# Chapter 3. linear transformation

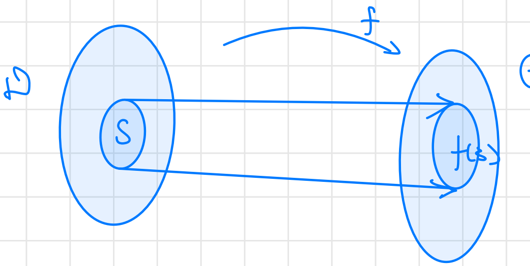
Let $f: \mathcal{D} \to \mathcal{C}$ be a function.

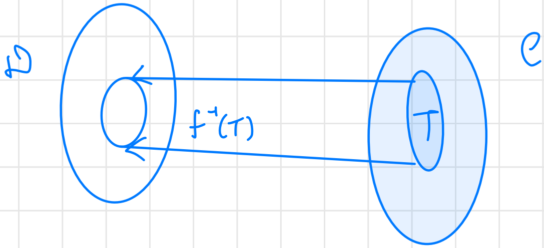
Injective: $\forall\ x, y \in \mathcal{D}$, $f(x) = f(y) \Rightarrow x = y$.
$x \neq y \Rightarrow f(x) \neq f(y)$.

Subjective: $\forall y \in \mathcal{C}$, $\exists x \in \mathcal{D}$ s.t. $f(x) = y$
range $f = \mathcal{C}$.

bijective: both injective and subjective.

def. Let $S \subseteq \mathcal{D}$. Then the image of $S$ under $f$ is
$f(S) = \{ f(s) \in \mathcal{C} \mid s \in S \}$.
Let $T \subseteq \mathcal{C}$, Then the pre image of $T$ under $f$ is
$f^{-1}(T) = \{ x \in \mathcal{D} \mid f(x) \in T \}$.




Def Let $\bar{V}, \bar{W}$ be two v.s. over field $\mathbb{R}$
Then function $T$ from $\bar{V}$ to $\bar{W}$ is a linear map
if ⓐ $\forall x, y \in \bar{V}$, $T(x+y) = T(x) + T(y)$ additivity
ⓑ $\forall x \in \bar{V}$, $\forall \alpha \in \mathbb{R}$, $T(\alpha x) = \alpha T(\alpha)$ homogeniety.

Homomorphism: a map that preserves the structure under 2 sets.

e.g. $f: \mathbb{R}_n[x] \mapsto \mathbb{R}$.

$p(x) \longmapsto \int_0^1 p(x)\,dx$.

proof: Let $p(x), q(x) \in \mathbb{R}_n[x]$, arbitrary

Let $k \in \mathbb{R}$, arbitrary

② $f(p(x)+q(x)) = \int_0^1 (p(x)+q(x))\,dx$

$= \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx$

$f(kp(x)) = \int_0^1 kp(x)\,dx = k\int_0^1 p(x)\,dx$

e.g. ②. Let $A \in M_{m\times n}(\mathbb{R})$

$T_A: \mathbb{R}^n \to \mathbb{R}^m$.

$x \longmapsto Ax$

$T_A$ is a linear map.

e.g. ③.

$\mathbb{R}[x] \to \mathbb{R}[x]$

$p(x) \mapsto x^2 p(x)$

$T$ is a linear map?

$p(x)+q(x) \mapsto x^2(p(x)+q(x)) = x^2p(x) + x^2q(x)$.

$k\,p(x) \mapsto x^2(kp(x)) = kx^2(p(x))$.

e.g. ④ $T: \bar{V} \to \bar{W}$

$x \mapsto 0_{\bar{W}}$.

zero-map

Notation: Let $\bar{V}, \bar{W}$ be two vector spaces over $\mathbb{R}$, then the set of all linear maps from $\bar{V}$ to $\bar{W}$ is denoted by $\mathcal{L}(\bar{V}, \bar{W})$.

Let $T: \bar{V} \to \bar{W}$ be a linear map.

def. $\ker T = \{ x \in \bar{V} \mid T(x) = 0_{\bar{W}} \}$ subspace of $\bar{V}$

$\mathrm{Im}\,T = \{ y \in \bar{W} \mid T(x) = y,$ for some $x \in \bar{V} \}$. subspace of $\bar{W}$

Theorem: Let $\bar{V}$ be a vector space with basis $\langle b_1, \dots, b_n \rangle$, and $\bar{W}$ be a v.s. with basis $\langle d_1, \dots, d_n \rangle$ over $\mathbb{R}$. There exists a unique linear map from $\bar{V}$ to $\bar{W}$ s.t. $T(b_i) = d_i \ \forall i \in \{1, \dots, n\}$.

proof $\uparrow$: Let $T: \bar{V} \to \bar{W}$ defined as $T\left(\sum_{i=1}^{n} c_i b_i\right) = \sum_{i=1}^{n} c_i d_i$

for all $\{c_i\}_{i=1}^{n}$ in $\mathbb{R}$.

Note: $T(b_i) = d_i \quad \forall i \in \{1, \dots, n\}$

NTS. T is linear ... unique.

$f: \bar{V} \to \bar{W}$

domain codomain.

abiyen Gevani

linear map:

① $f(x+y) = f(x) + f(y), \ \forall x, y \in \bar{V}$

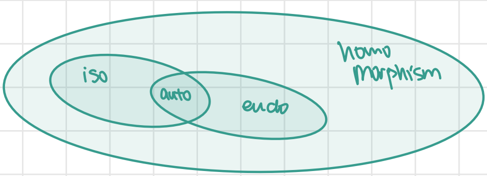
② $f(\alpha x) = \alpha f(x), \ \forall x \in \bar{V}, \alpha \in \mathbb{R}$.

$f \in \mathcal{L}(\bar{V}, \bar{W})$

$\bar{W} \geq \mathrm{Im} f := \{ y \in \bar{W} \mid f(x)\, y$ for some $x \in \bar{V} \}$

$\bar{V} \geq \mathrm{Ker} f := \{ x \in \bar{V} \mid f(x) = 0_{\bar{W}} \}$.

Endo: $\mathcal{L}(\bar{V}, \bar{V})$ linear operator. to itself.



Thm: Let $\bar{V}$ be a vector space with basis $\langle b_1, \dots, b_n \rangle$
$\bar{W}$ $\langle d_1, \dots, d_n \rangle$

There exists a unique linear map from $\boxed{\bar{V} \to \bar{W}}$ Same dimension
s.t. $T(b_i) = d_i$, $\forall i \in \{1, \dots, n\}$.

Existence: Let $T: \bar{V} \to \bar{W}$,
$T\left(\sum_{i=1}^{n} c_i b_i\right) = \sum_{i=1}^{n} c_i d_i$ where $\{c_i\}_{i=1}^{n}$ are arbitrary scales in $\mathbb{R}$.

Now check if $T$ is linear: Let $x = \sum_{i=1}^{n} a_i b_i$, $y = \sum_{i=1}^{n} c_i b_i$ be elements in $\bar{V}$

what are we trying to prove?

Let $\alpha \in \mathbb{R}$

consider $T(x+y) = T\left(\sum_{i=1}^{n} (a_i + c_i) b_i\right)$

$= \sum_{i=1}^{n} (a_i + c_i) d_i$

$= \sum_{i=1}^{n} a_i d_i + \sum_{i=1}^{n} c_i d_i = T(x) + T(y)$.

$T(\alpha x) = T\left(\alpha \sum_{i=1}^{n} a_i b_i\right)$

$= T\left(\sum_{r=1}^{n} \alpha a_i b_i\right)$

$= \sum_{i=1}^{n} \alpha a_i d_i = \alpha \sum_{i=1}^{n} a_i d_i = \alpha T(x)$.

Uniqueness: Sps $\tilde{T} \in \mathcal{L}(\bar{V}, \bar{W})$ s.t. $\tilde{T}(b_i) = d_i$ $\forall i \in \{1, \dots, n\}$.

suppose $\tilde{T}$ and $T$ are both linear maps from $\bar{V} \to \bar{W}$, proof $\tilde{T}(x)$ and $T(x)$ are the same.

Let $x$ be an arbitrary element in $\bar{V}$, i.e. $x = \sum_{i=1}^{n} c_i b_i$ for some $\{c_i\}_{i=1}^{n}$ scalar.

Consider $T(x) - \tilde{T}(x) = T\left(\sum_{i=1}^{n} c_i b_i\right) - \tilde{T}\left(\sum_{i=1}^{n} c_i b_i\right)$

$= \sum c_i d_i - \sum c_i d_i = 0$.

$\therefore$ The image of $x$ under $T$ and $\tilde{T}$ are the same

$\Rightarrow T = \tilde{T}$

Prop: Let $T \in \mathcal{L}(\bar{V}, \bar{W})$, then $\ker T = \{\vec{0}\}$ iff $T$ is injective.

What does this mean?

if $\ker T \{0\} \Rightarrow$ injective?

Let $x, y \in \bar{V}$, s.t. $T(x) = T(y)$.

$\Rightarrow T(x) - T(y) = \vec{0}$

$T(x-y) = \vec{0}$. (linear)

$x - y \in$ Kernel of $T$, which only contains $0$.

$\therefore x - y = 0$

$\therefore x = y$.

if injective $\Rightarrow \ker T \{0\}$?

Let $\vec{v} \in \ker T$ i.e. $T(\vec{v}) = \vec{0}$

$T(\vec{0}_v) = T(\bar{v} - \bar{v}) = T(\bar{v}) - T(\bar{v}) = \vec{0}_{\bar{w}}$.

$\Rightarrow T(\vec{0}) = T(\bar{v})$.

$\Rightarrow \vec{v} = \vec{0}$ by injectivity of $T$.

① $\ker T$ "measures" how injective $T$ is

② $\forall T \in \mathcal{L}(\bar{V}, \bar{W})$, $T(\vec{0}_{\bar{v}}) = 0_{\bar{w}}$

Function compositions:

Let $f: A \to B$, $g: B \to C$ be two functions.

then $g \circ f: A \to C$.

$x \mapsto g(f(x))$.

facts: $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} d$

$h \circ (g \circ f) = (h \circ g) \circ f$.

def. Let $S$ be a non-empty set, then $id: S \to S$

$x \mapsto x$.

is called the identity function $S$.

e.g. $f: \mathbb{R} \to \mathbb{R}$ unique.

$f(x) = x$ is $id_{\mathbb{R}}$.

prop. Let $f: \mathcal{D} \to C$ a function.

Then $f \circ id_{\mathcal{D}} = f$

$id_C \circ f = f$.

$$\begin{tikzcd} \mathcal{D} \arrow[loop left, "id_{\mathcal{D}}"] \arrow[rr, "f"] & & C \arrow[loop right, "id_C"] \end{tikzcd}$$

( a way to map to itself ).

prop. let $f: A \to B$ be a function that is bijective,

$\exists$ a function from $B \to A$ s.t.

$f \circ g = id_B$

$g \circ f = id_A$

$g$ is the inverse of $f$, denoted as $f^{-1}(x)$.

prop. The linear map from earlier is a bijection.

$$\boxed{\begin{array}{l} T: \bar{V} \to \bar{W} \\ \bar{V} \text{ has basis } \langle b_1, \dots, b_n \rangle \\ W \quad \dots \quad \langle d_1, \dots, d_n \rangle \\ T(b_i) = d_i \ \forall i \in \{1, \dots, n\} \end{array}}$$

Proof injective:

KerT is trivial.

consider $x \in \ker T$ i.e. $x = \sum_{i=1}^{n} c_i b_i$ for some $c_1, \dots, c_n$.

$T(x) = T\left(\sum_{i=1}^{n} c_i b_i\right) = 0$.

linear. $\downarrow$

$\sum_{i=1}^{n} c_i \underbrace{T(b_i)}_{= d_i} = \vec{0}$. $\Rightarrow \sum_{i=1}^{n} c_i d_i = \vec{0}$.

$\downarrow$ basis

$\downarrow$ linearly independent.

$\downarrow$

$c_1 = \dots = c_n = 0$

$\downarrow$

$x = \sum_{i=0}^{n} 0 b_i = \vec{0}$

proof surjection:

Let $y \in \bar{W}$, i.e. $y = \sum_{i=1}^{n} c_i a_i$

for $c_1, \dots, c_n$ scalar from $\mathbb{R}$

Note: $T(c_i b_i) = c_i a_i = \sum_{i=1}^{n} c_i a_i = y$.

This means $\sum_{i=1}^{n} c_i b_i$ is a pre-image of $y$

def. Let $T \in \mathcal{L}(\bar{V}, \bar{W})$ ① . We say $T$ is isomorphism if it's bijective. ②

a form of homomorphism.

def Let $\bar{V}, \bar{W}$ be vector spaces. We say $\bar{V}$ is isomorphic to $\bar{W}$ if $\exists$ an isomorphism from $\bar{V}$ to $\bar{W}$.

Notation: if $\bar{V}$ & $\bar{W}$ are isomorphic, we denote $\bar{V} \cong \bar{W}$.

Cor. If $\bar{V}, \bar{W}$ are vector spaces such that $\dim \bar{V} = \dim \bar{W} < \infty$, then $\bar{V} \cong \bar{W}$.

ex. $\mathbb{R}^n \cong P_{n-1}(\mathbb{R})$. $M_{m \times n}(\mathbb{R}) \cong \mathbb{R}^{m \times n}$

Fact: If $T \in \mathcal{L}(\bar{V}, \bar{W})$, and $T$ is an isomorphism, then $T^{-1}$ is also an isomorphism.

Matrix:

- matrix addition:
  Let $A, B \in M_{m \times n}(\mathbb{R})$, then $(A+B) \in M_{m \times n}(\mathbb{R})$
  and $(A+B)_{i,j} = (A)_{i,j} + (B)_{i,j}$
- matrix scalar multiplication:
  Let $A, B \in M_{m \times n}(\mathbb{R})$, $k \in \mathbb{R}$.
  $kA \in M_{m \times n}(\mathbb{R})$,
  and $(kA)_{i,j} = k(A)_{i,j}$.

- Matrix multiplication :

  Let $A \in M_{m\times n}(\mathbb{R})$ , $B \in M_{n\times p}(\mathbb{R})$.

  Then $AB \in M_{m\times p}(\mathbb{R})$

  and $(AB)_{i,j} = (A)_{i,\cdot}(B)_{\cdot,j}$ dot product.

- Let $A$ be a square matrix, then $tr(A) = \sum_{i=1}^{n} (A)_{i,i}$.

What is the pre-image of $b$ under $A$? $\rightarrow$ $A\vec{v}$. $A$ acting on $\vec{v}$.

$A$ sends $\vec{x}$ to $\vec{b}$

maps $\downarrow$

$A\vec{x} = \vec{b}$

Short Review :

| system linear equations. | augmented matrix | vector equations | |
|---|---|---|---|
| | $(A\vert b)$ | $x_1a_1 + \cdots + x_na_n$ | |
| has no solution | inconsistent, rref$(A\vert b)$ has pivot in last row | $\vec{b} \notin \operatorname{span}(a_1,\dots,a_n)$ | $T_A$ not surjective $\vec{b} \notin Im(T_A)$ or $\vec{b} \notin Col(T_A)$ |
| has exactly 1 solution | Pivot in every row in rref$(A\vert b)$, Consistent | $\vec{b} \in \operatorname{span}(a_1,\dots,a_n)$ $(a_1,\dots,a_n)$ is linearly independent | $\vec{b} \in Im(T_A)$ $T_A$ is injective |
| has $\infty$ly many solutions | has free variable | $\vec{b} \in \operatorname{span}(a_1,\dots,a_n)$ $(a_1,\dots,a_n)$ is linearly dependent. | $\vec{b} \in Im(T_A)$ $T_A$ is not injective |

Let $A \in M_{m\times n}(\mathbb{R})$.

$T_A : \mathbb{R}^n \to \mathbb{R}^m$.

$x \mapsto Ax$

$m\left(\quad\right)_{n}\left(\quad\right)_{n\times 1} =$

$T_A$ is injective if each col has a pivot in rref

$T_A$ is surjective if each row has a pivot in rref.

