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Unit - 1 ⇝ Matrix Theory

Method 1 ⇝ Introduction

Introduction

→ In 1884 British mathematician J. J. Sylvester first introduced the term matrix.

→ Matrices are one of the most powerful tools in mathematics. Matrix notation and operations are used in electronic spreadsheet programs for personal computer, which is used in different areas of business and science like budgeting, sales projection, cost estimation, analyzing the results of an experiment etc.

→ In graphic design, digital images are referred as matrices.

→ Matrices are also used in cryptography.

→ This mathematical tool is not only used in certain branches of sciences, but also in genetics, economics, sociology, modern psychology and industrial management.

Matrix

→ An ordered rectangular array of **objects** arranged in rows and columns is known as matrix.

→ Objects can be number, expressions, symbols, etc.

→ These numbers, expressions or symbols are known as **elements** or entries of matrix.

→ It is denoted by upper case letters A, B, C, etc.

→ Following notations are used for matrices:

$$\begin{bmatrix} & & \end{bmatrix} \quad \& \quad \begin{pmatrix} & & \end{pmatrix}$$

→ For Example:

$$A = \begin{bmatrix} 1 & -1 \\ 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} + & - \\ \% & \cup \\ \Delta & \mathbb{N} \end{bmatrix}, \quad C = \begin{bmatrix} x+y & x^2+3 & x \\ 5 & y^2 & z \end{bmatrix}$$

→ In any matrix,

- horizontal line of elements is known as **row** of the matrix.
- vertical line of elements is known as **column** of the matrix.

$$\begin{array}{ccc} \begin{bmatrix} 15 & 56 & 76 \\ 7 & 70 & 92 \end{bmatrix} & \begin{array}{l} \longleftarrow \text{First Row} \\ \longleftarrow \text{Second Row} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{First Column} & \text{Second Column} & \text{Third Column} \end{array} & \end{array}$$

Order OR Size of a Matrix

→ A matrix with **m** rows and **n** columns is known as a matrix of order $\mathbf{m \times n}$.

→ It is read as "m by n".

→ For Example:

$$A = \begin{bmatrix} 1 & -1 \\ 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} + & - \\ \% & \cup \\ \Delta & \mathbb{N} \end{bmatrix}, \quad C = \begin{bmatrix} x+y & x^2+3 & x \\ 5 & y^2 & z \end{bmatrix}$$

Here, order of matrix A is 2×2,

order of matrix B is 3×2,

order of matrix C is 2×3.

→ In general, matrix of order $m \times n$ can be written as follow:

$$X = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \quad \text{or} \quad X = \left[a_{ij} \right]_{m \times n} \; ; \; 1 \le i \le m, \; 1 \le j \le n$$

Where, $\mathbf{a_{ij}}$ is element of $\mathbf{i^{th}}$ row and $\mathbf{j^{th}}$ column of the matrix X.

→ The number of elements of matrix of order $m \times n$ is $\mathbf{m \cdot n}$.

Example of Method-1: Introduction

<table>
<tr><td>C</td><td>1</td><td>
Consider the following information regarding the number of men and women workers in three factories I, II and III:

	Men workers	Women workers
I	30	25
II	25	31
III	27	25

Represent the above information in the form of a 3×2 matrix. What does the entry in the third row and second column represent?

Answer: $\mathbf{A = \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 25 \end{bmatrix}}$

The entry in the third row and second column represents the number of women workers in factory III.
</td></tr>
<tr><td>C</td><td>2</td><td>
In the matrix, $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 6 & 9 & 0 & -5 \\ \sqrt{7} & 7 & 2.3 & 5 \\ 12 & -2 & 8 & 6 \end{bmatrix}$

(1) What is order of matrix A?

(2) How many elements are in matrix A?

(3) What is the value of elements a_{12}, a_{23}, a_{33} & a_{42}?

Answer: **(1)** $\mathbf{4 \times 4}$ **(2) 16**

(3) $\mathbf{a_{12} = 2}$, $\mathbf{a_{23} = 0}$, $\mathbf{a_{33} = 2.3}$, $\mathbf{a_{42} = -2}$
</td></tr>
</table>

C	3	If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements? **Answer: If it has 18 elements** $\mathbf{1\times 18,\quad 2\times 9,\quad 3\times 6,\quad 6\times 3,\quad 9\times 2,\quad 18\times 1.}$ **If it has 5 elements** $\mathbf{1\times 5,\quad 5\times 1.}$
C	4	Construct a 2×2 matrix $A=\left[\,a_{ij}\,\right]$, whose elements are given by: $(1)\ \ a_{ij}=\dfrac{(i+j)^2}{2}\quad (2)\ \ a_{ij}=\dfrac{i+2j}{2}\quad (3)\ \ a_{ij}=\dfrac{\lvert -3i+j\rvert}{2}$ **Answer:** $\mathbf{(1)\ \ A=\begin{bmatrix} 2 & \frac{9}{2} \\ \frac{9}{2} & 8 \end{bmatrix},\qquad (2)\ \ A=\begin{bmatrix} \frac{3}{2} & \frac{5}{2} \\ \frac{5}{2} & 3 \end{bmatrix},}$ $\mathbf{(3)\ \ A=\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{5}{2} & 2 \end{bmatrix}}$

Method 2 ⇝ Types of Matrices

Row Matrix

→ A matrix which has only **one row** is known as a row matrix.

→ For Example:

$A = [\,1 \quad 2 \quad 3\,]$ is a row matrix of order 1×3.

→ Order of row matrix is $\mathbf{1 \times n}$.

Column Matrix

→ A matrix which has only **one column** is known as a column matrix.

→ For Example:

$$A = \begin{bmatrix} -1 \\ 6 \\ 0 \end{bmatrix} \text{ is a column matrix of order } 3 \times 1.$$

→ Order of column matrix is $\mathbf{m \times 1}$.

Square Matrix

→ A matrix is known as square matrix if the number of **rows** and **columns** are **equal**.

→ For Example:

$$A = \begin{bmatrix} 1 & -1 & 6 \\ \sqrt{2} & 6 & -4 \\ 5 & 0 & 5 \end{bmatrix} \text{ is a square matrix of order } 3 \times 3.$$

→ For any square matrix, $\mathbf{m = n}$.

→ Square matrix of order $n \times n$ is written as A_n.

Principal Diagonal of Matrix

→ The elements a_{ij} of a square matrix, for which $\mathbf{i = j}$ are known as diagonal elements or principal diagonal elements.

