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# Calculus - Line Integrals

## Topic: Scalar Line Integrals

**Given:**

1. The mass of the curve given by $\mathbf{r}(t) = [3 \cos(t), 5 \sin(t)], 0 \leq t \leq \pi$ is to be computed with a density $\rho(x, y) = x^2 + y^2$.
   *Hint:* Recall integration by parts: $$\int_a^b f(x) \cdot g'(x) dx = [f(x) \cdot g(x)]_a^b - \int_a^b f'(x) \cdot g(x) dx$$
2. The perimeter of a cardioid parameterized by $\mathbf{r}(t) = (\cos(t) (1 - \cos(t)), \sin(t) (1 - \cos(t)))$ is to be computed for $-\pi \leq t \leq \pi$.
   *Hint:* $\cos \left( \frac{\theta}{2} \right) = \sqrt{\frac{1 + \cos(\theta)}{2}}$

## Solution:

### Part (a):

**Step 1: Introduction and Given**

Given the parameterized curve $\mathbf{r}(t) = [3 \cos(t), 5 \sin(t)], 0 \leq t \leq \pi$.

The density function is $\rho(x, y) = x^2 + y^2$.

**Step 2: Determine the differential length $ds$**

Compute $\mathbf{r}'(t)$ and then find $ds$.

$$\mathbf{r}(t) = [3 \cos(t), 5 \sin(t)]$$

$$\mathbf{r}'(t) = [-3 \sin(t), 5 \cos(t)]$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3 \sin(t))^2 + (5 \cos(t))^2} = \sqrt{9 \sin^2(t) + 25 \cos^2(t)}$$

Explanation: The derivative $\mathbf{r}'(t)$ represents the rate of change of $\mathbf{r}(t)$, and its magnitude gives the differential length $ds$.

**Step 3: Simplify the expression for $ds$**

$$\|\mathbf{r}'(t)\| = \sqrt{9 \sin^2(t) + 25 \cos^2(t)}$$

Explanation: The differential length $ds$ is required for the line integral calculation. The expression is simplified to use in the integral.

**Step 4: Calculate the density $\rho(\mathbf{r}(t))$**

$$\rho(\mathbf{r}(t)) = \rho(3 \cos(t), 5 \sin(t)) = (3 \cos(t))^2 + (5 \sin(t))^2 = 9 \cos^2(t) + 25 \sin^2(t)$$

Explanation: The density function is evaluated at $\mathbf{r}(t)$ to integrate along the curve.

**Step 5: Set up the integral for the mass**

$$m = \int_0^\pi \rho(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt$$

$$m = \int_0^\pi (9 \cos^2(t) + 25 \sin^2(t)) \sqrt{9 \sin^2(t) + 25 \cos^2(t)} dt$$

Explanation: The total mass is the line integral of the density function times the differential length.

**Final Solution for Part (a):**

$$m = \int_0^\pi (9 \cos^2(t) + 25 \sin^2(t)) \sqrt{9 \sin^2(t) + 25 \cos^2(t)} dt$$

### Part (b):

**Step 1: Introduction and Given**

Given the parameterized curve $\mathbf{r}(t) = (\cos(t) (1 - \cos(t)), \sin(t) (1 - \cos(t))), -\pi \leq t \leq \pi$.

**Step 2: Determine the differential length $ds$**

Compute $\mathbf{r}'(t)$ and then find $ds$.

$$\mathbf{r}(t) = (\cos(t) (1 - \cos(t)), \sin(t) (1 - \cos(t)))$$

$$\mathbf{r}'(t) = \left[-\sin(t)(1 - \cos(t)) - \cos(t) \sin(t), \cos(t)(1 - \cos(t)) \right]$$

$$\|\mathbf{r}'(t)\| = \sqrt{[-\sin(t)(1 - \cos(t)) - \cos(t) \sin(t)]^2 +[\cos(t)(1 - \cos(t))]^2} = \sqrt{[sec(t)]^3- tan(t)}$$

Explanation: The derivative $\mathbf{r}'(t)$ represents the rate of change of $\mathbf{r}(t)$, and its magnitude gives the differential length $ds$.
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Step 3: Simplify the expression for ds

$$ \|\mathbf{r}'(t)\| = \sqrt{[\sin^2(t)(sec(t))]} $$

Explanation: The simplification helps to set up the integral for the perimeter calculation.

Step 4: Set up the integral for the perimeter

$$ P = \int_{-\pi}^\pi \|\mathbf{r}'(t)\| dt $$

$$ P = \int_{-\pi}^\pi \sqrt{[\sin^2(t)(sec(t))]} dt $$

Explanation: The total perimeter is the line integral of the differential length.

Final Solution for Part (b):

$$ P = \int_{-\pi}^\pi \sqrt{[\sin^2(t)(sec(t))]} dt $$

Summary:

The mass of the curve is `\( m = \int 0^\pi (9 \cos^2(t) + 25 \sin^2(t)) \sqrt{9 \sin^2(t) + 25 \cos^2(t)} dt \)`.

The perimeter of the cardioid is `\( P = \int_{-\pi}^\pi \sqrt{\sin^2(t)(sec(t))} dt \)`.

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