Course Materials for GEN-AI

Northeastern University

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Thank you for your understanding and collaboration.

Convex Conjugate Functions

1 Definition of Convex Conjugate

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semi-continuous function. The **convex conjugate** (also known as the **Fenchel conjugate**) of f is defined as:

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \left(\langle y, x \rangle - f(x) \right), \tag{1}$$

where $\langle y, x \rangle$ denotes the inner product (dot product) of y and x. Intuitively, $f^*(y)$ captures the maximum difference between the linear functional $\langle y, x \rangle$ and the function $f(x)$ for all x.

1.1 Geometric Intuition

The convex conjugate $f^*(y)$ can be seen as the "envelope" of all the tangent hyperplanes (or lines in the 2D case) to the graph of $f(x)$. It represents the steepest linear function that lies below the graph of $f(x)$ at a given slope y.

1.2 Key Properties of the Convex Conjugate

- **Duality:** $(f^*)^* = f$ if f is convex, proper, and lower semi-continuous.
- **Fenchel Inequality:** For all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we have

$$f(x) + f^*(y) \geq \langle x, y \rangle. \tag{2}$$

- **Convexity:** $f^*(y)$ is always a convex function, even if $f(x)$ is not strictly convex.

2 Examples

2.1 Example 1: Convex Conjugate of a Quadratic Function

Step 1: Definition of Function
Let $f(x) = \frac{1}{2}x^2$. We compute the conjugate using the definition:

$$f^*(y) = \sup_{x \in \mathbb{R}} \left(yx - \frac{1}{2}x^2 \right). \tag{3}$$

Step 2: Solve for x

To maximize $yx - \frac{1}{2}x^2$ with respect to x, we take the derivative and set it to zero:

$$\frac{d}{dx}\left(yx - \frac{1}{2}x^2\right) = y - x = 0 \implies x = y. \tag{4}$$

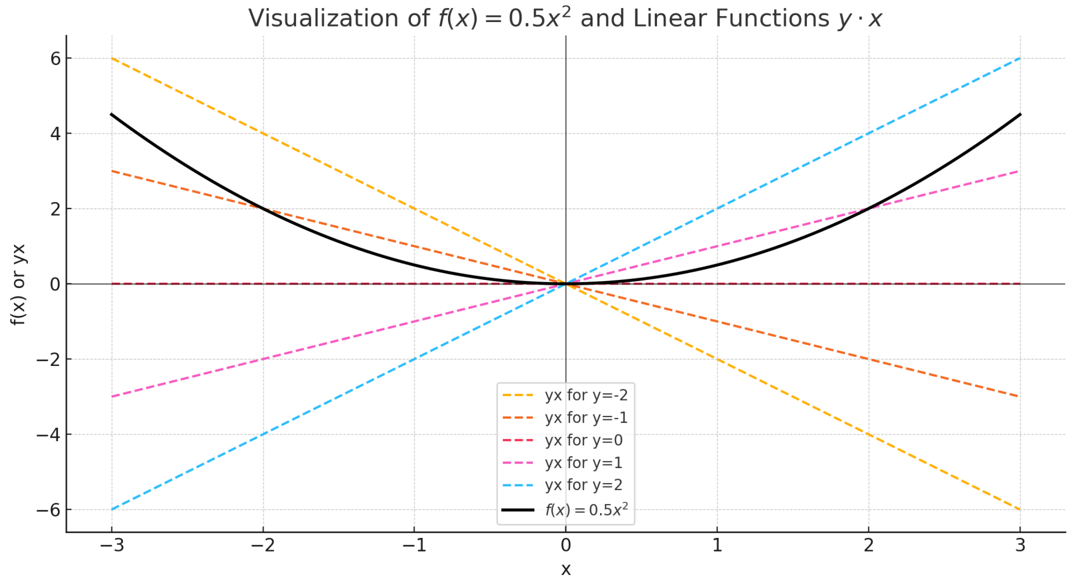


Figure 1: Visualization of $f(x) = \frac{1}{2}x^2$ and its conjugate $f^*(y) = \frac{1}{2}y^2$.

