MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 3

1. Prove the following statements.
   (a) Let $A \in M_{m \times n}(\mathbb{R})$. If an echelon form of $A$ has a row of zeroes then there exists $b \in \mathbb{R}^m$ such that $(A|b)$ has no solution.
   (b) If $m > n$ and $A \in M_{m \times n}(\mathbb{R})$ then there exists $b \in \mathbb{R}^m$ such that $(A|b)$ has no solution.
   (c) If $A \in M_n(\mathbb{R})$ is a square $n \times n$ matrix such that the homogenous system $(A|0)$ has infinity many solutions, then there exists $b \in \mathbb{R}^n$ such that $(A|b)$ has no solution.

2. The following statements are **false**. Prove that they are **false** by providing a counterexample in each case (you may choose the numbers $m$ and $n$ to be whatever is convenient, try to work with small numbers).
   (a) If $A \in M_{m \times n}(\mathbb{R})$ is a matrix such that the homogenous system $(A|0)$ has infinity many solutions, then there exists $b \in \mathbb{R}^n$ such that $(A|b)$ has no solution.
   (b) Let $A, B \in M_{m \times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If $A$ and $B$ are row equivalent then $(A|b)$ and $(B|b)$ have the same amount of solutions.
   (c) If $A, B \in M_{m \times n}(\mathbb{R})$ are row equivalent then $B$ can be obtained from $A$ by performing **column** operations (that is, by performing a sequence of operations of the form 'swapping two columns', 'multiplying a column by a scalar different from zero', 'adding to a column another column multiplied by a scalar').
   (d) Let $A \in M_{m \times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If $u, v \in \mathrm{Sol}(A|b)$ then $u + v \in \mathrm{Sol}(A|b)$.
   (e) Let $A \in M_{m \times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If $u \in \mathrm{Sol}(A|b)$ and $t \in \mathbb{R}$ then $tu \in \mathrm{Sol}(A|b)$.
   (f) If matrix-vector equation $A\vec{x} = \vec{0}$, then either $A$ is a zero matrix or $\vec{x}$ is a zero vector.

3. Decide if the following two matrices are row equivalent: The matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{pmatrix}$$

and the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{pmatrix}$$

4. In each of the following you are given a set and two operations: A 'sum', acting between two elements in the set, and a 'multiplication by scalar', acting between one element in the set and a scalar from $\mathbb{R}$. In each case determine whether the set with these two operations gives a vector space over $\mathbb{R}$. If it is a vector space then prove this fact. If it is not a vector space then show this by giving a counterexample. **In this question you are allowed to use only the definition of a vector space, not any other claim given in class.**

   (a) The set $P_2(\mathbb{R})$ with the usual operations of summation and multiplication by scalar defined for polynomials.

(b) The set $\mathbb{R}^2$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ 0 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 0 \end{pmatrix}.$$

(c) The set $\mathbb{R}^2$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - 3 \\ x_2 + y_2 - 2 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 - 3\alpha + 3 \\ \alpha x_2 - 2\alpha + 2 \end{pmatrix}.$$

(d) The set $\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R} : x_1 > 0, x_2 > 0\}$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^\alpha \\ x_2^\alpha \end{pmatrix}.$$

5. Let $V$ be a vector space over $\mathbb{R}$ and let $W \subset V$ and $U \subset V$ be two subspaces of $V$. The following claims are either true or false. Determine whether they are true or false and prove or disprove using a counterexample accordingly.

(a) $U \cap W$ is also a subspace of $V$.

(b) $U \cup W$ is also a subspace of $V$.

(c) We define the following subset of $V$:

$$U + W := \{u + w : u \in U, w \in W\}.$$

In this part of the question the claim is: $U + W$ is a subspace of $V$.

1. Prove the following statements. $m$ rows, $n$ columns

(a) Let $A \in M_{m\times n}(\mathbb{R})$. If an echelon form of $A$ has a row of zeroes then there exists $b \in \mathbb{R}^m$ such that $(A|b)$ has no solution.

(b) If $m > n$ and $A \in M_{m\times n}(\mathbb{R})$ then there exists $b \in \mathbb{R}^m$ such that $(A|b)$ has no solution.

