MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 4

1. In each of the following you are given a statement, which may be true or false. Determine whether the statement is correct and show how you reached this conclusion.
   (a) $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \in \mathrm{span}\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$
   (b) $2 + 3x + 2x^2 - x^3 \in \mathrm{span}\{1 - x^3, 2 + x + x^2, 3 - x\}$
   (c) $\mathrm{span}\{\begin{pmatrix} 5 & -2 \\ -5 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}\} \subseteq \mathrm{span}\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$
   (d) $\mathrm{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\} = \mathrm{span}\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}$
   (e) $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\}$ spans $\mathbb{R}^2$.
   (f) $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ spans $\mathbb{R}^2$.
   (g) $\{1 - x + x^2, x - x^2 + x^3, 1 + x^2 - x^3, x^3\}$ spans $\mathbb{R}_3[x]$.

2. In each of the following you are given a vector space (you do not need to prove that this is indeed a vector space). Find a spanning set for each of these vector spaces.
   (a) $\{\begin{pmatrix} a+b+c \\ a-2b \\ 3a-2c \\ 4c-b \end{pmatrix} : a, b, c \in \mathbb{R}\}$
   (b) $\{A \in M_2(\mathbb{R}) : A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$
   (c) $\{p(x) \in \mathbb{R}_3[x] : p'(1) = 0\}$
   (d) $\{p(x) \in \mathbb{R}_n[x] : p(1) = p(-1)\}$

3. In each of the following you are given a set, determine whether it is linearly independent or linearly dependent, show how you reach your conclusion.
   (a) $\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$
   (b) $\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}$
   (c) $\{1 - x^3, 2 + x + x^2, 3 - x, 1 + x + x^2 + x^3\}$
   (d) $\{f(x) = \sin^2 x, g(x) = \cos^2(x), h(x) = 1\}$ (Note that $h(x)$ is the constant function which is equal to 1 for every x).

4. Let V be a vector space and w_1, w_2, w_3 in V be such that $\{w_1, w_2, w_3\}$ is linearly independent. Prove or disprove the following claims.
   (a) The set $\{w_1 + w_2 + w_3, w_2 + w_3, w_3\}$ is linearly independent.

(b) The set $\{w_1 + 2w_2 + w_3, w_2 + w_3, w_1 + w_2\}$ is linearly independent.

5. Let V be a vector space and let $S \subset V$ and $T \subset V$ be two finite subsets of V. Prove or disprove the following claims.

   (a) If $S \subset T$ and S is linearly independent then T is linearly independent.

   (b) If $S \subset T$ and T is linearly independent then S is linearly independent.

   (c) If S and T are linearly independent then $S \cap T$ is either empty or linearly independent (Remark: sometimes people consider an empty set to be linearly independent).

   (d) If S and T are linearly independent then $S \cup T$ is linearly independent.

   (e) If $W = \text{span}S$ and $U = \text{span}T$ then $W + U = \text{span}(S \cup T)$.

6. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.

   (a) Let V be a vector space which satisfies dim$V$=3. Then there exist, a subspace W of V and a subspace U of W (that is, $U \subset W \subset V$) such that dim$U$=1 and dim$W$=2.

   (b) Let V be a vector space which satisfies dim$V$=3 and let W be a **non–trivial** subspace of V and U be a **non–trivial** subspace of W (that is, $U \subset W \subset V$) then dim$U$=1 and dim$W$=2.

   (c) Let V be a vector space which satisfies dim$V$=3 and let $v_1, v_2, v_3 \in V$ be such that $\{v_1, v_2\}$ are linearly independent, $\{v_2, v_3\}$ are linearly independent, and $\{v_3, v_1\}$ are linearly independent. Then $\{v_1, v_2, v_3\}$ is a basis for V.

   (d) Let V be a vector space and $v_1, ..., v_n \in V$ then: $\{v_1, ..., v_n\}$ is linearly independent iff $\dim\big(\text{span}\{v_1, ..., v_n\}\big) = n$.

   (e) Let V be a vector space and let $V_1, V_2, V_3 \subset V$ be such that $V_1 + V_2 = V_1 + V_3$ and dimV_2 =dimV_3 then $V_2 = V_3$. (The sum of two subspaces was defined in previous HW's).

