MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 7

1. Consider the following linear transformations $T, S : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ given by

$$Tp(x) = p'(x) \quad \text{and} \quad Sp(x) = p(x+1)$$

and consider the following bases of $\mathbb{R}^3[x]$:

$$\mathcal{E} = \{1, x, x^2, x^3\}$$

and

$$\mathcal{B} = \{1, 1+x, (1+x)^2, (1+x)^3\}$$

(a) Find $[T]_{\mathcal{E}\to\mathcal{B}}, [T]_{\mathcal{B}\to\mathcal{B}}, [T]_{\mathcal{B}\to\mathcal{E}}, [T]_{\mathcal{E}\to\mathcal{E}}$.

(b) Find $[S]_{\mathcal{E}\to\mathcal{E}}, [S]_{\mathcal{B}\to\mathcal{B}}$.

(c) Find $[T \circ S]_{\mathcal{E}\to\mathcal{E}}$.

(d) $[T \circ S]_{\mathcal{B}\to\mathcal{B}}$.

(e) Use $[T]_{\mathcal{E}\to\mathcal{E}}$ to find a basis for the kernel and image of L.

2. Consider the following ordered bases of $\mathbb{R}^3$:

$$\mathcal{B} = < \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} >$$

$$\mathcal{C} = < \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} >$$

$$\mathcal{E} = < \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} >$$

Find the following matrices of transition from basis to basis:

$$[id]_{\mathcal{B}\to\mathcal{E}}, [id]_{\mathcal{C}\to\mathcal{E}}, [id]_{\mathcal{E}\to\mathcal{B}}, [id]_{\mathcal{E}\to\mathcal{C}}, [id]_{\mathcal{B}\to\mathcal{C}}, [id]_{\mathcal{C}\to\mathcal{E}}.$$

3. Consider the transformations S and T and the bases B and E from Q1. Find the following matrices of transition from basis to basis:

$$[id]_{\mathcal{B}\to\mathcal{E}}, [id]_{\mathcal{E}\to\mathcal{B}}$$

Check that the formula of transition from basis to basis holds in the following cases:

$$[T]_{\mathcal{B}} = [id]_{\mathcal{E}\to\mathcal{B}}[T]_{\mathcal{E}}[id]_{\mathcal{B}\to\mathcal{E}}$$
$$[S]_{\mathcal{E}} = [id]_{\mathcal{B}\to\mathcal{E}}[S]_{\mathcal{B}}[id]_{\mathcal{E}\to\mathcal{B}}$$

4. Consider curve defined by $49x^2 - 30\sqrt{3}xy + 19y^2 = 64$ on $\mathbb{R}^2$.

(a) Show that with respect to basis

$$\mathcal{B} =< \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} >$$

the curve is an ellipse.

(b) Show that with respect to basis

$$\mathcal{C} =< \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} >$$

the curve is the unit circle.

5. True or false. Remember to justify your answer.
   (a) There exists a non-zero upper-triangular matrix $A \in M_2(\mathbb{R})$ such that A^2 is the zero matrix.
   (b) Let $A \in M_n(\mathbb{R})$. If $AB = BA$ for every $B \in M_n(\mathbb{R})$ then $B = \lambda I_n$ for some $\lambda \in \mathbb{R}$.
   (c) Let $A, B \in M_n(\mathbb{R})$. Then AB is invertible if and only if both A and B are invertible.
   (d) Let $A \in M_n(\mathbb{R})$. A is NOT invertible if and only if there exists $B \in M_n(\mathbb{R})$ such that $AB = 0$.
   (e) Let $A, B \in M_n(\mathbb{R})$. If both A and B are invertible then $AB = BA$.
   (f) Let $A \in M_n(\mathbb{R})$. If A is invertible then $A + I$ is also invertible.
   (g) If $A^2 - I$ is invertible then $A - I$ is invertible.

6. In class we mentioned that $< A, B >_1 = tr(AB^T)$ defines an inner product on on $M_{m\times n}(\mathbb{R})$ and in studio covered that $< A, B >_2 = tr(A^T B)$ is an inner product. Is there a typo in terms of where the transpose operation is on?

