MATH-1564-K1,K2,K3 Linear Algebra with Abstract Vector Spaces

Homework 10

1. Show that similarity defines an equivalence relation on $M_n(\mathbb{R})$.
2. Let A, B be matrices similar to each other.
   (a) Show that they have the same eigenvalues. Do they have the same eigenvectors?
   (b) Show that they have the same rank.
   (c) Show that they have the same trace.
3. Consider the following matrices.

$$A = \begin{pmatrix} 8 & 3 & -3 \\ -6 & -1 & 3 \\ 12 & 6 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & -4 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad F = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

   For each of these matrices solve the following:
   (a) Find all of the eigenvalues of the matrix, determine the geometric multiplicity and the algebraic multiplicity of each eigenvalue.
   (b) Determine if the matrix is diagonalizable. If it is then find a diagonal matrix D and an invertible matrix P so that the matrix is equal to PDP^{-1}.
4. Consider the matrix E from Q1
   (a) Find the eigenvalues of E^2. Is E^2 diagonalizable?
   (b) Find the eigenvalues of E^{10}. Is E^{10} diagonalizable?
   (c) Find the eigenvalues of $E^3 - 5E^2 + 2E + 3I$. Is $E^3 - 5E^2 + 2E + 3I$ diagonalizable?
   (d) Is E invertible? If so, find the eigenvalues of E^{-1}. Is E^{-1} diagonalizable?
   (e) Compute E^5.
5. Each of the following you are given a linear map. Determine whether it is diagonalizable.
   (a) $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by

$$TA = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} A$$

   $\{1, x, x^2\}$.
   (b) $T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ given by $Tp(x) = x(p(x+1) - p(x))$
   (c) Let V be a vector space and $B = (v_1, v_2, v_3)$ a basis for V. Here we consider the linear transformation $T : V \to V$ which satisfies $Tv_1 = 5v_1$, $Tv_2 = v_2 + 2v_3$ and $Tv_3 = 2v_2 + v_3$.
6. Let V be a vector space of dimension 5. Does there exist a linear map $T : V \to V$ such that $dim Im T = 3$ and:
   (a) T has 5 distinct eigenvalues?
   (b) T has 4 distinct eigenvalues?

(c) T has 4 distinct eigenvalues and T is not diagonalizable?

7. Prove or disprove the following claims.

   (a) If $A \in M_3(\mathbb{R})$ has rows equal to v $2v$ $3v$ for some $v \in \mathbb{R}^3$ and A has a nonzero eigenvalue then A is diagonalizable.

   (b) If $A \in M_4(\mathbb{R})$ has characteristic polynomial $q_A(x) = x^2(x+5)(x+6)$ and

$$\begin{pmatrix} 0 \\ -1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 3 \\ 4 \end{pmatrix} \in null(A)$$

   then A is diagonalizable.

   (c) Let $A \in M_n(\mathbb{R})$. If 0 is an eigenvalue of A then its geometric multiplicity is equal to $n - rankA$.

   (d) There exists $A \in M_5(\mathbb{R})$ which is diagonalizable and satisfies $rankA = 1$ and $trA = 0$.

   (e) If $A \in M_n(\mathbb{R})$ is diagonalizable and 2 is the only eigenvalue of A then $A = 2I$.

   (f) If $A, B \in M_n(\mathbb{R})$ have the same eigenvalues and A is diagonalizable then so is B.

1. Show that similarity defines an equivalence relation on $M_n(\mathbb{R})$.

Equivalence : ① reflexive

② symmetric

③ transitive .

①. $A = I_n A I_n^{-1}$ $\therefore$ ✓ $A \sim A$

② Let $B = SAS^{-1}$ Need to show $A = S'BS'^{-1}$

$S^{-1}BS = S^{-1}(SAS^{-1})S = I_n A I_n = A$.

$\therefore$ ✓ $B \sim A \Rightarrow A \sim B$

③ Let $A \sim B$ and $B \sim C$, Need to show $A \sim C$.

