# Capstone Project: How Thick is a Three-Sided Coin?

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**Abstract**

A coin-flip is often used as a determiner of choices, with a 50% chance of landing heads and a 50% chance of landing tails. However, this percentage is theoretically inaccurate , as there is a chance, albeit slim, that the coin lands on its non-circular surface. Let's compare this to a long cylindrical rod. The probability that when flipped, the rod lands on one of its circular faces is comparatively slim when compared to the probability that it lands on its longer, round section.

The goal of this project is to determine the relationship between the diameter-height ratio of a coin and the probability it lands on either face, and using that information to answer the research question: How thick is a three-sided coin? This experiment used three sets of 10 3D-printed coins, each with different diameters and a cup to throw them onto a flat glass table to simulate random coin flips. Then one-sample Z-test for proportions was performed on the data to determine whether the proportion of coins that landed on its side was 1/3.

Then, a regression analysis was used to determine two diameters of coins that should land on its side at a 1/3 rate, using the previous data to determine the relationship between diameter and probability that a given coin lands on its side.

Future research could have a better procedure where each coin is flipped instead of being tumbled and tossed like a die, being a better representation of a conventional coin flip. In this experiment, n = 500 was used for each sample size, but in future research larger sample sizes can be used to better capture the true probability of each coin landing on its side. Another possibility for future research is a physics simulation, where potential confounding variables like a slightly sloped table or inconsistencies in 3d printed coins, or mass distributions within the coin can be eliminated.

# 1 Introduction

Whether that be who gets the ball at the beginning of an NFL football game or to determine what restaurant a group of friends decides to dine in at, Coin flips are generally seen as a random determiner of a binary choice, with 1/2 probability heads and 1/2 probability tails. However, this is not completely true, as it can land on its side.

Although one could potentially use even numbers and odd numbers on a die to make a random choice between 3 possibilities, it is still interesting to think about a cylinder, a three-sided solid as a three-sided coin.

I hypothesize that the true proportion of coin tosses that land on its side of the determined final thickness is 1/3.

# 2 Experimental Design and Procedure

## 2.1 Design

This experiment is designed with a Block design, where I 3D Printed 3 blocks of 10 coins, each block with coins of a different thickness. The diameter was held to a constant of 3/4 inches or 19.05 millimeters for all coins. Each set of 10 coins will be tossed onto the glass table using a cup and the coins that land on their side will be counted. After 50 trials, creating a sample size of $n = 500$ for each block, a two-tailed one-sample Z-test will be performed on each sample to determine whether the probability of landing on its side is 1/3. Each coin will be printed in red Hatchbox PLA on my Artillery Sidewinder X1 3D printer with identical settings: 20% infill with a cubic infill pattern with bed temperature 60° Celsius and Nozzle temperature 200° degrees Celsius. Each coin was printed individually, with Elmer's PVA glue to improve bed adhesion (ability of extruded plastic to stick to the bed), reducing potential warping with each print. An image of the 3D printer can be found in Figure 1.

Later, using these 3 data points, I will construct regression models and interpolate to determine the true thickness required for a coin to land on its side with a probability of 1/3, and repeat two-tailed one-sample Z-tests on the newly printed blocks of coins to determine if they are truly three-sided coins.

The first block contained coins whose spherical "band surface area" above each face is equal (see Figure 2). The formula for the "band surface area" of the side section of a coin is $S_B = 2\pi rh$ where $r$ is the diameter of the sphere and $h$ is the height/thickness of the cylinder. The band area $S_B$ should be equal to 1/3 of the surface area of the entire sphere, so we get the equation $2\pi rh = \frac{4\pi r^2}{3}$ and solve for $h$ using polar identities to put it in terms of $d$, the diameter of the coin. $h = \frac{d}{2\sqrt{2}}$. When we substitute 19.05 mm for $d$, we are left with a height of 6.735 mm.

The second block contained coins whose height was determined by the arc lengths above the side-section of the coins were equal to the arc-lengths above the circular faces of the coins, leading to a thickness that is $h = \frac{d}{\sqrt{3}}$. When we substitute 19.05 mm for $d$, we are left with a height 10.999 mm.

The third block of cylinders contained coins with thickness 8 mm and diameter 19.05 mm, a guess that I made in between to more accurately determine the relationship between thickness and probability of landing on its side.