7. Let $M \in M_n(\mathbb{R})$. Characterize M such that $< \cdot, \cdot >_M : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $< x, y >_M = (Mx)^T(My)$ is an inner product. Justify your answer.

8. Consider the subspace of $\mathbb{R}^4$

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a - b + c - d = 0 \text{ and } a + d = 0 \right\}$$

of $\mathbb{R}^4$.
   (a) Find a basis for U.
   (b) Find an orthonormal basis $\mathcal{C}$ for U.
   (c) Let $x = (1, 2, 3, 4)^T \in \mathbb{R}^4$. Find the orthogonal projection of x onto the space U : $\mathrm{Proj}_U\, x$.
   (d) Find an orthonormal basis of $\mathbb{R}^4$ that contains the vectors from $\mathcal{C}$ from (b).
   (e) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the basis that you obtained from (d).
   (f) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the standard basis of $\mathbb{R}^4$.
   (g) Use the answer from (f) to calculate $\mathrm{Proj}_U\, x$

1. Consider the following linear transformations $T, S : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ given by

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and

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(a) Find $[T]_{\mathcal{E}\to\mathcal{B}}$, $[T]_{\mathcal{B}\to\mathcal{B}}$, $[T]_{\mathcal{B}\to\mathcal{E}}$, $[T]_{\mathcal{E}\to\mathcal{E}}$. x^2+1+2x

(b) Find $[S]_{\mathcal{E}\to\mathcal{E}}$, $[S]_{\mathcal{B}\to\mathcal{B}}$.

(c) Find $[T \circ S]_{\mathcal{E}\to\mathcal{E}}$.

(d) $[T \circ S]_{\mathcal{B}\to\mathcal{B}}$.

(e) Use $[T]_{\mathcal{E}\to\mathcal{E}}$ to find a basis for the kernel and image of L.

$$(x^2+1+2x)(1+x) = x^2 + x^3 + 1 + x + 2x + 2x^2 = x^3 + 3x^2 + 3x + 1$$

a. $[T]_{\mathcal{E}\to B} = \begin{pmatrix} | & & | \\ T[\mathcal{E}_1]_B & \cdots & T[\mathcal{E}_n]_B \\ | & & | \end{pmatrix}$

$= ([0]_B, [1]_B, [2x]_B, [3x^2]_B) = \begin{pmatrix} 0 & 1 & -2 & 3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$[T]_{B\to B} = \begin{pmatrix} | & & | \\ T[B_1]_B & \cdots & T[B_n]_B \\ | & & | \end{pmatrix}$

$= ([0]_B, [1]_B, [2x+2]_B, [3x^2+6x+3]_B)$

$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$[T]_{B\to\mathcal{E}} = \begin{pmatrix} | & & | \\ T[B_1]_\mathcal{E} & \cdots & T[B_n]_\mathcal{E} \\ | & & | \end{pmatrix} = ([0]_\mathcal{E}, [1]_\mathcal{E}, [2x+2]_\mathcal{E}, [3x^2+6x+3]_\mathcal{E})$

$= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$[T]_{\mathcal{E}\to\mathcal{E}} = \begin{pmatrix} | & & | \\ T[\mathcal{E}_1]_\mathcal{E} & \cdots & T[\mathcal{E}_n]_\mathcal{E} \\ | & & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

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(e) Use $[T]_{\mathcal{E}\to\mathcal{E}}$ to find a basis for the kernel and image of L.

$1 + x^2 + 2x$

$x^3 + 3x^2 + 3x + 1$

$(x+1)^2 = (x^2+2x+1)(x+1)$
$= x^3 + x^2 + 2x^2 + 2x + x + 1$
$= x^3 + 3x^2 + 3x + 1$

$(x+2)(x^2+4x+4)$
$= x^3 + 4x^2 + 4x + 2x^2 + 8x + 8$
$= x^3 + 6x^2 + 12x + 8$

b. $[S]_{\mathcal{E}\to\mathcal{E}} = \begin{pmatrix} | & & | \\ S[\mathcal{E}_1]_{\mathcal{E}} & \cdots & S[\mathcal{E}_n]_{\mathcal{E}} \\ | & & | \end{pmatrix}$