1. In each of the following you are given a statement, which may be true or false. Determine whether the statement is correct and show how you reached this conclusion.

(a) $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \in \text{span}\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$

check if

(a) $\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\left(\begin{array}{ccc|c} 2 & -1 & -2 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ -1 & 0 & -1 & -1 \end{array}\right)$ is consistent

switch. $\left(\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & -2 & 1 \\ -1 & 0 & -1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -7 & -6 & -3 \\ 0 & 3 & 1 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & -2 & -5 \end{array}\right)$

$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 17 \end{array}\right)$

$\therefore$ it is inconsistent.

$\therefore$ doesn't exists $c_1, c_2, c_3 \in \mathbb{R}$

s.t. $c_1\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} + c_2\begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} + c_3\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

$\therefore$ incorrect

(b) $2 + 3x + 2x^2 - x^3 \in \text{span}\{1 - x^3, 2 + x + x^2, 3 - x\}$

let $c_1, c_2, c_3 \in \mathbb{R}$.

we need to check if $\exists\, c_1, c_2, c_3$ s.t.

$c_1(1-x^3) + c_2(2+x+x^2) + c_3(3-x) = 2+3x+2x^2-x^3$

LHS:

$c_1 - \underline{c_1x^3} + 2c_2 + \underline{c_2x} + \underline{c_2x^2} + 3c_3 - \underline{c_3x}$

$= -c_1x^3 + c_2x^2 + (c_2 - c_3)x + (c_1 + 2c_2 + 3c_3)$

$? \begin{cases} -c_1 = -1 \\ c_2 = 2 \\ c_2 - c_3 = 3 \\ c_1 + 2c_2 + 3c_3 = 2 \end{cases}$

another way to do is to check if

$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & -1 \end{array}\right)$ is consistent.

$c_1 = 1 \quad c_2 = 2. \quad c_3 = -1$

$\therefore$ ✓ It's true.

(c) $\text{span}\{\begin{pmatrix} 5 & -2 \\ -5 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}\} \subseteq \text{span}\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$

check if ① $\left(\begin{array}{ccc|c} 2 & -1 & -2 & 5 \\ 0 & 1 & 1 & -2 \\ 1 & 3 & 2 & -5 \\ -1 & 0 & -1 & -3 \end{array}\right)$ and ② $\left(\begin{array}{ccc|c} 2 & -1 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ -1 & 0 & -1 & -1 \end{array}\right)$ are consistent.

①

$$\left(\begin{array}{ccc|c} 2 & -1 & -2 & 5 \\ 0 & 1 & 1 & -2 \\ 1 & 3 & 2 & -5 \\ -1 & 0 & -1 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & -5 \\ 0 & 1 & 1 & -2 \\ -1 & 0 & -1 & -3 \\ 2 & -1 & -2 & 5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & 3 & 1 & -8 \\ 0 & -7 & -6 & 15 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

∴ It's consistent.

②

$$\left(\begin{array}{ccc|c} 2 & -1 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ -1 & 0 & -1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ 2 & -1 & -2 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 1 & 3 \\ 0 & -7 & -6 & -7 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right)$$

∴ it's also consistent.

∴ The statement is true.

(d) $\text{span}\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\} = \text{span}\{\begin{pmatrix}3\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\-1\end{pmatrix}\}$

Check: ① $\left(\begin{array}{ccc|c}3&1&1&1\\2&0&2&0\\1&1&-1&1\end{array}\right)$ and ② $\left(\begin{array}{ccc|c}3&1&1&1\\2&0&2&1\\1&1&-1&0\end{array}\right)$ are consistent?

① $\left(\begin{array}{ccc|c}1&1&-1&1\\2&0&2&0\\3&1&1&1\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&1&-1&1\\0&-2&4&-2\\0&-2&4&-2\end{array}\right)$ $\therefore$ consistent

② $\left(\begin{array}{ccc|c}1&1&-1&0\\2&0&2&1\\3&1&1&1\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&1&-1&0\\0&-2&4&1\\0&-2&4&1\end{array}\right)$ $\therefore$ consistent.

$\therefore \text{span}\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\} \subseteq \text{span}\{\begin{pmatrix}3\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\}$

Check ③ $\left(\begin{array}{cc|c}1&1&3\\0&1&2\\1&0&1\end{array}\right)$, ④ $\left(\begin{array}{cc|c}1&1&1\\0&1&0\\1&0&1\end{array}\right)$, ⑤ $\left(\begin{array}{cc|c}1&1&1\\0&1&2\\1&0&-1\end{array}\right)$ are consistent?

