MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 5

1. Answer the following questions.
   (a) Do elementary row operations affect a matrix's column space? Justify your response.
   (b) Do elementary row operations affect a matrix's null space? Justify your response.

2. Find a basis and then the dimension of the following subspace:

$$\text{span}\{2 + x^2 - 2x^3, 1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}$$

3. Let V be a vector space over $\mathbb{R}$ and $U, W \subseteq V$ be two subspaces of V.
   (a) Prove that there exist a basis B of U and a basis C of W such that $B \cap C$ is a basis for $U \cap W$.
   (b) Is it true that for every basis B of U and every basis C of W the set $B \cap C$ is a basis for $U \cap W$?
   (c) Recall the definition of $U + W$. Prove the following dimension formula:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

   (d) Let $U, W \subset \mathbb{R}_4[x]$ be two subspaces which satisfy $\dim(U) = \dim(W) = 3$ prove that $U \cap W \neq \{\mathbf{0}\}$.
   (e) Find the dimension of the following space:

$$\text{span}\{\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}\} \cap \text{span}\{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}\}$$

4. Let V, W be vector spaces over $\mathbb{R}$. Consider $(\mathcal{L}(V, W), +, \cdot)$ where $+$ is defined as

$$(f_1 + f_2)(v) = f_1(v) + f_2(v), \forall v \in V$$

for any two linear maps $f_1, f_2 \in \mathcal{L}(V, W)$ and $\cdot$ is defined to be

$$(kf)(v) = k(f(v), \forall v \in V$$

where k is any scalar from $\mathbb{R}$ and f is any linear map in $\mathcal{L}(V, W)$. Show that $(\mathcal{L}(V, W), +, \cdot)$ is a vector space over $\mathbb{R}$.

5. Let $T \in \mathcal{L}(V, W)$. Show that
   (a) $\ker T$ is a subspace of V.
   (b) $\text{Im} T$ is a subspace of W.
   (c) If $T \in \mathcal{L}(V, V)$, is it possible that $\ker T \cap \text{Im} T \neq \{\mathbf{0}\}$? Explain your answer.

6. In each of the following you are given two vector spaces and a function between them. Determine whether the function is a linear transformation or not. Prove your claim.

(a)

$$T : \mathbb{R}^3 \to M_2(\mathbb{R})$$

given by,

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y & y-2z \\ 3x+z & 0 \end{pmatrix}$$

(b)

$$T : \mathbb{R}_2[x] \to \mathbb{R}^3$$

given by,

$$Tp = \begin{pmatrix} p(2) \\ p'(2) \\ p''(2) \end{pmatrix}$$

(c)

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$

given by,

$$TA = A^2$$

(d) Fix $B \in M_3(\mathbb{R})$ and consider the function:

$$T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$$

given by,

$$TA = AB$$

(e)

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$

given by,

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c+1 & 2a+3b+2 \\ d-b-8 & 2a \end{pmatrix}$$

7. In each of the following you are given a linear transformation (you don't need to prove that it is a linear transformation). Follow the following directions for each such transformation:

- Find a basis for the kernel and the image of this transformation.
- Find the dimension of the kernel and the image of this transformation. (Remark: This question will continue in the next HW, you may want to keep a copy of your solution to this part of the question).
- Determine whether the transformation is surjective. Explain your answer.
- Determine whether the transformation is injective. Explain your answer.

(a) For

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 7 & -6 \\ 0 & 5 & -5 \end{pmatrix}$$

consider the linear map

$$T_A : \mathbb{R}^3 \to \mathbb{R}^4$$

where the notation T_A was defined in class.

(b)

$$S : M_2(\mathbb{R}) \to \mathbb{R}^2$$

given by

$$SA = A \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

(c)

$$L : \mathbb{R}_3[x] \to \mathbb{R}^2$$

given by

$$Lp = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}$$

(d)

$$\Phi : \mathbb{R}^3 \mapsto \mathbb{R}_3[x]$$

given by

$$\Phi \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b) + (a-2b+c)x + (b-3c)x^2 + (a+b+c+d)x^3$$

1. Answer the following questions.

   (a) Do elementary row operations affect a matrix's column space? Justify your response.

