An Investigation into the 2x2x2 Rubik's Cube Group

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Rationale

The rules that govern the solving of a Rubik's cube have always eluded me. In middle school, I began "speed-cubing", learning many algorithms and techniques to get certain pieces of the cube into certain positions without changing the already "solved" portions to layer-by-layer solve the cube. The math behind the algorithm, apart from simple corner rotation permutations never made sense to me, only stored in my muscle memory. In this paper, I seek to demystify the 2x2x2 Rubik's cube group while discovering its other properties.

Introduction

The Rubik's cube is a 3D puzzle developed by Hungarian professor Ernő Rubik in 1974 and since has become the most popular puzzle toy in the world, with more than 350 million sold as of 2009. The cube consists of six faces, each divided into a 3x3 grid. The Rubik's cube starts out "solved", with each face having a uniform color. Each layer can be rotated independently by increments of $\pi/2$ radians. The Rubik's cube has 8 corner pieces and 12 edge pieces, with the edge pieces having 2 color panels and the corner pieces having three. However, for the purposes of this paper, we will scale the cube down to a 2x2x2 cube. The 2x2x2 puzzle was invented in 1970 by Larry D. Nichols, who created a cube with 8 corner pieces that could rotate and lock into increments of $\pi/2$ radians using magnets (*Smithsonian,* Reese) (Figure 1).

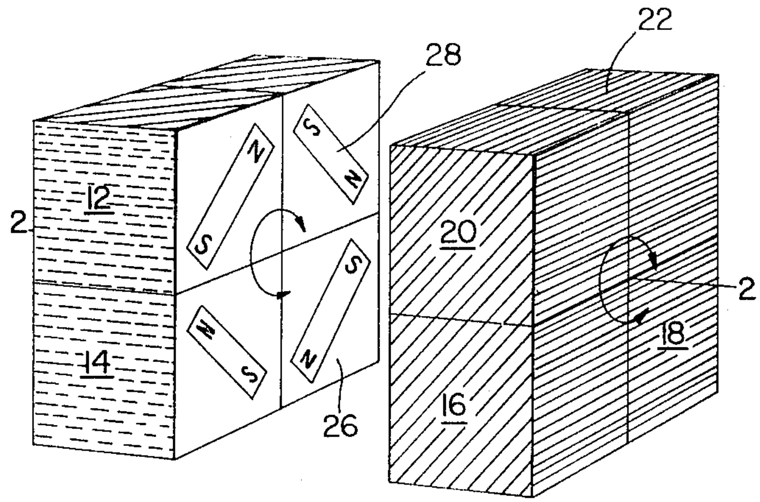

Figure 1: Diagram from Nichols' U.S. Patent (Larry D. Nichols)

Modern 2x2x2 cubes use a spherical plastic core and 8 plastic pieces, and may use magnets to similarly lock the cube when it rotates.

Conventions

The naming conventions for rotations on the cube for "speedcubers" consists of the moves: U (upper layer), D (down layer), L (left layer), R(right layer), F (Front Layer) and B(Back Layer), rotating its corresponding face in the clockwise direction. Adding ‘ to a move indicates rotating the corresponding face in the counterclockwise direction (Figure 2).