③ $\left(\begin{array}{cc|c}1&1&3\\0&1&2\\1&0&1\end{array}\right) \Rightarrow \left(\begin{array}{cc|c}1&1&3\\0&1&2\\0&-1&-2\end{array}\right) \Rightarrow \left(\begin{array}{cc|c}1&1&3\\0&1&2\\0&0&0\end{array}\right)$ $\therefore$ consistent

④ $\left(\begin{array}{cc|c}1&1&1\\0&1&0\\1&0&1\end{array}\right) \Rightarrow \left(\begin{array}{cc|c}1&1&1\\0&1&0\\0&-1&0\end{array}\right) \Rightarrow \left(\begin{array}{cc|c}1&1&1\\0&1&0\\0&0&0\end{array}\right)$ $\therefore$ consistent

⑤ $\left(\begin{array}{cc|c}1&1&1\\0&1&2\\1&0&-1\end{array}\right) \Rightarrow \left(\begin{array}{cc|c}1&1&1\\0&1&2\\0&-1&-2\end{array}\right) \Rightarrow \left(\begin{array}{cc|c}1&1&1\\0&1&2\\0&0&0\end{array}\right)$ $\therefore$ consistent

$\therefore \text{span}\{\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\} \subseteq \text{span}\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\}$.

$\therefore$ two spans equal to each other. $\therefore$ True.

(e) $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\}$ spans $\mathbb{R}^2$.

$\nexists\, c \in \mathbb{R}$ s.t. $c\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

$\therefore$ These are two vectors are linearly independent

$\therefore$ yes, the set spans $\mathbb{R}^2$.

(f) $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ spans $\mathbb{R}^2$.

$-1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\therefore$ They are linearly dependent

$\therefore$ The set only spans a line but not the entire $\mathbb{R}^2$.

(g) $\{1-x+x^2, x-x^2+x^3, 1+x^2-x^3, x^3\}$ spans $\mathbb{R}_3[x]$.

Check if $\exists\ C_1, C_2, C_3, C_4 \in \mathbb{R}$ s.t.
$C_1(1-x+x^2)+C_2(x-x^2+x^3)+C_3(1+x^2-x^3)+C_4(x^3)=0$
while at least one of C_1, C_2, C_3, C_4 is not 0.

$\underline{C_1-C_1x+C_1x^2}+\underline{C_2x-C_2x^2+C_2x^3}+\underline{C_3+C_3x^2-C_3x^3}+\underline{C_4x^3}$
$=(C_4+C_2-C_3)x^3+(C_1-C_2+C_3)x^2+(-C_1+C_2)x$
$+(C_1+C_3)$

And $\mathbb{R}_3[x]$
$= a_1x^3+a_2x^2+a_3x+a_4$

$a_1, a_2, a_3, a_4 \in \mathbb{R}$.

$\therefore$ it is possible that

$$\begin{cases} C_4+C_2-C_3 = a_1 \\ C_1-C_2+C_3 = a_2 \\ -C_1+C_2 = a_3 \\ C_1+C_3 = a_4 \end{cases}$$

$\therefore$ statement is true.

2. In each of the following you are given a vector space (you do not need to prove that this is indeed a vector space). Find a spanning set for each of these vector spaces.

(a) $\left\{\begin{pmatrix} a+b+c \\ a-2b \\ 3a-2c \\ 4c-b \end{pmatrix} : a,b,c \in \mathbb{R}\right\}$

(b) $\{A \in M_2(\mathbb{R}) : A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$

(c) $\{p(x) \in \mathbb{R}_3[x] : p'(1) = 0\}$

(d) $\{p(x) \in \mathbb{R}_n[x] : p(1) = p(-1)\}$

a. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 4 \end{pmatrix} \right\}$

b. A needs to be in the form $\begin{pmatrix} 2a & -a \\ 2b & -b \end{pmatrix}$.

$\therefore$ span $\left\{ \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix} \right\}$

c. $p(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 : a_1, a_2, a_3, a_4 \in \mathbb{R}$.

