CheggSolutions - Thegdp

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## Subject: Differential Equations

### Topic: Solving Linear Differential Equations with Initial Conditions

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**Given:**

The linear differential equation:

$$\frac{d^2 y(t)}{dt^2} + 2 \frac{d y(t)}{dt} = \frac{d x(t)}{dt} + x(t)$$

Initial conditions:

$$y(0) = 2, \quad \dot{y}(0) = 1$$

Input function:

$$x(t) = u(t)$$

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**Step 1: Introduction and Initial Setup**

To solve the given differential equation, it is necessary to find both the homogeneous solution (response to the characteristic equation) and particular solution (response to the input $x(t)$).

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**Step 2: Homogeneous Solution**

First, solve the homogeneous part of the differential equation:

$$\frac{d^2 y(t)}{dt^2} + 2 \frac{d y(t)}{dt} = 0$$

Assume the solution is:

$$y_h(t) = e^{rt}$$

Substitute into the homogeneous equation:

$$r^2 e^{rt} + 2r e^{rt} = 0$$

Divide by $e^{rt}$:

$$r^2 + 2r = 0$$

Solve for $r$:

$$r(r + 2) = 0$$

$$r = 0, \quad r = -2$$

Homogeneous solution:

$$y_h(t) = C_1 + C_2 e^{-2t}$$

Explanation: Homogeneous solutions solve differential equations without considering external input. The
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characteristic equation $r^2 + 2r = 0$ determines the values of r.

Step 3: Particular Solution

The particular solution $y_p(t)$ responds to the input $x(t)$. Given $x(t) = u(t)$, a suitable particular solution can be guessed. Assume:

$$y_p(t) = A e^{\alpha t}$$

Given $x(t) = u(t)$, differentiate and substitute into the original equation:

$$x(t) = u(t)$$

For $t \geq 0$:

$$x(t) = u(t) = 1$$

(which implies: $\frac{dx(t)}{dt} = 0$ for $t > 0$).

Substitute values into differential equation:

$$\frac{d^2 y_p}{dt^2} + 2 \frac{d y_p}{dt} = x(t) + \frac{dx(t)}{dt}$$

Particular solution assumed constant $y_p(t) = A$:

The equation simplifies to:

$$0 + 0 = A$$

$$A = 1$$

$$y_p(t) = 1$$

Step 4: Total Solution

Combine homogeneous and particular solutions:

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = (C_1 + C_2 e^{-2t}) + 1$$

$$y(t) = C_1 + C_2 e^{-2t} + 1$$

Explanation: Total solution is the superposition of the homogeneous and particular solutions.

Step 5: Apply Initial Conditions

Use initial conditions to determine C_1 and C_2.

1. $y(0) = 2$:

$$2 = C_1 + C_2 e^{0} + 1$$

$$2 = C_1 + C_2 + 1$$

$$C_1 + C_2 = 1$$

2. $\dot{y}(0) = 1$:

$$\dot{y}(t) = -2 C_2 e^{-2t}$$

$$1 = \dot{y}(0) = -2 C_2 e^{0}$$

$$1 = -2 C_2$$

$$C_2 = -\frac{1}{2}$$

Solve for C_1:

$$C_1 - \frac{1}{2} = 1$$

$$C_1 = 1.5$$

Explanation: Given initial conditions allow solving constants to provide a specific solution.

Step 6: Final Solution

Substituting C_1 and C_2 into total solution:

$$y(t) = 1.5 - \frac{1}{2} e^{-2t} + 1$$

$$y(t) = 2.5 - 0.5 e^{-2t}$$

Explanation: The final solution incorporates all findings and meets initial conditions.

Final Solution:

$$y(t) = 2.5 - 0.5 e^{-2t}$$