$= \left([1]_{\mathcal{E}}, [1+x]_{\mathcal{E}}, [x^2+2x+1]_{\mathcal{E}}, [x^3+3x^2+3x+1]_{\mathcal{E}} \right)$

$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$[S]_{\mathcal{B}\to\mathcal{B}} = \begin{pmatrix} | & & | \\ S[B_1]_{B} & \cdots & S[B_n]_{B} \\ | & & | \end{pmatrix}$

$= \left([1]_B, [x+2]_B, [x^2+4x+4]_B, [x^3+6x^2+12x+8]_B \right)$

$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

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c. $[T \circ S]_{\mathcal{E}\to\mathcal{E}} = T_{\mathcal{B}\to\mathcal{E}} S_{\mathcal{E}\to\mathcal{B}}$

$$T_{\mathcal{B}\to\mathcal{E}} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_{\mathcal{E}\to\mathcal{B}} = \begin{pmatrix} | & & | \\ S[\mathcal{E}_1]_{\mathcal{B}} & \cdots & S[\mathcal{E}_n]_{\mathcal{B}} \\ | & & | \end{pmatrix}$$

$$= \left([0]_{\mathcal{B}}, [1+x]_{\mathcal{B}}, [x^2+2x+1]_{\mathcal{B}}, [x^3+3x^2+3x+1]_{\mathcal{B}} \right)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\therefore [T \circ S]_{\mathcal{E}\to\mathcal{E}} = T_{\mathcal{B}\to\mathcal{E}} S_{\mathcal{E}\to\mathcal{B}} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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(e) Use $[T]_{\mathcal{E}\to\mathcal{E}}$ to find a basis for the kernel and image of L.

x^2+1+2x

$(x^2+2x+1)(1+x)$

$= x^2+x^3+2x+2x^2$
$+1+x$

x^3+3x^2+3x+1

$3x^2+6x+3$

$3x+3$

d. $[T\circ S]_{B\to B} = T_{\mathcal{E}\to B}\, S_{B\to\mathcal{E}}$.

$$T_{\mathcal{E}\to B} = \begin{pmatrix} 0 & 1 & -2 & -3 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_{B\to\mathcal{E}} = \left([1]_B,\ [x+1]_B,\ [x^2+4x+4]_B,\ [x^3+6x^2+12x+8]_B\right)$$

$$= \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$2-2$

$\therefore [T\circ S]_{B\to B} = T_{\mathcal{E}\to B}\, S_{B\to\mathcal{E}}$

$$= \begin{pmatrix} 0 & 1 & -2 & -3 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1. Consider the following linear transformations $T, S : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ given by

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(c) Find $[T \circ S]_{\mathcal{E}\to\mathcal{E}}$.

(d) $[T \circ S]_{\mathcal{B}\to\mathcal{B}}$.

(e) Use $[T]_{\mathcal{E}\to\mathcal{E}}$ to find a basis for the kernel and image of L.

$$[T]_{\Sigma\to\Sigma} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{kernel}(T) = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \text{span}\{(1)\}.$$

$$\text{image}(T) = \text{span}\{1, x, x^2\}.$$

2. Consider the following ordered bases of $\mathbb{R}^3$:

$$\mathcal{B} = < \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} >$$

$$\mathcal{C} = < \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} >$$

$$\mathcal{E} = < \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} >$$

Find the following matrices of transition from basis to basis:

$$\underline{[id]_{\mathcal{B}\to\mathcal{E}}}, \underline{[id]_{\mathcal{C}\to\mathcal{E}}}, [id]_{\mathcal{E}\to\mathcal{B}}, [id]_{\mathcal{E}\to\mathcal{C}}, [id]_{\mathcal{B}\to\mathcal{C}}, \underline{[id]_{\mathcal{C}\to\mathcal{E}}}.$$ the same.