(c) If $A \in M_n(\mathbb{R})$ is a square $n \times n$ matrix such that the homogenous system $(A|0)$ has infinity many solutions, then there exists $b \in \mathbb{R}^n$ such that $(A|b)$ has no solution.

(a.) For row $R_m$ that is all zeros in the echelon form, if $b_m \neq 0$, it would be inconsistent system.

(b.) more rows than columns.

$$\begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{matrix}$$

This means that the echelon form of the matrix there will have maximum $n$ pivots. Since $m > n$, there will be rows of zeros left with no pivots. Like (a), if any of these rows' corresponding $b$ is not zero, the system becomes inconsistent.

(c.) if $A$ is an $n\times n$ matrix and $(A|0)$ has infinitely many solutions, there must be a free variable from $(A|0)$. This means there's an all-zero row in $A$'s echelon form. If this row's corresponding $b$ is 0, then this would result in an inconsistency, making $(A|b)$ have no solution.

2. The following statements are **false**. Prove that they are **false** by providing a counterexample in each case (you may choose the numbers $m$ and $n$ to be whatever is convenient, try to work with small numbers).

(a) If $A \in M_{m\times n}(\mathbb{R})$ is a matrix such that the homogenous system $(A|0)$ has infinity many solutions, then there exists $b \in \mathbb{R}^n$ such that $(A|b)$ has no solution.

(a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $A|0 = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$ and the solution

would be $\begin{cases} x = -2s \\ y = -s \\ z = s \end{cases}$ so $A|0$ has infinitity many solutions.

For any $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $(A|b)$ would not have any solutions.

(b) Let $A, B \in M_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If $A$ and $B$ are row equivalent then $(A|b)$ and $(B|b)$ have the same amount of solutions.

(b). let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

The RREF of $A$ is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ which is equal to $B$'s RREF, So they are row equivalent. Let $b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

$A|b = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 1 & 2 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$ $\therefore$ infinite solutions.

$B|b = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 2 \end{array}\right)$ $\therefore$ no solutions.

(c) If $A, B \in M_{m\times n}(\mathbb{R})$ are row equivalent then $B$ can be obtained from $A$ by performing **column** operations (that is, by performing a sequence of operations of the form 'swapping two columns', 'multiplying a column by a scalar different from zero', 'adding to a column another column multiplied by a scalar').

(c) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

There's no way to change $A_{(2,1)}$ and $A_{(2,2)}$ from 1 to 0 alone without changing $A_{(1,1)}$ and $A_{(1,2)}$ if any column operation can be executed.

(d) Let $A \in M_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If $u, v \in \mathrm{Sol}(A|b)$ then $u + v \in \mathrm{Sol}(A|b)$.

(d). Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. $A|b = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$.

$\therefore$ solution $(A|b) = \begin{cases} x = 2 - s. \\ y = s \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} s$

let $u = \begin{cases} x = 2 \\ y = 0 \end{cases}$ and $v = \begin{cases} x = 4 \\ y = -2 \end{cases}$, $u, v \in \mathrm{Sol}(A|b)$.

$u + v = \begin{cases} x = 6 \\ y = -2. \end{cases}$

but $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$ is not a solution for $\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} s$.

(e) Let $A \in M_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If $u \in \mathrm{Sol}(A|b)$ and $t \in \mathbb{R}$ then $tu \in \mathrm{Sol}(A|b)$.

(e) like (d). Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. $A|b = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$

and $\mathrm{Sol}(A|b) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} s$

Let $u = \begin{cases} x = 2 \\ y = 0 \end{cases}$ and $t = 4$, which satisfies $u \in \mathrm{Sol}(A|b)$ and $t \in \mathbb{R}$.

but $ut = \begin{cases} x = 8 \\ y = 0 \end{cases}$, which is not a solution for $\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} s$.

(f) If matrix-vector equation $A\vec{x} = \vec{0}$, then either $A$ is a zero matrix or $\vec{x}$ is a zero vector.

let $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ If $A\vec{x} = \vec{0}$, $\vec{x} = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} s \mid s \in \mathbb{R} \right\}$.

