Total Response of a Linear Differential System

Given and Introduction

The given second-order linear differential equation is

$$
\frac{d^2 y(t)}{dt^2} + 2 \frac{d y(t)}{dt} = \frac{d x(t)}{dt} + x(t)
$$

with initial conditions $y(0) = 2$ and $\dot{y}(0) = 1$, and input function $x(t) = u(t)$ (where $u(t)$ is the unit step function). The objective is to find the total response $y(t)$.

Step-by-Step Solution

Step 1: Find the homogeneous solution

Consider the homogeneous equation:

$$
\frac{d^2 y_h(t)}{dt^2} + 2 \frac{d y_h(t)}{dt} = 0
$$

To solve this, assume $y_h(t) = e^{rt}$ and substitute into the homogeneous equation:

$$
r^2 e^{rt} + 2r e^{rt} = 0
$$
$$
e^{rt} (r^2 + 2r) = 0
$$
$$
r(r + 2) = 0
$$

Hence, the roots are $r = 0$ and $r = -2$. Therefore, the homogeneous solution is:

$$
y_h(t) = C_1 + C_2 e^{-2t}
$$

Explanation: Solving the characteristic equation provides the fundamental solutions that form linear combinations for free response.

Step 2: Find the particular solution

For $x(t) = u(t)$:

$$
\frac{d x(t)}{dt} = \delta(t)
$$

The differential equation becomes:

$$
\frac{d^2 y_p(t)}{dt^2} + 2 \frac{d y_p(t)}{dt} = \delta(t) + u(t)
$$

To solve for the particular solution, assume a response based on the input behavior. For $u(t)$, look for a particular solution of the form:

$$
y_p(t) = A + Bt \quad \text{for} \quad t \geq 0
$$

Taking derivatives:

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\[
\frac{d y_p(t)}{dt} = B
\]
\[
\frac{d^2 y_p(t)}{dt^2} = 0
\]
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Plug in these into the differential equation:

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\[
0 + 2B = \delta(t) + u(t)
\]
```

For \(t \geq 0 \), ignoring \(\delta(t) \),

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\[
2B = 1 \quad \Rightarrow \quad B = \frac{1}{2}
\]
```

For initial conditions and steady state:

```
\(
K = 0
\)
```

Thus, \(y_p(t) = \frac{t}{2} \).

Explanation: Focusing on the form of particular solution \(t\) solves coefficients by balancing the input type in the steady state response.

Step 3: Apply initial conditions

Combine the total solution:

```
\[
y(t) = y_h(t) + y_p(t) = C_1 + C_2 e^{-2t} + \frac{t}{2}
\]
```

Use the initial conditions \(y(0) = 2 \) and \(\dot{y}(0) = 1 \):

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1) \( y(0) = 2 \):
\[
2 = C_1 + C_2
\]
2) \( \dot{y}(0) = 1 \):
\[
\dot{y}(t) = -2C_2 e^{-2t} + \frac{1}{2}
\]
\[
\dot{y}(0) = 1 = -2C_2 + \frac{1}{2}
\]
```

Solving these:

```
- From \( y(0) \):
\[
C_2 = -\frac{1}{4}
\]
- From \( \dot{y}(0) \):
\[
C_1 = \frac{9}{4}
\]
```

Final Solution

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\[
y(t) = \frac{9}{4} - \frac{1}{4} e^{-2t} + \frac{t}{2}
\]
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Explanation: The solution is the sum of the homogenous and particular solutions with initial conditions applied to find parameters.