$T_A$ is bijective if each row and col has a pivot in rref.

$\rightarrow$ $\boxed{\text{Square matrix}}$

$M_{m\times n}(\mathbb{R}) \cong \mathbb{R}^{m\times n}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$$

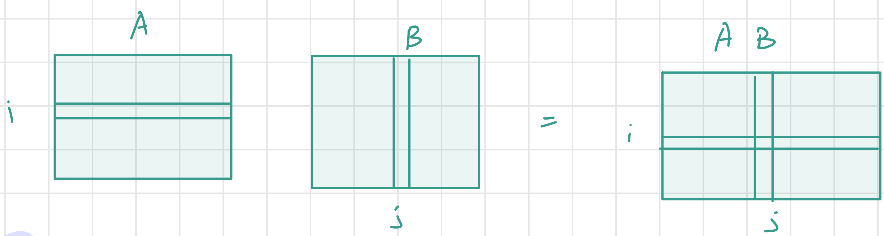
different: multiplication of 2 $M_{m\times n}(\mathbb{R})$ : $\rightarrow$ matrix

... 2 $\mathbb{R}^{m\times n}$ : $\rightarrow$ scaler.

Let $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{n\times p}(\mathbb{R})$, then

$(AB) \in M_{m\times p}(\mathbb{R})$.

$(AB)_{i,j} = (A)_{i,\bullet} \bullet (B)_{\bullet,j}$




e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -1 & 10 \end{pmatrix}$

RM: $A(b_1, \dots, b_n) = (Ab_1, \dots, Ab_n)$

↓ - linear combination of cols in A

- A sends $b_1$ to sth..

$$\begin{pmatrix} \text{—} & a_1 & \text{—} \\ & \vdots & \\ \text{—} & a_m & \text{—} \end{pmatrix} B = \begin{pmatrix} \text{—} & a_1 B & \text{—} \\ & \vdots & \\ \text{—} & a_m B & \text{—} \end{pmatrix}$$

↓ - linear combination of rows in A.

def. Let $A \in M_{m\times n}(\mathbb{R})$. The transpose A is $A^T$ s.t. $(A^T)_{i,j} = A_{j,i}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

prop. $(AB)^T = B^T A^T$

$(AX)^T = X^T A^T$

$$\left( (1\ 2\ 3) \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} \right)^T = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

linear combination of rows

linear combination of columns

def. row space and col space.

Let $A \in M_{m \times n}(\mathbb{R})$

The row space of A is

[ $Row(A) = span(A_{1,\cdot}, \ldots, A_{m,\cdot})$ ]

[ $Col(A) = span(A_{\cdot,1}, \ldots, A_{\cdot,n})$ ]

e.x. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ $row(A) = span\{(1\ 2), (3\ 4), (5\ 6)\} \subseteq M_{1\times 2}(\mathbb{R}) \cong \mathbb{R}^2$

$col(A) = span\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}\} \subseteq M_{3\times 1}(\mathbb{R}) \cong \mathbb{R}^3$

row rank = 2.

col rank = 2

def. Let $A \in M_{m\times n}(\mathbb{R})$ The column rank of A is the dim of of Col(A). The row rank of A is dim of Row(A).

Thm: For matrix A, its column rank is equal to its row rank. In particular, rank(A) is either row rank or col rank.

Proof based on 2 lemmas:

Lemma ①: Elementary row operation doesn't change row space.

②: Elementary row operation doesn't change the column rank.

reduce A to RREF. the row rank = # of pivoted rows = # of pivoted colums = column rank.

def. Let $T \in \mathcal{L}(V, W)$, then $\operatorname{rank} T = \dim(\operatorname{Im} T)$

e.x. $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix}$ $\quad T_A : \mathbb{R}^2 \to \mathbb{R}^3.$

$x \mapsto Ax$

$\operatorname{col} A = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$

$\operatorname{null} A = \operatorname{Span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$

$\underbrace{1 \text{ pivot}}_{\text{col}} + \underbrace{1 \text{ free var}}_{\text{var}} = 2$ unknowns.

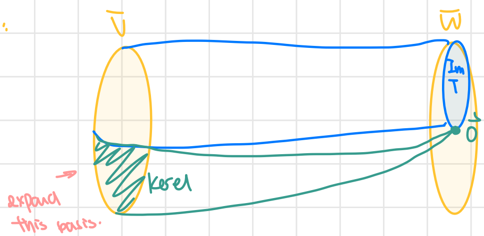
RN: $\dim(\operatorname{Col}(A)) + \dim(\operatorname{Nul}(A)) = \boxed{2 = \text{dim of domain.}}$

Rank - Nullity Theorem:

Let $T \in \mathcal{L}(\bar{V}, \bar{W})$. where $\dim \bar{V} < \bar{W}$,

$\underbrace{\dim(\operatorname{Im}(T))}_{\text{rank}} + \underbrace{\dim(\ker(T))}_{\text{dim}} = \dim(\bar{V})$

proof:




Let $\{v_1, \dots, v_n\}$ be a basis of $\ker T$. This means $\dim \ker T = n$.
Extend $\{v_1, \dots, v_n\}$ to $\{v_1, \dots, v_n, w_1, \dots, w_m\}$ to form a basis for $\bar{V}$. This means $\dim \bar{V} = n + m$.

claim: $\{T(w_1) \dots T(w_m)\}$ is a basis for $\operatorname{Im} T$.

span. – pf of claim.

let $y \in \operatorname{Im} T$ $\therefore$ $\exists\, x = \sum_{i=1}^{n} c_i v_i + \sum_{i=}^{n} d_i w_i \in \bar{V}$

Im T can be hit by x

s.t. $T(x) = y$

$$y = T(x) = T\left(\sum_{i=1}^{n} c_i v_i + \sum_{i=1}^{m} d_i w_i\right)$$

$$= \underbrace{\sum_{i=1}^{n} c_i \underbrace{(T v_i)}_{\substack{0. \\ \text{since } v_i \text{ is} \\ \text{kernel}}}}_{0} + \underbrace{\sum_{i=1}^{n} d_i (T w_i)}_{\sum_{i=1}^{n} d_i (T w_i).}$$

$\therefore$ $y$ is a linear combination of $(T(w_i))_{i=1}^{n}$.

**proof linearly independent:**

$$\sum_{i=1}^{m} a_i T(w_i) = \vec{0} \qquad ?\ a_1 = \dots = a_m = 0$$

$$T\Big(\underbrace{\sum_{i=1}^{m} a_i w_i}_{\text{is in the kernel of } T.}\Big) = \vec{0}$$

$\therefore \exists\ c_i$ where $\sum_{i=1}^{m} a_i w_i = \sum_{i=1}^{n} c_i v_i$

$\sum_{i=1}^{m} a_i w_i - \sum_{i=1}^{n} c_i v_i = 0.$ (by definition)

$\therefore \Rightarrow a_i = c_i = 0$

$\therefore \{T(w_i)\}_{i=1}^{m}$ are linearly independent.

linear map: cannot go from high to low.

e.g. show $\nexists$ no injective map from $\mathbb{R}^3 \to \mathbb{R}^2$.

SBWOC: that $\exists T$, injective, from $\mathbb{R}^3 \to \mathbb{R}^2$,

$\dim(\ker T)$ 3 + $\dim(\operatorname{Im} T)$ 0 = $\dim(\mathbb{R}^3)$ 3

but 3 cannot be $\dim(\mathbb{R}^2)$.

def. $I_n$ is the identity matrix of $M_n(\mathbb{R})$ if

$$I_n = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & & 1 \end{pmatrix}$$

RM: $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$$\underset{n\times m}{I_n A} = A$$

$$\underset{m\times n}{B I_n} = B$$

$$I_n v = v \qquad v \in \mathbb{R}^n$$

$$T_{I_n} : \mathbb{R}^n \to \mathbb{R}^n \qquad (\text{identity map of } \mathbb{R}^n)$$
$$v \mapsto v$$

| | Spaces matrices | Space of linear maps. |
|---|---|---|
| closure: "+". | $A+B$. | $f+g$. |
| Scalar multi: | $kA$. | $kf$ |
| additive identity: | 0 matrix | 0 map |
| | identity matrix | identity map |
| distributive law: | $A(B+C) = AB+AC$. | $f\circ(g+h) = f\circ g + f\circ h$ |
| | $(A+B)C = AC+BC$ | $(f+g)\circ h = f\circ h + g\circ h$ |
| additive inverse: | $-A$ | $-f$ |
| associativity: | $(AB)C = A(BC)$ | $f\circ(g\circ h) = (f\circ g)\circ h$ |

Finite case, linear maps can be represented by matries.

$$\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \cong M_{m\times n}(\mathbb{R})$$

Every matrix $A \in M_{m\times n}(\mathbb{R})$ defines a linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$
$x \mapsto Ax$.

Question: can every l.m. from $\mathbb{R}^n \to \mathbb{R}^m$ be represented by a matrix? (matrix-vector multiplication).

Let $T: \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear map, $\exists A \in M_{m\times n}(\mathbb{R})$ s.t. $T_A = T$.

Pf. Let $T$ be a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$.

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$.

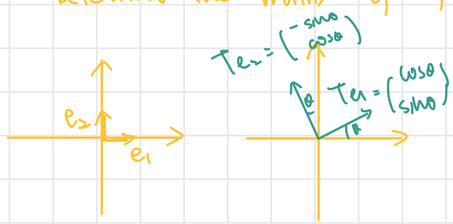
$x = x_1 e_1 + \cdots + x_n e_n$ where $\{e_i\}_{i=1}^{n}$ is the standard basis of $\mathbb{R}$.

$\therefore\ Tx = x_1 T(e_1) + \cdots + x_n T(e_n)$.

$$= \underbrace{\begin{pmatrix} | & & | \\ T(e_1) & \cdots & T(e_n) \\ | & & | \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

then. $T_A = T$.

e.g. Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ that rotates everything by $\theta$. ccw.
Determine the matrix of $T$.




$\therefore \Rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

rotation

e.g. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ Explain what $T_A$ is graphically?


horizontal shift

reflection

projection

Let $\bar{V}$ be a v.s. with $B = \langle b_1, \ldots, b_n \rangle$ as a basis.
$\bar{W}$ - - - - $C = \langle c_1, \ldots, c_n \rangle$ as a basis

Let $T \in \mathcal{L}(\bar{V}, \bar{W})$

$$\begin{tikzcd}
\bar{V} \arrow[r, "T"] \arrow[d, "L_B"'] & \bar{W} \cong \mathbb{R}^m \\
\mathbb{R}^n \arrow[r, "M"] & \mathbb{R}^m \arrow[u, "L^{-1}_C"']
\end{tikzcd}$$

$L_B: \bar{V} \to \mathbb{R}^n$

$\bar{V} \mapsto [V]_B$.

Let $v \in \bar{V}$,

$$Tv = \underbrace{L_C^{-1} \circ \underbrace{M \circ \underbrace{L_B\, v}_{v\text{'s coordinate in } \mathbb{R}^n}}_{\text{image's coordinate in } \mathbb{R}^m}}_{\text{image of } v \text{ in } \bar{W}}$$

$$T = L^{-1}{C} \circ M \circ L_B \quad \text{or} \quad L^{-1}_C\, M\, L_B$$

e.g. Consider $T: \mathbb{R}_2[x] \to \mathbb{R}^2$

$$p \mapsto \begin{pmatrix} p'(0) \\ p'(1) \end{pmatrix}$$

$4 + 3x + 5x^2$

rank

$$\begin{tikzcd}
\underline{\mathbb{R}_2[x]} = \langle 1, x, x^2 \rangle \arrow[r, "T"] \arrow[d, "{[\cdot]_B}"'] & \mathbb{R}^2 : \langle e_1, e_2 \rangle \\
\mathbb{R}^3 \arrow[ur, "M_{2\times 3}"'] &
\end{tikzcd}$$

$(4, 3, 5)$

def Let $\bar{V}$ be a v.s. with basis $B = \langle b_1, \dots, b_n \rangle$

$\bar{W}$ ... $C = \langle c_1, \dots, c_m \rangle$

$T = L_C^{-1} \circ M \circ L_B$.

Let $T \in \mathcal{L}(\bar{V}, \bar{W})$, then the matrix of $T$ is

$$[T]_{B\to C} = \begin{pmatrix} | & & | \\ [T(b_1)]_C & \cdots & [T(b_n)]_C \\ | & & | \end{pmatrix}$$

$\ker(T) \cong \operatorname{nul}([T]_{B\to C})$

'' and same $\left\{\begin{array}{l}\text{injectivity} \\ \text{surjectivity}\end{array}\right.$.

e.g. Let $B = \langle 1, x, x^2 \rangle$, let $C = \langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rangle$

$$T(1) = \begin{pmatrix}0\\0\end{pmatrix} \quad T(x) = \begin{pmatrix}1\\1\end{pmatrix} \quad T(x^2) = \begin{pmatrix}0\\2\end{pmatrix}$$

Rank

$$[T]_{B\to C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

surjective.

$$\underbrace{\text{image}(T) \cong \operatorname{col}([T]_{B\to C})}_{\text{have same rank.}}$$

∴ The study of $T$ can be reduced to the study of $[T]_{B\to C}$

$\text{rank}[T]_{B\to C} = \text{rank}[T]$

e.g. Consider $T: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$.

$$p \mapsto p'$$

① determine $[T]_B$ for a $B$ of your choice

② check if $T$ is inj / sur / bi

③ Determine $\text{Im}\,T$ and $\ker T$ by considering $\text{Col}([T]_B)$, $\text{Nul}([T]_B)$.

Sol. $[T]^B_C$, $[T]_{B\to C}$, $[T]_{C\leftarrow B}$.

$[T]_{B\to B} \to [T]_B$

Let $B = \langle x^2, x, 1 \rangle$

$$[T]_{B\to B} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\text{Col}([T]_B) = \text{span}\left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \text{span}\left\{ \underset{x}{\underline{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}}, \underset{1}{\underline{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}} \right\}$$

$$\therefore \text{Im}([T]_B) = \text{span}\{x, 1\}.$$

$$\text{Nul}([T]_B) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\therefore \ker(T) = 1.$$

e.g. Consider $U := \text{span}\{\cos 2x, \sin 2x\} \subset C^\infty$.

Consider $T: U \to U$

$$p \mapsto p'$$

Question: $T$ bijective?

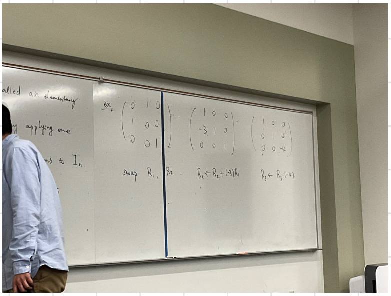
$$[T]_B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

rank $= 2$.

∴ bijective.

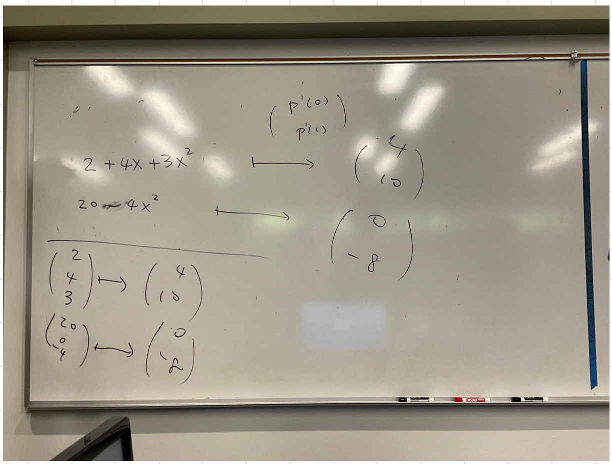
def. $E \in M_n(\mathbb{R})$ is called an elementary matrix if $E$ is obtained by applying one of the 3 elementary row operations to $I_n$.

