

i.e., $E_\lambda = \{ X \mid X \text{ is eigen vector of A corresponding to eigen value } \lambda \}$

Characteristic Equation

→ Let A be a square matrix of order n and I be an identity matrix of order n, then $\mathbf{| A - \lambda I | = 0}$ is known as characteristic equation, where λ is a scalar.

→ The characteristic equation of matrix A of order 2×2 is

$$\boldsymbol{\lambda^2 - S_1\lambda + S_2 = 0}$$

Where $S_1 = tr(A) =$ Sum of eigen values

$S_2 = | A | =$ Determinant of A

→ The characteristic equation of matrix A of order 3×3 is

$$\boldsymbol{\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0}$$

Where $S_1 = tr(A) =$ Sum of eigen values

$S_2 =$ Sum of minors of pricipal diagonal elements $= M_{11} + M_{22} + M_{33}$

$S_3 = |A| =$ Determinant of A

Properties of Eigen Value and Eigen Vector

(1) $\text{Trace}(A) =$ Sum of all the eigen values of matrix A.

(2) $|A| =$ Product of all the eigen values of matrix A.

(3) If one of the eigen value of matrix A is zero, then $|A| = 0$, hence A^{-1} does not exist.

(4) The eigen values of triangular matrix are the elements on its principal diagonal.

(5) The square matrix A and A^T have the same eigen values but eigen vectors need not to be same.

(6) If λ is an eigen value of a non-singular matrix A and X is an eigen vector correspond to λ, then

Matrix	**Eigen value**
A^T	λ
A^k	λ^k
A^{-1}	$\frac{1}{\lambda}$
$A \pm kI$	$\lambda \pm k\,;\, k \in \mathbb{R}$
kA	$k\lambda\,;\, k \in \mathbb{R}$

$\rightarrow$ Procedure to find the eigen values, eigen vectors for a matrix A of order 3×3

(1) Write the characteristic equation $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ and find S_1, S_2 and S_3.

(2) Find the roots of the characteristic equation. These are the Eigen Values of A, say $\lambda_1, \lambda_2, \lambda_3$.

(3) Solve the homogeneous system $(A - \lambda I) \cdot X = O$ for each eigen value by using Gauss Elimination Method.

(4) Eigen vector is the solution X of above system.

Examples of Method-1: Eigen Values and Eigen Vectors

C	1	Find the eigen values and corresponding eigen vectors of the matrix $A=\begin{bmatrix}5 & 0 & 1\\ 1 & 1 & 0\\ -7 & 1 & 0\end{bmatrix}$. **Answer:** $\boldsymbol{\lambda = 2,\ \ 2,\ \ 2,\qquad X=\left[-\frac{1}{3}\quad -\frac{1}{3}\quad 1\right]^T}$
C	2	Find the eigen values and eigen bases for the eigen space of the matrix $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$. **Answer:** $\boldsymbol{\lambda = -1,\ \ -1,\ \ 2}$ **Eigen Bases:** $\boldsymbol{\{(-1\quad 1\quad 0)^T, (-1\quad 0\quad 1)^T, (1\quad 1\quad 1)^T\}}$
C	3	Find the eigen values of $A=\begin{bmatrix}-5 & 4 & 34\\ 0 & 0 & 4\\ 0 & 0 & 4\end{bmatrix}$. Is it invertible? **Answer:** $\boldsymbol{-5,\ \ 0,\ \ 4}$ & **No**
C	4	Find the eigen values and eigen vectors of the matrix $A=\begin{bmatrix}3 & 1 & 4\\ 0 & 2 & 6\\ 0 & 0 & 5\end{bmatrix}$. **Answer:** $\boldsymbol{\lambda = 2 \rightsquigarrow X=[-1\quad 1\quad 0]^T,\qquad \lambda = 3 \rightsquigarrow X=[1\quad 0\quad 0]^T}$ $\boldsymbol{\lambda = 5 \rightsquigarrow X=[3\quad 2\quad 1]^T}$
C	5	If $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, then find the eigen values and eigen vectors corresponding to each eigen value of A. Also, write the eigen space for each eigen values. **Answer:** $\boldsymbol{\lambda = -1 \rightsquigarrow X=[-1\quad 1]^T,\qquad \lambda = 1 \rightsquigarrow X=[1\quad 1]^T}$ $\boldsymbol{E_{-1}=\left\{\begin{bmatrix}-1\\ 1\end{bmatrix}\cdot t \mid t\in\mathbb{R}\right\},\quad E_1=\left\{\begin{bmatrix}1\\ 1\end{bmatrix}\cdot t \mid t\in\mathbb{R}\right\}}$

C	6	Let $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, the find eigen values of $A,\ A^{25},\ 3A,\ A^{-1},\ A^{T},$ $A + 2I\ \ \&\ \ A^3 - 5I.$ **Answer:** $\mathbf{A \rightsquigarrow -1,\ 3,\quad A^{25} \rightsquigarrow -1,\ 3^{25},\quad 3A \rightsquigarrow -3,\ 9}$ $\mathbf{A^{-1} \rightsquigarrow -1,\ \frac{1}{3},\quad A^{T} \rightsquigarrow -1,\ 3,\quad A + 2I \rightsquigarrow 1,\ 5}$ $\mathbf{A^3 - 5I \rightsquigarrow -6,\ 22}$

Method 2 ⇝ Power Method

→ In many engineering problems, it is required to compute the numerically largest eigen value and corresponding eigen vector. In such cases, power method is convenient which is also well-suited for machine computations.