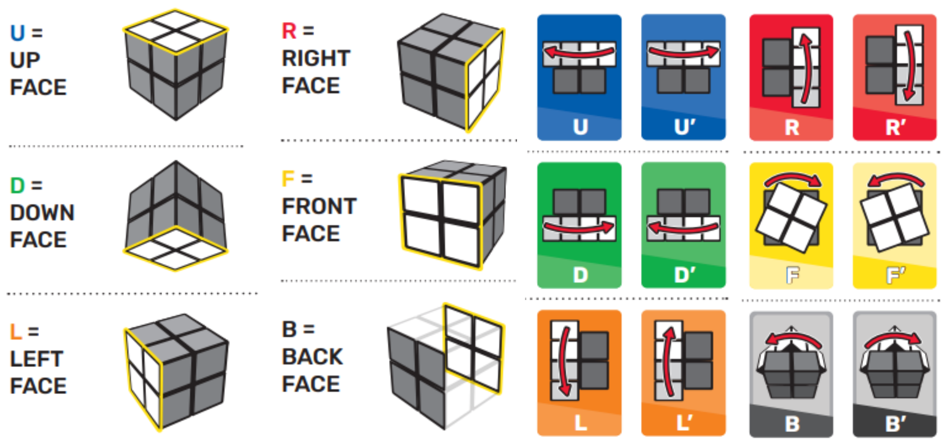

Figure 2: Graphics from Rubiks.com's official solve guide (Rubik's Official Website)

For the purpose of this paper, we will simplify these moves, as we can look at a simple cube's three dimensions. (Figure 3)

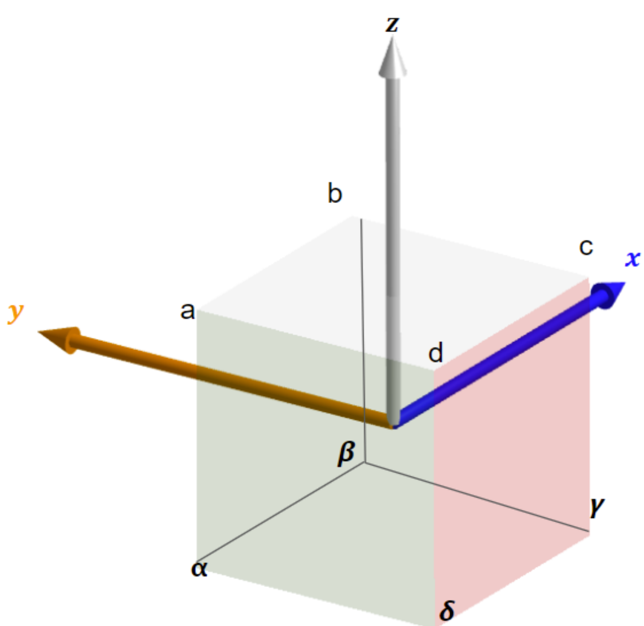

Figure 3

This cube displays the three axes, $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and the corners, labeled $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta}$. Note there are no divisions in the cube. We can describe the operation X as a clockwise rotation while "looking down" the x axis, as shown in Figure 4. X^{-1} can be described as the counterclockwise rotation.

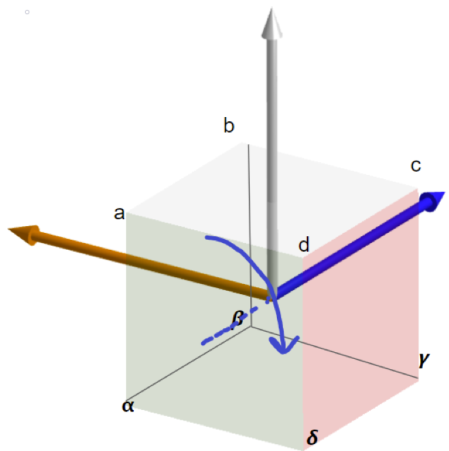

Figure 4: operation X

Representing Operations in a Matrix

This operation can be described in a matrix: as shown below:

X=

	x	y	z
x	1	0	0
y	0	0	1
z	0	-1	0

Where going down the column, the x axis is and is mapped to the positive x axis, the y axis is "sent to" the z axis, and the z axis is sent to the negative y axis, the rest of the matrix is filled with zeros. We can similarly describe the operations Y and Z as:

Y=

	x	y	z
x	0	0	-1
y	0	1	0
z	1	0	0

and Z=

	x	y	z
x	0	1	0
y	-1	0	0
z	0	0	1

Rotating the cube twice with operations X, Y, or Z should reverse the two axes that are not being "preserved", as we observe:

$$X^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, Y^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, Z^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotating the cube three times with operations $X^3,\ Y^3,\ Z^3$ gives us the inverse of each operation