Fact: let $A \in M_{n \times m}(\mathbb{R})$, then $EA$ is applying row operation to $A$ that corresponds to $E$.




e.g.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$$




$[v]_B$.

coordinate vector of $v$ in relation to $B$.

Let $E \in M_n(\mathbb{R})$ that is obtained by one of the elementary row operations to $Id_n$. Then $E$ is called an elementary matrix.

RM: Let $A \in M_{n\times m}(\mathbb{R})$, $E$ be an elementary matrix in $M_n(\mathbb{R})$, Then $EA$ is $A$ with $E$'s elementary operation applied.

Left multiplication.

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

same row operation.

right multiplication:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

same column operation.

def. Let $A \in M_n(\mathbb{R})$. We say $A$ is invertible if $\exists\, B \in M_n(\mathbb{R})$ s.t. $AB = BA =$ Identity matrix $n$

$B$ is the inverse of $A$, denoted as $A^{-1}$

e.g. Is $I_n$ invertible? yes. $I_n \cdot I_n = I_n \cdot I_n = I_n$

Is $\vec{0}$ invertible? No. $\vec{0} \cdot$ anything $= \vec{0} \neq I_n$.

Is elementary matrix invertible? yes.

prop: The inverse of $A$ is unique.

proof: Let $B$ and $\hat{B}$ be 2 inverses of $A$. $\in M_n(\mathbb{R})$

$B = BI_n = BA\hat{B} = I_n\hat{B} = \hat{B}$

lemma: Let $A \in M_n(\mathbb{R})$, then $A$ is invertible if $A$ is row equivalent to identity.

pf. $\exists$ a sequence of $E_1, \dots, E_m$, elementary matrices s.t.

$(E_m \cdots E_1)A = \text{rref}(A) = I_n$

this is the matrix that is the left inverse of $A$.

$\therefore (E_m \dots E_1) = B$. in $BA = AB = I_n$.

$\therefore$ indeed invertible.

Lemma : let $A, B \in M_n(\mathbb{R})$, if $AB = I_n$, then $BA = I_n$ (vice versa)

$\therefore$ left inverse = right inverse.

If $AB = I_n$, $Col(AB) = \mathbb{R}^n$

$\therefore Col(A) \subseteq Col(AB) = \mathbb{R}^n$.

$\therefore Col(A) = \mathbb{R}^n$. $\Rightarrow$ $Null(A) = \{\vec{0}\}$

rank $A = n$

linearly independent

Let $BA\vec{x} = \vec{y}$

$\underbrace{AB}_{I_n}A\vec{x} = A\vec{y}$

$\therefore A\vec{x} = A\vec{y}$

injective

$\therefore \vec{x} = \vec{y}$ $\therefore BA = I_n$

Theorem : Let $A \in M_n(\mathbb{R})$, The following are equivalent :

1. $A$ is invertible.
2. $\exists C \in M_n(\mathbb{R})$ s.t. $CA = I_n$.
3. $\exists B \in M_n(\mathbb{R})$ s.t. $AB = I_n$
4. $A$ is row equivalent to $I_n$
5. RREF of $A$ has $n$ - pivots.
6. rank of $A = n$
7. $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is injective
8. $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is surjective
9. $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is bijective
10. Columns of $A$ form a basis of $\mathbb{R}^n$ ( $Col(A) = \mathbb{R}^n$ )
11. $Nul(A) = \{\vec{0}\}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Algorithm to find $A^{-1}$.

1. Adjoint $I_n$ to $A$ to form $(\underset{n\times n}{A} \mid \underset{n\times n}{I_n})$
2. Row reduce it to its RREF. Call it $(\underset{n\times n}{M} \mid \underset{n\times n}{B})$

③ If $M = I_n$, then $B = A^{-1}$
If $M \neq I_n$, then $A$ is not invertible.

$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \overset{①}{\Rightarrow} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right) \therefore A^{-1} = \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix}$

Review: $T : \bar{V} \to \bar{W}$
basis $B$ (for $\bar{V}$), basis $C$ (for $\bar{W}$)

$[T]_{B \to C} = \begin{pmatrix} | & & | \\ [T(b_1)]_C & \cdots & [T(b_n)]_C \\ | & & | \end{pmatrix}$

prop: Let $T : \bar{V} \to \bar{W}$, finite dimensional space.
If $T$ is an isomorphism, $[T]_{B \to C}$ is an invertible matrix,
in particular, $[T^{-1}]_{C \to B} = ([T]_{B \to C})^{-1}$

ex. $B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle$, $E = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$

$[id]_B = [id]_{B \to B} = \begin{pmatrix} | & | \\ [id(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})]_B & [id(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix})]_B \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$[id]_E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$[id]_{B \to E} = \left( [(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})]_E \quad [(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix})]_E \right) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$[id]_{E \to B} = \left( [(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix})]_B \quad [(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix})]_B \right) = \left( \begin{pmatrix} 2 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$= ([id]_{B \to E})^{-1}$

$$[id]_{E\to B}\,[id]_{B\to E} = [id]_{B\to B}$$

$$\underbrace{[id]_{E\to B}\,\underbrace{[id]_{B\to E}\,[v]_B}_{[v]_E}}_{[v]_B\,.} = \underbrace{[id]_{B\to B}\,[v]_B}_{[v]_B}$$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "{[T]_{C\to C}}"] & & \mathbb{R}^n \arrow[dd, bend left=60, no head, "{[id]_{C\to B}}"] \\
V \arrow[u, "{[\cdot]_C}"] \arrow[d, "{[\cdot]_B}"'] \arrow[rr, "T"] & & V \arrow[u, "{[\cdot]_C}"'] \arrow[d, "{[\cdot]_B}"] \\
\mathbb{R}^n \arrow[rr, "{[T]_{B\to B}}"] \arrow[uu, bend left=60, no head, "{[id]_{B\to C}}"] & & \mathbb{R}^n
\end{tikzcd}$$

e.g. $[S\circ T]_{B\to D} = [S]_{C\to D}\cdot[T]_{B\to C}.$

$$\underset{\text{basis } B}{V} \xrightarrow{T} \underset{\text{basis } C}{W} \xrightarrow{S} \underset{\text{basis } D}{Z}$$

let $\bar{V}$ be a vector space with basis $\mathcal{B} = \langle b_1, \dots, b_n \rangle$

$\mathcal{C} = \langle c_1, \dots, c_n \rangle$

let $v \in \bar{V}$. then

$$[id]_{B \to C} [v]_B = [v]_C.$$

In particular, we call $[id]_{B \to C}$ the change of basis matrix from $B$ to $C$.

e.g. standard basis $= \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$ $\quad B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle.$

$\mathcal{E}$

$$v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad [v]_{\mathcal{E}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \quad [v]_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$[id]_{B \to \mathcal{E}} = \left( [id(b_1)]_{\mathcal{E}} \quad [id(b_2)]_{\mathcal{E}} \right)$$
$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$[id]_{\mathcal{E} \to B} = \left( [id(\mathcal{E}_1)]_B \quad [id(\mathcal{E}_2)]_B \right)$$
$$= \left( \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_B \quad \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_B \right)$$
$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

RM: Let $B \in M_n(\mathbb{R})$ be a matrix whose columns form a basis for $\mathbb{R}^n$, $B$ is invertible.
Then $B^{-1} v$ for $v \in \mathbb{R}^n$ is $v$'s representation with respect to the basis formed by the columns of $B$: $[v]_B = B^{-1} v$.

$B \cdot [v]_B = v$

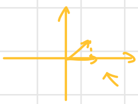
Let $T \in \mathcal{L}(\bar{V}, \bar{V})$ linear transformation | algorithm.

Let $\mathcal{B} = \langle b_1, \dots, b_n \rangle$, $\mathcal{C} = \langle c_1, \dots, c_n \rangle$

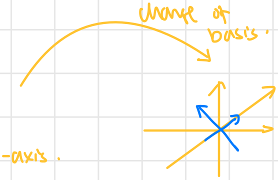
$$[T]_{B \to B} = [id]_{B \to C} \, [T]_{C \to C} \, [id]_{C \to B}.$$

$$\underbrace{[T]_{B\to B}\,[v]_B}_{[Tv]_B} = \underbrace{[id]_{C\to B}\,\underbrace{[T]_{C\to C}\,\underbrace{[id]_{B\to C}\,[v]_B}_{[v]_C}}_{[Tv]_C}}_{[Tv]_B}.$$

e.g. Consider $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$



$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projection to x-axis.




Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ T projects $\vec{v}$ to span $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

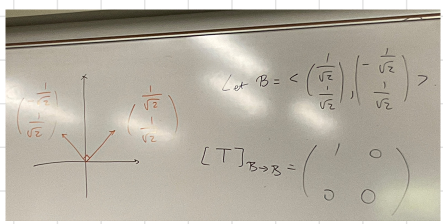
$[T]_{\mathcal{E}\to\mathcal{E}}$.

Let $B = \left\langle \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\right.$ $\quad \mathcal{E} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$

$$[T]_{B\to B} \stackrel{?}{=} [id]_{\mathcal{E}\to B}\,[T]_{\mathcal{E}\to\mathcal{E}}\,[id]_{B\to\mathcal{E}}.$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



let $T \in \mathcal{L}(\bar{V}, \bar{V})$

let $p(x) \in \mathbb{R}_n[x]$. polynomial.

$p(T) : \bar{V} \to \bar{V}$. $\quad ( v \in \bar{V} \mapsto p(T)(v) )$.

e.g. $T \in \mathcal{L}(\bar{V}, \bar{V})$.

if $\quad T^2 + 2T + 3\,id \quad \bar{V} \to \bar{V}$.

$$v \mapsto (T^2 + 2T + 3\,id)(v)$$
$$= T^2(v) + 2T(v) + 3(v).$$
$$= T \circ T(v) + 2T(v) + 3v.$$

define addition:

$\cdot + \cdot : \quad \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$\qquad (a, b) \to a+b$

$tr(\cdot) : \quad M_n(\mathbb{R}) \to \mathbb{R}$.

$\qquad M \mapsto tr(M)$

Dot product : let $u, v \in \mathbb{R}^n$, $u \cdot v = \sum_{i=1}^{n} u_i v_i$

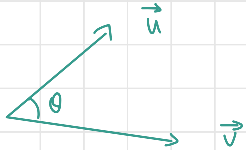
$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \qquad = u^T v$$

length of $u$ = $\sqrt{u_1^2 + \dots + u_n^2} = \sqrt{u \cdot u} = \sqrt{u^T u}$

norm of $u$

$\|u\|$

$u \cdot v = \|u\| \, \|v\| \cos\theta$.




by law of cosine.

def. Let $\bar{V}$ be a vector space over $\mathbb{R}$. Consider map. $\langle \cdot , \cdot \rangle$ : $\bar{V} \times \bar{V} \to \mathbb{R}$

that satisfies
① $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in \bar{V}$
② $\langle x, ky \rangle = k \langle x, y \rangle$ for all $x, y \in \bar{V}$, $k \in \mathbb{R}$
③ $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in \bar{V}$

bilinear.

④ $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.

Then $\langle \cdot , \cdot \rangle$ is called an inner product.
$\bar{V}$ together with the $\langle \cdot , \cdot \rangle$ is called inner product space.

e.g. Dot product on $\mathbb{R}^n$ is an inner product.
$(\mathbb{R}^n, \_ \cdot \_)$ is an inner product space.
↑ dot product.

pf. Let $x, y \in \mathbb{R}^n$

because it's a number

① $x \cdot y = \underline{x^T y = (x^T y)^T} = y^T x = y \cdot x$

② $x \cdot ky = x^T k y = k x^T y = k(x \cdot y)$

③ $x \cdot (y+z) = x^T(y+z) = x^T y + x^T z = (x \cdot y) + (x \cdot z)$

④ $x \cdot x = x_1^2 + \dots + x_n^2 \geq 0$. ✓
if $x_1^2 + \dots + x_n^2 = 0$, every $x_i$ needs to be zero. ∴ ✓

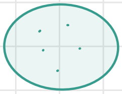
e.g. Consider $M_n(\mathbb{R})$.
$tr(AB^T)$ is an inner product on $M_n(\mathbb{R})$

e.g. Consider $\mathbb{R}_n[x]$, then $\int_0^1 p(x) q(x)\, dx$ is an inner product.

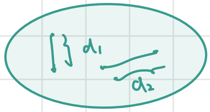
prop: Let $\langle \cdot , \cdot \rangle$ be an inner product on $\bar{V}$ over $\mathbb{R}$. Then $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$ is a norm.

prop: $\langle x, y \rangle = \|x\| \|y\| \cos\theta$

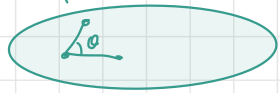
v. s.



normed space.




inner product space.




Metric space

distance.

measure distance between things.

distance + angle.

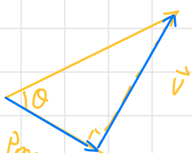
def. let $u, v \in (\bar{V}, \langle \cdot, \cdot \rangle)$

Then $u, v$ are orthognal iff $\langle u, v \rangle = 0$

① $\vec{0}$ is orthognal to itself

② $\vec{0}$ is orthognal to everything.

Obs. If $\vec{x} \neq 0$, $\vec{x} = \underbrace{\frac{\vec{x}}{\|\vec{x}\|}}_{\text{dir of } \vec{x} \text{ in unit vector}} \underbrace{\|\vec{x}\|}_{\text{norm of } \vec{x}}$




$= \|\vec{v}\| \cdot \cos\theta$

$\vec{v} = Proj_{\vec{u}}(\vec{v}) + (\vec{v} - proj_{\vec{u}}(\vec{v}))$

$\langle \vec{v}, \frac{\vec{u}}{\|\vec{u}\|} \rangle \cdot \frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle \vec{v}, \vec{u} \rangle \vec{u}}{\|\vec{u}\|^2}$

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$\vec{v} =$ length · direction.

$= \underbrace{\|\vec{v}\| \cdot \cos\theta}_{\text{write this as inner product.}} \cdot \frac{\vec{u}}{\|\vec{u}\|}$

$= \|\vec{v}\| \quad 1 \cdot \cos\theta \cdot \frac{\vec{u}}{\|\vec{u}\|}$

$= \|\vec{v}\| \cdot \frac{\|\vec{u}\|}{\|\vec{u}\|} \cdot \cos\theta \cdot \frac{\vec{u}}{\|\vec{u}\|}$

$= \langle \vec{v}, \frac{\vec{u}}{\|\vec{u}\|} \rangle \cdot \frac{\vec{u}}{\|\vec{u}\|}$

**6.3 Definition** *inner product*

An ***inner product*** on $V$ is a function that takes each ordered pair $(u, v)$ of elements of $V$ to a number $\langle u, v\rangle \in \mathbf{F}$ and has the following properties:

**positivity**
$\langle v, v\rangle \geq 0$ for all $v \in V$;

**definiteness**
$\langle v, v\rangle = 0$ if and only if $v = 0$;

**additivity in first slot**
$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$;

**homogeneity in first slot**
$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$;

**conjugate symmetry**
$\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

Review:

① $V$ s.t. $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$.

ⓐ $\langle x, y\rangle = \langle y, x\rangle$

ⓑ $\langle x, ky\rangle = k\langle x, y\rangle$

ⓒ $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

ⓓ $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = \vec{0}$

② $x \perp y$ iff $\langle x, y\rangle = 0$

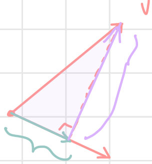
③ if $\langle \cdot , \cdot \rangle$ is an inner product, then $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$ is a norm.

