MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 9

1. Compute the determinants of the following matrices:

$$\begin{pmatrix} 2 & 6 & 16 \\ -3 & -6 & 18 \\ 5 & 12 & 35 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 3 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 0 & 1 \\ -2 & 2 & -1 \\ 0 & 4 & -3 \end{pmatrix}, \begin{pmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

2. (a) Let $a, b, c \in \mathbb{R}$. Prove that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (c-a)(c-b)(b-a)$$

(b) Find the values of a for which the following set is a basis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} a-1 \\ -3 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ a+5 \\ 6 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ a-4 \end{pmatrix} \right\}$$

(c) Assume that,

$$\begin{vmatrix} a & x & l \\ b & y & m \\ c & z & n \end{vmatrix} = 2$$

Find:

$$\begin{vmatrix} 2a+3x & 2b+3y & 2c+3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{vmatrix}$$

3. Let $A, B \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Prove or disprove the following claims:

(a) $|A+B| = |A| + |B|$

(b) $|\lambda A| = \lambda |A|$

(c) $|\lambda A| = \lambda^n |A|$

(d) If A is anti-symmetric (that is, $A^T = -A$) and n is odd then A is not invertible.

(e) If A is anti-symmetric (that is, $A^T = -A$) and n is even then A is not invertible.

(f) If $AB = 0$ then $|A^2| + |B^2| = 0$.

(g) If $|A+B| = |A|$ then B is the zero matrix.

4. (a) Compute the determinant of the following $n \times n$ matrix:

$$\begin{pmatrix} 4 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 4 \end{pmatrix}$$

(b) For the matrix,

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Prove that $|A| = 1 + (-1)^{(n+1)}$. (Note the 1 on the left lowest corner).

5. For the following exercises $i = \sqrt{-1}$. And the field we are working with is $\mathbb{C}$.
   (a) Determine all complex numbers z such that $z^2 = i$.
   (b) Write $3 + 4i$ in its polar form.
   (c) Determine call $z \in \mathbb{C}$ such that $z^4 = 1$.
   (d) If $z \in \mathbb{C}$ is non-zero, what is z^{-1} in polar form? Compare $z, \bar{z}, z^{-1}$ in a sketch.

6. Let $f(x)$ be a polynomial with real coefficients. Show that if $z \in \mathbb{C}$ is a root of $f(x)$ then so is its complex conjugate $\bar{z}$.

7. Let $A \in M_n(\mathbb{R})$ that is invertible with eigen-pair (λ, v).
   (a) Is v an eigenvector of A^5? What's its corresponding eigenvalue? Generalize.
   (b) Is v an eigenvector of A^{-1}? What's its corresponding eigenvalue?
   (c) Is v an eigenvector of $A^2 + 3A + 6I_n$? What's its corresponding eigenvalue?

1. Compute the determinants of the following matrices:

$$\begin{pmatrix} 2 & 6 & 16 \\ -3 & -6 & 18 \\ 5 & 12 & 35 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 3 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 0 & 1 \\ -2 & 2 & -1 \\ 0 & 4 & -3 \end{pmatrix}, \begin{pmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 2 & 6 & 16 \\ -3 & -6 & 18 \\ 5 & 12 & 35 \end{vmatrix} = \begin{matrix} 2 & 6 & 16 & 2 & 6 \\ -3 & -6 & 18 & -3 & -6 \\ 5 & 12 & 35 & 5 & 12 \end{matrix}$$

$$= -12\times 35 + 6\times 18\times 5 - 3\times 12\times 16 + 5\times 6\times 16 - 2\times 12\times 18 + 3\times 6\times 35$$

$$= 222$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 3 & 2 & 0 \end{vmatrix} = \begin{matrix} 1 & 2 & 3 & 1 & 2 \\ -1 & 5 & 2 & -1 & 5 \\ 3 & 2 & 0 & 3 & 2 \end{matrix}$$

$$= 12 - 6 - 45 - 4 = -43$$

$$\begin{vmatrix} 4 & 0 & 1 \\ -2 & 2 & -1 \\ 0 & 4 & -3 \end{vmatrix} = \begin{matrix} 4 & 0 & 1 & 4 & 0 \\ -2 & 2 & -1 & -2 & 2 \\ 0 & 4 & -3 & 0 & 4 \end{matrix}$$