$A = SBS^{-1}$ $B = TCT^{-1}$

$A = S(TCT^{-1})S^{-1}$

$= (ST)\,C\,(T^{-1}S^{-1})$

$\therefore$ ✓ $A \sim C$.

2. Let A, B be matrices similar to each other.

(a) Show that they have the same eigenvalues. Do they have the same eigenvectors?

(b) Show that they have the same rank.

(c) Show that they have the same trace.

a. $A \sim B$, means $B = P^{-1}AP$, so $A = PBP^{-1}$

Since $A\vec{v} = \lambda\vec{v}$, $PBP^{-1}\vec{v} = \lambda\vec{v}$

$BP^{-1}\vec{v} = \lambda P^{-1}\vec{v}$

$\therefore P^{-1}\vec{v}$ is the eigenvector of B with the same eigenvalue λ. $\therefore$ same eigenvalue but dif eigenvector.

b. $A = SBS^{-1}$ for $\vec{v} \in \ker(A)$, $SBS^{-1}\vec{v} = 0$.

$S^{-1}SBS^{-1}\vec{v} = S^{-1}0 = 0$

$BS^{-1}\vec{v} = 0$ $\therefore S^{-1}\vec{v} = \ker(B)$.

Since S^{-1} is invertible, it's injective.

$\therefore \dim(S^{-1}\vec{v}) = \dim(\vec{v})$

$\therefore \dim(\ker B) = \dim(\ker A)$.

$\therefore \text{rank } B = \text{rank } A$.

c. From previous HWs, we know that $tr(AB) = tr(BA)$.

$tr(A) = tr(SBS^{-1})$

$= tr((SB)(S^{-1}))$

$= tr((S^{-1})(SB))$

$= tr(B)$.

3. Consider the following matrices.

$$A = \begin{pmatrix} 8 & 3 & -3 \\ -6 & -1 & 3 \\ 12 & 6 & -4 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & -4 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad F = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

For each of these matrices solve the following:

(a) Find all of the eigenvalues of the matrix, determine the geometric multiplicity and the algebraic multiplicity of each eigenvalue.

(b) Determine if the matrix is diagonalizable. If it is then find a diagonal matrix D and an invertible matrix P so that the matrix is equal to PDP^{-1}.

$$A = \begin{pmatrix} 8 & 3 & -3 \\ -6 & -1 & 3 \\ 12 & 6 & -4 \end{pmatrix} \qquad \begin{pmatrix} 8-\lambda & 3 & -3 \\ -6 & -1-\lambda & 3 \\ 12 & 6 & -4-\lambda \end{pmatrix}$$

$$(8-\lambda)\begin{vmatrix} -1-\lambda & 3 \\ 6 & -4-\lambda \end{vmatrix} - 3\begin{vmatrix} -6 & 3 \\ 12 & -4-\lambda \end{vmatrix} - \begin{vmatrix} -6 & -1-\lambda \\ 12 & 6 \end{vmatrix}$$

$$= (8-\lambda)[(-1-\lambda)(-4-\lambda) - 18] - 3[(-6)(-4-\lambda) - 36]$$
$$- [(-36) - 12(-1-\lambda)]$$
$$= -\lambda^3 + 3\lambda^2 - 4 = (\lambda+1)(\lambda-2)^2$$

$$E\{\lambda=-1\} = \left(\begin{array}{ccc|c} 9 & 3 & -3 & 0 \\ -6 & 0 & 3 & 0 \\ 12 & 6 & -3 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}\right\} \leftarrow \dim = 1.$$

$$E\{\lambda=2\} = \left(\begin{array}{ccc|c} 6 & 3 & -3 & 0 \\ -6 & -3 & 3 & 0 \\ 12 & 6 & -6 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}\right\} \leftarrow \dim = 2.$$

$\lambda = -1$: alg: 1 geo: 1

$\lambda = 2$: alg: 2. geo: 2.

since $\forall \lambda$, alg mult. = geo. mult. $\therefore$ diagnolizable.