④ $(\mathbb{R}^n, x^T y)$, $\left(C[-1,1], \int_{-1}^{1} p(x) q(x)\, dx\right)$
continuous function

$(M_{m\times n}(\mathbb{R}), \operatorname{tr}(A^T B))$

| | | |
|---|---|---|
| $\|v\|$ | norm of vector $v$ | non-negative scaler |
| $\lvert\lambda\rvert$ | absolute value of $\lambda \in \mathbb{R}$ | non-negative scaler |
| $\langle x, y\rangle$ | inner product of $x$ and $y$ | scaler |
| $\lvert\langle x, y\rangle\rvert$ | absolute value of the inner product | non-negative scaler |

Projection:




projection of $\vec{v}$ onto direction of $\vec{u}$
$\text{Proj}_{\vec{u}} \vec{v}$

$\text{Proj}_{\vec{u}} \vec{v} = k\vec{u} \quad k \in \mathbb{R}$.
( $\text{Proj}_{\vec{u}} \vec{v}$ and $\vec{u}$ same direction )

also, $\langle \vec{v} - k\vec{u}, \vec{u}\rangle = 0$.
( $\vec{v} - \text{Proj}_{\vec{u}}(\vec{v})$ and $\text{Proj}_{\vec{u}}(\vec{v})$ $\perp$. )

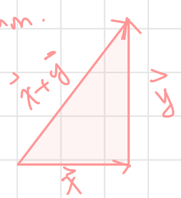
$\langle \vec{v}, \vec{u}\rangle - k\langle \vec{u}, \vec{v}\rangle = 0$.

$$k = \frac{\langle \vec{v}, \vec{u}\rangle}{\langle \vec{u}, \vec{u}\rangle}$$

formula of projection: $\therefore \text{proj}_{\vec{u}} \vec{v} = \dfrac{\langle \vec{v}, \vec{u}\rangle}{\langle \vec{u}, \vec{u}\rangle} \vec{u}$

RM: if $\vec{u} \neq 0$, $\text{proj}_{\vec{u}}(\vec{v}) = \dfrac{\langle \vec{v}, \vec{u}\rangle}{\langle \vec{u}, \vec{u}\rangle} \vec{u}$.

P - theorum.




$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

LHS: $\langle x+y, x+y \rangle$

$$= \langle x, x+y \rangle + \langle y, x+y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \|x\|^2 + \underbrace{2\langle x, y \rangle}_{\hookrightarrow = 0 \text{ because orthogonal.}} + \|y\|^2$$

Cauchy - Schwarz Inequality.

$$|\langle x, y \rangle| \le \|x\| \cdot \|y\|.$$

Pf: case ①: $y = 0$. LHS = 0, == RHS ✓

case ②: Otherwise, consider orthogonal decomposition of $\vec{x}$.

$y \neq 0$ $\quad \vec{x} = \dfrac{\langle x, y \rangle}{\langle y, y \rangle} \cdot y + w.$

$$\|x\|^2 = \left\| \frac{\langle x, y \rangle}{\langle y, y \rangle} y \right\|^2 + \underbrace{\|w\|^2}_{\text{must } \ge 0.}$$

Only equal if one's a scaler multiplication of the other.

$$\therefore \|x\|^2 \le \left\| \frac{\langle x, y \rangle}{\langle y, y \rangle} y \right\|^2$$

$$\|x\|^2 \ge \left( \frac{\langle x \cdot y \rangle}{\langle y \cdot y \rangle} \right)^2 \|y\|^2$$

$$\ge \left( \frac{\langle x \cdot y \rangle}{\|y\|^2} \right)^2 \|y\|^2$$

$$\ge \frac{\langle x, y \rangle^2}{\|y\|^4} \|y\|^2$$

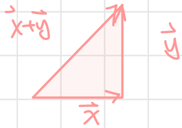
$$\|x\|^2 \ge \frac{\langle x, y \rangle^2}{\|y\|^2}$$

$$\therefore \|x\| \cdot \|y\| \ge \langle x, y \rangle$$

Triangle inequality.

**prop.** Let $\|\cdot\|$ be the induced norm from $\langle \cdot , \cdot \rangle$

then $\|x+y\| \le \|x\| + \|y\|$.




$$\|x+y\|^2 = \langle x+y, x+y \rangle$$
$$= \|x\|^2 + \underline{2\langle x,y \rangle} + \|y\|^2.$$

$$\langle x,y \rangle \le \frac{\|x\| \cdot \|y\|}{2}$$

$$\le \|x\|^2 + 2\|x\| \cdot \|y\| + \|y^2\| \quad (C.S).$$
$$= (\|x\| + \|y\|)^2 \Rightarrow \|x+y\| \le \|x\| + \|y\|$$

only equal if $\langle u, v \rangle = \|u\|\|v\|$ (one's a multiple of the other). (on the same line).

**def.** Let $\bar{V}$ be a v.s. over $\mathbb{R}$.

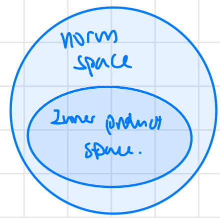
Considers function $\|\ \| : \bar{V} \to \mathbb{R}$. s.t.

① $\|x\| \ge 0$ and $\|x\| = 0$ iff $x = \vec{0}$

② $\|\lambda x\| = |\lambda| \, \|x\|$ for $\forall \lambda \in \mathbb{R}$, $x \in \bar{V}$

③ $\|x+y\| \le \|x\| + \|y\|$

Then $(\bar{V}, \|\cdot\|)$ is called a norm space.




**R.m.** Because C.S. inequality, $-\|x\|\cdot\|y\| \le \langle x,y \rangle \le \|x\| \cdot \|y\|$

$$\therefore \underbrace{-1 \le \frac{\langle x,y \rangle}{\|x\| \ \|y\|} \le 1}$$

we can use $\arccos\left(\frac{\langle x,y \rangle}{\|x\| \cdot \|y\|}\right)$ to describe

the angle between $x$ and $y$.

**def.** Set of vectors $\{v_1, \ldots, v_n\}$ is an orthonormal set if

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \ne j \end{cases}$$

- orthogonal to each other
- normal (unit vectors) individually.

**e.g.** $\{ = \langle e_1, e_2, \ldots, e_n \rangle$ is an orthonormal set

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

def. Let $\bar{V}$ be a vector space, then $\langle q_1, \dots, q_n \rangle$ is an basis if ① It is an orthonormal set
② $\text{Span}(q_1, \dots, q_n) = \bar{V}$

e.g. Standard basis: $\mathcal{E} = \langle e_1, \dots, e_n \rangle$ is an orthonormal basis for $\mathbb{R}^n$.

Theorem: let $\langle q_1, \dots, q_n \rangle$ be a basis for subspace $\bar{u}$, Then $\forall u \in \bar{u}$,
$u = c_1 q_1, \dots, c_n q_n$

Then $c_i = \langle u, q_i \rangle$ $\qquad \|u\|^2 = \sum_{i=1}^{m} c_i^2$

$$c_i = \frac{\langle u, q_i \rangle}{\boxed{\langle q_i, q_i \rangle} = 1} \quad \boxed{q_i} = 1.$$
$$= \langle u, q_i \rangle$$

multiple degree P-theorm.

pf. Since $\langle q_1, \dots, q_n \rangle$ is an basis,
$u$ can be uniquely written as a l.c. of $(q_1, \dots q_n)$
$u = c_1 q_1 + \dots + c_n q_n$
Now consider $\langle u, q_i \rangle = \langle c_1 q_1 + \dots + c_n q_n \rangle$
$= c_1 \langle q_1, q_i \rangle + \dots + c_i \langle q_i, q_i \rangle + \dots + c_n \langle q_n q_i \rangle$

$\langle u, q_i \rangle = c_i \quad \forall i \in \{1, \dots, n\}$.
$$\|u\|^2 = \underbrace{\|c_1 q_1\|^2}_{\vec{x}} + \underbrace{\|c_2 q_2 + \dots + c_m q_m\|^2}_{\vec{y}}$$

$\vec{x}$ and $\vec{y}$ are orthognal to each other

$= c_i^2 + \|c_2 q_2\|^2 + \|c_3 q_3 + \dots + c_m q_m\|^2$.

⋮ keep expanding...

$= c_1^2 + c_2^2 + \dots + c_m^2$

# Gram-Schmidt Process.

linearly independent set $\rightarrow$ $\boxed{\phantom{xxxx}}$ $\rightarrow$ orthonormal set.

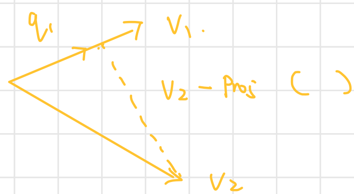
Let $\{v_1, \cdots, v_n\}$ be a linearly independent set of vectors.

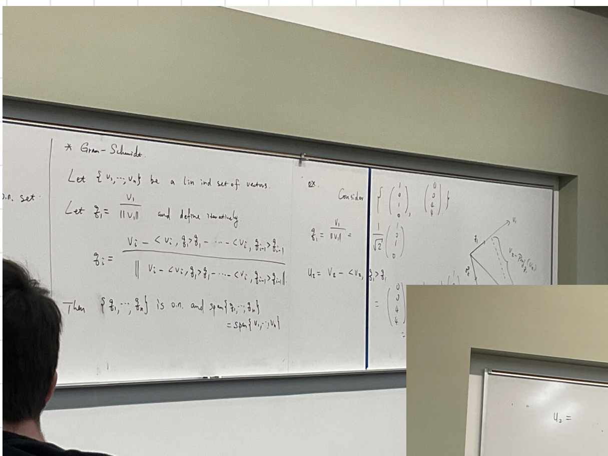
Let $q_1 = \dfrac{v_1}{\|v_1\|}$ and define iteratively.

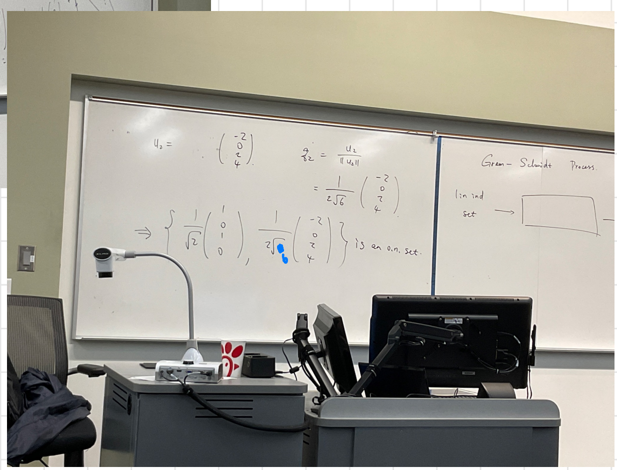
$$q_i = \frac{v_i - \langle v_i, q_1\rangle q_1 - \cdots - \langle v_i, q_{i-1}\rangle q_{i-1}}{\| v_i - \langle v_i, q_1\rangle q_1 - \cdots - \langle v_i, q_{i-1}\rangle q_{i-1} \|}$$

Then $\{q_1, \cdots, q_n\}$ is orthonormal and $\operatorname{span}\{q_1, \cdots, q_n\} = \operatorname{span}\{v_1, \cdots, v_n\}$

e.x. Consider $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \\ 4 \end{pmatrix} \right\}$.



Facts : 1. If $\bar{V}$ is finite dimensional, then $\bar{V}$ always have an o.n. basis.

2. Any o.n. set in $\bar{V}$ can be expanded to an o.n. basis. if $\bar{V}$ is finite dimensional.

Orthogonal projection matrix :

Let $\vec{u} \in \mathbb{R}^n$ be a unit vector, consider $\text{Proj}_{\vec{u}}(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$

$x \longmapsto \text{Proj}_{\vec{u}}(\vec{x})$

$$[\text{Proj}_{\vec{u}}(\cdot)]_{\varepsilon \to \varepsilon} = \vec{u}\,\vec{u}^T$$

RM: If $\vec{v}$ is nonzero and not normal, then the orthogonal projection is given by $\frac{1}{\|v\|^2} \vec{v}\,\vec{v}^T$

def. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$, the outer product of $\vec{u}, \vec{v}$ is

$$\vec{u} \otimes \vec{v} = \vec{u}\,\vec{v}^T$$

RM: $\text{rank}(\vec{u}\,\vec{v}^T) = 1$

prop. Let $\bar{u} \in \bar{V}$ be spanned by orthornormal set $\langle q_1, \dots, q_n \rangle$,

$$\text{Proj}_{\bar{u}} \vec{v} = \sum_{i=1}^{m} \langle \vec{v}, q_i \rangle q_i$$

RM: How to find $\text{Proj}_{\bar{u}} \vec{v}$ ?

① find a basis $B$ for $\bar{u}$.

② use G.S. to turn $B$ to $Q$ consisting of orthonormal basis.

③ $\text{Proj}_{\bar{u}}(\vec{v}) = \sum_{i=1}^{m} \langle \vec{v}, q_i \rangle q_i$.

when $Q = \langle q_1, \dots, q_n \rangle$.

e.x. Consider the subspace $\mathbb{R}_1[x]$ of continuous function defined on $\mathbb{R}$. Let $\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x) q(x)\, dx$.

Determine the best approximation of $e^x$ with a linear function.

Note that a linear function has the form of $a + bx$, which is spanned by $\langle 1, x \rangle$.

orthogonal but not normal

$\mathrm{Proj}_{\mathbb{R}_1[x]}(e^x)$

$\|1\| = \langle 1, 1\rangle^{\frac{1}{2}} = \left(\int_{-1}^{1} 1\cdot 1\, dx\right)^{\frac{1}{2}} = \sqrt{2}$

$\|x\| = \langle x, x\rangle^{\frac{1}{2}} = \left(\int_{-1}^{1} x^2\, dx\right)^{\frac{1}{2}} = \sqrt{\frac{2}{3}}$.

An o.n. basis for span $(1, x)$ is $\left(\frac{1}{\sqrt{2}}, \frac{x}{\sqrt{\frac{2}{3}}}\right)$

$$\mathrm{Proj}_{\mathbb{R}_1[x]}(e^x) = \langle e^x, \tfrac{1}{\sqrt{2}}\rangle \cdot \tfrac{1}{\sqrt{2}} + \langle e^x, \tfrac{x}{\sqrt{\frac{2}{3}}}\rangle \frac{x}{\sqrt{\frac{2}{3}}}$$

$$= \int_{-1}^{1} e^x \tfrac{1}{\sqrt{2}}\, dx\, \tfrac{1}{\sqrt{2}} + \int_{-1}^{1} e^x \frac{x}{\sqrt{\frac{2}{3}}}\, dx\, \frac{x}{\sqrt{\frac{2}{3}}}$$

$$= \tfrac{1}{2}\left(e - \tfrac{1}{e}\right) + \tfrac{3}{e} x$$

The matrix of orthogonal projection onto a subspace.

$$[A]_{\varepsilon\to\varepsilon} = q_1 q_1^T + \cdots + q_m q_m^T$$

$$= \sum_{i=1}^{m} q_i \otimes q_i$$

(if $\{q_1, \ldots, q_m\}$ is an orthonormal basis for subspace $\bar{u}$)

observations: let $A$ be a matrix that represents an orthogonal projection:

$A^T = A$

$A^2 = A$ Project it once: on space. project it again: it's already on the space, nothing changes.

Let $\bar{u}$ be a subspace in $\bar{V}$, then the orthogonal complements of $\bar{u}$ is

$\bar{u}^{\perp} = \{ v \in \bar{V} \mid \langle \vec{v}, \vec{u}\rangle = 0, \ \forall \vec{u} \in \bar{u} \}$

Fact:
1. $\{\vec{0}\}^{\perp} = \bar{V}$
2. $\bar{V}^{\perp} = \{\vec{0}\}$
3. $(\bar{u}^{\perp})^{\perp} = \bar{u}$ is $\bar{u}$ is a subspace.
4. $\dim(\bar{u}) + \dim(\bar{u}^{\perp}) = \dim(\bar{V})$
5. $\bar{u} \cap \bar{u}^{\perp} = \{\vec{0}\}$

Prop. Let $P \in \mathcal{L}(\bar{V})$ which is an orthogonal projection onto subspace $\bar{U}$. $\bar{V}$ is finite-dimensional.

