Tentative Course Outline:

Week	Dates	Tue	Thu	Fri	Remark
1	8/21-8/25	1st Lec.			
2	8/28-9/1	HW1 Due		WW2 Due	WW1 Due on 8/28
3	9/4-9/8	HW2 Due		WW3 Due	9/4 Recess
4	9/11-9/15	HW3 Due		WW4 Due	
5	9/18-9/22	HW4 Due	Midterm 1		
6	9/25-9/29			WW5 Due	
7	10/2-10/6	HW5 Due		WW6 Due	
8	10/9-10/13			WW7 Due	10/9-10/10 Recess
9	10/16-10/20	HW6 Due		WW8 Due	
10	10/23-10/27	HW7 Due		WW9 Due	10/28 W day
11	10/30-11/3	HW8 Due	Midterm 2		
12	11/6-11/10			WW10 Due	
13	11/13-11/17	HW9 Due		WW11 Due	
14	11/20-11/24	HW10 Due			Thanksgiving
15	11/27-12/1			WW12 Due	
16	12/4-12/8	HW11 Due	Final		

theories in lectures, examples in studios.

piezza : extra credit.

$\mathbb{Z}$: set of integer. $\{0, \pm 1, \pm 2, \pm 3, \dots\}$

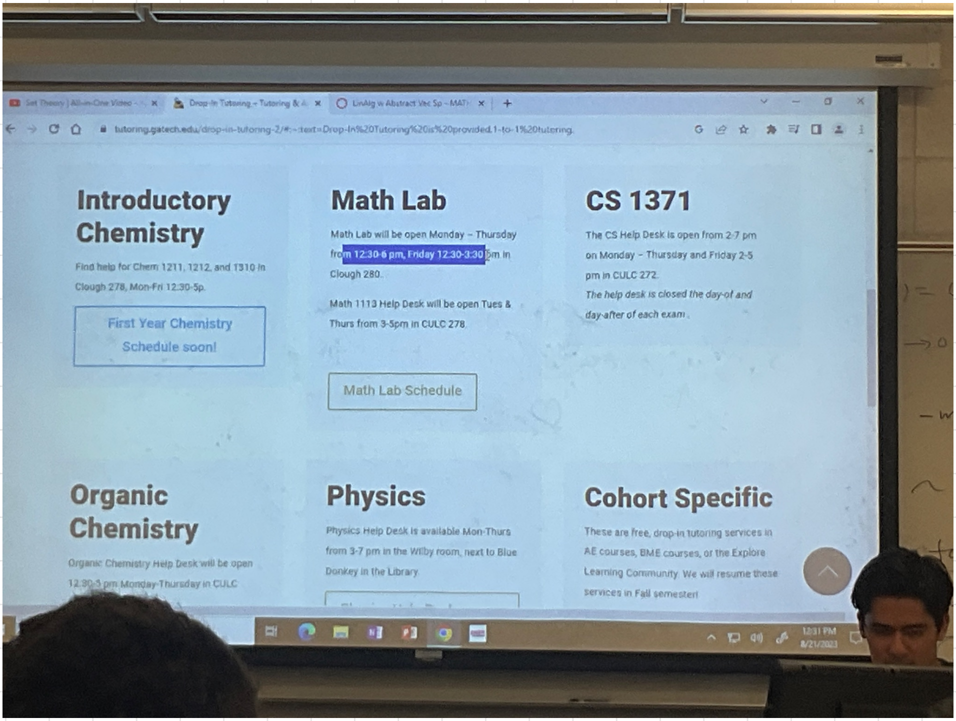
$\mathbb{Q}$ rational number $\{ p/q \mid p, q \in \mathbb{Z}, q \neq 0 \}$

such that

$\mathbb{R}$: real number

$\mathbb{C}$: irrational number $\{ a+bi \mid a, b \in \mathbb{R} \quad i^2 = -1 \}$

08/21



set $\{1, 2, 3\}$ $1 \in A$, $4 \notin A$

$P := \{ p \mid p$ is a prime $\}$

predicate

$A = \{2, 3, 6\}$

$B = \{1, 2, 3, 4, 5, 6\}$, A is a subset of B.

if $A \subseteq B$ and $B \subseteq A$, $A = B$.

if $A \subseteq B$ but $A \neq B$, A is a proper subset ($\subsetneq$)

$\emptyset$ empty set , all empty sets are equal.

$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$

$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

$A \cup \emptyset = A$

$A \cup A = A$

Cardinality

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|A \cup B| \leq |A| + |B|$$

distributive property.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

proof :
$A \cup (B \cap C) \in (A \cup B) \cap (A \cup C).$
$(A \cup B) \cap (A \cup C). \in A \cup (B \cap C)$
$\therefore$ they are equal.

if $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$
$A \setminus B = \{1, 3, 5\}$
$B \setminus A = \{6, 8\}$

U : universal set depends on the context.

$A = \{x \in B \mid P\}$
$A^c = \{x \in B \mid \neg P\}$

$\emptyset^c = U$
$U^c = \emptyset$
if $A \subseteq B$, then $B^c \subseteq A^c$

De Morgan's Law .
if A and B are subsets of U
$(A \cup B)^c = A^c \cup B^c$

8/22 Lecture.

A math statement can be assigned a true value of True / False.

And : $\wedge$

Or : $\vee$

Not : $\neg$

De Morgan's Law. $\neg(P \vee Q) = \neg P \wedge \neg Q$

$\neg(P \wedge Q) = \neg P \vee \neg Q$

e.g. Negate $1 \le x < 5$

sol. $\neg(1 \le x < 5) =$

$\neg(1 \le x \wedge x < 5)$

$= \neg(1 \le x) \vee \neg(x < 5)$

$= (1 < x) \vee (x \geq 5)$

implication : form. $P \Rightarrow Q$

(P implies Q), (P only if Q), (if P then Q)

truth table :

negation. [counter example.]

P	Q	$P \Rightarrow Q$	$P \wedge \neg Q$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

if the first statements are not satisfied we just assume it's true

$P \wedge \neg Q$ has opposite truth value with $P \Rightarrow Q$

ex. if a person's from GATech, they are math major.

counter example : a person is not math major but in GATech

implication 2.0 $P \Leftrightarrow Q$. double implication.

which means step 1 $P \Rightarrow Q$, step 2. $Q \Rightarrow P$

P if and only if Q.

when (P and Q are true) or (P and Q are false)

case work!
↓
a way to proof

e.x. Show that $P \Rightarrow Q$ and $Q \Rightarrow P$ cannot be false at the same time

pf. if $P \Rightarrow Q$ is true, we are done.

if $P \Rightarrow Q$ is false, then P is true and Q is false,

which means $Q \Rightarrow P$ is true.

converse / counter positive.

the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$

the contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$

e.g. $P \Rightarrow Q$ has the same truth value with its contrapositive.

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

odd. an integer that can be written as $2k+1$ for $k \in \mathbb{Z}$

even: $2k$

question: Show if $x^2 - 4x + 3$ is even, then x is odd.

sol. ∴ proof if $x^2 - 4x + 3$ is odd, then x is even

proof contra-positive!

x is even, x can be written as $2k$. ($k \in \mathbb{Z}$)

then $(2k)^2 - 4(2k) + 3 = 4k^2 - 8k + 3$

$$= 2\underbrace{(2k^2 - 4k + 1)}_{\text{new } k \in \mathbb{Z}} + 1$$

odd

← assume the

proof by contradiction.

e.g. Show that the $\sqrt{2}$ is irrational

let $m, n \in \mathbb{Z}$, if mn is even, then at least one of them is even.

$\sqrt{2} \notin \mathbb{Q}$

Sol. Suppose by way of contradiction, that $\sqrt{2}$ is Rational.

$\sqrt{2} = p/q$ for some $p, q \in \mathbb{Z}$ and $q \neq 0$.

∴ $2 = \frac{p^2}{q^2}$, p^2 is even, p is even. $p = 2k$. ($k \in \mathbb{Z}$)

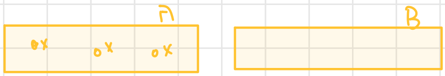
$4k^2 = 2q^2$. $2k^2 = q \Rightarrow q^2$ is even $\Rightarrow q$ is even.

