MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 6

1. Consider bijective map $f : \mathcal{D} \to \mathcal{C}$. In class, we have stated that there exists map $g : \mathcal{C} \to \mathcal{D}$ such that $f \circ g = \mathrm{id}_{\mathcal{D}}$ and $g \circ f = \mathrm{id}_{\mathcal{D}}$. g in this case is called the inverse of f, denoted by f^{-1}. In the context of linear maps, we say linear map $T \in \mathcal{L}(V, W)$ is **invertible** if there exists linear map $S \in \mathcal{L}(W, V)$ such that $T \circ S = \mathrm{id}_W$ and $S \circ T = \mathrm{id}_V$. In this case, we denote S by T^{-1}.
   (a) Let $T \in \mathcal{L}(V, W)$. Show that its inverse T^{-1}, if it exists, is unique.
   (b) Show that $T \in \mathcal{L}(V, W)$ is invertible iff it is bijective, i.e., an isomorphism.
   (c) Give two examples of linear maps that are not invertible for two different reasons.

2. In class, we have shown that if two finite-dimensional vector spaces are of the same dimension, then they are isomorphic. Show the converse is also true.

3. Let $A \in M_{m\times n}(\mathbb{R})$ and consider the linear map T_A associated with A defined as $T_A : \mathbb{R}^n \to \mathbb{R}^m$ where $T_A(x) = Ax$. Show that
   (a) T_A is injective if there is a pivot in every col of rref(A).
   (b) T_A is surjective if there is a pivot in every row of rref(A).

4. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

   Compute the following expressions if defined: AB, BA, D^2, B^2, DC, CB, BC, FE, EF, CE, EC.

5. Here are some facts about matrices that will come in handy in the future.
   (a) A square matrix $A \in M_n(\mathbb{R})$ is called **diagonal** if all of its entries that are not on the main diagonal are equal zero, that is, A is diagonal if $(A)_{ij} = 0$ for all $i \neq j$. Here is an example of a diagonal matrix:
$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
   Prove that if $A, B \in M_n(\mathbb{R})$ are both diagonal then both $A + B$ and AB are diagonal as well.
   (b) For a square matrix $A \in M_n(\mathbb{R})$ the **trace** of A, denoted $tr(A)$, is the sum of all of its entries on the main diagonal, that is $tr(A) = \sum_{i=1}^{n} (A)_{ii}$. Here is an example of a trace computation:
$$tr \begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix} = 3 + 2 + (-1) = 4.$$

i. For $A, B \in M_n(\mathbb{R})$ prove that $tr(AB) = tr(BA)$.

ii. Show that $tr(\cdot) : M_n(\mathbb{R}) \to \mathbb{R}$ is a linear map.

(c) For a square matrix $A \in M_n(\mathbb{R})$ the **transposed** of A, denoted A^T, is the matrix obtained by turning each row of A into a column by order, that is $(A^T)_{i,j} = (A)_{j,i}$. Here is an example of a transposed computation:

$$\begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 3 & 0 & 8 \\ 5 & 2 & 1 \\ 7 & 4 & -1 \end{pmatrix}.$$

For $A, B \in M_n(\mathbb{R})$ prove that $(AB)^T = B^T A^T$.

(d) Let $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{n\times k}(\mathbb{R})$, and $C \in M_{k\times \ell}(\mathbb{R})$. Prove that

$$A(BC) = (AB)C.$$

6. Consider each of the following there is a claim, which might be **true or false**. If the claim is true then prove it, and if it is false then provide a counterexample. (For counter examples you may choose any n you wish, but if you want to prove a claim then you should prove it for all possible n's).

   (a) If $A \in M_n(\mathbb{R})$ satisfies $A^2 = 0$ then $A = 0$. (Here 0 is the zero matrix).

   (b) If $A, B \in M_n(\mathbb{R})$ are such that $AB = BA$ then $AB^2 = B^2 A$.

   (c) Let $A, B, C \in M_n(\mathbb{R})$. If $AB = CB$ then $A = C$.

   (d) Let $A \in M_n(\mathbb{R})$, then $(A + I)^2 = A^2 + 2A + I$.

   (e) Let $A, B \in M_n(\mathbb{R})$, then $(A + B)^2 = A^2 + 2AB + B^2$.

7. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.

   (a) For any two matrices $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$ we have $\text{rank}(AB) = \text{rank}(A) \cdot \text{rank}(B)$.

   (b) If A is a square matrix then its column space is equal to its null space.

