MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 8

1. Consider $\mathbb{R}_2[x]$ with inner product

$$< f, g >= \int_0^1 f(x)g(x)dx$$

. Apply Gram-Schmidt to $1, x, x^2$ to get an orthonormal basis for $\mathbb{R}_2[x]$.

2. Consider $C[-\pi, \pi]$, vector space of continuous functions defined on interval $[-\pi, \pi]$. Define inner product

$$< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$$

.

   (a) Show that the set

$$F_n = \{\frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots, \sin(nx), \cos(nx)\}$$

   where $n \in \mathbb{N}$ is an orthonormal set.

   (b) Determine the orthogonal projection of function $f(x) = x$ onto the space spanned by F_n. This is usually called the n-th order Fourier approximation of function $f(x)$. If we represent this projection as

$$a_0 \frac{1}{\sqrt{2}} + b_1 \sin(x) + c_1 \cos(x) +, \ldots, + b_n \sin(nx) + c_n \cos(nx)$$

   then $a_0, b_1, c_1, \ldots, b_n, c_n$ are called the Fourier coefficients of function $f(x)$.

3. Consider $v \in \mathbb{R}^n$ and subspace $U \subseteq \mathbb{R}^n$. We know that we can write v as a sum of $v_1 \in U$ and $v_2 \in U^{\perp}$. Show that this decomposition is unique.

4. Four data points in $\mathbb{R}^3$ with coordinates are given as follows.

$$(-1, 2, 9), (0, 1, 1), (2, 0, 0), (1, 2, -1)$$

.

   Determine coefficients c_1, c_2 such that the plane $z = c_1 x + c_2 y$ best fits the data.

5. Let $P \in \mathcal{L}(V)$ be an orthogonal projection map in inner product space V that projects vectors into subspace U. Show from first principle that $< x, Py >=< Px, y >=< Px, Py >$ for all $x, y, z \in V$.

6. True or False.

   (a) If A, B are symmetric matrices, then so are their product AB.

   (b) If A admits a QR factorization, i.e., $A = QR$, then $R = Q^T A$.

   (c) If $A \in M_{m \times n}(\mathbb{R})$, then rank$(A) =$ rank$(A^T A)$.

   (d) Least square solution x^* to system $Ax = b$ is chosen so that Ax^* is as close as possible to b.

(e) If the cols of A are linearly independent, then the least square solution to system $Ax = b$ is unique.

(f) If $b \in \text{ Col}(A)$, then the least square solution x^* to system $Ax = b$ satisfies $Ax^* = b$.

(g) If $AA^T = A^T A$ for a square matrix, then A must be orthogonal.

(h) Let $A \in M_3(\mathbb{R})$ that represents an orthogonal projection with respect to standard basis in $\mathbb{R}^3$. There exists an orthogonal matrix $Q \in M_3(\mathbb{R})$ sucht that $Q^T AQ$ is diagonal.

7. Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = (v_1 \,|v_2|\, v_3)$ with v_1, v_2, v_3 are the columns of A.

(a) Use Gram-Schmidt process to construct an orthonormal set $\{q_1, q_2, q_3\}$ such that for $j = 1, 2, 3$,

$$\text{span}\,\{q_1, \ldots, q_j\} = \text{span}\,\{v_1, \ldots, v_j\}\,.$$

(b) Use the answer from (i), find r_{ij}, for $1 \le i \le j \le 3$ such that

$$v_1 = r_{11} q_1, \quad v_2 = r_{12} q_1 + r_{22} q_2, \quad v_3 = r_{13} q_1 + r_{23} q_2 + r_{33} q_3.$$

(c) Denote $Q = (q_1 \,|q_2|\, q_3)$ and $R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$. Show that indeed $A = QR$ and $\text{Col}(A) = \text{Col}(Q)$.

(d) Show that $Q^T Q = I_3$ and $QQ^T = q_1 q_1^T + q_2 q_2^T + q_3 q_3^T$. Therefore QQ^T is the orthogonal projection onto $\text{Col}(Q) = \text{Col}(A)$.