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$$

$$B = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{pmatrix} \cdot \begin{matrix} -1-\lambda & 2 \\ 2 & -1-\lambda \\ 2 & 2 \end{matrix}$$

$$(-1-\lambda)^3 + 8 + 8 - 4(-1-\lambda) - 4(-1-\lambda) - 4(-1-\lambda) = 0.$$

$$(-1-\lambda)^3 - 12(-1-\lambda) + 16 = 0$$

$$-(1+\lambda^2+2\lambda)(1+\lambda) + 12 + 12\lambda + 16 = 0.$$

$$-1-\lambda-\lambda^2-\lambda^3-2\lambda-2\lambda^2+28+12\lambda = 0$$

$$-\lambda^3 - 3\lambda^2 + 9\lambda - 27 = 0$$

$$(\lambda+3)^2(\lambda-3) = 0$$

$$E\{\lambda=3\} = \left(\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

$$E\{\lambda=-3\} = \left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\therefore$ $\lambda = 3$ alg : 1 geo : 1

$\lambda = -3$ alg : 2 geo : 2

Since $\forall \lambda$, alg mult. = geo mult. $\therefore$ diagnolizable.

$$B = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$$

$$C = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & -4 \end{pmatrix} \qquad \begin{pmatrix} 4-\lambda & 0 & 3 \\ 0 & 5-\lambda & 0 \\ 3 & 0 & -4-\lambda \end{pmatrix}$$

$$(5-\lambda)\begin{vmatrix} 4-\lambda & 3 \\ 3 & -4-\lambda \end{vmatrix} = (5-\lambda)\left[(4-\lambda)(-4-\lambda) - 9\right]$$

$$= (5-\lambda)(\lambda^2-25) = (5-\lambda)(\lambda+5)(\lambda-5) = 0.$$

$$E\{\lambda=5\} = \left(\begin{array}{ccc|c} -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -9 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$$E\{\lambda=-5\} = \left(\begin{array}{ccc|c} 9 & 0 & 3 & 0 \\ 0 & 10 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right) = \text{Span}\left\{ \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\lambda_1 = 5$ alg $= 2$ geo $= 2$

$\lambda_2 = -5$ alg $= 1$ geo $= 1$

Since $\forall \lambda$, alg mult = geo mult $\therefore$ diagnolizable.

$$C = \begin{pmatrix} 3 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} 3 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$$

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{pmatrix}$$

$$(-1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda)(2-\lambda)$$

$$E\{\lambda = -1\} : \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}\right\}.$$

$$E\{\lambda = 1\} : \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}.$$

$$E\{\lambda = 2\} : \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

$\lambda = -1$ alg: 1 geo: 1

$\lambda = 1$ alg: 1 geo: 1

$\lambda = 2$ alg: 1 geo: 1

Since $\forall \lambda$, alg = geo. $\therefore$ diagonalizable.

$$E = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 6 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 6 & 0 & 0 \end{pmatrix}^{-1}$$

$$F = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 2-\lambda & 1 & 0 & 0 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{pmatrix}$$

$$(2-\lambda)^3 \quad (3-\lambda)$$

$$E\{\lambda = 2\} = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$$E\{\lambda = 3\} = \left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\lambda = 2$ alg: 3 geo: 1

$\lambda = 3$ alg: 1 geo. 1

for $\lambda = 2$: alg mult. $\neq$ geo mult

$\therefore$ not diagnolizable.

