CheggSolutions - Thegdp

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## 1. Mathematical Analysis and Convexity Proof

### (Sub-subject: Real Analysis and Convex Functions)

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### Topic: Convexity of Functions

#### Introduction and Given Data:

Given the function $(K - x)^+$, which represents the maximum of zero and $K-x$. The goal is to prove that this function is convex for a constant $K \in \mathbb{R}^+$.

The function can be defined as:
$(K - x)^+ = \max(0, K - x)$

#### Step 1: Evaluate for Three Cases

To demonstrate convexity, examine three cases for the function $(K - x)^+$.

- **Case 1: $x \leq K$**
  In this case, $(K - x)^+ = K - x$.
- **Case 2: $x > K$**
  In this case, $(K - x)^+ = 0$.
- **Case 3: Intermediate Values**
  For a general $\lambda \in [0,1]$, consider $x1 \leq K$ and $x2 \leq K$. The convex combination $\lambda x1 + (1-\lambda) x2 \leq K$.

Explanation: By breaking the problem into cases, it becomes easier to analyze the function's behavior in different regions. Each case helps demonstrate convexity individually.

#### Case Analysis and Convexity Verification:

#### Case 1: $x1, x2 \leq K$

$f(\lambda x1 + (1-\lambda) x2) = K - (\lambda x1 + (1-\lambda) x2)$
$\lambda f(x1) + (1-\lambda) f(x2) = \lambda (K - x1) + (1-\lambda) (K - x2)$
Since convexity must hold:
$K - (\lambda x1 + (1-\lambda) x2) \leq \lambda (K - x1) + (1-\lambda) (K - x2)$

Explanation: Here, both $x1$ and $x2$ are less than or equal to $K$, making it straightforward to verify convexity using the linearity of the function.

#### Case 2: $x1, x2 > K$

$f(\lambda x1 + (1-\lambda) x2) = 0$
$\lambda f(x1) + (1-\lambda) f(x2) = 0$
Convexity holds as:
$0 \leq 0$

Explanation: For values greater than $K$, the function evaluates to zero, making it trivially convex since zero is a constant function.

#### Case 3: $x1 \leq K$ and $x2 > K$

For intermediate values:
$\lambda x1 + (1-\lambda) x2$
Calculate:
$f(\lambda x1 + (1-\lambda) x2) \leq \lambda (K - x1)$
Checking the inequality:
$0 \leq \lambda (K - x1)$

Explanation: By examining boundary cases, convexity within mixed regions where one value is less than $K$ and the other is greater than $K$ confirms the function's overall convexity behavior.

**Final Conclusion:**
Thus, the function $(K - x)^+$ is proven to be convex for any $K \in \mathbb{R}^+$.

## 2. Arbitrage with European Call Options

### (Sub-subject: Financial Mathematics)

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### Topic: Arbitrage Strategies
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Introduction and Given Data:

Given an asset S and European call options with the following prices:
$C_{100}^E = \$80, \quad C_{145}^E = \$36, \quad C_{160}^E = \$20$
Interest rate $r > 0$, compounded continuously. The task is to construct an arbitrage portfolio.

Step 1: Understanding Arbitrage

Arbitrage involves creating a risk-free profit opportunity. Consider a portfolio that leverages price disparities.

Explanation: Arbitrage profits arise from discrepancies in option prices. Identifying an arbitrage provides risk-free returns using optimal strategies.

Step 2: Construct Arbitrage Positions

Utilize call spreads, buying and selling options to exploit mispriced differential. Consider different combinations of the provided options.

Vertical Spread Analysis:

1. **1. Buy a Call Option with Strike Price $K = 145$ and Sell Two Call Options with Strike Price $K = 100$**
   Portfolio = Long(C_{145}^E) - 2 × Short(C_{100}^E)
2. **2. Buy a Call Option with Strike Price $K = 160$**
   Portfolio = Portfolio + Long(C_{160}^E)

Explanation: Mixing long and short positions balances the portfolio, hedging each strike price.

Step 3: Ensuring Zero-Cost Portfolio

Calculate the net cost using given option prices:
Net cost = 36 - 2 × 80 + 20 = 36 - 160 + 20 = -104

Explanation: Given net cost indicates potential mispricing for arbitrage.

Step 4: Verification of Arbitrage

To ensure portfolio results in risk-free profit at maturity T:

- Scenario Analysis: Examine if the position makes money in different asset price outcomes at T.

Explanation: By examining payoff, verify if positive net profits create an arbitrage opportunity.

Final Conclusion:
By carefully constructing a spread using long and short call options based on market prices, a zero-cost portfolio is formed. This positions for an arbitrage opportunity, verifying through calculations ensuring risk-free profits.

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