MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 11

1. Prove or disprove the following claims.
   (a) The representation of quadratic form $q(\vec{x}) = \vec{x}^T A\vec{x}$, where A is a symmetric matrix, is unique.
   (b) Consider $q(\vec{x}) = \vec{x}^T A\vec{x}$ where $A \in M_2(\mathbb{R})$. $q(\vec{x})$ is not a quadratic form if A is not symmetric.
   (c) If $A \succ 0$ then $A^7 \succ 0$.
   (d) If $A \prec 0$, then $A^4 \prec 0$.
   (e) If $A \succ 0$ and $B \prec 0$ then $A - B \succ 0$.
   (f) Every matrix has a singular value decomposition.
   (g) Similar matrices must have the same singular values.
   (h) If A and B are real symmetric matrices such that $A^3 = B^3$, then A must be equal to B.

2. Orthogonally diagonalize the following matrices

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

3. Determine all matrices C such that $C^2 = B$ in problem 2.

4. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$. Show that

$$\max_{\|\vec{x}\| \neq 0} \frac{\vec{x}^T A\vec{x}}{\vec{x}^T\vec{x}} = \lambda_1$$

$$\min_{\|\vec{x}\| \neq 0} \frac{\vec{x}^T A\vec{x}}{\vec{x}^T\vec{x}} = \lambda_p$$

?

Comment at what vectors x the max and min values are attained.

5. Consider matrix

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

   (a) Determine an SVD of A.
   (b) Write A in the form of $\sum_{i=1}^{r} \sigma_i u_i v_i^T$, a sum of several rank-1 matrices.
   (c) Notice that A is invertible. Determine an SVD of A^{-1}. Do you need to start from scratch?
   (d) Determine an SVD of A^T. Do you need to start from scratch?

6. What can one say about a matrix's eigenvalues and its singular values? Consider the following. Note that here σ_1 is the largest singualr value and σ_n the smallest of matrix A.

(a) Let $A \in M_2(\mathbb{R})$. Let $\boldsymbol{w} \in \mathbb{R}^2$ be a unit vector. Show that

$$\sigma_2 \leq \|A\boldsymbol{w}\| \leq \sigma_1$$

by tracing what happens to $\boldsymbol{w}$ under the three matrices in A's SVD.

(b) Show part (a) algebraically.

(c) Let $A \in M_{m\times n}(\mathbb{R})$. Show that

$$\sigma_n\|v\| \leq \|Av\| \leq \sigma_1\|v\|$$

for any $v \in \mathbb{R}^n$.

(d) Let λ be a real eigenvalue of matrix $A \in M_n(\mathbb{R})$. Show that

$$\sigma_n \leq |\lambda| \leq \sigma_1$$

.

(e) Consider the matrix A in Q5. Determine $\min_{\|x\|=1} \|Ax\|$. Comment at what vectors x the min value is attained.

(f) Consider the matrix A in Q5. Determine $\max_{\|x\|=1} \|Ax\|$. Comment at what vectors x the max value is attained.

1. Prove or disprove the following claims.

(a) The representation of quadratic form $q(\vec{x}) = \vec{x}^T A\vec{x}$, where A is a symmetric matrix, is unique.

True. Let A and $\check{A}$ be the symmetric matrices for $q(\vec{x}) = \vec{x}^T A \vec{x}$ and $q(\vec{x}) = \vec{x}^T \check{A} \vec{x}$.

Since A and $\check{A}$ are symmetric, they are orthonormally diagonalizable.

$\therefore A = QDQ^{-1}$ $\quad \check{A} = \check{Q}\check{D}\check{Q}^{-1}$.

where D and $\check{D}$ are diagonal matrices and their diagonals contain the eigenvalues of the quadratic form $q(\vec{x})$.

Since orthogonal transformation $(Q, Q^{-1}, \check{Q}, \check{Q}^{-1})$ preserve spectrum, the eigenvalues of a linear transformation is the same.

Therefore, $D = \check{D}$.

$$\vec{x}^T QDQ^{-1}\vec{x} = \vec{x}^T \check{Q}\check{D}\check{Q}^{-1}\vec{x}$$
$$QDQ^{-1} = \check{Q}\check{D}\check{Q}^{-1}$$
$$D = Q^{-1}\check{Q}\check{D}\check{Q}^{-1}Q$$
$$A = QDQ^{-1} = Q(Q^{-1}\check{Q}\check{D}\check{Q}^{-1}Q)Q^{-1}$$
$$= \check{Q}\check{D}\check{Q}^{-1}$$
$$= \check{A}.$$

(b) Consider $q(\vec{x}) = \vec{x}^T A\vec{x}$ where $A \in M_2(\mathbb{R})$. $q(\vec{x})$ is not a quadratic form if A is not symmetric.