The black curve represents the function $f(x) = \frac{1}{2}x^2$. The dashed lines are the linear functions $y \cdot x$ for different values of y (like $y = -2, -1, 0, 1, 2$).

Step 3: Compute the Supremum Substituting $x = y$ into the objective function, we get:

$$f^*(y) = y \cdot y - \frac{1}{2}y^2 = \frac{1}{2}y^2. \tag{5}$$

Result: The conjugate of $f(x) = \frac{1}{2}x^2$ is:

$$f^*(y) = \frac{1}{2}y^2. \tag{6}$$

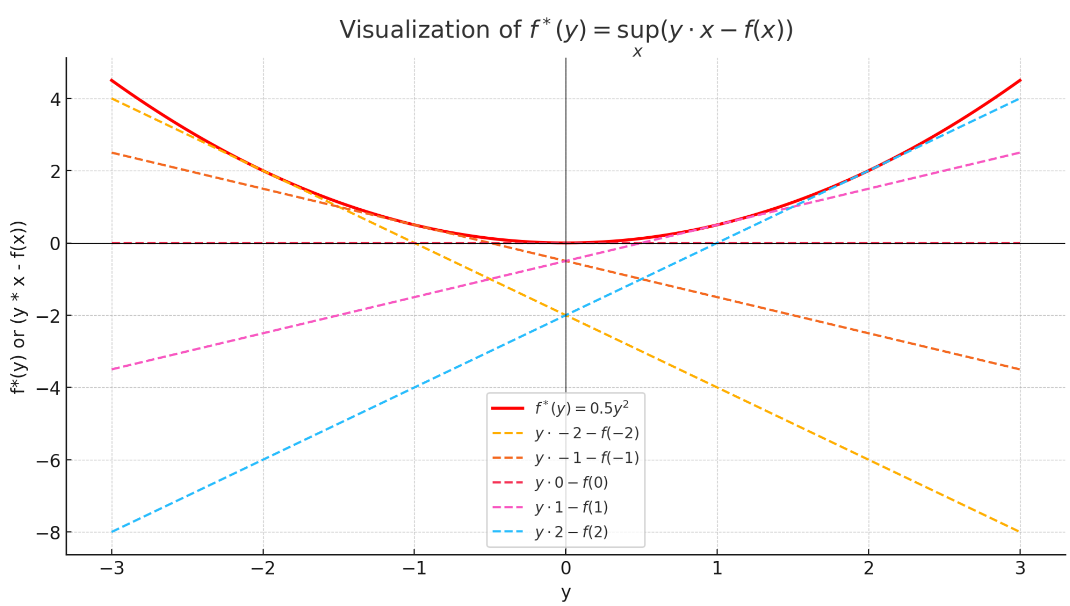


Figure 2: Visualization of $f^*(y)$

The red curve shows the conjugate $f^*(y) = \frac{1}{2}y^2$, which is the result of optimizing the supremum:

$$f^*(y) = \sup_x (y \cdot x - f(x)) \tag{7}$$

for every y.

The dashed curves show $y \cdot x - f(x)$ for different values of x (like $x = -2, -1, 0, 1, 2$). The curve $f^*(y)$ is the "upper envelope" of all these dashed lines, which means it's the supremum of these values for each y.

Fenchel Inequality

The heatmap visualizes the Fenchel inequality:

$$f(x) + f^*(y) \geq \langle x, y \rangle \implies f(x) + f^*(y) - x \cdot y \geq 0 \tag{8}$$

The quantity $f(x) + f^*(y) - x \cdot y$ (called the Fenchel gap) is plotted as a function of x and y.