$X^3 = X^{-1},\ Y^3 = Y^{-1},\ Z^3 = Z^{-1}$

In addition, rotating the cube 4 times, with operations $X^4,\ Y^4,\ Z^4$ gives us the Identity.

$$I = X^4 = Y^4 = Z^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These 3 x 3 permutation matrices also have the relations:

$$XY^2 = Z^2X, XZ^2 = Y^2X,\ YX^2 = Z^2Y,\ YZ^2 = X^2Y,\ ZX^2 = Y^2Z, ZY^2 = X^2Z$$

$$YZ = X^3Y = ZX^3,\ ZX = Y^3Z = XY^3,\ XY = Z^3X = YZ^3$$

$$XZ = YX = ZY,$$

$$X^2Y^2 = Y^2X^2 = Z^2$$

$$X^2Y^3 = XZX, Y^2Z^3 = YXY,\ Z^2X^3 = ZYZ$$

The operations X, Y, Z have order 4.

The Group Generated by $X,\ Y,\ Z$

The symmetric group containing the rotational symmetries of a cube, S_4 is known to have 24 elements. We can generate this group using the matrices $X,\ Y,\ Z$ with elements. Note I have already stated that the binary operations $X,\ Y,\ Z$ are closed, have inverses $X^{-1},\ Y^{-1},\ Z^{-1}$, and have identity I. The operations are also associative, as is matrix multiplication (dot product).

Here are the 24 elements of the group and their corresponding permutation matrices:

$$\left[I,\ X,\ Y,\ Z,\ X^2,\ Y^2,\ Z^2,\ X^3,\ Y^3,\ Z^3, XY,\ XY^2,\ XY^3,\ XZ,\ XZ^2,\ XZ^3,\ X^2Y,\ X^2Y^3, X^3Y,\ X^3Y^3,\ X^2Z,\ X^2Z^3,\ X^3Z^3,\right]$$

I	X	X^2	X^3	Y	Y^2
$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$

Y^3	Z	Z^2	Z^3	XY	XY^2
$\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&0&-1\\1&0&0\\0&-1&0\end{bmatrix}$	$\begin{bmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}$

XY^3	XZ	XZ^2	XZ^3	X^2Y	X^2Y^3
$\begin{bmatrix}0&0&1\\-1&0&0\\0&-1&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$	$\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&-1&0\\0&0&1\\-1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix}$

X^3Y	X^3Y^3	X^2Z	X^2Z^3	X^3Z	X^3Z^3
$\begin{bmatrix}0&0&-1\\-1&0&0\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}$	$\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&-1\\-1&0&0\end{bmatrix}$	$\begin{bmatrix}0&-1&0\\0&0&-1\\1&0&0\end{bmatrix}$