→ The elements $a_{11},\ a_{22},\ a_{33},\ \ldots,\ a_{nn}$ are principal diagonal elements.

→ The line passing through diagonal elements is known as **principal** diagonal of matrix.

→ For Example:

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A = \begin{bmatrix} \mathbf{1} & 7 & 9 \\ \sqrt{2} & \mathbf{6} & -4 \\ 5 & 0 & \mathbf{3} \end{bmatrix}$$

Principal Diagonal

The elements $1,\ 6\ \&\ 3$ are principal diagonal elements.

The elements $7,\ 9,\ -4,\ \sqrt{2},\ 5\ \&\ 0$ are non-diagonal elements of matrix A.

Diagonal Matrix

→ A **square** matrix in which all the elements **above** the principal diagonal & **below** the principal diagonal are **zero** is known as diagonal matrix.

→ $A = [\,a_{ij}\,]_{n\times n}$ is diagonal matrix, if $\mathbf{a_{ij} = 0}$ whenever $\mathbf{i \neq j}$.

→ For Example:

$$A = [\,2\,], \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Scalar Matrix

→ A **diagonal** matrix in which all the principal diagonal elements are **same** is known as scalar matrix.

→ For Example:

$$A = [\,2\,], \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Identity Matrix

→ A **diagonal** matrix in which all the principal diagonal elements are **1** is known as identity matrix.

→ Identity matrix of order n is denoted by $\mathbf{I_n}$ or $\mathbf{I_{n\times n}}$ or **I**.

→ For Example:

$$I_1 = [\,1\,],\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Zero Matrix **OR** Null Matrix

→ A matrix in which **all elements** are **zero** is known as a zero matrix.

→ It is denoted by $\mathbf{O_{m\times n}}$ or $\mathbf{O_n}$ (In case of square matrix) or **O**.

→ For Example:

$$O_1 = [\,0\,],\quad O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\quad O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Upper Triangular Matrix

→ A **square** matrix in which all the elements **below** the principal diagonal are **zero** is known as upper triangular matrix.

→ For Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ \mathbf{0} & 9 & 2 \\ \mathbf{0} & \mathbf{0} & 6 \end{bmatrix},\quad B = \begin{bmatrix} 4 & 5 & 3 & 0 \\ \mathbf{0} & -2 & 8 & 5 \\ \mathbf{0} & \mathbf{0} & 6 & 9 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -2 \end{bmatrix}$$

Lower Triangular Matrix

→ A **square** matrix in which all the elements **above** the principal diagonal are **zero** is known as lower triangular matrix.

→ For Example:

$$A = \begin{bmatrix} 5 & \mathbf{0} & \mathbf{0} \\ 6 & 9 & \mathbf{0} \\ 7 & 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 4 & 3 & \mathbf{0} & \mathbf{0} \\ 1 & 0 & 5 & \mathbf{0} \\ 8 & 2 & -1 & 5 \end{bmatrix}$$

Equality of matrices

→ Two matrices $A = [\,a_{ij}\,]$ and $B = [\,b_{ij}\,]$ are known as equal matrices if

(1) **Order** of matrix A = **Order** of matrix B.

(2) $a_{ij} = b_{ij}$, for all i and j.

→ For Example:

If $\begin{bmatrix} a & b \\ c & x \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -3 & 2 \\ 8 & 7 \end{bmatrix}$, then $a = 0,\ \ b = 4,\ \ c = -3,\ \ x = 2,\ \ y = 8\ \ \&\ \ z = 7.$

Example of Method-2: Types of Matrices

C	1	Determine the type of given matrices. $A=\begin{bmatrix}0 & 1 & 0\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\ 5 & -1\end{bmatrix}$, $C=\begin{bmatrix}x & 0\\ 0 & x\end{bmatrix}$, $P=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 5\end{bmatrix}$, $Q=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$, $R=\begin{bmatrix}1\\ -2\\ 8\end{bmatrix}$, **Answer: A is Row matrix;** **B is Square matrix and Lower triangular matrix;** **C is Square matrix, Diagonal matrix and scalar matrix;** **P is Square matrix and Diagonal matrix;** **Q is Zero OR Null matrix;** **R is Column matrix**
C	2	Find x and y if $\begin{bmatrix}2x & 3\\ 0 & y\end{bmatrix}=\begin{bmatrix}x-4 & 3\\ 0 & 3\end{bmatrix}$. **Answer:** $\mathbf{x=-4,\quad y=3}$
C	3	Find the value of a, b, c and d from the equation: $\begin{bmatrix}a-b & 2a+c\\ 2a-b & 3c+d\end{bmatrix}=\begin{bmatrix}-1 & 5\\ 0 & 13\end{bmatrix}$ **Answer:** $\mathbf{a=1,\quad b=2,\quad c=3,\quad d=4}$

Method 3 ⇝ Matrix Operations

Addition of Matrices

→ Let $A = [\,a_{ij}\,]$ and $B = [\,b_{ij}\,]$ are two matrices with **same order** $m \times n$, then addition of matrices A and B is written as $\mathbf{A + B}$ & defined as $A + B = [\,\mathbf{a_{ij}} + \mathbf{b_{ij}}\,]$.

→ For Example:

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 9 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 8 & -5 \\ 0 & 2 & 11 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 9 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 8 & -5 \\ 0 & 2 & 11 \end{bmatrix} = \begin{bmatrix} 1+2 & -2+8 & 3+(-5) \\ 6+0 & 9+2 & 2+11 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & -2 \\ 6 & 11 & 13 \end{bmatrix}$$

Difference of Matrices

→ Let $A = [\,a_{ij}\,]$ and $B = [\,b_{ij}\,]$ are two matrices with **same order** $m \times n$, then difference of matrices A and B is written as $\mathbf{A - B}$ & defined as $A - B = [\,\mathbf{a_{ij}} - \mathbf{b_{ij}}\,]$.

→ For Example:

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 9 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 8 & -5 \\ 0 & 2 & 11 \end{bmatrix}$, then

$$A - B = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 9 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 8 & -5 \\ 0 & 2 & 11 \end{bmatrix} = \begin{bmatrix} 1-2 & (-2)-8 & 3-(-5) \\ 6-0 & 9-2 & 2-11 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -10 & 8 \\ 6 & 7 & -9 \end{bmatrix}$$

Multiplication of Matrix by a Scalar

→ Let $A = [\,a_{ij}\,]$ be an $m \times n$ matrix and **k** be any **non-zero real number**, then scalar multiplication k and A, written as $k \cdot A$ or **kA** & defined as $k \cdot A = [\,\mathbf{k} \cdot \mathbf{a_{ij}}\,]$.