$\Rightarrow p, q$ both have a factor of 2. a contradiction.

quantifiers. ① for all = for each = for every $\forall$

② there exist. = at least 1 such ... = $\exists$

e.g.



For each dot x in box, x is yellow.

↳ true for both A and B. ⇒ an implication in disguise.

There exist a yellow dot in the box.

↳ true for A, false for B.

e.g. $\neg(\forall x \in S$ such that $p(x)) = \exists x \in S$ such that $\neg p(x))$

e.g. negate the following.

$\underbrace{\exists \text{ a integer } x}_{\text{quantifier}}$ such that $\underbrace{n > x \text{ for every integer } n.}_{\text{statement.}}$

Sol. $\forall$ integer x, $\exists$ an integer n such that $n \le x$

e.g. write as quantifier

1 is the smallest positive integer.

For every positive integer n, we have $n \ge 1$.

Let A, B be 2 non-empty sets, the Cartisian product of A, B is

$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$

e.g. $\{1, 2\} \times \{3, 4, 5\}$

$= \{ (1, 3), (1, 4), (1, 5),$
$(2, 3), (2, 4), (2, 5) \}$

$\mathbb{Z}^2$

$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{ (x, y) \mid x, y \in \mathbb{R} \}$ $\mathbb{R}^n$

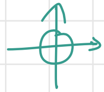
$\mathbb{Q}^2$

A relation $\mathcal{R}$ is a subset on $\underset{\text{domain}}{D} \times \underset{\text{codomain.}}{C}$

we say x is related to y on $\mathcal{R}$, i.e. $x \mathcal{R} y$,

if $(x, y) \in \mathcal{R}$

$\{(x,y) \mid x^2+y^2=1\} \subseteq \mathbb{R}^2$



$Px \to Qx$

$\equiv (\neg Px \cup Qx)$

A function from domain D to codomain C is a relation on $D \times C$ such that for each element X in D, it maps it to exactly one element in C.

* contrapositive :

- $P \Rightarrow Q$. contrapositive would be $\neg Q \Rightarrow \neg P$.
- Same truth value

e.g. If it rains then I take an umbrella

↓ If I don't take an umbrella then it doesn't rain.

* negation :

- $P \Rightarrow Q$ negating the statement would be $\neg (P \Rightarrow Q)$

$$(P \Rightarrow Q) \equiv (\neg P \Rightarrow Q)$$
$$\neg (P \Rightarrow Q) \equiv \neg (\neg P) \wedge \neg Q$$
$$\therefore \equiv (P \wedge \neg Q)$$

- negates | reverse the truth-value.

- e.g. If r is rational then the decimal expansion of r is repeating.

negation : r is rational

* converse : hypothesis and conclusion are reversed

$P \Rightarrow Q$'s converse would be $Q \Rightarrow P$

* inverse $f^{-1}(x)$. undo the function.

$P \Rightarrow Q$'s inverse is $\neg P \Rightarrow \neg Q$

e.g. If I am on time for work then I catch the 8:05 bus

↓ If I am late for work then I miss the 8:05 bus.

$\mathbb{R}\times\mathbb{R} = \{(x,y) \mid x\in\mathbb{R},\ y\in\mathbb{R}\}$ (vector spaces)

$\mathbb{R}^n = \{(x_1,\cdots,x_n) \mid x_1,\cdots,x_n\in\mathbb{R}\}$

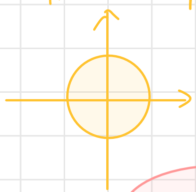
def: A relation from set $\mathcal{D}$ to set C is a subset $R\subseteq \mathcal{D}\times C$. We say $x\in\mathcal{D}$ is related to $y\in C$ if $(x,y)\in R$, denoted by $x\,R\,y$.

e.g. relation: if x is a child of y, $\overline{W}$ = world population.

$\mathcal{R}^2 \subseteq \overline{W}^2$ s.t. $(x,y)\in\mathcal{R}$

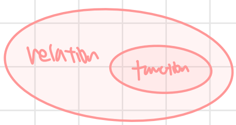
then (Niki, Niki's dad) $\in \mathcal{R}$

e.g. $\mathbb{R}^2$, $\mathcal{R} = \{(x,y) \mid x^2+y^2=1\}$



$1\,\mathcal{R}\,0$ since $(1,0)\in\mathcal{R}$

$(4,3)\notin\mathcal{R}$ $\Rightarrow$ 4 is not related to 3 with relation to $\mathcal{R}$.



def. A function f from domain D to codomain C is a relation $\mathcal{R}\subseteq D\times C$ s.t. for every element $x\in\mathcal{D}$, f maps x to exactly 1 element $y\in C$

(只要 input 里所有值都有 output 就 legal. output 有多不管;
有 input 无结果为 illegal.)

def Let f be a function from domain $\mathcal{D}$ to codomain C, then the range / image of f is

$$\mathrm{Range}(f) := \mathrm{Im}(f) = \{\underbrace{y\in C}_{\text{check condition 1}} \mid \underbrace{y=f(x),\ \text{for some } x\in\mathcal{D}}_{\text{check condition 2}}\}$$

问题: Is $2\in\mathrm{Range}(f)$? check condition 1, check condition 2 def An equivalent relation $\sim$ on set S is a relation from S to S such that (tilde $\sim$)

① reflexive:

If $x \in S$, then $x \sim x$

② symmetric:

For all $x, y \in S$, if $x \sim y$, then $y \sim x$

③ transitive

For all $x, y, z \in S$, if $x \sim y$, $y \sim z$, then $x \sim z$

e.g. Consider people in this room. Call it S

Define $x \sim y$ if x and y have the same weight

sol: ① Same person has the same weight w/ him(herself).

② for all $x, y \in S$, $x \sim y \Rightarrow y \sim x$

③ for all $x, y, z \in S$, $x \sim y$, $y \sim z$, then $x \sim z$.

having the same weight.

e.g. Consider $\mathbb{Z}$ and $x \sim y$ defined as $x \sim y$ if $x - y = 3k$ for some $k \in \mathbb{Z}$.

Sol. ① for any x in $\mathbb{Z}$, $x - x = 0 = 3k$.

② if $x - y = 3k$

Consider $y - x = 3(-k) \Rightarrow y \sim x$.

③ Let $x, y, z \in \mathbb{Z}$ Further assume $x \sim y$, $y \sim z$.

ⓐ $x - y = 3k$ $(k \in \mathbb{Z})$

ⓑ $y - z = 3m$ $(m \in \mathbb{Z})$

ⓐ + ⓑ $= x - z = 3(k - m)$

$x - z = 3n$ $(n \in \mathbb{Z})$

$x \sim z$

def Given equivalent relation $\sim$ on set S, and an element $a \in S$, the equivalent class of a is

$$[a] = \{x \in S \mid x \sim a\}$$

weight e.g.: $[\text{Ben}] = \{$ all people who have the same weight as Ben $\}$

representative

3k e.g. : $[0] = \{x \in \mathbb{Z} \mid x \sim 0\}$

$= \{x \in \mathbb{Z} \mid x = 3k,\ k \in \mathbb{Z}\}$ $\quad 0, \pm 3, \pm 6 \ldots$

$[1] = \{x \in \mathbb{Z} \mid x \sim 1\}$ $\quad$ x 和 1 一样 除 3 …1.

$= \{x \in \mathbb{Z} \mid x = 3k+1,\ k \in \mathbb{Z}\}$ $\quad -2, 1, 4, 7 \ldots$

$\boxed{x=1 \text{ 时得到 } 3k,\ x = \text{其他什么数时也可以得到 } 3k?}$ $\Leftarrow$

$[2] = \{x \in \mathbb{Z} \mid x \sim 2\}$

$= \{x \in \mathbb{Z} \mid x = 3k+2,\ k \in \mathbb{Z}\}$ $\quad -1, 2, 5 \ldots$

Theorem. Consider equivalent relation $\sim$ on set S. Then the collection of all equivalent classes under $\sim$ form a partition of set S, denoted by $S/\sim$.

$$S/\sim \; = \{[a] \mid a \in S\}$$

def Let $S_1, S_2 \ldots$ be subsets of U. Then we say $S_1, S_2, \ldots$ form a partition of U.