→ This method can be used when the matrix A of order n has n linearly independent eigen vectors.

→ The eigen values can be ordered in magnitude as $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$.

Here, λ_1 is known as dominant eigen value of A and the remaining eigen values $\lambda_2, \lambda_3, \dots, \lambda_n$, are known as sub-dominant eigen values of A.

i.e., if the eigenvalues of a matrix are 2, 5, and −13, then −13 is the dominant eigenvalue as $|-13| > |5| \geq |2|$.

Procedure for Finding Largest Eigen Value and Corresponding Eigen Vector

→ For a square matrix A,

(1) Apply the formula $AX_i = \lambda_{i+1}X_{i+1};\ i = 0, 1, 2, \dots$

(2) Let us consider initial eigen vector X_0 (X_0 is an arbitrary vector) as

$$X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \textbf{OR} \quad X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{OR} \quad X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(3) Find X_1 by calculating $AX_0 = \lambda_1 X_1$; where λ_1 is dominant eigen value

X_2 by calculating $AX_1 = \lambda_2 X_2$; where λ_2 is dominant eigen value

$\vdots$

(4) Repeat this process until $X_r = X_{r-1}$ up to desired level of accuracy.

(5) We get λ_r largest eigen value and X_r the corresponding eigen vector.

→ This iterative procedure for finding the largest eigen value of a matrix is known as Rayleigh's power method.

→ For Example:

We find largest eigen value and corresponding eigen vector of the matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

Let us consider, initial eigen vector $X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Now we find $AX_0 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \mathbf{5}\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \lambda_1 X_1$

$$AX_1 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \mathbf{5.8}\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = \mathbf{5.966}\begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.249 \end{bmatrix} = \mathbf{5.994}\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = \mathbf{5.999}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \mathbf{6}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda_6 X_6$$

Hence the largest eigen value is 6 and eigen vector is $\begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$.

Examples of Method-2: Power Method

C	1	Find the largest eigen value and the corresponding eigen vector of the following matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. **Answer: 6,** $\mathbf{[1 \quad 0.25\,]^T}$
C	2	Find the largest eigen value and the corresponding eigen vector of the following matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. **Answer: 2.41,** $\mathbf{[\,0.71 \quad 0.5 \quad 1\,]^T}$
C	3	Find the largest eigen value and the corresponding eigen vector of the following matrix $A = \begin{bmatrix} 15 & -4 & 3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$. **Answer: 13.52,** $\mathbf{[0.68 \quad -0.5 \quad -1\,]^T}$
C	4	Find the largest eigen value and the corresponding eigen vector of the following matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ 2 & 4 & 4 \end{bmatrix}$. **Answer: 2,** $\mathbf{[0.33 \quad -0.67 \quad 1\,]^T}$

Method 3 ⇝ Gauss Jacobi Method

Introduction

→ Numerical methods provide a way to solve problems quickly and easily as compare to analytic methods.

→ Numerical methods are useful to solve problems whose analytic solution is not available or the analytic solution can't be generalized.

→ Numerical methods are used to solve system of linear equations, to find roots of non-linear equations, to solve ODE, etc.

→ In general, an equation is solved by factorization but in some cases, the method of factorization fails. In such cases, numerical methods can be used.

Iterative Method

→ In this method, an approximation to the true solution is assumed initially to start method. By applying the method repeatedly, better and better approximation are obtained.

→ For large systems, iterative methods are faster than direct methods and round-off error are also smaller.

→ In fact, iteration is a self-correcting process and any error made at any stage of computation gets automatically corrected in the subsequent steps.

→ There are two iterative methods which is

(1) Gauss Jacobi Method

(2) Gauss Seidel Method

Gauss Jacobi Method

→ This method is applicable to the system of equations in which leading diagonal elements of the coefficient matrix are dominant (large in magnitude) in their respective rows.

→ Procedure to find solution system of simultaneous linear equation:

(1) Consider the system of equations:

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \dots\dots (1)$$

Where co-efficient matrix A must be diagonally dominant,

$$|\, a_1 \,| \geq |\, b_1 \,| + |\, c_1 \,|$$

i.e., $|\, b_2 \,| \geq |\, a_2 \,| + |\, c_2 \,|$ and the inequality is strictly greater than for **at least**

$$|\, c_3 \,| \geq |\, a_3 \,| + |\, b_3 \,|$$

one row.