   (c) If $A \in M_{m\times n}(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent.

   (d) If $A \in M_n(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent

8. For any two matrices $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$ which satisfy $AB = 0$ prove that $\text{rank}(B) + \text{rank}(A) \le n$.

9. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.

   (a) For any two $m \times n$ matrices A and B we have $\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$.

   (b) For any two $m \times n$ matrices A and B we have $\text{rank}(A + B) \le \text{rank}(A) + \text{rank}(B)$.

1. Consider bijective map $f : \mathcal{D} \to \mathcal{C}$. In class, we have stated that there exists map $g : \mathcal{C} \to \mathcal{D}$ such that $f \circ g = \mathrm{id}_{\mathcal{D}}$ and $g \circ f = \mathrm{id}_{\mathcal{D}}$. g in this case is called the inverse of f, denoted by f^{-1}. In the context of linear maps, we say linear map $T \in \mathcal{L}(V, W)$ is **invertible** if there exists linear map $S \in \mathcal{L}(W, V)$ such that $T \circ S = \mathrm{id}_W$ and $S \circ T = \mathrm{id}_V$. In this case, we denote S by T^{-1}.

(a) Let $T \in \mathcal{L}(V, W)$. Show that its inverse T^{-1}, if it exists, is unique.

(b) Show that $T \in \mathcal{L}(V, W)$ is invertible iff it is bijective, i.e., an isomorphism.

(c) Give two examples of linear maps that are not invertible for two different reasons.

a. let T^{-1} and $\tilde{T}^{-1}$ be T's inverses.

$T^{-1} = T^{-1}\,\mathrm{Id}_V = T^{-1}\,T\,\tilde{T}^{-1} = \mathrm{Id}\,\tilde{T}^{-1} = \tilde{T}^{-1}$

$\therefore$ The inverse for T is unique.

b. Need to show: ① bijective $\Rightarrow$ invertible

② invertible $\Rightarrow$ bijective.

① If a linear map is bijective, it's surjective and injective. Surjective means that the linear map's range equals codomain, i.e. all of the elements in W can be mapped from some element in V. Injective means that $T(v)$ for $\forall\, v \in \bar{V}$ is different.

Combined, it means that $\forall\, w \in \bar{W}$ is mapped from exactly one unique $v \in \bar{V}$. Therefore, T^{-1} exists.

$\therefore$ invertible.

② If a linear map is invertible, $\exists\, T^{-1}$ s.t. $TT^{-1} = T^{-1}T = \mathrm{Id}_{\bar{V}}$.

If we want to show it's bijective, N.T.S. ① injective

② surjective.

①. let $T(v_1) = T(v_2)$

$T^{-1}T(v_1) = T^{-1}T(v_2)$

$v_1 = v_2$.

$\therefore$ injective.

② $T(v) = w$.

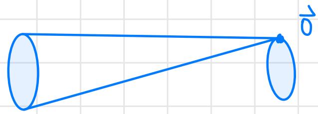
$T^{-1}(T(v)) = T^{-1}(w)$

$\therefore\ T^{-1}(w) = v. \quad \forall\, w \in \bar{W}$

$\therefore$ surjective.

$\therefore$ bijective.

C. 1. zero map. zero map maps everything to zero map, and nothing to the identity map.



it's not injective.
and not surjective.

2. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ surjective but not inject

projection to x-axis.

$\mathbb{R}^2 \mapsto \mathbb{R}^1$

2. In class, we have shown that if two finite-dimensional vector spaces are of the same dimension, then they are isomorphic. Show the converse is also true.

Let $\bar{V}$ and $\bar{W}$ be two vector spaces. Suppose by contradiction, Assume they don't have the same dimension : $\dim(\bar{V}) = m$, $\dim(\bar{W}) = n$ s.t. $m \neq n$. If $\bar{V}$ and $\bar{W}$ are isomorphic, there's an isomorphism from $\bar{V} \to \bar{W}$, and this transformation T is bijective.

$\therefore$ $\ker T = 0$. $\dim(\operatorname{im} T) = \dim(\bar{W})$

Since $\dim(\ker(T)) + \dim(\operatorname{im}(T)) = 0 + n = n \neq m = \dim(\bar{V})$,

it violates the rank-nullity theorem ..

So if they've same dimension, they must be isomorphic.