~~集合体.~~ 一种分组.

if ① $S_i \cap S_j = \emptyset$ $\quad$ ($\forall i \neq j$ any two $\times$ overlaps)

② $\bigcup_{i=1} S_i = U$ $\quad$ (if all of them form the entire U)

proof.

Lemma: let $\sim$ be an equivalent relation on set S and let $a, b, \in S$, then either

① $[a] = [b]$ or

② $[a] \cap [b] = \emptyset$

$\emptyset \subset \emptyset$

$\emptyset \in \{\emptyset\}$

$\emptyset \subset \{\emptyset\}$: $\emptyset$ is a subset of

$\emptyset \notin \emptyset$

def : Let A be a set then its power set, denoted by $\mathcal{P}(A)$ is $\mathcal{P}(A) = \{ x \mid x \subseteq A \}$

e.g. let $A = \{1, 3\}$, $\mathcal{P}(A) = \{ \emptyset, \{1\}, \{3\}, \{1, 3\} \}$

prop Let A be a finite set, then $|\mathcal{P}(A)| = 2^{|A|}$

Proof by induction : (if the statement is about natural numbers or integers)

Step 1 : base case : check the smallest number when $n = n_0$ (smallest).

Step 2 : induction hypothesis : assume statement is true when $n = k$

Step 3 : induction step : if statement is true for $n = k+1$

e.g. proof by induction :

① pf. Base case : if true $|A| = 0$
if A is an empty set, $|\mathcal{P}(A)| = 2^0 = 1$. $A = \emptyset$.
$\mathcal{P}(A) = \{\emptyset\}$, which has 1 element.

② induction hypothesis.
Assume statement is true when $|A| = k$.
i.e. if $|A| = k$, then $|\mathcal{P}(A)| = 2^k$

③ if $|A| = k+1$, $|\mathcal{P}(A)| = 2^{k+1}$

let a be an element in A,
then $|A \setminus \{a\}| = k$

by IH $|\mathcal{P}(A\setminus\{a\})| = 2^k$

all subsets that have a

$\Rightarrow \mathcal{P}(A) = S_{\{a\}} \cup S_{\overline{\{a\}}}$ all subsets that do not have a.

任何这里的 element 加上 a 就成了最后的集

where $S_{\{a\}} = \{X \subseteq A \mid a \in X\}$

$S_{\overline{\{a\}}} = \{X \subseteq A \mid a \notin X\} \Rightarrow |\mathcal{P}(S_{\overline{\{a\}}})| = 2^k$

$\therefore S_{\{a\}} = \{X \cup \{a\} \mid X \in S_{\overline{\{a\}}}\} \Rightarrow |S_{\{a\}}| = 2^k$

Further $S_{\{a\}} \cap S_{\overline{\{a\}}} = \emptyset$.

$|\mathcal{P}(A)| = 2^k + 2^k = 2^{k+1}$.

System of linear equations.

$$\begin{cases} a_1 x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\ \vdots \\ a_m x_1 + a_m x_2 + \cdots + a_{mn} x_n = b_m \end{cases}$$

m equations.

We say $(s_1, \ldots, s_n)$ is a solution to the system if it satisfie all equations.

e.g. $\begin{cases} 2x + 3y = 8. \\ x - y = -1 \end{cases}$ $\begin{cases} x = 1 \\ y = 2. \end{cases}$ is a solution

2 lines overlap and have 1 intersert

$\begin{cases} 2x + 4y = 5 \\ 2x + 4y = -1 \end{cases}$ no solutions.

they don't overlap.

$\begin{cases} x + y + z = 1 \\ x - z = 2 \end{cases}$

$x = 2 + z$ $\quad 2z + y = -1$ $\quad y = -1 - 2z$

a line built from 2 intersecting planes

$$\begin{cases} x = t+2 & \text{plane} \\ y = -1-2t & \text{plane} \\ z = t & t \in \mathbb{R} \end{cases}$$

def. The solution set of linear system $\mathcal{L}$ is the set of all its solutions.

def. If linear system $\mathcal{L}$ does not have a solution, $\mathcal{L}$ is said to be inconsistent. Otherwise it's consistent.

$$\begin{cases} x+2y+z=6 \\ \quad 2y-z=7 \\ \qquad z=3. \end{cases} \qquad \begin{cases} x=2 \\ y+6x=7 \end{cases}$$

backward substitution | forward substitution.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 1 & 5 \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 6 & 1 & 7 \end{array}\right)$$

Augmented matrices to the corresponding linear system.

def. the first non-zero entry in a row of a matrix is called the leading coefficient or pivot

def We say matrix M is in Row Echelon Form if

① For each row with a pivot, the pivot is to the right of the row above it except for the first row.

② All rows of zeros, if any, are at the bottom of the matrix.

def We say a matrix is in Reduced Row Echlon form, if

① it is in Row Echlon form

② all pivots are 1

③ For each column with a pilot, all other entries are 0s.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right) \rightarrow \text{solutions } (5, 6, 7)$$

Thm Let $\mathcal{L}$ be a system of linear equations and let $\mathcal{L}'$ be a system obtained by performing any of the following equations:

① multiplying non-zero scaler

② swap 2 equations

③ construct a new equation by adding to an equation a scaler multiple of another

$\Rightarrow$ $\mathcal{L}$ and $\mathcal{L}'$ have the same solution set.

def A $m \times n$ matrix in $\mathbb{R}$ is of the form

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{pmatrix} \quad \text{where } a_{ij} \in \mathbb{R}.$$

If $n=1$, we call this matrix a column vector, or a vector.

If $m=1$, we call it a row vector.

* sum and scaler multiplication of vectors

Let $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ $\quad c \in \mathbb{R}$

$$u + v = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \qquad cu = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}.$$

def Let $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$

$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$

then $\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n$

(scaler) $= \sum_{j=1}^{n} u_j v_j$

Product of a matrix and a vector

Let $A \in M_{m \times n}(\mathbb{R})$ all real-valued matrices of dimension $m \times n$.

Let A's row vectors be $w_1, \ldots, w_m$.

Let $\vec{v} \in \mathbb{R}^n$.

then $\underset{(m \times n)(n \times 1)}{A \vec{v}} = \begin{pmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_m & - \end{pmatrix} \vec{v} = \begin{pmatrix} \vec{w}_1 \cdot \vec{v} \\ \vdots \\ \vec{w}_m \cdot \vec{v} \end{pmatrix}$

$\underset{(m \times n)\ (n \times 1)}{A \vec{x}} = \underset{(m \times 1)}{\vec{b}}$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{cases} = \underset{A}{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{pmatrix}} \underset{\vec{x}}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \underset{\vec{b}}{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}$$

Thm Consider a system of linear equations and its $M = RREF\,(A|b)$

The system has no solutions iff M has a row of the form $[\,0 \cdots 0 \mid c\,]$, $c \neq 0$

has exactly 1 solution iff # of of pivots of $RREF\,(A)$ = # of unknowns

has ∞ly many solutions iff ... < # of unknowns

Augmented matrix for this system is $(A|b)$

- this algorithm should keep the solutions the same.
- $RREF\,(A|b)$ is unique
- $RREF\,(A|b)$ exists.

prop. ① All elementary row operations are reversible

② Elementary row operations do not change the solution of the underlying system.

Thm Consider $M_{m\times n}(\mathbb{R})$. Define $\sim$ as for $A, B \in M_{m\times n}(\mathbb{R})$, $A \sim B$ if there exists a sequence of elementary row operations that turn A to B. Then $\sim$ is an equivalence relation

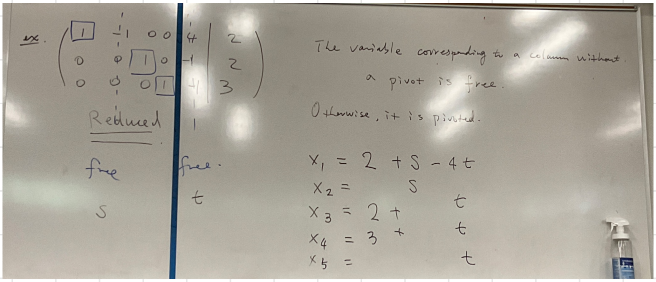
e.g.