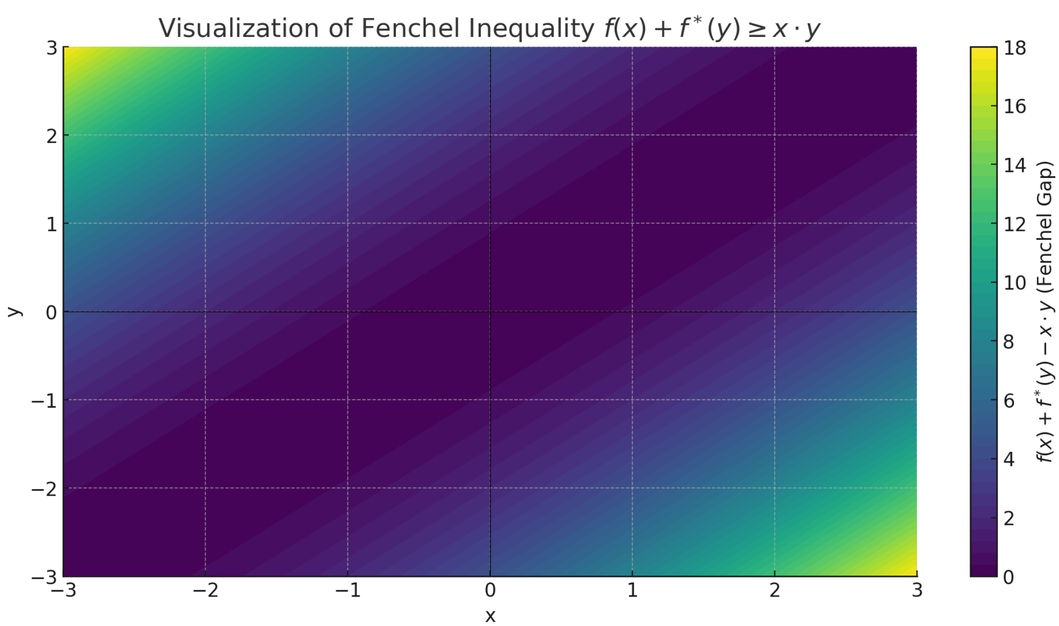
The Fenchel gap is:

$$\text{Fenchel gap} = f(x) + f^*(y) - x \cdot y \tag{9}$$

Using the specific forms of $f(x)$ and $f^*(y)$, we have:

$$\text{Fenchel gap} = \frac{1}{2}x^2 + \frac{1}{2}y^2 - x \cdot y \tag{10}$$

This is a quadratic function in x and y. We can rewrite it to understand its structure.



The heatmap shows the "Fenchel gap" $f(x) + f^*(y) - x \cdot y$ as a function of x and y.

By definition, Fenchel's inequality states that this quantity is always non-negative:

$$f(x) + f^*(y) \geq \langle x, y \rangle \tag{11}$$

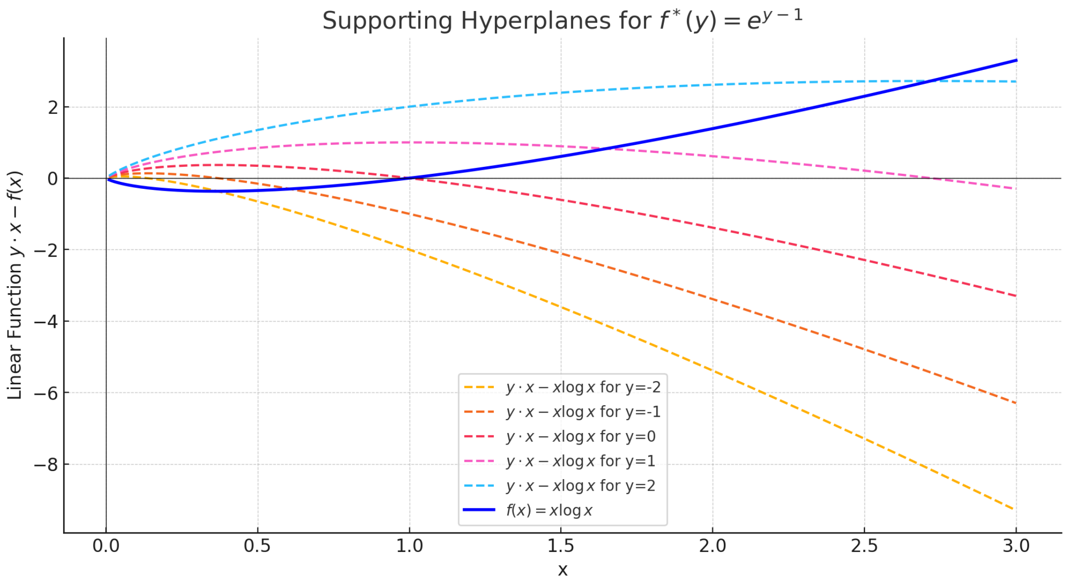
Darker regions correspond to smaller values, and lighter regions indicate larger values of the gap. This visualization illustrates that the Fenchel gap is always non-negative, in accordance with the Fenchel inequality.