   (b) Do elementary row operations affect a matrix's null space? Justify your response.

let $A \in M_{2\times 2}$. $= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(a). $Col(A) = \{ y \in \mathbb{R}^m \mid Ax = y$. for some $x \in \mathbb{R}^n \}$

$\begin{array}{c} R_1: \\ R_2: \end{array} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$ $\quad col(A) = \{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \}$

if we change A with row operation like $R_2 \leftarrow R_2 - R_1$.

$A' = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ $\quad Col(A') = \{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \}$

$\therefore$ $col(A) \neq col(A')$

$\therefore$ yes, elementary row operations can affect a matrix's column space.

(b). For $A \in M_{m\times n}$, $Null(A) = \{ x \in \mathbb{R} \mid Ax = \vec{0}_{\mathbb{R}^m} \}$

Since row operations do not change row space of matrix, so there exists a matrix N, s.t. NA is A's echelon form.

$MA(x) = M(Ax) = M0 = 0$. ($y \in Null(A)$).

Let $y \in Null(A)$. $Ay = N^{-1}(N(Ay)) = N^{-1}0 = 0$.

$\therefore$ null space of A and NA are the same.

$\therefore$ Row operation doesn't change the null space.

2. Find a basis and then the dimension of the following subspace:

$$\text{span}\{2 + x^2 - 2x^3, 1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}$$

Put the span in matrix form:

$$\begin{pmatrix} -2 & -1 & -5 & -3 \\ 1 & 1 & 2 & 0 \\ 0 & -2 & 2 & 6 \\ 2 & 1 & 5 & 3 \end{pmatrix}$$

Row reduce it:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & -2 & 2 & 6 \\ 0 & -1 & 1 & 3 \end{pmatrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{these three rows are the same.}$$

$$\therefore \Rightarrow \begin{array}{cccc} \text{pivot} & \text{pivot} & & \\ \downarrow & \downarrow & & \end{array}$$

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\therefore$ The first 2 polynomials form the basis:

span $\{(2 + x^2 - 2x^3), (1 - 2x + x^2 - x^3)\}$

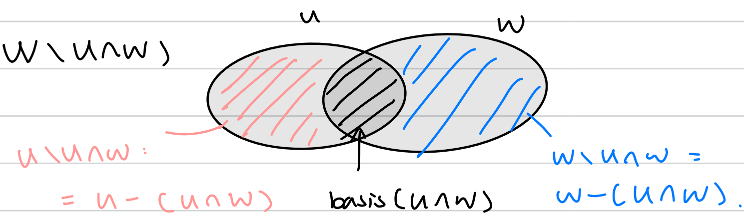
$\therefore$ dimension $= |\text{span}| = 2$.

3. Let V be a vector space over $\mathbb{R}$ and $U, W \subseteq V$ be two subspaces of V.

(a) Prove that there exist a basis B of U and a basis C of W such that $B \cap C$ is a basis for $U \cap W$.

(a.) let D be the basis for $u \cap w$.
construct the basis B from D for $u \in u \cap w$,
and construct the basis C from D for $W \in w \cap w$.
B and C exist, and $B \cap C$ exist and equals to D.

$D = B \cup C \setminus (u \setminus u \cap w) \setminus (w \setminus u \cap w)$



(b) Is it true that for every basis B of U and every basis C of W the set $B \cap C$ is a basis for $U \cap W$?

b. Let $B_1 = \{e_1, e_2\}$,
$B_2 = \{e_1 + e_2, e_1 - e_2\}$.
They both spans $\mathbb{R}^2$, but $B_1 \cap B_2 = \{0\}$
and $\{0\}$ doesn't span anything.
$\therefore$ proved false by counter-example.

(c) Recall the definition of $U + W$. Prove the following dimension formula:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

let

$V = (U+W) = \{u + w \mid u \in U$ and $w \in W\}$.

$\forall$ $v \in V$, $v =$ a linear combination of u and w $=$

$\sum_i \alpha_i u_i + \sum_j \beta_j w_j$, which is the $\dim(U) + \dim(W)$ part.