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{array}\right) \Rightarrow \text{reduced REF}$$

free   free

The variable corresponding to a column without a pivot is free

Otherwise, it's pivoted.



$\Rightarrow$ parametric form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} -4 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} t$$

if no rows w/ no free variables ⇒ unique solution.

free variables ⇒ infinite many solutions.

e.g. $\begin{cases} x + y + 2z = -2 \\ \quad\;\; y + 3z = 7 \\ x \quad\;\; - z = -1 \end{cases} \Rightarrow$

pivot.

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 2 & -2 \\ 0 & \boxed{1} & 3 & 7 \\ 1 & 0 & -1 & -1 \end{array}\right)$$

get rid of this:

$R_3 \leftarrow R_3 - R_1$

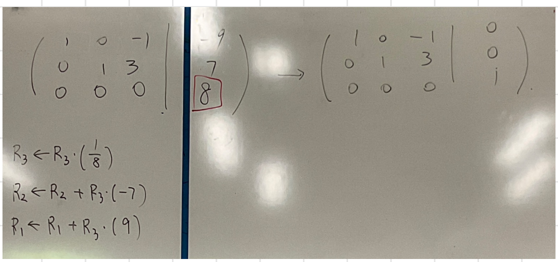
$= \;\; 0 \quad -1 \quad -3 \;|\; 1$

$R_1 \leftarrow R_1 - R_2$

$R_3 \leftarrow R_3 + R_2.$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -9 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 8 \end{array}\right)$$

inconsistency.



def System of the form $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{0}}$ is called homogeneous.

$$\begin{cases} a_{11}x_1 + \cdots a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$$

① $\Leftarrow$ $\underset{n\times 1}{\vec{0}}$ is a solution

trivial solution.

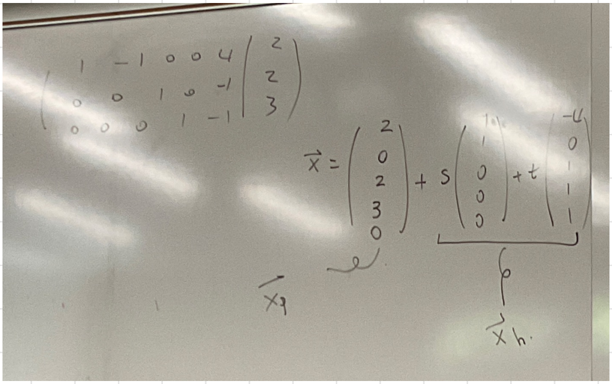
② if $A\vec{x} = \vec{0}$ has a non-trivial solution, it has infinitely many sols.

proof of ②: let $\vec{x}_0$ be a non-trivial solution,
Then $k\vec{x}_0$ for $k \in \mathbb{R}$ is a solution to $A\vec{x} = \vec{0}$
since $A(k\vec{x}_0) = k(A\vec{x}_0) = k\vec{0} = \vec{0}$.
$\therefore$ infinite many solution.

Thm: Let $A\vec{x} = \vec{b}$ be a system with a particular solution $\vec{x}_p$,
i.e. $A\vec{x}_p = \vec{b}$
Then its solution set S is
$\{\vec{x}_p + \vec{x}_h \mid A\vec{x}_h = \vec{0}\}$ S'

particular solution

homogeneous solution.



proof: $S = S'$
$S' \subseteq S$
let $\vec{x} \in S'$ (arbitrary)
$\vec{x} = \vec{x}_p + \vec{x}_h$ for some $\vec{x}_h$
$A\vec{x} = A(\vec{x}_p + \vec{x}_h) = A\vec{x}_p + A\vec{x}_h = b$
$\therefore A\vec{x} = \vec{b}$
$\Rightarrow x \in S$

$S \subseteq S'$

Let $\vec{x}$ be a solution to the system.

consider $\vec{x} = \vec{x}_p + (\vec{x} - \vec{x}_p)$

let $\vec{x}_h = (\vec{x} - \vec{x}_p)$

Note $A\vec{x}_h = A\vec{x} - A\vec{x}_p = \vec{b} - \vec{b} = 0$.

$\Rightarrow \vec{x} = \vec{x}_p + \vec{x}_h$ when $A\vec{x}_h = \vec{0}$

$\Rightarrow x \in S$

Coro : For a given system of linear equations, either it has

① exactly 1 solution

② no solution

or ③ infinitely many solution.

$(A|b)$: consistent : $\left\langle \begin{array}{l} \text{one solution} \\ \text{infinitely many solutions.} \end{array} \right.$ – free variable.

inconsistent : $[0, 0 \ldots 0 \mid c]$ $c \neq 0$.

prop. A system can either have exactly one solution or many solutions or no solution.

proof: no 4th scenario.

Suppose by contradiction: there's a finite collection of solutions. $\{S_1, S_2 \ldots S_w\}$. to $A\vec{x} = b$.

Let $S_1 \neq S_2$.

$As_1 = b$

$As_2 = b$.

$A(s_1 - s_2) = \vec{0}$ $\therefore$ $S_1 - S_2$ is a non trivial solution to $A\vec{x} = \vec{0}$

$\Rightarrow$ $A\vec{x} = \vec{0}$ has infinitely many solutions

But solution set to $A\vec{x} = \vec{b}$ is of the form

$\{\vec{x_p} + \vec{x_h} \mid \vec{x_h} \text{ is homogeneous solution to } A\vec{x} = \vec{0}\}$

$\therefore$ there will be infinite # of solutions.

Thm. Existence & Uniqueness of RREF.

Given matrix A there exists a sequence of row operations that would turn to A is RREF, and this RREF is unique.

proof.

- existence: proof by induction.

Base case: 1 column vector.

If it's a column vector of all 0's, it is in RREF.

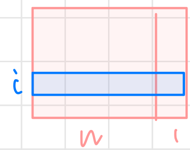
Otherwise, it has at least one entry that's not all 0's.

$\hookrightarrow$ then move to first row and turn into 1 and get rid of all other rows.

Induction Hypothesis: Suppose for matrix $A_{m\times n}$, there exists a sequence of steps to turn it into RREF.

Induction step: NTS this holds for $A_{m\times(n+1)}$.

For left part: $(A_{m\times n})$: ↑ Satisfies RREF check



Assume the $\boxed{\text{ith}}$ row of the $m\times n$ block is a zero row.
If the last column is all zeros, this row is in RREF.
Otherwise, pick the last non-zero entry in the last column in row range (i, m) and let it be the candidate for pivot.

Categorize all 2×2 RREFs:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

2 pivots　　1 pivots　　0 pivots.

Matrix · vector

$$\underset{m\times n}{A}\ \underset{n\times 1}{\vec{x}} = \begin{pmatrix} - w_1 - \\ \vdots \\ - w_m - \end{pmatrix} \vec{x} = \underset{m\times 1}{\begin{pmatrix} w_1\cdot\vec{x} \\ \vdots \\ w_m\cdot\vec{x} \end{pmatrix}}.$$

e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$

if ⇓ $= b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. ?

⇓
$$\begin{cases} x_1 + 2x_2 = 4 \\ 3x_1 + 4x_2 = 5 \end{cases}$$

parametric form : $x_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Vector equation.