→ For Example:

Let $A = \begin{bmatrix} 0 & 4 & -3 \\ -7 & 5 & 10 \end{bmatrix}$ and $k = 5$ be any non-zero real number, then

$$5A = 5 \cdot \begin{bmatrix} 0 & 4 & -3 \\ -7 & 5 & 10 \end{bmatrix} = \begin{bmatrix} \mathbf{5} \cdot 0 & \mathbf{5} \cdot 4 & \mathbf{5} \cdot (-3) \\ \mathbf{5} \cdot (-7) & \mathbf{5} \cdot 5 & \mathbf{5} \cdot 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 20 & -15 \\ -35 & 25 & 50 \end{bmatrix}$$

Properties of Matrix Addition, Difference & Scalar Multiplication

→ Let A, B and C are three matrices with same order & α and β are scalars, then

(1) $A + B = B + A$ (Commutative Law)

(2) $A - B \neq B - A$

(3) $A + (B + C) = (A + B) + C$ (Associative Law)

(4) $A - (B - C) \neq (A - B) - C$

(5) $A + O = O + A = A$ (Existence of additive identity)

(6) $A + (-A) = (-A) + A = O$ (The existence of additive inverse)

(7) $\alpha(A \pm B) = \alpha A \pm \alpha B$

(8) $(\alpha \pm \beta)A = \alpha A \pm \beta A$

Multiplication of Matrices

→ Multiplication of two matrices A and B is denoted as $A \times B$ or **AB** and it is defined if following condition satisfied:

No. of **columns** of A = No. of **rows** of B

→ Similarly, Multiplication of two matrices B and A is denoted as $B \times A$ or **BA** and it is defined if following condition satisfied:

No. of **columns** of B = No. of **rows** of A

→ Let $A = [\,a_{ik}\,]$ be $m \times p$ matrix and $B = [\,b_{kj}\,]$ be a $p \times n$ matrix, then

Resultant matrix $AB = [\,c_{ij}\,]$ is $m \times n$ matrix

Where **ij − entry** is obtained by multiplying the $\mathbf{i^{th}}$ **row** of A by the $\mathbf{j^{th}}$ **column** of B.

i. e.,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \vdots & \cdots & a_{2p} \\ \mathbf{a_{i1}} & \mathbf{a_{i2}} & \cdots & \mathbf{a_{ip}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & \mathbf{b_{1j}} & \cdots & b_{1n} \\ b_{21} & \cdots & \mathbf{b_{2j}} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{p1} & \cdots & \mathbf{b_{pj}} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \cdots & \vdots \\ \vdots & \mathbf{c_{ij}} & \vdots \\ \vdots & \cdots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}$$

$$\mathbf{c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{ip} \cdot b_{pj}}.$$

→ For Example:

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 9 & 2 \end{bmatrix}_{2\times 3}$ and $B = \begin{bmatrix} 2 & 8 & 1 \\ 0 & 2 & 4 \\ -7 & -5 & 2 \end{bmatrix}_{3\times 3}$

Here, No. of columns of matrix A (3) = No. of rows of matrix B (3)

Therefore, AB is defined.

Order of resultant matrix will be 2×3.

Order of Matrix **A**	×	Order of Matrix **B**	=	Order of Matrix **AB**
$2 \times \cancel{3}$		$\cancel{3} \times 3$		2×3

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 9 & 2 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 & 8 & 1 \\ 0 & 2 & 4 \\ -7 & -5 & 2 \end{bmatrix}_{3\times 3} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}_{2\times 3}$$

11-entry

$$AB = \begin{bmatrix} \boxed{1 \quad -2 \quad 3} \\ 6 \quad 9 \quad 2 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 0 \\ -7 \end{matrix}} & \begin{matrix} 8 \\ 2 \\ -5 \end{matrix} & \begin{matrix} 1 \\ 4 \\ 2 \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{1 \cdot 2 + (-2) \cdot 0 + 3 \cdot (-7)} & ? & ? \\ ? & ? & ? \end{bmatrix}$$

12-entry

$$AB = \begin{bmatrix} \boxed{1 \quad -2 \quad 3} \\ 6 \quad 9 \quad 2 \end{bmatrix} \begin{bmatrix} \begin{matrix} 2 \\ 0 \\ -7 \end{matrix} & \boxed{\begin{matrix} 8 \\ 2 \\ -5 \end{matrix}} & \begin{matrix} 1 \\ 4 \\ 2 \end{matrix} \end{bmatrix} = \begin{bmatrix} -19 & \boxed{1 \cdot 8 + (-2) \cdot 2 + 3 \cdot (-5)} & ? \\ ? & ? & ? \end{bmatrix}$$

13-entry

$$AB = \begin{bmatrix} \boxed{1 \quad -2 \quad 3} \\ 6 \quad 9 \quad 2 \end{bmatrix} \begin{bmatrix} \begin{matrix} 2 \\ 0 \\ -7 \end{matrix} & \begin{matrix} 8 \\ 2 \\ -5 \end{matrix} & \boxed{\begin{matrix} 1 \\ 4 \\ 2 \end{matrix}} \end{bmatrix} = \begin{bmatrix} -19 & -11 & \boxed{1 \cdot 1 + (-2) \cdot 4 + 3 \cdot 2} \\ ? & ? & ? \end{bmatrix}$$

21-entry

$$AB = \begin{bmatrix} 1 \quad -2 \quad 3 \\ \boxed{6 \quad 9 \quad 2} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 0 \\ -7 \end{matrix}} & \begin{matrix} 8 \\ 2 \\ -5 \end{matrix} & \begin{matrix} 1 \\ 4 \\ 2 \end{matrix} \end{bmatrix} = \begin{bmatrix} -19 & -11 & -1 \\ \boxed{6 \cdot 2 + 9 \cdot 0 + 2 \cdot (-7)} & ? & ? \end{bmatrix}$$

22-entry

$$AB = \begin{bmatrix} 1 \quad -2 \quad 3 \\ \boxed{6 \quad 9 \quad 2} \end{bmatrix} \begin{bmatrix} \begin{matrix} 2 \\ 0 \\ -7 \end{matrix} & \boxed{\begin{matrix} 8 \\ 2 \\ -5 \end{matrix}} & \begin{matrix} 1 \\ 4 \\ 2 \end{matrix} \end{bmatrix} = \begin{bmatrix} -19 & -11 & -1 \\ -2 & \boxed{6 \cdot 8 + 9 \cdot 2 + 2 \cdot (-5)} & ? \end{bmatrix}$$

23-entry

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 9 & 2 \end{bmatrix} \begin{bmatrix} 2 & 8 & 1 \\ 0 & 2 & 4 \\ -7 & -5 & 2 \end{bmatrix} = \begin{bmatrix} -19 & -11 & -1 \\ -2 & 56 & \boxed{6 \cdot 1 + 9 \cdot 4 + 2 \cdot 2} \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 9 & 2 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 & 8 & 1 \\ 0 & 2 & 4 \\ -7 & -5 & 2 \end{bmatrix}_{3\times 3} = \begin{bmatrix} -19 & -11 & -1 \\ -2 & 56 & 46 \end{bmatrix}_{2\times 3}$$

While, No. of columns of matrix B (3) $\neq$ No. of rows of matrix A (2)

Therefore, BA is not defined.

