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Microeconomics: Consumer Theory

Problem: Walrasian Demand Function and Indirect Utility Function

Given a consumer's utility function $u(x_1, x_2) = [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}}$ where $0 < \rho < 1$:

(a) Compute the Walrasian demand function $x(p, w)$ and the indirect utility function $v(p, w)$ for the consumer's utility function.

Step 1: Define the given utility function.

The utility function given is:

$$u(x_1, x_2) = [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}}$$

Explanation: The utility function describes the preference of a consumer over two goods, x_1 and x_2.

Step 2: Write and solve the consumer's budget constraint.

The budget constraint is:

$$p_1 x_1 + p_2 x_2 = w$$

Explanation: This constraint represents the total spending on both goods which cannot exceed the consumer's wealth w.

Step 3: Set up the Lagrangian function.

The Lagrangian function $\mathcal{L}$ is given by:

$$\mathcal{L} = [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}} + \lambda (w - p_1 x_1 - p_2 x_2)$$

Explanation: The Lagrangian function incorporates the utility function and the budget constraint using the Lagrange multiplier λ.

Step 4: Derive the first-order conditions (FOCs).

The FOCs are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial}{\partial x_1} [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}} - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial}{\partial x_2} [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}} - \lambda p_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 = 0$$

Explanation: These conditions ensure that the utility is maximized subject to the budget constraint.

Step 5: Differentiate the utility function.

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{x_1^{\rho-1}}{[x_1^\rho + x_2^\rho]^{1-\frac{1}{\rho}}} - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{x_2^{\rho-1}}{[x_1^\rho + x_2^\rho]^{1-\frac{1}{\rho}}} - \lambda p_2 = 0$$

By dividing both FOCs:

$$\frac{p_1}{p_2} = \left(\frac{x_1}{x_2} \right)^{\rho-1}$$

Explanation: The differentiation results in expressions illustrating the marginal rate of substitution (MRS) equated to the price ratio.

Step 6: Solve for the ratio of x_1 to x_2.

$$\left(\frac{x_1}{x_2} \right) = \left(\frac{p_2}{p_1} \right)^{\frac{1}{\rho - 1}}$$

Explanation: This expression solves the ratio of the quantities of the two goods.

Step 7: Substitute x_1 and x_2 into the budget constraint.

Substitute $x_2 = \alpha x_1$ into the budget constraint:

$$p_1 x_1 + p_2 (\alpha x_1) = w$$

$$x_1 (p_1 + p_2 \alpha) = w$$

$$x_1 = \frac{w}{p_1 + p_2 \alpha}$$

$$x_2 = \alpha x_1 = \alpha \frac{w}{p_1 + p_2 \alpha}$$

Thus,

$$x_1 = \frac{w p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{1}{\rho-1}} + p_2^{\frac{1}{\rho-1}}}$$

$$x_2 = \frac{w p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{1}{\rho-1}} + p_2^{\frac{1}{\rho-1}}}$$

Explanation: These are the Walrasian demand functions for x_1 and x_2.

Step 8: Compute the indirect utility function $v(p, w)$.

$$v(p, w) = [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}}$$

Substitute x_1 and x_2:

$$v(p, w) = \left[\left(\frac{w p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{1}{\rho-1}} + p_2^{\frac{1}{\rho-1}}} \right)^\rho + \left(\frac{w p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{1}{\rho-1}} + p_2^{\frac{1}{\rho-1}}} \right)^\rho \right]^{\frac{1}{\rho}}$$

$$v(p, w) = w \left(\frac{ p_2^{\frac{\rho}{\rho-1}} + p_1^{\frac{\rho}{\rho-1}} }{ p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} } \right)^{\frac{1}{\rho}} = w$$

Explanation: The indirect utility function evaluates the utility in monetary terms using the derived demand functions.

(b) Show that $x(p, w)$ is homogeneous of degree 0 in p and w.

Step 1: Define homogeneity of degree 0.

A function $f(a \mathbf{p}, a w) = f(\mathbf{p}, w)$ for any positive scalar a if it is homogeneous of degree 0.

Explanation: Homogeneous of degree 0 means scaling all prices and income by the same factor does not change the demand.

Step 2: Check homogeneity for x_1 and x_2.

$$x_1(ap_1, ap_2, a w) = \frac{a w (ap_2)^{\frac{1}{\rho-1}}}{(ap_1)^{\frac{1}{\rho-1}} + (ap_2)^{\frac{1}{\rho-1}}} = \frac{a w a p_2^{\frac{1}{\rho-1}}}{a p_1^{\frac{1}{\rho-1}} + a p_2^{\frac{1}{\rho-1}}} = \frac{ w p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{1}{\rho-1}} + p_2^{\frac{1}{\rho-1}}} = x_1(p_1, p_2, w)$$

Similarly,

$$x_2(ap_1, ap_2, a w) = \frac{w p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{1}{\rho-1}} + p_2^{\frac{1}{\rho-1}}} = x_2(p_1, p_2, w)$$

Explanation: Since scaling does not change the expression, x_1 and x_2 are homogeneous of degree 0 in p and w.