Any square matrix can be considered as the sum of a symmetric part and a anti-symmetric part.

$$M = \tfrac{1}{2}(M + M^T) + \tfrac{1}{2}(M - M^T) = M_S + M_A$$

$M_S = M_S^T$ and $M_A = -M_A^T$

for an antisymmetric matrix,

$$q(\vec{x}) = \vec{x}^T M_a \vec{x} = (\vec{x}^T M_a^T \vec{x})^T = -(x^T M_a \vec{x})^T = -q(x).$$

So M_a has to be zero.

So for a general linear transformation,

$$q(\vec{x}) = \vec{x}^T M \vec{x} = \vec{x}^T M_s \vec{x} + \vec{x}^T M_a \vec{x}$$
$$= \vec{x}^T M_s \vec{x} + \vec{x}_T \, 0 \, \vec{x}$$
$$= \vec{x}^T M_s \vec{x}.$$

$\therefore$ $q(\vec{x}) = \vec{x}^T A \vec{x}$, A has to be symmetric.

(c) If $A \succ 0$ then $A^7 \succ 0$.

For $A \succ 0$, col $(\vec{x}^T A \vec{x}) > 0$.
let $\lambda_i \in$ spectrum of A,
$(\lambda_i)^7 \in$ spectrum of A^7
given $\lambda_i > 0$. $(\lambda_i)^7 > 0$.
$\therefore$ true.

(d) If $A \prec 0$, then $A^4 \prec 0$.

For $A \prec 0$, let $\lambda_i \in$ spectrum of A
then $(\lambda_i)^4 \in$ spectrum of A^4
given $\lambda_i \in < 0$, $(\lambda_i)^4 > 0$
$\therefore$ $A^4 \succ 0$
$\therefore$ false.

(e) If $A \succ 0$ and $B \prec 0$ then $A - B \succ 0$.

$A \succ 0$. $\forall \lambda_i^A > 0$
$B \prec 0$. $\forall \lambda_i^B < 0$
$\exists \lambda^A{}_i - \lambda^B{}_i > 0$ because (positive − negative) > 0.
$\therefore$ true.

(f) Every matrix has a singular value decomposition.

For every matrix $A \in M_{m \times n}(\mathbb{R})$, $A^T A$ is always symmetric positive semidefinite. Based on the spectrum theorum, A^TA can be decomposed into QDQ^T, where D's entries on the diagonal are the eigenvalues of A^TA. and $\bar{V}$'s column vectors are corresponding eigenvectors.
By these, we can get the Σ and $\bar{V}$ for $A = \bar{U} \Sigma \bar{V}^T$.

Because $\frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sqrt{\lambda_i}} = \frac{A\vec{v}_i}{\sigma_i}$, we can construct $\bar{U}$ with

$\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}$. Therefore, every matrix can have a singular value decomposition.

(g) Similar matrices must have the same singular values.

False. let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$

$A \sim B$ because $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$

$A^TA = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$ and $\sigma_{A1} = 3$ $\sigma_{A2} = 2$.

$B^TB = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 1 & 10 \end{pmatrix}$ $\begin{pmatrix} 4-\lambda & 4 \\ 1 & 10-\lambda \end{pmatrix} \Rightarrow (4-\lambda)(10-\lambda) - 4 = 0$

$40 - 14\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 14\lambda - 36 = 0.$

$\lambda_1 = 7+2\sqrt{13}$ $\lambda_2 = 7-2\sqrt{13}$

clearly, $\sigma_{A1} \neq \sigma_{B1}$ and $\sigma_{A2} \neq \sigma_{B2}$.

(h) If A and B are real symmetric matrices such that $A^3 = B^3$, then A must be equal to B.

$A = Q_A D_A Q_A^{-1}$ and $B = Q_B D_B Q_B^{-1}$

$A^3 = B^3$ $\therefore Q_A D_A^3 Q_A^{-1} = Q_B D_B^3 Q_B^{-1}$

$\therefore Q_A D_A Q_A^{-1} = Q_B D_B Q_B^{-1}$

$\therefore$ Yes, $A = Q D_A Q^{-1} = Q D_B Q^{-1} = B.$

2. Orthogonally diagonalize the following matrices

$A = PDP^{-1}$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda)$$

$$= -\lambda(\lambda+1)(\lambda-1) + 2(\lambda+1) = 0$$

$(\lambda+1)[(-\lambda)(\lambda-1)+2] = 0$

$(\lambda+1)[-\lambda^2 + \lambda + 2] = 0$

$(\lambda+1)(-\lambda+2)(\lambda+1) = 0$

$\lambda_1 = 2 \quad \lambda_2 = -1 \quad \lambda_3 = -1$

$E\{\lambda_1 = 2\} = \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$

$E\{\lambda_2 = \lambda_3 = -1\} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$

