CSOR 4231, Section 3: Analysis of Algorithms I - Problem Set #2

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Collaborators

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1 DPV 2.12

Let $L(n)$ be the total numbers of lines printed, according to the description, we have

$$L(n) = 2L(n/2) + 1, L(1) = 0$$

Using the master theorem, $L(n) = \Theta(n)$

2 DPV 2.8 and 2.9

2.8

(a) There are 4 coefficients, choose $\omega = e^{2\pi i/4} = i$. Thus:

$$FFT((1,0,0,0),\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

To find out which sequence is $(1,0,0,0)$ the FFT, we calculate the inverse FFT:

$$\frac{1}{4}FFT((1,0,0,0),w^{-1}) = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

(b) choose the same $\omega = i$.

$$FFT((1,0,1,-1),\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \\ 3 \\ -i \end{bmatrix}$$

To find out which sequence is $(1,0,1,-1)$ the FFT, we calculate the inverse FFT:

$$\frac{1}{4}FFT((1,0,1,-1),w^{-1}) = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} = 1/4 \begin{bmatrix} 1 \\ -i \\ 3 \\ i \end{bmatrix}$$

2.9

(a) $x+1 = 1+x+0x^2+0x^3$, we represent this polynomial with $(1,1,0,0)$.
$x^2+1 = 1+0x+1x^2+0x^3$, represented as $(1,0,1,0)$.
Since $n=4$, $\omega = e^{2\pi i/4} = i$. The FFT of these two sequence is:

$$FFT((1,1,0,0),\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1+i \\ 0 \\ 1-i \end{bmatrix}$$

$$FFT((1,0,1,0),\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

multiply the two results, we have the values $(4,0,0,0)$. Then the inverse FFT:

$$\frac{1}{4}FFT((4,0,0,0),w^{-1}) = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore the final result is $1+x+x^2+x^3$.

(a) $1+x+2x^2 = 1+x+2x^2+0x^3$, we represent this polynomial with $(1,1,2,0)$.
$2+3x = 2+3x+0x^2+0x^3$, represented as $(2,3,0,0)$.
Since $n=4$, $\omega = e^{2\pi i/4} = i$. The FFT of these two sequence is:

$$FFT((1,1,2,0),\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -1+i \\ 2 \\ -1-i \end{bmatrix}$$

$$FFT((2,3,0,0),\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2+3i \\ -1 \\ 2-3i \end{bmatrix}$$

multiply the two results, we have the values $(20, -5-i, -2, -5+i)$. Then the inverse FFT:

$$\frac{1}{4}FFT((20,-5-i,-2,-5+i),w^{-1}) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 20 \\ -5-i \\ -2 \\ -5+i \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \\ 6 \end{bmatrix}$$

Therefore the final result is $2 + 5x + 7x^2 + 6x^3$.

3 DPV 2.27

(a) Let $A = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}$, then $A^2 = \begin{bmatrix} a_0^2 + a_1a_2 & a_0a_1 + a_1a_3 \\ a_0a_2 + a_2a_3 & a_1a_2 + a_3^2 \end{bmatrix}$

By inspection, $A^2 = \begin{bmatrix} a_0^2 + M & a_1N \\ a_2N & M + a_3^2 \end{bmatrix}$, where $M = a_1a_2, N = a_0 + a_3$, now only 5 multiplications are sufficient to compute A^2.

(b) Using the same notation for a $n \times n$ matrix M, we write $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are all $n/2 \times n/2$ matrices (assume n is even). In this case,

$$M^2 = \begin{bmatrix} A^2 + BC & AB + BD \\ CA + CD & CB + D^2 \end{bmatrix}$$

By dividing the problem, we cannot get 5 subproblems as we get 5 multiplications for 2×2 matrix, because for scalars it holds that $a_0a_1 + a_1a_3 = a_2(a_0 + a_3)$, while for matrices, $AB + BD = B(A + D)$ does not always hold since AB is not equivalent to BA.

(c) i. Note that, $(A+B)^2 = A^2 + B^2 + AB + BA$, so $AB + BA = (A+B)^2 - A^2 - B^2$, addition and subtraction of these matrices can be done in $O(n^2)$. If $n \times n$ matrices can be squared in time $S(n) = O(n^c)$, then we can compute $AB + BA$ in time $3S(n) + O(n^2)$.

ii.

$$AB + BA = \begin{bmatrix} 0 & XY \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & XY \\ 0 & 0 \end{bmatrix}_{2n \times 2n}$$

iii. Given conclusions and results in i. and ii., for two $n \times n$ matrices X, Y, we can embed them in $2n \times 2n$ matrices A, B (as described in ii.), the result of XY is in matrix $AB + BA$, which can be computed in time $3S(2n) + O(n^2)$. Therefore the product XY can be computed in time $3S(2n) + O(n^2) = 3 \times 2^c O(n^c) + O(n^2) = O(n^c) + O(n^2) = O(n^c)$ (since we are consider possible $c < \log_2 7$, also $c > 2$ for sure).