Cayley Table

	I	X^2	X^3	Y	Y^2	Y^3	Z	Z^2	Z^3	XY	XY^2	XY^3
I	I	X^2	X^3	Y	Y^2	Y^3	Z	Z^2	Z^3	XY	XY^2	XY^3
X^2	X^2	X^3	Y	I	XY^2	XY^3	XZ	XZ^2	XZ^3	X^2Y	X^2Y^3	Z^3
X^3	X^3	Y	I	X^2	X^2Y^3	Z^3	X^3Y	X^2Z^3	Y^3	X^2Z^3	X^3Y^3	XZ^3
Y	Y	I	X^2	X^3	X^3Y^3	XZ^3	X^2Z	X^3Z	XY^3	X^3Z^3	Y^2	Y^3
Y^2	Y^2	XZ^2	X^3Y	X^3Z^3	Y^3	Z	I	X^3Y^3	X^2Y^3	XY^2	X^2Z^3	X^3Z
Y^3	Y^3	XZ^3	Z^3	XY^3	Z	I	Y^2	X^2Z^3	X^3	X^2Z^3	X^2Z	Y
Z	Z	X^2Y	X^2Y^3	X^3Z	I	Y^2	Y^3	XZ	X^3Y	X^2Z	XY	X^3Z^3
Z^2	Z^2	XZ	X^2Z^3	X^3Y^3	XZ^2	X^2Z^3	X^3Z	Z^3	XY	I	X^2	XY^2
Z^3	Z^3	XY^3	Y^3	XZ^3	X^3Y	X^3	X^2Y^3	XY	I	Z^2	XZ	X^2
XY	XY	XY^2	X^2Z^3	X^2Z	X^3Z^3	X^2Z^3	X^2Y	I	Z^2	Z^3	XY^3	XZ
XY^2	XY^2	X^2Z^3	X^2Z	XY	XY^3	XZ	X^2	Y^2	X^3Y^3	X^2Y^3	X^3Z	Z^2
XY^3	XY^3	Y^3	XZ^3	Z^3	XZ	X^2	XY^2	X^3Z^3	Y	X^3Z	Z	I
XZ	XZ	X^2Z^3	X^3Y^3	Z^2	X^2	XY^2	XY^3	X^3Y	X^2Z	Z	X^2Y	XY
XZ^2	XZ^2	X^3Y	X^3Z^3	Y^2	X^2Z^3	X^3Z	Z^2	XZ^3	X^2Y	X^2	X^3	X^2Y^3
XZ^3	XZ^3	Z^3	XY^3	Y^3	X^2Z	Y	X^3Y^3	X^2Y	X^2	XZ^2	X^3Y	X^3
X^2Y	X^2Y	X^2Y^3	X^3Z	Z	XY	X^3Z^3	X^2Z^3	X^2	XZ^2	XZ^3	Z^3	X^3Y
X^2Y^3	X^2Y^3	X^3Z	Z	X^2Y	Z^3	X^3Y	X^3	XY^2	Y^2	X^3Y^3	Z^2	XZ^2
X^3Y	X^3Y	X^3Z^3	Y^2	XZ^2	X^3	X^2Y^3	Z^3	X^2Z	Z	XZ	X^2Z^3	X^2Y
X^3Y^3	X^3Y^3	Z^2	XZ	X^2Z^3	XZ^3	X^2Z	Y	X^2Y^3	XY^2	Y^2	XZ^2	X^2Z^3
X^2Z	X^2Z	XY	XY^2	X^2Z^3	Y	X^3Y^3	XZ^3	Z	XZ	X^3Y	X^3Z^3	X^2Z^3
X^2Z^3	X^2Z^3	X^2Z	XY	XY^2	X^3Z	Z^2	XZ^2	Y^3	X^2Z^3	X^3	Y	X^3Y^3
X^2Z^3	X^2Z^3	X^3Y^3	Z^2	XZ	X^2Y	XY	X^3Z^3	X^3	X^2Z^3	Y^3	XZ^3	X^2Z
X^3Z	X^3Z	Z	X^2Y	X^2Y^3	Z^2	XZ^2	X^2Z^3	XY^3	X^3Z^3	Y	I	Y^2
X^3Z^3	X^3Z^3	Y^2	XZ^2	X^3Y	X^2Z^3	X^2Y	XY	Y	X^3Z	XY^3	Y^3	Z