$$[id]_{B\to\mathcal{E}} = \begin{pmatrix} | & & | \\ id[b_1]_{\mathcal{E}} & \cdots & id[b_n]_{\mathcal{E}} \\ | & & | \end{pmatrix}$$

$$= \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{E}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}_{\mathcal{E}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}_{\mathcal{E}} \right) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$[id]_{C\to\mathcal{E}} = \left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}_{\mathcal{E}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{E}} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}_{\mathcal{E}} \right) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

$$[id]_{\mathcal{E}\to B} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{B} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{B} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{B} \right) = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$[id]_{\mathcal{E}\to C} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{C} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{C} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{C} \right) = \begin{pmatrix} -\frac{3}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix}$$

$$[id]_{B\to C} = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{C} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}_{C} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}_{C} \right) = \begin{pmatrix} \frac{3}{8} & \frac{7}{8} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{5}{8} & \frac{1}{4} \end{pmatrix}$$

$$[id]_{C\to\mathcal{E}} = \left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}_{\mathcal{E}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{E}} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}_{\mathcal{E}} \right) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

3. Consider the transformations S and T and the bases B and E from Q1. Find the following matrices of transition from basis to basis:

$$[id]_{\mathcal{B}\to\mathcal{E}},\ [id]_{\mathcal{E}\to\mathcal{B}}$$

Check that the formula of transition from basis to basis holds in the following cases:

$$[T]_{\mathcal{B}} = [id]_{\mathcal{E}\to\mathcal{B}}[T]_{\mathcal{E}}[id]_{\mathcal{B}\to\mathcal{E}}$$
$$[S]_{\mathcal{E}} = [id]_{\mathcal{B}\to\mathcal{E}}[S]_{\mathcal{B}}[id]_{\mathcal{E}\to\mathcal{B}}$$

$$\mathcal{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[id]_{\mathcal{B}\to\mathcal{E}} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{E}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{E}} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{E}} \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}_{\mathcal{E}} \right) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[id]_{\mathcal{E}\to\mathcal{B}} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{B}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_{\mathcal{B}} \right)$$

$$= \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(sanity check: $[id]_{\mathcal{E}\to\mathcal{B}} == [id]^{-1}_{\mathcal{B}\to\mathcal{E}}$)

$$T_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad T_{\mathcal{E}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

RHS =

$[id]_{\mathcal{E}\to\mathcal{B}}[T]_{\mathcal{E}}[id]_{\mathcal{B}\to\mathcal{E}} =$

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -2 & 3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = [T]_{\mathcal{B}}$$

$= \text{LHS}.$

$\therefore \checkmark$

$$[S]_{\mathcal{E}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad [S]_{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{RHS} = [id]_{B \to \mathcal{E}} [S]_B [id]_{\mathcal{E} \to B}.$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = [S]_{\mathcal{E}} = \text{LHS} \quad \therefore \checkmark$$

4. Consider curve defined by $49x^2 - 30\sqrt{3}xy + 19y^2 = 64$ on $\mathbb{R}^2$.

(a) Show that with respect to basis

$$\mathcal{B} =< \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} >$$

the curve is an ellipse.

(b) Show that with respect to basis

$$\mathcal{C} =< \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} >$$

the curve is the unit circle.

(a)

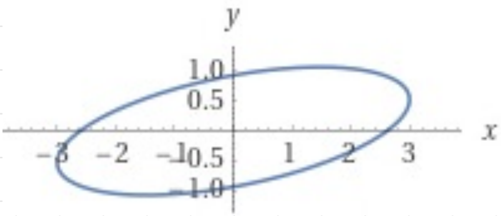
$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} + \beta \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$$

Sub x and y in, check if ellipse.

$$\begin{cases} x = \frac{1}{2}\alpha - \frac{\sqrt{3}}{2}\beta \\ y = \frac{\sqrt{3}}{2}\alpha + \frac{1}{2}\beta \end{cases}$$

$$49\left(\tfrac{1}{2}\alpha - \tfrac{\sqrt{3}}{2}\beta\right)^2 - 30\left(\tfrac{1}{2}\alpha - \tfrac{\sqrt{3}}{2}\beta\right)\left(\tfrac{\sqrt{3}}{2}\alpha + \tfrac{1}{2}\beta\right) + 19\left(\tfrac{\sqrt{3}}{2}\alpha + \tfrac{1}{2}\beta\right)^2 = 64$$

I graphed it in a calculator →



and it's indeed an ellipse.

