1. i

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 5 & 4 & -1 & 0 \\ 2 & -4 & 1 & 1 \\ -7 & -14 & 5 & 10 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 5 & 4 & -1 & 0 \\ 2 & -4 & 1 & 1 \\ 0 & -14 & 5 & 11 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 5 & 4 & -1 & 0 \\ \cancel{7}\ 1 & 0 & 0 & \cancel{x}\ \frac{1}{7} \\ 0 & -14 & 5 & 11 \end{array}\right)$$

switch row sequences:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{7} \\ 0 & -14 & 5 & 11 \\ 5 & 4 & -1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{7} \\ 0 & 1 & -\frac{5}{14} & -\frac{11}{14} \\ 5 & 4 & -1 & 0 \end{array}\right) \xRightarrow{R_3 - R_1 \times 5} 0 \quad 4 \quad -1 \mid -\frac{5}{7}$$

$R_3 - R_2 \times 4$: $\quad R_2 \times 4 = 0 \quad 4 \quad -\frac{20}{14} \quad -\frac{44}{14}$

new $R_3 = 0 \quad 0 \quad \frac{-14+20}{14} \quad \frac{-10+44}{14}$

$= 0 \quad 0 \quad \frac{6}{14} \mid \frac{34}{14}$

$= 0 \quad 0 \quad 1 \mid \frac{34}{6} = \frac{17}{3}$

$$\boxed{\therefore \begin{cases} x = \frac{1}{7} \\ y = \frac{26}{21} \\ z = \frac{17}{3} \end{cases}}$$

One unique solution

ii

$R_2 \leftarrow R_1 + 2 \cdot R_2$; $R_1 \leftarrow R_1 / 2$; $R_3 \leftarrow R_3 - 3 \cdot R_1$

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 2 \\ -1 & 1 & -1 & 2 \\ 3 & 2 & 1 & 2 \\ 5 & 4 & -1 & 2 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 1 \\ 0 & 3 & 0 & 6 \\ 0 & \frac{1}{2} & -2 & -1 \\ 0 & \frac{3}{2} & -6 & -3 \end{array}\right)$$

R_3, R_4 double; R_2 half. $\qquad R_3 \leftarrow R_3 - R_2 \quad R_1 \leftarrow R_1 - \frac{1}{2} R_2$

$R_4 \leftarrow R_4 - 3R_2$.

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -4 & -2 \\ 0 & 3 & -12 & -6 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & -12 & -12 \end{array}\right) \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} R_3 = R_4 = (0 \quad 0 \quad 1 \mid 1)$$

$$\boxed{\therefore \begin{cases} x = -1 \\ y = 2 \\ z = 1 \end{cases}}$$

One unique solution

$y = -1 \quad z = -\frac{1}{2} - \frac{1}{2}$

$b = 1 \quad a = 2$

iii

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 2 & -2 & 4 & 1 & 2 & 4 \\ 3 & 1 & 6 & 0 & 1 & -3 \\ 1 & 0 & 2 & 2 & 1 & 4 \end{array}\right) \xRightarrow{\text{Use } R_1 \text{ to cancel } R_2 - R_4.} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 4 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 2 & 1 & 4 \end{array}\right) \xRightarrow{\text{change sequence}} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 4 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 8 & 3 & 19 \end{array}\right) \xRightarrow{\text{cancel } R_4 \text{ w/ } R_3} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & -13 & -13 \end{array}\right) \quad \therefore \ x_5 = 1.$$

entire matrix in RREF:

$$\left(\begin{array}{ccccc|c} 1 & 0 & \boxed{2} & 0 & 0 & -1 \\ 0 & 1 & \boxed{0} & 0 & 0 & -1 \\ 0 & 0 & \boxed{0} & 1 & 0 & 2 \\ 0 & 0 & \boxed{0} & 0 & 1 & 1 \end{array}\right)$$

let $x_3 = t$.

$$\begin{cases} x_1 = -1 - 2t \\ x_2 = -1 \\ x_3 = t \\ x_4 = 2 \\ x_5 = 1 \end{cases}$$

infinite many solutions.

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$$

iv.

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 5 \\ 0 & 2 & 3 & 4 & 5 & 5 \\ 0 & -5 & -5 & -9 & -9 & -10 \\ 0 & 3 & 2 & 5 & 4 & 5 \end{array}\right) \Rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 5 & 2 & 7 & 5 \\ 0 & 1 & -4 & 1 & -1 & 0 \\ 0 & 0 & -10 & 4 & -14 & 10 \\ 0 & 0 & 5 & 2 & 7 & 5 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 5 & 2 & 7 & 5 \\ 0 & 1 & -4 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{7}{5} & -1 \\ 0 & 0 & 0 & 1 & 0 & \frac{5}{2} \end{array}\right)$$

in RREF:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{5} & -\frac{5}{2} \\ 0 & 0 & 1 & 0 & \frac{7}{5} & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{5}{2} \end{array}\right)$$

$$\therefore \begin{cases} x_1 = 0 \\ x_2 = -\frac{5}{2} - \frac{2}{5}t \\ x_3 = -\frac{7}{5}t \\ x_4 = \frac{5}{2} \\ x_5 = t \end{cases}$$

$$\therefore \boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{5}{2} \\ 0 \\ \frac{5}{2} \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -\frac{2}{5} \\ -\frac{7}{5} \\ 0 \\ 1 \end{pmatrix}}$$

infinitely many solutions

2. a.

$$\left(\begin{array}{ccc|c} 1 & 1 & a & -1 \\ -1 & (a-1) & (2-a) & (a+1) \\ 6 & (5a+6) & (7a+7) & a^2 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 6R_1]{R_2 \leftarrow R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & a & -1 \\ 0 & a & 2 & a \\ 0 & 5a & a+7 & a^2+6 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & a & -1 \\ 0 & a & 2 & a \\ 0 & 0 & a-3 & (a-2)(a-3) \end{array}\right)$$

(i) no solution: since there can't be an a to form $(0 \quad 0 \quad 0 \mid \text{not } 0)$ row, there aren't any ways to make the system have no solutions.