	XZ	XZ^2	XZ^3	X^2Y	X^2Y^3	X^3Y	X^3Y^3	X^2Z	X^2Z^3	X^2Z^3	X^3Z	X^3Z^3
I	XZ	XZ^2	XZ^3	X^2Y	X^2Y^3	X^3Y	X^3Y^3	X^2Z	X^2Z^3	X^2Z^3	X^3Z	X^3Z^3
X^2	X^3Y	X^2Z^3	Y^3	X^2Z^3	X^3Y^3	X^2Z	Y^2	Z	X^3Z	X^3Z^3	Z^2	XY
X^3	X^2Z	X^3Z	XY^3	X^3Z^3	Y^2	Z	XY^2	XZ	Z^2	XY	XZ^2	X^2Y
Y	Z	Z^2	Z^3	XY	XY^2	XZ	X^2Y^3	X^3Y	XZ^2	X^2Y	X^2Z^3	X^2Z^3
Y^2	Z^2	XZ^3	X^2Y	X^2	X^3	Z^3	X^2Z^3	XY	X^2Z	XZ	Y	XY^3
Y^3	X^3Y^3	X^2Y	X^2	XZ^2	X^3Y	X^2Y^3	XZ	XY^2	XY	Z^2	X^3Z^3	X^3Z
Z	X^2Z^3	X^2	XZ^2	XZ^3	Z^3	X^3	Z^2	X^2Z^3	XY^2	X^3Y^3	XY^3	Y
Z^2	XY^3	X^3Y	X^2Z	Z	X^2Y	X^3Z^3	XZ^3	Y	X^3	Y^3	X^2Y^3	Y^2
Z^3	XY^2	X^3Z^3	Y	X^3Z	Z	Y^2	X^2Z	X^3Y^3	X^2Z^3	X^2Z^3	X^2Y	XZ^2
XY	X^2	Y^2	X^3Y^3	X^2Y^3	X^3Z	XZ^2	Y	XZ^3	Y^3	X^3	Z	X^3Y
XY^2	XZ^2	Y^3	X^2Z^3	X^3	Y	XZ^3	X^3Z^3	X^2Y	Z	X^3Y	I	Z^3
XY^3	Y^2	X^2Z^3	X^3	X^2Z^3	X^2Z	X^3Y^3	X^3Y	X^2Y^3	X^2Y	XZ^2	XY	Z^2
XZ	X^3Z^3	X^3	X^2Z^3	Y^3	XZ^3	Y	XZ^2	X^3Z	X^2Y^3	Y^2	Z^3	I
XZ^2	Z^3	X^2Z	Z	XZ	X^2Z^3	XY	Y^3	I	Y	XY^3	X^3Y^3	XY^2
XZ^3	X^2Y^3	XY	I	Z^2	XZ	XY^2	Z	Y^2	X^3Z^3	X^3Z	X^2Z^3	X^2Z^3
X^2Y	X^3	XY^2	Y^2	X^3Y^3	Z^2	X^2Z^3	I	Y^3	XY^3	Y	XZ	X^2Z
X^2Y^3	X^2Z^3	XY^3	X^3Z^3	Y	I	Y^3	XY	X^2Z^3	XZ	X^2Z	X^2	XZ^3
X^3Y	XY	Y	X^3Z	XY^3	Y^3	I	X^2Z^3	Z^2	X^3Y^3	XY^2	XZ^3	X^2
X^3Y^3	X^3Z	Z^3	XY	I	X^2	XY^3	X^2Y	X^3Z^3	X^3Y	Z	X^3	Y^3
X^2Z	X^2Y	I	Z^2	Z^3	XY^3	X^2	X^3Z	XZ^2	Y^2	X^2Y^3	Y^3	X^3
X^2Z^3	XZ^3	Z	XZ	X^3Y	X^3Z^3	X^2Y	XY^3	X^2	I	Z^3	Y^2	X^2Y^3
X^2Z^3	Y	X^2Y^3	XY^2	Y^2	XZ^2	X^3Z	X^2	XY^3	Z^3	I	X^3Y	Z
X^3Z	Y^3	XZ	X^3Y	X^2Z	XY	X^2Z^3	Z^3	X^3	X^2	XZ^3	XY^2	X^3Y^3
X^3Z^3	I	X^3Y^3	X^2Y^3	XY^2	X^2Z^3	Z^2	X^3	Z^3	XZ^3	X^2	X^2Z	XZ

Notice how each row and column contains every element in the group.