$$= -24 - 8 + 16 = -16$$

$$\left| \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 & 1 \\ -2 & 2 & -1 \\ 0 & 4 & -3 \end{pmatrix} \right| = -43 \times -16 = 688.$$

$$\begin{vmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 2 & 1 \end{vmatrix} = -1 \begin{pmatrix} 4 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & 3 & 4 \end{pmatrix} - 2 \begin{pmatrix} 4 & -4 & 1 \\ 1 & 2 & 3 \\ 2 & 0 & 4 \end{pmatrix} + 1 \begin{pmatrix} 4 & -4 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

$$= -1 \times -29 - 2\times 20 + 1 \times 28$$

$$= 29 - 40 + 28 = 17$$

2. (a) Let $a, b, c \in \mathbb{R}$. Prove that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (c-a)(c-b)(b-a)$$

$$\text{LHS} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} : \begin{matrix} 1 & a & a^2 & 1 & a \\ 1 & b & b^2 & 1 & b \\ 1 & c & c^2 & 1 & c \end{matrix}$$

$$= bc^2 + ab^2 + a^2c - a^2b - b^2c - ac^2$$

$$\begin{aligned} \text{RHS} &= (c-a)(c-b)(b-a) \\ &= (c^2 - bc - ac + ab)(b-a) \\ &= bc^2 - ac^2 - b^2c + abc - abc + a^2c + ab^2 - a^2b \\ &= bc^2 - ac^2 - b^2c + a^2c + ab^2 - a^2b \end{aligned}$$

$\therefore$ LHS = RHS

$\therefore$ $\checkmark$

(b) Find the values of a for which the following set is a basis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} a-1 \\ -3 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ a+5 \\ 6 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ a-4 \end{pmatrix} \right\}$$

$$x \begin{pmatrix} a-1 \\ -3 \\ -6 \end{pmatrix} + y \begin{pmatrix} 3 \\ a+5 \\ 6 \end{pmatrix} + z \begin{pmatrix} -3 \\ -3 \\ a-4 \end{pmatrix} = 0. \Rightarrow x = y = z = 0.$$

$$\therefore \det \begin{pmatrix} a-1 & 3 & -3 \\ -3 & a+5 & -3 \\ -6 & 6 & a-4 \end{pmatrix} \neq 0.$$

$$\begin{matrix} a-1 & 3 & -3 & a-1 & 3 \\ -3 & a+5 & -3 & -3 & a+5 \\ -6 & 6 & a-4 & -6 & 6 \end{matrix} \quad \begin{aligned} & (a-1)(a+5)(a-4) + 54 + 54 \\ & - 18(a+5) + 18(a-1) + 9(a-4) \\ = {} & a^3 - 12a - 16. \\ = {} & (a-4)(a+2)^2. \end{aligned}$$

if $(a-4)(a+2)^2 = 0$,

$a = 4$ or $a = -2$. $\therefore$ a = anything other than -2 or 4.

(c) Assume that,

$$\begin{vmatrix} a & x & l \\ b & y & m \\ c & z & n \end{vmatrix} = 2$$

Find:

$$\begin{vmatrix} 2a+3x & 2b+3y & 2c+3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{vmatrix}$$

$\det(A) = \det(A^T)$.

$$\therefore \begin{vmatrix} 2a+3x & 2b+3y & 2c+3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{vmatrix} = \begin{vmatrix} 2a+3x & l+x & 7l \\ 2b+3y & m+y & 7m \\ 2c+3z & n+z & 7n \end{vmatrix}.$$

according to rules of det:

$$\begin{vmatrix} a & x & l \\ b & y & m \\ c & z & n \end{vmatrix} = 2 \Rightarrow \begin{vmatrix} 2a & x & 7l \\ 2b & y & 7m \\ 2c & z & 7n \end{vmatrix} = 2 \times 2 \times 7 = 28.$$

$\because$ adding a scaled column onto another doesn't change det,

$$\begin{vmatrix} 2a+3x & l+x & 7l \\ 2b+3y & m+y & 7m \\ 2c+3z & n+z & 7n \end{vmatrix} = \begin{vmatrix} 2a & x & 7l \\ 2b & y & 7m \\ 2c & z & 7n \end{vmatrix} = 28.$$

3. Let $A, B \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Prove or disprove the following claims:

(a) $|A+B| = |A| + |B|$

False. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $A+b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

$|A| = 1$ $|B| = 0$ $|A+B| = 0$

$|A+B| \neq |A| + |B|$.