1. Consider $\mathbb{R}_2[x]$ with inner product

$$< f, g >= \int_0^1 f(x)g(x)dx$$

. Apply Gram-Schmidt to $1, x, x^2$ to get an orthonormal basis for $\mathbb{R}_2[x]$.

$u_1 = 1 \quad , \quad u_2 = x \quad , \quad u_3 = x^2$

$q_1 = 1$

$q_2 = x - \langle q_1, u_2 \rangle q_1 = \int_0^1 x \, dx = x - \frac{1}{2}$

$\|q_2\| = \sqrt{\langle x - \frac{1}{2}, x - \frac{1}{2} \rangle} = \sqrt{\int_0^1 (x - \frac{1}{2})^2 dx} = \sqrt{\frac{1}{12}} = \frac{1}{\sqrt{12}}$

$q_2 = \sqrt{12}\, x - \frac{\sqrt{12}}{2}$

$q_3 = x^2 - \langle q_1, u_3 \rangle q_1 - \langle q_2, u_3 \rangle q_2$

$= x^2 - \int_0^1 x^2 \, dx - \int_0^1 (\sqrt{12}\, x - \frac{\sqrt{12}}{2})(x^2) \, dx \, (\sqrt{12}\, x - \frac{\sqrt{12}}{2})$

$= x^2 - \frac{1}{3} - \frac{2x - 1}{2} = x^2 - x + \frac{1}{6}$

$\|q_3\| = \sqrt{\int_0^1 (x^2 - x + \frac{1}{6})^2 dx} = \frac{1}{6\sqrt{5}}$

$q_3 = 6\sqrt{5}\,(x^2 - x + \frac{1}{6})$

orthonormal basis $= \{1, \sqrt{12}\, x - \frac{\sqrt{12}}{2}, 6\sqrt{5}\,(x^2 - x + \frac{1}{6})\}$.

2. Consider $C[-\pi, \pi]$, vector space of continuous functions defined on interval $[-\pi, \pi]$. Define inner product

$$< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$$

.

(a) Show that the set

$$F_n = \{\frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots, \sin(nx), \cos(nx)\}$$

where $n \in \mathbb{N}$ is an orthonormal set.

Orthogonal part : need to show : $(m, n \in \mathbb{N})$.

① $\langle \frac{1}{\sqrt{2}}, \sin(nx) \rangle = 0$

② $\langle \frac{1}{\sqrt{2}}, \cos(nx) \rangle = 0$.

③ $\langle \sin(nx), \sin(mx) \rangle = 0$ if $n \neq m$.

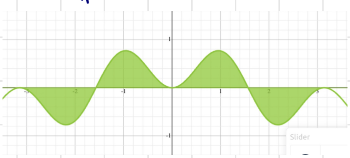
④ $\langle \cos(nx), \cos(mx) \rangle = 0$ if $n \neq m$.

⑤ $\langle \sin(nx), \cos(mx) \rangle = 0$.

①: $\frac{1}{\sqrt{2}} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx)\, dx = k \cdot \left[-\frac{1}{n} \cos(nx) \right]_{-\pi}^{\pi} = 0$. ✓

② . $\frac{1}{\sqrt{2}} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx)\, dx = k \cdot \left[\frac{1}{n} \sin(nx) \right]_{-\pi}^{\pi} = 0$ ✓

③ $\int_{-\pi}^{\pi} \sin(nx) \sin(mx)\, dx$ is shown as below: (odd)

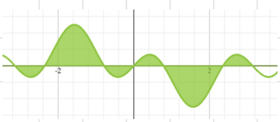


$\therefore \langle \sin(nx), \sin(mx) \rangle = 0$. ✓

④ $\int_{-\pi}^{\pi} \cos(nx) \cos(mx)\, dx =$

$\underbrace{\left[\cos(nx) \frac{1}{m} \sin(mx)\right]}_{=0} - \left[\cos(nx) \frac{1}{m} \sin(mx)\, dx\right]$ is odd $\therefore$ 0 ✓

⑤ $\int_{-\pi}^{\pi} \sin(nx) \cos(mx)\, dx$ is show as below: (odd).



$\therefore \langle \sin(nx), \cos(mx) \rangle = 0$ ✓.