$W \to V$ isomorphic $\Rightarrow$ same dim

bijective $\therefore$ $\ker(T) = \{0\}$

$\underbrace{\dim(\ker(T))}_{0} + \dim(\underbrace{\operatorname{Im} T}_{V}) = \dim(W)$

3. Let $A \in M_{m\times n}(\mathbb{R})$ and consider the linear map T_A associated with A defined as $T_A : \mathbb{R}^n \to \mathbb{R}^m$ where $T_A(x) = Ax$. Show that

   (a) T_A is injective if there is a pivot in every col of rref(A).

   (b) T_A is surjective if there is a pivot in every row of rref(A).

(a). If $A \in M_{m\times n}(\mathbb{R})$ has a pivot every column.
Then its RREF looks like $\begin{bmatrix} P & & \\ & P & \\ & & P \\ 0 & 0 & 0 \end{bmatrix}$ or a square matrix like $\begin{bmatrix} P & & \\ & P & \\ & & P \end{bmatrix}$. So $A|\vec{0}$ would have 1 solution,

which would be the trivial solution of this homogenous matrix.
Therefore, $\ker A = \{\vec{0}\}$ and A is injective.

(b). If $A \in M_{m\times n}(\mathbb{R})$ has a pivot every row, Then its RREF would look like $\begin{bmatrix} P & & & \\ & P & & \\ & & P & \end{bmatrix}$ or $\begin{bmatrix} P & & \\ & P & \\ & & P \end{bmatrix}$.

$A|\vec{b}$ would have at least 1 solution. Which means every $\vec{b}$ in the co-domain has at least 1 solution.
So it's surjective.

4. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Compute the following expressions if defined: AB, BA, D^2, B^2, DC, CB, BC, FE, EF, CE, EC.

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+4 & 2 & 3+2 \\ 2-2 & -1 & 6-1 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 5 \\ 0 & -1 & 5 \end{pmatrix}$$

2×2 $\quad 2\times 3$ $\quad 2\times 3$.

$$BA = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$ not defined.

2×3 $\quad 2\times 2$

$$D^2 = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1+0+6 & 0+0-3 & 3+0+3 \\ 1+1-4 & 0+1+2 & 3-2-2 \\ 2-1+2 & 0-1-1 & 6+2+2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -3 & 6 \\ -2 & 3 & -1 \\ 3 & -2 & 9 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}$$ not defined

2×3 $\quad 2\times 3$

$$DC = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1+0+6 & 2+0-3 \\ 1+3-4 & 2+1+2 \\ 2-3+2 & 4-1-1 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ 0 & 5 \\ 1 & 2 \end{pmatrix}$$

3×3 $\quad 3\times 2$ $\quad 3\times 2$

4. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Compute the following expressions if defined: AB, BA, D^2, B^2, DC, $\underline{CB}$, $\underline{BC}$, $\underline{FE}$, $\underline{EF}$, $\underline{CE}$, $\underline{EC}$.

$$CB = \underset{3\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}} = \underset{3\times 3}{\begin{pmatrix} 1+4 & 0+2 & 3+2 \\ 3+2 & 0+1 & 9+1 \\ 2-2 & 0-1 & 6-1 \end{pmatrix}} = \begin{pmatrix} 5 & 2 & 5 \\ 5 & 1 & 10 \\ 0 & -1 & 5 \end{pmatrix}$$

$$BC = \underset{2\times 3}{\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}} \underset{3\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 1+0+6 & 2+0-3 \\ 2+3+2 & 4+1-1 \end{pmatrix}} = \begin{pmatrix} 7 & -1 \\ 7 & 4 \end{pmatrix}$$

$$FE = \underset{1\times 3}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} = \underset{1\times 1}{1+4-3} = 2$$

$$EF = \underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} \underset{1\times 3}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}} = \underset{3\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{pmatrix}}$$

$$CE = \underset{3\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} \quad \text{not defined}$$

$$EC = \underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} \underset{3\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}} \quad \text{not defined}$$

5. Here are some facts about matrices that will come in handy in the future.

(a) A square matrix $A \in M_n(\mathbb{R})$ is called **diagonal** if all of its entries that are not on the main diagonal are equal zero, that is, A is diagonal if $(A)_{ij} = 0$ for all $i \neq j$. Here is an example of a diagonal matrix:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Prove that if $A, B \in M_n(\mathbb{R})$ are both diagonal then both $A + B$ and AB are diagonal as well.

let A, B be represented by $\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & a_n \end{pmatrix}$ and $\begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & b_n \end{pmatrix}$

since matrix addition is by entre, $A+B = \begin{pmatrix} a_1+b_1 & 0 & \cdots & 0 \\ 0 & a_2+b_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & a_n+b_n \end{pmatrix}$.

so A+B is diagnol

$$AB = \begin{pmatrix} a_1b_1 & 0 & \cdots & 0 \\ 0 & a_2b_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & a_nb_n \end{pmatrix}$$

so AB is also diagnol.