(b) $|\lambda A| = \lambda |A|$

False. let $A \in M_{m \times n}(\mathbb{R})$. $A = \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{pmatrix}$.

$\lambda A = \begin{pmatrix} | & | & & | \\ \lambda c_1 & \lambda c_2 & \cdots & \lambda c_n \\ | & | & & | \end{pmatrix}$.

$\therefore \det(\lambda A) = \lambda^n \det(A)$. $\neq \lambda \det(A)$.

(c) $|\lambda A| = \lambda^n |A|$

True. Refer to 3.b.

(d) If A is anti-symmetric (that is, $A^T = -A$) and n is odd then A is not invertible.

True.

$A^T = -A$.

$|A^T| = A$

since A is odd, $|-A| = (-1)^{\overset{\text{odd.}}{n}} |A| = -|A|$.

$|A^T| = |A|$

if $|-A| = |A|$

$-|A| = |A|$

$\therefore |A| = 0$

$\therefore$ not invertible.

(e) If A is anti-symmetric (that is, $A^T = -A$) and n is even then A is not invertible.

$$\begin{vmatrix} a & b \\ -b & a \end{vmatrix} = a^2 + b^2 \qquad \text{let } a = 0 \qquad b = 1$$

$$\det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1 \neq 0 \quad \therefore \text{ invertible.} \qquad \therefore \text{ false}$$

(f) If $AB = 0$ then $|A^2| + |B^2| = 0$.

let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\qquad AB = 0.$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$|A^2| = 1 \qquad |B^2| = 0. \qquad |A^2| + |B^2| = 1 + 0 = 1 \neq 0.$

$\therefore$ false. proved by counterexample.

(g) If $|A + B| = |A|$ then B is the zero matrix.

let $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad |A| = 1$

$\qquad |A+B| = 1.$

$|A| = |A+B|$ but $B \neq 0.$

$\therefore$ false. proved by counterexample.

4. (a) Compute the determinant of the following $n \times n$ matrix:

$$\begin{pmatrix} 4 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 4 \end{pmatrix}$$

Try smaller scales:

$n=2$: $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} = 16 - 1 = 15$

$n=3 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} = 4\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}$

$= 15 \times 4 - 1 \times (3) + 1 \times (-3).$

$= 54.$

$n=4 = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}$

$n=3:\ 4 \times 15 - 2 \times 3^1$

$4 \times 15 - (n-1)3^{(n-2)}$

$= 4\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & 4 \\ 1 & 1 & 1 \end{vmatrix}$

$= 4 \times 54 - 1 \times 9 - 1 \times 9 - 1 \times 9.$

$= 216 - 3 \times 9 = 189.$

$n=4:\ 4 \times (4 \times 15 - 2 \times 3) - 3 \times 3^2$

$4 \times (4 \times 15 - 2 \times 3) - (n-1)3^{(n-2)}$

$n=5 = \begin{pmatrix} 4 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$

$= 4\begin{vmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 \end{vmatrix}$

$$+1\begin{vmatrix} 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 4\times 189 - 4\times 27.$$

$n=5$: $4\times[(4\times 4\times 15 - 2\times 3) - (3\times 3^2)] - 4\times 3^3$

$n=2$ $\quad 15$ $\leftarrow$ base case.

$n=3$ $\quad 4\times 15 - 2\times 3^1$

$n=4$ $\quad 4(4\times 15 - 2\times 3^1) - 3\times 3^2$

$n=5$ $\quad 4(4(4\times 15 - 2\times 3^1) - 3\times 3^2) - 4\times 3^3$

n . $\quad 4(\det(n-1)) - (n-1)\, 3^{(n-2)}$

If row reduce :

$n=2$: $4\times \frac{15}{4}$

$n=3$: $4\times \frac{15}{4} \times \frac{18}{5}$

$n=4$: $4\times \frac{15}{4} \times \frac{18}{5} \times \frac{21}{6}$.

$\vdots$

n . $4\cdot \left(\prod_{p=2}^{n} \frac{3(p+3)}{p+2} \right)$

(b) For the matrix,

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Prove that $|A| = 1 + (-1)^{(n+1)}$. (Note the 1 on the left lowest corner).