$\therefore A\vec{x} = \begin{pmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_n} \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{v_1} + \cdots + x_n \vec{v_n}$

$\therefore A\vec{x} = b \implies x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_n} = b.$

how much portion of each vector to create b?

def. We say vector b is a linear definition of vectors $V_1, V_2, \ldots, V_n$ if the exist scaler $c_1, c_2, \ldots, c_n$ such that

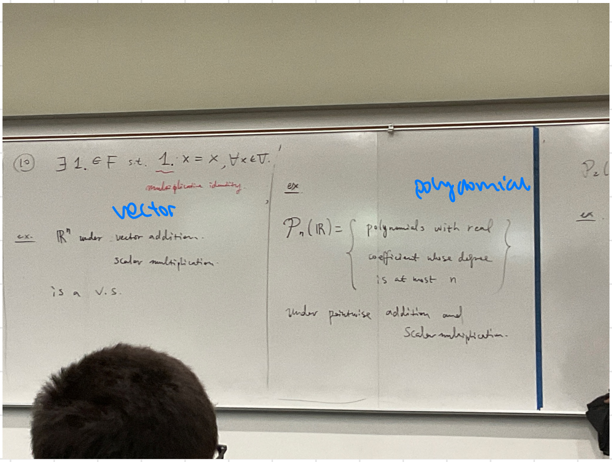
$b = c_1 v_1 + \cdots + c_n v_n$

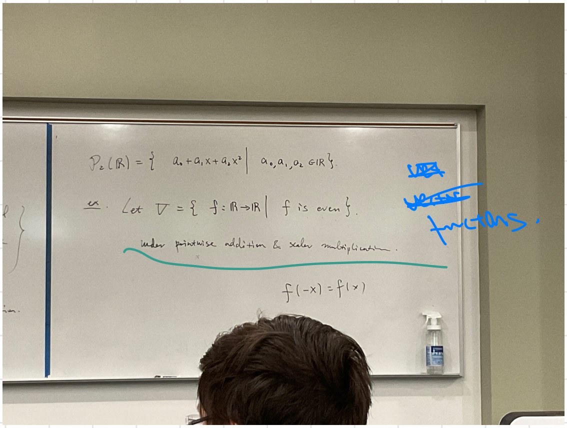
ex. Is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$?

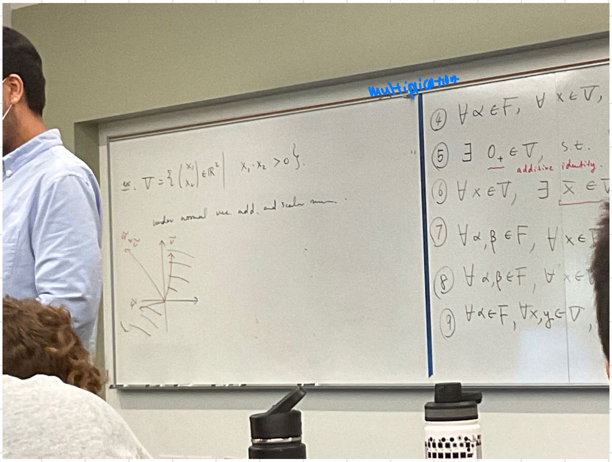
$\implies$ solve $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right) \implies (A|b)$ Aug. matrix.

We say set $\vec{V}$ together with 2 operations "+" and "$\cdot$" over field F is a vector space if

① $\forall x, y \in \vec{V}$ $x+y \in \vec{V}$ — close under addition
② $\forall x, y \in \vec{V}$, $x+y = y+x$ — commandative
③ $\forall x, y, z \in \vec{V}$, $x+y+z = x+(y+z)$. — associative
④ $\forall \alpha \in F$, $\forall x \in \vec{V}$, $\alpha x \in \vec{V}$ — close under scaler multiplication
⑤ $\exists 0_+ \in \vec{V}$, s.t. $\forall x \in \vec{V}$, $x + 0_+ = x$. — additive identity
⑥ $\forall x \in \vec{V}$, $\exists \bar{x} \in \vec{V}$ s.t. $x + \bar{x} = 0_+$ — addition inverse
⑦ $\forall \alpha, \beta \in F$, $\forall x \in \vec{V}$, $\alpha\beta x = \alpha(\beta x)$
⑧ $\forall \alpha, \beta \in F$, $\forall x \in \vec{V}$, $(\alpha+\beta)x = \alpha x + \beta x$
⑨ $\forall \alpha \in F$, $\forall x, y \in \vec{V}$, $\alpha(x+y) = \alpha x + \alpha y$
⑩ $\exists 1 \in F$ s.t. $1x = x$, $\forall x \in \vec{V}$ — multiplication identity


(10) $\exists\, 1 \in F$ s.t. $1 \cdot x = x,\ \forall x \in V$.
multiplicative identity
vector
ex. $\mathbb{R}^n$ under vector addition.
scalar multiplication.
is a v.s.
ex
polynomial
$P_n(\mathbb{R}) = \{$ polynomials with real coefficient whose degree is at most n $\}$
under pointwise addition and scalar multiplication.


$P_2(\mathbb{R}) = \{ a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$.
ex. Let $V = \{ f: \mathbb{R} \to \mathbb{R} \mid f$ is even $\}$.
under pointwise addition & scalar multiplication.
$f(-x) = f(x)$
functions.


multiplication
ex. $V = \{ \binom{x_1}{x_2} \in \mathbb{R}^2 \mid x_1 \cdot x_2 > 0 \}$.
under normal vec. add. and scalar mu.
(4) $\forall \alpha \in F,\ \forall x \in V,$
(5) $\exists\, 0_+ \in V,$ s.t.
additive identity
(6) $\forall x \in V,\ \exists\, \bar{x} \in V$
(7) $\forall \alpha, \beta \in F,\ \forall x \in$
(8) $\forall \alpha, \beta \in F,\ \forall x \in$
(9) $\forall \alpha \in F,\ \forall x, y \in V,$

Facts: ① 0_+ is unique

② $\vec{x}$, the additive inverse is unique $\forall x \in \vec{V}$

③ $0\vec{v} = \vec{0}$

④ $a\vec{0} = \vec{0}$.

proof of ①: suppose they aren't unique ⇒ show equal.

Let 0_+, $0_+'$ be two additive identity:

$$\underbrace{0_+ = 0_+}_{\text{additive identity}} + \underbrace{0_+' = 0_+'}_{\text{additive identity.}}$$

$\mathbb{F}$: $\mathbb{R}$ real numbers

$(\bar{V}, +_{\bar{V}}, \cdot_{\mathbb{F},\bar{V}})$.

Scaler multiplication

① ~ ⑩ Axioms.

Let $\bar{V}$ be a vector space, Then a subset $\bar{u} \subseteq \bar{V}$ is a subspace if it itself is a vector space.

A subset $\bar{u} \subseteq \bar{V}$ is a subspace if

① $\bar{u}$ is not empty

② $\bar{u}$ is closed under addition $\left(\begin{array}{c}\forall x, y \in \bar{u} \\ x+y \in \bar{u}\end{array}\right)$

③ $\bar{u}$ is closed under scaler multiplication. $\left(\begin{array}{c}\forall x \in \bar{u} \quad \forall \alpha \in \mathbb{F} \\ \alpha x \in \bar{u}\end{array}\right)$

e.g. $(\mathbb{R}^2, +, \cdot)$

$\bar{V} = \mathbb{R}^2 \qquad \bar{u} = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 + 3x_2 = 0\}$

① note $\vec{0} = \begin{pmatrix}0\\0\end{pmatrix} \in \vec{u}$

② Let $\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}y_1\\y_2\end{pmatrix} \in \bar{u}$ $\quad 2x_1 + 3x_2 = 0$, $\quad 2y_1 + 3y_2 = 0$ prove,

to prove: $\begin{pmatrix}x_1+y_1\\x_2+y_2\end{pmatrix} \in u$ $\qquad 2(x_1+y_1) + 3(x_2+y_2) = 0$?

$\therefore$ ② ✓.

③ Let $\begin{pmatrix}x_1\\x_2\end{pmatrix} \in \bar{u} \qquad \alpha \in \mathbb{R}$

i.e. $2x_1 + 3x_2 = 0$.

$2(\alpha x_1) + 3(\alpha x_2) = 0$.