Example 2: Convex Conjugate of Negative Entropy

Step 1: Definition of Function

Let $f(x) = x \log x$ (for $x > 0$ and $f(x) = +\infty$ for $x \leq 0$). We compute its conjugate:

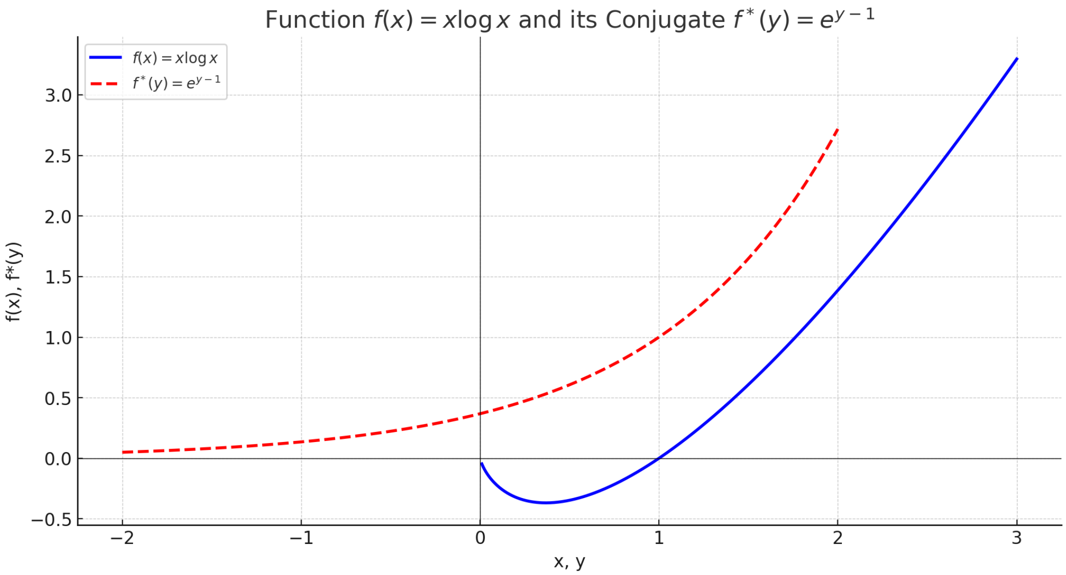
$$f^*(y) = \sup_{x>0} (yx - x \log x) . \tag{12}$$



Step 2: Solve for x

To maximize $yx - x \log x$, we compute the derivative and set it to zero:

$$\frac{d}{dx} (yx - x \log x) = y - \log x - 1 = 0 \implies x = e^{y-1}. \tag{13}$$



Step 3: Compute the Supremum

Substituting $x = e^{y-1}$ into the objective function:

$$f^*(y) = y \cdot e^{y-1} - e^{y-1} \log e^{y-1} = e^{y-1}(y - (y-1)) = e^{y-1}. \tag{14}$$

Result: The conjugate of $f(x) = x \log x$ is:

$$f^*(y) = e^{y-1}. \tag{15}$$