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ -1 & -\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 5 \\ 0 & 0 & 0 & 3-\lambda \end{pmatrix}$$

$$(3-\lambda) \begin{vmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} \begin{matrix} -\lambda & 1 \\ -1 & -\lambda \\ 0 & 0 \end{matrix}$$

$$= (3-\lambda)\left[(-\lambda)(-\lambda)(2-\lambda) + (2-\lambda)\right]$$
$$= (3-\lambda)\left[(2\lambda^2 - \lambda^3) + (2-\lambda)\right] = (3-\lambda)(-\lambda^3 + 2\lambda^2 - \lambda - 2)$$
$$= \lambda^4 - 5\lambda^3 + 7\lambda^2 - 5\lambda + 6$$

$$E\{\lambda = 3\} = \left(\begin{array}{cccc|c} -3 & 1 & 0 & 0 & 0 \\ -1 & -3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 5 \\ 1 \end{pmatrix}\right\}$$

alg 1 geo 1

$$E\{\lambda = 2\} = \left(\begin{array}{cccc|c} -2 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

alg 1 geo 1

$$E\{\lambda = i\} = \left(\begin{array}{cccc|c} -i & 1 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 & 0 \\ 0 & 0 & 2-i & 5 & 0 \\ 0 & 0 & 0 & 3-i & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

alg 1 geo 1

$$E\{\lambda = -i\} = \left(\begin{array}{cccc|c} i & 1 & 0 & 0 & 0 \\ -1 & i & 0 & 0 & 0 \\ 0 & 0 & 2+i & 5 & 0 \\ 0 & 0 & 0 & 3+i & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

alg 1 geo 1

diagonalisable.

$$F = \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & 1 & 1 \\ 5 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & 1 & 1 \\ 5 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1}$$

4. Consider the matrix E from Q1

(a) Find the eigenvalues of E^2. Is E^2 diagonalizable?

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad E^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 1-\lambda & 3 & 1 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 3 \\ 0 & 4-\lambda \end{vmatrix} = (1-\lambda)^2(4-\lambda)$$

$$E\{\lambda=1\} = \left(\begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$$E\{\lambda=4\} = \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\forall \lambda$, alg mult = geo mult

$\therefore$ diagonalizable.

(b) Find the eigenvalues of E^{10}. Is E^{10} diagonalizable?

$$E^{10} = \begin{pmatrix} 1 & 1023 & 341 \\ 0 & 1024 & 341 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1-\lambda & 1023 & 341 \\ 0 & 1024-\lambda & 341 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

$$\det = (1-\lambda)^2 (1024-\lambda).$$

$$\lambda = 1 \quad \text{or} \quad \lambda = 1024.$$

$$E\{\lambda=1\} = \left(\begin{array}{ccc|c} 0 & 1023 & 341 & 0 \\ 0 & 1023 & 341 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right\}.$$

$$E\{\lambda = 1024\} = \left(\begin{array}{ccc|c} -1023 & 1023 & 341 & 0 \\ 0 & 0 & 341 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\because$ $\forall \lambda$, alg mult. = geo mult

$\therefore$ diagonolizable.

(c) Find the eigenvalues of $E^3 - 5E^2 + 2E + 3I$. Is $E^3 - 5E^2 + 2E + 3I$ diagonalizable?

$1^3 - 5\times 1^2 + 2\times 1 + 3\times 1 = 1$ alg $= 1$.

$4^3 - 5\times 4^2 + 2\times 4 + 3\times 1 = -5$. alg $= 2$

$E\{\lambda = 1\} = \left(\begin{array}{ccc|c} 0 & -6 & -3 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ $\leftarrow$ dim $= 1$

geo $= 1$

$E\{\lambda = -5\} = \left(\begin{array}{ccc|c} -6 & -6 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$. $\leftarrow$ dim $= 2$.

geo $= 2$.

$\therefore$ diagnolizable.

(d) Is E invertible? If so, find the eigenvalues of E^{-1}. Is E^{-1} diagonalizable?