(b) For a square matrix $A \in M_n(\mathbb{R})$ the **trace** of A, denoted $tr(A)$, is the sum of all of its entries on the main diagonal, that is $tr(A) = \sum_{i=1}^{n}(A)_{ii}$. Here is an example of a trace computation:

$$tr\begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix} = 3 + 2 + (-1) = 4.$$

i. For $A, B \in M_n(\mathbb{R})$ prove that $tr(AB) = tr(BA)$.

ii. Show that $tr(\cdot) : M_n(\mathbb{R}) \to \mathbb{R}$ is a linear map.

i let A, B be $\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \\ a_{n,1} & & a_{n,n} \end{pmatrix}$ and $\begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \\ b_{n,1} & & b_{n,n} \end{pmatrix}$

$$AB = \begin{pmatrix} (A_{1,\cdot})\cdot(B_{\cdot,1}) & \cdots & & \\ & (A_{2,\cdot})\cdot(B_{\cdot,2}) & & \\ \vdots & & (A_{3,\cdot})\cdot(B_{\cdot,3}) & \\ & & & \ddots \\ & & & (A_{n,\cdot})\cdot(B_{\cdot,n}) \end{pmatrix}$$

$$BA = \begin{pmatrix} (B_{1,\cdot})\cdot(A_{\cdot,1}) & \cdots & \\ & (B_{2,\cdot})\cdot(A_{\cdot,1}) & \\ \vdots & & \ddots \\ & & (B_{n,\cdot})\cdot(A_{\cdot,1}) \end{pmatrix}$$

$$tr(AB) = \sum_{i=1}^{n} [(A_{i,\cdot})\cdot(B_{\cdot,i})]$$

$(A_{1,\cdot})\cdot(B_{\cdot,1}) = (a_{1,1} \times b_{1,1}) + (a_{1,2} \times b_{2,1}) + \cdots + (a_{1,n} \times b_{n,1})$

$(A_{n,\cdot})\cdot(B_{\cdot,n}) = (a_{n,1} \times b_{1,n}) + (a_{n,2} \times b_{2,n}) + \cdots + (a_{n,n} \times b_{n,n})$

If we take out the first element of each $[(A_{i,\cdot})\cdot(B_{\cdot,i})]$

and add them together, the sum would be $(a_{1,1} \times b_{1,1}) + (a_{2,1} \times b_{1,2}) + \dots + (a_{n,1} \times b_{1,n}) = (A_{\cdot,1}) \cdot (B_{1,\cdot})$, which is the first element of $\sum_{i=1}^{n} [(A_{\cdot,i}) \cdot (B_{i,\cdot})]$.

Therefore, the sum of all nth element in the expansion of $(A_{i,\cdot}) \cdot (B_{\cdot,i})$ would be the nth element of $\sum_{i=1}^{n} [(A_{\cdot,i}) \cdot (B_{i,\cdot})]$.

$$\therefore \sum_{i=1}^{n} [(A_{i,\cdot}) \cdot (B_{\cdot,i})] = \sum_{i=1}^{n} [(A_{\cdot,i}) \cdot (B_{i,\cdot})].$$

$$\therefore \operatorname{tr}(AB) = \operatorname{tr}(BA).$$

ii. To prove is a linear map, we need to show $\operatorname{tr}(\cdot): M_n(\mathbb{R}) \to \mathbb{R}$'s additivity and homogeneouty.

① additivity:

$$\operatorname{tr}(A) + \operatorname{tr}(B) = \sum_{i=1}^{n} (A_{i,i}) + \sum_{i=1}^{n} (B_{i,i})$$

$$= \sum_{i=1}^{n} (A_{i,i} + B_{i,i}).$$

$$= \operatorname{tr}(A+B). \quad \therefore \checkmark$$

② homogeneouty: for $\alpha \in \mathbb{R}$, $A \in M_n(\mathbb{R})$

$$\operatorname{tr}(\alpha A) = \sum_{i=1}^{n} (\alpha A_{i,i}) = \alpha \sum_{i=1}^{n} (A_{i,i}) = \alpha \operatorname{tr}(A).$$

Therefore, it's a linear map. $\therefore \checkmark$

(c) For a square matrix $A \in M_n(\mathbb{R})$ the **transposed** of A, denoted A^T, is the matrix obtained by turning each row of A into a column by order, that is $(A^T)_{i,j} = (A)_{j,i}$. Here is an example of a transposed computation:

$$\begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 3 & 0 & 8 \\ 5 & 2 & 1 \\ 7 & 4 & -1 \end{pmatrix}.$$

For $A, B \in M_n(\mathbb{R})$ prove that $(AB)^T = B^T A^T$.