$p'(1) = 0a_1 + 1a_2 + 2a_3 + 3a_4 = 0$.

$\therefore \begin{pmatrix} 0 & 1 & 2 & 3 & | & 0 \end{pmatrix}$

$a_2 = -2a_3 - 3a_4$

$\left\{\begin{aligned} a_1 &= s_1 \\ a_2 &= -2s_2 - 3a_3. \\ a_3 &= s_2 \\ a_4 &= s_3 \end{aligned}\right.$

$\therefore \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} s_1 + \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} s_2 + \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} s_3$

$\therefore$ spanning set $=$ span $\{1, -2x + x^2, -3x + x^3\}$

d. Consider a less abstract example of $p(x) \in \mathbb{R}_n[x]$,

$P_3(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$, where $a_i \in \mathbb{R}$.

If $P_3(1) = P_3(-1)$, $a_0 + a_1 + a_2 + a_3 = a_0 - a_1 + a_2 - a_3$.

$\therefore\ a_1 + a_3 = -(a_1 + a_3)$

$\therefore\ a_1 + a_3 = 0$. and coefficients of even degree terms do not matter.

$\therefore$ spanning set $= \{ p(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n \mid \sum_{k=0}^{n/2} a_{2k+1} = 0 \}$

3. In each of the following you are given a set, determine whether it is linearly independent or linearly dependent, show how you reach your conclusion.

(a) $\left\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\right\}$

(a). check if $\exists$ $c_1, c_2, c_3 \in \mathbb{R}$ s.t.

$c_1 \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} + c_2 \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} + c_3 \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} = \vec{0}$ implies $c_1 = c_2 = c_3 = 0$.

$$\left(\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -7 & -6 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

three pivots

$\therefore$ exactly 1 solution, which is the trivial solution.

$\therefore c_1 = c_2 = c_3 = 0$.

$\therefore$ a is linearly independent

(b) $\left\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right\}$

check if $\left(\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right)$ has only one solution, the trivial solution.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 0 & 2 & 0 \\ 3 & 1 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

2 pivots for 3 columns.

$\therefore$ infinite solutions.

$\therefore$ linearly dependent.

(c) $\{1 - x^3, 2 + x + x^2, 3 - x, 1 + x + x^2 + x^3\}$

Let $c_1, c_2, c_3, c_4 \in \mathbb{R}$

check if $c_1(1-x^3) + c_2(2+x+x^2) + c_3(3-x)$
$+ c_4(1+x+x^2+x^3) = 0$ implies $c_1 = c_2 = c_3 = c_4 = 0$

LHS $= \underline{c_1} - \underline{c_1 x^3} + \underline{2c_2} + \underline{c_2 x} + \underline{c_2 x^2} + \underline{3c_3} - \underline{c_3 x} +$
$\underline{c_4} + \underline{c_4 x} + \underline{c_4 x^2} + \underline{c_4 x^3}$
$= (c_4 - c_1)x^3 + (c_2 + c_4)x^2 + (c_2 - c_3 + c_4)x$
$+ (c_1 + 2c_2 + 3c_3 + c_4)$

$$\begin{cases} c_4 - c_1 = 0 \\ c_2 + c_4 = 0 \\ c_2 - c_3 + c_4 = 0 \\ c_1 + 2c_2 + 3c_3 + c_4 = 0. \end{cases} \Rightarrow \begin{aligned} c_4 &= c_1 \\ c_2 &= -c_4 = -c_1 \\ c_2 &= c_3 - c_4 \end{aligned}$$

Only solution is $c_1 = c_2 = c_3 = c_4 = 0$
$\therefore$ linearly independent.

(d) $\{f(x) = \sin^2 x, g(x) = \cos^2(x), h(x) = 1\}$ (Note that $h(x)$ is the constant function which is equal to 1 for every x).

? $\exists$ $c_1, c_2 \in \mathbb{R}$. s.t. $c_1 \sin^2(x) + c_2 \cos^2(x) + 1 = 0$
that c_1, c_2 don't have to be 0s?

since $\sin^2(x) + \cos^2(x) = 1$,
$- \sin^2(x) - \cos^2(x) = -1$
$\therefore$ if $c_1 = c_2 = -1$,

$c_1 f(x) + c_2 g(x) + h(x) = 0$ doesn't imply $c_1 = c_2 = 0$.

