CheggSolutions - Thegdp

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## Q2. Proof of Measurability Using a Dense Subset

**Given:** Let $(X, \mathcal{S}, \mu)$ be a measure space and let $A$ be a dense subset of $\mathbb{R}$. Show that a function $f: X \rightarrow \mathbb{R}$ is measurable if and only if $\{x \in X : f(x) \geq a\}$ is measurable for each $a \in A$.

### Step 1: Given and Introduction

Given a function $f: X \rightarrow \mathbb{R}$.

Need to show $f$ is measurable if and only if $\{x \in X : f(x) \geq a\}$ is measurable for each $a \in A$.

**Measurability Definition:** A function $f$ is measurable if for every Borel set $B \subset \mathbb{R}$, $f^{-1}(B)$ is in $\mathcal{S}$.

### Step 2: Necessity

Assume $f$ is measurable.

Show $\{x \in X : f(x) \geq a\}$ is measurable for each $a \in A$.

If $f$ is measurable, then $\{x \in X : f(x) \geq a \} = f^{-1}([a, \infty))$ is measurable for every $a \in \mathbb{R}$.

### Step 3: Sufficiency

Assume $\{x \in X : f(x) \geq a\}$ is measurable for each $a \in A$.

Need to show $\{x \in X : f(x) \geq t\}$ is measurable for each $t \in \mathbb{R}$.

Since $A$ is dense in $\mathbb{R}$, for any $t \in \mathbb{R}$, a sequence $\{a_n\}$ exists such that $a_n \rightarrow t$.

$\{ x \in X : f(x) \geq t \} = \bigcup_{n=1}^{\infty} \{ x \in X : f(x) \geq a_n \}$ or $\{ x \in X : f(x) \geq t \} = \bigcap_{n=1}^{\infty} \{ x \in X : f(x) \geq a_n \}$.

### Step 4: Conclusion for Proof

As $\{ x \in X : f(x) \geq a \}$ is measurable for each $a \in A$, each $\{x \in X : f(x) \geq t \}$ is measurable for any $t \in \mathbb{R}$.

Hence, $f$ is measurable, i.e., $f^{-1}(B) \in \mathcal{S}$ for any Borel set $B \subset \mathbb{R}$.

### Final Solution for Q2:

The function $f: X \rightarrow \mathbb{R}$ is measurable if and only if $\{ x \in X : f(x) \geq a \}$ is measurable for each $a \in A$, where $A$ is a dense subset of $\mathbb{R}$.

## Q3. Example of Non-Lebesgue Measurable Function

**Given:** Provide an example of a non-Lebesgue measurable function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $|f|$ is measurable, and $f^{-1}(\{a\})$ is measurable for each $a \in \mathbb{R}$.
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Step 1: Given and Introduction

Provide an example satisfying the given properties.

Let the Vitali set V be a subset of $\mathbb{R}$ that is not Lebesgue measurable.

Construct a function $f: \mathbb{R} \rightarrow \mathbb{R}$ that is based on V.

Step 2: Defining the Function

Define $f(x) = \chi_V(x)$, where χ_V is the characteristic function of V.

$\chi_V(x) = 1$ if $x \in V$

$\chi_V(x) = 0$ if $x \not\in V$

Step 3: Checking Measurability of $|f|$

Compute $|f|(x) = | \chi_V(x)|$.

Since $\chi_V(x)$ only takes values 0 or 1, $|f|(x)$ is always 0 or 1, both of which are Borel sets.

Hence, $|f|(x)$ is measurable.

Step 4: Checking Preimage of Points

For any $a \in \mathbb{R}$,

If $a = 1$, $f^{-1}(\{1\}) = \{x \in \mathbb{R} : \chi_V(x) = 1\} = V$, which is not measurable.

If $a = 0$, $f^{-1}(\{0\}) = \{x \in \mathbb{R} : \chi_V(x) = 0\} = \mathbb{R} \setminus V$, which is measurable since it is the complement of a non-measurable set.

Final Solution for Q3:

The function $f(x) = \chi_V(x)$, where V is a Vitali set, is a non-Lebesgue measurable function that satisfies $|f|$ being measurable and $f^{-1}(\{a\})$ is measurable for each $a \in \mathbb{R}$.

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