(2) Solving the system (1) for x, y, z respectively,

We obtain, $$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \dots\dots (2)$$

(3) We start with $x_0 = 0,\ y_0 = 0\ \&\ z_0 = 0$ in equation (2) and we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0) = \frac{d_1}{a_1}$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0) = \frac{d_2}{b_2}$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0) = \frac{d_3}{c_3}$$

(4) Again, substituting these value x_1, y_1, z_1 in eq. (2), the next approximation is obtained.

(5) Repeat this process until the values of x, y, z are obtained to desire level of accuracy.

Examples of Method-3: Gauss Jacobi Method

C	1	Solve the following system by using Gauss Jacobi method. $6x + 2y - z = 4, \quad x + 5y + z = 3, \quad 2x + y + 4z = 27$ **Answer**: $(\mathbf{2}, \quad \mathbf{-1}, \quad \mathbf{6})$
C	2	Solve the following system by using Gauss Jacobi method. $2x + y + z = 0, \quad 3x + 5y + 2z = 15, \quad 2x + y + 4z = 8$ **Answer**: $(\mathbf{0.5}, \quad \mathbf{0.5}, \quad \mathbf{0.5})$
C	3	Solve the following system by using Gauss Jacobi method, correct up to two decimal places. $10x + y - z = 11.19, \quad x + 10y + z = 28.08, \quad -x + y + 10z = 35.61$ **Answer**: $(\mathbf{1.23}, \quad \mathbf{2.34}, \quad \mathbf{3.45})$
C	4	Solve by Gauss Jacobi method, correct up to two decimal places. $20x + 2y + z = 30, \quad x - 40y + 3z = -75, \quad 2x - y + 10z = 30$ starting with $(1,2,-1)$. **Answer**: $(\mathbf{1.14}, \quad \mathbf{2.13}, \quad \mathbf{2.99})$
C	5	Solve the following system by using Gauss Jacobi method. $27x + 6y - z = 85, \quad x + y + 54z = 110, \quad 6x + 15y + 2z = 72$ **Answer**: $(\mathbf{2.425}, \quad \mathbf{3.573}, \quad \mathbf{1.926})$

Method 4 ⇝ Gauss Seidel Method

Gauss Seidel Method

→ This is a modification of Gauss Jacobi method. In this method, we replace the approximation by the corresponding new ones as soon as they are calculated.

→ Procedure to find solution system of simultaneous linear equation:

(1) Consider the system of equations:

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \dots\dots(1)$$

Where co-efficient matrix A must be diagonally dominant,

$$|\,a_1\,| \geq |\,b_1\,| + |\,c_1\,|$$

i.e., $|\,b_2\,| \geq |\,a_2\,| + |\,c_2\,|$ and the inequality is strictly greater than for **at least**

$$|\,c_3\,| \geq |\,a_3\,| + |\,b_3\,|$$

one row

(2) Solving the system (1) for x, y, z respectively,

We obtain,
$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \dots\dots(2)$$

(3) We Start with $x_0 = 0, y_0 = 0$ & $z_0 = 0$ in the equation of x,

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0) = \frac{d_1}{a_1}$$

(4) Substituting $x = x_1$ & $z = z_0$ in the equation of y,

$$y_1 = \frac{1}{b_2}(d_2 - a_2\mathbf{x_1} - c_2z_0)$$

(5) Substituting $x = x_1$ & $y = y_1$ in the equation of z,

$$z_1 = \frac{1}{c_3}(d_3 - a_3\mathbf{x_1} - b_3\mathbf{y_1})$$

(6) Repeat this process until the values of x, y, z are obtained to desire level of accuracy.

Examples of Method-4: Gauss Seidel Method

C	1	Solve the following system by using Gauss Seidel method. $10x_1 + x_2 + x_3 = 6, \quad x_1 + 10x_2 + x_3 = 6, \quad x_1 + x_2 + 10x_3 = 6$ **Answer**: $(\mathbf{0.5}, \quad \mathbf{0.5}, \quad \mathbf{0.5})$
C	2	Solve the following system by using Gauss Seidel method. $9x + y + z = 10, \quad 2x + 10y + 3z = 19, \quad 3x + 4y + 11z = 0$ **Answer**: $(\mathbf{1}, \quad \mathbf{2}, \quad \mathbf{-1})$
C	3	Solve the following system by using Gauss Seidel method, correct up to three decimal places. $27x + 6y - z = 85, \quad x + y + 54z = 110, \quad 6x + 15y + 2z = 72$ **Answer**: $(\mathbf{2.425}, \quad \mathbf{3.573}, \quad \mathbf{1.926})$
C	4	By using Gauss Seidel method solve the following system up to seven iterations. $12x_1 + 3x_2 - 5x_3 = 1, \quad x_1 + 5x_2 + 3x_3 = 28, \quad 3x_1 + 7x_2 + 13x_3 = 76$ Use initial condition: $(x_1^0, x_2^0, x_3^0) = (1, 0, 1)$. **Answer**: $(\mathbf{1}, \quad \mathbf{3}, \quad \mathbf{4})$

*** * * * * End of the Unit * * * * ***