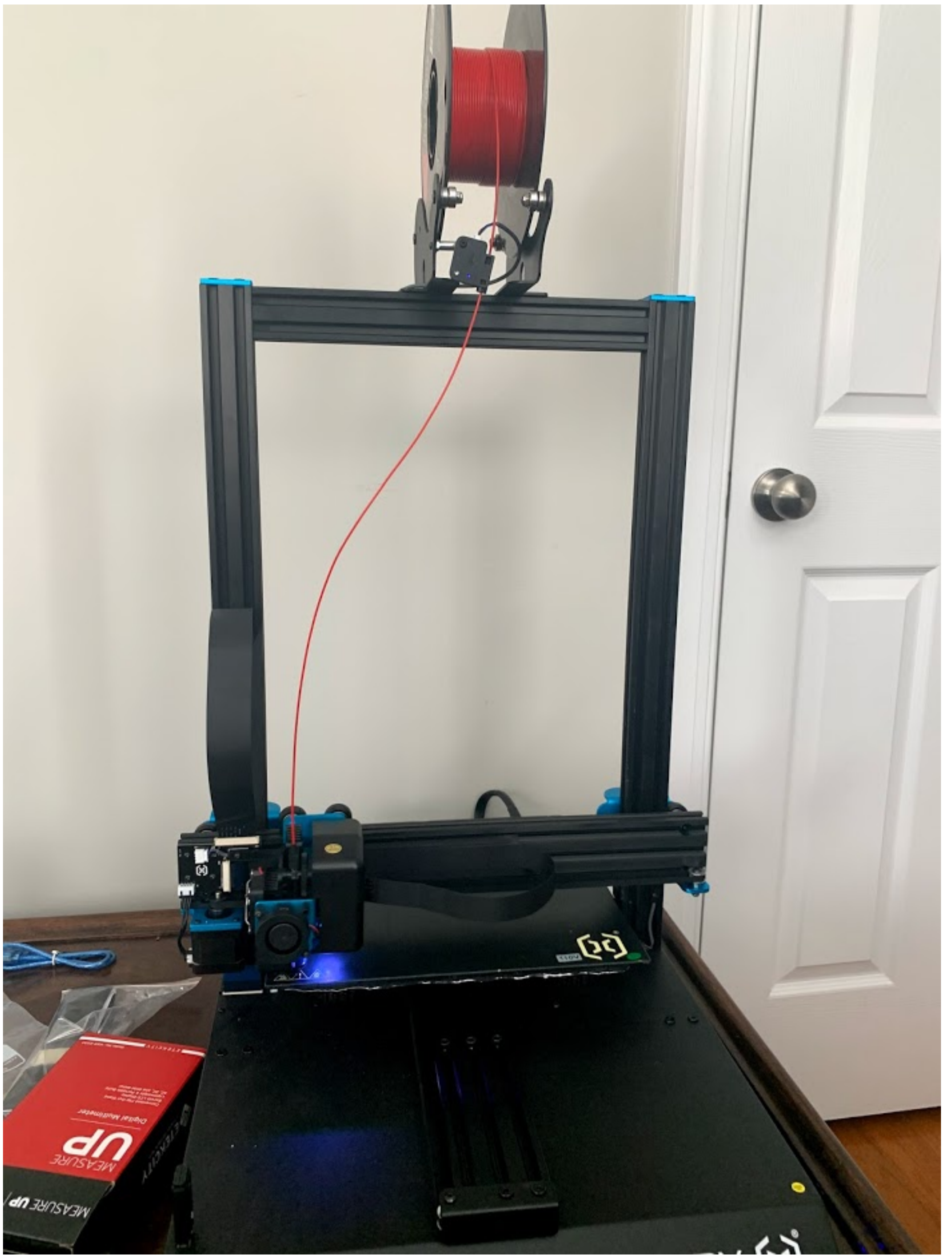


Figure 1: A picture of the Artillery Sidewinder X1 Printer I used.

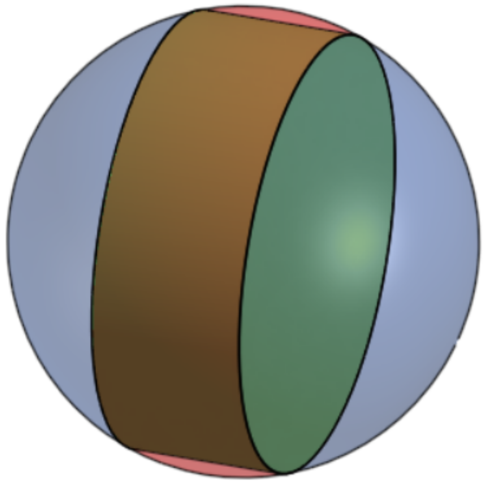

Figure 2: 3D model I created with a visual representation of the spherical band surface areas for coins in block 1.