① $P^2 = P$ ② $P(u) = u$ if $u \in \bar{U}$

③ $\text{image}(P) = \bar{U}$ ④ $P(v) = 0$ if $v \in \bar{U}^\perp$

⑤ $\ker(P) = \bar{U}^\perp$

Let $T \in \mathcal{L}(\bar{V}, \bar{W})$. then adjoint of $T$, denoted by $T^*$ is map from $\bar{W}$ to $\bar{V}$ s.t. $\langle Tx, y \rangle_{\bar{W}} = \langle x, T^* y \rangle_{\bar{V}}$ for $x \in \bar{V}$, $y \in \bar{W}$.

Rm. In $\mathbb{R}^n$, with the dot product,
let $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $T^* = ?$ Let $A$ be matrix representation.

$$\langle Ax, y \rangle_{\mathbb{R}^m} = \langle x, A^* y \rangle_{\mathbb{R}^n}.$$

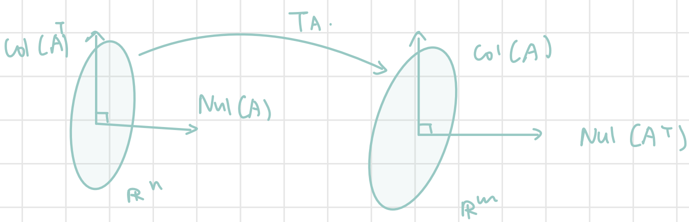
dot product

$$= (Ax)^T y$$
$$= x^T A^T y$$
$$= \langle x, A^T y \rangle$$

$\therefore$ The adjoint of $T$ is $A^T$. $(T^* = A^T)$.

Thm: Let $A \in M_{m \times n}(\mathbb{R})$.
If $A\vec{v} = \vec{0}$, then $A = \begin{pmatrix} - a_1 - \\ \vdots \\ - a_m - \end{pmatrix} \vec{v} = \vec{0}$

$\text{Nul}^\perp(A) = \text{Col}(A^T)$ $\text{Nul}^\perp(A^T) = \text{Col}(A)$



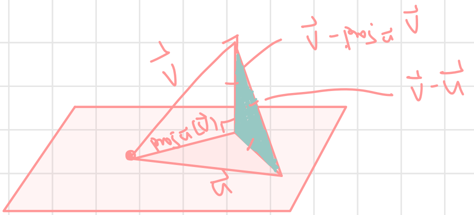
def. let $P \in M_n(\mathbb{R})$, then $P$ is an orthognal proj.
iff $P = P^T$, $P = P^2$.

let $\bar{V}$ be a finite-dimensional v.s.
let $\bar{u} \subseteq \bar{V}$ be a subspace. $\bar{u}^\perp$: orthogonal complement of $\bar{u}$
then any vector $v \in \bar{V}$ can be written as $\underline{v = v_1 + v_2}$
where $v_1 \in \bar{u}$ and $v_2 \in \bar{u}^\perp$ unique
We say $P \in \mathcal{L}(\bar{V})$ is an orthognal projection map
if $Pv = v_1$.

prop: $P = P^2$
$\operatorname{Im} P = \bar{u}$
$\ker P = \bar{u}^\perp$

prop. let $Q \in M_{m \times n}(\mathbb{R})$ whose cols form a o.n. basis for $\operatorname{col}(Q)$, then the orthognal projection map onto the space $\operatorname{col}(Q)$ is given by
$$QQ^T = \sum q_i q_i^T \quad \text{where} \quad Q = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix}.$$

prop: let $\bar{u}$ be a finite-dimensional subspace in $\bar{V}$,
$\forall \vec{v} \in \bar{V}$:
$$\|\vec{v} - \operatorname{Proj}_{\bar{u}} \vec{v}\| \leq \|\vec{v} - \vec{u}\|$$
for any $\vec{u} \in \bar{u}$.



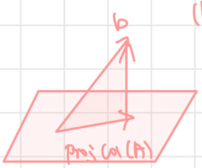

$$\|\vec{v} - \operatorname{Proj}_{\bar{u}}(\vec{v})\|^2 \leq \|\vec{v} - \operatorname{Proj}_{\bar{u}}(\vec{v})\|^2 + \|\operatorname{Proj}_{\bar{u}}(\vec{v}) - \vec{u}\|^2$$
P. theorem.
$$= \|\vec{v} - \operatorname{Proj}_{\bar{u}}(\vec{v}) + \operatorname{Proj}_{\bar{u}}(\vec{v}) - \vec{u}\|^2$$
$$= \|\vec{v} - \vec{u}\|^2$$

When $A x = \vec{b}$ doesn't have a solution. $\Rightarrow$ inconsistency.

$\vec{b} \notin \text{Col}(A)$

def. We say $\vec{x}^*$ is a least square solution to system $Ax = b$. If

$$\|\vec{b} - A\vec{x}^*\| \le \|\vec{b} - A\vec{x}\|$$




RM: $\vec{x}^* = \text{argmin} \|\vec{b} - A\vec{x}\| \quad \vec{x} \in \mathbb{R}^2$.

argument of $\|\vec{b} - A\vec{x}\|$ when $\vec{b} - A\vec{x}$ is min.

Col(A)

RM: $\vec{x}^*$ is a least square solution to $A\vec{x} = \vec{b}$

If $A\vec{x}^* = \text{Proj}_{\text{Col}(A)} \vec{b}$

Thm: $\vec{x}^*$ is a least square solution to $A\vec{x} = \vec{b}$ if

$$A^T A \vec{x}^* = A^T \vec{b}$$

Prop. Let $A \in M_{m \times n}(\mathbb{R})$.

- $\text{null}(A) = \text{null}(A^T A)$
- $\text{null}(A) = \{\vec{0}\}$ if and only if $A^T A$ is invertible.

if $\text{null}(A) = \{\vec{0}\}$, then. $A^T A$ is invertible.

pf: $\text{null}(A) \subseteq \text{Null}(A^T A)$

① Let $x \in \text{null}(A)$. $\Rightarrow A\vec{x} = \vec{0}$.

Let $A^T Ax$. $A^T(Ax) = A^T \vec{0} = \vec{0}$

② Let $x \in \text{null}(A^T A)$. $\Rightarrow A^T A \vec{x} = 0$

This means $Ax \in \text{Null}(A^T) = \text{Col}^{\perp}(A)$

but $Ax$ is also in the column space of $A$.

$\Rightarrow A\vec{x} = \vec{0}$

$\vec{b} - A\vec{x}^* \in \text{Col}^{\perp}(A)$

$\iff A^T(\vec{b} - A\vec{x}^*) = \vec{0}$

$\iff A^T A \vec{x}^* = A^T \vec{b}$

$\therefore$ If $A$ has linearly independent columns, then

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

if $\vec{x}^*$ is the projection, $(A^TA)^{-1}A^T\vec{b}$ is projection.

Corr. Let $A \in M_{m\times n}(\mathbb{R})$ with linearly independent columns, then the orthogonal projection onto $\mathrm{col}(A)$ is given by the matrix $A(A^TA)^{-1}A^T$.

2 ways $\rightarrow$ $q_i q_i^T$
for projection. $\rightarrow$ combine them $\rightarrow$ $A$,
do $A(A^TA)^{-1}A^T$,

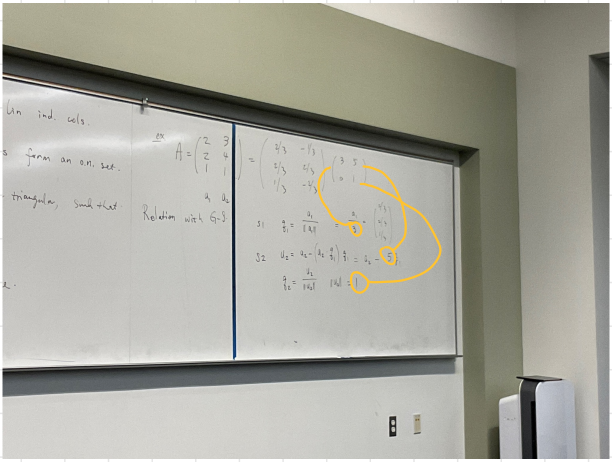
Let $A \in M_{m\times n}(\mathbb{R})$. with linearly independent columns, then $\exists\, Q \in M_{m\times n}(\mathbb{R})$, whose columns form an orthonormal set and $\exists\, R \in M_{m\times n}(\mathbb{R})$ which is upper triangular s.t. $A = QR$. Further, this decomposition is unique.

matrix - representation of Gram Schmidt.

e.g. $A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}$

↑ orthonormal basis

↑ upper triangle



Prop. Let $A \in M_{m\times n}(\mathbb{R})$ with linear independent columns. Consider $A = QR$, then $Q$'s columns are an orthonormal basis for $\mathrm{col}(A)$ and

$$\begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$$

and

$$\begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix}$$

$$R_{1,1} = \|a_1\|$$
$$R_{1,2} = a_2 \cdot q_1$$
$$R_{2,2} = \| v_2 - R_{1,2}\, q_1 \|$$
$$R_{3,1} = a_3 \cdot q_1$$
$$R_{3,2} = a_3 \cdot q_2$$
$$R_{3,3} = \| a_3 - R_{1,3}\, q_1 - R_{2,3}\, q_2 \|$$
$$\vdots$$

If $A$ has linearly independent columns, $\begin{cases} A = QR. \\ A^T A \vec{x} = A^T \vec{b} \end{cases}$

$$(QR)^T (QR)\vec{x} = (QR)^T \vec{b}$$
$$R^T \underbrace{Q^T Q}_{=I} R \vec{x} = R^T Q^T \vec{b}$$
$$R^T R \vec{x} = R^T Q^T \vec{b}$$
$$\downarrow$$
$$R\vec{x} = Q^T \vec{b}$$

$R^T$. x zero, row-reducable $\rightarrow I_n$.
$\rightarrow$ invertible.
(non-trivial subspace).

This means the least square solution can be found by considering $Rx = Q^T \vec{b}$

def. Let $T \in \mathcal{L}(\mathbb{R}^n)$, we say $T$ is an orthogonal transformation if $T$ preserves norms, i.e
$$\|T\vec{x}\| = \|\vec{x}\| \quad \text{for all } \vec{x} \in \mathbb{R}^n.$$

def. $Q \in M_n(\mathbb{R})$ is an orthogonal matrix if its columns form an orthonormal set.

- If $Q$ is an orthogonal matrix, $T_Q : \mathbb{R}^n \to \mathbb{R}^m$ is an orthogonal transformation.

Prop. Orthogonal transformation perserves orthogonality and the inner product
(preserves all angles)

Determine area of parallelogram formed by $\vec{v}_1, \vec{v}_2$

$\text{Area} = \{\vec{v} \mid \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 \mid 0 \le c_1 \le 1,\ 0 \le c_2 \le 1\}$.

$\text{Volum} = \{\vec{v} \in \mathbb{R}^3 \mid \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3,\ 0 \le c \le 1,\ \forall i \in \{1, 2, 3\}\}$

$\text{Parallelogram Piped} = \{\vec{v} \in \mathbb{R}^n \mid \vec{v} = \sum_{i=1}^{n} c_i v_i,\ 0 \le v_i \le 1,\ \forall i \in \{1, \dots, n\}\}$.

Let $\Phi$ denote the $\boxed{\text{Vol}}$ of parallelopiped formed by $v_1, \dots, v_n$

① $\Phi(v_1, \dots, kv_i, \dots, v_n) = k\Phi(v_1, \dots, v_i, \dots, v_n)$

② $\Phi(v_1, \dots, v_i + kv_j, \dots, v_j, \dots, v_n) = \Phi(v_1, \dots, v_n)$. $\forall i \ne j$

③ $\Phi(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -\Phi(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$

④ $\Phi(e_1, \dots, e_n) = 1$

Def. We say determinate $: M_n(\mathbb{R}) \to \mathbb{R}$ is the unique $n \times n$ determinate if the provided 4 conditions are satisfied.

elementary column operators.

- $\det(A) = \det(A^T)$.
  $\therefore$ we can translate column operation to row operations.
- $\det(AB) = \det(A) \cdot \det(B)$

lemma: ① If $E$ is elementary, then $\det(E) = \det(E^T)$

② $\det(AE) = \det(A) \cdot \det(E)$

③ if $A$ is invertible, then $A$ is a product of elementary matrices.

proof of ③: consider the 3 possibilities of $E$.

Theorem: Let $A \in M_n(\mathbb{R})$ then $\det A = \det A^T$.

pf. ① If $A$ is not invertible: then $\det(A) = 0$.

note that $A^T$ is also not invertible. , and $\det(A^T) = 0$

$\therefore \det(A) = \det(A^T)$

② If $A$ is invertible,

$A$ can be written as a product of a series of elementary matrices.

$$A = \prod_{i=1}^{n} E_i$$

$$\det(A^T) = \det\left(\prod_{i=1}^{n} E_i\right)^T$$

$$= \det\left(\prod_{i=n}^{1} (E_i)^T\right)$$

$$= \prod_{i=n}^{1} \det(E_i)^T$$

$$= \prod_{i=n}^{1} \det(E_i)$$

$$= \prod_{i=1}^{n} \det(E_i) = \det(A).$$

Thm: Let $A, B \in M_n(\mathbb{R})$ then $\det(AB) = \det(A) \cdot \det(B)$

if either one is invertible, the product will also be invertible.

Pf: If either of $A, B$ is singular, then $AB$ is singular.

This means $\det(AB) = 0 = \det(A) \cdot \det(B)$.

Otherwise, if both $A$ and $B$ are invertible,

let $B = \prod_{i=1}^{n} E_i$ where $\{E_i\}_{i=1}^{n}$ are elementary matrices.

then $\det(AB) = \det\left(A \prod_{i=1}^{n} E_i\right) = \det(A) \cdot \prod_{i=1}^{n} \det(E_i) = \det(A)\det(B)$

RM. If $A$ is invertible, the det, $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Thm: for fixed $n \in \mathbb{N}$, the $n \times n$ determinent function exists and is unique.

pf for uniqueness.

Let $\det_1(\cdot)$ and $\det_2(\cdot)$ be two determinent functions for $M_n(\mathbb{R}) \to \mathbb{R}$. let $A$ be arbitrary matrix in $M_n(\mathbb{R})$. need to show $\det_1(A) = \det_2(A)$.

Row reduce $A$ to its RREF.
keeping track of sign changes and scaling: $\det_1(\cdot)$, $\det_2(\cdot)$ should give us the same answer

- full ranked: $\det_1(\cdot) = \det_2(\cdot) = 1$ (identity).
- not full ranked: $\det_1(\cdot) = \det_2(\cdot) = 0$.

$\therefore$ at the end, $\det_1(\cdot) = \det_2(\cdot)$

def. Let $f: \underset{1}{\bar{V}} \times \cdots \times \underset{n}{\bar{V}} \to \mathbb{R}$, $\bar{V}$ is a vector space.

We say multilinear (linear in every component) $f$.

homogeneouty. ① $f(a_1, \ldots, ka_j, \ldots, a_n) = k f(a_1, \ldots, a_j, \ldots, a_n)$

additivity ② $f(a_1, \ldots, \underset{j}{v+w}, \ldots, a_n) = f(a_1, \ldots, \underset{j}{v}, \ldots, a_n) +$

$f(a_1, \ldots, \underset{j}{w}, \ldots, a_n)$

$\forall j \in \{1, \ldots, n\}, k \in \mathbb{N}.$
$\forall j \in \{1, \ldots, n\}, \forall v, w \in \bar{V}$

RM: inner product is bilinear (2 components)
determinent is multilinear.

Thm: determinent is multilinear in column and row.

e.x

$$\begin{vmatrix} 1 & 0 & 5 \\ 3 & 2 & 1 \\ 1 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 5 \\ 3 & 1 & 1 \\ 1 & 4 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 5 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

We say $f : V^n \to \mathbb{R}$ is multilinear if

$$\underbrace{V \times V \times \cdots \times V}_{n}$$

$f(v_1, \dots, \alpha v_k, \dots, v_n) = \alpha f(v_1, \dots, v_k, \dots, v_n)$ $\forall k \in \{1, \dots, n\}$ $\forall \alpha \in \mathbb{R}$.