4 DPV 3.14

A linear time algorithm goes as follow (assume the graph is represented by adjacency list):

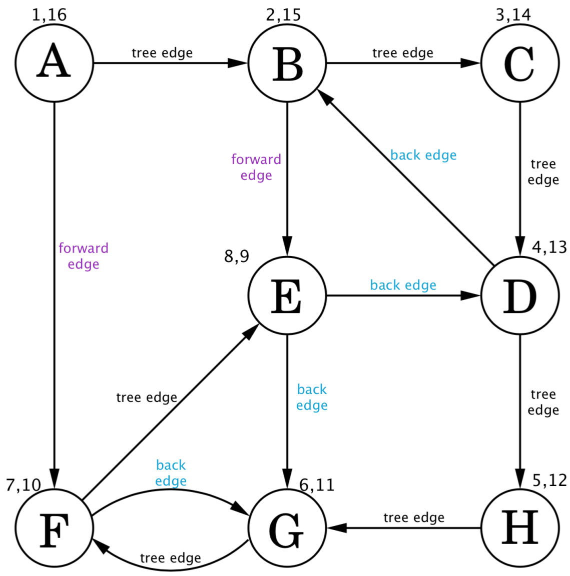
1. First, scan all edges. For each edge $e = (u, v)$, we increase the indegree(initialized to 0) of vertex v by 1.
2. Second, go through the indegrees of each vertex, add a vertex to *sources* (a list that contains all source vertices) if it has indegree of 0.
3. We loop until the *sources* list is empty, in each iteration, we pick one vertex v from the *sources* list and remove it from the list. This vertex is emitted. Then for each edges goes out from v, $e = (v, w)$, we decrease the indegree of w, if the indegree of w reaches 0, we also add it to the *sources* list.
4. When the loop ended, every vertex is emitted in a linearized order. Otherwise the graph is not topological sortable.

Note that 1. 2. 3. all run in linear time, for 3., each vertex will be in the *sources* list once and each iteration will remove one vertex from the list.

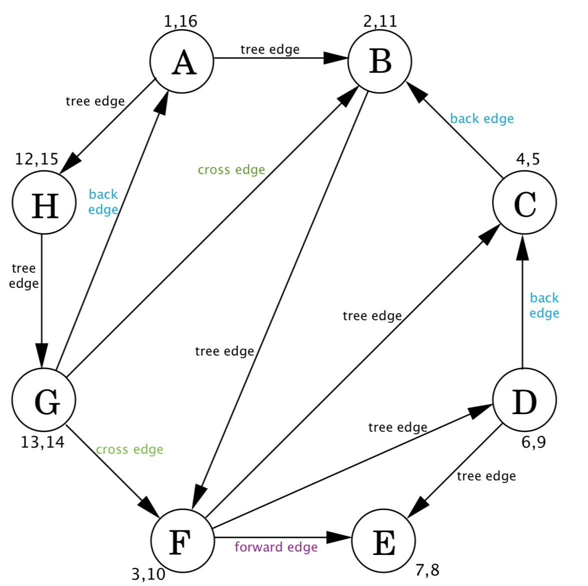
Therefore we have a linear time algorithm for linearization (topological sorting).

5 DPV 3.2

(a) See below.



(b) See below.



6

(a) *Proof.* Let f be any $monotone function$ in the grid $G(d,n)$.

Let $x_0 = (0,0,...,0)$, if $f(x_0) = (0,0,...,0) = x_0$, then x_0 is a fix point for f, we are done.

If $f(x_0) = x_1 \neq x_0$, then it must be true that $x_1 > x_0$, because according to the definition, in $G(d,n)$, x_0 is the minimum (each component of x_0 is 0). Thus

$f(x_1) \geq f(x_0)$, i.e., $f(x_1) = x_2 \geq x_1$. Then we can know that $x_3 = f(x_2) \geq f(x_1) = x_2$. In this manner, it is trivial to see that

$$\begin{aligned} x_2 = f(x_1) &\geq f(x_0) = x_1 \\ x_3 = f(x_2) &\geq f(x_1) = x_2 \\ x_4 = f(x_3) &\geq f(x_2) = x_3 \\ &\cdots \\ x_i = f(x_{i-1}) &\geq f(x_{i-2}) = x_{i-1} \end{aligned}$$

There must be some point x_k in this sequence, s.t. $f(x_k) = x_k$. I'll prove by contradiction. If no such x_k exists, then we can keep calculate x_i based on x_{i-1}:

$$\begin{aligned} x_2 = f(x_1) &> f(x_0) = x_1 \\ x_3 = f(x_2) &> f(x_1) = x_2 \\ x_4 = f(x_3) &> f(x_2) = x_3 \\ &\cdots \\ x_i = f(x_{i-1}) &> f(x_{i-2}) = x_{i-1} \\ &\cdots \end{aligned}$$

where $x_i > x_{i-1}$ is defined as at least 1 component of x_i is larger than x_{i-1}'s components and other components stay equal. since $x_i > x_{i-1}$, there is at least one component of x_i that is larger than the corresponding component of x_{i-1}. We cannot increase components of x_i infinitely, because the maximum we can have here is $(n-1, n-1, ..., n-1)$. Therefore, at some point x_k, we must have that $x_{k+1} = f(x_k) = f_(x_{k-1}) = x_k$. That x_k is a fix point of f. □