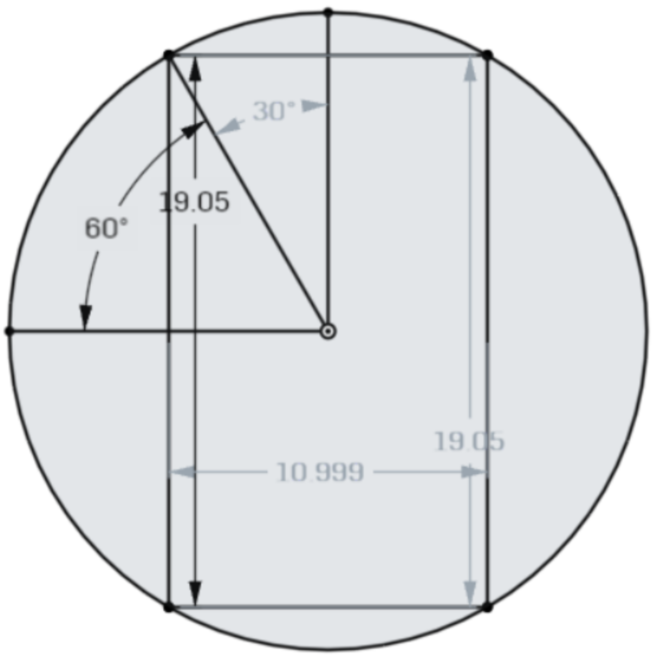


Figure 3: Model I created with a visual derivation of the thickness of the ”arc length” coins in block 2.

