CheggSolutions - Thegdp

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## Linear Algebra

### Quadratic Forms, Eigenvalues, and Eigenvectors

**Given**

$$ f(\mathbf{x}) = 3(x_1^2 + x_2^2 + x_3^2) + 4(x_1x_2 + x_1x_3 + x_2x_3) $$

Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$.

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**(a) Find the symmetric matrix $A$ to let $f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$.**

**Step 1: Introduction and Matrix Representation**

The given quadratic form can be represented in the form $f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$, where $A$ is a symmetric matrix.

**Step 2: Constructing Matrix $A$**

The given function is

$$ f(\mathbf{x}) = 3(x_1^2 + x_2^2 + x_3^2) + 4(x_1x_2 + x_1x_3 + x_2x_3) $$.

This can be written as:

$$ f(\mathbf{x}) = 3x_1^2 + 3x_2^2 + 3x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 $$.

By comparing with $\mathbf{x}^T A \mathbf{x}$, the symmetric matrix $A$ is formed:

$$ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} $$

Since $A$ is symmetric, $A = A^T$, we have:

$$ A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} $$.

**Explanation:** Each coefficient of $x_i^2$ represents the diagonal elements, scaled by the given quantity, and each cross term $x_ix_j$ represents off-diagonal elements with appropriate scaling to maintain symmetry.

**(b) Find eigenvalues and eigenvectors of $A$. Then find a nonsingular matrix $P$ and a diagonal matrix $D$ to diagonalize $A$.**

**Step 1: Introduction to Eigenvalues and Eigenvectors**

To diagonalize $A$, find its eigenvalues and corresponding eigenvectors.

**Step 2: Finding Eigenvalues**

The eigenvalues $\lambda$ are found by solving the characteristic equation:

$$ \det(A - \lambda I) = 0 $$.

For $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$, the characteristic polynomial is:

$$ \begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 3-\lambda & 2 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0 $$.

Expanding the determinant:

$$ (3-\lambda)[(3-\lambda)(3-\lambda) - 2\cdot2] - 2[2(3-\lambda) - 2\cdot2] + 2[2(3-\lambda) - 2\cdot2] = 0, $$

$$ (3-\lambda)[(3-\lambda)^2 - 4] - 2[2(3-\lambda) - 4] + 2[2(3-\lambda) - 4] = 0, $$

$$ (3-\lambda)[\lambda^2 - 6\lambda + 9 - 4] = 0, $$
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$$ (3-\lambda)[\lambda^2 - 6\lambda + 5] = 0, $$

So the eigenvalues are:

$$ \lambda_1 = 1, \quad \lambda_2 = 5, \quad \lambda_3 = 5 $$.

Step 3: Finding Eigenvectors

For \(\lambda_1 = 1 \):

$$ (A - \lambda_1 I)\mathbf{v}_1 = 0 $$,

$$ \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ v_{1,3} \end{bmatrix} = 0 $$,

$$ 2v_{1,1} + 2v_{1,2} + 2v_{1,3} = 0 $$.

Eigenvector \(\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \).

Similarly, for \(\lambda_2 = 5 \):

$$ (A - \lambda_2 I)\mathbf{v}_2 = 0 $$,

$$ \begin{bmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{bmatrix} = 0 $$,

Eigenvectors \(\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \).

Together, the nonsingular matrix \(P \) (largest set of eigenvectors) and the diagonal matrix \(D \) (matrix with eigenvalues) are:

$$ P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} $$,

$$ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} $$.

Explanation: The matrix \(A \) is diagonalized by finding its eigenvalues and corresponding eigenvectors. Eigenvalues are the roots of the characteristic polynomial, and eigenvectors are solutions to \((A - \lambda I)\mathbf{v} = 0 \).

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