$(U+W)$ has a basis of $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$.

However, all the basis elements in U or W would exist

in $U \vee W$, so the repeated vectors, $(U \cap W)$, should

be eliminated. (so the $\dim(U \cap W)$ should be

subtracted).

$\therefore \dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W.)$

(d) Let $U, W \subset \mathbb{R}_4[x]$ be two subspaces which satisfy $\dim(U) = \dim(W) = 3$ prove that $U \cap W \neq \{\mathbf{0}\}$.

(e) Find the dimension of the following space:

$$\text{span}\{\begin{pmatrix}1\\1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\-3\\1\end{pmatrix}\} \cap \text{span}\{\begin{pmatrix}0\\1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\2\\4\\1\end{pmatrix}\}$$

d. $\mathbb{R}_4[x]$'s basis is $\{1, x, x^2, x^3\}$

Since $\dim(U) = \dim(W) = 3$, U and W contain 3 elements of the 4 elements in the basis.

If they are the same 3 elements, $U \cap W =$ these 3 elements from the basis. If they are two different sets of 3 elements, then $U \cap W =$ the two overlapping elements from the basis. $\therefore U \cap W \neq \{0\}$.

e. $$\text{span}\{\begin{pmatrix}1\\1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\-3\\1\end{pmatrix}\} \cap \text{span}\{\begin{pmatrix}0\\1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\\3\end{pmatrix}, \cancel{\begin{pmatrix}1\\2\\4\\1\end{pmatrix}}\}$$

$$= \text{span}\{\begin{pmatrix}1\\1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\-3\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\\3\end{pmatrix}\}.$$

$= 4.$

4. Let V, W be vector spaces over $\mathbb{R}$. Consider $(\mathcal{L}(V,W), +, \cdot)$ where $+$ is defined as

$$(f_1 + f_2)(v) = f_1(v) + f_2(v), \forall v \in V$$

for any two linear maps $f_1, f_2 \in \mathcal{L}(V,W)$ and $\cdot$ is defined to be

$$(kf)(v) = k(f(v), \forall v \in V$$

where k is any scalar from $\mathbb{R}$ and f is any linear map in $\mathcal{L}(V,W)$. Show that $(\mathcal{L}(V,W), +, \cdot)$ is a vector space over $\mathbb{R}$.

Need to check the 10 Axioms.

① closed under addition: for f_1 and $f_2 \in \mathcal{L}(V,W)$,
$(f_1+f_2)(v) = f_1(v) + f_2(v) \quad \forall v \in V$
f_1 and f_2 are linear maps, $\therefore$ f_1+f_2 is also linear map $\mathcal{L}$.
$\therefore \checkmark$

② closed under scalar multiplication: For $k \in \mathbb{R}$, and $f \in \mathcal{L}(V,W)$, $(kf) =$
$kf(v) = f(kv)$. $\forall v \in V$,

③ commutative: $(f_1+f_2)(v) = f_1(v) + f_2(v) = f_2(v) + f_1(v) \quad \therefore \checkmark$

④ associative: $((f_1+f_2)+f_3)(v) = (f_1+f_2)(v) + f_3(v) = f_1(v) + f_2(v)$
$+ f_3(v) = f_1(v) + (f_2+f_3)(v) = (f_1+(f_2+f_3))(v)$.
$\therefore \checkmark$

⑤ $\forall f \in \mathcal{L}(V,W)$, $\exists -f$ s.t. $(-f)(v) = -f(v) \quad \therefore \checkmark$

⑥ $\exists \vec{0}$ in $\mathbb{R}$, s.t. $\forall v \in V$, $(0)(v) = 0$. $\therefore \checkmark$

⑦ scalar multiplication is associative: $\forall a, b \in \mathbb{R}$ and $\forall f \in \mathcal{L}(V,W)$,
$(a \cdot b \cdot f)(v) = a(b \cdot f(v))$ by definition of $\cdot$. $\therefore \checkmark$

⑧ $\forall f \in \mathcal{L}(V,W)$, $(1 \cdot f)(v) = 1 \cdot f(v) = f(v)$.