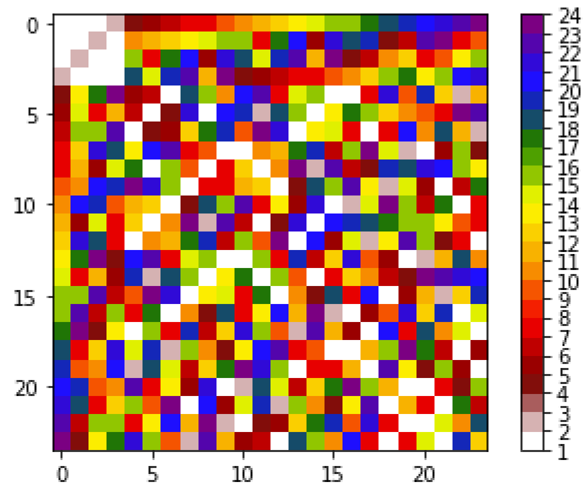

Figure 5: Visual color representation of cayley table, 1 representing the first element in the group (could not resolve 1,2,3 looking identical)

Operations on a 2x2x2 Rubik’s Cube

To tackle the 2x2x2 Rubik’s cube, we must look at the corners of the cube, labeled $a, b, c, d, \alpha, \beta, \gamma, \delta$ Each base operation X, Y, Z has been “split” into two operations: $X_+, X_-, Y_+, Y_-, Z_+, Z_-$. We have the identities $X_+X_- = X$, $Y_+Y_- = Y$, and $Z_+Z_- = Z$. The operations X_+ and X_- are shown below.

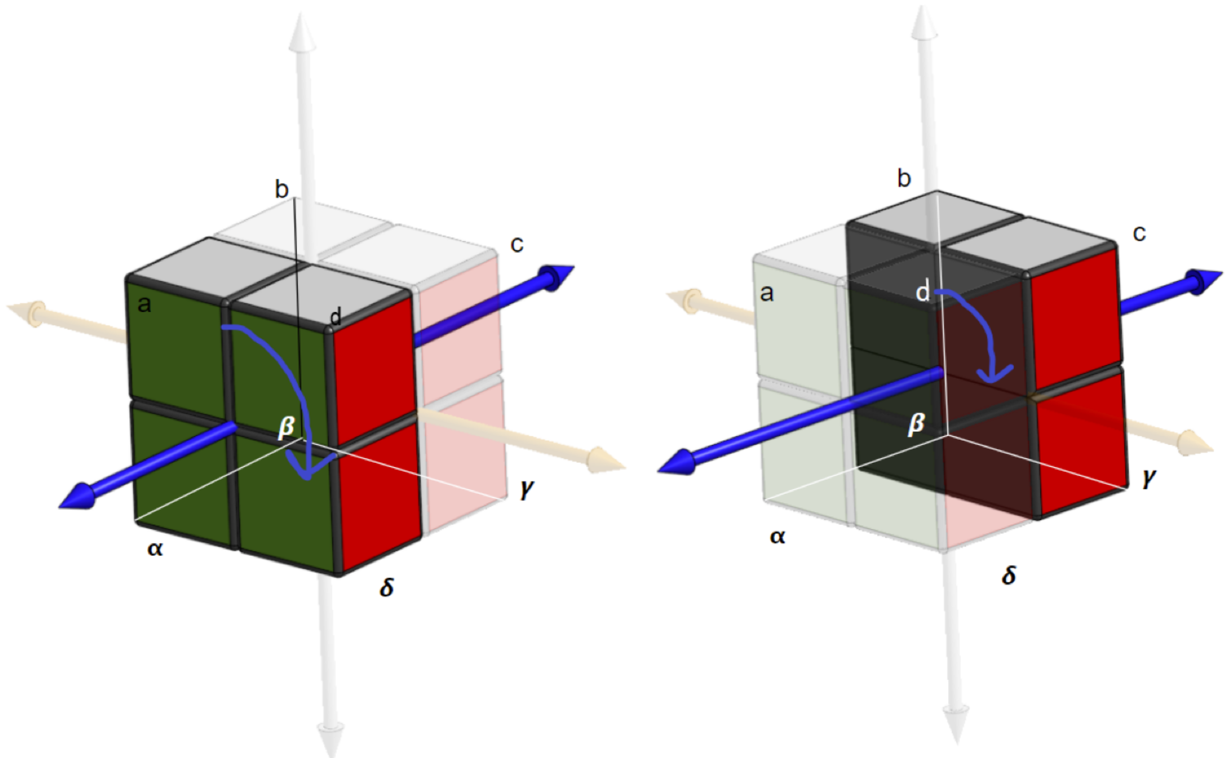

Figure 6: Operations X_-(left) and X_+ (right)

Note that operation X_- is the same operation as X, just the half of the cube that corresponds to the negative x axis. It is still in the same clockwise direction. Similarly, X_- corresponds to the positive x axis, but in the same clockwise direction.