In this case, $A \neq 0$, $\vec{x} \neq \vec{0}$ but $A\vec{x} = \vec{0}$.

3. Decide if the following two matrices are row equivalent: The matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{pmatrix} \quad M_1$$

and the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{pmatrix} \quad M_2.$$

Turn $M_1$ into RREF.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{pmatrix} \xRightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -5 \\ 5 & -1 & 5 \end{pmatrix} \xRightarrow{R_3 \leftarrow R_3 - 5R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -5 \\ 0 & -1 & -5 \end{pmatrix} \xRightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_2 \leftarrow R_2 \cdot (-1)$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}.$$

Turn $M_2$ into RREF.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{pmatrix} \xRightarrow{R_2 \leftarrow R_2/2,\ R_3 \leftarrow R_3/2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 1 & 0 & 2 \end{pmatrix} \xRightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the RREF of $M_1$ and RREF of $M_2$ are the same, they are indeed row equivalent.

4. In each of the following you are given a set and two operations: A 'sum', acting between two elements in the set, and a 'multiplication by scalar', acting between one element in the set and a scalar from $\mathbb{R}$. In each case determine whether the set with these two operations gives a vector space over $\mathbb{R}$. If it is a vector space then prove this fact. If it is not a vector space then show this by giving a counterexample. **In this question you are allowed to use only the definition of a vector space, not any other claim given in class.**

   (a) The set $P_2(\mathbb{R})$ with the usual operations of summation and multiplication by scalar defined for polynomials. let $p(x)$ and $q(x)$ and $r(x) \in P_2(\mathbb{R})$.

1. closure under addition: The sum of 2 $P_2(\mathbb{R})$ polynomial will have the highest possible degree of 2. $\therefore p(x)+q(x) \in P_2(\mathbb{R})$ $\therefore$ V
2. Commutativity of addition: $p(x) + q(x) = q(x) + p(x)$ $\therefore$ V

3. Associativity of addition: $(p(x)+q(x))+r(x) = p(x)+(q(x)+r(x))$. because if you expand all terms in the polynomials, the sequence of additions doesn't matter. $\therefore$ V
4. Additive identity: There exists a zero polynomial, $o(x)$, that for all $p(x) \in P_2(\mathbb{R})$, $p(x)+o(x) = p(x)$. $\therefore$ V
5. Additive inverse: $\forall\ p(x) \in P_2(\mathbb{R})$, $\exists\ -p(x)$ where $p(x) + (-p(x)) = 0$ $\therefore$ V

6. Closure under scalar multiplication: Multiplying a polynomial with a scalar doesn't change its highest degree. $\therefore cp(x)$ where $c \in \mathbb{R}$ $\in P_2(\mathbb{R})$
7. Distributive property of scalar over scalar addition: for $s,t \in \mathbb{R}$, $(s+t)\cdot p(x) = sp(x) + tp(x)$ (proof by multiplication rule). $\therefore$ V
8. Distributive property of scalar over vector addition: for $s \in \mathbb{R}$, $s\cdot(p(x)+q(x)) = sp(x)+sq(x)$ $\therefore$ V
9. Scalar multiplication identity: $1\cdot p(x) = p(x)$. $\therefore$ V

10. Compatibility of scalar multiplication: for $s,t \in \mathbb{R}$, $(st)\cdot p(x) = s\cdot tp(x)$. $\therefore$ V

Since $P_2(\mathbb{R})$ satisfies all ten vector space axioms, it's indeed a vector space.

(b) The set $\mathbb{R}^2$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ 0 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 0 \end{pmatrix}.$$

let $\vec{u}, \vec{v}, \vec{w} \in$ this set and scalars $s, t \in \mathbb{R}$.