(b) Show that with respect to basis

$$\mathcal{C} = < \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} >$$

the curve is the unit circle.

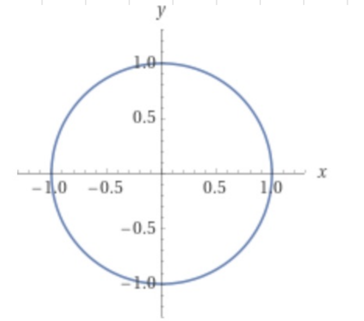
let $\begin{bmatrix} x \\ y \end{bmatrix} = \left(\alpha \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}, \beta \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \right)$

$x = 2\alpha - \frac{\sqrt{3}}{2}\beta$

$y = 2\sqrt{3}\alpha + \frac{1}{2}\beta$.

plug in :

$$49\left(2\alpha - \frac{\sqrt{3}}{2}\beta\right)^2 - 30\sqrt{3}\left(2\alpha - \frac{\sqrt{3}}{2}\beta\right)\left(2\sqrt{3}\alpha + \frac{1}{2}\beta\right) + 19\left(2\sqrt{3}\alpha + \frac{1}{2}\beta\right)^2 = 64$$



Simplifying this will produce a unit circle.

← Also shown by graphing calculator.

5. True or false. Remember to justify your answer.

(a) There exists a non-zero upper-triangular matrix $A \in M_2(\mathbb{R})$ such that A^2 is the zero matrix.

(b) Let $A \in M_n(\mathbb{R})$. If $AB = BA$ for every $B \in M_n(\mathbb{R})$ then $A = \lambda I_n$ for some $\lambda \in \mathbb{R}$.

(c) Let $A, B \in M_n(\mathbb{R})$. Then AB is invertible if and only if both A and B are invertible.

(a) true: let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) Let $B = E_{i,j}$ (1 at (i,j), 0 at everywhere else).

$$AE_{i,j} = \begin{pmatrix} | & & | & & | \\ 0 & \cdots & a_{\bullet,i} & \cdots & 0 \\ | & & | & & | \end{pmatrix}$$

$$E_{ij}A = \begin{pmatrix} — & 0 & — \\ & \vdots & \\ — & a_{j,\bullet} & — \\ & \vdots & \\ — & 0 & — \end{pmatrix}$$

If $AE_{i,j} == E_{i,j}A$, then $a_{\bullet,i} == a_{j,\bullet}$.
So A must be diagonal.

Let $B = E_{i,i}$ $\quad AE_{i,i} == E_{i,i}A \Rightarrow$ every entry in the diagonal is the same. $\therefore A = \lambda I_n$. $\therefore$ true.

(c) ① If A or B isn't invertible, that means A or B is not full rank and there's a free variable in that matrix. Therefore, AB would have free variable, and thus AB wouldn't be invertible. So AB is invertible implies A and B are invertible.

(Another proof: $AB(AB)^{-1} = I$ $\quad \therefore A(B(AB)^{-1}) = I$.

$\therefore B(AB)^{-1} = A^{-1}$.

Same for B.)

② If both A and B have inverses A^{-1} and B^{-1},

$AA^{-1} = I$ and $BB^{-1} = I$.

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = I$.

$\therefore (B^{-1}A^{-1})$ is a valid inverse for (AB).

$\therefore$ true.

(d) Let $A \in M_n(\mathbb{R})$. A is NOT invertible if and only if there exists $B \in M_n(\mathbb{R})$ such that $AB = 0$.

let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ but A is invertible

$\therefore$ proved false by counter example.

(e) Let $A, B \in M_n(\mathbb{R})$. If both A and B are invertible then $AB = BA$.
(f) Let $A \in M_n(\mathbb{R})$. If A is invertible then $A + I$ is also invertible.
(g) If $A^2 - I$ is invertible then $A - I$ is invertible.