⑨ $\forall \alpha, \beta \in \mathbb{R}$, $(\alpha+\beta) \times f(v) = [(\alpha+\beta)f](v) = (\alpha f)(v) + (\beta f)(v)$. $\therefore \checkmark$

⑩ $\forall \alpha \in \mathbb{R}$, and $f_1, f_2 \in \mathcal{L}(V,W)$, $\alpha \cdot (f_1+f_2)(v) = (\alpha f_1 + \alpha f_2)(v)$
$= \alpha f_1(v) + \alpha f_2(v)$. $\therefore \checkmark$

$\therefore$ $\boxed{\text{it's indeed a V.S.}}$

5. Let $T \in \mathcal{L}(V, W)$. Show that

(a) ker$T$ is a subspace of V.

(b) ImT is a subspace of W.

a Need to show : ① kerT is not empty

② close under addition

③ close under multiplication

① $T(0) = 0$. ✓

② let $a, b \in \ker T$

$T(a) = 0$, $T(b) = 0$. $\therefore$ $T(a) + T(b) = 0 + 0 = 0$.

$\therefore$ $a + b \in \ker T$. ✓

③ let $a \in \ker T$, $c \in \mathbb{R}$.

$ca = c \cdot 0 = 0$. $\therefore$ $ca \in \ker T$.

$\therefore$ kerT is a subspace of V

b. Need to show : ① ImT is not empty

② close under addition

③ close under multiplication

① $T(\vec{0}_v) = \vec{0}_w$ so there exists the 0 vector. $\therefore$ ✓

② Because $\mathcal{L}(V, W)$ is a linear transformation, for $a, b \in V$,

$T(a) + T(b) = T(a+b)$

$\therefore$ ✓

③ Because $\mathcal{L}(V, W)$ is a linear transformation, for $a \in V$,

$c \in \mathbb{R}$, $cT(a) = T(ca)$ $\therefore$ ✓

(c) If $T \in \mathcal{L}(V, V)$, is it possible that $\ker T \cap \operatorname{Im} T \neq \{\mathbf{0}\}$? Explain your answer.

C. Yes. It is possible, proved by one example.

define $T \in \mathcal{L}(V, V)$, where $T\left(s\binom{1}{0} + r\binom{0}{1}\right) = \left(s\binom{0}{0} + r\binom{1}{0}\right)$.

This linear map sends $T\binom{1}{0}$ to $\binom{0}{0}$, so by definition, $\binom{1}{0}$ is in the kernel of T. Since $\binom{1}{0}$ is a possible output of the linear map, it's also in the Image of T.

$\therefore$ In this case, $\ker T \cap \operatorname{Im} T = \binom{1}{0} \neq \{0\}$.

6. In each of the following you are given two vector spaces and a function between them. Determine whether the function is a linear transformation or not. Prove your claim.

(a)

$$T : \mathbb{R}^3 \to M_2(\mathbb{R})$$

given by,

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y & y-2z \\ 3x+z & 0 \end{pmatrix}$$

check additivity: for $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in \mathbb{R}^3$,

$$T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z \end{pmatrix} = \begin{pmatrix} x_1+y_1 & y_1-2z_1 \\ 3x_1+z_1 & 0 \end{pmatrix} + \begin{pmatrix} x_2+y_2 & y_2-2z_2 \\ 3x_2+z_2 & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} x_1+x_2+y_1+y_2 & y_1+y_2-2z_1-2z_2 \\ 3x_1+3x_2+z_1+z_2 & 0 \end{pmatrix}.$$

$$T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2+y_1+y_2 & y_1+y_2-2z_1-2z_2 \\ 3x_1+3x_2+z_1+z_2 & 0 \end{pmatrix}.$$

$$\therefore T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}. \quad \checkmark$$

check homogeneity: for α in $\mathbb{R}$,

$$\alpha T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha\begin{pmatrix} x+y & y-2z \\ 3x+z & 0 \end{pmatrix} = \begin{pmatrix} \alpha(x+y) & \alpha(y-2z) \\ \alpha(3x+z & 0 \end{pmatrix}$$

$$T\begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} = \begin{pmatrix} \alpha(x+y) & \alpha(y-2z) \\ \alpha(3x+z) & 0 \end{pmatrix}$$

$$\therefore \alpha T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T\begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} \quad \therefore \checkmark$$

$\boxed{\therefore \text{ it is}}$

(b)

$$T : \mathbb{R}_2[x] \to \mathbb{R}^3$$

given by,

$$Tp = \begin{pmatrix} p(2) \\ p'(2) \\ p''(2) \end{pmatrix}$$

additivity:

Let $p_1 = a_1x^2 + b_1x + c_1$ and $p_2 = a_2x^2 + b_2x + c_2 \in \mathbb{R}_2[X]$.