$\therefore A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$

$B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix} \qquad (3-\lambda)^2 = 4$

$3 - \lambda = \pm 2$

$\lambda_1 = 5 \qquad \lambda_2 = 1$

$E\{\lambda_1 = 5\} = \left(\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \qquad \therefore B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$

$E\{\lambda_2 = 1\} = \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}.$

3. Determine all matrices C such that $C^2 = B$ in problem 2.

$$B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .$$

$$C = d \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \pm\sqrt{5} & 0 \\ 0 & \pm\sqrt{1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$C_1 = d \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = d \begin{pmatrix} \sqrt{5} & -1 \\ \sqrt{5} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{5}+1 & \sqrt{5}-1 \\ \sqrt{5}-1 & \sqrt{5}+1 \end{pmatrix}$$

$$C_2 = d \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{5} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = d \begin{pmatrix} -\sqrt{5} & 1 \\ -\sqrt{5} & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\sqrt{5}-1 & -\sqrt{5}+1 \\ -\sqrt{5}+1 & -\sqrt{5}-1 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{5}-1 & \sqrt{5}+1 \\ \sqrt{5}+1 & \sqrt{5}-1 \end{pmatrix}$$

$$C_4 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{5} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\sqrt{5}+1 & -\sqrt{5}-1 \\ -\sqrt{5}-1 & -\sqrt{5}+1 \end{pmatrix}$$

4. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$. Show that

$$\max_{\|\vec{x}\| \neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \lambda_1$$

$$\min_{\|\vec{x}\| \neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \lambda_p$$

Comment at what vectors x the max and min values are attained.

Let $B = \operatorname{span}\{b_1, \ldots, b_n\} \in \mathbb{R}^n$ ← orthonormal basis for $\mathbb{R}^n$.

$x = \sum_i c_i b_i$, $Ab_i = \lambda b_i$

$$x^T A x = x^T A \sum_i c_i b_i$$

$$= \left(\sum_i c_i b_i^T\right)\left(\sum_i x_i (Ab_i)\right)$$

$$= \left(\sum_i c_i b_i^T\right)\left(\sum_i c_i (\lambda_i b_i)\right)$$

$$= \sum_i (c_i)^2 (\lambda_i \cancel{b_i b_i^T}) \quad I_n$$

$$= \sum_i \lambda_i (c_i)^2$$

$$x^T x = \sum_i c_i b_i^T \sum c_i b_i$$

$$= \sum_i c_i^2$$

$$\therefore \frac{x^T A x}{x^T x} = \frac{\sum_i \lambda_i (c_i)^2}{\sum_i (c_i)^2} = \lambda_i$$

$$\lambda_p \leq \lambda_i \leq \lambda_1$$

$$\therefore \quad \lambda_p \leq \frac{x^T A x}{x^T x} \leq \lambda_1$$

$A\vec{v} = \lambda \vec{v}$.

$\therefore$ If $A = \lambda_1$, $\vec{v}$ has to be the corresponding eigenvector.

Same thing for if $A = \lambda_n$.

$\therefore$ max value : $\vec{x} = \vec{v}_1$

min value : $\vec{x} = \vec{v}_n$

5. Consider matrix

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

(a) Determine an SVD of A.

a. $A = U\Sigma V^T$ $\quad A^TA = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}$

$$\begin{vmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{vmatrix} = (4-\lambda)(13-\lambda) - 36 = 0$$

$$52 - 17\lambda + \lambda^2 - 36 = 0$$

$$\lambda^2 - 17\lambda + 16 = 0$$

$$(\lambda - 16)(\lambda - 1) = 0$$

$\therefore \lambda_1 = 16, \quad \lambda_2 = 1$

$\sigma_1 = 4 \quad \sigma_2 = 1$

$E\{\lambda_1 = 16\} = \left(\begin{array}{cc|c} -12 & 6 & 0 \\ 6 & -3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}. \quad v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$E\{\lambda_2 = 1\} = \left(\begin{array}{cc|c} 3 & 6 & 0 \\ 6 & 12 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\} \quad v_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{4}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \frac{8}{\sqrt{5}} \\ \frac{4}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

$u_2 = \frac{1}{\sigma_2} A v_2 = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

$\therefore A = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$

(b) Write A in the form of $\sum_{i=1}^{r} \sigma_i u_i v_i^T$, a sum of several rank-1 matrices.