(e) let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. $A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$

$BA = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

$AB \neq BA$ $\therefore$ false.

(f) let $A = \begin{pmatrix} -3 & 2 \\ 5 & -6 \end{pmatrix}$. A is invertible.

However, $A + I = \begin{pmatrix} -2 & 2 \\ 5 & -5 \end{pmatrix}$, and $A + I$ is not invertible.

$\therefore$ proved false by counter example.

(g) $(A^2 - I)$ is invertible $\Rightarrow \exists (A^2 - I)^{-1}$ s.t. $(A^2 - I)^{-1}(A^2 - I) = I$.

Since matrix of same dimension is distributive.

$(A^2 - I) = (A + I)(A - I)$.

$\therefore (A^2 - I)(A^2 - I)^{-1}$

$= (A + I)(A - I)(A^2 - I)^{-1}$

$\therefore (A + I)(A^2 - I)^{-1}$ is the inverse of $(A - I)$.

6. In class we mentioned that $< A, B >_1 = tr(AB^T)$ defines an inner product on on $M_{m\times n}(\mathbb{R})$ and in studio covered that $< A, B >_2 = tr(A^T B)$ is an inner product. Is there a typo in terms of where the transpose operation is on?

$tr(A) = tr(A^T)$.

$tr(AB) = tr(BA)$

$\therefore \langle A, B\rangle_2 = tr(A^T B) = tr((A^T B)^T) = tr(B^T A^{TT})$

$= tr(B^T A) = tr(AB^T) = \langle A, B\rangle_1$

so they are actually the same, so it's not a typo.

7. Let $M \in M_n(\mathbb{R})$. Characterize M such that $< \cdot, \cdot >_M: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $< x, y >_M = (Mx)^T(My)$ is an inner product. Justify your answer.

If $\langle x, y \rangle_M = (Mx)^T(My)$ is an inner product,
it has to pass the following checks:
① $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in \bar{V}$
② $\langle x, ky \rangle = k\langle x, y \rangle$ for all $x, y \in \bar{V}$, $k \in \mathbb{R}$
③ $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in \bar{V}$
④ $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.

①. $(Mx)^T(My) = (My)^T(Mx)$
This is always satisfied since M is a square matrix.

② $(Mx)^T(M\lambda y) = \lambda (Mx)^T(My)$
③ $\langle x, y+z \rangle_M = \langle x, y \rangle_M + \langle x, z \rangle_M$.
$(Mx)^T(M(y+z)) = (Mx)^T(My) + (Mx)^T(Mz)$
④ $(Mx)^T(Mx) \geq 0 \Rightarrow M^T M \geq 0$

If $(Mx)^T(Mx) = 0$ implies $x = 0$, $\ker(M) = \{\vec{0}\}$.
Since M is a square matrix, it's all following:

① A is invertible.
② $\exists\, C \in M_n(\mathbb{R})$ s.t. $CA = I_n$.
③ $\exists\, B \in M_n(\mathbb{R})$ s.t. $AB = I_n$
④ A is row equivalent to I_n
⑤ RREF of A has n - pivots.
⑥ rank of $A = n$
⑦ $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is injective
⑧ $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is surjective
⑨ $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is bijective
⑩ Columns of A form a basis of $\mathbb{R}^n$ ($\mathrm{Col}(A) = \mathbb{R}^n$)
⑪ $\mathrm{Nul}(A) = \{\vec{0}\}$.

8. Consider the subspace of $\mathbb{R}^4$

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a - b + c - d = 0 \text{ and } a + d = 0 \right\}$$

$a =$

of $\mathbb{R}^4$.

$a + c = b + d.$

$a = -d$

(a) Find a basis for U.

(b) Find an orthonormal basis $\mathcal{C}$ for U.

(c) Let $x = (1, 2, 3, 4)^T \in \mathbb{R}^4$. Find the orthogonal projection of x onto the space U : $\operatorname{Proj}_U x$.

(d) Find an orthonormal basis of $\mathbb{R}^4$ that contains the vectors from $\mathcal{C}$ from (b).

(e) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the basis that you obtained from (d).

(f) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the standard basis of $\mathbb{R}^4$.