(ii) exactly one solution: as long as $a \neq 3$

(iii) infinitely many solution: $a = 3$. So that Row 3 would be

b.

$$\left(\begin{array}{ccc|c} (a+1) & a & -a & (2+a) \\ (a+1) & (a+2) & -(a+2) & (a+4) \\ (a+1) & a & (a^2-6) & (a^2-2a+4) \\ (2a+2) & 2a & (a-3)(a+2) & (a^2-a+6) \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} (a+1) & a & -a & (2+a) \\ 0 & 2 & -2 & 2 \\ 0 & 0 & (a^2+a-6) & (a^2-3a+2) \\ 0 & 0 & (a^2+a-6) & (a^2-3a+2) \end{array}\right)$$

simplify:

$$\left(\begin{array}{ccc|c} (a+1) & a & -a & (2+a) \\ 0 & 2 & -2 & 2 \\ 0 & 0 & (a+3)(a-2) & (a-2)(a-1) \\ \cancel{0} & \cancel{0} & \cancel{(a+3)(a-2)} & \cancel{(a-2)(a-1)} \end{array}\right)$$ the same.

(i) no solution: $a = -3$. last row would be $(0 \quad 0 \quad 0 \mid 20)$.

(ii) exactly one solution: $a \neq -3$ and $a \neq 2$

(iii) infinitely many solutions: $a = 2$. So last row $= (0 \quad 0 \quad 0 \mid 0)$

3. (i) to show $b+d \in L(A)$, we need to show that $(A|b+d)$ has at least one solution.
Since $b \in L(A)$, $d \in L(A)$, then $(A|b)$ and $(A|d)$ both have at least one solution. $(A|(b+d)) = (A|b) + (A|d)$
so one solution of $(A|(b+d))$ would be the sum of $(A|b)$ and $(A|d)$.

(ii) to show that $t \cdot b \in L(A)$, we need to show $t \cdot b$ has at least one solution.
Since $b \in L(A)$, b has at least one solution.
Since t is a scaler, multiplying the matrix b with the scaler is an elementary operation which doesn't make the solution invalid. So $t \cdot b$ still has at least one solution.

4. a. This type of matrix does exist.
For example, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$: there must be one free variable to ensure infinitely may solutions.

b. This type of matrix does exist.
For example, one of the rows of this matrix $A \in M_{m \times n}$ is all zero, like $A_{m \times n} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$.

b_1 could be $\begin{pmatrix} x \\ 0 \end{pmatrix}$, where $x \in \mathbb{R}$.
b_2 could be $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where $x_1, x_2 \in \mathbb{R}$. and $x_2 \neq 0$.

c. Suppose by contradiction. let there be a matrix A that there exists $b_2, b_3 \in \mathbb{R}$ where $(A|b_2)$ has infinite amount of solutions and $(A|b_3)$ has no solution. Then at least one row in A's echelon form needs to be zeros. This will create a free variable that would result in infinite solutions for any $b \in \mathbb{R}$.
Therefore, there wouldn't be any b that can make the $(A|b)$ have only one solution.

d. There cannot be a matrix like this because a row of zeros means there's a free variable, which suggests that there's infinitely amount of solutions.

e. There's no such matrix.
If this echelon form has one row of zeros, then if the corresponding b_n is 0, it has infinitely many solutions. If the corresponding b_n is not 0, then it has no solutions.

f. It doesn't exist, for a homogeneous system, if $(A|0)$ has one non-trivial solution, then it has infinite amount of solutions. Then there must be a free variable, so the echelon form of the system must have at least one row of zeros.

g. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \left(\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $A_{3\times3}$ could be $\rightarrow \begin{pmatrix} 3 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

$\therefore$ it exists

h. Yes, there exists such matrix.

let $A = (1 \ -1 \ -1)$

then $(1 \ -1 \ -1 \mid 0) =$

$x - y - z = 0$.

$x = y + z$. if $y = s$, $z = t$, then $x = s+t$.

i. There exists such A. $A = (0 \ 0)$.

the solution of $(A|0)$ is $\{ s \ t \mid s, t \in \mathbb{R} \}$

If I can prove $(s \ t) = (s+t \quad s-t \mid s, t \in \mathbb{R})$,

then the solution of $(A|0)$ could also be $\{(s+t \quad s-t) \mid s, t \in \mathbb{R}\}$

① for $(x, y) \in \{(s, t) \mid s, t \in \mathbb{R}\}$

$(x, y) = \left(\frac{x+y}{2} + \frac{x-y}{2}, \ \frac{x+y}{2} - \frac{x-y}{2}\right)$ $\left(\frac{x+y}{2}, \ \frac{x-y}{2}\right) \in \{(s, t) \mid s, t \in \mathbb{R}\}$

$\therefore (s, t) \subseteq \{(s+t), (s-t)\}$

② for $(x, y) \in \{((s+t), (s-t)) \mid s, t \in \mathbb{R}\}$

$x = s+t$, $y = s-t$,

so $(x, y) \in \{(s, t) \mid s, t \in \mathbb{R}\}$

$\therefore \{(s+t), (s-t)\} \subseteq (s, t)$

$\therefore (s, t) = (s+t, \ s-t)$.

$\therefore$ Solution of $(0 \ 0 \mid 0) = \{(s+t \quad s-t) \mid s, t \in \mathbb{R}\}$.