$\therefore$ it's indeed orthognol.

to check normal, need to check: $(n \in \mathbb{N})$

① $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = 1$

② $\langle \sin(nx), \sin(nx) \rangle = 1$

③ $\langle \cos(nx), \cos(nx) \rangle = 1$

①. $\frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{2}}\right)^2 dx = \frac{1}{\pi} \left[\frac{1}{2}x\right]_{-\pi}^{\pi} = \frac{1}{\pi} \cdot \left(\frac{1}{2}\pi + \frac{1}{2}\pi\right) = 1.$ ✓

②

$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(nx)\, dx = 1 - \frac{\sin(2\pi n)}{2\pi n}.$ $\because \sin(2\pi n) = 0.$

$\therefore 1 - \frac{\sin(2\pi n)}{(2\pi n)} = 1.$ ✓

③ $\frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(nx)\, dx = 1 + \frac{\sin 2\pi n}{2\pi n}$ $\because \sin(2\pi n) = 0,$

$\therefore 1 + \frac{\sin(2\pi n)}{(2\pi n)} = 1.$ ✓

$\therefore$ it's also normal.

(b) Determine the orthogonal projection of function $f(x) = x$ onto the space spanned by F_n. This is usually called the n-th order Fourier approximation of function $f(x)$. If we represent this projection as

$$a_0 \frac{1}{\sqrt{2}} + b_1 \sin(x) + c_1 \cos(x) +, \ldots, + b_n \sin(nx) + c_n \cos(nx)$$

then $a_0, b_1, c_1, \ldots, b_n, c_n$ are called the Fourier coefficients of function $f(x)$.

$\mathrm{Proj}_{F_n} u = \sum_{v \in F_n} \langle u, v \rangle v$

$= \langle x, \frac{1}{\sqrt{2}} + \sin(x) + \cos(x) + \cdots + \sin(nx) + \cos(nx) \rangle$

$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx$

$= \langle x, \frac{1}{\sqrt{2}} \rangle + \langle x, \sin x \rangle + \langle x, \cos x \rangle + \cdots + \langle x, \sin nx \rangle + \langle x, \cos nx \rangle$

$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} x\, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x\, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos x\, dx$

$+ \cdots + \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx\, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx\, dx.$

$\frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx = -\frac{2(-1)^n}{n}$ $\frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx)\, dx = 0.$

$\therefore = 2\sin(x) - \sin(2x) + \frac{2}{3} \sin(3x) - \cdots + \left(-\frac{2(-1)^n}{n}\right) \sin(nx).$

3. Consider $v \in \mathbb{R}^n$ and subspace $U \subseteq \mathbb{R}^n$. We know that we can write v as a sum of $v_1 \in U$ and $v_2 \in U^\perp$. Show that this decomposition is unique.

for $\vec{v} \in \mathbb{R}^n$, it can be written as $\vec{v}_1 \in \bar{u} + \vec{v}_2 \in \bar{u}^\perp$

for $\vec{w} \in \mathbb{R}^n$, it can be written as $\vec{w}_1 \in \bar{u} + \vec{w}_2 \in \bar{u}^\perp$

need to show that $\vec{v}_1 = \vec{w}_1$ and $\vec{v}_2 = \vec{w}_2$

$\vec{v} = \vec{v}_1 + \vec{v}_2 = \vec{w}_1 + \vec{w}_2$

$\therefore \vec{v}_1 - \vec{w}_1 = \vec{w}_2 - \vec{v}_2 \Rightarrow (\vec{v}_1 - \vec{w}_1) - (\vec{w}_2 - \vec{v}_2) = 0$

$\vec{v}_1 - \vec{w}_1 \in \bar{u}$, (closed under addition.)

$\vec{w}_2 - \vec{v}_2 \in \bar{u}^\perp$. (also closed under addition.)

$(\vec{v}_1 - \vec{w}_1)$ and $(\vec{w}_2 - \vec{v}_2)$ must be each other's additive inverse. but since they are orthogonal, they must both be zero.