(g) Use the answer from (f) to calculate $\operatorname{Proj}_U x$

(a.) $\begin{cases} a = -t \\ b = s - 2t \\ c = s \\ d = t \end{cases}$ basis: $\operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(b). $q_1 = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$. Using the G.S : $q_2 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$\therefore$ orthonormal basis $C = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}$.

(c) $X = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ $\vec{u}_1 = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$ $\vec{u}_2 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$$\operatorname{Proj}_{\vec{u}_1} \vec{X} = \frac{\langle \vec{u}_1, \vec{x} \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1 = \frac{2\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} \vec{u}_1 = \frac{5\sqrt{2}}{2} \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{5}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix}$$

$$\operatorname{Proj}_{\vec{u}_2} \vec{X} = \frac{\langle \vec{u}_2, \vec{x} \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle} \vec{u}_2 = 2 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{Proj}_u X = \begin{pmatrix} 0 \\ \frac{5}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{pmatrix}$$

(d) We need to find 4 linearly independent vectors
2 could be $\begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. Another 2 could be $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

The first 2 vectors are already orthogonal and normal.
Now is to orthogonalize and normalize $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

$$q_1 = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \quad \text{and} \quad q_2 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$q_3 = \frac{v_3 - \langle v_3, q_1 \rangle q_1 - \langle v_3, q_2 \rangle q_2}{\text{norm of denominator}} = \frac{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}}{\cdot\cdot}$$

$$= \text{normalized} \begin{pmatrix} \frac{3}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}. \quad \text{norm} = \sqrt{\frac{9}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}} = \frac{12}{16} = \frac{3}{4}.$$
$$= \frac{\sqrt{3}}{2}$$

$$\begin{pmatrix} \frac{3}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \Big/ \frac{\sqrt{3}}{2} = \begin{pmatrix} \frac{3}{4} \times \frac{2}{\sqrt{3}} \\ -\frac{1}{4} \times \frac{2}{\sqrt{3}} \\ \frac{1}{4} \times \frac{2}{\sqrt{3}} \\ \frac{1}{4} \times \frac{2}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \end{pmatrix}$$

$$q_4 = \frac{v_4 - \langle v_4, q_1 \rangle q_1 - \langle v_4, q_2 \rangle q_2 - \langle v_4, q_3 \rangle q_3}{\text{norm of denominator.}}$$

$$= \frac{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - 0 - \frac{1}{2}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - \frac{\sqrt{3}}{6}\begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \end{pmatrix}}{\cdot\cdot} = \frac{\begin{pmatrix} 0 \\ \frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}}{\frac{\sqrt{3}}{3}} = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{pmatrix}$$

$\therefore$ orthonormal basis $Q = \left\{ \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{2}{\sqrt{3}\sqrt{2}} \\ \frac{1}{\sqrt{3}\sqrt{2}} \\ -\frac{\sqrt{3}}{3} \\ 0 \end{pmatrix} \right\}.$

(d) Find an orthonormal basis of $\mathbb{R}^4$ that contains the vectors from $\mathcal{C}$ from (b).

(e) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the basis that you obtained from (d).

(f) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space U with respect to the standard basis of $\mathbb{R}^4$.

(g) Use the answer from (f) to calculate $\text{Proj}_U\, x$

(e)

$$\text{Proj}_{\vec{u}}(q) = \left([P(q_1)]_e \;\; [P(q_2)]_e \;\; \cdots \;\; [P(q_4)]_e \right)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(f)

$$\text{Proj}_{\vec{u}}(\cdot) = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

(g)

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{2}{4} - \frac{3}{4} - \frac{4}{4} \\ \frac{1}{4} + \frac{6}{4} + \frac{3}{4} - \frac{4}{4} \\ -\frac{1}{4} + \frac{2}{4} + \frac{9}{4} + \frac{4}{4} \\ -\frac{1}{4} - \frac{2}{4} + \frac{3}{4} + \frac{4}{4} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{4} \\ \frac{6}{4} \\ \frac{14}{4} \\ \frac{4}{4} \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{pmatrix}$$