CheggSolutions - Thegdp

```html

## Sub-Subject: Differential Equations - Initial Value Problems

### Given and Introduction

Given an initial value problem modeling the motion of a spring-mass-dashpot system, the task is to solve the differential equation:

$$ m y'' + c y' + k y = F(t), $$

$$ y(0) = 0, \quad y'(0) = 0 $$

Where the parameters are:

- $m = 2$ kilograms
- $c = 8$ kilograms per second
- $k = 80$ Newtons per meter
- $F(t) = \begin{cases} 30 & \text{if} \ 0 \le t \le \frac{\pi}{2}, \\ 0 & \text{if} \ t > \frac{\pi}{2} \end{cases}$

### Step 1: Formulate the Differential Equation

Starting with the given differential equation:

$$ 2 y'' + 8 y' + 80 y = F(t) $$

The differential equation governs the system's motion where the mass m, damping coefficient c, and spring constant k are given.

### Step 2: Solve for $0 \le t \le \frac{\pi}{2}$

For $0 \le t \le \frac{\pi}{2}$:

Given $F(t) = 30$,

The differential equation becomes:

$$ 2 y'' + 8 y' + 80 y = 30 $$

### Step 3: Find the Complementary Solution

First, solve the homogeneous equation:

$$ 2 y'' + 8 y' + 80 y = 0 $$

The characteristic equation is:

$$ 2r^2 + 8r + 80 = 0 $$

$$ r^2 + 4r + 40 = 0 $$

Solving for r using the quadratic formula $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$ r = \frac{-4 \pm \sqrt{16 - 160}}{2} = \frac{-4 \pm \sqrt{-144}}{2} = -2 \pm 6i $$

Therefore, the complementary solution is:

$$ y_c(t) = e^{-2t} ( C_1 \cos(6t) + C_2 \sin(6t) ) $$

### Step 4: Particular Solution

Assuming the particular solution $y_p$ of the non-homogeneous equation, try a constant solution since the right-hand side is a constant 30:

$$ y_p = A $$

Substitute $y_p$ into $2 y'' + 8 y' + 80 y = 30$:

$$ 80A = 30 $$

$$ A = \frac{30}{80} = \frac{3}{8} $$

Thus, the particular solution is:

$$ y_p = \frac{3}{8} $$

### Step 5: General Solution

The general solution is:
```

$$ y(t) = y_c(t) + y_p = e^{-2t} (C_1 \cos(6t) + C_2 \sin(6t)) + \frac{3}{8} $$

Step 6: Applying Initial Conditions for $0 \le t \le \frac{\pi}{2}$

At $t = 0$:

$$ y(0) = 0 \Rightarrow C_1 + \frac{3}{8} = 0 \Rightarrow C_1 = -\frac{3}{8} $$

$$ y'(0) = 0 \Rightarrow -2 C_1 + 6 C_2 = 0 \Rightarrow -2 \left(-\frac{3}{8} \right) + 6 C_2 = 0 \Rightarrow \frac{3}{4} + 6 C_2 = 0 \Rightarrow C_2 = -\frac{1}{8} $$

Thus,

$$ y(t) = e^{-2t} \left(-\frac{3}{8} \cos (6t) - \frac{1}{8} \sin (6t) \right) + \frac{3}{8} $$

This gives:

$$ y(t) = \frac{3}{8} \left(1 - e^{-2t} \left(\cos (6t) + \frac{1}{3} \sin (6t) \right) \right) $$

Step 7: For $t > \frac{\pi}{2}$

For $t > \frac{\pi}{2}$,

The forced response $F(t) = 0$, so,

$$ 2y'' + 8y' + 80y = 0 $$

The complementary solution for the homogeneous equation in this region remains:

$$ y(t) = e^{-2t} (A \cos (6t) + B \sin (6t)) $$

Step 8: Determining Coefficients for Continuity

At $t = \frac{\pi}{2}$:

$$ y \left(\frac{\pi}{2} \right) = \frac{3}{8} \left(1 - e^{-\pi} \left(\cos(3 \pi) + \frac{1}{3} \sin (3 \pi) \right) \right) $$

$$ y' \left(\frac{\pi}{2} \right) = \text{Derivative of } \frac{3}{8} \left(1- e^{-2t} \left(\cos (6t) + \frac{1}{3} \sin (6t) \right) \right) $$

Step 9: Long-term Behavior ($t \rightarrow \infty$)

$$ \lim_{t \to \infty} y(t) = 0 $$

Hence, the long term behavior for large positive values:

Final Solution

For $0 \le t \le \frac{\pi}{2}$:

$$ y(t) = e^{-2t} \left(-\frac{3}{8} \cos(6t) - \frac{1}{8} \sin(6t) \right) + \frac{3}{8} $$

For $t > \frac{\pi}{2}$:

$$ y(t) = e^{-2t} (A \cos(6t) + B \sin(6t)) $$

Where A and B are determined for continuity at $t = \frac{\pi}{2}$:

Long term behavior:

$$ \lim_{t \to \infty} y(t) = 0 $$