Order of Matrix **B** 3×3 $\times$ Order of Matrix **A** 2×3 $=$ **BA** is not defined

Results on Matrix Multiplication

→ Let A, B and C are three matrices & k be a scalar, then

(1) $(AB)\,C = A\,(BC)$ (The associative law)

(2) $A\,(B \pm C) = AB \pm AC$ (Left distributive law)

(3) $(B \pm C)\,A = BA \pm CA$ (Right distributive law)

(4) $IA = AI = A$ (The existence of multiplicative identity)

(5) $k(A \pm B) = kA \pm kB$

Example of Method-3: Matrix Operations

C	1	Given, $A = \begin{bmatrix} \sqrt{3} & 1 & -3 \\ 6 & 9 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 8 & 3 \\ 0 & 2 & \frac{3}{2} \end{bmatrix}$. Find $A + B$ and $A - B$. **Answer**: $\mathbf{A + B = \begin{bmatrix} \sqrt{3}+2 & 9 & 0 \\ 6 & 11 & \frac{7}{2} \end{bmatrix}}$, $\mathbf{A - B = \begin{bmatrix} \sqrt{3}-2 & -7 & -6 \\ 6 & 7 & \frac{1}{2} \end{bmatrix}}$
C	2	Given, $A = \begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & -3 \\ 12 & 7 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 3 & -5 \\ 11 & 0 & 9 \\ -7 & 8 & -2 \end{bmatrix}$. Find $5A + 3B$ and $2A - 3B + 5I$. **Answer**: $\mathbf{5A + 3B = \begin{bmatrix} 38 & 14 & -5 \\ 33 & -5 & 12 \\ 39 & 59 & 34 \end{bmatrix}}$, $\mathbf{2A - 3B + 5I = \begin{bmatrix} -5 & -7 & 19 \\ -33 & 3 & -33 \\ 45 & -10 & 27 \end{bmatrix}}$
C	3	If $A = \begin{bmatrix} 6 & 1 & -7 \\ -8 & -6 & 2 \\ 11 & 6 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & -10 & 9 \\ 8 & -4 & -1 \\ -6 & 4 & 1 \end{bmatrix}$, then find the matrix X such that $4B + 3X = 2A$. **Answer**: $\mathbf{X = \frac{1}{3}\begin{bmatrix} -24 & 42 & -50 \\ -48 & 4 & 8 \\ 46 & -4 & -20 \end{bmatrix}}$

C	4	Find values of x and y from the following equation: $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix}+\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$ **Answer: $x = 2$, $y = 9$**
C	5	Let $A=\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$. Find AB and BA, if possible. **Answer: $AB=\begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$, BA is not defined.**
C	6	Let $A=\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$. Show that $AB \neq BA$. **Answer: $AB=\begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$, $BA=\begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$** **$\therefore$ $AB \neq BA$.**
C	7	If $A=\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$, then find $A^2-3A+9I$. **Answer: $A^2-3A+9I=\begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$**

C	8	If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40I = O$.
C	9	If $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ and $A + 2B = A^2$, then find B. **Answer:** $\mathbf{B} = \begin{bmatrix} \mathbf{6} & \mathbf{4} \\ \mathbf{3} & \mathbf{7} \end{bmatrix}$

Method 4 ⇝ Determinant

Determinant

→ For every square matrix $A = [\, a_{ij} \,]$ of order n, we can associate a number with matrix A is known as determinant of matrix A.

→ It is denoted by **| A |** or **det(A)** and read as determinant of A.

(1) **Determinant of matrix of order one**

If $A = [\, a \,]$ be a matrix of order 1, then $|\, A \,| = a$.

(2) **Determinant of matrix of order two**

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2, then

$$|\, A \,| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

(3) **Determinant of matrix of order three**

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a matrix of order 3, then

Expansion along with first row (R_1)

$$|\, A \,| = \begin{vmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Also, we can expand determinant along with first row (C_1)

$$|\, A \,| = \begin{vmatrix} \mathbf{a_{11}} & a_{12} & a_{13} \\ \mathbf{a_{21}} & a_{22} & a_{23} \\ \mathbf{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

→ For Example:

Consider a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, then

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 1(3 \cdot 2 - 1 \cdot 1) - 2(2 \cdot 2 - 1 \cdot 3) + 3(2 \cdot 1 - 3 \cdot 3)$$

$$= 1(6 - 1) - 2(4 - 3) + 3(2 - 9)$$

$$= 1(5) - 2(1) + 3(-7)$$

$$= 5 - 2 - 21$$

$$= -18$$

Example of Method-4: Determinant

C	1	Evaluate the determinant of following matrices. $A = \begin{bmatrix} \sqrt{2} & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & \sqrt{5} \end{bmatrix}$, $C = \begin{bmatrix} -\frac{2}{\sqrt{7}} & 5 \\ 3 & 4 \end{bmatrix}$ **Answer:** $\mathbf{|A| = 2\sqrt{2} - 12}$, $\mathbf{|B| = \sqrt{5} + 6}$, $\mathbf{|C| = -\frac{8}{\sqrt{7}} - 15}$
C	2	Find the value of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$ **Answer:** $\mathbf{x = \pm 2\sqrt{2}}$
C	3	Evaluate the determinant of following matrices. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$ **Answer:** $\mathbf{|A| = -18}$, $\mathbf{|B| = 0}$, $\mathbf{|C| = 5}$

Method 5 ⇝ Adjoint of Square Matrix

Transpose of a matrix

→ A matrix obtained by **interchanging** the **rows** of matrix A into **columns** or the columns of matrix A into rows is known as transpose of a matrix A.

→ It is denoted by $\mathbf{A^T}$ and read as transpose of A.