$A = (\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T)$

$= 4\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} + 1\begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$

$$= 4\begin{pmatrix} \frac{2}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} + \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{4}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{8}{5} & \frac{16}{5} \\ \frac{4}{5} & \frac{8}{5} \end{pmatrix} + \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{4}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{10}{5} & \frac{15}{5} \\ 0 & \frac{10}{5} \end{pmatrix}$$

(check :)

$$= \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = AV.$$

(c) Notice that A is invertible. Determine an SVD of A^{-1}. Do you need to start from scratch?

$$A^{-1} = U\Sigma^{-1}V^T = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

(d) Determine an SVD of A^T. Do you need to start from scratch?

$$A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

6. What can one say about a matrix's eigenvalues and its singular values? Consider the following. Note that here σ_1 is the largest singualr value and σ_n the smallest of matrix A.

(a) Let $A \in M_2(\mathbb{R})$. Let $\boldsymbol{w} \in \mathbb{R}^2$ be a unit vector. Show that

$$\sigma_2 \leq \|A\boldsymbol{w}\| \leq \sigma_1$$

by tracing what happens to $\boldsymbol{w}$ under the three matrices in A's SVD.

$A\vec{w} = \bar{U}\Sigma\bar{V}^T\vec{w}$.

$\underbrace{V^T\vec{w}}$ put in unit circle. and change w_i into $\vec{e}_i$

$\underbrace{\Sigma}$ scale each $\vec{e}_i$ by a factor of σ_i, change them into $\sigma_i e_i$

$\underbrace{U}$ change of basis back into $(A\vec{w})_i$

$\|A\vec{w}_i\|$ would be equal to $\|\sigma_i e_i\| = |\sigma_i|\|e_i\| = \sigma_i$.

$\therefore$ $\sigma_2 \leq \|A\vec{w}\| \leq \sigma_1$.

(b) Show part (a) algebraically.

$w_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad w_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$A = U\Sigma V^T$. $\quad Aw = U\Sigma V^T w$. since U and V are orthogonal, they preserve norm so $\|Aw\| = \|U\Sigma V^T w\| = \|\Sigma w\|$

$\Sigma w_1 = \sigma_2$ and $\Sigma w_2 = \sigma_1$

$\therefore$ $\sigma_2 \leq \|\Sigma w\| \leq \sigma_1$

$\sigma_2 \leq \|Aw\| \leq \sigma_1$

(c) Let $A \in M_{m\times n}(\mathbb{R})$. Show that

$$\sigma_n\|v\| \leq \|Av\| \leq \sigma_1\|v\|$$

for any $v \in \mathbb{R}^n$.

$\vec{v}$ can be written as $\bar{V}c$, where $\bar{V}$ is orthogonal.

$A\vec{v} = \bar{U}\Sigma\bar{V}^T\bar{V}c = \bar{U}\Sigma c$

$\|\vec{v}\| = \|Vc\| = c$. and $\|A\vec{v}\| = \|\Sigma c\|$

$\sigma_n\|c\| \leq \|\Sigma c\| \leq \sigma_1\|c\|$

$\therefore \sigma_n\|\vec{v}\| \leq \|A\vec{v}\| \leq \sigma_n\|\vec{v}\|$.

(d) Let λ be a real eigenvalue of matrix $A \in M_n(\mathbb{R})$. Show that

$$\sigma_n \leq |\lambda| \leq \sigma_1$$

Since $A\vec{v} = \lambda\vec{v}$, $\|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$,

The equation from c can be written as

$\sigma_n\|\vec{v}\| \leq |\lambda|\|\vec{v}\| \leq \sigma_1\|\vec{v}\|$

so $\sigma_n \leq |\lambda| \leq \sigma_1$

(e) Consider the matrix A in Q5. Determine $\min_{\|x\|=1}\|Ax\|$. Comment at what vectors x the min value is attained.

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$\therefore$ when $x = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$, the corresponding eigenvalue is 1,

so $Ax = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$ $\qquad \|Ax\| = \sqrt{\frac{1}{5}+\frac{4}{5}} = 1$.

Therefore, the eigenvector corresponding to the smallest eigenvalue is the x that would make $\|Ax\|$ the smallest.

(f) Consider the matrix A in Q5. Determine $\max_{\|x\|=1} \|Ax\|$. Comment at what vectors x the max value is attained.

$A\vec{x} = \lambda \vec{x}$. the biggest λ is 4.

$\max_{\|x\|=1} \|Ax\| = \|\lambda \vec{x}\| = |\lambda| \, \|\vec{x}\| = 4 \times 1 = 4$.

the corresponding $\vec{x}$ would be the corresponding eigenvector of $\lambda = 4$, which is $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$.