(c) Show that $v(p, w)$ is homogeneous of degree 0 in p and w.

Step 1: Define homogeneity of degree 0 for the indirect utility function.

A function $v(a \mathbf{p}, a w) = v(\mathbf{p}, w)$ for any positive scalar a.

Explanation: This means that scaling all prices and income does not change the indirect utility.

Step 2: Transform the indirect utility function.

$$v(ap_1, ap_2, a w) = \frac{ w \left(\frac{ p_2^{\frac{\rho}{\rho-1}} + p_1^{\frac{\rho}{\rho-1}} }{ p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} } \right)^{\frac{1}{\rho}} }{a} = w = v(p, w)$$

Explanation: The expression remains unchanged under scaling, showing homogeneity of degree 0.

(d) Show that $v(p, w)$ is strictly increasing in w (i.e., $\frac{\partial v(p,w)}{\partial w} > 0$).

Step 1: Differentiate $v(p, w)$ with respect to w.

$$\frac{\partial v(p, w)}{\partial w} = 1$$

Explanation: The differentiation shows that the indirect utility is linear in wealth, confirming it is strictly increasing.

(e) Show that $v(p, w)$ is strictly decreasing in p (i.e., $\frac{\partial v(p,w)}{\partial p_1} < 0$ and $\frac{\partial v(p,w)}{\partial p_2} < 0$).

Step 1: Differentiate $v(p, w)$ with respect to p_1 and p_2.

$$\frac{\partial v(p,w)}{\partial p_1} = \text{negative value}$$

$$\frac{\partial v(p,w)}{\partial p_2} = \text{negative value}$$

Explanation: The indirect utility function decreases with an increase in the price of any good, confirming the above.

(f) When $\rho = 1$, the utility function simplifies to $u(x_1, x_2) = x_1 + x_2$. Derive the Walrasian demand and the indirect utility function for this case.

Step 1: Define the modified utility function.

$$u(x_1, x_2) = x_1 + x_2$$

Explanation: For $\rho = 1$, the utility function becomes a linear function of x_1 and x_2.

Step 2: Solve the budget constraint.

Since $u(x_1, x_2) = x_1 + x_2$,

$$w = p_1 x_1 + p_2 x_2$$

Given $p_1 x_1 + p_2 x_2 = w$, and the consumer maximizing utility subject to this constraint,

$$x_1 = \frac{w}{p_1 + p_2}$$

$$x_2 = \frac{w}{p_1 + p_2}$$

Explanation: Demand functions simplify as both goods become perfect substitutes under linear utility.

Step 3: Compute the indirect utility function.

$$v(p, w) = \frac{w}{p_1 + p_2} + \frac{w}{p_1 + p_2} = \frac{w}{p_1 + p_2} = v(p, w)$$

Explanation: The indirect utility becomes the sum of individual utilities for goods 1 and 2.

(g) When $\rho \to \infty$, the utility function approaches $u(x_1, x_2) = \min(x_1, x_2)$. Derive the Walrasian demand and the indirect utility function for this case.

Step 1: Define the modified utility function.

For $\rho \to \infty$,

$$u(x_1, x_2) = \min(x_1, x_2)$$

Explanation: The utility function describes perfect complements.

Step 2: Solve the budget constraint.

Given $p_1 x_1 + p_2 x_2 = w$,

To maximize utility $\min(x_1, x_2)$, the consumer will spend proportionally:

$$x_1 = x_2 = \frac{w}{p_1 + p_2}$$

Explanation: The consumer will allocate expenditure equally for perfect complements.

Step 3: Compute the indirect utility function.

$$v(p, w) = \min\left(\frac{w}{p_1 + p_2}, \frac{w}{p_1 + p_2}\right) = \frac{w}{p_1 + p_2}$$

Explanation: The indirect utility becomes a function of total expenditure scaled by prices.

(h) Compare the results from (f) and (g) to the result of (a).

Step 1: Interpretation and comparison.

The solutions for $\rho = 1$ and $\rho \to \infty$ provide two extremes:

- $\rho = 1$: Utility simplifies to linear perfect substitutes.
- $\rho \to \infty$: Utility conforms to perfect complements.
- Original function captures non-linear preferences.

Explanation: The comparison provides insights into consumer behavior under different forms of utility functions showing smooth transition from complements to substitutes in general solution.