$\begin{pmatrix}\alpha x_1\\\alpha x_2\end{pmatrix} = \alpha \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \bar{u}$

$\bar{V} = \mathbb{R}^3$, $\bar{U} \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid \begin{matrix} 3x_1 + 2x_2 = 0 \\ 4x_2 - 3x_3 = 0 \end{matrix} \}$

is $\bar{U}$ a subspace in $\mathbb{R}^3$

$\bar{V} = \mathbb{R}^n$, $\bar{U} = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}, A \in M_{m \times n}(\mathbb{R}) \}$.

① not empty because the trivial solution is one solution.

② Let $\vec{u}$ and $\vec{v}$ be 2 arbitary elements in $\bar{U}$

$$\left. \begin{matrix} A\vec{u} = \vec{0} \\ A\vec{v} = \vec{0}. \end{matrix} \right\} \Rightarrow A(\vec{u} + \vec{v}) = \vec{0}$$

$\therefore \vec{u} + \vec{v} \in \bar{U}$

③ Let $\vec{u} \in \bar{U}$, $\alpha \in \mathbb{R}$

i.e. $A\vec{u} = \vec{0}$

$\alpha A\vec{u} = \vec{0}$

$\therefore A(\alpha\vec{u}) = \vec{0} \Rightarrow \alpha\vec{u} \in \bar{U}$

def Let $A \in M_{m \times n}(\mathbb{R})$ The null space of A is

$Null(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$. $Ker(A)$.

ex. $\bar{V} = \{ f : [0,1] \to \mathbb{R} \}$.

$\bar{U} = \{ f \in \bar{V} \mid f$ is twice differentiable and $2f'' + 3f' - 4f = 0 \}$.

① $f_0 : [0:1] \to \mathbb{R}$

$x \mapsto 0$

$\therefore f_0$ is a function. $\therefore f_0 \in \bar{U}$

②③ Let f, g be arbituary elements in $\bar{U}$

and $\alpha, \beta \in \mathbb{R}$.

check $\alpha f + \beta g \in \bar{U}$

$2f'' + 3f' - 4f = 0$ ... ⓐ.

$2g'' + 3g' - 4g = 0$ ... ⓑ.

$\alpha ⓐ + \beta ⓑ = 2\alpha f'' + 3\alpha f' - 4\alpha f + 2\beta g'' + 3\beta g' - 4\beta g = 0$

$= 2(\alpha f'' + \beta g'') + 3(\alpha f' + \beta g') - 4(\alpha f + \beta g) = 0$

$= 2(\alpha f+\beta g)'' + 3(\alpha f+\beta g)' - 4(\alpha f+\beta g) = 0$

$\Rightarrow \therefore \alpha f + \beta g \in \bar{u}$

def. a function $p: \mathbb{F} \Rightarrow \mathbb{F}$ is a polynomial with coefficients in $\mathbb{F}$ if
$p(x) = a_0 + a_1 x + \cdots + a_n x^n$

def. a polynomial $a_0 + a_1 x + \cdots a_n x^n$ is of degree n if $a_n \neq 0$.

$P_2(\mathbb{R}) = \{ a_0 + a_1 x + a_2 x^2 \mid a_0 . a_1 , a_2 \in \mathbb{R} \}$. $\mathbb{R}_2[x]$

def. Let $\bar{u}, \bar{w}$ be two subspaces of $\bar{V}$.
Then the sum of $\bar{u}$ and $\bar{w}$ is
$\bar{u} + \bar{w} = \{ \vec{u} + \vec{w} \mid \vec{u} \in \bar{u}, \vec{w} \in \bar{w} \}$

VS ?

def A linear combination of vectors $v_1 . v_2, \cdots v_n$ is of the form $c_1 v_1 + \cdots + c_n v_n$ for some scalers $c_1, \ldots, c_n$.

def Let $S = \{ v_1, \cdots v_n \} \subseteq \bar{V}$
$\text{span}(S) = \{ c_1 v_1 + \cdots c_n v_n \mid c_1, \cdots c_n \in \mathbb{R} \}$
span(S) is the collection of all its linear combination

def If $\text{span}(S) = \bar{u}$, we say S spans $\bar{u}$

e.x. $\text{span}\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix} \right) \stackrel{?}{=} \mathbb{R}^2$

$\subseteq$ ✓

$\supseteq$: Let $\begin{pmatrix}x\\y\end{pmatrix}$ be an arbitary element in $\mathbb{R}^2$
We need to check if $\begin{pmatrix}x\\y\end{pmatrix}$ is a linear combination of $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}$

$\Downarrow$ $\stackrel{?}{\exists}$ s, t s.t. $\begin{pmatrix}x\\y\end{pmatrix} = s\begin{pmatrix}1\\1\end{pmatrix} + t\begin{pmatrix}2\\3\end{pmatrix}$

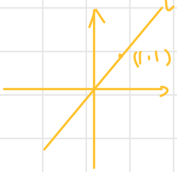
$\Downarrow$ have solution? $\left(\begin{array}{cc|c} 1 & 2 & x \\ 1 & 3 & y \end{array}\right)$

$\downarrow$ RREF. $\left(\begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & y-x \end{array}\right)$

$\Rightarrow$ has exactly one solution.

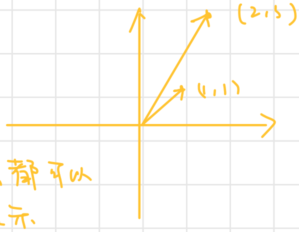
$\mathbb{R}^2$. Span $\begin{pmatrix}1\\1\end{pmatrix}$ or span $\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right)$

$\{t\begin{pmatrix}1\\1\end{pmatrix} \mid t \in \mathbb{R}\}$



span $\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\right)$.

$= \mathbb{R}^2$



因为 coordinate 上任意一点都可以用这两个 vectors 的和表示.

$\text{span}(1, x, x^2) = P_2(\mathbb{R})$

LHS: $\{c_1 + c_2 x + c_3 x^2 \mid c_1, c_2, c_3 \in \mathbb{R}\}$

RHS: $\{a_0 + a_1 x + a_2 x^2 \mid a_1, a_2, a_3 \in \mathbb{R}\}$

$\text{span}(x, x-2, (x-2)^2) \overset{?}{=} P_2(\mathbb{R})$

prop. Let $v_1, \cdots, v_n$ be elements from V

span $(v_1, \ldots, v_n)$ is the smallest subspace that contains all $v_1, \cdots, v_n$.

This is the subspace "generated" by $v_1, \ldots, v_n$.

pf. ① it's a subspace

(1.1) not empty:

consider $0v_1 + \cdots + 0v_n = \vec{0} \in \text{span}(v_1, \cdots, v_n)$

(1.2) x, y arbitrary $\cdot\cdot$ in span $(v_1, \cdots, v_n)$

$x = \sum_{i=1}^{n} c_i v_i$ for some $\{c_i\}_{i=1}^{n} \subseteq \mathbb{R}$

$y = \sum_{i=1}^{n} d_i v_i$ for ... $\{d_i\}_{i=1}^{n} \subseteq \mathbb{R}$

consider $x+y = \sum_{i=1}^{n} (c_i + d_i) v_i$

$\Rightarrow (x+y) \in \text{span}(v_1, \cdots, v_n)$

(1.3) let x be an arbitrary element in $\text{span}(v_1, \cdots, v_n)$

i.e. $x = \sum_{i=1}^{n} c_i v_i$ for some $\{c_i\}_{i=1}^{n} \subseteq \mathbb{R}$

let $\alpha \in \mathbb{R}$, arbitrary.

consider $\alpha x = \alpha \sum_{i=1}^{n} c_i v_i$

$= \sum_{i=1}^{n} (\alpha c_i) v_i$

$\therefore \alpha x \in \text{span}(v_1, \cdots, v_n)$

(2) Smallest.

We need to show: for every subspace $\bar{u}$ that contains $\{v_1, \cdots, v_n\}$, $\bar{u}$ also contains $\text{span}(v_1, \cdots, v_n)$

This is true because $\bar{u}$ needs to contain all linear combination of $v_1, \cdots, v_n \Rightarrow \bar{u}$ contains $\text{span}(v_1, \cdots, v_n)$.