For $A \in M_n(\mathbb{R})$, $A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \\ a_{n,1} & & a_{n,n} \end{pmatrix}$, $A^T = \begin{pmatrix} a_{1,1} & \cdots & a_{n,1} \\ \vdots & \ddots & \\ a_{1,n} & & a_{n,n} \end{pmatrix}$

i.e. $A_{i,\cdot} = A^T_{\cdot,i}$ and $A_{\cdot,i} = A^T_{i,\cdot}$.

For $B \in M_n(\mathbb{R})$ it's the same thing.

$$\therefore B^T A^T = \begin{pmatrix} b^T_{1,\cdot} \cdot a^T_{\cdot,1} & \cdots & b^T_{1,\cdot} \cdot a^T_{\cdot,n} \\ b^T_{2,\cdot} \cdot a^T_{\cdot,1} & & b^T_{2,\cdot} \cdot a^T_{\cdot,n} \\ \vdots & & \vdots \\ b^T_{n,\cdot} \cdot a^T_{\cdot,1} & & b^T_{n,\cdot} \cdot a^T_{\cdot,n} \end{pmatrix}$$

$$= \begin{pmatrix} a_{1,\cdot} \cdot b_{\cdot,1} & \cdots & a_{n,\cdot} \cdot b_{\cdot,1} \\ a_{1,\cdot} \cdot b_{\cdot,2} & \cdots & a_{n,\cdot} \cdot b_{\cdot,2} \\ \vdots & & \vdots \\ a_{1,\cdot} \cdot b_{\cdot,n} & & a_{n,\cdot} \cdot b_{\cdot,n} \end{pmatrix}$$

$$= (AB)^T$$

(d) Let $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{n\times k}(\mathbb{R})$, and $C \in M_{k\times \ell}(\mathbb{R})$. Prove that

$$A(BC) = (AB)C.$$

$$((AB)C)_{i,j} = \sum_{k}^{n} \left((AB)_{i,k} C_{k,j} \right)$$

$$= \sum_{k}^{n} \left(\sum_{\ell} (A_{i,\ell} B_{\ell,k}) C_{k,j} \right)$$

$$(A(BC))_{i,j} = \sum_{k}^{n} \left(A_{i,\ell} \sum_{k}^{n} (B_{\ell,k} C_{k,j}) \right).$$

$$\therefore ((AB)C)_{i,j} = (A(BC))_{i,j}$$

$$\therefore (AB)C = A(BC).$$

6. Consider each of the following there is a claim, which might be **true or false**. If the claim is true then prove it, and if it is false then provide a counterexample. (For counter examples you may choose any n you wish, but if you want to prove a claim then you should prove it for all possible n's).

(a) If $A \in M_n(\mathbb{R})$ satisfies $A^2 = 0$ then $A = 0$. (Here 0 is the zero matrix).

(b) If $A, B \in M_n(\mathbb{R})$ are such that $AB = BA$ then $AB^2 = B^2A$.

(c) Let $A, B, C \in M_n(\mathbb{R})$. If $AB = CB$ then $A = C$.

(d) Let $A \in M_n(\mathbb{R})$, then $(A + I)^2 = A^2 + 2A + I$.

(e) Let $A, B \in M_n(\mathbb{R})$, then $(A + B)^2 = A^2 + 2AB + B^2$.

(a) let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore$ False. Proved by counterexample.

(b) True. $AB^2 = ABB = BAB = BBA = B^2A$.

(c) If $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $CB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$AB = CB$, but $A \neq C$.