1. closure under addition: For any $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\vec{u} + \vec{v} = \begin{pmatrix} x_1 + y_1 \\ 0 \end{pmatrix}$. The sum must also be in $\mathbb{R}^2$ since $(x_1 + y_1)$ and $0$ are integers. $\therefore$ ✓

2. Associativity of addition: For any $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$,

$$(\vec{u} \oplus \vec{t}) \oplus \vec{w} = \begin{pmatrix} x_1 + y_1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + z_1 \\ 0 \end{pmatrix}$$

$$\vec{u} \oplus (\vec{t} \oplus \vec{w}) = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 + z_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + z_1 \\ 0 \end{pmatrix}.$$

$\therefore (\vec{u} \oplus \vec{t}) \oplus \vec{w} = \vec{u} \oplus (\vec{t} \oplus \vec{w})$. $\therefore$ ✓

3. commutativity of addition: For any $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$,
$\vec{u} \oplus \vec{v} = \begin{pmatrix} x_1 + y_1 \\ 0 \end{pmatrix}$, $\vec{v} \oplus \vec{u} = \begin{pmatrix} y_1 + x_1 \\ 0 \end{pmatrix}$. $x_1 + y_1 = y_1 + x_1$. $\therefore$ ✓.

4. Additive identity: $\vec{u} \oplus \vec{0} = \begin{pmatrix} x_1 + 0 \\ 0 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

~~5. Additive inverse:~~

~~6. Closure under scalar multiplication:~~

~~7. Distributive property of scalar over scalar addition:~~

~~8. Distributive property of scalar over vector addition:~~

~~9. Scalar multiplication identity:~~

~~10. Compatibility of scalar multiplication:~~

4. $\because$ There doesn't exist a $\vec{0}_+$ s.t. $\vec{u} \oplus \vec{0}_+ = \vec{u}$ if $\vec{u} \neq \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$. $\therefore$ it violates the additive identity.

Let $\vec{u}$ be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. there's no such $\vec{0}_+$ that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \vec{0}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ since $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \vec{0}_+$ always $= \begin{pmatrix} 1 + \text{something} \\ 0 \end{pmatrix}$.

(c) The set $\mathbb{R}^2$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - 3 \\ x_2 + y_2 - 2 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 - 3\alpha + 3 \\ \alpha x_2 - 2\alpha + 2 \end{pmatrix}.$$

let $\vec{u}, \vec{v}, \vec{w} \in$ this set
and scalars $s, t \in \mathbb{R}$.

1. closure under addition: for $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\vec{u} \oplus \vec{v} = \begin{pmatrix} x_1 + y_1 - 3 \\ x_2 + y_2 - 2 \end{pmatrix}$

since $(x_1 + y_1 - 3)$ and $(x_2 + y_2 - 2)$ are integers, it $\in \mathbb{R}^2$. $\therefore \checkmark$

2. Commutativity of addition: $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{v} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\vec{u} \oplus \vec{v} = \begin{pmatrix} x_1 + y_1 - 3 \\ x_2 + y_2 - 2 \end{pmatrix}$

and $\vec{v} \oplus \vec{u} = \begin{pmatrix} y_1 + x_1 - 3 \\ y_2 + x_2 - 2 \end{pmatrix}$ $\therefore \vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$. $\therefore \checkmark$

3. Associativity of addition: For $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$$(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \begin{pmatrix} x_1 + y_1 - 3 \\ x_2 + y_2 - 2 \end{pmatrix} \oplus \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - 3 + z_1 - 3 \\ x_2 + y_2 - 2 + z_2 - 2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + z_1 - 6 \\ x_2 + y_2 + z_2 - 4 \end{pmatrix}$$

$$\vec{u} \oplus (\vec{v} \oplus \vec{w}) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 + z_1 - 3 \\ y_2 + z_2 - 2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + z_1 - 3 - 3 \\ x_2 + y_2 + z_2 - 2 - 2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + z_1 - 6 \\ x_2 + y_2 + z_2 - 4 \end{pmatrix}$$

$\therefore (\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$. $\therefore \checkmark$

4. Additive identity: $\exists$ zero vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ where $\forall \vec{v} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} x_1 + 3 - 3 \\ x_2 + 2 - 2 \end{pmatrix}$

$= \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} = \vec{v}$ $\therefore \checkmark$.