$f(v_1, \dots, v_k + w_k, \dots, v_n) = f(v_1, \dots, v_k, \dots, v_n)$

$+ f(v_1, \dots, w_k, \dots, v_n)$

tensor

Real

Inner product.

$\langle a+b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$

$\langle ka, c \rangle = k \langle a, c \rangle$

$\langle a, b+c \rangle = \langle a, b \rangle + \langle a, c \rangle$

$\langle a, kb \rangle = k \langle a, b \rangle$

lemma: let $A \in M_n(\mathbb{R})$. Fix $i \in \{1, \dots, n\}$.
Then by adding a linear combination of the rest of the columns or rows to $i$th column or row will not change the determinent.

pf of column:
we want to show $\det(a_1, \dots, a_{k-1}, v+w, a_{k+1}, \dots, a_n)$
$= \det(a_1, \dots, a_{k-1}, v, a_{k+1}, \dots, a_n)$
$+ \det(a_1, \dots, a_{k-1}, w, a_{k+1}, \dots, a_n)$
if this whole thing are linearly dependent, then
LHS = 0 = RHS
and we are done.
otherwise, if they are linearly independent, we can form
$(a_1, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_n)$ as
a basis for $\mathbb{R}^n$.

then $v$ must be a liner combination of $b_i a_i$
i.e. $v = \sum_{i=1}^{n} b_i a_i$ for some $\{b_i\} \in \mathbb{R}$.

$w = \sum_{i=1}^{n} c_i a_i$ for some $\{c_i\} \in \mathbb{R}$.

$\therefore$ LHS $= \det(a_1, \dots, a_{k-1}, \sum (b_i + c_i) a_i, a_{k+1}, \dots, a_n)$
$= \det(a_1, \dots, a_{k-1}, a_k(b_k + c_k), a_{k+1}, \dots, a_n)$.
$= (b_k + c_k) \det(a_1, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_n)$
$= b_k \det(a_1, \dots, a_n) + c_k \det(a_1, \dots, a_n)$
$= \det(a_1, \dots b_k a_k, \dots a_n) + \det(a_1, \dots, c_k a_k, \dots a_n)$

Adding liner combination of the cols to the $k$-th col to form
$v$, $w$. $= \det(a_1, \dots, v, \dots, a_n) + \det(a_1, \dots, w, \dots, a_n)$

Sarrus's Thm: (only 3x3 Matrix)

$$\det \text{ of } \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{array}{ccccc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \\ - & - & - & + & + & + \end{array}$$

$= aei + bfg + cdh - gec - hfa - idb.$

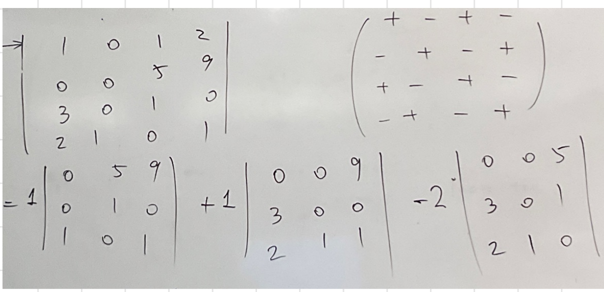
Laplace / Cofactor Expansion.

def. let $A \in M_n(\mathbb{R})$ Then its minor @ $(i, j)$ is a $(n-1) \times (n-1)$ matrix with the ith row, jth column removed from $A$, denote it by $M(i, j)$

det(A) can be found by expanding either one row or one column.

$$\det(A) = \sum (-1)^{i+j} A_{i,j} \cdot |M(i,j)|$$

ex. $\begin{vmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 5 & 9 \\ 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{vmatrix}$



determinants

1. row operation = col operation
2. Det of upper triangular matrix = product of main diagonal
3. $\det(I_n) = 1$
4. $\det(A) = \det(A^T)$
5. $\det(AB) = \det(A) \cdot \det(B)$
6. $\det(A^{-1}) = \dfrac{1}{\det(A)}$ if A is non-singular
7. A is invertible iff $\det(A) \neq 0$.
8. $\det(A) = 0$ if rows / cols are linearly dependent.
9. Q is orthogonal. $\det(Q) = \pm 1$

   (converse is not true!)
10. $|\det(A)| = \dfrac{|\text{area of region}|}{|\text{original region}|}$ (as expansion factor)

Similarity:

diagnol matrix

def. let $A \in M_n(\mathbb{R})$.

A is diagnol if $A_{i,j} = 0 \quad \forall i \neq j$

e.x. $T: \mathbb{R}^2 \to \mathbb{R}^3$

$\vec{x} \mapsto D\vec{x}$

when

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$D\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 3x_2 \\ 5x_3 \end{pmatrix} \qquad D^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 25 \end{pmatrix}$$

def let $A, B \in M_n(\mathbb{R})$. We say A is similar to B if $\exists\, S \in GL_n(\mathbb{R})$ s.t. $A = SBS^{-1}$

e.g. $I_n$ is similar to ?

$I_n = SBS^{-1}$ for some S invertible

$I_n S = SB \quad \therefore B = I_n.$

$\therefore I_n$ is similar to $I_n$ ; 0 is similar to 0.

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$$

$A \quad S \quad B \quad S^{-1}$

$A \sim B$.

$A^n = SB^nS^{-1}$

scale anything along $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ by $-1$
and ... $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ by $5$.

def. Let $T \in \mathcal{L}(\bar{V})$ where $\bar{V}$ is finite dimensional.
$T$ is diagonalizable if $T(b)$ is diagonal w.r.t. some basis $\mathcal{B}$.

def. let $A \in M_n(\mathbb{R})$, we say $A$ is diagonalizable if $A \sim$ a diagonal matrix.

i.e. $A = PDP^{-1}$ for some $P \in M_n(\mathbb{R})$ invertible and some $D$ diagonal.

e.x. $I_n$ diagonalizable ?

$I_n$ and $O$ are diagonalizable.
any diagonal matrices are diagonalizable.

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ invertible, not diagonalizable.

def Let $T \in \mathcal{L}(\bar{V})$ we say non-zero vector $v \in \bar{V}$ is an eigenvector of $T$ if $T(v) = \lambda v$ for some $\lambda$.

def Let $A \in M_n(\mathbb{R})$, then non-zero vector $\vec{v} \in \mathbb{R}^n$ is eigenvector of $A$ if $A\vec{v} = \lambda\vec{v}$ for some $\lambda$.

① $A$ acting on $\vec{v}$ = scaling $\vec{v}$
② $\vec{v}$'s direction remains the same under $A$.

$A\vec{v}$ and $\lambda\vec{v}$ is an eigen pair
$\lambda$ is the eigen value.

e.g. Any vector in $\mathbb{R}^n$ is an eigenvector.
The eigenvalue of $I_n$ is $1$.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is an orthogonal projection onto the $x$-axis.

$\begin{cases} \text{Anything on } x_1\text{-axis is an eigen vector.} \\ \text{Its corresponding eigenvalue} = 1. \end{cases}$

$\begin{cases} \text{Anything on } x_2\text{-axis is an eigen vector.} \\ \text{Its corresponding eigenvalue} = 0. \end{cases}$

prop. Let $A \in M_n(\mathbb{R})$

$\lambda$ is an eigenvalue of $A \iff \det(A - \lambda I_n) = 0$.

pf. Let $\lambda, v$ be an eigen pair., i.e.

$A\vec{v} = \lambda\vec{v}$ and $\vec{v} \neq 0$.

$A\vec{v} - \lambda\vec{v} = 0$

$(A - \lambda)\vec{v} = 0$

$\vec{v}$ in kernel, $\vec{v} \neq 0$.

$\therefore$ none trivial

not invertible.

$\therefore (A - \lambda)\vec{v} = 0 \iff \det(A - \lambda I_n) = 0$.

def Let $A \in M_n(\mathbb{R})$ the characteristic polynomial of $A$ is a polynomial of degree $n$ of the form

$P_A(\lambda) = \det(A - \lambda I_n)$

where $\lambda$ is seen as the unknown.

ex. $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

Determine. $P_A(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8$

To find its eigenvalue, we solve for $\lambda$:

$\lambda^2 - 4\lambda - 5 = 0$

$(\lambda - 5)(\lambda + 1) = 0$. $\therefore \lambda_1 = 5, \lambda_2 = -1$

to find eigenvectors:

when $\lambda = -1$, $A - \lambda I_n = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$.

$E_{\{\lambda = -1\}} = \mathrm{Nul}\,(A - \lambda I_2) = \mathrm{span}\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$.

when $\lambda = 5$, $A - \lambda I_n = \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}$

$E_{\{\lambda = 5\}} = \mathrm{Nul}\,(A - \lambda I_n) = \mathrm{span}\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\}$

def Let $\lambda$ be an eigenvalue of $A \in M_n(\mathbb{R})$, the eigenspace for $\lambda$ is $E_\lambda = \mathrm{Nul}\,(A - \lambda I_n)$

Any vector in $E_\lambda$ is an eigenvector for $\lambda$.

Rm. every non-zero vector in $E_\lambda$ is an eigenvector of $\lambda$.

Theorem: (Fundamental theorem of Algebra)
Polynomial $p(x) \in \mathbb{R}_n[x]$ splits over $\mathbb{C}$, that is
$p(x) = a(x - x_1)^{n_1}(x - x_2)^{n_2}(x - x_3)^{n_3} \cdots (x - x_p)^{n_p}$
where $x_i \in \mathbb{C}$, $a$ is the leading coefficient of $p(x)$
$\sum_{i=1}^{p} n_i = n$. This means counting multiplicities.
$p(x)$ has $n$ roots in $\mathbb{C}$

ex. $p(x) = (x-2)^2(x+i)(x-i)$
$p(x)$ has 4 roots in $\mathbb{C}$.

def. Consider characteristic polynomial $P_A(\lambda)$ of matrix $A \in M_n(\mathbb{R})$
$P_A(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_p)^{n_p}$
Then. $n_i$ for the term $(\lambda - \lambda_i)^{n_i}$ is called the algebraic multiplicity of eigenvalue $\lambda_i$.

Thm. (Fundamental theorem of Algebra I over $\mathbb{R}$)

$$p(x) = a(x-x_1)^{n_1}(x-x_2)^{n_2}\cdots(x-x_p)^{n_p}(x^2+A_1x+B_1)^{m_1}\cdots(x^2+A_qx+B_q)^{m_q}$$

$$\sum_{i=1}^{p} n_i + \sum_{j=1}^{q} 2m_j = n$$

and $A_j^2 - 4B_j < 0$ for all $(x^2+A_jx+B_j)^{m_j}$ terms ← unreal roots

ex. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\theta = \frac{\pi}{2}$

$A\vec{v} = \lambda\vec{v}$

$p_A(\lambda) = \det(A-\lambda I_2) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix}.$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$= \lambda^2 + 1$

Note: no real eigenvalues.
But over $\mathbb{C}$, we have $\lambda = i$

When $\lambda = i$, $A - \lambda I_2 = \begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix}$

Eigenspace for $\lambda = i$
$E(\lambda=i) = \mathrm{Nul}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}.$

Eigenspace for $\lambda = -i$
$E(\lambda=-i) = \mathrm{Null}\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}.$

complex conjugate of each other.

consider $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Note that $A^2 = -I_2$
$\sim i^2 = -1$

modulus

$z_1 = r_1 e^{i\theta_1}$ – argument

$z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = \underline{r_1 r_2}\, e^{i(\theta_1+\theta_2)}$

$T: \mathbb{C} \to \mathbb{C}$
$z \mapsto (1+\sqrt{3})z.$
$z \mapsto 2e^{i\frac{\pi}{3}} z$

$S: \mathbb{R}^2 \to \mathbb{R}^2$
$\vec{x} \mapsto A\vec{x}$

$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

$a+bi = a \cdot 1 + b \cdot i$

$\qquad\qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$= \sqrt{a^2+b^2} \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} \end{pmatrix}}_{\text{scaling}} \underbrace{\begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}}_{\text{rotation.}}$$

polar form: modulus part argument part

geometric multiplicity:

def. Let $A \in M_n(\mathbb{R})$, and $\lambda$ be one of its eigenvalues,

Then $\dim \operatorname{Nul}(A - \lambda I_n)$ is called $\lambda$'s geometric multiplicity

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ char poly: $P_A(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$

$\lambda = 1.$

Algebraic multiplicity $= 2.$

Geometric multiplicity $= 1.$

$\because E\{\lambda = 1\} = \operatorname{Nul}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}.$

$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $P_A(\lambda) = (x-3)(x-2)^2$

$\lambda = 3$ algebraic multiplicity $= 1$

$\lambda = 2$ algebraic multiplicity $= 2.$

$E\{\lambda = 3\} = \operatorname{Nul}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

$E_{\{\lambda=2\}} = \mathrm{Nul}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$

$\lambda = 3$ geometric multiplicity $= 1$

$\lambda = 2$ geometric multiplicity $= 2$.

prop!: let $A \in M_n(\mathbb{R})$, with distinct eigenvalues, $\lambda_1, \ldots, \lambda_p$. Their corresponding eigenvectors $v_1, \ldots, v_p$. Then $v_1, \ldots, v_p$ are linearly independent.

proof for two:

claim: If $(\lambda_1, v_1), (\lambda_2, v_2)$ are two eigen pairs such that $\lambda_1 \neq \lambda_2$, then $v_1, v_2$ are linearly independent.

pf. $A\vec{v}_1 = \lambda_1 \vec{v}_1$

$A\vec{v}_2 = \lambda_2 \vec{v}_2$

consider $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. $\circledast$

need to prove $c_1 = c_2 = 0$.

apply $A$ on both sides:

$Ac_1\vec{v}_1 + Ac_2\vec{v}_2 = \vec{0}$

$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$ ... $\circledast\circledast$

$\circledast \cdot \lambda_1 = c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 = \vec{0}$ ... ①

$\circledast \cdot \lambda_2 = c_1\lambda_2\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$ ... ②

① $- \circledast\circledast = c_1(\lambda_1 - \lambda_2)\vec{v}_1 = \vec{0}$

② $- \circledast\circledast = c_2(\lambda_2 - \lambda_1)\vec{v}_2 = \vec{0}$

$\lambda_1 \neq \lambda_2 \therefore \lambda_1 - \lambda_2 \neq 0$ and $\lambda_2 - \lambda_1 \neq 0$.

and $v_1 \neq \vec{0}$ and $v_2 \neq \vec{0}$. (eigenvectors aren't zero.)

$\therefore c_1 = c_2 = 0$.

can use proof by induction to prove the case for $n$.

Prop 2 : Let $A \in M_n(\mathbb{R})$ and $\lambda_1, \dots, \lambda_p$ are distinct eigenvalues of $A$ where

$v_1^{(1)}, v_2^{(1)}, \dots, v_{k_1}^{(1)}$ are linearly independent of $\lambda_1$

$\vdots$

$v_1^{(p)}, v_2^{(p)}, \dots, v_{k_1}^{(p)}$ are linearly independent of $\lambda_p$.

Then $v_1^{(1)}, \dots, v_{k_1}^{(1)}, \dots, v_1^{(p)}, \dots, v_{k_p}^{(p)}$ linearly independent.

Prop 1 :

$$\begin{array}{cccccc} \lambda_1, & \lambda_2, & \dots, & \lambda_n & & \boxed{\text{distinct}} \\ \Updownarrow & \Updownarrow & & \Updownarrow & & \\ v_1 & v_2 & & v_n & & \boxed{\text{lin ind}} \end{array}$$

Prop 2 :

$$\begin{array}{cccccc} \lambda_1, & \lambda_2, & \dots, & \lambda_n & & \boxed{\text{distinct}} \\ \Updownarrow & \Updownarrow & & \Updownarrow & & \\ v_1^{(1)} & v_1^{(2)} & & v_1^{(n)} & & \\ v_2^{(1)} & v_2^{(2)} & & v_2^{(n)} & & \boxed{\text{are lin. ind.}} \\ \vdots & \vdots & & \vdots & & \\ v_k^{(1)} & v_k^{(2)} & & v_k^{(n)} & & \\ \uparrow & \uparrow & & \uparrow & & \\ \text{basis of } E_1 & \text{basis of } E_2 & & \text{basis of } E_n & & \end{array}$$

Cor. Let $E_i$ be eigenspace for $\lambda_i$, eigenvalues for $A \in M_n(\mathbb{R})$.

