Gaussian Elimination:

$$\begin{cases} x + y - z = 9 \\ y + 3z = 3 \\ -x - 2z = 2 \end{cases}$$

Objective: Solve $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Gaussian elimination:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ -1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 2 \end{pmatrix}$$

$$\rightarrow \quad R_1 + R_3 \rightarrow R_3$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 11 \end{pmatrix}$$

$$\rightarrow \quad -1R_2 + R_3 \rightarrow R_3$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -6 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 8 \end{pmatrix}$$

$$\rightarrow \quad -1/6R_3 \rightarrow R_3$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ -4/3 \end{pmatrix}$$

$$\rightarrow \quad \begin{cases} x + y - z = 9 \\ y + 3z = 3 \\ z = -4/3 \end{cases}$$

$$\rightarrow \quad y + 3\left(-\frac{4}{3}\right) = 3, \;\; y = 7$$

$$\rightarrow \quad x + 7 - \left(-\frac{4}{3}\right) = 9, \;\; x = 2/3$$

$$Solution: (x, y, z) = (\frac{2}{3}, 7, -4/3)$$

Going from Gaussian elimination to find the inverse matrix

$$A.B = I$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -2 & 0 & 3 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$Solution\colon \;\; B = A^{-1} = \begin{pmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

Determinant:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \; |A| = ad - bc$$