Consider smaller ns:

$n=2$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\det = 0$.

$n=3$ $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ $\det = 2$.

$n=4$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} = 1 \times \underbrace{\begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \end{array}}_{u} - 1 \underbrace{\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \end{array}}_{v}$$

$$V_1 = u_1 - u_2 + u_3.$$

$$\therefore \det(u) = \det(v).$$

$$\therefore \det \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} = 0.$$

$n=5$.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} = 1 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 + 1 = 2.$$

In general,

$$|A| = \underbrace{\det \begin{pmatrix} \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{\cdots} & \cancel{0} & \cancel{0} \\ \cancel{0} & 1 & 1 & 0 & \cdots & 0 & 0 \\ \cancel{0} & 0 & 1 & 1 & \cdots & 0 & 0 \\ \cancel{\vdots} & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \cancel{0} & 0 & 0 & 0 & \cdots & 1 & 1 \\ \cancel{1} & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}}_{\text{always } = 1 \text{ because it's an upper triangular matrix with the diagonal full of 1.}} - \underbrace{\det \begin{array}{c} \begin{pmatrix} \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{\cdots} & \cancel{0} & \cancel{0} \\ 0 & \cancel{1} & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cancel{0} & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \cancel{\vdots} & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cancel{0} & 0 & 0 & \cdots & 1 & 1 \\ 1 & \cancel{0} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \\ \begin{matrix} \downarrow & & \downarrow & \downarrow & & & \downarrow \\ c_1 & & c_2 & c_3 & \cdots & & c_n \end{matrix} \end{array}}_{\text{to change } c_1 \text{ from } \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \text{ to } \begin{pmatrix} \pm 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{, need to add/subtract other rows. to/from } c_1.}$$

if $n = \text{odd}$: $c_1 - c_n + c_{(n-1)} \cdots - c_3 + c_2 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

if $n = \text{even}$: $c_1 - c_n + c_{(n-1)} - \cdots + c_3 - c_2 = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

$\therefore$ if $n = \text{odd}$,

$$\det \begin{pmatrix} \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{\cdots} & \cancel{0} & \cancel{0} \\ 0 & \cancel{1} & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cancel{0} & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \cancel{\vdots} & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cancel{0} & 0 & 0 & \cdots & 1 & 1 \\ 1 & \cancel{0} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = 1$$

and if $n = \text{even}$,

$$\det \begin{pmatrix} \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{\cdots} & \cancel{0} & \cancel{0} \\ 0 & \cancel{1} & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cancel{0} & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \cancel{\vdots} & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cancel{0} & 0 & 0 & \cdots & 1 & 1 \\ 1 & \cancel{0} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = -1.$$

$\therefore$ if $n = \text{odd}$, $|A| = 1 - 1 = 0$

$n = \text{even}$, $|A| = 1 - (-1) = 2$

5. For the following exercises $i = \sqrt{-1}$. And the field we are working with is $\mathbb{C}$.

(a) Determine all complex numbers z such that $z^2 = i$.

(b) Write $3 + 4i$ in its polar form.

(c) Determine call $z \in \mathbb{C}$ such that $z^4 = 1$.

(d) If $z \in \mathbb{C}$ is non-zero, what is z^{-1} in polar form? Compare $z, \bar{z}, z^{-1}$ in a sketch.

a. let $z = x + yi$

$z^2 = i$

$(x+yi)^2 = i$

$x^2 - y^2 + 2ixy = i$

$x^2 - y^2 + \underbrace{(2xy-1)i}_{=0} = 0.$

$\therefore x^2 - y^2 = 0.$

$(x+y)(x-y) = 0$

$\therefore y = \pm x.$

if $y = x$: $2x^2 = 1.$ $\quad x = \pm\left(\frac{\sqrt{2}}{2}\right)$

if $y = -x$: $2x^2 = -1$ $\quad x = \pm \frac{\sqrt{2}}{2} i$ ~~1~~.

