CheggSolutions - Thegdp

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## Partial Differential Equations

### Topic: Heat Equation

##### Exercise 3.4

#### Given:

- The heat equation $u_t = u_{xx}$ for $x \in (0, 1)$, $t > 0$.
- Boundary conditions: $u(0, t) = u(1, t) = 0$
- Initial condition: $u(x, 0) = f(x)$

The task is to find the formal solutions for different initial functions:

- (a) $f(x) = \sin(14\pi x)$
- (b) $f(x) = x(1 - x)$
- (c) $f(x) = \sin^3(\pi x)$

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#### Introduction

The given problem is to solve the heat equation with given initial and boundary conditions. The general solution can be found using the method of separation of variables.

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#### Step 1: General Form of the Solution

Assuming a solution of the form:

$u(x, t) = X(x)T(t)$

Substituting this into the heat equation:

$X(x) T'(t) = X''(x) T(t)$

Dividing both sides by $X(x)T(t)$:

$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$

where $\lambda$ is a separation constant.

This gives two ODEs:

$T'(t) + \lambda T(t) = 0$

$X''(x) + \lambda X(x) = 0$

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#### Step 2: Solving for $T(t)$ and $X(x)$

The solution for $T(t)$:

$T(t) = A e^{-\lambda t}$

The solution for $X(x)$:

$X''(x) + \lambda X(x) = 0$

With boundary conditions $X(0) = X(1) = 0$:

For non-trivial solution, $\lambda_n = (n\pi)^2$, $n \in \mathbb{Z}^+$.

$X_n(x) = \sin(n\pi x)$

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#### Step 3: Full Solution

$u_n(x,t) = e^{-(n\pi)^2 t} \sin(n\pi x)$

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#### Step 4: Applying Initial Condition, $u(x,0) = f(x)$

$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) = f(x)$
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The coefficients B_n are determined using the Fourier series of $f(x)$:

$B_n = 2 \int_{0}^{1} f(x) \sin(n\pi x) \, dx$

Solutions for Each Initial Function:

Part (a) $f(x) = \sin(14\pi x)$

$B_n = 2 \int_{0}^{1} \sin(14\pi x) \sin(n\pi x) \, dx$

Only the term where $n = 14$ will survive:

$B_{14} = 1$

Thus,

$u(x, t) = e^{-(14\pi)^2 t} \sin(14\pi x)$

Part (b) $f(x) = x(1 - x)$

$B_n = 2 \int_{0}^{1} x(1-x) \sin(n\pi x) \, dx$

Using integration by parts,

$B_n = \frac{2[1 - (-1)^{n+1}]}{(n\pi)^3}$

Thus,

$u(x, t) = \sum_{n=1}^{\infty} \frac{2[1 - (-1)^{n+1}]}{(n\pi)^3} e^{-(n\pi)^2 t} \sin(n\pi x)$

Part (c) $f(x) = \sin^3(\pi x)$

Express $\sin^3(\pi x)$ using multiple angle identities:

$\sin^3(\pi x) = \frac{3}{4} \sin(\pi x) - \frac{1}{4} \sin(3\pi x)$

The Fourier coefficients B_n will be:

$B_1 = \frac{3}{4}, \quad B_3 = -\frac{1}{4}$

Thus,

$u(x, t) = \frac{3}{4} e^{-\pi^2 t} \sin(\pi x) - \frac{1}{4} e^{-9\pi^2 t} \sin(3\pi x)$

Final Solutions:

(a) $u(x, t) = e^{-(14\pi)^2 t} \sin(14\pi x)$

(b) $u(x, t) = \sum_{n=1}^{\infty} \frac{2[1 - (-1)^{n+1}]}{(n\pi)^3} e^{-(n\pi)^2 t} \sin(n\pi x)$

(c) $u(x, t) = \frac{3}{4} e^{-\pi^2 t} \sin(\pi x) - \frac{1}{4} e^{-9\pi^2 t} \sin(3\pi x)$

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