→ For Example:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ -2 & 5 \end{bmatrix}_{3\times 2} \Rightarrow A^T = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 5 \end{bmatrix}_{2\times 3}$$

→ If order of matrix A is $\mathbf{m \times n}$, then the order of matrix A^T is $\mathbf{n \times m}$.

Minors

→ The **determinant** obtained by deleting $\mathbf{i^{th}}$ row and $\mathbf{j^{th}}$ column is known as minor of the element a_{ij}.

→ It is denoted by $\mathbf{M_{ij}}$ and read as minor of the element a_{ij}.

→ For Example:

$$\text{Consider a square matrix } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The minors $M_{11},\ M_{12}\ \&\ M_{22}$ can be obtained as follows:

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22} \cdot a_{33} - a_{32} \cdot a_{23} \qquad \left(\because \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix}\right)$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21} \cdot a_{33} - a_{31} \cdot a_{23}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11} \cdot a_{33} - a_{31} \cdot a_{13}$$

Cofactors

→ Cofactor of an element a_{ij} is defined as $\mathbf{A_{ij} = (-1)^{i+j} \cdot M_{ij}}$.

→ It is denoted by $\mathbf{A_{ij}}$ and read ad cofactor of a_{ij}.

→ For Example:

Consider a square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The cofactor A_{11}, A_{12} & A_{22} can be obtained as follows:

$A_{11} = (-1)^{1+1} \cdot M_{11} = a_{22} \cdot a_{33} - a_{32} \cdot a_{23}$

$A_{12} = (-1)^{1+2} \cdot M_{12} = -(a_{21} \cdot a_{33} - a_{31} \cdot a_{23})$

$A_{22} = (-1)^{2+2} \cdot M_{22} = a_{11} \cdot a_{33} - a_{31} \cdot a_{13}$

Cofactor Matrix

→ Cofactor matrix of a square matrix A is obtained by **replacing** each element of matrix A by its **cofactor**.

→ It is denoted as $[\mathbf{A_{ij}}]$ and read as cofactor matrix of A.

→ For Example:

Consider a square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\text{Cofactor matrix of } A = [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

Adjoint of a Square Matrix

→ The adjoint matrix of a square matrix $A = [\,a_{ij}\,]$ is **transpose** of cofactor matrix.

i.e., if $A = [\,a_{ij}\,]$, then $adj(A) = [\,A_{ij}\,]^T$

→ It is denoted by **adj(A)** or **adj A** and read as adjoint of A.

(1) **Adjoint of 2 × 2 matrix**

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ then adj } A = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

(2) **Adjoint of 3 × 3 matrix**

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

→ For Example:

$$\text{Let } A = \begin{bmatrix} 1 & -2 & 4 \\ 5 & 3 & 0 \\ -7 & 2 & 6 \end{bmatrix}$$

Minors of all nine elements can be obtained as follow:

$$M_{11} = \begin{vmatrix} 3 & 0 \\ 2 & 6 \end{vmatrix} = 3 \cdot 6 - 2 \cdot 0 = 18$$

$$M_{12} = \begin{vmatrix} 5 & 0 \\ -7 & 6 \end{vmatrix} = 5 \cdot 6 - (-7) \cdot 0 = 30$$

$$M_{13} = \begin{vmatrix} 5 & 3 \\ -7 & 2 \end{vmatrix} = 5 \cdot 2 - (-7) \cdot 3 = 31$$

$$M_{21} = \begin{vmatrix} -2 & 4 \\ 2 & 6 \end{vmatrix} = (-2) \cdot 6 - 2 \cdot 4 = -20$$

$$M_{22} = \begin{vmatrix} 1 & 4 \\ -7 & 6 \end{vmatrix} = 1 \cdot 6 - (-7) \cdot 4 = 34$$

$$M_{23} = \begin{vmatrix} 1 & -2 \\ -7 & 2 \end{vmatrix} = 1 \cdot 2 - (-7) \cdot (-2) = -12$$

$$M_{31} = \begin{vmatrix} -2 & 4 \\ 3 & 0 \end{vmatrix} = (-2) \cdot 0 - 3 \cdot 4 = -12$$

$$M_{32} = \begin{vmatrix} 1 & 4 \\ 5 & 0 \end{vmatrix} = 1 \cdot 0 - 5 \cdot 4 = -20$$

$$M_{33} = \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix} = 1 \cdot 3 - 5 \cdot (-2) = 13$$

$$\text{Cofactor matrix of A} = \left[A_{ij} \right] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 18 & -30 & 31 \\ 20 & 34 & 12 \\ -12 & 20 & 13 \end{bmatrix}$$

$$\therefore \operatorname{adj}(A) = \left[A_{ij} \right]^T = \begin{bmatrix} 18 & 20 & -12 \\ -30 & 34 & 20 \\ 31 & 12 & 13 \end{bmatrix}$$

Example of Method-5: Adjoint of Matrix

C	1	Find the transpose of the given matrix. $A=\begin{bmatrix}2 & -9 & 3\\ 13 & 11 & -17\\ 3 & 6 & 15\\ 4 & 13 & 1\end{bmatrix}$, $B=\begin{bmatrix}22 & -21 & -99\\ 85 & 31 & -2\sqrt{3}\\ 7 & -12 & 57\end{bmatrix}$ **Answer:** $A^T=\begin{bmatrix}2 & 13 & 3 & 4\\ -9 & 11 & 6 & 13\\ 3 & -17 & 15 & 1\end{bmatrix}$, $B^T=\begin{bmatrix}22 & 85 & 7\\ -21 & 31 & -12\\ -99 & -2\sqrt{3} & 57\end{bmatrix}$
C	2	Find the adjoint of the following matrices. $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$, $C=\begin{bmatrix}-1 & 5\\ -3 & 2\end{bmatrix}$ **Answer:** $A=\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}$, $B=\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}$, $C=\begin{bmatrix}2 & -5\\ 3 & -1\end{bmatrix}$
C	3	Find the adjoint of the following matrix $A=\begin{bmatrix}2 & -3 & 5\\ 6 & 0 & 4\\ 1 & 5 & -7\end{bmatrix}$ **Answer:** $A=\begin{bmatrix}-20 & 4 & -12\\ 46 & -19 & 22\\ 30 & -13 & 18\end{bmatrix}$

Method 6 ⇝ Inverse of a Matrix by Adjoint Method

Inverse of a Square Matrix

→ For any **square matrix** A of order n, if there exists a matrix B such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I_n}$, then the matrix B is known as inverse of A.