(b) *Proof.* In the proof of (a), we keep calculating x_i with the formula $x_{i+1} = f(x_i)$. In each step, we calculate $x_{i+1} = f(x_i)$, if $x_{i+1} = f(x_i) = x_i$, then we find our fixpoint. Otherwise we go next step with x_{i+2} base on x_{i+1}. Note that if $x_{i+1} = f(x_i) \neq x_i$, then at least one component of x_{i+1} is at least 1 larger than that component of x_i ($x_{i+1} > x_i$). If we don't see any fixpoint and we keep going on, then we run into $x_k = (n-1, n-1, ..., n-1)$, at this point, it must hold that $x_{k+1} = f(x_k) = x_k$, because there exists no such d-tuple x' in $G(d, n)$ s.t. $x' > x_k = (n-1, n-1, ..., n-1)$.

The maximum number of steps we can do until we get $(n-1, n-1, ..., n-1)$ is $O(nd)$. In each step, we only need to evaluate f on x_i and compare $f(x_i)$ with x_i, which can be done in $O(1)$. Therefore a fixpoint can be found in time $O(nd)$. □

(c) i) When $d = 1$, $x_0 = 0$, $x_{max} = n - 1$, also, $f(x_0) = f(0) \geq 0, f(x_{max}) = f(n-1) \leq n-1$. We can use binary search to find a fix point in such case. Note that, for any $0 \leq l \leq h \leq n-1$, if $f(l) \geq l, f(h) \leq h$, then there exists a fixpoint in $[l, h]$. This can be proved by induction.

- This is obvious if $l = h$.
- It $h = l + 1$, either $f(l) = l$ or $f(h) = h$ must holds because otherwise $f(h) \geq f(l) \geq l + 1 > l$, it will make $f(h) \geq h$, but $f(h) < h$.
- If $h > l$, this also holds when $f(l) = l$ or $f(h) = h$. Otherwise the case is $f(l) \geq l+1 > l$ and $f(h) < h$, then $f(l+1) \geq f(l) \geq l+1$, we reduce it to $[l + 1, h]$, which holds by induction hypothesis.

Binary Search:
In each step, we find the midpoint $m = l + \frac{(h-l+1)}{2}$, where initialy $l = 0, h = n - 1$. At any step, $f(l) \geq l, f(h) \leq h$ (this holds for $0, n - 1$).

If $f(m) = m$, then we are done.

If $f(m) > m$, then there must exist a fixpoint k in $[m + 1, h]$ because $f(m + 1) \geq f(m) \geq m + 1 > m$. We set $l = m + 1$

If $f(m) < m$, a fixpoint is in $[l, m-1]$, because $f(m-1) \leq f(m) \leq m-1 < m$. We set $h = m - 1$.

The **Binary Search** method can find the fixpoint in $\log n$ time.

ii) Let $p^{(k)}$ be the fix point (ignore the last coordinate)we can get by solving for $G(d - 1, n)$ with $x_d = k$ fixed.

$$p^{(k)} = (p_1^{(k)}, ..., p_{d-1}^{(k)}, k)$$

By definition:

$$f(p^{(k)}) = (p_1^{(k)}, ..., p_{d-1}^{(k)}, k')$$

Also, let $f_i(x)$ be the i th component of $f(x)$, i.e.

$$f_d(p^{(k)}) = k', p_d^{(k)} = k$$

Assumption: If we view $f_d(p^{(k)})$ as a function: $G(1, n) \to G(1, n)$, I'll assume it is monotone here.

Now I'll prove something similar to i):

for any $0 \leq l \leq h \leq n - 1$,
if $f_d(p^{(l)}) \geq l, f_d(p^{(h)}) \leq h$,
then there exists a fixpoint for f with the last coordinate in $[l, h]$

- if $l = h$, we have $f_d(p^{(l)}) = l$, which means $f_d(p^{(l)}) = p^{(l)}$, then this is a fixpoint we are looking for.
- if $h = l + 1$, either $f_d(p^{(l)}) = l$ or $f_d(p^{(h)}) = h$ must holds because otherwise $f_d(p^{(h)}) \geq f_d(p^{(l)}) \geq l + 1 > l$, it will make $f_d(p^{(h)}) \geq h$, but $f_d(p^{(h)}) < h$.
- If $h > l$, this also holds when $f_d(p^{(l)}) = l$ or $f_d(p^{(h)}) = h$. Otherwise the case is $f_d(p^{(l)}) \geq l+1 > l$ and $fph < h$, then $f_d(p^{(l+1)}) \geq f_d(p^{(l)}) \geq l+1$, we reduce it to $[l + 1, h]$, which holds by induction hypothesis.

Then we can use binary search similar to i) to find the right fix point where $f_d(p^{(x_d)}) = x_d$.