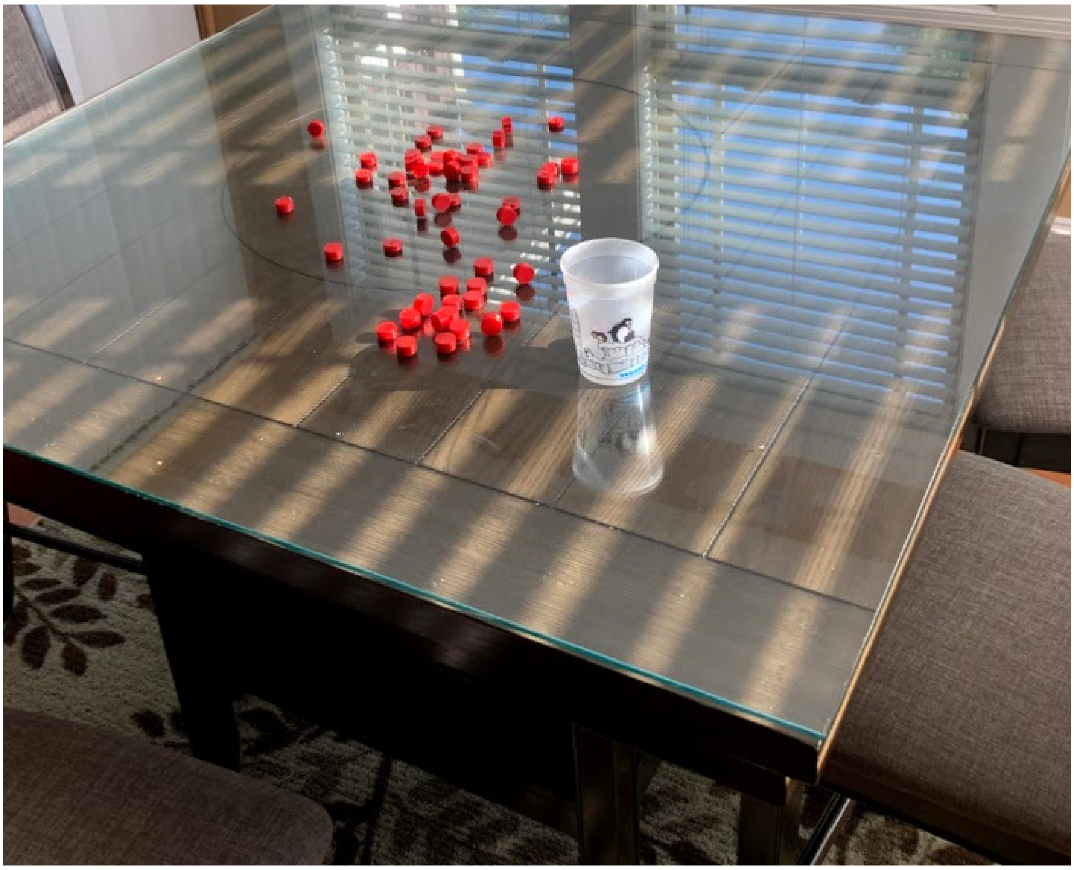

Figure 4: A picture of the cup and 3D-Printed coins I used.

## 2.2 Design and Implementation Concerns

The main concerns with the implementation of this experiment come with concerns regarding the material and experiment's setup. Coins that are used for coin flips in general are textured, and are made out of a metal alloy. However, the coins used in this setup are made of PLA plastic, and are printed on a 3d printer with 20% infill, leading to a different density, air resistance, rotational inertia, and potentially different mass distributions when compared to a coin that is solid metal.

When it comes to the experiment's setup, problems arise when coins roll off the table, I counted those coins as landing on their side, as though the surface that the coins landed on was infinitely large. However, I cannot be completely sure that these coins would not have fallen.

Due to time and convenience concerns, I did not decide to flick each coins as if it was a coin flip, as doing it in sets of 10 would drastically decrease the amount of time to have a given sample size. However, to do this experiment with more accurate conditions, it would be better to flip the coins individually.

# 3 Findings and Analysis

| Thickness (mm) | Proportion of coins on their side |
|---|---|
| 6.735 | $\frac{69}{500}$ |
| 10.999 | $\frac{219}{500}$ |
| 8.0 | $\frac{88}{500}$ |

These are the 3 blocks of coins' data. (Cumulative Data found in Appendix A)

## 3.1 Inference Testing

A one-sample Z-test for proportions is appropriate here to compare the sample proportions to the claimed proportion of $\frac{1}{3}$.

H0 - The true proportion of coin flips that land on their side is $\frac{1}{3}$

HA - The true proportion of coin flips that land on their side is not $\frac{1}{3}$

Let $p$ be the true proportion of coins of a given thickness that land on its side. I will conduct this test at a significance level of $\alpha = .05$

$$H_0 : p = 0.50$$

$$H_a : p \neq 0.50$$

$$\alpha = .05$$

Test statistic, $z$, is represented by the formula:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \tag{1}$$

Checking Conditions:

**Independence**: Each coin toss is independent from the others, coin tosses in which coins hit each other after contacting the table were omitted. A sample of 500 coin tosses can be assumed to be less than 10% of all coin tosses that are performed overall.

**Normality**: $np_0 > 10 = 500(\frac{1}{3}) > 10$, $n(1-p_0) > 10 = 500(\frac{2}{3}) > 10$, therefore our condition for normality is met. This condition was further met by all map samples.

**Randomness**: We can assume this sample to be random.

Plugging in the values for block 1 yields:

$$z = \frac{\frac{69}{500} - \frac{1}{3}}{\sqrt{\frac{\frac{1}{3}(\frac{2}{3})}{500}}}, z = -9.265, p =< .00001 \tag{2}$$

Plugging in the values for block 2 yields:

$$z = \frac{\frac{211}{500} - \frac{1}{3}}{\sqrt{\frac{\frac{1}{3}(\frac{2}{3})}{500}}}, z = 4.206, p =< .00001 \tag{3}$$

Plugging in the values for block 3 yields:

$$z = \frac{\frac{88}{500} - \frac{1}{3}}{\sqrt{\frac{\frac{1}{3}\left(\frac{2}{3}\right)}{500}}}, z = -7.463, p =< .00001 \tag{4}$$

All of the p-values fall in the rejection region, all of the p values are less than the significance level $\alpha = .05$. We have significant evidence that these coins' proportion of landing on the side is different than $\frac{1}{3}$.

## 3.2 Regression Models

Using the previous data to determine the relationship between coin thickness and probability of landing on the side, I construct a linear and quadratic regression model with the independent variable: Coin Thickness in mm on the x-axis and the dependent variable: Proportion of coins landing on the side on the y-axis.