→ It is denoted by $\mathbf{A^{-1}}$ and read as inverse of A.

→ Inverse of a matrix A is defined if and only if $|\mathbf{A}| \neq \mathbf{0}$.

Procedure to Find Inverse of Matrix

(1) Find determinant of matrix.

(2) Find cofactor matrix.

(3) Find adjoint of matrix.

(4) Use the following formula to find inverse of matrix.

$$A^{-1} = \frac{1}{|A|} \text{adj } A \;; \quad |A| \neq 0$$

→ For Example:

We find inverse of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Step–1: Find determinant of matrix A

Evaluating determinant by third row,

$$|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ \boxed{1 \quad 0 \quad 0} \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1\big(1 - (-1)\big) = 1(1 + 1)$$

$$= 2 \neq 0$$

Hence, inverse of matrix A exists.

Step–2: Find cofactor matrix A

$$A_{11} = M_{11} = \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} = 0 \qquad A_{12} = -M_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$$A_{13} = M_{13} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1 \qquad A_{21} = -M_{21} = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{22} = M_{22} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad A_{23} = -M_{23} = -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$$A_{31} = M_{31} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \qquad A_{32} = -M_{32} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$A_{33} = M_{33} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

$$\therefore A = [\,A_{ij}\,] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & -1 & -1 \end{bmatrix}$$

Step–3: Find Adjoint of matrix A

$$\operatorname{adj}(A) = [\,A_{ij}\,]^T = \begin{bmatrix} 0 & 0 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

Step– 4: Use $A^{-1} = \frac{1}{|A|}\,\operatorname{adj} A$

$$A^{-1} = \frac{1}{|A|}\,\operatorname{adj} A = \frac{1}{2} \cdot \begin{bmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Example of Method-6: Inverse of a Matrix by Adjoint Method

C	1	Find the inverse of the following matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ **Answer:** $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$, $B^{-1} =$ **Not exist**, $C^{-1} = \begin{bmatrix} \frac{2}{13} & -\frac{5}{13} \\ \frac{3}{13} & -\frac{1}{13} \end{bmatrix}$
C	2	Find the inverse of the following matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ **Answer:** $A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{5}{3} & -1 \\ \frac{4}{3} & -\frac{23}{3} & -4 \\ -\frac{1}{3} & \frac{11}{3} & 2 \end{bmatrix}$
C	3	If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$. Hence find A^{-1}. **Answer:** $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Linear Equations

→ In two dimensions, a line in a rectangular xy-coordinate system can be represented by an equation of the form $ax + by = c$ (a, b not both 0)

→ In three dimensions, a plane in a rectangular xyz-coordinate system can be represented by an equation of the form $ax + by + cz = d$ (a, b, c not all 0).

→ More generally, a linear equation in n variables $x_1, x_2, \dots, x_n$ can be expressed in the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where $a_1, a_2, \dots, a_n$ and b are constants, and the a_i's are not all zero.

→ linear equation does not involve any products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions. Also, the power of all variables is one.

- For Examples:

  (1) $3x - 2y = 7$

  (2) $\frac{1}{2}x - \pi z = \sqrt{2}$

  (3) $\left(\sin\frac{\pi}{2}\right)x_1 - x_2 = e^2$

→ Non-linear equation does involve any products or roots of variables and variables involved in trigonometric, exponential, or logarithmic functions.

- For Examples:

  (1) $xy + z = 2$

  (2) $e^x - 2y = 4$

  (3) $\sin x_1 + 2x_2 - x_3 = 0$

System of Simultaneous Linear Equations

→ A finite set of linear equations is known as **system of linear equations** or, more briefly, a linear system. The variables are also known as unknowns.

→ General linear system of m equations and n variables $x_1, x_2, \dots, x_n$ can be written as:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

→ Matrix representation of the above system is $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

→ For above system,

Matrix A is known as co-efficient matrix.

Matrix X is known as variable matrix.

Matrix B is known as constant matrix.

→ Augmented Matrix

The augmented matrix of the above system is denoted as $[\,A \mid B\,]$ or $[\,A : B\,]$ and defined as

$$[\,A \mid B\,] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right]_{m \times (n+1)}$$

→ Every system of linear equations has either **unique solution** (exactly one solution) or **infinitely many solutions** or **no solution**.

→ Consistent System

- If a system of linear equations has **at least one solution**, then that system is known as consistent system.

→ Inconsistent System

- If a system of linear equations has **no solution**, then that system is known as inconsistent system.

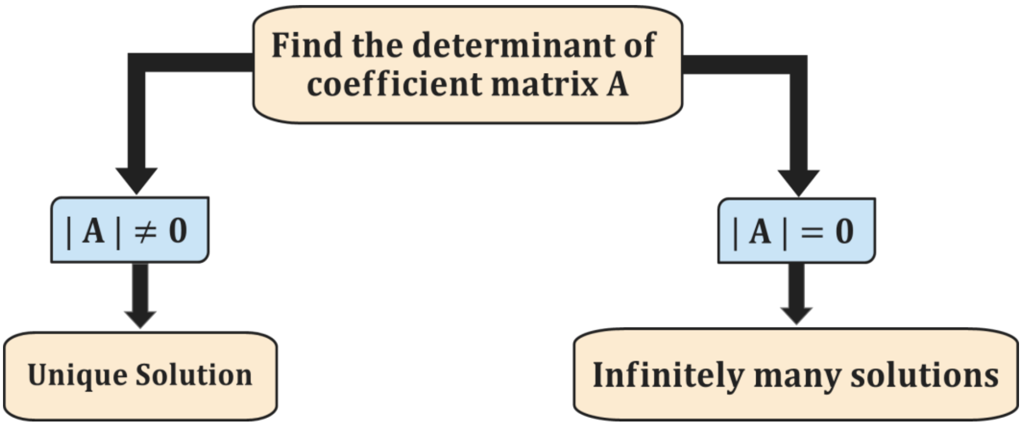
Homogeneous System of Linear Equations

→ In a system $AX = B$, if constant matrix B is zero matrix (O), then the system $AX = O$ is known as Homogeneous system of linear equations.

→ Homogeneous system always contains trivial solution, so this system is always **consistent**.

→ Solution by Determinant Method:

For a homogeneous system, in which number of equations and number of variables are equal.