$z = x + iy = \underline{x + ix} = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right).$

b. $(r, \theta) = 5e^{i \tan^{-1}\left(\frac{4}{3}\right)} = 5e^{i \cdot 53.13^\circ}$ or $3\cos\theta + 4\sin\theta\, i$

c. $z = 1 + 0i$

$= \cos(2\pi k) + i\sin(2\pi k)$

$= \cos(4 \times \frac{1}{2}\pi k) + i\sin(4 \times \frac{1}{2}\pi k)$

$z^4 = (\cos(\frac{1}{2}\pi k) + i\sin(\frac{1}{2}\pi k))^4$

$\therefore z = \cos(\frac{1}{2}\pi k) + i\sin(\frac{1}{2}\pi k)$

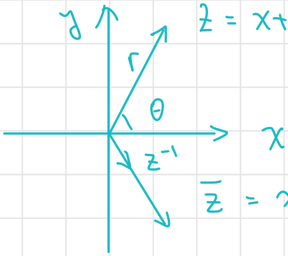
$\therefore z = \cos(\frac{\pi}{2} \cdot 0) + i\sin(\frac{\pi}{2} \cdot 0) = 1$

$z = \cos(\frac{\pi}{2} \cdot 1) + i\sin(\frac{\pi}{2} \cdot 1) = i$

$z = \cos(\frac{\pi}{2} \cdot 2) + i\sin(\frac{\pi}{2} \cdot 2) = -1$

$z = \cos(\frac{\pi}{2} \cdot 3) + i\sin(\frac{\pi}{2} \cdot 3) = -i$

d.



$z = x+iy.$

$\bar{z} = x-iy.$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{(x-iy)}{(x+iy)^2} = \frac{x-iy}{x^2-y^2}.$$

$z = (re^{i\theta})$

polar form: $z^{-1} = r^{-1}(\cos\theta + i\sin\theta).$

6. Let $f(x)$ be a polynomial with real coefficients. Show that if $z \in \mathbb{C}$ is a root of $f(x)$ then so is its complex conjugate $\bar{z}$.

$P(z) =$

Given $a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n = 0$. Need to show

$P(\bar{z}) = a_0 + a_1 \bar{z} + a_2 \bar{z}^2 + \dots + a_n \bar{z}^n = 0$.

$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n = 0$.

$\therefore \overline{P(z)} = \overline{a_0} + \overline{a_1 z} + \overline{a_2 z^2} + \dots + \overline{a_n z^n} = 0$.

$= \bar{a}_0 + \bar{a}_1 \bar{z} + \bar{a}_2 \bar{z}^2 + \dots + \bar{a}_n \bar{z}^n = 0$.

since $f(x)$ is a polynomial with real polynomials,

$a_0 = \bar{a}_0$, $a_1 = \bar{a}_1$, $\dots$, $a_n = \bar{a}_n$

$\therefore a_0 + a_1 \bar{z} + a_2 \bar{z}^2 + \dots + a_n \bar{z}^n = 0$.

$\therefore \bar{z}$ is also a root for $f(x)$.

7. Let $A \in M_n(\mathbb{R})$ that is invertible with eigen-pair (λ, v).

(a) Is v an eigenvector of A^5? What's its corresponding eigenvalue? Generalize.

(b) Is v an eigenvector of A^{-1}? What's its corresponding eigenvalue?

(c) Is v an eigenvector of $A^2 + 3A + 6I_n$? What's its corresponding eigenvalue?

a. Eigen-pair : $A\vec{v} = \lambda\vec{v}$

$$A(A(A(A(A(\vec{v}))))) = \lambda^5 \vec{v}.$$

$$\therefore A^5 \vec{v} = \lambda^5 \vec{v}$$

$\therefore$ (A^5, λ^5) is the eigen-pair.

b.

$$A\vec{v} = \lambda\vec{v}$$
$$A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$$
$$I\vec{v} = A^{-1}\lambda\vec{v}$$
$$\vec{v} = A^{-1}\lambda\vec{v}$$
$$\lambda^{-1}\vec{v} = A^{-1}\vec{v}$$

$\therefore$ (A^{-1}, λ^{-1}) is the eigen-pair.

c.

$$A\vec{v} = \lambda\vec{v}$$
$$(A^2 + 3A + 6I_n)\vec{v} = A^2\vec{v} + 3A\vec{v} + 6I_n\vec{v}$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$\lambda^2\vec{v} + 3\lambda\vec{v} + 6\vec{v}$$
$$= (\lambda^2 + 3\lambda + 6)\vec{v}$$

$\therefore$ $(A^2 + 3A + 6I, \lambda^2 + 3\lambda + 6)$ is the eigen-pair.