$\therefore$ linearly dependent.

4. Let V be a vector space and w_1, w_2, w_3 in V be such that $\{w_1, w_2, w_3\}$ is linearly independent. Prove or disprove the following claims.

(a) The set $\{w_1 + w_2 + w_3, w_2 + w_3, w_3\}$ is linearly independent.

(a) let $c_1, c_2, c_3 \in \mathbb{R}$.

$$c_1(w_1 + w_2 + w_3) + c_2(w_2 + w_3) + c_3 w_3$$
$$= c_1 w_1 + c_1 w_2 + c_1 w_3 + c_2 w_2 + c_2 w_3 + c_3 w_3$$

① $$= c_1 w_1 + (c_1 + c_2) w_2 + (c_1 + c_2 + c_3) w_3.$$

$\{w_1, w_2, w_3\}$ is linearly independent means $d_1 w_1 + d_2 w_2 + d_3 w_3 = 0$ implies $d_1 = d_2 = d_3 = 0$. ($d_1, d_2, d_3 \in \mathbb{R}$).

since $c_1 - c_3$ and $d_1 - d_3$ are all arbitrary.

$c_1 = d_1$, $c_1 + c_2 = d_2$, $c_1 + c_2 + c_3 = d_3$.

so $c_1 w_1 + (c_1 + c_2) w_2 + (c_1 + c_2 + c_3) w_3 = 0$

implies $c_1 = c_1 + c_2 = c_1 + c_2 + c_3 = 0$

$\therefore$ this set is also linearly independent.

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another approach: (matrix approach)

check how many solutions are there for

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

it only has a unique solution, which is the trivial one.

$\therefore$ it's linearly independent.

(b) The set $\{w_1 + 2w_2 + w_3, w_2 + w_3, w_1 + w_2\}$ is linearly independent.

check how many solutions are in $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$

$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\therefore$ There are infinitely many solutions

$\therefore$ The set is linearly dependent

$\therefore$ False.

5. Let V be a vector space and let $S \subset V$ and $T \subset V$ be two finite subsets of V. Prove or disprove the following claims.

(a) If $S \subset T$ and S is linearly independent then T is linearly independent.

(b) If $S \subset T$ and T is linearly independent then S is linearly independent.

(c) If S and T are linearly independent then $S \cap T$ is either empty or linearly independent (Remark: sometimes people consider an empty set to be linearly independent).

(d) If S and T are linearly independent then $S \cup T$ is linearly independent.

(e) If $W = \text{span}S$ and $U = \text{span}T$ then $W + U = \text{span}(S \cup T)$.

1a) False. If S is linearly independent, that means no vectors in S can be written as a linear combination of other vectors in S, but the other vectors in T that are not in S might be able to be written as linear combination of other vectors.

counter example : $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ $T = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$S \subset T$ and S is linearly independent. but T is not.

1b) True. If T is linearly independent, no vectors in T can be written as a linear combination of other vectors. If $S \subset T$, S contains some vectors in T, which are still unable to be expressed as linear combination of others. So S has to be linearly independent too.

1c) True. If S and T don't overlap, U is empty. If S and T overlap, U will not be empty.

Let $S \cap T = U$. All $\vec{u}$'s in U are in both S and T, or $U \subset S$ and $U \subset T$. If S and T are linearly independent, all the elements in each subset cannot be written as linear combination of other vectors in their subsets. So naturally, U, the subset of S and T, would only contain vectors that are linearly independent.

(d). False. Counterexample: $S = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ $T = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$.

$S \cup T = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$. and $\begin{pmatrix}1\\1\end{pmatrix}$ can be written as

$1 \cdot \begin{pmatrix}1\\0\end{pmatrix} + 1 \cdot \begin{pmatrix}0\\1\end{pmatrix}$. $\therefore$ S, T are linearly independent but $S \cup T$ is not.

(e) True.

$W = \{c_1 s_1 + c_2 s_2 + \cdots + c_n s_n \mid c_1, \ldots, c_n \in \mathbb{R}\}$

$U = \{d_1 T_1 + d_2 T_2 + \cdots + d_n T_n \mid d_1, \ldots, d_n \in \mathbb{R}\}$

$W + U = c_1 s_1 + \cdots + c_n s_n + d_1 T_1 + \cdots + d_n T_n$.