Also note that operation X sends $a\ to\ d,\ d\ to\ \delta, \delta\ to\ \alpha, \alpha\ to\ a,\ b\ to\ c,\ c\ to\ \gamma, \gamma\ to\ \beta,\ \beta\ to\ b$.
Similarly, X_-sends $a\ to\ d,\ d\ to\ \delta, \delta\ to\ \alpha, \alpha\ to\ a$ and X_+sends $a\ to\ d,\ d\ to\ \delta, \delta\ to\ \alpha, \alpha\ to\ a$.
These six operations $X_+, X_-, Y_+, Y_-, Z_+, Z_-$ correspond to the same six operations or their inverses(B', F, L', R, U', D respectively) described in the “Conventions” portion of this paper, but for the sake of consistency I will be using the X, Y, Z convention for the rest of this paper.

We have additional relations:

$$X\ =\ X_-X_+,\ Y\ =\ Y_-Y_+,\ Z\ =\ Z_-Z_+$$

As well as countless more that I will not painstakingly discover.

Representing Operations on a 2x2x2 Rubik's Cube

Rather than using the three axes that govern the cube, due to the nature of the 2x2x2 cube, we must consider the corners. We can write each move $X_+, X_-, Y_+, Y_-, Z_+, Z_-$ as a permutation matrix.

Recall:
The move X_- moves $a\ to\ d,\ d\ to\ \delta, \delta\ to\ \alpha, \alpha\ to\ a$. This can be represented by a permutation shown below, where the symbols are interchangeable with the numbers 1-8.

$$\begin{pmatrix} a & b & c & d & \alpha & \beta & \gamma & \delta \\ d & b & c & \delta & a & \beta & \gamma & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 3 & 8 & 1 & 6 & 7 & 5 \end{pmatrix}$$

This can also be represented in cycle notation:

$$(a, d, \delta, \alpha)$$

For our purposes, however we can represent this as a 8 x 8 permutation matrix:

$X_- =$

	a	b	c	d	α	β	γ	δ
a	0	0	0	1	0	0	0	0
b	0	1	0	0	0	0	0	0
c	0	0	1	0	0	0	0	0
d	0	0	0	0	0	0	0	1
α	1	0	0	0	0	0	0	0
β	0	0	0	0	0	1	0	0
γ	0	0	0	0	0	0	1	0
δ	0	0	0	0	1	0	0	0

Note how the unchanged corners have a 1 on the main diagonal of the matrix, but in contrast to the permutation matrices of operations X, Y, and Z, we do not have negative numbers.

Very similarly to the group generated by operations X, Y, and Z on a cube, one could find all the permutation matrices created by the group generated by $X_+, X_-, Y_+, Y_-, Z_+, Z_-$. However, one will realize this number becomes very large. Any permutation of all 8 corners is possible, combined with 24 orientations of the cube. This gives 8! * 24 = 967,680 permutations within this group. On a 2x2x2 Rubik's cube, there are also corner rotations (which I have not addressed with the corner permutations), multiplying that number by another 3^7. If one ignores the 24 orientations of the cube, there are 3,674,160 possible positions on a 2x2x2 Rubik's cube.

Conclusion

The 2x2x2 Rubik's cube presents multiple magnitude greater complexity than the group S_4, which we took a deep look at in this paper. A simple divide from a 1x1x1 cube creates a multitude of possibilities. In this paper we took a look at how permutation groups can be used to represent such a simultaneously simple and complex object. In the future, I hope to investigate methods to represent the 2x2x2 Rubik's cube including its corner twists as a group, and compute the cycle length of any given permutation, and potentially create a simple algorithm to position the corners and twist them into a "solved" position.

Works Cited

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