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}. \qquad \det(E) = -1 \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -1 \times 2 = -2$$

$\det(E) \neq 0$ $\therefore$ invertible.

$$E^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{pmatrix}$$

char. poly. $= (1-\lambda)(\frac{1}{2}-\lambda)(-1-\lambda)$

$\lambda_1 = 1, \quad \lambda_2 = \frac{1}{2}, \quad \lambda_3 = -1$

$$E^{-1}\{\lambda = 1\} = \left(\begin{array}{ccc|c} 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$$E^{-1}\{\lambda = \tfrac{1}{2}\} = \left(\begin{array}{ccc|c} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}.$$

$$E^{-1}\{\lambda = -1\} = \left(\begin{array}{ccc|c} 2 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ 1 \end{pmatrix}\right\}$$

$\lambda = 1$: alg $= 1$. geo $= 1$
$\lambda = \frac{1}{2}$: alg $= 1$ geo $= 1$
$\lambda = -1$: alg $= 1$ geo $= 1$.

$\therefore$ diagonalizable.

(e) Compute E^5.

$$E = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 6 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 6 & 0 & 0 \end{pmatrix}^{-1}$$

$$E^5 = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 6 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 6 & 0 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 31 & 10 \\ 0 & 32 & 11 \\ 0 & 0 & -1 \end{pmatrix}$$

5. Each of the following you are given a linear map. Determine whether it is diagonalizable.

(a) $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by

$$TA = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} A$$

char. poly. $= \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$

$= (1-\lambda)(4-\lambda) - 4$

$= 4 - \lambda - 4\lambda + \lambda^2 - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5).$

$\therefore \lambda = 0$ or $\lambda = 5$.

$E_{\{\lambda=0\}} = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$.

$E_{\{\lambda=5\}} = \left(\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right) = \text{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \right\}$.

$\because$ For $\lambda = 0$, alg $=$ geo $= 1$

For $\lambda = 5$, alg $=$ geo $= 1$

$\therefore$ diagnolizable.

(b) $T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ given by $Tp(x) = x(p(x+1) - p(x))$

new basis : $B = \{1, x, x^2\}$.

$$[T]_B = \left([T(1)]_B \quad [T(x)]_B \quad [T(x^2)]_B \right)$$

$$= \left([x(1-1)]_B \quad [x(x+1-x)]_B \quad [x[(x+1)^2 - x^2]]_B \right)$$

$$= \left([0]_B \quad [x]_B \quad [2x^2+x]_B \right)$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Need to check if $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ is diagnolizable.

$$\det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix} = -\lambda \begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = -\lambda(1-\lambda)(2-\lambda)$$

$$E_{\{\lambda=0\}} = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$E_{\{\lambda=1\}} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$E_{\{\lambda=2\}} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\therefore$ The matrix is diagnolizable, So is the T.

(c) Let V be a vector space and $B = (v_1, v_2, v_3)$ a basis for V. Here we consider the linear transformation $T : V \to V$ which satisfies $Tv_1 = 5v_1$, $Tv_2 = v_2 + 2v_3$ and $Tv_3 = 2v_2 + v_3$.

consider $[T]_B = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$

Need to check if $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ is diagonalizable

$$\det \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} = (5-\lambda)\left[(1-\lambda)^2 - 4\right]$$
$$= (5-\lambda)(1-\lambda-2)(1-\lambda+2)$$
$$= (5-\lambda)(-1-\lambda)(3-\lambda)$$

$$E\{\lambda = 5\} = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$$E\{\lambda = 3\} = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

$$E\{\lambda = -1\} = \left(\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right) = \text{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$$

$\forall \lambda$, alg = geo.

$\therefore$ diagonalizable.

6. Let V be a vector space of dimension 5. Does there exist a linear map $T : V \to V$ such that $dim Im T = 3$ and: rank $= 3$

(a) T has 5 distinct eigenvalues?

(b) T has 4 distinct eigenvalues?

a. Suppose by way of contradiction that T has 5 distinct eigenvalues, these eigenvectors would be linearly independent. The image would have a dimension of 5, instead of 3.
$\therefore$ It's not possible.

b. If T has 4 distinct eigenvalues, there are 4 linearly independent eigenvectors for T. If one of them is zero, the other 3 would be able to form a basis for the image. $\therefore$ It's possible that T has 4 distinct eigenvalues while $\dim Im T = 3$.
$\therefore$ True.