$p' = 2a_1x + b_1$ $\quad p'' = 2a_1$

$$Tp_1 = \begin{pmatrix} 4a_1 + 2b_1 + c_1 \\ 4a_1 + b_1 \\ 2a_1 \end{pmatrix} \qquad Tp_2 = \begin{pmatrix} 4a_2 + 2b_2 + c_2 \\ 4a_2 + b_2 \\ 2a_2 \end{pmatrix}$$

$$Tp_1 + Tp_2 = \begin{pmatrix} 4(a_1 + a_2) + 2(b_1 + b_2) + (c_1 + c_2) \\ 4(a_1 + a_2) + (b_1 + b_2) \\ 2(a_1 + a_2). \end{pmatrix}$$

$T(p_1+p_2)$: $\quad p_1 + p_2 = (a_1 + a_2)x^2 + (b_1 + b_2)X + (c_1 + c_2)$

$$T(p_1+p_2) = \begin{pmatrix} 4(a_1 + a_2) + 2(b_1 + b_2) + (c_1 + c_2) \\ 4(a_1 + a_2) + (b_1 + b_2) \\ 2(a_1 + a_2) \end{pmatrix}$$

$\therefore Tp_1 + Tp_2 = T(p_1 + p_2)$ $\quad \therefore \checkmark$

homogeneity: $\quad$ let $\alpha \in \mathbb{R}$. $\quad \alpha p = \alpha a x^2 + \alpha b x + \alpha c$

$$\alpha \cdot Tp = \begin{pmatrix} \alpha 4a + \alpha 2b + \alpha c \\ \alpha 4a + \alpha b \\ \alpha a \end{pmatrix}. \qquad T\alpha p = \begin{pmatrix} 4\alpha a + 2\alpha b + \alpha c \\ 4\alpha a + \alpha b \\ \alpha a. \end{pmatrix}$$

$\therefore \alpha Tp = T\alpha p.$ $\quad \therefore \checkmark$

$\therefore$ $\boxed{\text{linear map.}}$

(c)

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$

given by,

$$TA = A^2$$

additive : let $M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in M_2(\mathbb{R})$

$Tm_1 + Tm_2 = \begin{pmatrix} a_1^2 + a_2^2 & b_1^2 + b_2^2 \\ c_1^2 + c_2^2 & d_1^2 + d_2^2 \end{pmatrix}$. $T(m_1 + m_2) = \begin{pmatrix} (a_1 + a_2)^2 & (b_1 + b_2)^2 \\ (c_1 + c_2)^2 & (d_1 + d_2)^2 \end{pmatrix}$.

$Tm_1 + Tm_2 \neq T(m_1 + m_2)$.

$\therefore$ $\boxed{\text{not linear map.}}$

(d) Fix $B \in M_3(\mathbb{R})$ and consider the function:

$$T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$$

given by,

$$TA = AB$$

additivity:

let $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in M_3(\mathbb{R})$, A and $A' \in M_3(\mathbb{R})$.