The linear regression yielded a line with equation $y = 0.06939x - 0.3499$ with $R^2 - 0.9716$, and the quadratic regression yielded an equation $y = 0.0121x^2 + -0.1496x + 0.592$ with $R^2 = 1$.

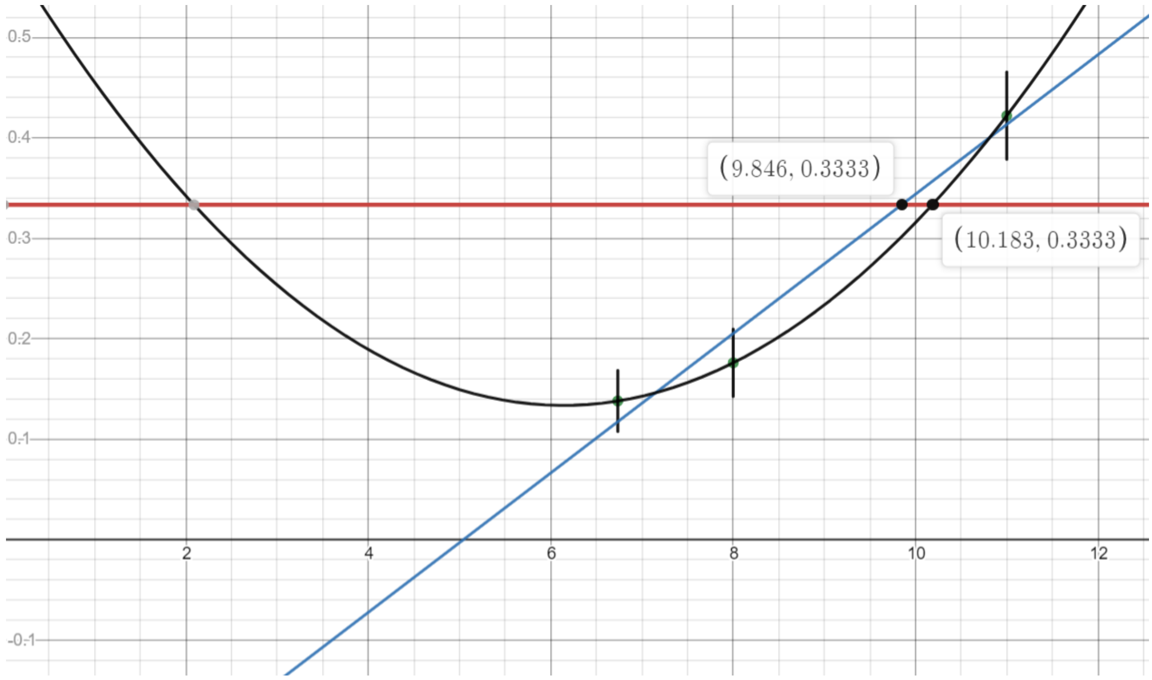


Figure 5: Caption

Here I decided to use data interpolation to find the theoretical thicknesses of a perfect 3-sided coin based on the 2 relationships found, and data interpolation in this case, especially for the quadratic model can be done, because the theoretical thicknesses are between the data already collected.

Based on the linear regression, represented as the positively sloped blue line in Figure 5, the theoretical thickness of a 3-sided coin based on a linear relationship is 9.846 millimeters. Based on the quadratic regression, represented by the black line going through all 3 points, the theoretical thickness of a 3-sided coin based on this relationship is 10.183 millimeters.

Using these 2 models, I printed another 2 blocks of 10 coins, Block 4 with thickness 9.846 mm and Block 5 with thickness 10.183 mm. Each block was rolled 50 times, leading to a sample with size $n = 500$.

The results are as shown:

| Thickness (mm) | Proportion of coins on their side |
|---|---|
| 9.846 | $\frac{146}{500}$ |
| 10.183 | $\frac{178}{500}$ |

Cumulative Data Found in Appendix A

## 3.3 Inference Testing

A one-sample Z-test for proportions is appropriate here to compare the sample proportions to the claimed proportion of $\frac{1}{3}$.