---

def. Vectors $v_1, \cdots, v_n$ are said to be $\boxed{\text{linearly independent}}$ if whenever vector equation $x_1 v_1 + \cdots + x_n v_n = \vec{0}$, $x_1 = \cdots = x_n = 0$

= If $x_1 v_1 + \cdots + x_n v_n = \vec{0}$, then $x_1 = x_2 = \cdots = x_n = 0$

$\Leftrightarrow$ the system $\left(\begin{array}{ccc|c} | & & | & 0 \\ v_1 & \cdots & v_n & \vdots \\ | & & | & 0 \end{array}\right)$

has only the trivial solution.

$\Leftrightarrow$ REF or RREF has n pivots.

def. We say vectors $v_1, \dots v_n$ are linearly independent if $x_1 v_1 + \dots + x_n v_n = \vec{0}$ implies $x_1 = \dots = x_n = 0$

e.x. Consider $\bar{V} = \{ f : \mathbb{R} \to \mathbb{R} \}$ and $\{ \cos 2x, \sin^2 x, \cos^2 x \}$

dependent, since $(-1)\sin^2 x + 1 \cos^2 x + (-1)\cos 2x = 0$
$\therefore$ $\vec{0}$ doesn't come from a non-trivial solution.

e.x. $\bar{V} = \mathbb{R}^4$. $v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\{ v_1, v_2, v_3 \}$ linearly dependent?

consider $x_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

check $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right)$. REF $\Rightarrow$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

3 pivots. $\therefore$ exactly 1 solution
$\therefore$ trivial solution
$\therefore$ independent.

Any collection of vectors that includes $\vec{0}$ is linearly dependent.

$\bar{V} = \{ f : \mathbb{R} \to \mathbb{R} \}$ $S = \{ e^x, x, x^2 \}$.

consider $c_1 e^x + c_2 x + c_3 x^2 = 0$, does it mean $c_1 = c_2 = c_3 = 0$?
Note this equation needs to hold for $\forall x \in \mathbb{R}$; which is only possible when $c_1 = c_2 = c_3 = 0$. $\therefore$ linearly independent.

Theorum: let $v_1, \dots, v_n$ be vectors in $\bar{V}$. If $\text{span}(v_1, \dots v_n) = \bar{V}$, and $\{w_1, \dots, w_k\}$ is a linearly independent set in $\bar{V}$, then $k \leq n$.

Subspace of $\mathbb{R}^2$: (1) $\text{span}(\vec{v})$ when $\vec{v} \neq \vec{0}$ ↗ a line that passes $(0,0)$.

(2) $\mathbb{R}^2$

(3) $\{\vec{0}\}$ origin.

↑ span by empty set ϕ.

proof: consider w_1 which is in $\text{span}(v_1, \cdots v_n)$.

This means $w_1 = \sum_{i=1}^{n} c_i v_i$ for some $\{c_i\}_{i=1}^{n}$.

$w_1 \neq \vec{0}$, since $\{w_1, \dots, w_n\}$ is linearly independent.

This implies at least one of $\{c_i\}_{i=1}^{n}$ is not zero.

Relabeling the matrices, call the non-zero entry c_1.

$$w_1 = c_1 v_1 + \dots + c_n v_n$$

$$v_1 = \frac{[w_1 - (c_2 v_2 + \dots + c_n v_n)]}{c_1}$$

$\Rightarrow \text{span}(w_1, v_2, v_3, \dots v_n) = \bar{V}$ $\Rightarrow$ still linearly independent.

Now consider w_2

Note $w_2 \in \bar{V}$, which means $w_2 = \alpha_1 w_1 + c_2 v_2 + \dots + c_2 v_n$.

At least one of $\{c_i\}_{i=2}^{n}$ is not zero, otherwise $w_2 = \alpha_1 w_1$, which is impossible. Relabeling, call the non-zero entry c_2.

$$\Rightarrow w_2 = \alpha_1 w_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n.$$

$$v_2 = \frac{\alpha_1 w_1 - (c_3 v_3 + \dots + c_n v_n)}{c_2} = \bar{V}$$

If $k > n$, then replacing the previous steps we have $w_k \in \text{span}(w_1, w_2, \dots, w_n)$ which is not possible.

def. We say vector space $\bar{V}$ is finite dimension if $\bar{V}$ is spanned by a finite collection of vectors.
Otherwise it's infinite dimensional.

e.g. $\mathbb{R}^2$. can be $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

$P_2(\mathbb{R})$ $\{x, x, x^2\}$ spans $P_2(\mathbb{R})$

finite

$P(\mathbb{R}) := \left\{ \begin{matrix} \text{polynomial} \\ \text{with real} \\ \text{coefficient} \end{matrix} \right\}$.

$\{(x_1, x_2, \cdots) \mid x_1, x_2, \cdots \in \mathbb{R}\}$

infinite.

def We say collection of vectors $\mathcal{B} = \{v_1, \cdots, v_n\} \subseteq \bar{V}$ is a basis of $\bar{V}$ if

① $\mathcal{B}$ is linearly independent

② $\mathcal{B}$ spans $\bar{V}$

e.x. a basis for $\mathbb{R}^n$ is

$\mathcal{B} = \{e_1, \cdots, e_n\}$

when e_i, $\forall i \in \{1, \cdots, n\}$ is of the form

$$e_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \cdots \text{ith entry.}$$

$\Leftarrow$ standard basis for $\mathbb{R}^n$

e.x. basis of Null(A). when $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

$\text{Null}(A) := \{\vec{x} \in \mathbb{R}^2 \mid A\vec{x} = \vec{0}\}$ $\therefore$ Solve $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right)$

$$\begin{cases} x_1 = -t \\ x_2 = t \end{cases} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \text{span}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\Rightarrow \text{Null}(A) = \underbrace{\text{span}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$.

① linearly independent.
② spans $\bar{V}$

e.x. find basis for $\{ p(x) \in P_2(\mathbb{R}) \mid p'(1) = 0 \}$

Let $a_0 + a_1 x + a_2 x^2$ be an arbitrary element in $\bar{u}$.
This means $(a_0 + a_1 x + a_2 x^2)' \Big|_{x=1}$.

$$= a_1 + 2a_2 x \Big|_{x=1} = 0.$$

$$a_1 + 2a_2 = 0$$

This means $a_0 + a_1 x + a_2 x^2 = a_0 + a_1 x + \left(-\frac{a_1}{2}\right) x^2$
$= a_0 + \left(x - \frac{1}{2} x^2\right) a_1$

which implies $\text{span}\left(1, x - \frac{1}{2} x^2\right) = \bar{u}$

Note that $\{ 1, x - \frac{1}{2} x^2 \}$ is also linearly independent.

Theorem: If $\bar{V}$ is finite dimensional, then all basis of $\bar{V}$ are of the same size.

proof: Let B_1, B_2 be two bases of $\bar{V}$.
B_1 is linearly independent and B_2 spans $\bar{V}$ $\Rightarrow |B_1| \le |B_2|$
B_2 $\qquad$ B_1 $\qquad$ $|B_2| \le |B_1|$
$\Rightarrow |B_1| = |B_2|$

def If B is a basis for finite dimensional vector space $\bar{V}$, then dimension of $\bar{V}$ is $|B|$, denoted by $\dim|\bar{V}|$.

We say vectors $\{v_1, \dots v_n\} \subseteq \bar{V}$ form a basis for $\bar{V}$ if ① $\text{span}(v_1, \dots, v_n) = \bar{V}$
② $\{v_1, \dots, v_n\}$ is linearly independent.

alt def. We say $\{v_1, \dots, v_n\} \subseteq \bar{V}$ is a basis for $\bar{V}$ if every element $v \in \bar{V}$ can be uniquely written as a linear combination of $\{v_1, \dots, v_n\}$.

e.x. prove they are equivalent.

$\Uparrow$ Since every element can be written as a linear combination of $v_1, \dots, v_n$, this shows $v_1, \dots, v_n$ span $\bar{V}$

Note that $\vec{0} = 0v_1 + \cdots + 0v_n$ and by uniqueness, the trivial solution is the only solution to $x_1v_1 + \cdots + x_nv_n = \vec{0}$
$\Rightarrow v_1, \dots, v_n$ are linearly independent.