$\therefore \vec{v}_1 - \vec{w}_1 = 0$ $\quad\quad$ $\vec{w}_2 - \vec{v}_2 = 0$

$\therefore \vec{v}_1 = \vec{w}_1$ $\quad\quad$ $\vec{w}_2 = \vec{v}_2$.

$\therefore$ they are indeed unique.

4. Four data points in $\mathbb{R}^3$ with coordinates are given as follows.

$$(-1, 2, 9), (0, 1, 1), (2, 0, 0), (1, 2, -1)$$

.

Determine coefficients c_1, c_2 such that the plane $z = c_1 x + c_2 y$ best fits the data.

$$\begin{cases} -1x + 2y = 9z \\ \quad 1y = 1z \\ 2x = 0 \\ 1x + 2y = -1z \end{cases}$$

$$A = \begin{pmatrix} -1 & 2 \\ 0 & 1 \\ 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = b = \begin{pmatrix} 9 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$4\times 3 \qquad 3\times 1 \qquad 4\times 1$

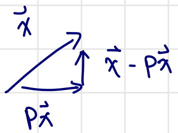
find x^* where $A^T A x^* = A^T b$.

$$A^T A = \begin{pmatrix} -1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & 1 \\ 2 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 9 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} -1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 9 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -10 \\ 17 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 6 & 0 & -10 \\ 0 & 9 & 17 \end{array}\right) = \begin{pmatrix} -\frac{5}{3} \\ \frac{17}{9} \end{pmatrix}$$

5. Let $P \in \mathcal{L}(V)$ be an orthogonal projection map in inner product space V that projects vectors into subspace U. Show from first principle that $< x, Py >=< Px, y >=< Px, Py >$ for all $x, y, z \in V$.



We know that $P\vec{x} \perp (\vec{x} - P\vec{x})$ $\quad P\vec{y} \perp (\vec{y} - P\vec{y})$.

$P\vec{x} \in \bar{U}$ and $P\vec{y} \in \bar{U}$.

$\therefore P\vec{y} \perp (\vec{x} - P\vec{x})$

$\therefore \langle P\vec{y}, \vec{x} - P\vec{x} \rangle = 0$.

$\langle P\vec{y}, \vec{x} \rangle = \langle P\vec{y}, P\vec{x} \rangle$

$P\vec{x} \perp (\vec{y} - P\vec{y})$

$\langle P\vec{x}, \vec{y} \rangle = \langle P\vec{x}, P\vec{y} \rangle$.

$\therefore \langle P\vec{y}, \vec{x} \rangle = \langle P\vec{y}, P\vec{x} \rangle = \langle P\vec{x}, P\vec{y} \rangle = \langle P\vec{x}, \vec{y} \rangle$

6. True or False.

(a) If A, B are symmetric matrices, then so are their product AB.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ (symmetric)

$AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ (not symmetric.)

$\therefore$ proved false by counterexample.

(b) If A admits a QR factorization, i.e., $A = QR$, then $R = Q^T A$.

True. In QR factorization, Q is an orthonormal matrix.

$\therefore$ $Q^T = Q^{-1}$ $\quad\therefore$ $R = Q^{-1} A = Q^T A$.

(c) If $A \in M_{m\times n}(\mathbb{R})$, then $\text{rank}(A) = \text{rank}(A^T A)$.

$\dim(\ker(A)) + \text{rank}(A) = n.$

$\dim(\ker(A^T A)) + \text{rank}(A^T A) = n$

$\therefore$ $\text{rank}(A) = \text{rank}(A^T A)$

$Ax = 0$

$A^T Ax = A^T 0 = 0$

$\therefore$ $\ker(A) = \ker(A^T A)$

(d) Least square solution x^* to system $Ax = b$ is chosen so that Ax^* is as close as possible to b.

True. (definition).

$\| b - Ax^* \| \leq \| b - \vec{v} \| \quad \forall \ \vec{v} \in \text{col}(A)$

(e) If the cols of A are linearly independent, then the least square solution to system $Ax = b$ is unique.

True. The least square solution is the closest solution to A|b. If cols of A are linearly independent, there's the only way to linearly combine those columns to form this solution. $\therefore$ There is only 1 x^* that would yield the least square solution.