$\dim(E_1) + \dots + \dim(E_n) \le n$.

(geometric multiplicity's sum $\le$ rank $(A)$.

Diagnolizable?

iff
1. $T \in \mathcal{L}(\bar{V})\ \exists B$ s.t. $[T]_B$ is diagonal
2. $\exists P$ invertible, $\exists D$ diagonal s.t. $A = PDP^{-1}$
3. $\exists$ eigenbasis
4. $\sum_i$ Geometric multiplicity $(\lambda_i) = n$
5. $\forall \lambda_i$, Alg. Mul. $(\lambda_i)$ = Geo. Mul. $(\lambda_i)$.

if 6. $\exists\, n$ distinct eigenvalue.

From the 2 props from last class,

Lemma : $T \in \mathcal{L}(\bar{V})$ is diagonalizable iff there exists a basis for $\bar{V}$ consisting of T's eigenvalue.

Sketch of pf : let $B = \langle v_1, \dots, v_n \rangle$ and $Tv_i = \lambda_i v_i$ for $i \in \{1, \dots, n\}$.

$$[T]_B = \begin{pmatrix} | & | & & | \\ [Tv_1]_B, & [Tv_2]_B, & \dots, & [Tv_n]_B \\ | & | & & | \end{pmatrix}.$$

$$= \begin{pmatrix} | & | & & | \\ [\lambda_1 v_1]_B, & [\lambda_2 v_2]_B, & \dots, & [\lambda_n v_n]_B \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & 0 & & \\ & \vdots & \ddots & 0 \\ 0 & 0 & & \lambda_n \end{pmatrix}$$

def This is the eigenbasis.

Cor. Let $T \in \mathcal{L}(\bar{V})$ Let $\lambda_1, \dots, \lambda_p$ be distinct eigenvalues. Consider their corresponding eigenspaces $E_1, \dots, E_p$

$$\sum_{i=1}^{p} \underset{\downarrow}{\dim(E_i)} \leq n.$$

geometric multiplicity of $\lambda_i$

pf. with the previous notation,

$$\bigcup_{i=1}^{p} \{ v_1^{(i)}, \dots, v_{k_i}^{(i)} \}$$ linearly independent.

However, the maximal # of linearly independent vectors in $\bar{V}$, whose dim is $n$, is $n$.

Cor. if $\sum_{i=1}^{p} \dim(E_i) = n$, then $T$ has an eigenbasis.

if $\sum_{i=1}^{p} \dim(E_i) < n$, then $T$ doesn't have an eigenbasis from T.

Thm: Let $T \in \mathcal{L}(\bar{V})$. For each eigenvalue $\lambda$, $\text{Alg Mul}(\lambda) \geq \text{Geo Mul}(\lambda)$.

Lemma 1: Consider block upper triangular matrix $C := \begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$

$$P_C(\lambda) = P_A(\lambda) \cdot P_B(\lambda)$$

pf: $P_C(\lambda) = \det(C - \lambda I_n)$

$$= \det \begin{pmatrix} A - \lambda I_n & * \\ 0 & B - \lambda I_n \end{pmatrix}$$

$$= \det(A - \lambda I_n) \times \det(B - \lambda I_n)$$

$$= P_A(\lambda) = P_B(\lambda)$$

Lemma 2: Let $A \in M_n(\mathbb{R})$

Let $\{B_1, \cdots, B_p, B_{p+1}, \cdots, B_n\}$ be a basis where the first $p$ vectors are eigenvectors of $A$ with eigenvalue $\lambda_0$. Then $A$ is similar to $\begin{pmatrix} \lambda_0 I_p & * \\ 0 & B \end{pmatrix}$

for some block matrix $B$.

Pf. Call this basis $\mathcal{B}$

$$[T_A]_{\mathcal{B}} = \begin{pmatrix} | & & | & | & & | \\ [Ab_1]_{\mathcal{B}}, & \cdots & [Ab_p]_{\mathcal{B}}, & [Ab_{p+1}]_{\mathcal{B}}, & \cdots, & [Ab_n]_{\mathcal{B}} \\ | & & | & | & & | \end{pmatrix}.$$

$$= \begin{pmatrix} | & & | & | & & | \\ [\lambda b_1]_{\mathcal{B}}, & \cdots & [\lambda b_p]_{\mathcal{B}}, & [\lambda b_{p+1}]_{\mathcal{B}}, & \cdots, & [\lambda b_n]_{\mathcal{B}} \\ | & & | & | & & | \end{pmatrix}.$$

$$= \left(\begin{array}{cccc|c} \lambda_0 & 0 & & 0 & \\ 0 & \lambda_0 & & \vdots & \\ \vdots & 0 & \ddots & 0 & * \\ 0 & 0 & & \lambda_0 & \\ \hline \vdots & \vdots & & \vdots & \\ 0 & 0 & & 0 & B. \end{array}\right)$$

pf. Thm : Alg Mul $(\lambda_i) \geq$ Geo Mul $(\lambda_i)$

Let $\lambda_0$ be an eigenvalue of T.

Let $\langle b_1, \dots, b_p, b_{p+1}, \dots, b_n \rangle$ be a basis for $\bar{V}$

where $b_1, \dots, b_p$ are eigenvectors corresponding to $\lambda_0$.

$$[T]_B = \begin{pmatrix} \lambda_0 I_p & * \\ 0 & B \end{pmatrix}$$

$$P_{[T]_B}(\lambda) = (\lambda_0 - \lambda)^p \, P_B(\lambda)$$

Note the eigenspace $E_{\lambda_0}$ has dim of $p$ this means the algebraic multiplicity of $\lambda$ is at least $p$.

Cor $T \in L(\bar{V})$ is diagnolizable iff $\forall \lambda_i$, an eigenvalue of T, its geometric multiplicity & algebraic multiplicity coincide.

ex. $A = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & -4 \end{pmatrix}$

① Char poly : $P_A(\lambda) = \begin{vmatrix} 4-\lambda & 0 & 3 \\ 0 & 5-\lambda & 0 \\ 3 & 0 & -4-\lambda \end{vmatrix}$

$$= (5-\lambda) \begin{vmatrix} 4-\lambda & 3 \\ 3 & -4-\lambda \end{vmatrix}$$

$$= -(\lambda+5)(\lambda-5)^2$$

When $\lambda = 5$. $E_{\{\lambda=5\}} = \text{Nul} \begin{pmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -9 \end{pmatrix} = \text{Nul} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$$= \text{Span} \left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

When $\lambda = -5$, $E_{\{\lambda = -5\}} = \mathrm{Nul}\begin{pmatrix} 9 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$$A = \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}}_{P^{-1}}$$

def. symmetric matrices:

Let $A \in M_n(\mathbb{R})$. Then $A$ is said to be symmetric if $A = A^T$

lemma: If $A$ is symmetric, $A$'s all eigenvalues are all real.

Lemma: Let $A$ be a symmetric matrix,
Let $\mu$, $\lambda$ be two distinct eigenvalues of $A$,
Then eigenspaces $E_\lambda$, $E_\mu$ are orthogonal.

Thm: If $A$ is symmetric, then $A$ is orthogonally diagonalizable with real eigenvalues
i.e. $\exists$ orthogonal matrix $Q$, $\exists$ diagonal matrix $D$.
s.t. $A = QDQ^T$.

Pf. Induction on size of $A$.

Base case: $n=1$
IH: Assume true for $n=k$.
IS: Need to show this works for $n=k+1$.

Let $\lambda$ be a real eigenvalue of $A$.
let $\vec{v}_1$ be its corresponding eigenvector of unit length.
extend $v_1$ to form $\langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n+1} \rangle$, an o.n. basis for $\mathbb{R}^{n+1}$

Let $B = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_{n+1} \\ | & & | \end{pmatrix}$ ← B is orthognal, $(B^{-1}AB)^T = B^T A B$
$= \begin{pmatrix} \lambda & 0 \\ 0 & C^T \end{pmatrix}$

(orthogonal to each other)

$B^{-1}AB = \begin{pmatrix} \lambda_{1\times1} & 0_{k\times1} \\ 0_{1\times k} & C_{k\times k} \end{pmatrix}$.

?

C is symmetric.

since $C$ is $k \times k$ and $C = C^T$, by IH, $\exists\, K \in M_k(\mathbb{R})$, $D \in M_k(\mathbb{R})$
$C = KDK^{-1}$ or $D = K^{-1}CK$.

Consider $R \in M_{n+1}(\mathbb{R})$ of the form

$$R = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & K \end{array}\right)$$

$$\underbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & K^{-1} \end{array}\right)}_{R^{-1}} \left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & KDK^{-1} \end{array}\right) \underbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & K \end{array}\right)}_{R} = \left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & D \end{array}\right)$$

$\uparrow$ diagonal.

$\Rightarrow$ $R^{-1}B^{-1}ABR$ is diagonal.

product of 2 orthogonal matrices is also orthogonal.

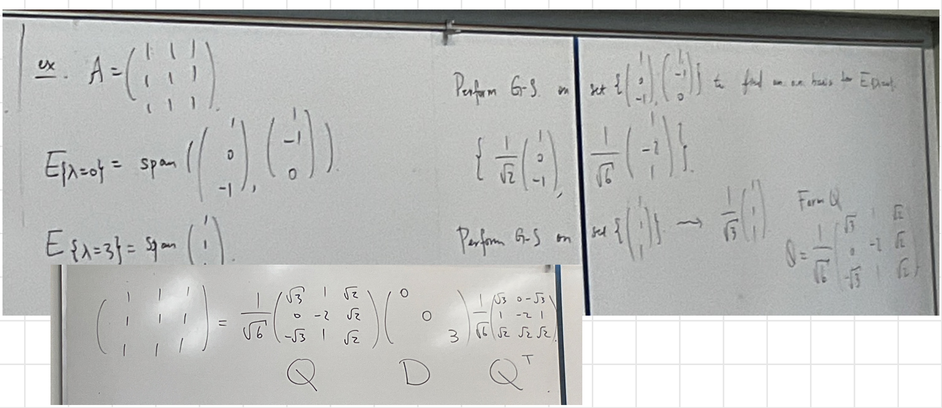
Let $Q = BR$, $Q$ is also orthogonal.
$\therefore$ $Q^{-1}AQ$ is also diagonal

R.M. For computation:

Step 1: Determine all eigenvalues of $A$ and the spanning set of all its eigenspaces.

Step 2: Use G-S. to determine o.n. bases for each eigenspace

Step 3: Form $Q$ by combining the eigenbasis from step 2.



def. Function $f: \mathbb{R}^n \to \mathbb{R}$ is a quadratic form if it is a linear combination of term $\underline{x_i x_j}$ where $i, j \in \{1, \ldots, n\}$.
can be the same or different.

ex. $f(x_1, x_2, x_3) = x_1^2 + 6x_2x_3 - 5x_3^2$
$f(x_1, x_2) = 6x_1^2 + 4x_1x_2 + 3x_2^2$

$$
\begin{aligned}
f(x_1, x_2) &= 6x_1^2 + 4x_1x_2 + 3x_2^2 \\
&= (x_1 \ x_2) \begin{pmatrix} 6x_1 + 2x_2 \\ 2x_1 + 3x_2 \end{pmatrix} \\
&= (x_1 \ x_2) \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{aligned}
$$

RM. For quadratic form $Q(\vec{x})$, we can always write it as $Q(\vec{x}) = \vec{x}^T A \vec{x}$ where $A$ is a symmetric matrix.

$$x^T A x = x^T Q D \underbrace{Q^{-1} x}_{[x]_{\mathcal{B}}}$$

prop. Let $Q(\vec{x}) = \vec{x}^T A \vec{x}$ be a quadratic form with $A$, a symmetric matrix. Let $\lambda_1, \ldots, \lambda_n$ be $A$'s real eigenvalues and $\mathcal{B}$ be the corresponding eigenbasis. Then $Q(x) = \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2$ where $(c_1, \ldots, c_n)^T$ is $[\vec{x}]_{\mathcal{B}}$

$$
\begin{aligned}
&\vec{x}^T \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}^{-1} \vec{x} \\
&= (c_1 \ c_2) \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&= 2c_1^2 + 7c_2^2
\end{aligned}
$$

Def. Let $A$ be a symmetric matrix, and $Q(\vec{x}) = \vec{x}^T A \vec{x}$. We say $A$ is positive semidefinite if $Q(\vec{x}) \geq 0$ for all $\vec{x} \in \mathbb{R}^n$. We say $A$ is positive definite if $Q(\vec{x}) > 0$ for non-zero $x \in \mathbb{R}^n$.

$A \succcurlyeq 0$ positive semidefinite
$A \succ 0$ positive definite

Cor. $A \succcurlyeq 0$ iff all its eigenvalues $\geq 0$
$A \succ 0$ iff all its eigenvalues $> 0$.

RM. Let $A$ be symmetric. spectral decomposition.

$$A = QDQ^T$$

$$\begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \begin{pmatrix} - & q_1 & - \\ & \vdots & \\ - & q_n & - \end{pmatrix}$$

$$= \lambda_1 q_1 q_1^T + \cdots + \lambda_n q_n q_n^T$$

$$= \sum_{i=1}^{n} \lambda_i q_i q_i^T.$$

orthogonal projection onto orthonormal eigenbasis.

Spectral theorem:

if $A \in M_n(\mathbb{R})$ that is symmetric, there exist

$Q \in M_n(\mathbb{R})$ orthogonal

$D \in M_n(\mathbb{R})$ diagonal s.t. $A = QDQ^T$

$\downarrow$

eigenvalue is always real.

Quadratic form: $f: \mathbb{R}^n \to \mathbb{R}$

a linear combination of $x_i, x_j$ where $i, j \in \{1, \dots, n\}$.

e.x. $Q(x_1, x_2) = 4x_1^2 - 5x_1x_2 + 6x_2^2$

$$= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 4 & -\frac{5}{2} \\ -\frac{5}{2} & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$= \underbrace{\vec{x}^T Q}\, D\, \underbrace{Q^T \vec{x}}$ $\leftarrow$ because $B[\vec{x}]_B = \vec{x}$. $[\vec{x}]_B = B^{-1}\vec{x}$

$= [\vec{x}]_Q^T D [\vec{x}]_Q$ $\quad$ $[\vec{x}]_Q$ is $\vec{x}$'s rep. w.r.t. eigenbasis

$= \begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ $\quad$ i.e. columns of $Q$

$= \lambda_1 c_1^2 + \lambda_2 c_2^2$ $\quad$ $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = [\vec{x}]_B$

Symmetric $A$: $\vec{x}^T A \vec{x}$.

| | | | | | |
|---|---|---|---|---|---|
| $A$ is p.d. | if $\vec{x}^T A \vec{x}$ | $> 0$ | $\forall x \neq 0$. | | all $\lambda > 0$ |
| $A$ is p.s.d. | if $\vec{x}^T A \vec{x}$ | $\geq 0$ | $\forall x$ | $\Leftrightarrow$ | all $\lambda \geq 0$ |
| n.d. | | $< 0$ | $\forall x \neq 0$ | | all $\lambda < 0$ |
| n.s.d | | $\leq 0$ | $\forall x$ | | all $\lambda \leq 0$ |

Otherwise, $A$ is indefinite.

Notation: $\sigma(A)$ : all eigenvalues of $A$. (spectrum of $A$)

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$.

Prop. Let $A \in M_{m \times n}(\mathbb{R})$

Then $A^T A$ is symmetric positive semidefinite.

$A = PDP^{-1}$ $\quad A^2 = PD^2P^{-1}$

Given matrix $B$, how to find the square root of $B$?
i.e. a matrix whose square is $B$?