$\therefore$ False. Proved by counterexample

(d) let A be $\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$ and $(A+I) = \begin{pmatrix} a_{1,1}+1 & \cdots & a_{1,n} \\ \vdots & \ddots & \\ a_{n,1} & & a_{n,n}+1 \end{pmatrix}$

$$(A+I)^2 = \begin{pmatrix} (a_{1,1}+1)^2 + \sum_{i=2}^{n}(a_{1,i})(a_{i,1}) & \cdots & (a_{1,1}+1)(a_{1,n}) + \sum_{i=2}^{n-1}(a_{1,i})(a_{i,n}) + (a_{1,n})(a_{n,n}+1) \\ & & \\ (a_{n,1})(a_{1,1}+1) + \sum_{i=2}^{n-1}(a_{n,i})(a_{i,1}) + (a_{n,n}+1)(a_{n,1}) & \cdots & \sum_{i=1}^{n-1}(a_{n,i})(a_{i,n}) + (a_{n,n}+1)^2 \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n}(a_{1,i})(a_{i,1}) + 2a_{1,1} + 1 & \cdots & \sum_{i=1}^{n}(a_{1,i})(a_{i,n}) + 2a_{1,n} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n}(a_{n,i})(a_{i,1}) + 2a_{n,1} & & \sum_{i=1}^{n}(a_{n,i})(a_{i,n}) + 2a_{n,n} + 1 \end{pmatrix}$$

$= A^2 + 2A + I$

$\therefore$ True.

e.

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$B^2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$

$A^2 + B^2 + 2AB = \begin{pmatrix} 2+1+2\times 2 & 2+0+2\times 0 \\ 2+1+2\times 2 & 2+0+2\times 0 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 7 & 0 \end{pmatrix}$

$(A+B) = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \qquad (A+B)^2 = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4+2 & 2+1 \\ 4+2 & 2+1 \end{pmatrix}$

$= \begin{pmatrix} 6 & 3 \\ 6 & 3 \end{pmatrix}.$

$\therefore (A+B)^2 \neq A^2 + B^2 + 2AB$

$\therefore$ proved false by counter ex.

7. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.

(a) For any two matrices $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$ we have $\text{rank}(AB) = \text{rank}(A) \cdot \text{rank}(B)$.

let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$\therefore$ $\text{col}(A) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $\text{col}(B) = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

$\therefore$ rank $A = 2$, and rank $B = 2$.

However, $AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and rank $(AB) = 2$.

$\therefore$ rank $(AB) \neq$ rank $(A) \cdot$ rank (B).

$\therefore$ proved false by counterexample.

(b) If A is a square matrix then its column space is equal to its null space.

let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Col $A = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

ker $A = \vec{0}$.

$\therefore$ col $A \neq$ ker A.

$\therefore$ proved false by counter example.

(c) If $A \in M_{m\times n}(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent.

let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

rows in A are linearly independent.

but columns are not : $1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\therefore$ proved false by counter example.

(d) If $A \in M_n(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent

If every row is linearly independent to each other, there's a pivot in every row. $\therefore$ number of pivot in RREF (A) is equal to n. Since it's a square matrix, this implies that each column has a pivot, too. So the columns are linearly independent.

8. For any two matrices $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$ which satisfy $AB = 0$ prove that $\text{rank}(B) + \text{rank}(A) \le n$.

See $AB = 0$ as a linear transformation. A is mapping B to $\vec{0}$, so $\text{Col}(B) \subseteq \ker(A)$

rank(B) is the number of linearly independent columns in matrix B; its maximum is the $\dim(\text{Col}(B))$.

according to the rank-nullity theorum,

$$\text{rank}(A) + \text{nullity}(A) = n.$$

since $\text{nullity}(A) = \dim(\ker(A)) \ge \dim(\text{Col}(B))$

$= \text{rank}(B)$,

$\text{rank}(A) + \text{rank}(B) \le n$

9. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.

(a) For any two $m \times n$ matrices A and B we have $\text{rank}(A+B) = \text{rank}(A) + \text{rank}(B)$.

(b) For any two $m \times n$ matrices A and B we have $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

a. let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $A+B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\text{rank}(A) = 2$, $\text{rank}(B) = 2$, and $\text{rank}(A+B) = 1$.

$\therefore \text{rank}(A+B) \neq \text{rank}(A) + \text{rank}(B)$.

$\therefore$ Proved false by counterexample.

b. For $A, B \in M_{m \times n}$, we have $\text{Col}(A)$ and $\text{Col}(B)$. let S be the set of linearly independent vectors that span $\text{Col}(A)$, and let T be the set of linearly independent vectors spanning $\text{Col}(B)$. Every element in $\text{Col}(A+B)$ can be written as a linear combination of some element from A and some element from B. So $\text{Col}(A+B) \subseteq S \cup T$. The rank is the number of linearly independent vectors in the column spaces.

$\therefore \text{rank}(A+B) \leq \text{rank}(S) + \text{rank}(T)$

$\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

$\therefore$ Proved true.