(f) If $b \in \text{Col}(A)$, then the least square solution x^* to system $Ax = b$ satisfies $Ax^* = b$.

True. If $b \in \text{Col}(A)$ then the solution for $Ax^* = b$ is x.

(g) If $AA^T = A^T A$ for a square matrix, then A must be orthogonal.

False. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $AA^T = A^T A$. A is not orthogonal.

(h) Let $A \in M_3(\mathbb{R})$ that represents an orthogonal projection with respect to standard basis in $\mathbb{R}^3$. There exists an orthogonal matrix $Q \in M_3(\mathbb{R})$ sucht that $Q^T AQ$ is diagonal.

True. Let $A \in M_3(\mathbb{R})$.
let q_1 and $q_2 \in$ orthorgonal basis (A)
and $q_3 \in$ orthorgonal basis $(\text{Nul}(A))$.

$AQ = (Aq_1 \quad Aq_2 \quad Aq_3) = (q_1 \quad q_2 \quad q_3)$.

$Q^T AQ = I_3$. $\leftarrow$ diagnol.

7. Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = (v_1\,|v_2|\,v_3)$ with v_1, v_2, v_3 are the columns of A.

(a) Use Gram-Schmidt process to construct an orthonormal set $\{q_1, q_2, q_3\}$ such that for $j = 1, 2, 3$,

$$\operatorname{span}\{q_1, \ldots, q_j\} = \operatorname{span}\{v_1, \ldots, v_j\}.$$

a. $u_1 = \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$ $\quad q_1 = \begin{pmatrix}1/2\\-1/2\\1/2\\-1/2\end{pmatrix}$

$q_2 = u_2 - \text{Proj}_{q_1} u_2 = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}$

$q_3 = u_3 - \text{Proj}_{q_1} u_3 - \text{Proj}_{q_2} u_3$

$= \begin{pmatrix}1\\1\\0\\0\end{pmatrix} - 0 - \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\-1/2\\-1/2\end{pmatrix}$

$\therefore \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

(b) Use the answer from (i), find r_{ij}, for $1 \le i \le j \le 3$ such that

$$v_1 = r_{11}q_1, \quad v_2 = r_{12}q_1 + r_{22}q_2, \quad v_3 = r_{13}q_1 + r_{23}q_2 + r_{33}q_3.$$

$V_1 = r_{11} q_1$ $\qquad V_2 = r_{12} q_1 + r_{22} q_2$

$\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = 2 \times \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}$ $\qquad \begin{pmatrix}1\\0\\1\\0\end{pmatrix} = 1 \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix} + 1 \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$

$\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = 0 \times \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix} + 1 \times \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix} + 1 \times \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}$

$\therefore r_{11} = 2.$ $\qquad r_{12} = 1$ $\qquad r_{13} = 0$

$r_{22} = 1$ $\qquad r_{23} = 1$

$r_{33} = 1$

(c) Denote $Q = (q_1 \,|q_2|\, q_3)$ and $R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$. Show that indeed $A = QR$ and $\mathrm{Col}(A) = \mathrm{Col}(Q)$.

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad R = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$QR = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = A. \qquad \therefore \checkmark$$

$$\mathrm{Col}(A) = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$$\mathrm{rref} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \therefore \text{basis } \mathrm{col}(A) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$$\mathrm{Col}(Q) = \left\{ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right\}.$$

$$\mathrm{rref} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \therefore \text{basis } \mathrm{col}(Q) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$$\therefore \quad \mathrm{Col}(A) = \mathrm{col}(Q).$$

(d) Show that $Q^TQ = I_3$ and $QQ^T = q_1q_1^T + q_2q_2^T + q_3q_3^T$. Therefore QQ^T is the orthogonal projection onto $\mathrm{Col}(Q) = \mathrm{Col}(A)$.

$$Q^TQ = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. = I_3.$$

$$QQ^T = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

$$q_1q_1^T + q_2q_2^T + q_3q_3^T$$

$$= \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{pmatrix} \quad \therefore \quad QQ^T = q_1q_1^T + q_2q_2^T + q_3q_3^T$$