$\operatorname{span}(S+T) = \{x_1(s_1 + t_1) + \cdots + x_n(s_n + t_n) \mid x_1, \ldots x_n \in \mathbb{R}\}$

$= x_1 s_1 + \cdots + x_n s_n + x_1 t_1 + \cdots x_n t_n$

Since c_x, d_x, and x_x are arbitrary.

$W + U = \operatorname{span}(S \cup T)$

6. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.

(a) Let V be a vector space which satisfies dim$V$=3. Then there exist, a subspace W of V and a subspace U of W (that is, $U \subset W \subset V$) such that dim$U$=1 and dim$W$=2.

True: If $\dim V = 3$, there will be 3 linearly independent vectors in the basis for V i.e. $\{v_1, v_2, v_3\}$ span V. As a subspace of V, W contains elements from V, which could be $W = \text{span}\{v_1, v_2\}$. And as a subspace for W, U can be $\text{span}\{v_1\}$. $\dim U = 1$, and $\dim W = 2$.

(b) Let V be a vector space which satisfies dim$V$=3 and let W be a **non–trivial** subspace of V and U be a **non–trivial** subspace of W (that is, $U \subset W \subset V$) then dim$U$=1 and dim$W$=2.

* If $\subset$ is equivalent to $\subseteq$, then this statement is false because it's possible that $U = W = V$, then $\dim U = \dim W = \dim V = 3$.

True.

If $\dim V = 3$, let V's basis be $\text{span}\{v_1, v_2, v_3\}$. If $W \subset V$, W must contain less than 3 elements from v_1, v_2, and v_3. Assume by way of contradiction, let $\dim W = 3$, then $\dim W$ needs to contain 3 linearly independent elements, which is not possible. So $\dim W < 3$. Also because W is a non-trivial subspace, it's not empty. $\therefore$ $\dim W$ is either 1 or 2. Since U also needs to be a subspace of W, $\dim W = 2$ and $\dim U = 1$.

(c) Let V be a vector space which satisfies dim$V$=3 and let $v_1, v_2, v_3 \in V$ be such that $\{v_1, v_2\}$ are linearly independent, $\{v_2, v_3\}$ are linearly independent, and $\{v_3, v_1\}$ are linearly independent. Then $\{v_1, v_2, v_3\}$ is a basis for V.

True. A basis for a vector space V is a set of vectors in V such that every vector in V can be written uniquely as a finite linear combination of elements of B. Since $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_1, v_3\}$ are linearly independent, any vector cannot be written as a linear combination of the two other vectors. So $\{v_1, v_2, v_3\}$ is a basis for V.

(d) Let V be a vector space and $v_1, ..., v_n \in V$ then: $\{v_1, ..., v_n\}$ is linearly independent iff $\dim(\text{span}\{v_1, ..., v_n\}) = n$.

True. First part is to prove that $\{v_1, \dots, v_n\}$ is linearly independent $\Rightarrow$ $\dim(\text{span}\{v_1, \dots, v_n\}) = n$.
By definition of dimension, the dimension of a vector space is the number of linearly independent vectors that span the vector space.
Second part is to prove $\dim(\text{span}\{v_1, \dots, v_n\}) = n \Rightarrow \{v_1, \dots, v_n\}$ is linearly independent.
Suppose by way of contradiction, if $\{v_1, \dots, v_n\}$ is linearly dependent, then at least 1 vector, v_k can be written as a linear combination of other vs.
So $\dim(\text{span}\{v_1, \dots, v_n\}) < n$, which is not true.

(e) Let V be a vector space and let $V_1, V_2, V_3 \subset V$ be such that $V_1 + V_2 = V_1 + V_3$ and $\dim V_2 = \dim V_3$ then $V_2 = V_3$. (The sum of two subspaces was defined in previous HW's).

False. A counterexample would be $V_1 = \mathbb{R}^2$. $\dim V_1 = 2$

$V_2 = \left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$, and $V_3 = \left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$. $\dim V_2 = \dim V_3 = 1$

A plane V_1 + a vector V_2 on this plane $= V_1$
$V_1 + V_3$ would also $= V_1$.
$\therefore V_1 + V_2 = V_1 + V_3$
However, $V_2 \neq V_3$, as V_2 and V_3 are two very different vectors.