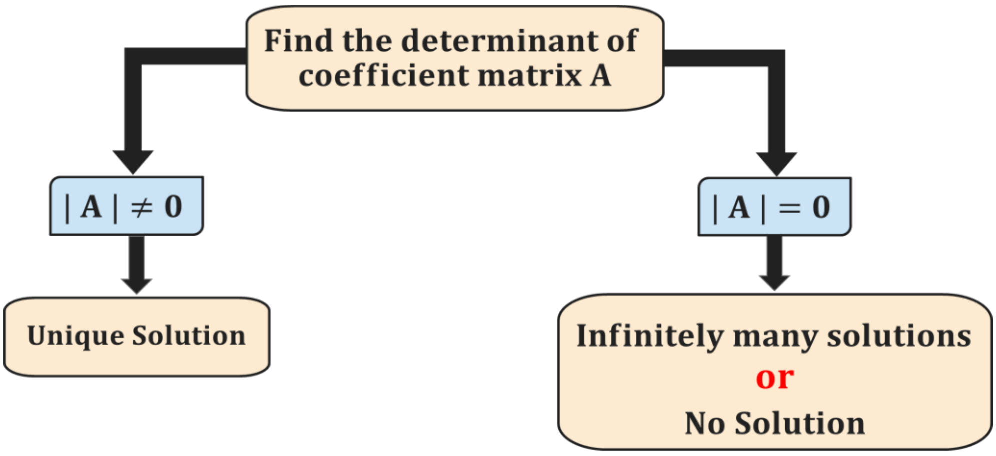
Non-homogeneous System of Linear Equations

→ In a system $AX = B$, if constant matrix B is not a zero matrix, then the system $AX = B$ is known as non-homogeneous system of linear equations.

→ Non-homogeneous system has either unique solution, infinitely many solutions or no solutions.

→ Solution by Determinant Method:

For a non-homogeneous system, in which number of equations and number of variables are equal.



Elementary Row Operations

(1) Interchange of any two rows

- Notation: $\mathbf{R_{ij}}$ or $R_i \leftrightarrow R_j$
- Meaning: **Interchange** of i^{th} row and j^{th} row.
- For Example:

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 4 & 10 \\ 0 & 3 & -6 \end{bmatrix} \xrightarrow{\text{Applying } R_{13}} \sim \begin{bmatrix} 0 & 3 & -6 \\ 2 & 4 & 10 \\ 1 & -2 & 5 \end{bmatrix}$$

(2) Multiplication of any row by a non-zero scalar k

- Notation: $\mathbf{k \cdot R_i}$
- Meaning: **Multiplication** of all the elements of i^{th} row by a non-zero scalar k
- For Example:

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 4 & 10 \\ 0 & 3 & -6 \end{bmatrix} \xrightarrow{\text{Applying } 2R_2} \sim \begin{bmatrix} 0 & 3 & -6 \\ 4 & 8 & 20 \\ 1 & -2 & 5 \end{bmatrix}$$

(3) Add a multiple of one row to another row

- Notation: $\mathbf{R_{ij}(k)}$ or $k \cdot R_i + R_j$
- Meaning: **Multiplication** of all the elements of i^{th} row by a non-zero scalar k and **add** it into j^{th} row.
- For Example:

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 4 & 10 \\ 0 & 3 & -6 \end{bmatrix} \xrightarrow{\text{Applying } R_{12}(-2)} \sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 8 & 0 \\ 0 & 3 & -6 \end{bmatrix}$$

$$\begin{array}{rrrrrr} 1 & -2 & 5 \times (-2) & = & -2 & 4 & -10 \\ & & & + & 2 & 4 & 10 \\ \hline & & & & 0 & 8 & 0 \end{array}$$

→ If we apply any elementary row operation on any matrix A, then the resultant matrix is **equivalent** to the matrix A.

It is denoted by $\sim A$ and read as "equivalent to the matrix A".

→ **Zero Row**

- A row in which all the elements are zero is known as zero row.
- For Example:

For $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix}$, second row is zero row.

→ **Leading OR Pivot Element**

- The **first non-zero** element of any row is known as leading element of that row.
- For example:

$$A = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 0 & 4 \\ 0 & 0 & -6 \end{bmatrix}$$

The leading element of 1^{st} row is -2, The leading element of 2nd row is 1,

The leading element of 3rd row is -6.

→ **Rows in decreasing order**

- To arrange rows of matrix in decreasing order, follow the steps:

  (1) Find the order of each row by counting the number of elements in that row starting from leading element.

  (2) To arrange rows in decreasing order, apply the elementary row operation(s).

- For Example:

$$A = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 0 & 4 \\ 0 & 0 & -6 \end{bmatrix} \begin{array}{l} \to 2 \\ \to 3 \\ \to 1 \end{array} \quad \xrightarrow{\text{Applying } R_{12}} \quad \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & -6 \end{bmatrix} \begin{array}{l} \to 3 \\ \to 2 \\ \to 1 \end{array}$$

Echelon Form

→ Procedure to find row echelon form of any matrix:

(1) If a matrix has a zero row, then it is at the bottom of the matrix.

(2) Arrange all rows of matrix in **decreasing** order.

(3) With the help of leading element of 1st row, make all the elements **zero** below the leading element of 1st row.

(4) **Repeat** Steps (1), (2) and (3) for remaining rows except last row.

→ Row echelon form of matrix is **not unique**.

→ For Example:

We convert the following matrix into Row Echelon form.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & 2 & -4 & 3 \\ 0 & 4 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{bmatrix}$$

Step 1: In given matrix A, first row is a zero row, which must be the last row of matrix A.

$$\sim \begin{bmatrix} 0 & 5 & 0 & 1 \\ 6 & 2 & -4 & 3 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (\text{Applying } R_{14})$$

Step 2: Arrange all the rows in decreasing order.

$$\sim \begin{bmatrix} 6 & 2 & -4 & 3 \\ 0 & 5 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (\text{Applying } R_{12})$$

Step 3: With the help of leading element of 1st row, make all the elements zero below the leading element of 1st row.

As all the elements below the leading element of 1st row are zero, there is nothing to do.

Step 4: Repeat Step 2 and Step 3 for **2nd row**.

As matrix after Step 3 is in decreasing order, let us make all the elements zero below the leading element of 2nd row.

$$\sim \begin{bmatrix} 6 & 2 & -4 & 3 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 1 & -\frac{4}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left(\text{Applying } R_{23}\left(-\frac{4}{5}\right)\right)$$

Step 5: Repeat Step 2 and Step 3 for **3rd row**.

As matrix after above operation is in decreasing order, let us make all the elements zero below the leading element of 3rd row.

As all the elements below the leading element 1 of 3rd row is zero, there is nothing to do.