(c) T has 4 distinct eigenvalues and T is not diagonalizable?

If rank $= 3$, then $\dim \ker T = 2$.
$\therefore$ geo. mult. $\{\lambda = 0\} = 2$

SBWOC that T is not diagnolizable, Σ geo < 5.
let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be 4 dif eigenvalues.
We know that geo mult $\{\lambda_1 = 0\} = 2$,
$\therefore$ Σ geo of $\lambda_2, \lambda_3, \lambda_4 < 5 - 2 = 3$

Since geo. mult. cannot be zero,
Σ geo of $\lambda_2, \lambda_3, \lambda \geq 3$.
$\therefore$ it's not possible.

7. Prove or disprove the following claims.

(a) If $A \in M_3(\mathbb{R})$ has rows equal to v $2v$ $3v$ for some $v \in \mathbb{R}^3$ and A has a nonzero eigenvalue then A is diagonalizable.

(b) If $A \in M_4(\mathbb{R})$ has characteristic polynomial $q_A(x) = x^2(x+5)(x+6)$ and

$$\begin{pmatrix} 0 \\ -1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 3 \\ 4 \end{pmatrix} \in null(A)$$

then A is diagonalizable.

(c) Let $A \in M_n(\mathbb{R})$. If 0 is an eigenvalue of A then its geometric multiplicity is equal to $n - rank A$.

a. true. A's rank = 1 $\therefore$ nullity = 2.

$\therefore$ geo mult $\{\lambda = 0\}$ = 2.

For the non-zero eigenvalue of A. geo mult has to be 1 because $A \in M_3(\mathbb{R})$.

geo for $\{\lambda = 0\}$ + geo $\{\lambda \neq 0\}$ = 3 = #col of A

$\therefore$ diagnolizable.

b. $\lambda = 0$: alg = 2 geo = 2 (from dim ker.)

$\lambda = -5$: alg = 1 geo = ?

$\lambda = -6$: alg = 1 geo = ?

$\uparrow$ from c.p.

geo $\{\lambda = -5\}$ + geo $\{\lambda = -6\}$ $\leq$ 4 - 2 = 2

$\therefore$ geo $\{\lambda = -5\}$ = 1

geo $\{\lambda = -6\}$ = 1.

$\therefore$ $\forall \lambda$, geo = alg.

$\therefore$ true.

c. geo $\{\lambda = 0\}$ = dim ker. A

dim ker A + dim Im A = n (R.N.T).

geo $\{\lambda = 0\}$ = n - dim Im A $\therefore$ true.

(d) There exists $A \in M_5(\mathbb{R})$ which is diagonalizable and satisfies $rank A = 1$ and $tr A = 0$.

(e) If $A \in M_n(\mathbb{R})$ is diagonalizable and 2 is the only eigenvalue of A then $A = 2I$.

(f) If $A, B \in M_n(\mathbb{R})$ have the same eigenvalues and A is diagonalizable then so is B.

d. rank $A = 1$ $\therefore$ $\dim\ker A = 4$.

$\therefore$ geo. mult $\{\lambda = 0\} = 4$.

If A is diagnolizable, $A = SBS^{-1}$. and alg $\{\lambda = 0\} = 4$.

So B will be $\begin{pmatrix} w & 0 & 0 & 0 & 0 \\ 0 & 0 & & & \\ 0 & & \ddots & & \\ \vdots & & & & \\ 0 & & & & 0 \end{pmatrix}$, so trace $A \neq 0$.

$\therefore$ false.

e. $A = SBS^{-1}$ and $B = \begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & & & 2 \end{pmatrix}$.

$\therefore A \sim 2I_n$.

$A = 2I$.

$\therefore$ true.

f. counterexample: $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

for A: $\lambda = 2$: alg $= 2$, geo $= 2$ diagnolizable

for B: $\lambda = 2$: alg $= 2$, geo $= 1$. not diagnolisable.

$\therefore$ false.