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## Subject: Calculus - Differentiation and Series Expansion

### Given Function:

$$f : \mathbb{R} \setminus \{-3\} \rightarrow \mathbb{R} : x \mapsto \frac{x}{x+3}$$

### Part (a): Determine the general expression for $f^{(n)}(x)$ for $n \geq 1$

Introduction and Given Information:

- The given function is $f(x) = \frac{x}{x+3}$.
- $\mathbb{R} \setminus \{-3\}$ denotes the domain of $f$.
- The goal is to find a general expression for the $n$-th derivative of $f$.

**First Derivative:** $f'(x)$

Using the quotient rule: $$f'(x) = \frac{(x + 3) \cdot 1 - x \cdot 1}{(x+3)^{2}} = \frac{x + 3 - x}{(x+3)^{2}} = \frac{3}{(x+3)^{2}}$$

**Second Derivative:** $f''(x)$

Using the quotient rule on $f'(x)$: $$f''(x) = \frac{d}{dx} \left( \frac{3}{(x+3)^{2}} \right) = 3 \cdot \frac{-2(x+3)}{(x+3)^4} = \frac{-6}{(x+3)^3}$$

**Third Derivative:** $f'''(x)$

Differentiate $f''(x)$: $$f'''(x) = \frac{d}{dx} \left( \frac{-6}{(x+3)^{3}} \right) = -6 \cdot \frac{-3(x+3)^2}{(x+3)^{6}} = \frac{18}{(x+3)^4}$$

**General Expression:**

$$f^{(n)}(x) = \frac{(-1)^{n+1} \cdot n! \cdot 3}{(x+3)^{n+1}}$$

**Supporting Statement:** The first and each subsequent derivative followed a recognizable pattern, leading to a quotient where the differentiation results in a multiplication by a negative power along with factorial terms.

### Part (b): Provide Taylor series of $f$ about 0

Introduction and Given Information:

Function Evaluation:

$$f(0) = \frac{0}{3} = 0$$

First Derivative:

$$f'(0) = \frac{3}{(3)^2} = \frac{1}{3}$$

Second Derivative:

$$f''(0) = \frac{-6}{3^3} = \frac{-2}{9}$$

Third Derivative:

$$f'''(0) = \frac{18}{3^4} = \frac{2}{27}$$

Taylor Series Expansion:

$$f(x) = 0 + \frac{x}{3} - \frac{2x^2}{18} + \frac{2x^3}{162} + \ldots = \frac{x}{3} - \frac{x^2}{9} + \frac{x^3}{81} + \ldots$$

**Convergence Radius:** The series converges for $|x| < 3$.

**Supporting Statement:** Each term was derived by evaluating the function and its derivatives at zero, giving a clear expansion reflecting a specific pattern with its convergence limited to $|x| < 3$.

### Part (c): Approximate $f(0.5)$ using Taylor approximation of order 3

Introduction and Given Information:

$$f(x) \approx T_3(x) = \frac{x}{3} - \frac{x^2}{9} + \frac{x^3}{81}$$

Computing Value:
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$$ f(0.5) \approx \frac{0.5}{3} - \frac{(0.5)^2}{9} + \frac{(0.5)^3}{81} $$

$$ = \frac{0.5}{3} - \frac{0.25}{9} + \frac{0.125}{81} $$

$$ \approx \frac{0.5}{3} - \frac{0.02778}{81} + \frac{0.00154}{} $$

$$ = \approx 0.16667 - 0.02778 + 0.00154 $$

$f(0.5) \approx 0.13889$

Supporting Statement: By substituting x with 0.5 in the third-order Taylor polynomial, the approximate value of $f(0.5)$ was computed demonstrating a relatively simple addition and subtraction of small fractions.

Part (d): Use Taylor's Mean Value theorem to show the maximum error <= 0.0008

Introduction and Given Information:

Taylor's Remainder Theorem:

$$ R_3(x) = \frac{f^{(4)}(c)}{4!} x^4 $$

For some c between 0 and 0.5.

$$ f^{(4)}(x) = \frac{-24}{(x+3)^{5}} $$

Bounding $f^{(4)}$ for $c \in [0, 0.5]$: \p

$$ |f^{(4)}(x)| \leq \frac{24}{3^5} = \frac{24}{243} \approx 0.09877 $$

Max Error:

$$ |R_3(x)| \leq \left(\frac{0.09877}{24} \right) \cdot (0.5^4) $$

$$ = 0.04124 \cdot 0.0625 = 0.000128 $$

|R_3(x)| <= 0.0008

Supporting Statement: Using the Taylor's remainder formula, the fourth derivative was bounded and evaluated over the specified interval, confirming the error remains below the required threshold.

Part (e): Sketch the graph of f

Intersections and Behavior Analysis:

- Zero at $x = 0$: $y = \frac{x}{x+1}$
- Crosses y-axis: $y = 0$
- Intervals of Increase/Decrease: $f'(x) = \frac{3}{(x+3)^{2}}$ Positive for $x \in \mathbb{R} \setminus \{-3\}$ i.e., increasing throughout its domain.
- Convexity/Concavity: $f''(x) = \frac{-6}{(x+3)^{3}}$ Negative $x > -3$, concavity downward.
- Asymptotes: Vertical asymptote at $x = -3$; Horizontal asymptote when \(y = 1; lim x->0 = \frac{x}{x+1} =1)

Supporting Statement: Analyzing the function and its derivatives led to the understanding of where intersections and the behavior over intervals in terms of increase and concavity. Observed asymptotic behavior was also highlighted.

Final Answer:

(a) $f^{(n)}(x) = \frac{(-1)^{n+1} n! 3}{(x+3)^{n+1}}$.
(b) Taylor Series: $\frac{x}{3} - \frac{x^2}{9} + \frac{x^3}{81} + \ldots$, Convergence Radius: $|x| < 3$.
(c) Approximation: $f(0.5) \approx 0.13889$.
(d) Maximum Error: $|R_3(x)| \leq 0.000128$.
(e) The graph intersects axes at (0,0), increasing in entire domain, concave downward $x > -3$ and asymptotes at $x = -3$ vertically and y = 1 horizontally showing the graph sketch.

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