$$TA + TA' = \begin{pmatrix} a_{11}b_{11} + a'_{11}b_{11} & a_{12}b_{12} + a'_{12}b_{12} & a_{13}b_{13} + a'_{13}b_{13} \\ a_{21}b_{21} + a'_{21}b_{21} & a_{22}b_{22} + a'_{22}b_{22} & a_{23}b_{23} + a'_{23}b_{23} \\ a_{31}b_{31} + a'_{31}b_{31} & a_{32}b_{32} + a'_{32}b_{32} & a_{33}b_{33} + a'_{33}b_{33} \end{pmatrix}$$

$$T(A+A') = \begin{pmatrix} b_{11}(a_{11} + a'_{11}) & b_{12}(a_{11} + a'_{11}) & b_{13}(a_{11} + a'_{11}) \\ b_{21}(a_{21} + a'_{21}) & b_{22}(a_{21} + a'_{21}) & b_{23}(a_{21} + a'_{21}) \\ b_{31}(a_{31} + a'_{31}) & b_{32}(a_{31} + a'_{31}) & b_{33}(a_{31} + a'_{31}) \end{pmatrix}$$

$\therefore TA + TA' = T(A+A')$

homogeneity:

let $\alpha \in \mathbb{R}$.

$$\alpha TA = \begin{pmatrix} \alpha a_{11}b_{11} & \alpha a_{12}b_{12} & \alpha a_{13}b_{13} \\ \alpha a_{21}b_{21} & \alpha a_{22}b_{22} & \alpha a_{23}b_{23} \\ \alpha a_{31}b_{31} & \alpha a_{32}b_{32} & \alpha a_{33}b_{33} \end{pmatrix} = T(\alpha A).$$

$\therefore$ ✓

$\therefore$ $\boxed{\text{linear map.}}$

(e)

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$

given by,

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c+1 & 2a+3b+2 \\ d-b-8 & 2a \end{pmatrix}$$

Additivity: let $M = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $N = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

$$TM + TN = \begin{pmatrix} a_1 - c_1 + 1 & 2a_1 + 3b_1 + 2 \\ d_1 - b_1 - 8 & 2a_1 \end{pmatrix} + \begin{pmatrix} a_2 - c_2 + 1 & 2a_2 + 3b_2 + 2 \\ d_2 - b_2 - 8 & 2a_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 - c_1 - c_2 + 2 & 2a_1 + 2a_2 + 3b_1 + 3b_2 + 4 \\ d_1 + d_2 - b_1 - b_2 - 16 & 2a_1 + 2a_2 \end{pmatrix}$$

$$(M+N) = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

$$T(M+N) = \begin{pmatrix} a_1 + a_2 - c_1 - c_2 + 1 & 2a_1 + 2a_2 + 3b_1 + 3b_2 + 2 \\ d_1 + d_2 - b_1 - b_2 - 8 & 2a_1 + 2a_2 . \end{pmatrix}.$$

$TM + TN \neq T(M+N)$.

$\therefore$ $\boxed{\text{not linear map.}}$

- Find a basis for the kernel and the image of this transformation.
- Find the dimension of the kernel and the image of this transformation. (Remark: This question will continue in the next HW, you may want to keep a copy of your solution to this part of the question).
- Determine whether the transformation is surjective. Explain your answer.
- Determine whether the transformation is injective. Explain your answer.

(a) For

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 7 & -6 \\ 0 & 5 & -5 \end{pmatrix}$$

consider the linear map

$$T_A : \mathbb{R}^3 \to \mathbb{R}^4$$

where the notation T_A was defined in class.

1. $\ker A = \{ x \in \mathbb{R}^3 \mid Ax = \vec{0} \}$. $\therefore$ find solutions for $(A|\vec{0})$.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 7 & -6 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \therefore \begin{cases} x_1 = -t \\ x_2 = t \\ x_3 = t \end{cases}$$

$\boxed{\ker A \text{ span } = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}}$

$\therefore$ $\boxed{\text{dimension} = 1}$

$\operatorname{Im} A = \{ y \in \mathbb{R}^4 \mid Ax = y,\ x \in \mathbb{R}^3 \}$

$= \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 7 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -6 \\ -5 \end{pmatrix} \right\}$. since $V_3 = V_1 - V_2$,

$\quad\ V_1 \qquad\quad V_2 \qquad\quad V_3.$ $\qquad V_3$ is a l. c. of V_1 and V_2.

$\therefore$ basis of $\boxed{\operatorname{Im} A = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 7 \\ 5 \end{pmatrix} \right\}}$. $\boxed{\therefore \text{dimension} = 2.}$

surjective: codomain $\neq \operatorname{Im} A$ $\therefore$ not

injective: Since $\ker A \neq$ trivial, it's not injective.

