CheggSolutions - Thegdp

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## Dirichlet Convolution

Dirichlet convolution is a significant operation in number theory. It combines two arithmetic functions $f$ and $g$ to form a new function $(f * g)(n)$. A function $f$ is completely multiplicative if $f(mn) = f(m)f(n)$ for all positive integers $m$ and $n$. The problem at hand is to provide a counterexample demonstrating that the Dirichlet convolution of two completely multiplicative functions is not necessarily completely multiplicative.

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### Step 1: Define Dirichlet Convolution

The Dirichlet convolution of two arithmetic functions $f$ and $g$ is defined as:

$$(f * g)(n) = \Sigma(d|n)\ f(d)g(n/d)$$

*Explanation:* This formula sums the product of $f(d)$ and $g(n/d)$ over all positive divisors $d$ of $n$. It's a common operation in multiplicative number theory.

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### Step 2: Define Completely Multiplicative Functions

Consider two completely multiplicative functions as $f(n) = 1$ and $g(n) = (-1)^n$.

*Explanation:* A function $f$ being completely multiplicative means $f(mn) = f(m)f(n)$. In this case, $f(n)$ is trivially 1 for any $n$ and $g(n) = (-1)^n$, is also completely multiplicative.

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### Step 3: Compute Dirichlet Convolution

Now, compute the Dirichlet convolution $(f * g)(n)$:

$$(f * g)(1) = \Sigma(d|1)\ f(d)g(1/d) = f(1)g(1) = 1 \cdot (-1)^1 = -1$$
$$(f * g)(2) = \Sigma(d|2)\ f(d)g(2/d) = f(1)g(2) + f(2)g(1) = 1 \cdot (-1)^2 + 1 \cdot (-1)^1 = 1 - 1 = 0$$

*Explanation:* Calculation of Dirichlet convolution for each $n$ involves summing over all divisors $d$. Here, calculations demonstrate the values for $n = 1$ and $n = 2$.

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### Step 4: Check Completely Multiplicative Property

Examine if $(f * g)(mn) = (f * g)(m)(f * g)(n)$ holds in general. Consider $m = 1$ and $n = 2$:

$$(f * g)(1 \cdot 2) = (f * g)(2) = 0$$
$$(f * g)(1) \cdot (f * g)(2) = (-1) \cdot 0 = 0$$

While for $m = 1$ and $n = 3$:

$$(f * g)(1 \cdot 3) = (f * g)(3) = \Sigma(d|3)\ f(d)g(3/d) = f(1)g(3) + f(3)g(1) = 1 \cdot (-1)^3 + 1 \cdot (-1)^1 = -1 - 1 = -2$$
$$(f * g)(1) \cdot (f * g)(3) = (-1) \cdot (-2) = 2$$

*Explanation:* This demonstrates that the convolution $(f * g)(2) = 0$ but $(f * g)(1) \cdot (f * g)(2) = 0$ which appears consistent until alternative values demonstrate inconsistency.

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### Final Solution

The Dirichlet convolution $f * g(n)$ for $f(n) = 1$ and $g(n) = (-1)^n$ is not completely multiplicative. Specifically, $(f * g)(3 \cdot 1) \neq (f * g)(3) \cdot (f * g)(1)$. Thus, this configuration serves as the required counterexample.

*Explanation:* The result shows that the convolution does not retain the completely multiplicative nature, as verified by our example calculations.

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