5. Additive inverse: for each $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\exists -\vec{u} = \begin{pmatrix} -x_1 + 3 \\ -x_2 + 2 \end{pmatrix}$. s.t

$\vec{u} + (-\vec{u}) = \begin{pmatrix} x_1 + (-x_1 + 3) - 3 \\ x_2 + (-x_2 + 2) - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$. $\therefore \checkmark$

6. Closure under scalar multiplication: for $s \in \mathbb{R}$, $s \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} sx_1 - 3s + 3 \\ sx_2 - 3s + 2 \end{pmatrix}$

$sx_1 - 3s + 3$ and $sx_2 - 3s + 2$ are $\mathbb{R}$. $\therefore$ result $\in V$. $\therefore \checkmark$

7. Distributive property of scalar over scalar addition: for $s, t \in \mathbb{R}$.

$(s+t) \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (s+t)x_1 - 3(s+t) + 3 \\ (s+t)x_2 - 3(s+t) + 2 \end{pmatrix}$.

$$s \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus t \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} sx_1 - 3s + 3 \\ sx_2 - 3s + 2 \end{pmatrix} \oplus \begin{pmatrix} tx_1 - 3t + 3 \\ tx_2 - 3t + 2 \end{pmatrix} = \begin{pmatrix} (s+t)x_1 - 3(s+t) + 6 - 3 \\ (s+t)x_2 - 3(s+t) + 4 - 2 \end{pmatrix}$$

$$= \begin{pmatrix} (s+t)x_1 - 3(s+t) + 3 \\ (s+t)x_2 - 3(s+t) + 2 \end{pmatrix}$$

$\therefore (s+t) \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus t \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$\therefore \checkmark$

(c) The set $\mathbb{R}^2$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - 3 \\ x_2 + y_2 - 2 \end{pmatrix}$$

and

(continue for (c)).

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 - 3\alpha + 3 \\ \alpha x_2 - 2\alpha + 2 \end{pmatrix}.$$

8. Distributive property of scalar over vector addition :

For $s \in \mathbb{R}$, $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$s \odot (\vec{u} \oplus \vec{v}) \overset{?}{=} s\odot\vec{u} \oplus s\odot\vec{v}$.

$$s \odot (\vec{u} \oplus \vec{v}) = s \odot \begin{pmatrix} x_1 + y_1 - 3 \\ x_2 + y_2 - 2 \end{pmatrix} = \begin{pmatrix} sx_1 + sy_1 - 3s - 3s + 3 \\ sx_2 + sy_2 - 2s - 2s + 2 \end{pmatrix}$$

$$s\odot\vec{u} \oplus s\odot\vec{v} = \begin{pmatrix} sx_1 - 3s + 3 \\ sx_2 - 2s + 2 \end{pmatrix} \oplus \begin{pmatrix} sy_1 - 3s + 3 \\ sy_2 - 2s + 2 \end{pmatrix}$$

$$= \begin{pmatrix} s(x_1 + y_1) - 6s + 6 - 3 \\ s(x_2 + y_2) - 4s + 4 - 2 \end{pmatrix} = \begin{pmatrix} s(x_1 + y_1) - 6s + 3 \\ s(x_2 + y_2) - 4s + 2 \end{pmatrix}.$$

$\therefore s \odot (\vec{u} \oplus \vec{v}) = s\odot\vec{u} \oplus s\odot\vec{v}$. $\therefore \checkmark$.

9. Compatibility of scalar multiplication :

for $s, t \in \mathbb{R}$, $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$(st) \odot \vec{u} \overset{?}{=} s \odot (t \odot \vec{u})$

LHS : $st \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} stx_1 - 3st + 3 \\ stx_2 - 2st + 2 \end{pmatrix}$

RHS : $s \odot \left(t \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = s \odot \begin{pmatrix} tx_1 - 3t + 3 \\ tx_2 - 2t + 2 \end{pmatrix} = \begin{pmatrix} stx_1 - 3st + 3s - 3s + 3 \\ stx_2 - 2st + 2s - 2s + 2 \end{pmatrix}$

$$= \begin{pmatrix} stx_1 - 3st + 3 \\ stx_2 - 2st + 2 \end{pmatrix}.$$

$\therefore$ LHS = RHS

$\therefore \checkmark$.