$\Downarrow$ l.c. part is immediate, we need to show uniqueness.
Let $\vec{v} = \sum_{i=1}^{n} c_i v_i \cdots ①$
$\vec{v} = \sum_{i=1}^{n} d_i v_i \cdots ②$
are two different linearly combination sets.

$① - ② = \vec{0} = \vec{v} - \vec{v} = \sum_{i=1}^{n} (c_i - d_i) v_i$

$\therefore$ all coefficients should be zero.
$(c_i - d_i) = 0 \quad \forall i \in \{1, \dots, n\}$.
$\therefore$ they are the same linear combination
$\therefore$ this representation is unique.

prop. Let $v_1, \dots, v_n$ be vectors from v.s. $\bar{V}$
$\mathrm{span}(v_1, \dots, v_n) = \mathrm{span}(v_1, \dots, v_n, w)$ iff $w \in \mathrm{span}(v_1, \dots, v_n)$

Cor. Let $\bar{V}$ be a finite-dimensional vector space. If set $S \subseteq \bar{V}$ is a spanning set, then S can be reduced to a basis.

$$\Big\{ \underbrace{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}}_{\substack{\text{linearly}\\\text{independent.}\\ \therefore\ \text{keep.}}}, \cancel{\begin{pmatrix}1\\1\end{pmatrix}}, \cancel{\begin{pmatrix}2\\3\end{pmatrix}} \Big\} \subseteq R^2$$

also l.d. $\therefore$ get rid of.

linearly dependent $\therefore$ X. get rid of it.

$\therefore \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\} \subseteq R^2$ is the basis.

Cor. Let V be a finite-dimensional vector space, Set $S \subseteq \bar{V}$ is a linearly independent
Then S can be extended to a basis of $\bar{V}$

use basis to find coordinate.

def. Let $\bar{V}$ be a vector space A sequence of vectors $v_1, \dots, v_n$ in $\bar{V}$ is an ordered basis if
① $v_1, \dots, v_n$ spans $\bar{V}$.
② $v_1, \dots, v_n$ are linearly independent.

Notation : $\langle v_1, \dots, v_n \rangle$

e.x. Consider $\bar{V} = \mathbb{R}^2$

$\mathcal{B}_1 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$

$\mathcal{B}_2 = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$

$\mathcal{B}_3 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$

we represent the same thing in respect with $\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$

def. Let $\bar{V}$ be a vector space with an ordered basis $\mathcal{B} = \langle b_1, \dots, b_n \rangle$

If $v \in \bar{V}$ can be written as

$$v = c_1 b_1 + \dots + c_n b_n$$

then $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is called v's representation with respect to basis $\mathcal{B}$ or v's coordinate vector with respect to basis $\mathcal{B}$.

$$\mathrm{Rep}_{\mathcal{B}}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \text{ or } [v]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

e.x. Consider $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \mathbb{R}^2$

Determine $[\vec{v}]_{\mathcal{B}_1}$, $[\vec{v}]_{\mathcal{B}_2}$, $[\vec{v}]_{\mathcal{B}_3}$.

$$\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]_{\mathcal{B}_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]_{\mathcal{B}_2} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]_{\mathcal{B}_3} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

e.x. $V = P_2(\mathbb{R})$ $\mathcal{B}_1 = \langle 1, x, x^2 \rangle$

$\mathcal{B}_2 = \langle 1, x-1, (x-1)^2 \rangle$

$p(x) = 9x^2 + 4x + 3.$

$[p(x)]_{\mathcal{B}_1} = \begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix}$ $\quad$ $[p(x)]_{\mathcal{B}_2} = \begin{pmatrix} 16 \\ 22 \\ 9 \end{pmatrix}$

$c_1 \cdot 1 + c_2 \cdot (x-1) + c_3 \cdot (x-1)^2$
$= 9x^2 + 4x + 3.$

$c_1 + c_2 x - c_2 + c_3 x^2 - 2c_3 x + c_3$
$= 9x^2 + 4x + 3.$
$= c_2 x^2 + ((c_2 - 2c_3)x) + (c_1 - c_2 + c_3)$

$c_3 = 9$ $\quad c_2 = 22 \quad c_1 = 16$

def Let $A \in M_{m \times n}(\mathbb{R})$ Then
A can be seen as $\begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$

$a_1, \ldots, a_n$ column vectors of A.
The column space of A is span $(a_1, \ldots, a_2)$
Notation $Col(A)$.

$A\vec{v} = \vec{b} \iff \vec{b} \in Col(A)$ | again
$\iff \vec{b} \in Span(a_1, \ldots, a_n)$
$\iff \vec{b}$ is a linear combination of $a_1, \ldots, a_n$
$\iff \vec{v}$ is a solution to $x_1 a_1 + \cdots + x_n a_n = \vec{b}$

if $a_1, \ldots, a_n$ are linearly independent, then $\langle a_1, \ldots, a_n \rangle$ form a ordered basis for $col(A)$.
This means $\vec{v}$ is the coordinate vector for $\vec{b}$ with respect to $\langle a_1, \ldots, a_n \rangle$

Algorithm of finding a basis?

$$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$$

Method 1:

$$A^T = \begin{pmatrix} — & a_1^T & — \\ & \vdots & \\ — & a_n^T & — \end{pmatrix}$$

Row reduce A^T to either REF / RREF

$\downarrow$

$$B^T = \begin{pmatrix} — & b_1^T & — \\ & \vdots & \\ — & b_n^T & — \end{pmatrix}$$

let I be all numbers s.t. b_i^T is pivoted row

$I := \{ i \in \mathbb{N} \mid b_i^T \text{ is a pivoted row} \}$

Then $\{ b_i \}_{i \in I}$ form a basis for col(A).

Method 2:

$$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} \xRightarrow{\text{REF} \,/\, \text{RREF}} B = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}$$

$I := \{ i \in \mathbb{N} \mid b_i \text{ is a pivoted column} \}$

Then $\{a_i\}_{i \in I}$ form a basis for column(A).

prop. let A be a Matrix $_{m \times n}$. then the pivoted colums of A form a basis of col(A).

proof. let $A, B \in M_{(m \times n)}$ s.t. B is RREF(A).

$$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}, \quad B = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}.$$

Let $I := \{ i \in \mathbb{N} \mid b_i \text{ is a pivoted column} \}$

NTS. $\{a_i\}_{i \in I}$ form a basis for col(A).

By definition of RREF: $\{b_i\}_{i \in I}$ is a subset of the standard basis of $\mathbb{R}^n$.

$\therefore$ $\{b_i\}_{i \in I}$ is a linearly independent set.

Claim: $\{a_i\}$ is also linearly independent.

Consider $A\vec{x} = \vec{0}$, $B\vec{x} = \vec{0}$

Note they share the same solution set.

$$a_1x_1 + \cdots + a_nx_n = \vec{0}$$

$$b_1x_1 + \cdots + b_nx_n = \vec{0}$$

This means any linear relations between $\{a_1, \ldots, a_n\}$ is preserved for $\{b_1, \ldots, b_n\}$ and vice versa.

(can be translated to each other..)

↓

all the a_is are linearly independent.

Let a_j be a non-pivoted column of A, $(j \notin I.)$

Consider b_j, which doesn't have a pivot

If there are m pivoted columns to the left of b_j,

then they are exactly $e_1, \ldots, e_m$

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ & & 0 \\ & & \vdots \end{matrix}$$

non-pivoted column.

2.

This means b_j is a linear combination of its column to the left.

$\Rightarrow$ a_j can also be written as a linear combination of its previous pivoted columns.

$\Rightarrow$ All non-pivoted columns are linear combination of the pivoted columns i.e. $\mathrm{span}(a_1, \ldots, a_n) = \mathrm{Span}\{\{a_i\}_{i \in I}\}$

$\Rightarrow \mathrm{Span}(\{a_i\}_{i \in I}) = \mathrm{Col}(A)$.