(b) given by

$$S : M_2(\mathbb{R}) \to \mathbb{R}^2$$

$$SA = A\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$ $\quad 3a = 2b.$ $\quad 3c = 2d.$

$\begin{pmatrix} 3a-2b \\ 3c-2d. \end{pmatrix}.$

$\ker A.$ $\quad \underline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3a-2b \\ 3c-2d \end{pmatrix}$ $\quad a = \frac{2}{3}b$ $\quad c = \frac{2}{3}d.$

$$\boxed{\begin{array}{l} \therefore \ker A = \text{span}\left\{\begin{pmatrix} \frac{2}{3} & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}\right\} \\ \therefore \text{dim is } 2. \end{array}}$$

$\boxed{\text{Im}A : \quad \mathbb{R}^2 \quad \dim = 2.}$ $\quad$ or span $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

$\because$ $\ker A \neq \{0\}$. $\therefore$ it's not injective.

$\because$ $\text{Im}A$ = Codomain $= \mathbb{R}^2$ $\therefore$ it's surjective.

$ax^3 + bx^2 + cx + d. \rightarrow \begin{pmatrix} e \\ f \end{pmatrix}$

(c)

$$L : \mathbb{R}_3[x] \to \mathbb{R}^2$$

given by

$$Lp = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}$$

$p(2) = 8a + 4b + 2c + d$
$p(1) = a + b + c + d.$
$p'(0) = 3ax^2 + 2bx + c$
$= c.$

$\therefore L(ax^3 + bx^2 + cx + d) = \begin{pmatrix} 7a + 3b + c \\ c \end{pmatrix}$

$\ker L = \{ p \in \mathbb{R}_3[x] \mid Lp = \vec{0} \}$

$\therefore L(ax^3 + bx^2 + cx + d) = \begin{pmatrix} 7a + 3b + c \\ c \end{pmatrix} = \vec{0}.$

$\therefore \begin{cases} a = -\frac{3}{7}t \\ b = t \\ c = 0 \\ d = s \end{cases}$

$\boxed{\text{basis} = \text{span}\{ (-\tfrac{3}{7}x^3 + x^2), (1) \} \hookrightarrow \dim(\ker) = 2.}$

$\text{Im} L = \{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \mid Lp = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, p \in \mathbb{R}_3[x] \}$

$\therefore$ the basis $= \begin{pmatrix} 7a + 3b + c \\ c \end{pmatrix}$, where $a, b, c \in \mathbb{R}$

any y_1 can be written as $7a + 3b + c$, $a, b, c \in \mathbb{R}$
and any y_2 can be written as a, $a \in \mathbb{R}$.

$\boxed{\therefore \text{Im} L = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}. \quad \therefore \dim(\text{Im}) = 2.}$

It's surjective because $\text{Im} L$ = Codomain.
It's not injective because $\ker L \neq \{0\}$.

(d)

$$\Phi : \mathbb{R}^3 \mapsto \mathbb{R}_3[x]$$

given by

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto ax^3 + bx^2 + cx + d.$

$$\Phi \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b) + (a-2b+c)x + (b-3c)x^2 + (a+b+c\cancel{\quad})x^3$$

✗ sur ✓ inject.

$\ker \Phi$: $\begin{cases} a+b = 0 \\ a-2b+c = 0 \\ b-3c = 0 \\ a+b+c = 0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0. \end{cases}$

$\boxed{\therefore \ker \Phi = \{0\}, \quad \dim(\ker \Phi) = 0.}$

$\mathrm{Im}\,\Phi$: $a(1+x+x^3) + b(1-2x+x^2+x^3) + c(x-3x^2+x^3)$

$\boxed{\begin{array}{l} \mathrm{span}\{(1+x+x^3), (1-2x+x^2+x^3), (x-3x^2+x^3) \\ \dim(\mathrm{Im}\,\Phi) = 3. \end{array}}$

It's injective because $\ker \Phi = \{0\}$.

It's not surjective because $\mathrm{Im}\,\Phi \neq \mathbb{R}_3[x]$