$\therefore$ The row echelon of given matrix A is

$$\begin{bmatrix} 6 & 2 & -4 & 3 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 1 & -\frac{4}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

$\rightarrow$ Procedure to find reduced row echelon form of any matrix:

(1) If a matrix has a zero row, then it is at the bottom of the matrix.

(2) Arrange all rows of matrix in **decreasing** order.

(3) Make leading element of 1st row **1(one)**.

(4) With the help of leading element of 1st row, make all the elements **zero** above & below the leading element of 1st row.

(5) **Repeat** Steps (1), (2), (3) and (4) for remaining rows.

$\rightarrow$ Reduced row echelon form of matrix is **unique**.

Remark

If the last row in row echelon form of [A | B] is

$[0 \quad 0 \quad \mathbf{p} \mid \mathbf{0} \text{ or } \mathbf{q}]$ $\Downarrow$ **Unique Solution**

$[0 \quad 0 \quad \mathbf{0} \mid \mathbf{p}]$ $\Downarrow$ **No Solution**

$[\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \mid \mathbf{0}]$ $\Downarrow$ **Infinitely Many Solution**

where, **p** and **q** are non $-$ zero number

Method 7 ⇝ Gauss Elimination Method

Gauss Elimination Method

→ Procedure to solve the system of linear equations using Gaussian Elimination Method:

(1) Convert the given system of linear equations into matrix form as $AX = B$, where A is coefficient matrix, X is variable matrix and B is constant matrix.

(2) Convert augmented matrix [A | B] into **row echelon form**.

(3) Apply back substitution to get equations.

(4) Solve the equations to find the unknown variables.

Examples of Method-7: Gauss Elimination Method

C	1	Using Gauss Elimination Method solve the following system: $-x + 3y + 4z = 30, \quad 3x + 2y - z = 9, \quad 2x - y + 2z = 10$ **Answer**: $(2, \ 4, \ 5)$
C	2	Solve the following system of linear equations by using Gauss Elimination Method: $-\frac{1}{x} + \frac{3}{y} + \frac{4}{z} = 30, \quad \frac{3}{x} + \frac{2}{y} - \frac{1}{z} = 9, \quad \frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 10$ **Answer**: $\left(\frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{5}\right)$
C	3	Solve the following system of linear equations by using Gaussian Elimination Method: $-2b + 3c = 1, \quad 3a + 6b - 3c = -2, \quad 6a + 6b + 3c = 5$ **Answer**: **The given system has no soluton.**
C	4	Solve the given system of linear equations by using Gauss Elimination Method. $x_1 - 2x_2 + 3x_3 = -2, \quad -x_1 + x_2 - 2x_3 = 3, \quad 2x_1 - x_2 + 3x_3 = -7$ **Answer**: $\{ (-4 - t, \ t - 1, \ t) \mid t \in \mathbb{R} \}$

C	5	Find the solution set of the given system of linear equations by using Gauss Elimination Method. $x_1 + 3x_2 - 2x_3 + 2x_5 = 0,\quad 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1,$ $5x_3 + 10x_4 + 15x_6 = 5,\quad 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6.$ **Answer**: $\left\{\left(-4s - 3t - 2r,\ t,\ -2s,\ s,\ r,\ \frac{1}{6}\right) \mid r,\ s,\ t \in \mathbb{R}\right\}$
C	6	The augmented matrix of a linear system has the form $\begin{bmatrix} -2 & 3 & 1 & : & a \\ 1 & 1 & -1 & : & b \\ 0 & 5 & -1 & : & c \end{bmatrix}$. (1) Determine when the linear system is consistent. (2) Determine when the linear system is inconsistent. (3) Does the linear system have a unique solution or infinitely many solutions? Here, $a, b, c \in \mathbb{R}$. **Answer**: (1) $a + 2b - c = 0$, (2) $a + 2b - c = 0 \neq 0$, (3) **Infinitely many solutions**
C	7	For which value of λ and k, the following system has (i) unique solution, (ii) no solution and (iii) infinitely many solutions? $x + y + z = 6,\quad x + 2y + 3z = 10,\quad x + 2y + \lambda z = k$ **Answer**: (i) $\lambda \neq 3,\ k \in \mathbb{R}$ ⇝ **unique solution** (ii) $\lambda = 3,\ k \neq 10$ (or $k \in \mathbb{R} \setminus \{10\}$) ⇝ **no solution** (iii) $\lambda = 3,\ k = 10$ ⇝ **infinitely many solutions**
C	8	Find the real value of λ for which given equations have solution other than $x = y = z = 0$. Also, find the solution for each real value of λ. $(1-\lambda)x - y + z = 0,\quad 2x + (1-\lambda)y = 0,\quad 2y - (1-\lambda)z = 0$ **Answer**: **For $\lambda = 1$, the given system has solution other than $x = y = z = 0$.** **Solution**: $(x,\ y,\ z) = (0,\ t,\ t);\ t \in \mathbb{R}$

Method 8 ⇝ Gauss Jordan Elimination Method

Gauss Jordan Elimination Method

→ Procedure to solve the system of linear equations using Gauss Jordan Method:

(1) Convert the given system of linear equations into matrix form as $AX = B$, where A is coefficient matrix, X is variable matrix and B is constant matrix.

(2) Convert augmented matrix [A | B] into **reduced row echelon form**.

(3) Apply back substitution to get equations.

(4) Solve the equations to find the unknown variables.

Examples of Method-8: Gauss Jordan Elimination Method

C	1	Solve the following system of linear equations using Gauss – Jordan Method. $x + y + 2z = 8, \quad -x - 2y + 3z = 1, \quad 3x - 7y + 4z = 10$ **Answer**: $(\mathbf{3}, \ \mathbf{1}, \ \mathbf{2})$
C	2	Solve the following system of linear equations using Gauss – Jordan Method. $-2y + 3z = 1, \quad 3x + 6y - 3z = -2, \quad 6x + 6y + 3z = 5$ **Answer**: **No solution**
C	3	Find the solution set of following system by using Gauss – Jordan Method. $2x_1 + 2x_2 - x_3 + x_5 = 0, \quad -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0,$ $x_1 + x_2 - 2x_3 - x_5 = 0, \quad x_3 + x_4 + x_5 = 0$ **Answer**: $\{\ (\mathbf{-s}, \ \mathbf{-t}, \ \mathbf{s}, \ \mathbf{-t}, \ \mathbf{0}, \ \mathbf{t}) \mid \mathbf{s}, \mathbf{t} \in \mathbb{R}\ \}$

* * * * * **End of the Unit** * * * * *