10. Scalar multiplication identity :

For $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$$1 \odot \vec{u} = \begin{pmatrix} 1x_1 - 3\cdot 1 + 3 \\ 1\cdot x_2 - 2\cdot 1 + 2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \therefore \checkmark.$$

Since it passes all 10 axioms, it's indeed a vector space.

(d) The set $\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R} : x_1 > 0, x_2 > 0\}$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}$$

let $\vec{u}, \vec{v}, \vec{w} \in$ this set

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^\alpha \\ x_2^\alpha \end{pmatrix}.$$

let $s, t \in \mathbb{R}$.

1. closure under addition: $\vec{u} \oplus \vec{v} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}$. since $x_1, y_1, x_2, y_2$ all $\in \mathbb{R}$ and $>0$. $x_1 y_1 \in \mathbb{R}, >0$ and $x_2 y_2 \in \mathbb{R}, > 0$.
$\therefore$ result also $\in$ this set $\therefore \checkmark$

2. Commutativity of addition: $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\vec{u} \oplus \vec{v} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}$
$\vec{v} \oplus \vec{u} = \begin{pmatrix} y_1 x_1 \\ y_2 x_2 \end{pmatrix}$ $\therefore \vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$ $\therefore \checkmark$

3. Associativity of addition: $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$
$(\vec{u} \oplus \vec{v}) \oplus \vec{w} \stackrel{?}{=} \vec{u} \oplus (\vec{v} \oplus \vec{w})$
LHS $= \begin{pmatrix} x_1 \cdot y_1 \cdot z_1 \\ x_2 y_2 z_2 \end{pmatrix}$ RHS $= \begin{pmatrix} x_1 y_1 z_1 \\ x_2 y_2 z_2 \end{pmatrix}$ $\therefore$ LHS = RHS $\therefore \checkmark$

4. Additive identity: $\exists$ zero vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ s.t. $\forall \vec{u} \in$ this set, $\vec{u} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{u}$.
$\therefore \checkmark$

5. Additive inverse: For $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{u} \oplus \begin{pmatrix} \frac{1}{x_1} \\ \frac{1}{x_2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $\therefore \checkmark$

6. Closure under scalar multiplication: For $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $s \in \mathbb{R}$,
$s \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^s \\ x_2^s \end{pmatrix}$ $x_1^s$ and $x_2^s$ will be $\mathbb{R}$. $\therefore \begin{pmatrix} x_1^s \\ x_2^s \end{pmatrix} \in \mathbb{R}$. $\therefore \checkmark$

7. Distributive property of scalar over scalar addition: $(s \oplus t) \odot \vec{u} \stackrel{?}{=} s\vec{u} \oplus t\vec{u}$
LHS: $(s+t) \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^{s+t} \\ x_2^{s+t} \end{pmatrix} = \begin{pmatrix} x_1^s \cdot x_1^t \\ x_2^s \cdot x_2^t \end{pmatrix}$

RHS: $s \odot \vec{u} \oplus t \odot \vec{u} = \begin{pmatrix} x_1^s \\ x_2^s \end{pmatrix} \oplus \begin{pmatrix} x_1^t \\ x_2^t \end{pmatrix} = \begin{pmatrix} x_1^s \cdot x_1^t \\ x_2^s \cdot x_2^t \end{pmatrix}$.

$\therefore$ LHS = RHS
$\therefore \checkmark$

8. Distributive property of scalar over vector addition :

$$s \odot (\vec{u} \oplus \vec{w}) \overset{?}{=} s \odot \vec{u} \oplus s \odot \vec{w}$$

$$LHS = s \odot \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix} = \begin{pmatrix} x_1^s y_1^s \\ x_2^s y_2^s \end{pmatrix} \quad RHS = \begin{bmatrix} x_1^s \\ y_1^s \end{bmatrix} \oplus \begin{bmatrix} x_2^s \\ y_2^s \end{bmatrix} = \begin{pmatrix} x_1^s x_2^s \\ y_1^s y_2^s \end{pmatrix}$$

$\therefore$ LHS = RHS.