H0 - The true proportion of coin flips that land on their side is $\frac{1}{3}$

HA - The true proportion of coin flips that land on their side is not $\frac{1}{3}$

Let $p$ be the true proportion of coins of a given thickness that land on its side. I will conduct this test at a significance level of $\alpha = .05$

$$H_0 : p = 0.50$$

$$H_a : p \neq 0.50$$

$$\alpha = .05$$

Test statistic, $z$, is represented by the formula:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \tag{5}$$

Checking Conditions:

**Independence**: Each coin toss is independent from the others, coin tosses in which coins hit each other after contacting the table were omitted. A sample of 500 coin tosses can be assumed to be less than 10% of all coin tosses that are performed overall.

**Normality**: $np_0 > 10 = 500(\frac{1}{3}) > 10$, $n(1 - p_0) > 10 = 500(\frac{2}{3}) > 10$, therefore our condition for normality is met. This condition was further met by all map samples.

**Randomness**: We can assume this sample to be random.

Plugging in the values for block 4 yields:

$$z = \frac{\frac{146}{500} - \frac{1}{3}}{\sqrt{\frac{\frac{1}{3}(\frac{2}{3})}{500}}}, z = -1.961, p = 0.04987 \tag{6}$$

Plugging in the values for block 5 yields:

$$z = \frac{\frac{178}{500} - \frac{1}{3}}{\sqrt{\frac{\frac{1}{3}(\frac{2}{3})}{500}}}, z = 1.075, p = .282 \tag{7}$$

The p-value of block 4 falls in the rejection region, as it is less than the significance level $\alpha = .05$. We have significant evidence that block 4's coins' proportion of landing on the side is different than $\frac{1}{3}$. However, for block 5, since the p value is greater than the significance level of $\alpha = .05$, we do not have significant evidence to reject the null hypothesis that the true proportion of coins landing on their side of thickness 10.183 mm is $\frac{1}{3}$.

# 4 Conclusion

This experiment's goal was to find the relationship between the thickness of a coin and the probability it lands on its side, and to construct a coin using that relationship that has a $\frac{1}{3}$ probability of landing on its side. In order to do this, 3 blocks of 10 coins were 3D printed in PLA plastic, each of different diameter based on different principles of predicting how a coin will fall, were tossed onto a table 50 times in order to have a sample size of $n = 500$ From there, 3 two-tailed one-sample Z-tests were performed to see if these blocks of coins could be considered 3-sided coins, however, all of the p values fell in the rejection region when a significance level of $\alpha = .05$ was used.

From there, linear regression and quadratic regression models were created to create equations that relate thickness of a coin to its probability of landing on its side. Using data interpolation, two thicknesses were yielded, and the trials of the new 2 blocks were repeated. Block 4's p value again fell in the rejection region at a significance level of $\alpha = .05$, however Block 5's p value did not fall in the rejection region.

Using this information, we can conclude that block 5's coin of diameter to thickness ratio of 19.05 mm to 10.183 mm is a thickness that can be used for a reasonable 3-sided coin. However, this experiment is not perfect, and if the sample size was magnitudes larger than

$n = 500$ and rather $n = 5000$ or even larger, the proportion found in the experiment will more closely follow the true proportion for a given coin. In addition, this experiment did not ”flip” coins, but rather tumble and toss them akin to dice.

Future research and experiments could use physics simulations to rapidly simulate coin flips of various diameters, or could use coins made of metal alloys instead.

# 5 Appendix A - Cumulative Data

| Thickness (mm) | Proportion of coins on their side |
|---|---|
| 6.735 | $\frac{69}{500}$ |
| 10.999 | $\frac{219}{500}$ |
| 8.0 | $\frac{88}{500}$ |
| 9.846 | $\frac{146}{500}$ |
| 10.183 | $\frac{178}{500}$ |

# Appendix B - Newspaper Article

## The thickness of a 3-sided coin has been found!

For years, mathematicians, physicists, and statisticians used the even and odd numbers on a 6 sided die to make a random choice between 3 options or better yet, a random number generator. However, Raleigh High school student, inspired by a video by mathematician Matt Parker in the United Kingdom, has used the knowledge learned in his AP statistics class to find a coin that has an equal probability of landing on either heads or tails, like a normal coin, but also on its side. The coin he created is one of around 2/5 inch thickness and 3/4 inch diameter (the same diameter as a conventional US penny). He conducted over 2500 coin flips in order to collect data to find this magical ratio of thickness to diameter, however he says not to get too excited : ”My experiment was done with 3D printed plastic and not metal, and I tossed the coins in sets of ten rather than flipped them like you would in a coin toss, but it is pretty cool that it works”.