Cor. There exists a unique p.s.d. matrix $B \in M_n(\mathbb{R})$
s.t. $B^2 = A^TA$. for $A \in M_{m\times n}(\mathbb{R})$

pf. $A^TA$ is symmetric positive semidefinite.
This means $A^TA = QDQ^T$

$$= Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} Q^T$$

Let $B = Q \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} Q^T$ then $B^2 = A^TA$.

① $B^T = B$.
② all $\lambda \geq 0$.
$\Rightarrow B$. symmetric p.s.d.

$B$ is unique since $\sqrt{\lambda_i} \geq 0$ for $i \in \{1, \dots, n\}$.

Lemma: Let $A \in M_{m\times n}(\mathbb{R})$ consider $T_A : \mathbb{R}^n \to \mathbb{R}^m$.
There exists an o.n. basis $v_1, \dots, v_n$ in $\mathbb{R}^n$ s.t.
$Av_1, \dots, Av_n$ form an orthogonal set in $\mathbb{R}^m$.

pf. Consider $A^TA$. Note $\exists Q \in M_n(\mathbb{R})$ orthogonal, $\exists D \in M_n(\mathbb{R})$ diagonal.
S.t. $A^TA = QDQ^T$.

Claim: Let $Q = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}$ $\quad Av_i \perp Av_j \quad \forall i \neq j$

$\uparrow$ orthonormal matrix in $\mathbb{R}^n$.

Consider $(Av_i)\cdot(Av_j)$
$= (Av_i)^T(Av_j)$
$= v_i A^T A v_j$ $\quad$ Note $v_j$ is an eigenvector of $A^TA$ with corresponding eigenvalue $\lambda_j$

$= \underbrace{v_i}_{\text{vector}} \underbrace{\lambda_j}_{\text{scalar}} \underbrace{v_j}_{\text{vector}}$

$= \lambda_j \underbrace{(v_i^T v_j)}_{\text{orthonormal.}} = 0$

def. Let $A \in M_{m\times n}(\mathbb{R})$ and $\lambda_1, \dots, \lambda_n$ be eigenvalues of $A^TA$ ($\lambda_1, \dots, \lambda_n$ are non-negative).

Then $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$ are called singular values of $A$.

denoted by $\sigma_1 \geq \dots \geq \sigma_n$.

Thm: Let $A \in M_{m\times n}(\mathbb{R})$, $\exists$ an orthonormal basis $v_1, \dots, v_n \in \mathbb{R}^n$ s.t. $Av_1, \dots, Av_n$ form an orthogonal set in $\mathbb{R}^m$ where $\|Av_i\| = \sigma_i$, the $i$-th singular value of $A$.

pf. With the same notation from previous result,

$$\begin{aligned}\|Av_i\|^2 &= (Av_i)^T(Av_i) \\ &= v_i^TA^TAv_i \\ &= v_i^T\lambda_i v_i = \lambda_i v_i^Tv_i = \lambda_i \underbrace{\|v_i\|^2}_{\text{orthonormal}} = \lambda_i\end{aligned}$$

Taking $\sqrt{\ }$, $\|Av_i\| = \sqrt{\lambda_i} = \sigma_i$.

orthogonal matrix

$$\underset{m\times n}{A} \underset{n\times n}{\underbrace{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}}_{\bar{V}}} = \underset{m\times m}{\underbrace{\begin{pmatrix} | & & | & | & & | \\ \underbrace{u_1 & \cdots & u_r}_{\substack{\text{im}(Av_1 \cdots Av_r) \\ \text{o.n.}}} & \underbrace{u_{r+1} & \cdots & u_m}_{\substack{\text{o.n. basis} \\ \text{of } \operatorname{Col}^{\perp}(A)}} \\ | & & | & | & & | \end{pmatrix}}_{\bar{U}}} \underset{m\times n}{\underbrace{\left(\begin{array}{c|c} \overset{r\times r}{\begin{matrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{matrix}} & 0 \\ \hline 0 & 0 \end{array}\right)}_{\Sigma}}$$

$Av_i = \sigma_i u_i$
where $1 \leq i \leq r$

$\downarrow$

some $Av_i$ are zeros.

$\therefore$ not enough $u_i$ to span the plane.

$\therefore$ need to expand by taking o.n. basis from $\operatorname{col}^{\perp}(A)$.

Thm. Let $A \in M_{m\times n}(\mathbb{R})$. There exists $\bar{U} \in M_m(\mathbb{R})$, orthogonal, $\bar{V} \in M_n(\mathbb{R})$ orthogonal, s.t. $A = \bar{U}\Sigma\bar{V}^T$, where

$\Sigma \in M_{m\times n}(\mathbb{R})$ and of the form $\left(\begin{array}{c|c} \begin{matrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right)$

$\sigma_1 \geq \dots \geq \sigma_r$ the first non-zero singular values of $A$.

SVD. Let $A \in M_{m\times n}(\mathbb{R})$

$$A = \bar{U}\,\Sigma\,\bar{V}^T$$

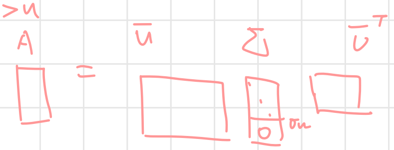
$m\times n \qquad m\times m \qquad \underset{m\times n}{\downarrow} \qquad n\times n$

$\bar{U}$ is orthogonal $\in M_m(\mathbb{R})$

$\bar{V}$ is orthogonal $\in M_n(\mathbb{R})$

$m<n$:


$m>n$




$A\bar{V} = \bar{U}\Sigma$

$Av_i = \sigma_i u_i \qquad 1 \le i \le r$

$$u_i = \frac{Av_i}{\sigma_i}$$

e.g. $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ $\qquad A^TA = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ orthogonally diagnolizable.

$\lambda_1 = 3 \qquad \lambda_2 = 1 \qquad \lambda_3 = 0.$

$\sigma_1 = \sqrt{3} \qquad \sigma_2 = 1 \qquad \sigma_3 = 0.$

$E\{\lambda=3\} = \text{span}\ \frac{1}{\sqrt6}\begin{pmatrix}1\\2\\1\end{pmatrix} \quad v_1 \qquad u_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{\frac{1}{\sqrt6}\begin{pmatrix}3\\3\end{pmatrix}}{\sqrt3}$

$E\{\lambda=1\} = \text{span}\ \frac{1}{\sqrt2}\begin{pmatrix}1\\0\\-1\end{pmatrix} \quad v_2 \qquad u_2 = \frac{A\vec{v}_2}{\sigma_1} = \frac{\frac{1}{\sqrt2}\begin{pmatrix}-1\\1\end{pmatrix}}{1}$

$E\{\lambda=0\} = \text{span}\ \frac{1}{\sqrt3}\begin{pmatrix}1\\-1\\1\end{pmatrix} \quad v_3$

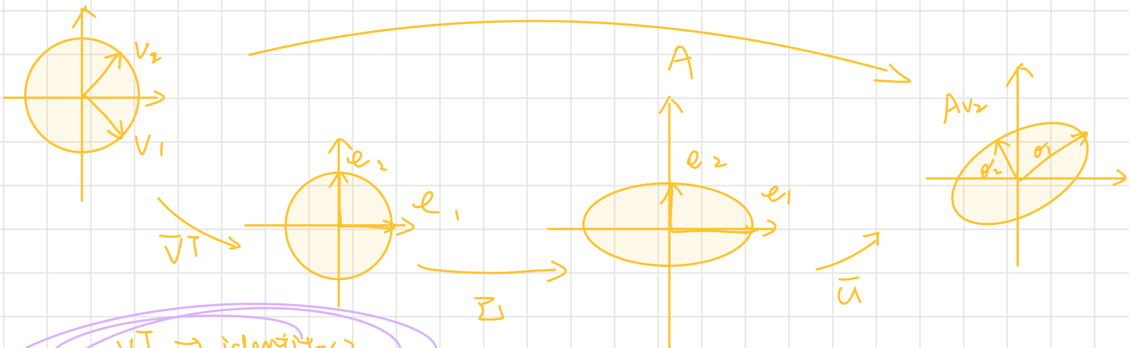
$$A = \bar{U}\Sigma\bar{V}^T$$

$$= \begin{pmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} \sqrt3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt6} & \frac{2}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt2} & 0 & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt3} & -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} \end{pmatrix}$$

e.x. $A = \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{array}{c} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \end{array} \begin{array}{l} v_1 \\ v_2 \end{array}$

$$A\vec{x} = \underbrace{\bar{U} \underbrace{\Sigma \underbrace{\bar{V}^T x}_{\text{rotation (change of basis)}}}_{\text{stretch it}}}_{\text{rotation (change of basis).}}$$

$PDP^{-1}$ vs $\bar{U}\underbrace{\Sigma}_{\text{not invertible.}}V^T$




$V^T \to$ identity?

?   orthogonal matrix : $\bar{V}$ : $\bar{V}^T\bar{V} = I_n$.

prop. let $\operatorname{rank} A = r$. Then singular values of A $\sigma_1, \ldots, \sigma_r$ are not zeros but $\sigma_{r+1}, \ldots, \sigma_n$ are zeros..

Pf: $Av_1, \ldots, Av_r$ span $\operatorname{col}(A)$

but $\operatorname{rank}(A) = \dim(\operatorname{col}(A)) = r$.

Relation between SVD and 4 fundamental spaces:

$\overset{\sigma_1 u_1}{Av_1}, \ldots, \overset{\sigma_r u_r}{Av_r}$ span $\operatorname{col}(A) \Rightarrow u_1, \ldots, u_r$ span $\operatorname{col}(A)$. $\Big\}$ from $Av_i = \sigma_i u_i$

$v_{r+1}, \ldots, v_n$ span $\operatorname{Nul}(A)$

$A = \bar{U}\Sigma\bar{V}^T$

$A^T = \bar{V}\Sigma\bar{U}^T$

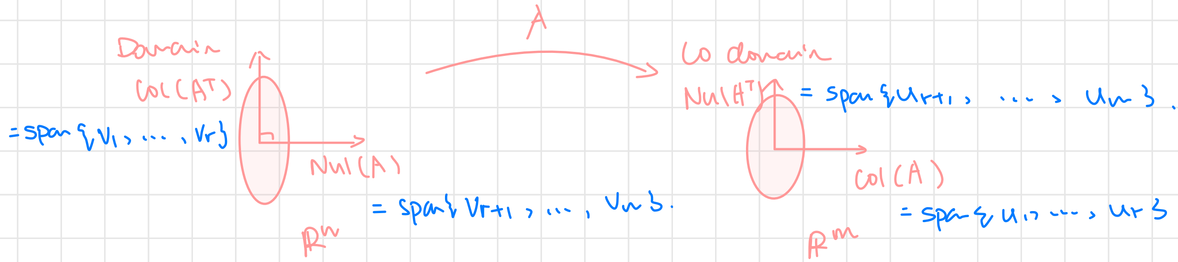
$A^T\bar{U} = \bar{V}\Sigma^T$

$A^T u_i = \sigma_i v_i$

$\Rightarrow \overset{=\sigma_1 v_1}{A^T u_1}, \ldots, \overset{\sigma_r v_r}{A^T u_r}$ span $\operatorname{Col}(A^T)$

$\Rightarrow v_1, \ldots, v_r$ span $\operatorname{Col}(A^T)$

$u_{r+1}, \ldots, u_m$ span $\operatorname{Nul}(A^T)$ $\Big\}$ from $A^T u_i = \sigma_i v_i$




Best low-rank approximation

$$A = \overset{U}{\begin{pmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{pmatrix}} \overset{\Sigma}{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}} \overset{V^T}{\begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix}}$$

$$= \sum_{i=1}^{r} \sigma_i \underbrace{\vec{u}_i \vec{v}_i^T}_{\text{matrix of rank 1.}}$$

?

$$\vec{u} \otimes \vec{v} = \vec{u}\,\vec{v}^T = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} (v_1 \cdots v_m)$$

$$= \underbrace{v_1 \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdots v_m \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}}_{\text{Span}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.}$$

$A \approx \sigma_1 u_1 \vec{v}_1^T$ rank-1 approximation

$A \approx \sigma_1 u_1 \vec{v}_1^T + \sigma_2 u_2 \vec{v}_2^T$ rank-2 approximation.

$\vdots$

Zero base

Least-Square Solution and SVD.

$A = U \Sigma V^T$

Prop. $\vec{x}^* = \dfrac{\vec{b} \cdot u_1}{\sigma_1} v_1 + \cdots + \dfrac{\vec{b} \cdot u_r}{\sigma_r} v_r$

is a least square solution to system $A\vec{x} = \vec{b}$

Pf. NTS. $A\vec{x}^* = \text{proj}_{Col(A)} \vec{b}$.

Consider $A\vec{x}^* = A \sum \dfrac{\vec{b} \cdot \vec{u}_i}{\sigma_1} v_i = \sum_{i=1}^{r} \dfrac{\vec{b} \cdot \vec{u}_i}{\sigma_i} \underbrace{A\vec{v}_i}_{= \sigma_i u_i}$

$= \sum_{i=1}^{r} (\vec{b} \cdot \vec{u}_i) u_i = \text{proj}_{\{u_i\}} \vec{b} = \text{proj}_{Col(A)} \vec{b}$

let $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ $D^2 = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix}$

$D^2 + 3D + 2I_n$ $\begin{pmatrix} \lambda_1^2 + 3\lambda_1 + 2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 + 3\lambda_n + 2 \end{pmatrix}$

if $P(x)$ is a polynomial of finite degree

$P(D) = \begin{pmatrix} P(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & P(\lambda_n) \end{pmatrix}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^D = I_n + D + \frac{D^2}{2!} + \cdots$

$= \begin{pmatrix} 1 + \lambda_1 + \frac{\lambda_1^2}{2!} \cdots & & \\ & \ddots & \\ & & 1 + \lambda_n + \frac{\lambda_n^2}{2!} + \cdots \end{pmatrix}$

Let $A = XDX^{-1}$ diagnolizable.

$e^A = X(e^D)X^{-1}$

Consider $y = y(t)$ $t \neq 0$.

Solve initial value problem: $\frac{dy}{dt} = ky$ where $y(0) = y_0$.

$$y = y_0 e^{kt}$$

Uncoupled system:

$$\begin{cases} \frac{dx}{dt} = -x \\ \frac{dy}{dt} = 5y \end{cases}$$

$x(0) = x_0$

$y(0) = y_0$.

$$\begin{cases} x = x_0 e^{-t} \\ y = y_0 e^{-t} \end{cases}$$

coupled system

$$\begin{cases} \frac{dx}{dt} = x + 2y \\ \frac{dy}{dt} = 4x + 3y. \end{cases}$$

let $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\frac{d}{dt}\vec{x} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \vec{x}$$

$A\vec{x}$. $= PDP^{-1}$

$\downarrow$

$\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$

$$\frac{d}{dt}\vec{x} = PDP^{-1}\vec{x} \qquad P^{-1}\frac{d}{dt}\vec{x} = DP^{-1}\vec{x}$$

let $y = P^{-1}\vec{x}$

$$\Rightarrow \frac{d\vec{y}}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \vec{y}$$

$$\vec{y} = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{5t} \end{pmatrix}$$

$\mathbb{R}$ $\mathbb{C}$

$A = A^T$ symmetric matrix. $A = A^*$

def. let $A \in M_n(\mathbb{C})$. We say $A$ is Hermitian if $A = \overline{A^T}$ ie. $(A)_{ij} = \overline{(A)_{ji}}$

We denote $\overline{A^T}$ by $A^*$

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \quad \begin{pmatrix} 5 & -i \\ i & 4 \end{pmatrix}$$

def. Let $V$ be a V.S. over $\mathbb{F}$ ($\mathbb{R}$, $\mathbb{C}$)

$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$

s.t. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ conjugate symmetry.

$\langle kx, y \rangle = k \langle x, y \rangle$

$\langle x+z, y \rangle = \langle x, y \rangle + \langle x, z \rangle$

$\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.