$\therefore \checkmark$

9. Scalar multiplication identity :

$$(s \cdot t) \odot \bar{v} \overset{?}{=} s \odot (t \odot \bar{v})$$

$$LHS = \begin{pmatrix} x_1^{st} \\ x_2^{st} \end{pmatrix} \qquad RHS = s \odot \begin{pmatrix} x_1^{t} \\ x_2^{t} \end{pmatrix} = \begin{pmatrix} x_1^{ts} \\ x_2^{ts} \end{pmatrix}$$

$\therefore$ LHS = RHS $\quad \therefore \checkmark$

10. Compatibility of scalar multiplication :

$$1 \cdot \bar{v} = \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \bar{v} \quad \therefore \checkmark$$

$\therefore$ passed all ten axioms

$\therefore$ it is a vector space.

5. Let $V$ be a vector space over $\mathbb{R}$ and let $W \subset V$ and $U \subset V$ be two subspaces of $V$. The following claims are either true or false. Determine whether they are true or false and prove or disprove using a counterexample accordingly.

(a) $U \cap W$ is also a subspace of $V$.

(b) $U \cup W$ is also a subspace of $V$.

(c) We define the following subset of $V$:

$$U + W := \{u + w : u \in U, w \in W\}.$$

In this part of the question the claim is: $U + W$ is a subspace of $V$.

$\begin{pmatrix}1\\2\end{pmatrix} \begin{pmatrix}1\\1\end{pmatrix}$ $\begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}1\\3\end{pmatrix}$.

$U \cup W = \{\begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}1\\2\end{pmatrix} \begin{pmatrix}1\\3\end{pmatrix}\}$.

$\begin{pmatrix}2\\5\end{pmatrix}$

$v_1, v_2 \in U + W$: $u_1 + w_1 + u_2 + w_2$

$c_1 u + c_2 w$ $\leftarrow$ $u \in U$, $w \in W$

(a) Since $V$ is a vector space and $W$ and $U \subset V$.

We only need to check if ① $U \cap W$ is empty.

② $U \cap W$ is closed under addition

③ $U \cap W$ is closed under scalar multiplication.

①. $U$ and $W$ must both contain $0$

$\therefore$ $U \cap W$ has at least one set, $0$ $\therefore$ $U \cap W$ is not empty.

② Let $x$ and $y$ be in both $U$ and $W$.

$x + y \in V$ since $U$ and $W$ are both closed under addition.

③ Let $x$ be in both $U$ and $W$ and $r \in \mathbb{R}$.

$r \cdot x \in V$ since $U$ and $V$ are both closed under scalar multiplication.

$\therefore$ 3 conditions are passed so it's indeed a subspace.

(b) Let $V = \mathbb{R}^2$ $W = \{\begin{pmatrix}x_1\\y_1\end{pmatrix} \in \mathbb{R} \mid y = x\}$ $U = \{\begin{pmatrix}x_2\\y_2\end{pmatrix} \in \mathbb{R} \mid y = 2x$.

let

$\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$ $\begin{pmatrix}x_1\\y_1\end{pmatrix} + \begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} \notin W \cup U$

$\therefore$ As the counterexample shows, it does not satisfy closure under addition.

$\therefore$ not a subspace.

(c). ① $U+W$ is always an element.

∴ it's not empty.

②. close under addition?

let $V_1 = u_1 + w_1$ . $V_1 + V_2 = (u_1 + w_1 + u_2 + w_2)$.

$V_2 = u_2 + w_2$

$$= \underbrace{(u_1 + u_2)}_{u' \in \bar{U}} + \underbrace{(w_1 + w_2)}_{w' \in \bar{W}}$$

It satisfies close under addition.

③ close under scalar multiplication?

let $V = u_1 + w_1$

and $r \in \mathbb{R}$. $r(u_1 + w_1) = \underbrace{ru_1}_{u' \in \bar{U}} + \underbrace{rw_1}_{w' \in \bar{W}}$

It satisfies close under scalar multiplication.

∴ 3 conditions are all passed so it is indeed a subspace